ON INCOMPACTNESS FOR CHROMATIC NUMBER OF GRAPHS
SH1006

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Abstract. We deal with incompactness. Assume the existence of non-reflecting stationary subset of the regular cardinal \( \lambda \) of cofinality \( \kappa \). We prove that one can define a graph \( G \) whose chromatic number is \( > \kappa \), while the chromatic number of every subgraph \( G' \subseteq G, |G'| < \lambda \) is \( \leq \kappa \). The main case is \( \kappa = \aleph_0 \).

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§ 0. Introduction

The questions and results. During the Hajnal conference (June 2011) Magidor asked me on incompactness of “having chromatic number $\aleph_0$”; that is, there is a graph $G$ with $\lambda$ nodes, chromatic number $> \aleph_0$ but every subgraph with $< \lambda$ nodes has chromatic number $\aleph_0$ when:

$(\ast)_1$ $\lambda$ is regular $> \aleph_1$ with a non-reflecting stationary $S \subseteq S_{\aleph_0}$, possibly though better not, assuming some version of GCH.

Subsequently also when:

$(\ast)_2$ $\lambda = \aleph_{\omega+1}$.

Such problems were first asked by Erdős-Hajnal, see [EH74]; we continue [Sh:347].

First answer was using BB, see [Sh:309, 3.24] so assuming

$\Box (a) \quad \lambda = \mu^+$

$\Box (b) \quad \mu^{\aleph_0} = \mu$

$\Box (c) \quad S \subseteq \{ \delta < \lambda : \text{cf}(\delta) = \aleph_0 \} \text{ is stationary not reflecting}$

or just

$\Box' (a) \quad \lambda = \text{cf}(\lambda)$

$\Box' (b) \quad \alpha < \lambda \Rightarrow |\alpha|^{\aleph_0} < \lambda$

$\Box' (c) \quad \text{as above.}$

However, eventually we get more: if $\lambda = \lambda^{\aleph_0} = \text{cf}(\lambda)$ and $S \subseteq S_{\aleph_0}$ is stationary non-reflective then we have $\lambda$-incompactness for $\aleph_0$-chromatic. In fact, we replace $\aleph_0$ by $\kappa = \text{cf}(\kappa) < \lambda$ using a suitable hypothesis.

Moreover, if $\lambda^\kappa > \lambda$ we still get $(\lambda^\kappa, \lambda)$-incompactness for $\kappa$-chromatic number. In §2 we use quite free family of countable sequences.

In subsequent work we shall solve also the parallel of the second question of Magidor, i.e.

$(\ast)_2$ for regular $\kappa \geq \aleph_0$ and $n < \omega$ there is a graph $G$ of chromatic number $> \kappa$ but every sub-graph with $< \aleph_{\kappa+n+1}$ nodes has chromatic number $\leq \kappa$.

In fact, considerably is proved, see [Sh:F1240]. We thank Menachem Magidor for asking, Peter Komjath for stimulating discussion and Paul Larson, Shimoni Garti and the referee for some comments.

§ 0(B). Preliminaries.

Definition 0.1. For a graph $G$, let $\text{ch}(G)$, the chromatic number of $G$ be the minimal cardinal $\chi$ such that there is colouring $c$ of $G$ with $\chi$ colours, that is $c$ is a function from the set of nodes of $G$ into $\chi$ or just a set of of cardinality $\leq \chi$ such that $c(x) = c(y) \Rightarrow \{x, y\} \notin \text{edge}(G)$.

{Preliminaries}
Definition 0.2. 1) We say “we have $\lambda$-incompactness for the $(< \chi)$-chromatic number” or $\text{INC}_{\text{chr}}(\lambda, < \chi)$ when: there is a graph $G$ with $\lambda$ nodes, chromatic number $\geq \chi$ but every subgraph with $< \lambda$ nodes has chromatic number $< \chi$.

2) If $\chi = \mu^+$ we may replace “$< \chi$” by $\mu$; similarly in 0.3.

We also consider

Definition 0.3. 1) We say “we have $(\mu, \lambda)$-incompactness for $(< \chi)$-chromatic number” or $\text{INC}_{\text{chr}}(\mu, \lambda, < \chi)$ when there is an increasing continuous sequence $\langle G_i : i \leq \lambda \rangle$ of graphs each with $\leq \mu$ nodes, $G_i$ an induced subgraph of $G_\lambda$ with $\text{ch}(G_\lambda) \geq \chi$ but $i < \lambda \Rightarrow \text{ch}(G_i) < \chi$.

2) Replacing (in part (1)) $\chi$ by $\bar{\chi} = (\chi_0, \chi_1)$ means $\text{ch}(G_\lambda) \geq \chi_1$ and $i < \lambda \Rightarrow \text{ch}(G_i) < \chi_0$; similarly in 0.2 and parts 3), 4) below.

3) We say we have incompactness for length $\lambda$ for $(< \chi)$-chromatic (or $\bar{\chi}$-chromatic) number when we fail to have $(\mu, \lambda)$-compactness for $(< \chi)$-chromatic (or $\bar{\chi}$-chromatic) number for some $\mu$.

4) We say we have $[\mu, \lambda]$-incompactness for $(< \chi)$-chromatic number or $\text{INC}_{\text{chr}}[\mu, \lambda, < \chi]$ when there is a graph $G$ with $\mu$ nodes, $\text{ch}(G) \geq \chi$ but $G^1 \subseteq G \land |G^1| < \lambda \Rightarrow \text{ch}(G^1) < \chi$.

5) Let $\text{INC}_{\text{chr}}^+([\mu, \lambda, < \chi])$ be as in part (1) but we add that even the $c\ell(G_i)$, the colouring number of $G_i$ is $< \chi$ for $i < \lambda$, see below.

6) Let $\text{INC}_{\text{chr}}^+[\mu, \lambda, < \chi]$ be as in part (4) but we add $G^1 \subseteq G \land |G^1| < \lambda \Rightarrow c\ell(G^1) < \chi$.

7) If $\chi = \kappa^+$ we may write $\kappa$ instead of “$< \chi$”.

Definition 0.4. 1) For regular $\lambda > \kappa$ let $S^\lambda_\kappa = \{ \delta < \lambda : \text{cf}(\delta) = \kappa \}$.

2) We say $C$ is a $(\geq \theta)$-closed subset of a set $B$ of ordinals when: if $\delta = \sup(\delta \cap B) \in B$, $\text{cf}(\delta) \geq \theta$ and $\delta = \sup(C \cap \delta)$ then $\delta \in C$.

Definition 0.5. For a graph $G$, the colouring number $c\ell(G)$ is the minimal $\kappa$ such that there is a list $\langle a_\alpha : \alpha \in \alpha(*) \rangle$ of the nodes of $G$ such that $\alpha < \alpha(*) \Rightarrow \kappa \geq |\{ \beta < \alpha : \{a_\beta, a_\alpha\} \in \text{edge}(G) \}|$. 

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{y6} {y8}
§ 1. FROM NON-REFLECTING STATIONARY IN COFINALITY $\aleph_0$}

Claim 1.1. There is a graph $G$ with $\lambda$ nodes and chromatic number $> \kappa$ but every subgraph with $< \lambda$ nodes have chromatic number $\leq \kappa$ when:

\[ \begin{align*}
(\text{a}) & \quad \lambda, \kappa \text{ are regular cardinals} \\
(\text{b}) & \quad \kappa < \lambda < \lambda^\kappa \\
(\text{c}) & \quad S \subseteq S^\kappa_\kappa \text{ is stationary, not reflecting.}
\end{align*} \]

Proof. Stage A: Let $\vec{X} = \langle X_i : i < \lambda \rangle$ be a partition of $\lambda$ to sets such that $|X_i| = \lambda$ or just $|X_i| = |i + 2|^\kappa$ and $\min(X_i) \geq i$ and let $X_{<i} = \bigcup\{X_j : j < i\}$ and $X_{\leq i} = X_{<i+1}$. For $\alpha < \lambda$ let $i(\alpha)$ be the unique ordinal $i < \lambda$ such that $\alpha \in X_i$. We choose the set of points = nodes of $G$ as $Y = \langle \{\alpha, \beta \} : \alpha < \beta < \lambda, i(\beta) \in S \text{ and } \alpha < i(\beta) \rangle$ and let $Y_{<i} = \{\{\alpha, \beta \} \in Y : i(\beta) < i\}$.

Stage B: Note that if $\lambda = \kappa^+$, the complete graph with $\lambda$ nodes is an example (no use of the further information in $\vec{X}$). So without loss of generality $\lambda > \kappa^+$.

Now choose a sequence satisfying the following properties, exists by [Sh:63, Ch.III]:

\[ \begin{align*}
(\text{a}) & \quad \vec{C} = \langle C_\delta : \delta \in S \rangle \\
(\text{b}) & \quad C_\delta \subseteq \delta = \sup(C_\delta) \\
(\text{c}) & \quad \text{otp}(C_\delta) = \kappa \text{ such that } (\forall \beta \in C_\delta)(\beta + 1, \beta + 2 \notin C_\delta) \\
(\text{d}) & \quad \vec{C} \text{ guesses } \vec{\text{ clubs.}}
\end{align*} \]

Let $\langle \alpha^*_{\delta, \varepsilon} : \varepsilon < \kappa \rangle$ list $C_\delta$ in increasing order.

For $\delta \in S$ let $\Gamma_\delta$ be the set of sequence $\vec{\beta}$ such that:

\[ \begin{align*}
(\text{a}) & \quad \vec{\beta} \text{ has the form } \langle \beta_\varepsilon : \varepsilon < \kappa \rangle \\
(\text{b}) & \quad \vec{\beta} \text{ is increasing with limit } \delta \\
(\text{c}) & \quad \alpha^*_{\delta, \varepsilon} < \beta_{2\varepsilon + 1} < \alpha^*_{\delta, \varepsilon + 1} \text{ for } i < 2, \varepsilon < \kappa \\
(\text{d}) & \quad \beta_{2\varepsilon + 1} \in X_{<\alpha^*_{\delta, \varepsilon + 1}} \setminus X_{<\alpha^*_{\delta, \varepsilon}} \text{ for } i < 2, \varepsilon < \kappa \\
(\text{e}) & \quad (\beta_{2\varepsilon}, \beta_{2\varepsilon + 1}) \in Y \text{ hence } \epsilon \in Y_{<\alpha^*_{\delta, \varepsilon + 1}} \subseteq Y_{<\delta} \text{ for each } \varepsilon < \kappa
\end{align*} \]

(can ask less).

So $|\Gamma_\delta| \leq |\delta|^\kappa \leq |X_\delta| \leq \lambda$ hence we can choose a sequence $\langle \beta_\gamma : \gamma \in X_\delta \subseteq X_\delta \rangle$ listing $\Gamma_\delta$.

Now we define the set of edges of $G$: $\text{edge}(G) = \{(\alpha_1, \alpha_2), (\min(C_\delta), \gamma) : \delta \in S, \gamma \in X_\delta \text{ hence the sequence } \beta_\gamma = \langle \beta_{\gamma, \varepsilon} : \varepsilon < \kappa \rangle \text{ is well defined and we demand } (\alpha_1, \alpha_2) \in \langle (\beta_{\gamma, 2\varepsilon}, \beta_{\gamma, 2\varepsilon + 1}) : \varepsilon < \kappa \rangle \}$.

Stage C: Every subgraph of $G$ of cardinality $< \lambda$ has chromatic number $\leq \kappa$.

For this we shall prove that:

$\oplus_1 \chi(G|Y_{<i}) \leq \kappa$ for every $i < \lambda$.

This suffice as $\lambda$ is regular, hence every subgraph with $< \lambda$ nodes is included in $Y_{<i}$ for some $i < \lambda$.

For this we shall prove more by induction on $j < \lambda$: $\oplus_1 \chi(G|Y_{<i}) \leq \kappa$ for every $i < \lambda$.

$^3$the guessing clubs are used only in Stage D.
\( \oplus_{2,j} \) if \( i < j, i \notin S, c_1 \) a colouring of \( G \mid Y_{<i} \), Rang \((c_1) \subseteq \kappa \) and \( u \in [\kappa]^{\kappa} \) then there is a colouring \( c_2 \) of \( G \mid Y_{<j} \) extending \( c_1 \) such that Rang \((c_2 \mid (Y_{<j} \setminus Y_{<i})) \subseteq u \).

**Case 1:** \( j = 0 \)
Trivial.

**Case 2:** \( j \) successor, \( j - 1 \notin S \)
Let \( i \) be such that \( j = i + 1 \), but then every node from \( Y_j \setminus Y_i \) is an isolated node in \( G \mid Y_{<j} \), because if \( \{(\alpha, \beta), (\alpha', \beta')\} \) is an edge of \( G \mid Y_j \) then \( i(\beta), i(\beta') \in S \) hence necessarily \( i(\beta) \neq j - 1 = i, i(\beta') \neq j - 1 = i \) hence both \( (\alpha, \beta), (\alpha', \beta') \) are from \( Y_i \).

**Case 3:** \( j \) successor, \( j - 1 \in S \)
Let \( j - 1 \) be called \( \delta \) so \( \delta \in S \). But \( i \notin S \) by the assumption in \( \oplus_{2,j} \) hence \( i < \delta \). Let \( \varepsilon(\ast) < \kappa \) be such that \( \alpha^\varepsilon(\ast) > i \).
Let \( \langle u : \varepsilon \leq \kappa \rangle \) be a sequence of subsets of \( u \), a partition of \( u \) to sets each of cardinality \( \kappa \); actually the only disjointness used is that \( u_\kappa \cap \bigcup_{\varepsilon < \kappa} u_\varepsilon = \emptyset \).

We let \( i_0 = i, i_1 + \varepsilon = \bigcup \{\alpha^\varepsilon(\ast) + 1 + \varepsilon : \varepsilon < 1 + \varepsilon \} \) for \( \varepsilon < \kappa, i_\kappa = \delta \) and \( i_{\kappa + 1} = \delta + 1 = j \).
Note that:

- \( \varepsilon < \kappa \Rightarrow i_\varepsilon \notin S_j \).

[Why? For \( \varepsilon = 0 \) by the assumption on \( i \), for \( \varepsilon \) successor \( i_\varepsilon \) is a successor ordinal and for \( i \) limit clearly \( \text{cf}(i_\varepsilon) = \text{cf}(\varepsilon) < \kappa \) and \( S \subseteq S^\kappa \).]

We now choose \( c_{2,\zeta} \) by induction on \( \zeta \leq \kappa + 1 \) such that:

- \( c_{2,0} = c_1 \)
- \( c_{2,\zeta} \) is a colouring of \( G \mid Y_{<i_\zeta} \)
- \( c_{2,\zeta} \) is increasing with \( \zeta \)
- \( \text{Rang}(c_{2,\zeta} \mid (Y_{<i_{\zeta + 1}} \setminus Y_{<i_\zeta})) \subseteq u_\zeta \) for every \( \zeta < \zeta \).

For \( \zeta = 0, c_{2,0} = c_1 \) so is given.

For \( \zeta = \varepsilon + 1 < \kappa \): use the induction hypothesis, possible as necessarily \( i_\varepsilon \notin S \).
For \( \zeta \leq \kappa \) limit: take union.

For \( \zeta = \kappa + 1, \) note that each node \( b \) of \( Y_{<i_\kappa} \setminus Y_{<i_\kappa} \) is not connected to any other such node and if the node \( b \) is connected to a node from \( Y_{<i_\kappa} \) then the node \( b \) necessarily has the form \( (\min(C_\delta), \gamma) \in X^l \), hence \( \beta_\gamma \) is well defined, so the node \( b = (\min(C_\delta), \gamma) \) is connected in \( G \), more exactly in \( G \mid Y_{<\delta} \) exactly to the \( \kappa \) nodes \( \{\langle \gamma, 2\xi, \beta_\gamma, 2\xi + 1 \rangle : \varepsilon < \kappa \} \), but for every \( \varepsilon < \kappa \) large enough, \( c_{2,\kappa} (\langle \gamma, 2\xi, \beta_\gamma, 2\xi + 1 \rangle) \in u_\varepsilon \) hence \( \notin u_\kappa \) and \( |u_\kappa| = \kappa \) so we can choose a colour.

**Case 4:** \( j \) limit
By the assumption of the claim there is a club \( e \) of \( j \) disjoint to \( S \) and without loss of generality \( \min(e) = i \). Now choose \( c_{2,\xi} \) a colouring of \( Y_{<\xi} \) by induction on \( \xi \in e \cup \{j\} \), increasing with \( \xi \) such that \( \text{Rang}(c_{2,\xi} \mid (Y_{<\xi} \setminus Y_{<i})) \subseteq u \) and \( c_{2,0} = c_1 \)

- For \( \xi = \min(e) = i \) the colouring \( c_{2,i} = c_{2,i} = c_1 \) is given,
for \( \xi \) successor in \( e \), i.e. \( \in \text{nacc}(e) \setminus \{i\} \), use the induction hypothesis with \( \xi, \max(e \cap \xi) \) here playing the role of \( j \), \( i \) there recalling \( \max(e \cap \xi) \in e, e \cap S = \emptyset \).

- for \( \xi = \sup(e \cap \xi) \) take union.

Lastly, for \( \xi = j \) we are done.

**Stage D**: \( \text{ch}(G) > \kappa \).

Why? Toward a contradiction, assume \( c \) is a colouring of \( G \) with set of colours \( \subseteq \kappa \). For each \( \gamma < \lambda \) let \( u_\gamma = \{c((\alpha, \beta)) : \gamma < \alpha < \beta < \lambda \text{ and } (\alpha, \beta) \in Y\} \). So \( \langle u_\gamma : \gamma < \lambda \rangle \) is a decreasing sequence of subsets of \( \kappa \) and \( \kappa < \lambda = \text{cf}(\lambda) \), hence for some \( \gamma(*) < \lambda \) and \( u_* \subseteq \kappa \) we have \( \gamma \in (\gamma(*), \lambda) \Rightarrow u_\gamma = u_* \).

Hence \( E = \{\delta < \lambda : \delta \text{ is a limit ordinal } > (\gamma(\delta)) \text{ and } (\forall \alpha < \delta)((i(\alpha) < \delta) \text{ and } c((\alpha, \beta)) = i) \text{ is a club of } \lambda \} \).

Now recall that \( \bar{C} \) guesses clubs hence for some \( \delta \in S \) we have \( C_\delta \subseteq E \), so for every \( \varepsilon < \kappa \) we can choose \( \bar{\beta}_c < \beta_{2c+1} \) from \( (\alpha_{3,0}, 0) \) such that \( \bar{\beta}_c \subseteq \kappa \) and \( \beta_{2c} \subseteq \kappa \) we have \( \gamma \in (\gamma(\delta), \lambda) \Rightarrow u_{\gamma} = u_* \).

Similarly Claim 1.2. There is an increasing continuous sequence \( \langle G_i : i \leq \lambda \rangle \) of graphs each of cardinality \( \lambda^\kappa \) such that \( \text{ch}(G_\lambda) > \kappa \) and \( i < \lambda \) implies \( \text{ch}(G_i) \leq \kappa \) and even \( c_\ell(G_i) \leq \kappa \).

**Proof.** Like 1.1 but the \( X_i \) are not necessarily \( \subseteq \lambda \) or use 2.2.

\[ \square_{1.2} \]
§ 2. From almost free

{Fromalmostfree}

Definition 2.1. Suppose $\eta_\beta \in {}^\kappa \text{Ord}$ for every $\beta < \alpha(*)$ and $u \subseteq \alpha(*)$, and $\alpha < \beta < \alpha(*) \Rightarrow \eta_\alpha \neq \eta_\beta$.

1) We say $\{ \eta_\alpha : \alpha \in u \}$ is free when there exists a function $h : u \to \kappa$ such that $\{ \eta_\alpha(\varepsilon) : \varepsilon \in [h(\alpha), \kappa) \} : \alpha \in u \}$ is a sequence of pairwise disjoint sets.

2) We say $\{ \eta_\alpha : \alpha \in u \}$ is weakly free when there exists a sequence $\langle \varepsilon, \zeta < \kappa \rangle$ of subsets of $u$ with union $u$, such that the function $\eta_\alpha \mapsto \eta_\alpha(\varepsilon)$ is a one-to-one function on $u_{\varepsilon, \zeta}$, for each $\varepsilon, \zeta < \kappa$.

{c3}

Claim 2.2. 1) We have $\text{INC}_{\chi\mu}(\mu, \lambda, \kappa)$ and even $\text{INC}^+_{\chi\mu}(\mu, \lambda, \kappa)$, see Definition 0.3(1),(5) when:

- $(a)$ $\alpha(*) \notin [\mu, \mu^+]$ and $\lambda$ is regular $\leq \mu$ and $\mu = \mu^\kappa$
- $(b)$ $\bar{\eta} = \langle \eta_\alpha : \alpha < \alpha(*) \rangle$
- $(c)$ $\eta_\alpha \in {}^{\kappa}\mu$
- $(d)$ $\langle u_i : i \leq \lambda \rangle$ is an increasing continuous sequence of subsets of $\alpha(*)$
- $\lambda \subseteq \alpha(*)$
- $(e)$ $\bar{\eta}|u_\alpha$ is free if $\alpha < \lambda$ iff $\bar{\eta}|u_\alpha$ is weakly free.

{c3}

2) We have $\text{INC}_{\chi\mu}(\mu, \lambda, \kappa)$ and even $\text{INC}^+_{\chi\mu}(\mu, \lambda, \kappa)$, see Definition 0.3(4) when:

- $(a), (b), (c)$ as in $(a)$ from 2.2
- $(d)$ $\bar{\eta}$ is not free
- $(e)$ $\bar{\eta}|u$ is free when $u \in [\alpha(*)]^{<\lambda}$.

Proof. We concentrate on proving part (1) the chromatic number case; the proof of part (2) and of the colouring number are similar. For $\mathcal{A} \subseteq {}^\kappa \text{Ord}$, we define $\tau_\mathcal{A}$ as the vocabulary $\{ P_\eta : \eta \in \mathcal{A} \} \cup \{ F_\varepsilon : \varepsilon < \kappa \}$ where $P_\eta$ is a unary predicate, $F_\varepsilon$ a unary function (will be interpreted as possibly partial).

Without loss of generality for each $i < \lambda$, $u_i$ is an initial segment of $\alpha(*)$ and let $\mathcal{A} = \{ \eta_\alpha : \alpha < \alpha(*) \}$ and let $\prec_\mathcal{A}$ be the well ordering $\{ (\eta_\alpha, \eta_\beta) : \alpha < \beta < \alpha(*) \}$ of $\mathcal{A}$.

We further let $K_\mathcal{A}$ be the class of structures $M$ such that (pedantically, $K_\mathcal{A}$ depend also on the sequence $\langle \eta_\alpha : \alpha < \alpha(*) \rangle$):

- $(a)$ $M = (|M|, F^M_\varepsilon, P^M_\eta)_{\varepsilon < \kappa, \eta \in \mathcal{A}}$
- $(b)$ $\langle P^M_\eta : \eta \in \mathcal{A} \rangle$ is a partition of $|M|$, so for $a \in M$ let $\eta_a = \eta_a^M$ be the unique $\eta \in \mathcal{A}$ such that $a \in P^M_\eta$
- $(c)$ if $a_\ell \in P^M_\eta$ for $\ell = 1, 2$ and $F^M_\varepsilon(a_1) = a_2$ then $\eta_1(\varepsilon) = \eta_2(\varepsilon)$ and $\eta_1 <_\mathcal{A} \eta_2$.

Let $K^\prime_\mathcal{A}$ be the class of $M$ such that

- $(a)$ $M \in K_\mathcal{A}$
- $(b)$ $|M| = \mu$
- $(c)$ if $\eta \in \mathcal{A}$, $u \subseteq \kappa$ and $\eta_\varepsilon <_\mathcal{A} \eta$, $\eta_\varepsilon(\varepsilon) = \eta(\varepsilon)$ and $a_\varepsilon \in P^M_\eta$ for $\varepsilon \in u$ then $\eta \in u$ we have $\varepsilon \in u \Rightarrow F^M_\varepsilon(a) = a_\varepsilon$ and $\varepsilon \in \kappa \setminus u \Rightarrow F^M_\varepsilon(a)$ not defined.
Clearly

\[ \exists \alpha \text{ there is } M \in K_{\mathcal{A}}. \]

[Why? As } \mu = \mu^\kappa \text{ and } |\mathcal{A}| = \mu.]

\[ \exists \beta \text{ for } M \in K_{\mathcal{A}} \text{ let } G_M \text{ be the graph with:} \]

\begin{itemize}
  \item set of nodes \(|M|\)
  \item set of edges \(\{\{a, F^M_\varepsilon(a)\} : a \in |M|, \varepsilon < \kappa \text{ when } F^M_\varepsilon(a) \text{ is defined}\}\).
\end{itemize}

Now

\[ \exists \gamma \text{ if } \mu \subseteq \alpha(*) \land \mathcal{A}_\alpha = \{\eta_\alpha : \alpha \in u\} \subseteq \mathcal{A} \text{ and } \eta_\mu |u \text{ is free, and } M \in K_{\mathcal{A}} \text{ then} \]

\[ G_{M, \mathcal{A}_\alpha} := G_M |(\cup \{P^M_\eta : \eta \in \mathcal{A}_\alpha\}) \text{ has chromatic number } \leq \kappa \text{; moreover has colouring number } \subseteq \kappa. \]

[Why? Let } h : u \rightarrow \kappa \text{ witness that } \eta_\mu |u \text{ is free and for } \varepsilon < \kappa \text{ let } \mathcal{B}_\varepsilon := \{\eta_\alpha : \alpha \in u \text{ and } h(\alpha) = \varepsilon\}, \text{ so } \mathcal{B} = \cup \{\mathcal{B}_\varepsilon : \varepsilon < \kappa\}, \text{ hence it is enough to prove for each } \varepsilon < \kappa \text{ that } G_{M, \mathcal{B}_\varepsilon} \text{ has chromatic number } \leq \kappa. \text{ To prove this, by induction on } \alpha \leq \alpha(*) \text{ we choose } c_\alpha \text{ such that:} \]

\begin{enumerate}
  \item[(a)] \(c_\alpha^\varepsilon\) is a function
  \item[(b)] \(\langle c_\beta : \beta \leq \alpha \rangle\) is increasing continuous
  \item[(c)] \(\operatorname{Dom}(c_\alpha^\varepsilon) = B_\alpha^\varepsilon := \cup \{P^M_\eta : \beta < \alpha \text{ and } \eta_\beta \in \mathcal{B}_\varepsilon\}\)
  \item[(d)] \(\operatorname{Rang}(c_\alpha^\varepsilon) \subseteq \kappa\)
  \item[(e)] if } a, b \in \operatorname{Dom}(c_\alpha) \text{ and } \{a, b\} \in \text{edge}(G_M) \text{ then } c_\alpha(a) \neq c_\alpha(b). \]
\end{enumerate}

Clearly this suffices. Why is this possible?

If } \alpha = 0 \text{ let } c_\alpha^\varepsilon \text{ be empty, if } \alpha \text{ is a limit ordinal let } c_\alpha^\varepsilon = \cup \{c_\beta^\varepsilon : \beta < \alpha\} \text{ and if } \alpha = \beta + 1 \land \alpha(\beta) \neq \varepsilon \text{ let } c_\alpha = c_\beta.

Lastly, if } \alpha = \beta + 1 \land h(\beta) = \varepsilon \text{ we define } c_\alpha^\varepsilon \text{ as follows for } a \in \operatorname{Dom}(c_\alpha^\varepsilon), c_\alpha^\varepsilon(a) \text{ is:} \]

**Case 1**: } a \in B_\beta^\varepsilon.

Then } c_\alpha^\varepsilon(a) = c_\beta^\varepsilon(a). \]

**Case 2**: } a \in B_\alpha^\varepsilon \setminus B_\beta^\varepsilon.

Then } c_\alpha^\varepsilon(a) = \min(\kappa \setminus \{c_\beta^\varepsilon(F^M_\zeta(a)) : \zeta < \varepsilon \text{ and } F^M_\zeta(a) \in \operatorname{Dom}(c_\beta^\varepsilon)\}). \]

This is well defined as:

\begin{enumerate}
  \item[(a)] } B_\alpha^\varepsilon = B_\beta^\varepsilon \cup P^M_{\eta_\alpha}
  \item[(b)] if } a \in B_\beta^\varepsilon \text{ then } c_\alpha^\varepsilon(a) \text{ is well defined (so case 1 is O.K.)}
  \item[(c)] if } \{a, b\} \in \text{edge}(G_M), a \in P^M_{\eta_\alpha} \text{ and } b \in B_\alpha^\varepsilon \text{ then } b \in B_\beta^\varepsilon \text{ and } \text{then } b \in \{F^M_\zeta(a) : \zeta < \varepsilon\}
  \item[(d)] } c_\alpha^\varepsilon(a) \text{ is well defined in Case 2, too}
  \item[(e)] } c_\alpha^\varepsilon \text{ is a function from } B_\alpha^\varepsilon \text{ to } \kappa
  \item[(f)] } c_\alpha^\varepsilon \text{ is a colouring.} \]
Why? Toward contradiction assume witnesses \(\bar{a}, b\) for some \(\varepsilon < \kappa\), contradiction to \(\varepsilon < \kappa\) in Case 2 above.

Next, clause (d) holds as

\[
\begin{align*}
B & = A \\
\varepsilon & < \kappa,
\end{align*}
\]

so indeed \(B\) is free iff \(A\) is free.

Proof. Clause (A): By 2.2 and [Sh:g, Ch.III], [Sh:g, Ch.IX, §1].

Clause (B): Follows from (A) by [Sh:g, Ch.VIII, §1].

Clause (C): Follows from (B) by [Sh:g, Ch.IX, §1]. □
ON INCOMPACTNESS FOR CHROMATIC NUMBER OF GRAPHS

REFERENCES

[Sh:F1240] ________, More on compactness of chromatic numbers.

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