COMPACTNESS OF CHROMATIC NUMBER II
SH1018

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Abstract. We try to look again at results of the form. There is a graph with chromatic number $> \aleph_0$ but every subgraph of cardinality $< \mu$ has chromatic number $\leq \aleph_0$.

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§ 0. Introduction

This continues [Sh:1006] but does not rely on it.

In [Sh:1006] we prove that if there is $\mathscr{F} \subseteq {}^\kappa\mathrm{Ord}$ of cardinality μ, λ-free not free then we can get a failure of λ-compactness for the chromatic number being κ. This gives (using [Sh:g, Ch.II]) that if μ is strong limit singular of cofinality κ and $2^\mu > \mu^+$ then we get the above for $\lambda = \mu^+$ (and more).

Our original objective is to answer a problem of Magidor: $\aleph_\omega$-compactness fails for being $\aleph_0$-chromatic, however lately Magidor prove the consistency. An earlier version was wrong and a new proof will be presented in a version under preparation.

We thank Komjath and Kojman for pointing out a terminal error in a previous attempt. Komjath also asked on the case $\mu = \lambda > \chi = \aleph_0$ when λ singular. {y21} Definition 0.2 tries to have a more general frame.

We intend to continue in [Sh:F1296].

Another problem on incompactness is about the existence of λ-free Abelian groups G which with no non-trivial homomorphism to $\mathbb{Z}$, in [Sh:883], for $\lambda = \aleph_n$ using $n - BB$. In [Sh:898] we get more λ's, almost in ZFC by 1-BB (black box). This proof suffices here (but not in ZFC). This is continued in [Sh:F1200] which originally we use here, but presently is not connected.

{y8}

Definition 0.1. 1) Assume $\mu \geq \lambda = \mathrm{cf}(\lambda) \geq \chi$. We say "we have (μ, λ)-incompactness for the $(< \chi)$-chromatic number" or $\mathrm{INC}_{\mathrm{chr}}(\mu, \lambda, < \chi)$ <u>when</u> there is an increasing continuous sequence $\langle G_i : i \leq \lambda \rangle$ of graphs each with $\leq \mu$ nodes, G_i an induced subgraph of G_λ with $\mathrm{ch}(G_\lambda) \geq \chi$ but $i < \lambda \Rightarrow \mathrm{ch}(G_i) < \chi$.
2) Replacing (in part (1)) χ by $\bar{\chi} = (< \chi_0, \chi_1)$ means $\mathrm{ch}(G_\lambda)) \geq \chi_1$ and $i < \lambda \rightarrow \mathrm{ch}(G_i) < \chi_0$; similarly in parts 3),4) below.
3) We say we have incompactness for length λ for $(< \chi)$-chromatic (or $\bar{\chi}$-chromatic) number <u>when</u> we fail to have (μ, λ)-compactness for $(< \chi)$-chromatic (or $\bar{\chi}$-chromatic) number for some μ.
4) We say we have $[\mu, \lambda]$-incompactness for $(< \chi)$-chromatic number or $\mathrm{INC}_{\mathrm{chr}}[\mu, \lambda, < \chi]$ <u>when</u> there is a graph G with μ nodes, $\mathrm{ch}(G) \geq \chi$ but $G^1 \subseteq G \wedge |G^1| < \lambda \Rightarrow \mathrm{ch}(G^1) < \chi$.
5) Let $\mathrm{INC}^+_{\mathrm{chr}}(\mu, \lambda, < \chi)$ be as in part (1) but we add that there is a partition $\langle A_{1,\varepsilon} : \varepsilon < \kappa \rangle$ of the set of nodes of G_i such that $c\ell(G_i \restriction A_{i,\varepsilon})$, the colouring number of $G_i \restriction A_{i,\varepsilon}$ is $< \chi$ for $i < \lambda$, see below.
6) Let $\mathrm{INC}^+_{\mathrm{chr}}[\mu, \lambda, < \chi]$ be as in part (4) but we add: if $G^1 \subseteq G$ and $|G^1| < \lambda$ <u>then</u> there is a partition $\langle A_\varepsilon : \varepsilon < \varepsilon_* \rangle$ of the nodes of G^1 to $\varepsilon_* < \chi$ sets such that $\varepsilon < \varepsilon_* \Rightarrow c\ell(G^1 \restriction A_\varepsilon) < \chi$.
7) If $\chi = \kappa^+$ we may write κ instead of "$< \chi$".
8) Let $\mathrm{INC}(\lambda, < \chi)$ means $\mathrm{INC}(\lambda, \lambda, < \chi)$, and similarly in the other cases.

{y21}
{y8}

Definition 0.2. In Definition 0.1 we allow λ similar when we replace $\bar{G}$ by $(G, \bar{A}), G$ a graph with $\leq \mu$ nodes, $\bar{A} = \langle A_i : i < \lambda \rangle$ a partition of the set of nodes, $\mathrm{Ch}(G_u) \leq \chi$ for $u \in [\lambda]^{<\lambda}$ where $G_\eta = G \restriction \bigcup\limits_{i \in u} A_i$.

§ 1. A sufficient criterion and relations to transversals

{r2}

Definition 1.1. 1) Let $\mathrm{Inc}[\mu,\lambda,\kappa]$ mean that we can find $\mathbf{a} = (\mathscr{A}, \bar{R})$ witnessing it which means that:

(a) $|\mathscr{A}| = \mu$

(b) $\bar{R} = \langle R_\varepsilon : \varepsilon < \kappa \rangle$

(c) R_ε is a two-place relation on $\mathscr{A}$, so we may write $\nu R_\varepsilon \eta$

(d) $\mathscr{A}$ is not free (for $\mathbf{a}$), see $(*)_1$ below or just not strongly free, see $(*)_2$ below

(e) $\mathbf{a} = (\mathscr{A}, \bar{R})$ is λ-free which means $\mathscr{B} \subseteq \mathscr{A} \wedge |\mathscr{B}| < \lambda \Rightarrow \mathscr{B}$ is $\mathbf{a}$-free

where

$(*)_1$ if $\mathscr{B} \subseteq \mathscr{A}$ then $\mathscr{B}$ is $\mathbf{a}$-free means that there is a witness $(h, <_*)$ which means
- (α) $<_*$ a well ordering of $\mathscr{B}$
- (β) h is a function from $\mathscr{B}$ to κ
- (γ) if $h(\eta) = h(\nu)$ and $\nu R_\zeta \eta$ for some ζ then $\nu <_* \eta$ (so really only $<_* \restriction \{\eta \in \mathscr{B} : h(\eta) = \varepsilon\}$ for $\varepsilon < \kappa$ count); so it is reasonable to assume each R_ε is irreflexive
- (δ) for any $\eta \in \mathscr{B}$ the set[1] $\exp(\eta, h, <_*)$ has cardinality $< \kappa$ where (recall that $\mathscr{B} = \mathrm{Dom}(h)$)
  - $\exp(\eta, h, <_*) = \exp(\eta, h, <_*, \mathbf{a}) = \{\zeta < \kappa$: there is $\nu <_* \eta$ such that $\nu R_\zeta \eta$ and $h(\nu) = h(\eta)\}$

$(*)_2$ if $\mathscr{B} \subseteq \mathscr{A}$ then $\mathscr{B}$ is strongly $\mathbf{a}$-free means that for every well ordering $<_*$ of $\mathscr{B}$ there is a function $h : \mathscr{B} \to \kappa$ such that $(h, <_* \restriction \mathscr{B})$ witness $\mathscr{B}$ is $\mathbf{a}$-free

$(*)_3$ if $\mathscr{B} \subseteq \mathscr{A}$ then $\mathscr{B}$ is weakly free means that there is a witness h which means
- (α) h is a function from $\mathscr{B}$ to κ
- (β) for every $\eta \in \mathscr{B}$ the set $\exp(\eta, h)$ has cardinality $< \kappa$ where
  - $\exp(\eta, h) = \exp(\eta, h, \mathbf{a}) = \{\zeta < \kappa$: there is $\nu \in \mathscr{B}$ such that $\nu R_\zeta \eta$ and $h(\nu) = h(\eta)\}$.

2) Let $\mathrm{Inc}(\mu,\lambda,\kappa)$ mean that we can find $(\mathscr{A}, \bar{\mathscr{A}}, \bar{R})$ witnessing it which means that:

$(a) - (d)$ as above

$(e)'$ $\bar{\mathscr{A}} = \langle \mathscr{A}_\alpha : \alpha < \lambda \rangle$ is a partition[2] of union $\mathscr{A}$ such that for each $u \in [\lambda]^{<\lambda}$ the set $\cup\{\mathscr{A}_\alpha : \alpha \in u\}$ is free (i.e. for $(\mathscr{A}, \bar{R})$).

3) We call $\mathbf{a} = (\mathscr{A}, \bar{R})$ a pre-witness for $[\mu,\lambda,\kappa]$ or $[\mu,\kappa)$ when it satisfies clauses (a),(b),(c) of part (1). For such $\mathbf{a}$ let $G_{\mathbf{a}}$ be the graph with set of notes $\mathscr{A}$ and set of edges $\{\{\eta,\nu\} : \eta R_\varepsilon \nu$ for some $\varepsilon < \kappa\}$.

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[1] exp stands for exceptional

[2] If λ is regular we can use $\langle \bigcup_{\alpha<\beta} \mathscr{A}_\alpha : \beta < \lambda \rangle$, so an increasing sequence of length λ with union $\mathscr{A}$ each set is free.

Claim 1.2. *We have* $\mathrm{INC}_{\mathrm{chr}}(\mu,\lambda,\kappa)$ *or* $\mathrm{INC}_{\mathrm{chr}}[\mu,\lambda,\kappa]$, *see Definition 0.1(4)* *<u>when</u>* : {r6} {y8}

$\boxplus$ (a) $\mathrm{Inc}(\chi,\lambda,\kappa)$ *or* $\mathrm{Inc}[\chi,\lambda,\kappa]$ *respectively*
(b) $\chi \le \mu = \mu^\kappa$.

Proof. Fix $\mathbf{a} = (\mathscr{A}, \bar{\mathscr{A}}, \bar{R})$ or $\mathbf{a} = (\mathscr{A}, \bar{R})$ witnessing $\mathrm{Inc}(\mu,\lambda,\kappa)$ or $\mathrm{Inc}[\mu,\lambda,\kappa]$ respectively. Now we define $\tau_{\mathscr{A}}$ as the vocabulary $\{P_\eta : \eta \in \mathscr{A}\} \cup \{F_\varepsilon : \varepsilon < \kappa\}$ where P_η is a unary predicate, F_ε a unary function (but it may be interpreted as a partial function).

We further let $K_{\mathbf{a}}$ be the class of structures M such that:

$\boxplus_1$ (a) $M = (|M|, F^M_\varepsilon, P^M_\eta)_{\varepsilon<\kappa,\eta\in\mathscr{A}}$
(b) $\langle P^M_\eta : \eta \in \mathscr{A}\rangle$ is a partition of $|M|$, so for $a \in M$ let $\eta[a] = \eta^M_a$ be the unique $\eta \in \mathscr{A}$ such that $a \in P^M_\eta$
(c) if $a_\ell \in P^M_{\eta_\ell}$ for $\ell = 1,2$ and $F^M_\zeta(a_2) = a_1$ then $\eta_1 R_\zeta \eta_2$.

Let $K^*_{\mathbf{a}}$ be the class of M such that:

$\boxplus_2$ (a) $M \in K_{\mathbf{a}}$
(b) $\|M\| = \mu$
(c) if $\eta \in \mathscr{A}, u \subseteq \kappa$ and for $\zeta \in u$ we have $\nu_\zeta \in \mathscr{A}, \nu_\zeta R_\zeta \eta$ and $a_\zeta \in P^M_{\nu_\zeta}$ <u>then</u> for some $a \in P^M_\eta$ we have $\zeta \in u \Rightarrow F^M_\zeta(a) = a_\zeta$ and $\zeta \in \kappa \backslash u \Rightarrow F^M_\zeta(a)$ not defined.

Clearly

$\boxplus_3$ there is $M \in K^*_{\mathbf{a}}$.

[Why? Obvious as we are assuming $|\mathscr{A}| = \chi \le \mu = \mu^\kappa$.]

$\boxplus_4$ for $M \in K_{\mathbf{a}}$ let G_M be the graph with:
- set of nodes $|M|$
- set of edges $\{\{a, F^M_\varepsilon(a)\} : a \in |M|, \varepsilon < \kappa$ when $F^M_\varepsilon(a)$ is defined$\}$.

{y8} We shall show that the graph G_M is as required in Definition 0.1(1) or 0.1(4)
{y8} (recalling κ^+ here stands for χ there, see 0.1(7)). Clearly G_M is a graph with μ
{r2} nodes so recalling Definition 1.1(2) or 1.1(1) it suffices to prove $\boxplus_5$ and $\boxplus_7$ below.

$\boxplus_5$ if $\mathscr{B} \subseteq \mathscr{A}$ is free, and $M \in K_{\mathbf{a}}$ <u>then</u> $G_{M,\mathscr{B}} := G_M {\restriction} (\cup\{P^M_\eta : \eta \in \mathscr{B}\})$ has chromatic number $\le \kappa$.

{r2} [Why? Let the pair $(h, <_*)$ witness that $\mathscr{B}$ is free (for $\mathbf{a} = (\mathscr{A}, \bar{R})$, see 1.1(1)$(*)_1$) so $h : \mathscr{B} \to \kappa$ and let $\mathscr{B}_\varepsilon = \{\eta \in \mathscr{B} : h(\eta) = \varepsilon\}$ for $\varepsilon < \kappa$.

Clearly

$\boxplus_{5.1}$ it suffices for each $\varepsilon < \kappa$ to prove that $G_{M,\mathscr{B}_\varepsilon}$ has chromatic number $\le \kappa$.

Let $\langle \eta_\alpha : \alpha < \alpha(*)\rangle$ list $\mathscr{B}$ in $<_*$-increasing order. We define $\mathbf{c}_\varepsilon : G_{M,\mathscr{B}_\varepsilon} \to \kappa$ by defining a colouring $\mathbf{c}_{\varepsilon,\alpha} : G_{M,\{\eta_\beta:\beta<\alpha\}\cap\mathscr{B}_\varepsilon} \to \kappa$ by induction on $\alpha \le \alpha(*)$ such that $\mathbf{c}_{\varepsilon,\alpha}$ is increasing continuous with α. For $\alpha = 0$, let $\mathbf{c}_{\varepsilon,\alpha} = \emptyset$, and for α limit take union. If $\alpha = \beta + 1$ and $\eta_\beta \notin \mathscr{B}_\varepsilon$ then we let $\mathbf{c}_\alpha = \mathbf{c}_\beta$.

Lastly, assume $\alpha = \beta + 1, \eta_\beta \in \mathscr{B}_\varepsilon$ then note that the set $u_{\varepsilon,\beta} = \{\zeta < \kappa$: there is $\nu <_* \eta_\beta$ such that $\nu \in \mathscr{B}_\varepsilon$ and $\nu R_\zeta \eta\}$ has cardinality $< \kappa$ because the pair $(<_*, h)$ witness "$\mathscr{B}$ is free". Hence, recalling $M \in K_{\mathbf{a}}$, for each $a \in P^M_{\eta_\beta}$, the set $u_{\varepsilon,\beta,a} := \{\zeta < \kappa_\varepsilon : F^M_\zeta(a) \in \{P^M_\nu : \nu <_* \eta_\beta$ and $\nu \in \mathscr{B}_\varepsilon\}\}$ is $\subseteq u_{\varepsilon,\beta}$
{r2} hence has cardinality $\le |u_{\varepsilon,\beta}| < \kappa$. But by $(*)_1(\gamma)$ of 1.1 and the definition of $K_{\mathbf{a}}, A_a := \{b \in G_{M,\{\eta_\gamma:\gamma<\beta\cap\mathscr{B}_\varepsilon\}} : \{b,a\}$ is an edge of $G_M\}$ is $\subseteq \{F^M_\zeta(a) : \zeta \in u_{\varepsilon,\beta,a}\}$ hence the set A_a has cardinality $\le |u_{\varepsilon,\beta,a}| < \kappa$. So define $\mathbf{c}_{\varepsilon,\alpha}$ extending $\mathbf{c}_{\varepsilon,\beta}$ by, for $a \in P^M_{\eta_\beta}$ letting $\mathbf{c}_{\varepsilon,\alpha}(a) = \min(\kappa \backslash \{\mathbf{c}_{\varepsilon,\beta}(b) : b \in P^M_\nu$ for some $\nu <_* \eta_\beta$ from $\mathscr{B}_\varepsilon$ and $\{b,a\}$ is an edge of $G_M\})$. Recalling there is no edge $\subseteq P_{\eta_\beta}$ this is a colouring.

So we can carry the induction. So indeed $\boxplus_5$ holds.]

$\boxplus_6$ if $\mathscr{B} \subseteq \mathscr{A}$ is free and $M \in K_{\mathbf{a}}$ then $G_{M,\mathscr{B}}$ is the union of $\le \kappa$ sets each with colouring number $\le \kappa$ hence also chromatic number $\le \kappa$.

[Why? By the proof of $\boxplus_5$.]

$\boxplus_7$ $\mathrm{chr}(G_M) > \kappa$ if $M \in K^*_{\mathbf{a}}$.

Why? Toward contradiction assume $\mathbf{c} : G_M \to \kappa$ is a colouring and let $<_*$ be a well ordering of $\mathscr{A}$. For each $\eta \in \mathscr{A}$ and $\varepsilon, \zeta < \kappa$ let $\Lambda_{\eta,\varepsilon,\zeta} = \{\nu : \nu \in \mathscr{A}, \nu <_* \eta, \nu R_\zeta \eta$ and $\varepsilon \in \mathscr{H}_\nu\}$ where for $\nu \in \mathscr{A}$ we define $\mathscr{H}_\nu = \{\varepsilon$: for some $a \in P^M_\nu$ we have $\mathbf{c}(a) = \varepsilon\}$.

<u>Case 1</u>: There is $\eta \in \mathscr{A}$ such that $(\forall \varepsilon \in \mathscr{H}_\eta)(\exists^\kappa \zeta < \kappa)[\Lambda_{\eta,\varepsilon,\zeta} \ne \emptyset]$.

So we can find a one-to-one function $g : \mathscr{H}_\eta \to \kappa$ such that $\Lambda_{\eta,\varepsilon,g(\varepsilon)} \ne \emptyset$ for every $\varepsilon \in \mathscr{H}_\eta \subseteq \kappa$. For each $\varepsilon \in \mathscr{H}_\eta \subseteq \kappa$ choose $\nu_\varepsilon \in \Lambda_{\eta,\varepsilon,g(\varepsilon)}$; possible as $\Lambda_{\eta,\varepsilon,g(\varepsilon)} \ne \emptyset$ by the choice of the function g. By the definition of "$\nu_\varepsilon \in \Lambda_{\eta,\varepsilon,g(\varepsilon)}$" there is $a_\varepsilon \in P^M_{\nu_\varepsilon}$ such that $\mathbf{c}(\nu_\varepsilon) = \varepsilon$; recalling $\nu_\varepsilon \in \Lambda_{\eta,\varepsilon,\zeta}$ we have $\nu_\varepsilon R_\zeta \eta$ holds. So as $M \in K^*_{\mathbf{a}}$ there is $a \in P^M_\eta$ such that $\varepsilon \in \mathscr{H}_\eta \subseteq \kappa \Rightarrow F^M_{g(\varepsilon)}(a) = a_\varepsilon$, but then $\{a, a_\varepsilon\} \in \mathrm{edge}(G_M)$ hence $\mathbf{c}(a) \ne \mathbf{c}(a_\varepsilon) = \varepsilon$ for every $\varepsilon \in \mathscr{H}_\eta \subseteq \kappa$, contradiction to the definition of $\mathscr{H}_\eta$.

<u>Case 2</u>: Not Case 1

So for every $\eta \in \mathscr{A}$ there is $\varepsilon \in \mathscr{H}_\eta \subseteq \kappa$ such that there are $< \kappa$ ordinals $\zeta < \kappa$ such that $\Lambda_{\eta,\varepsilon,\zeta} \ne \emptyset$. This means that there is $h : \mathscr{A} \to \kappa$ such that:

- $\bullet_1$ $\eta \in \mathscr{A} \Rightarrow h(\eta) \in \mathscr{H}_\eta$ and
- $\bullet_2$ $\eta \in \mathscr{A} \Rightarrow \kappa > |\{\zeta < \kappa : \Lambda_{\eta,h(\eta),\zeta} \ne \emptyset\}|$.

This implies that:

- $\bullet_3$ $\eta \in \mathscr{A} \Rightarrow \kappa > |\exp(\eta, h, \mathbf{a}, <_*)|$

because (we have $\bullet_2$ and):

- $\bullet_4$ if $\eta \in \mathscr{A}$ and $\varepsilon = h(\eta)$ <u>then</u> $\exp(\eta, h, <_*, \mathbf{a}) \subseteq \{\zeta < \kappa : \Lambda_{\eta,\varepsilon,\zeta} \ne \emptyset\}$.

[Why? As $h : \mathscr{A} \to \kappa$ and if $\zeta \in \exp(\eta, h, <_*, \mathbf{a})$ let ν exemplify this, that is, $\nu <_* \eta, \nu R_\zeta \eta$ and $h(\nu) = h(\eta) = \varepsilon$ and recall $h(\nu) = \varepsilon$ implies $\varepsilon \in \mathscr{H}_\nu$ by $\bullet_1$. But this means that $\nu \in \Lambda_{\eta,\varepsilon,\zeta}$ hence $\Lambda_{\eta,\varepsilon,\eta} \neq \emptyset$ as required.]

{r2} As $<_*$ was any well ordering of $\mathscr{A}$, this means, see 1.1$(*)_2$ holds, that $\mathscr{A}$ is
{r2} strongly free, contradiction to 1.1(d). $\square_{1.2}$

We can now reprove a result from [Sh:1006].

{r7}
Conclusion 1.3. *1) We have* $\mathrm{Inc}(\mu, \lambda, \kappa)$ *<u>when</u>*

$(*)$ *for some $\mathscr{F}$ and natural number $\mathbf{k} > 0$ we have*
- (a) *$\mathscr{F} \subseteq {}^\kappa\mu$ has cardinality μ and is tree like (i.e. $f_1(\bar{d}) = f_2(j) \wedge \{f_1, f_2\} \subseteq \mathscr{F} \Rightarrow f_1{\restriction}i = f_2{\restriction}j$*
- (b) *$\mathscr{F}$ is not free where*
  - *$\mathscr{F}' \subseteq \mathscr{F}$ is free means:*
  - *there is a sequence $\langle \mathscr{F}'_i : i < \kappa\rangle$ such that $\mathscr{F}' = \cup\{\mathscr{F}'_i : i < \kappa\}$ and for each $i, \mathscr{F}'_i$ has a transversal which means that $\{\mathrm{Rang}(\eta) : \eta \in \mathscr{F}'_i\}$ has a transversal (= one-to-one choice function)*
- (c) *$\mathscr{F}$ is the increasing union of $\langle \mathscr{F}_\alpha : \alpha < \lambda\rangle$ such that each $\mathscr{F}_\alpha$ is free.*

2) We have $\mathrm{Inc}[\mu, \lambda, \kappa]$ *<u>when</u>*

$(*)$ *as above but replacing clause (c) by:*
$(c)'$ *every $\mathscr{F}' \subseteq \mathscr{F}$ of cardinality $< \lambda$ has a transversal.*

Proof. 1), 2) We define $\mathbf{a}$ by choosing (for our $\mathscr{F}$):

- $\mathscr{A}_{\mathbf{a}} = \mathscr{F}$
- $<_{\mathscr{A}}$ any well ordering of $\mathscr{F}$; not part of $\mathbf{a}$
- R_ε is defined by: $f R_\varepsilon g$ iff $f <_{\mathscr{A}} g \wedge f(\varepsilon) = g(\varepsilon)$
- for part (1) let $\bar{\mathscr{A}}$ be a sequence witnessing clause (c).

So it suffices to prove $\mathrm{Inc}(\mu, \lambda, \kappa)$ or $\mathrm{Inc}[\mu, \lambda, \kappa]$; hence it suffices to prove that $\mathbf{a}$ witness it.

{r2} Now in Definition 1.1, clauses (a),(b),(c) are obvious. For clause (e), assume
{r7} $\mathscr{F}_2 \subseteq \mathscr{F}$ is free in the sense of 1.3(1)(b), and we shall prove that $\mathscr{F}_2$ is $\mathbf{a}$-free, this suffices for clause (e). By the assumption on $\mathscr{F}_2$, clearly $\mathscr{F}_2$ is the union of $\langle \mathscr{F}_{2,\zeta} : \zeta < \kappa\rangle, \mathscr{F}_{2,\zeta}$ has a transversal $\mathbf{h}_\zeta$. Now we define $h : \mathscr{F}_2 \to \kappa$ by: $h(f) = \mathrm{pr}(\zeta, \varepsilon)$ where $\zeta = \min\{\xi : f \in \mathscr{F}_{2,\xi}\}$ and ε is minimal such that $\mathbf{h}_\zeta(\mathrm{Rang}(f)) = f(\varepsilon)$, now the pairs $(h, <_{\mathscr{A}} {\restriction}\mathscr{F}_2)$ witness that $\mathscr{F}_2$ is free (for $\mathbf{a}$).

For clause (d) toward contradiction assume that $h : \mathscr{F} \to \kappa$ and well ordering $<_*$ of $\mathscr{A}$ witness $\mathscr{F}$ is free for $\mathbf{a}$, hence $\bar{\mathscr{B}} = \langle \mathscr{B}_\varepsilon : \varepsilon < \kappa\rangle$ is a partition of $\mathscr{F}$ when we let $\mathscr{B}_\varepsilon = \{f \in \mathscr{F} : h(f) = \varepsilon\}$.

{r2} By Definition 1.1, for each $\varepsilon < \kappa$ and $f \in \mathscr{B}_\varepsilon$ the set $u_f = \{\zeta < \kappa$: for some $g \in \mathscr{B}_\varepsilon$ we have $g R_\zeta f\}$ has cardinality $< \kappa$ and let $\zeta_f \in \kappa \backslash u_f$. For $\varepsilon, \zeta < \kappa$ let $\mathscr{B}_{\varepsilon,\zeta} = \{f \in \mathscr{B}_\varepsilon : \zeta_f = \zeta\}$ so $\langle \mathscr{B}_{\varepsilon,\zeta} : \varepsilon, \zeta < \kappa\rangle$ is a partition of $\mathscr{A}$. Now for each $\varepsilon, \zeta < \kappa$, if $f \neq g \in \mathscr{B}_{\varepsilon,\zeta}$ <u>then</u> $f(\zeta) \neq g(\zeta)$. Why? By symmetry we can assume $g <_{\mathscr{A}} f$ now $\zeta = \zeta_f \in \kappa \backslash u_f$, so g cannot witness $\zeta \in u_f$. So $\langle \mathscr{B}_{\varepsilon,\zeta} : \varepsilon, \zeta < \kappa\rangle$ contradicts clause (b) of the claim's assumption. $\square_{1.3}$

{r18}

Claim 1.4. *If* INC$[\mu, \lambda, \kappa]$ *or* INC(μ, λ, κ) *<u>then</u>* Inc$[\mu, \lambda, \kappa]$ *or* Inc(μ, λ, κ) *respectively.*

Proof. As the two cases are similar we do the INC(μ, λ, κ) case, so let $G, \langle G_i : i < \lambda \rangle$ witness it.

Let $<_*$ be a well ordering of the set of nodes of G. Define $\mathbf{a} = (\mathscr{A}, \bar{\mathscr{A}}, \bar{R})$ by:

- $\mathscr{A}$ is the set of nodes of G
- $\bar{\mathscr{A}} = \langle \mathscr{A}_i : i < \lambda \rangle$ with $\mathscr{A}_i$ the set of nodes of G_i
- $R_\varepsilon = \{(\nu, \eta) : \{\nu, \eta\}$ an edge of G and $\nu <_* \eta\}$.

Now check, noting when checking, that e.g. in $(*)_1$ of Definition 1.1, $\exp(\eta, \alpha, <_*)$ {r2}
is equal to κ or to $\emptyset$ as $\bigwedge\limits_{\varepsilon} R_\varepsilon = R_0$. $\square_{1.4}$

Private Appendix

§ 2

{b1}

Definition 2.1. For a linear order I let the graph $G_{I,k}$ be defined by: set of nodes $[I]^k$

$$\text{set of edges} : \{\{u_1, u_2\} : \quad \begin{array}{l} u_1, u_2 \in [\mathscr{U}]^k \text{ and } |u_1 \triangleleft u_2| = 2 \\ \text{for some } t_0 <_I \ldots <_I t_k \text{ we have} \\ u_1 = \{t_0, \ldots, t_{k-1}\}, u_2 = \{t_1, \ldots, t_k\}\}. \end{array}$$

Toward resolving $\aleph_\omega$: warm up.

We delay the problems of "$\beth_k(\kappa)$ is singular"; not only $\beth_k(\kappa), S \notin \check{I}_\theta[\lambda]$ (not not the C_α's not exactly a tree, each is union on θ set over an initial segment.

{b3}

Theorem 2.2. *We have* $\mathrm{INC}(2^{<\lambda}, \lambda, \kappa)$ *when:*

$\boxplus$ *for some* θ

- (a) $\theta = \beth_k(\kappa)$ *and* $k \geq 1$
- (b) $S \subseteq S^\lambda_\theta$ *is stationary, so* λ *is regular* $> \theta$
- (c) (α) S *is non-reflecting and* $S \in \check{I}_\theta[\lambda]$ *or just*
  (β) $\bar{C} = \langle C_\delta : \alpha < \lambda \rangle, C_\alpha \subseteq \delta = \mathrm{otp}(C_\alpha) \leq \theta, C_{\alpha_1} = C_{\alpha_2} \cap \alpha_1$ *and* $\delta \in S \Rightarrow \sup(C_\delta)$ *and* $\bar{C} \restriction S$ *is* λ*-free.*

Proof.

$(*)_1$ let $\mathscr{T} = \cup\{\mathscr{T}_\alpha : \alpha < \lambda\}$ where $\mathscr{T}_\alpha$ be the set η such that:
- (a) $\eta = \langle a_\beta : \beta \leq \alpha \rangle = \langle a_{\eta,\beta} : \beta < \alpha \rangle$
- (b) $a_\beta \subseteq \beta$ has order type $< \theta^+$
- (c) if $\mathrm{cf}(\delta) = \theta$ and $\delta = \sup(a_\beta \cap \delta) < \sup(a_\beta)$ then $\delta \in a_\beta \cap S$
- (d) if $\gamma \in C_\beta$ then $a_\gamma = a_\beta \cap \gamma$

$(*)_2$ (a) let G be the graph with <u>set of nodes</u> $\{u$: for some $\alpha < \lambda$ and $\eta \in \mathscr{T}_{\alpha_{k-1}}$ we have $u \subseteq a_{\eta,\alpha_{k-1}} \cup \{\alpha_{k-1}\}]$ <u>set of edges</u> as in 2.1

{b1}

$(*)_3$ for $\alpha < \lambda$ let $G_\alpha = G \restriction \{u \in G : u \subseteq \mathscr{T}_{\leq \alpha}\}$.

Clearly $\langle G_\alpha : \alpha < \lambda \rangle$ is $\subseteq$-increasing with union G and G has $2^{<\lambda}$ models hence it is enough to prove $() *_7 + (*)_8$ below.

For the free part we first define:

$(*)_4$ choose I satisfying:
- (a) I is a linear order of cardinality λ
- (b) $\mathrm{cf}(I) = \theta$
- (c) every interval is isomorphic to I

$(*)_5$ for $\alpha < \lambda$ we let f solve $\mathscr{T}_{<\alpha}$ when:
- (a) f is a function from $\mathscr{T}_{<\alpha}$ into I
- (b) if $\eta \in \mathscr{T}_\alpha$ then $f \restriction [\{\eta \restriction (\gamma+1) : \gamma + 1 < \alpha\}]^k$ is one to one and into some proper initial segment of I and then prove

$(*)_6$ if (A) then (B) where:
- (A) (a) $\alpha \in \lambda \backslash S$ and $\alpha < \beta < \lambda$

(b) f solves $\mathscr{T}_\alpha$

(c) $\bar{\mathscr{U}} = \langle \mathscr{U}_\eta : \eta \in \mathscr{T}_\alpha \rangle$

(d) if $\eta \in \mathscr{T}_\alpha$ then $\mathscr{U}_\eta \subseteq I$ is convex bounded stationary subset of θ disjoint to $\{f(u) : u \in [\{\eta{\restriction}(\gamma+1) : \gamma+1 < \alpha\}]^k\}$

(B) there is g such that

(a) g solves $\mathscr{T}_\beta$

(b) g extends f

(c) if $\eta \in \mathscr{T}_\beta$ and $u \in [\{\eta{\restriction}(\gamma+1) : \gamma+1 < \alpha\}]^k, u \notin \mathrm{Dom}(f)$ then $g(u) \in \mathscr{U}_{\eta{\restriction}(\alpha+1)}$.

[Why? We prove this by induction on β.]

Case 1: β is a successor ordinal.

So without loss of generality $\beta = \alpha + 1$ and easy.

Case 2: $\mathrm{cf}(\beta) in [\aleph_0, \theta]$

We choose an increasing continuous sequence $\langle \alpha_\varepsilon : \varepsilon \le \mathrm{cf}(\beta)\rangle$ with $\alpha_0 = \alpha, \alpha_{\mathrm{cf}(\beta)} = \beta$. For each $\eta \in \mathscr{T}_\alpha$ let $\langle \mathscr{U}_{\eta,\varepsilon} : \varepsilon < \mathrm{cf}(\beta)\rangle$ be a partition of $\mathscr{U}_\eta$ each $\mathscr{U}_{\eta,\varepsilon}$ of cardinality θ.

Now we choose f_ε by induction on $\varepsilon \le \mathrm{cf}(\beta)$ such that:

- f_ε solves $\mathscr{T}_{<\alpha_\varepsilon}$
- $f_\varepsilon = f$ if $\varepsilon = 0$
- $\langle f_\zeta : z\eta \le \varepsilon\rangle$ is $\subseteq$-increasing continuous
- if $\eta \in \mathscr{T}_{\alpha_\varepsilon}$ and $u \in [\{\eta{\restriction}(\gamma+1) : \gamma+1 < \alpha_\varepsilon\}]^k$ and $u \notin \mathrm{dom}(f)$ then $f_\varepsilon(u) \in \cup\{\mathscr{U}_{\eta{\restriction}\alpha,\zeta} : \zeta < \varepsilon\}$.

For $\varepsilon = 0$ we let $f_\varepsilon = f$ for ε successor say $\varepsilon = \zeta + 1$ use the induction hypothesis with $f_\zeta, \alpha_\zeta, \alpha_\varepsilon$ here standing for f, α, β there

- for ε limit ordinal let $f_\varepsilon = \cup\{f_\zeta : \zeta < \varepsilon\}$.

This is obviously possible and $f_{\mathrm{cf}(\beta)}$ is as promised, so $(*)_6$ holds.

$(*)_7$ $G_\alpha = G{\restriction}\mathscr{T}_{<\alpha}$ has chromatic number $\le \kappa$.

[Why? Let $\mathbf{c} : [\theta]^k \to \kappa$ be a colouring of the EH graphs.

As trivially there is a solution f_0 of $\mathscr{T}_1$, by $(*)$ there is a solution f_α of $\mathscr{T}_{<\alpha}$. Let $\mathbf{c}_\alpha : G_\alpha \to \kappa$ be $\mathbf{c}_\alpha(u) = boldc(\{f(\nu) : \nu \in u\})$ for u a role of G_α. Now check.]

$(*)_8$ G has no colouring with κ colours.

[Why? Toward contradiction assume $\mathbf{c} : G \to \kappa$ is a colouring. For $\alpha < \lambda$ let $\mathscr{A}_{<\alpha} = \{a : a$ a subset of order types $< \theta^+$ such that if $\mathrm{cf}(\delta) = \theta \wedge \delta < \sup(a) \wedge \delta = \sup(a \cap \delta)$ then $\delta \in a \wedge \delta \in S\}$. Let $\mathscr{P}_\alpha \subseteq [\alpha]^{<\theta}$ be of cardinality $< \theta$ such that $\langle \mathscr{P}_\alpha : \alpha < \lambda\rangle$ witness $S \in \check{I}_\theta[\lambda]$.

For $\alpha < \lambda$ and $a \in \mathscr{A}_{<\alpha}$ let $\mathrm{hrtg}_a = \{p : p$ is a set of cardinality $\le \beth_{k-1}(\kappa)$ equation of the form $c(\{\alpha_0, \ldots, \alpha_{k-2}, x\}) = i$ for some $\alpha_0 < \ldots < \alpha_{k-2}$ are from $a, i < \kappa\}$.]

$(*)_{8.1}$ let $\mathbf{Y}$ be the set of $(\eta, \mathscr{S}, g)$ such that

$(*)$ (a) $\eta \in \mathscr{T}$

$(b)(\alpha)$ $\mathscr{X} \subseteq \ell g(\eta)$ is of cardinality $< \theta$

(β) if $\beta \in \mathscr{X}, \beta \cap \mathscr{S}$ is non-empty with no last member then $X \cap \beta$ is unbounded in a_β

(c) g is a function with domain $\mathscr{X}$

(d) if $\beta \in \mathrm{Dom}(g)$ then

(α) $g(\beta)$ is of the form $\langle p_{g(\beta),\varepsilon} : \varepsilon \in w_g \rangle$

(β) $w_{g,\beta}$ is a non-stationary subset of θ

(γ) $p_{g,\varepsilon} \in \mathscr{P}_{a_\eta,\eta}$

(e) if $\beta_1 \in a_{\eta,\beta_2}$ so $\beta_1 < \beta_2 < \ell g(\eta)$ <u>then</u> $w_{g,\beta_1} \subseteq w_{g,\beta_2}$; moreover $g(\beta_1) = g(\beta_2) \restriction w_{g,\beta_2}$

$(*)_{8.2}$ we define the two-place realtion $\le_\eta$ on $\mathscr{Y} : (\eta_1, \mathscr{X}_1, g_1) \le_{\bar{x}} (\eta_2, \mathscr{X}_2, g_2)$ <u>iff</u>

(a) both belong to Y

(b) $\eta_1 \le \eta_2$

(c) $\mathscr{X}_1 = \mathscr{X}_2 \cap \ell g(\eta_1)$

(d) $g_1 = g_2 \restriction X_1$

$(*)_{8.3}$ (a) for $(\eta_1, \mathscr{X}_1, g_1) \in \mathbf{Y}$ and $\beta \in \mathscr{X}_1$ let $(\xi, \beta, \eta, x, g) = \min\{\varepsilon : \varepsilon = \theta$ <u>or</u> $\varepsilon \in w_g$ for some $(\eta_2, \beta_2, X_3, g_2)$ which is $<_*$ above $(\eta_1, \mathscr{X}_1, g_2)$ and $\beta_a \in \mathscr{X}_2$ we have $\beta_1 \in a_{\eta_2,\beta_2}$ and β_1 realizes $p_{g(\beta_1),\varepsilon}$ but no $\alpha < \ell g(\eta)$ realize the type $p_{g(\eta),\varepsilon}$, see below$\}$

(b) we say $\alpha < \ell g(\eta)$ realizes the type p in η when: if $c(\{\alpha_0, \dots, \alpha_{k-1}, x\}) = i$ then $\mathbf{c}(\{\alpha_0, \dots, \alpha_{k-1}, \alpha\}) = i$edge$\alpha_{k-1}$ and so necessarily $\{\alpha_0, \dots, \alpha_{k-1}\} \subseteq a_{\eta,\alpha}$.

$(*)_{8.4}$ we define a two-place realtion $\le_2$ on $\mathbf{Y}$ by; $(\eta_1, \mathscr{X}_1, g_1) <_2 (\eta_2, \mathscr{X}_2, g_2)$ iff $(\eta_1, \mathscr{X}_1, g_1) <_1 (\eta_2, \mathscr{X}_2, g_2)$ and

(e) $p_{g(\beta_1),\xi}$ is realized by some $\gamma \in a_{\beta_2}$ in η_2 <u>when</u>

(α) $\beta_2 \in \mathscr{X}_2$ hence $< \ell g(\eta_2)$ but $\beta_2 \ge \ell g(\eta_1)$

(β) $\beta_1 \in a_{\eta_2,\beta_2} \cap \mathscr{X}_1$

(γ) $\xi = \xi(\beta_1, \eta_1, \mathscr{X}_1, g_1) < \theta$

(δ) if $\mathrm{otp}(a_{\beta_2} \cap \mathscr{X}_1)$ is a limit ordinal then $|\mathscr{X}_2 \backslash \mathscr{X}_1| > 1$

(ε) there is no $\beta_1' \in a_{\eta_2,\beta_2} \cap \mathscr{X}_1$ such that $\xi(\beta_1', \eta_1, \mathscr{X}_1, g_1) < \xi$

$(*)_{8.5}$ (a) $\le_1$ partially order $\mathbf{Y}$

(b) any $\le_1$-increasing sequence of length $< \lambda$ in $\mathbf{Y}$ has an upper bound

(c) $\le_2$ partially order $\mathbf{Y}$

(d) for any $\le_2$-increasing sequence of length $< \lambda$ in $\mathbf{Y}$ has an upper bound

$(*)_{8.6}$ if $\delta < \theta$ is a limit ordinal, $(\eta_1, \mathscr{X}_1, g_1) \in \mathbf{Y}, \beta_\varepsilon \in \mathscr{X}_1$ for $\varepsilon < \delta, \varepsilon < \zeta < \delta \Rightarrow \beta_\varepsilon \in a_{\eta_1,\beta_\zeta}$ <u>then</u> for some $(\eta_2, \mathscr{X}_2, g_2) \in Y$ which is $\le_2$-above $(\eta_1, \mathscr{X}_1, g_2)$ and $\beta \in \mathscr{X}_2$ we have $\varepsilon < \delta \rightarrow \beta_\varepsilon \in a_{\eta_2,\beta_\zeta}$.

[Why? Easily.]

$(*)_{8.7}$ if $(\eta_1, \mathscr{X}_1, g_1) \in \mathbf{Y}$ and $\beta \in \mathscr{X}_1$ <u>then</u> there is $(\eta_2, \mathscr{X}_2, g_2) \in \mathbf{Y}$ which is $\le_2$-above $(\eta_1, \mathscr{X}_1, g_2)$ and for some $\beta_2 \in [\ell g(\eta_1), \ell g(\eta_2))$ from $\mathscr{X}_2$ we have $a_{\eta_2,\beta_2} = a_{\eta_1,\beta_1} \cup \{\beta_1\}$

$(*)_{8.8}$ there is a $\leq_2$-increasing sequence $\{\mathbf{y}_\alpha = \langle(\eta_\alpha, \mathscr{X}_\alpha, g_\alpha) : \alpha < \lambda\rangle$ of members of $\mathbf{Y}$ which is generic enough, i.e.

- (a) let $\chi > 2^\lambda$
- (b) $N_\alpha \prec (\mathscr{H}(\chi), \in)$ is of cardinality $< \lambda$ for $\alpha < \lambda$ be such that $\bar{N} = \langle N_\beta : \beta < \lambda\rangle$ is $\prec$-increasing continuous and $\bar{H}\restriction(\alpha+1) \in N_{\alpha+1}$ for $\alpha < \lambda$
- (c) $\bar{\mathscr{P}} \in N_0$ and $\{S, \bar{a}\} \subseteq N_0$
- (d) $\langle(\eta_\beta, \mathscr{X}_\beta, g_\beta) : \beta < \alpha\rangle \in N_{\alpha+1}$.

$\square$

Discussion 2.3. We have two drawbacks in 2.2: {b3}

(A) the demand "θ is regular" and moreover "$\theta = \beth_k$".

For this it is desirable that instead of using the EH graph (2.1) on $[\theta]^k$, really $[I]^\theta$, we use a graph that exists for more θ's, so it seems helpful to point out a sufficient condition for such graphs. {b1}

It will help to have three successive cardinals $\theta^+, \theta^{+2}, \theta^{+3}$ and if θ is singular, $\theta^{+2}, \theta^{+3}, \theta^{+4}$. An easy solution is Definition 2.4 below, more profound is considering appropriate games, but it is not clear if this really helps, i.e. enable us to induct {b12}

(B) eliminating "$S \in \check{I}_\theta[\lambda]$, this is helpful as then we have existence theorems, we intend to weaken the way the "memory for the a_i's.

We intend to allow for

- $a \triangleleft c_i$ for $i < \theta, i < j \Rightarrow \sup(a_i) < \min(a_2 \backslash a), \delta = \sup(\bigcup\limits_i a_i), \langle \alpha_{\delta,j} : j < \theta\rangle$ list C_δ (guessing clubs) and $a_i a \subseteq [\alpha_{\delta,i}, \alpha_{\delta,i+1})$.

But we have to allow some edges between the $\langle a_{\delta,j} \backslash a : j < \theta\rangle$. Is it helpful to have $|a_\delta \backslash a| = \sigma, \sigma \geq \theta$. Note that here the presentation via k-tuples is not helpful. {b12}

Definition 2.4. 1) $G^{k,m}_\theta$ is the graph such that:

- (a) the set of nodes is $[\theta]^{k+m}$
- (b) the set of edges is the set of pairs $\{u_1, u_2\}$ such that
  - (α) $u_\ell \in [\theta]^{k+m}$ let $\alpha_{\ell,0} < \ldots < \alpha_{\ell,k+m}$ list it in increasing order
  - (β) $\alpha_{1,i+1} = \alpha_{2,i}$ for $i = 0, \ldots, n-1$
  - (γ) $\alpha_{2,m} < \alpha_{1,n+1} < \alpha_{2,n+1} < \alpha_{1,n+2} < \ldots$.

2) Probably better to use the tree indiscernibility for $(\aleph_n, \aleph_0)$

(C) maybe letting $\sigma = \beth_{k-2}(\kappa), \theta = \beth_k(\kappa)$ if $\theta > \sigma^{+3}$ or so, for $\ell(*) \in \{1,2,3\}, S \subseteq S^\lambda_{\sigma+\ell(\delta)}$ stationary not reflecting $2^\mu = \lambda = \mu^+, \mu^{<\sigma} = \mu$.

So we have "$C_\delta \subseteq \delta = \sup(C_\delta)$", $\delta \in S \subseteq S^\lambda_\theta, \alpha_\delta$ and earlier "$\alpha \in C_\delta \Rightarrow C_\delta \cap \alpha \in \mathscr{P}_\alpha$ or at least $\mathscr{P}_{<\delta}$" and now $\{u \cap C_\delta : u \in \mathscr{P}_{<\delta}, |u| \leq \sigma\}$ is equal to $[C_\delta]^{\leq\sigma}$ or just cofinal for $\subseteq$ and $\mathrm{cf}(\mu)^{+\ell(*)+1(?)} = \theta^{+\ell(*)}$.

(D) So if we fix $\sigma = \mathrm{cf}(\sigma) < \beth_\omega$ and is $\lambda = \aleph_\sigma$

(E) we forgot to deal with $\mathscr{F} \subseteq {}^{\theta}\mu$ of cardinality $\mu^+, \mathrm{cf}(\mu) = \theta, \mathscr{F}$ is μ^+-free. This calls for using $S \subseteq S^{\mu^+}\sigma$ and without loss of generality $\langle C_\delta : \delta \in S \rangle$ is μ^+-free even if S reflects.

There is $\bar{C} = \langle C_\alpha : \alpha \in S \rangle, \mathrm{otp}(C_\alpha) \leq \theta^+, C_\delta$ is θ-closed. Now $\bar{C}$ forms a tree and {b3} the proof of 2.2 should deal with this "memory".

{k3} This is an important part; it is better to combine it with **??** but have not elaborated (use convergence along with some C_δ of θ sets defined from a sequence of "lower" bound subsets of δ, etc...)

We only have $[C_\delta]^{<\sigma}$ in the memory, more complicated.

Let N_* be $(N_\lambda, \{(\mathbf{y}_\alpha : \alpha < \lambda\})$, where $N_\lambda = \bigcup\limits_{\alpha<\lambda} N_\alpha$. As $\mathscr{S} \subseteq \lambda$ is stationary, necessary for some $\delta \in S$ we hvae $N^*_\delta := (N_\delta, \{\mathbf{y}_\alpha : \alpha < \lambda\}) \prec N_*$. Now let $\mathbf{y}' = \bigcup\limits_{\alpha<\delta} \mathbf{y}_\alpha$ and using $\bar{\mathscr{P}}_*$ we easily get contradiction.

§ 3

Below the case θ singular is on the one hand opaque; on the other hand we arrive to θ singular in the inductive construction. The aim of defining $\mathbf{M}$ is to use $\mathbf{m} \in \mathbf{M}$ to build examples for $\mathrm{INC}(\lambda, \kappa)$ for regular λ with some stationary $S \subseteq S^\lambda_\theta$ which does not reflect. It would be better if the property for θ implies the property for θ^+, for 2^θ or $(2^\theta)^+$ and for $\aleph_\theta$ it implies $\mathrm{INC}(\theta, \kappa)$.

Definition 3.1. Assume θ, σ are regular cardinals, $\mu \geq \theta \geq \sigma \geq \ kappa$. {c3}

Let $\mathbf{M} = M_{\mu,\theta,\sigma,\kappa}$ be the class of objects $\mathbf{m}$ consisting of (so $\tau = \tau_{\mathbf{m}}, \ldots$ omitting μ means for some μ) the cardinals $\mu, \theta, \sigma, \kappa$ and:

(a) (α) a vocabulary τ but the function symbols will be interpreted as partial

(β) τ has cardinality $\leq \mu$

(γ) $R \in T$ a two-place predicate intended for a graph

(b) (α) a normal tree $\mathscr{T}$ with root $\mathrm{rt}(\mathscr{T})$

(β) $\mathscr{T}$ has $\leq \theta^+$ levels, is $(< \theta)$-compelte and has no θ^+-branch

(γ) $\mathrm{lev}(\eta) = \mathrm{lev}_{\mathscr{T}}(\eta) = \mathrm{lev}_{\mathbf{m}}(\eta)$ is the level of η

(δ) for $\varepsilon < \mathrm{lev}(\eta)$ let $\eta{\restriction}\varepsilon$ be the unique $\nu <_{\mathscr{T}} \eta, \mathrm{lev}(\nu) = \varepsilon$

(ε) $\mathscr{T}_\varepsilon = \{\eta \in \mathscr{T} : \mathrm{lev}(\eta) = \varepsilon\}$

(ζ) for $\eta \in \mathscr{T}$ let $\mathrm{pre}(\eta)$ be its predecessor if $\mathrm{lev}(\eta)$ is a successor ordinal and is η otherwise

(η) $\langle \mathscr{S}_\ell : \ell < 4(?)\rangle$ is a partition of $\mathscr{S}_* = \cup\{\mathscr{T}_{\varepsilon+1} : \varepsilon < e^+\}$

(c) (α) for $\eta \in \mathscr{T}, M_\eta$ a τ-model

(β) M_η is increasing continuous with η

(γ) M_η is generated by P^{M_η}

(δ) $G_\eta = (|M_\eta|, R^{M_\eta})$ is a graph

(ε) option: $\mathbf{c}_\eta$ is a colouring of G_η (or $\bigcup\limits_{\nu <_I \eta} G_\nu$) is increasing with η [helpful? omit?]

(d) (α) $u_\eta \subseteq v_\eta$ are subsets of P^{M_η}

(β) • u_η is of cardinality $< \sigma$

• v_η is of cardinality $< \theta$

• if $v_\eta \neq 0$ then $\mathrm{lev}(\eta)$ is a successor ordinal

(γ) $P^{M_\eta} = \cup\{u_\nu : \nu \leq_{\mathscr{T}} \eta\}$

(e) (α) $\bar{a}_\eta$ lists u_η with no repetitions

(β) $A_\eta \subseteq M_{\mathrm{pre}(\eta)}$

(γ) if $\eta \in \mathscr{S}_\eta$ then A_η has cardinality $< \sigma$

(δ) let $N^-_\eta = M_\eta{\restriction}c\ell_{M_\eta}(A_\eta)$ and $N_\eta = M_\eta{\restriction}c\ell_{M_\eta}(A_\eta \cup v_\eta)$

(ε) if $A_\eta \neq \emptyset$ then $\ell g(\eta)$ is a successor ordinal

(η) [optional, used?] if $\eta \in \mathrm{suc}_{\mathscr{T}}(\nu)$ then M_η is the union of $N^+_\eta, M_{\mathrm{pre}(\eta)}$ over N^-_η, so in particular for every function symbol $F \in \tau, F^{M_\eta} = F^{N^+_\eta} \cup F^{M_{\mathrm{pre}(\eta)}}$

(f) (α) $<_{\mathbf{m}}$ is a (linear) well ordering of $\mathscr{T}$ such that $\eta <_{\mathscr{T}} \nu \Rightarrow \eta <_{\mathbf{m}} \nu$

(β) if $\eta \in \mathscr{S}_1$ and $\nu = \mathrm{pre}(\eta)$ <u>then</u> for some ρ we have $(\eta, \nu, \varrho, \rho) \in \mathbf{Q}_{\mathbf{m}}$ which implies (means (?))

- $\bullet_1$ $\nu <_{\mathscr{T}} \rho$ and $\eta \not\leq_{\mathscr{T}} \rho$ and $\rho <_{\mathbf{m}} \eta$
- $\bullet_2$ $f = f_{\eta,\rho}$ is an embedding of M_ρ into M_η
- $\bullet_3$ f is the identity on N^-_ν
- $\bullet_4$ if $x \in M_\rho$ then $\mathbf{c}_\eta(f(x)) = c_\rho(x)$

{c6} $\bullet_5$ more? put the colouring as in Definition 3.2?

* * *

{c6}

Definition 3.2. We say $\mathbf{m} \in \mathbf{M}_{\mu,\theta,\sigma,\kappa}$ is colourable <u>when</u> for every $\lambda > \theta$ in the following game $\eth = \eth_{\mathbf{m},\lambda}$ the colouring player has a winning strategy:

(A) (a) a play lasts up to θ moves

(b) if a player has no legal choice, it is a loss

(B) before the ε-th move a pair $(\eta_\varepsilon, \mathbf{c}_\varepsilon)$ is given or has been produced such that

(a) $\eta_\varepsilon \in \mathscr{T}_{\mathbf{m}}$ and $\varepsilon = 0 \Rightarrow \eta_\varepsilon = \mathrm{tr}(\mathscr{T}_{\mathbf{m}})$

(b) $\mathbf{c}_\varepsilon$ is a κ-colouring of the graph G_{η_ε} (no connection to $\mathbf{c}_{\eta_\varepsilon}$ is required!)

(c) if $\zeta < \varepsilon$ then $\mathbf{c}_\zeta, \mathbf{c}_\varepsilon$ agree on $M_{\eta,\eta_\varepsilon}$

(d) if $\varepsilon < \theta$ is a limit ordinal then

(α) if for some unbounded subset S of ε, $\langle \eta_\zeta : \zeta \in S \rangle$ is $<_I$-increasing then $\eta_\varepsilon = \bigcup\limits_{\zeta \in S} \eta_\zeta, \mathbf{c}_\varepsilon = \bigcup\limits_{\zeta \in S} \mathbf{c}_\zeta$

(β) if there is no such S then by (A)(b) the play stops and the colouring plays

(e) if $\varepsilon = \zeta + 1$ then $(\eta_\varepsilon, \mathbf{c}_\varepsilon)$ was chosen during the ζ-th move

(C) in the ε-th move the anti-colouring player has some possibilities

<u>Pos(a)</u>: (a) choose $\eta_{\varepsilon+1} \in \mathscr{T}$ such that $\eta_\varepsilon <_{\mathscr{T}} \eta_{\varepsilon+1}$ (but not necessary $\eta_\varepsilon = \mathrm{pre}(\eta_{\varepsilon+1})$!)

(b) then the colouring player has to choose a legal $\mathbf{c}_{\varepsilon+1}$

<u>Pos(b)</u>: choose $\nu_\varepsilon <_{\mathscr{T}} \eta_\varepsilon, \eta'_{\varepsilon+1} \in \mathrm{suc}_{\mathscr{T}}(\nu_\varepsilon)$ and $\eta_{\varepsilon+1}$ such that

- $\eta_\varepsilon <_{\mathbf{m}} \eta_{\varepsilon+1}$
- $x \in M_{\eta_\varepsilon} \wedge f_{\eta_{\varepsilon+1},\eta_\varepsilon}(x) \in M_{\nu_\varepsilon} \Rightarrow \mathbf{c}_\varepsilon(f_{\eta_{\varepsilon+1},\eta_\varepsilon}(x)) = \mathbf{c}_\varepsilon(x)$.

Then the colouring player has to choose a colouring $\mathbf{c}_{\varepsilon+1}$ of $G_{\eta_{\varepsilon+1}}$ extending $\mathbf{c}_\varepsilon \restriction G_{\nu_\varepsilon}$ and such that $x \in M_{\eta_\varepsilon} \Rightarrow \mathbf{c}_{\varepsilon+1}(f_{\eta_{\varepsilon+1},\eta_\varepsilon}(x)) = \mathbf{c}_\varepsilon(x)$.

Discussion 3.3. For θ singular maybe consider.

{c6} **Observation 3.4.** *(How a play goes) In 3.2(B)(d), if we have two such S's they give the same result (or waive the normality of $\mathscr{T}_{\mathbf{m}}$).*

Proof. FILL! □

* * *

Of course, it is easy to find $\mathbf{m} \in \mathbf{M}_{\mu,\theta,\sigma,\kappa}$ which is colourable, e.g. if all the graphs $G_{\mathbf{m},\eta}$ are empty. So we like to have a property in the direction.

Definition 3.5. We say $\mathbf{m} \in \mathbf{M}_{\mu,\theta,\sigma,\kappa}$ is full <u>when</u>: the anti-colouring player has a winning strategy in the game $\eth = \eth^{\rm c}_{\mathbf{m}}$ where {c10}

- (A) a play of the game last θ moves
- (B) (a) just before the ε move a pair $\eta_\varepsilon(\eta_\varepsilon, \mathbf{c}_\varepsilon)$ is chosen
  - (b) $\eta_\varepsilon \in \mathscr{T}_\varepsilon$ is $<_T$-increasing continuous with ε
  - (c) $\mathbf{c}_\varepsilon$ is a κ-colouring of increasing continuous with ε
  - (d) if $\varepsilon = 0$ then $\eta_\varepsilon = \mathrm{rt}(\mathscr{T})$, so M_{η_ε} is empty and $\mathbf{c}_\varepsilon$ the empty function
- (C) in the ε-th move:
  - (a) the anti-colouring player chooses a set $\mathscr{X}_\varepsilon$ such that
    - (α) $\mathscr{X}_\varepsilon \subseteq \{(\eta, \mathbf{c}) : \eta_\varepsilon \le_{\mathscr{T}} \eta$ and $\mathbf{c}$ is a κ-colouring of G_η extending $\mathbf{c}_{\eta_\varepsilon}\}$
    - (β) $\mathscr{X}_\varepsilon$ is closed under isomorphisms: if $(\mathbf{c}) \in \mathscr{X}_\varepsilon, \nu \in \mathscr{T}_{\rm lev)}$ a κ-colouring and there is an isomorphism from G_ρ onto G_ν over G_{η_ε} mapping $\mathbf{c}$ to $\mathbf{d}$ then $(\nu, \mathbf{d}) \in \mathscr{X}_\varepsilon$
  - (b) then the colouring player chooses $\eta_{\varepsilon+1} \in \mathrm{suc}_{\mathscr{T}_\varepsilon}(\eta_\varepsilon)$ such that one of the following occurs:
    - (α) there is no pair $(\eta, \mathbf{c}) \in \mathscr{X}_\varepsilon$ satisfying $\eta_{\varepsilon+1} \le_{\mathscr{T}} \eta$
    - (β) there is a pair $(\eta, \mathbf{c}) \in cX_\varepsilon$ such that $(\eta_{\varepsilon+1}, \eta_\varepsilon, \eta, (\mathbf{c}))$ are as in 3.2(f) and $\mathbf{c}_{\eta_{\varepsilon+1}}(f_{\eta_{\varepsilon+1},\eta}(x)) = \mathbf{c}(x)$ {c6}
- (D) in the end of the play, the anti-colouring palyer wins <u>when</u> letting $\eta_\theta = \bigcup\limits_{\varepsilon<\theta} \eta_\varepsilon, \mathbf{c}_\theta = \bigcup\limits_{\varepsilon<\theta} \mathbf{c}_\varepsilon$ there is $\eta \in \mathrm{suc}(\eta_\theta)$ such that no κ-colouring $\mathbf{c}$ of G_η extending $\mathbf{c}_\theta$ such that: for some...?

Discussion 3.6. So we like to prove:

- $\bullet_1$ if $\mathbf{M}_{\theta,\sigma,\kappa} \neq \emptyset$ then $\mathrm{INC}[\theta^+, \sigma, \kappa]$
- $\bullet_2$ if $\mathbf{M}_{\theta,\sigma,\kappa} \neq \emptyset$ then $\mathbf{M}_{\theta^+,\sigma,\kappa}$
  [at least if θ is regular this is similar to the $\aleph_\theta$ case]
- $\bullet_3$ $\mathbf{M}_{\mu^+,\sigma,\kappa} \neq \emptyset$ when $\mu > \mathrm{cf}(\mu) = \theta^+, \mathbf{M}_{\theta,\lambda,\kappa}$ and
  - $(a)_1$ $\mathrm{pp}(\mu) > \theta^+$ or at least
  - $(a)_2$ there is $\bar\lambda = \langle \lambda_\varepsilon : \varepsilon < \theta^+ \rangle$ is an increasing sequence of regualrs with limit $\mu, \mu^+ = \mathrm{tcf}(\prod\limits_\varepsilon \lambda_\varepsilon, <_{J^{\rm bd}_{\theta^+}})$ and for some $\bar f$ witnessing this, $\mathrm{bad}(\bar f) = \emptyset$ or at least
  - $(a)_3$ there is $S \subseteq S^{\mu^+}_\theta$ and $\bar C = \langle C_\delta : \delta \in S \rangle$ a treee-like strict club system which is free and $S \in \check I_\theta[\mu^+]$
- $\bullet_4$ assume $\mathbf{M}_{\theta,\sigma,\kappa} \neq \emptyset, \theta$ regular, $\lambda = \mathrm{cf}(\lambda) > \theta, S \subseteq S^\lambda_\theta$ stationary not reflecting and $\alpha < \lambda \Rightarrow |\alpha|^{<\sigma} < \lambda$ <u>then</u> $\mathbf{M}_{\lambda,\sigma,\kappa} \neq \emptyset$.

Even more if

$\bullet_5$ if $\mathbf{M}_{\theta,\sigma,\kappa}$ and ∂ is the first regular cardinalily $\geq 2^\theta$ then $\mathbf{M}_{\partial,\sigma,\kappa}$ or $M_{\partial,\theta,\kappa}$.

§ 4. Private Appendix

Moved 4/2013 from §1, pg.6:

{r8}

Conclusion 4.1. *For any κ, for arbitrarily large regular $\lambda < \min\{\theta : \theta = \aleph_\theta > \kappa\}$ we have* $\mathrm{INC}^+(\lambda, \lambda, \kappa)$.

{r12}

Remark 4.2. Note the variant of transversal we use for our purpose is equivalent by [Sh:161].

Proof. If $\kappa = \aleph_0$, by Magidor-Shelah [MgSh:204]. Generally similar. $\square_{4.1}$

§ 5. 10 Older material

Discussion 5.1. Do we have $\mathrm{IC}_{\mathrm{chr}}(\lambda, \aleph_\omega, \aleph_0)$ for some λ? Assume not.
1) If $\mu \in \mathbf{C}_{\aleph_0}$, then necessarily $\mathrm{pp}(\mu) = \mu^+$ hence $2^\mu = \mu^+$.
2) Let $\mu = \sum\limits_n \lambda_n, \lambda_n = \mathrm{cf}(\lambda_n) < \lambda_{n+1}, \bar{f} = \langle f_\alpha : \alpha < \mu^+ \rangle$ witness $\mu^+ = \mathrm{tcf}(\prod \lambda_n, <_{J^{\mathrm{bd}}_{\aleph_0}})$.

If $S_* = \mathrm{bad}_{\leq \aleph_\omega}(\bar{f}) = \mathrm{bad}(\bar{f} \cap S^{\mu^+}_{<\aleph_\omega})$ non-stationary, we are done. Otherwise for some $\ell \in \{1,2,3\}$ there is $S \subseteq S^{\mu^+}_{\aleph_\ell}$ stationary not reflecting in any $\delta \in S^{\mu^+}_{<\aleph_\omega}$, so we have $\diamondsuit_S$.
3) Recall for every $n, \mathrm{IC}_{\mathrm{chr}}(\beth^+_n(\kappa), \beth^+_n(\kappa), \kappa)$; so we can assume $\bigwedge\limits_n \beth_n < \aleph_\omega$, so $\beth_\omega = \aleph_\omega$.
4) Still we can prove: for some $\ell \in \{0,1,2,3\}, (\forall \lambda), \mathrm{IC}_{\mathrm{chr}}(\lambda, \aleph_\omega, \aleph_\ell)$. Can we use $X \subseteq [\mu]^n$. Use [Sh:620]?
5) Can we use an example of part (3) on some $\kappa = \mathrm{cf}(\kappa) < \beth_u$ and use...
6) Can we take stationary $S \subseteq S^\lambda_\theta$ not reflecting $S \notin \check{I}_\theta[\lambda]$, but on a partial square of it.

We use [Sh:1006, 0.3=y8].

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