Classifying classes of structures in model theory

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Overview of the talk

We shall try to explain a new and surprising result, the Recounting Theorem, which strongly indicates that there is more to be discovered about so-called dependent classes; and we introduce some basic definitions, results and themes of model theory needed to explain it.

We shall not mention history nor any of the illustrious researchers who have contributed. Apart from the recounting theorem, all results are quite old.

The talk will have two rounds:

- First round: A presentation of the result without details.
- Intermezzo: Some very basic notions of first order logic.
- Second round: Stability and Dependence again.

So let us start round one.

Classes of Structures

Group theory investigates groups. Model theory investigates classes of structures, such as:

- K_{ring}, the class of rings,
- K_{field} the class of fields,
- K_{group}, the class of groups.

Central prototypical examples (which will appear later) are:

1. K_{dlo}, the class of dense linear orders without endpoints.
2. K_{rg}, the class of random graphs. (I.e., the graphs such that any two disjoint finite sets A, B of vertices can be separated by a vertex x, i.e., x is connected to every member of A and not connected to any member of B.)

Note that in this talk we will only be interested in infinite structures.

Dividing Lines

Meta-Question

Can we find "useful/strong" dividing lines for the family of "reasonable" classes?

Our expectations:

- A high class has to contain many members (up to isomorphism), or complicated ones, or members which are rigid in suitable sense;
- On the low side, we can prove strong negations of these properties (i.e., few members, not complicated)
  moreover we should understand the members of this class, they have a structure theory or classification (such as dimension: e.g., algebraically closed fields are completely determined by giving the size of a transcendence basis.)

A priori it is not clear that such dividing lines exist.
This very general setup covers a lot of ground, but it seems that we can say very little. Anyhow, we restrict ourselves to so-called elementary classes, still a very comprehensive context:

Elementary classes

We concentrate exclusively on elementary classes, which are classes defined by first order logic, a formal language in which we can quantify over elements of our structures, but not over any of the following: subsets, infinite sequences, natural numbers. Also, sentences have to be of finite length.

For example, when talking about fields, we can say "the characteristic is 7", but not "the characteristic is > 0."

Definition

- The theory of a structure M is the set of first order sentences that are true in M.
- The elementary class K_M of a structure M consists of all structures N that have the same theory as M.

In this talk, we will only study elementary classes. (We have much to say on other situations, but not here and now.)

Examples of elementary classes

- In the language of orders: The order $\mathbb{Q}$ defines an elementary class $K_{\mathrm{dlo}} := K_{\mathbb{Q}}$ called "dense linear orders".
  This class also contains the order $\mathbb{R}$, i.e., $K_{\mathbb{R}} = K_{\mathrm{dlo}}$.
- In the language of fields: $\mathbb{C}$ defines the elementary class $K_{\mathbb{C}}$, which is the class of algebraically closed fields of characteristic 0.
  This class also contains the field of algebraic complex numbers.

Stability

Definition

We say K is stable if it is

- neither as bad (i.e., as complicated) as K_{rg} (random graphs)
- nor as bad as K_{dlo} (dense linear orders).

A structure M is stable iff K_M is.

We know that this is in fact an excellent and central dividing line:

- If an (elementary) class K is unstable, then it is complicated and has "non-structure" by various yardsticks.
- If K is stable, we have some simple structure (similar to dimension, and a kind of free amalgamation, called non-forking).

Examples:

- The field $\mathbb{C}$ is stable.
- The linear order $\mathbb{Q}$ is unstable.
- The ring $\mathbb{Z}$ (i.e., number theory) is unstable.

Counting types

One reason why stability is such a good dividing line, is that it is connected with counting so-called complete types. More on types later, for now just an example:
Every real $r \in \mathbb{R} \setminus \mathbb{Q}$ defines a type over the dense linear order $\mathbb{Q}$:
Basically, the type consists of the statements "$x < q$" for $q > r$, as well as "$x > q$" for $q < r$.
There are other types over $\mathbb{Q}$, such as "$x > q$" for all q (i.e., $+\infty$).
If we define types appropriately, we get:

Theorem

K is stable iff for every $M \in K$ there are "few" complete types over M.

(Few means: at most $||M||^{\aleph_0}$ many. We write $||M||$ for the cardinality of the universe of the structure M, and $||M||^{\aleph_0}$ for the number of countable subsets of this universe.)

Dependent classes

As much as the stable/unstable dividing line is great, we would like the positive (or: "low") side to cover more ground. This motivates

Definition

K is dependent, if it is not as bad/complicated as K_{rg} (random graphs).

So dependent meets "half the requirement for being stable".
On this family we know much less, still

Thesis

The dividing line dependent/independent is important.

The theorem promised in the beginning says:

The Recounting Theorem

If we count the complete types suitably (i.e., count them modulo some equivalence), then the dependent classes K are exactly the ones with few complete types over nice enough $M \in K$.

An example: fields

Question

For which fields F is their elementary class K_F stable? dependent?

Stable fields include:

- For any p, the class of algebraically closed fields of characteristic p.
- For $p > 0$, the class of separably closed fields of characteristic p.
- Any finite field (but this is dull, since the elementary class only has one element modulo isomorphism).

Are there any more stable fields? We do not know.
Is the family of dependent fields significantly wider family of classes?
Dependent fields include (in addition to all the stable ones) the following unstable fields:

- The reals,
- Many formal power series fields,
- the p-adics.

Applications

Disclaimer: For me, applications are not the aim, or "the test" for the merits of a theory, but naturally applications are expected; so here is a

Fact

There are substantial applications.

E.g., see the work on the Mordell-Lang conjecture.

Thesis

For elementary classes K,
looking at the behaviour of structures M in K of size κ will help us find such dividing lines.
(A typical situation is that all large enough κ behave uniformly, where "large enough" may mean "much bigger than continuum $= |\mathbb{R}| = 2^{\aleph_0}$".)
This often turns out to be helpful — even for those who (unlike me) have no interest in such large cardinals for their own sake.

Other examples of dividing lines that are easily explained:

Categoricity: For every (elementary) class K, one of the following occurs:

1. For every uncountable cardinal λ there is (modulo isomorphism) exactly one structure M in K which has cardinality λ.
2. For no uncountable cardinal λ does the above hold.

Main gap For every (elementary) class K, one of the following occurs:

1. for every cardinal λ, the number of structures in K which have cardinality λ is maximal, i.e., there are 2^λ many.
2. For every cardinal $\lambda = \aleph_\alpha$, the number above is bounded by a fixed function of α (which is much smaller than 2^λ for "typical" λ).

Such uniform dichotomic behaviour indicate it is a real dividing line.
Usually, proving there are few structures indicates that we can understand them.

Intermezzo

The first order language exemplified on fields. Some basic notions of first order logic: elementary substructures and types.

First order language for fields: definable sets

Given a field M we consider the naturally defined subsets of M, and more generally of M^n.

- Most widely used: The set of those $\bar{x}$ solving an equation $\sigma(\bar{x}) = 0$ where σ is a polynomial with integer coefficients.
- But we may also look at the set of solutions of k many equations, the set of non-solutions and, e.g., the set of $\bar{y}$ for which the following equation is solvable: $\sigma_1(\bar{x}, \bar{y}) = 0$.
- Generally, the family of first order definitions $\varphi = \varphi(\bar{x})$ is the closure of the family of "roots of polynomials" by intersection (of two), complement and projections (i.e., the set on n-tuples which can be lengthen to an $n + m$-tuples satisfying a first order definition φ).
- Again, note that conditions speaking about infinite sequences and about "for every subset of M" are not allowed.

This family has better closure properties than, say, the roots of polynomials; hence sometimes you better investigate it, even if you are interested just in polynomials.

The elementary class of a field

Definition

Let M be a field.

1. $\varphi[M]$ is the set of tuples (of appropriate length) satisfying φ in M.
2. The elementary class K_M is the class of the fields N such that for every φ we have $\varphi[N] = \emptyset$ iff $\varphi[M] = \emptyset$.
3. $M \prec N$, i.e., M is elementary substructure of N, iff M is a subfield of N, and $\varphi[M] = \varphi[N] \restriction M$ for every relevant φ.

It is easy to see that $M \prec N$ implies $N \in K_M$.

Types

Here is one definition for **S**(M), the family of complete types of M.

First, we define "f is an elementary embedding of M into N" by : f is an isomorphism for M onto some M' such that $M' \prec N$.

Definition

S(M) consists of all (a, M, N) with $M \prec N$ and $a \in N$, where we identify (a_1, M, N_1) and (a_2, M, N_2) iff there is a mapping fixing M which takes $a_1 \mapsto a_2$.

In more detail: if there are M^+, f_1, f_2 such that

- $M \prec M^+$,
- f_1 is an elementary embedding of N_1 into M^+ over M
- f_2 is an elementary embedding of N_2 into M^+ over M
- and $f_1(a_1) = f_2(a_2)$.

We write $\mathrm{tp}(a, M, N)$ for the equivalence class of (a, M, N) and call it the complete type of a over M.

Examples of types over fields and other structures

Let $M \subseteq N$ be two structures, and let $a \in N$.
Generally, the type of a over M, $tp(a, M, N)$, is characterized by the set of those $\varphi(x)$ with parameters in M such that $\varphi(a)$ holds in N.
Let us ignore the types of elements $a \in M$, as they are easy to understand; so assume $a \in N \setminus M$.

- Assume that $M \subseteq N$ are algebraically closed fields.
  Then all elements $b \in N \setminus M$ have the same type, so there is only one nontrivial type over M.
- If M is a dense linear order, there are always many nontrivial types: e.g., every irrational number determines a different type over $\mathbb{Q}$.
- Similarly for random graphs; every subset A of a random graph M determines a type (describing a vertex connected to all vertices in A, and to no other vertex in M).
- Consider the structure $(\mathbb{N}, +, \cdot, 0, 1)$; every partition $A \dot\cup B$ of the prime numbers determines the type $\{p|x : p \in A\} \cup \{p \not| x : p \in B\}$.

Round 2

We again look at stability and dependency.

The stable/unstable division

This is a major, well researched dividing line and, as mentioned, very useful. Recall that $\mathbf{S}(M) = \{\mathrm{tp}(a, M, N) : M \preceq N, a \in N\}$

Thesis

If in K there is an M with large $\mathbf{S}(M)$, say $|\mathbf{S}(M)| > |M|$, it is a sign of complexity.

If $\mathbf{S}(M)$ is small for all M in K, then we can expect to understand K.

Definition/Theorem

- K is stable in an infinite cardinal λ iff:
  ($M \in K$, M has λ elements) implies ($\mathbf{S}(M)$ has λ elements).
- K is stable iff it is stable in some λ.
- Equivalently, K is stable iff ($M \in K$, M has λ elements) implies ($\mathbf{S}(M)$ has at most $\lambda^{\aleph_0}$ elements) (for all λ).
- Equivalently, K is stable iff it is neither as complicated as K_{dlo} nor as K_{rg}.

Dependence

Recall that K is independent if it is as bad as K_{rg}, more formally:

Definition

K is independent iff for some definition $\varphi = \varphi(\bar{x}, \bar{y})$ and some $M \in K$: considering $\varphi[M]$ as a graph, it has an induced sub-graph which is random.

Question

But is dependent/independent a significant dividing line? E.g., can we understand dependent classes? Are independent ones complicated?

Dense linear orders: Few types modulo conjugacy

Lately have tried to recount $\mathbf{S}(\mathbb{Q}, <)$; recall there were continuum many members (one for each irrational, at least). But this time I succeed to count only up to 6!
How come? This time we count only up to conjugacy. Now for any two irrational numbers b, c there is an automorphism of the rational order taking the cut induced by b to the cut induced by c. So all the irrationals contribute just one type up to conjugacy.
What about others? there are

1. the trivial types ($x_0 = a, a \in \mathbb{Q}$),
2. $+\infty$,
3. $-\infty$,
4. $a + \epsilon$, ϵ "infinitesimal"
5. $a - \epsilon$.

Altogether six families, giving six conjugacy classes

Random graphs: Many types modulo conjugacy

Generally we can consider only structures with lots of automorphisms, so-called saturated models.
So maybe for all elementary classes we get few types up to conjugacy?
But consider the class K_{rg} of random graphs:
For any $M \in K_{\mathrm{rg}}$ and $A \subseteq M$ recall that there is a type coding A, so we should count the number of isomorphism types of the pairs (M, A) , and it is not hard to see that it is large.

For transparency assume *GCH*, the generalized continuum Hypothesis, i.e., assume that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all α. Then every elementary class K has in cardinality $\lambda = \aleph_{\alpha+1}$ a (unique) so-called saturated model $M_{K,\lambda}$.
So our question is:

Question

Given K, when is $\mathbf{S}(M_{K,\aleph_{\alpha+1}})$/conj small?

It turns out that there are three possibilities:

- small = constant (and at most $2^{\aleph_0}$),
- medium $\sim |\alpha|$,
- large = $2^{\aleph_{\alpha+1}}$: We have just seen an example for this behaviour: the random graph.

Independent classes

Not only the random graph has many types mod conjugacy, but more generally:

Theorem

If K is independent ($=$ as bad as K_{rg}) then

$$|\mathbf{S}(M_K, \aleph_{\alpha+1})/\, conj| = 2^{\aleph_{\alpha+1}}$$

Stable classes

Let K be an algebraically closed field (of characteristic 0, say). We have the following types modulo conjugacy:

- the algebraic elements, (countably many types)
- transcendental elements inside $M_{K,\lambda}$, ($||M||$ many types, but only one conjugacy class)
- transcendental elements outside of $M_{K,\lambda}$. (Only one type)

So:

Example

For the class K of algebraically closed fields, we get $|\mathbf{S}(M_{K,\aleph_{\alpha+1}})/\text{ conj}| = \aleph_0$.

In fact:

Theorem

If K is stable, then the number of types/conj is $\leq 2^{\aleph_0}$ and is constant.

Dependent but unstable classes

We are left with the main question: What about the (unstable but) dependent classes?
The obvious example is K_{dlo}: A cut has two cofinalities. So we have two cardinals, one is λ by saturation, the other is any cardinal $\aleph_\beta \leq \aleph_{\alpha+1}$. Hence $|\mathbf{S}(K_{\text{dlo}}, \aleph_{\alpha+1})/\operatorname{conj}| \geq |\alpha|$. A more careful analysis shows:

Example

$$|\mathbf{S}(K_{\text{dlo}}, \aleph_{\alpha+1})/\operatorname{conj}| \sim |\alpha|$$

It turns out that there is a general theorem:

Main Theorem: Recounting Theorem

Let K be dependent and unstable, and let $\lambda = \aleph_{\alpha+1}$ be large enough ($> \aleph_\omega$). Then

$$|\alpha| \leq |\mathbf{S}(M_{K,\aleph_{\alpha+1}})/\operatorname{conj}| \leq |\alpha|^{\aleph_0}.$$

Summary

For every elementary class K, exactly one of the following holds for the number $s_{\alpha+1} := |\mathbf{S}(M_{K,\aleph_{\alpha+1}})/\text{conj}|$:

- K is stable, and $s_{\alpha+1} \leq 2^{\aleph_0}$ is constant.
- K is unstable but dependent, and $|\alpha| \leq s_{\alpha+1} \leq |\alpha|^{\aleph_0}$.
- K is independent, and $s_{\alpha+1} = 2^{\aleph_{\alpha+1}}$ for all sufficiently large α.

Thesis

The theorem above is a strong indication that being dependent is a major dividing line, that there is much to be understood on dependent classes and more non-structure about independent classes

Proving this we are forced to understand structures in such classes.