

COMPACTNESS OF CHROMATIC NUMBER II
SH1018

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ABSTRACT. We try to look again at results of the form. There is a graph with chromatic number $> \aleph_0$ but every subgraph of cardinality $< \mu$ has chromatic number $\leq \aleph_0$.

Date: August 7, 2013.

2010 *Mathematics Subject Classification.* Primary: 03E05; Secondary: 05C15.

Key words and phrases. set theory, graphs, chromatic number, compactness, almost free Abelian groups, non-reflecting stationary sets.

The author would like to thank the Israel Science Foundation for partial support of this research (Grant No. 1053/11). The author thanks Alice Leonhardt for the beautiful typing. First typed July 3, 2011.

§ 0. INTRODUCTION

This continues [Sh:1006] but does not rely on it.

In [Sh:1006] we prove that if there is $\mathcal{F} \subseteq {}^\kappa\text{Ord}$ of cardinality μ, λ -free not free then we can get a failure of λ -compactness for the chromatic number being κ . This gives (using [Sh:g, Ch.II]) that if μ is strong limit singular of cofinality κ and $2^\mu > \mu^+$ then we get the above for $\lambda = \mu^+$ (and more).

Our original objective is to answer a problem of Magidor: \aleph_ω -compactness fails for being \aleph_0 -chromatic, however lately Magidor prove the consistency. An earlier version was wrong and a new proof will be presented in a version under preparation.

We thank Komjath and Kojman for pointing out a terminal error in a previous attempt. Komjath also asked on the case $\mu = \lambda > \chi = \aleph_0$ when λ singular.

{y21} Definition 0.2 tries to have a more general frame.

We intend to continue in [Sh:F1296].

Another problem on incompactness is about the existence of λ -free Abelian groups G which with no non-trivial homomorphism to \mathbb{Z} , in [Sh:883], for $\lambda = \aleph_n$ using $n - BB$. In [Sh:898] we get more λ 's, almost in ZFC by 1-BB (black box). This proof suffices here (but not in ZFC). This is continued in [Sh:F1200] which originally we use here, but presently is not connected.

{y8}

Definition 0.1. 1) Assume $\mu \geq \lambda = \text{cf}(\lambda) \geq \chi$. We say “we have (μ, λ) -incompactness for the $(< \chi)$ -chromatic number” or $\text{INC}_{\text{chr}}(\mu, \lambda, < \chi)$ when there is an increasing continuous sequence $\langle G_i : i \leq \lambda \rangle$ of graphs each with $\leq \mu$ nodes, G_i an induced subgraph of G_λ with $\text{ch}(G_\lambda) \geq \chi$ but $i < \lambda \Rightarrow \text{ch}(G_i) < \chi$.

2) Replacing (in part (1)) χ by $\bar{\chi} = \langle \chi_0, \chi_1 \rangle$ means $\text{ch}(G_\lambda) \geq \chi_1$ and $i < \lambda \rightarrow \text{ch}(G_i) < \chi_0$; similarly in parts 3),4) below.

3) We say we have incompactness for length λ for $(< \chi)$ -chromatic (or $\bar{\chi}$ -chromatic) number when we fail to have (μ, λ) -compactness for $(< \chi)$ -chromatic (or $\bar{\chi}$ -chromatic) number for some μ .

4) We say we have $[\mu, \lambda]$ -incompactness for $(< \chi)$ -chromatic number or $\text{INC}_{\text{chr}}[\mu, \lambda, < \chi]$ when there is a graph G with μ nodes, $\text{ch}(G) \geq \chi$ but $G^1 \subseteq G \wedge |G^1| < \lambda \Rightarrow \text{ch}(G^1) < \chi$.

5) Let $\text{INC}_{\text{chr}}^+(\mu, \lambda, < \chi)$ be as in part (1) but we add that there is a partition $\langle A_{1,\varepsilon} : \varepsilon < \kappa \rangle$ of the set of nodes of G_i such that $\text{cl}(G_i \upharpoonright A_{i,\varepsilon})$, the colouring number of $G_i \upharpoonright A_{i,\varepsilon}$ is $< \chi$ for $i < \lambda$, see below.

6) Let $\text{INC}_{\text{chr}}^+[\mu, \lambda, < \chi]$ be as in part (4) but we add: if $G^1 \subseteq G$ and $|G^1| < \lambda$ then there is a partition $\langle A_\varepsilon : \varepsilon < \varepsilon_* \rangle$ of the nodes of G^1 to $\varepsilon_* < \chi$ sets such that $\varepsilon < \varepsilon_* \Rightarrow \text{cl}(G^1 \upharpoonright A_\varepsilon) < \chi$.

7) If $\chi = \kappa^+$ we may write κ instead of “ $< \chi$ ”.

8) Let $\text{INC}(\lambda, < \chi)$ means $\text{INC}(\lambda, \lambda, < \chi)$, and similarly in the other cases.

{y21}

{y8}

Definition 0.2. In Definition 0.1 we allow λ similar when we replace \bar{G} by (G, \bar{A}) , G a graph with $\leq \mu$ nodes, $\bar{A} = \langle A_i : i < \lambda \rangle$ a partition of the set of nodes, $\text{Ch}(G_u) \leq \chi$ for $u \in [\lambda]^{< \lambda}$ where $G_\eta = G \upharpoonright \bigcup_{i \in u} A_i$.

§ 1. A SUFFICIENT CRITERION AND RELATIONS TO TRANSVERSALS

{r2}

Definition 1.1. 1) Let $\text{Inc}[\mu, \lambda, \kappa]$ mean that we can find $\mathbf{a} = (\mathcal{A}, \bar{R})$ witnessing it which means that:

- (a) $|\mathcal{A}| = \mu$
- (b) $\bar{R} = \langle R_\varepsilon : \varepsilon < \kappa \rangle$
- (c) R_ε is a two-place relation on \mathcal{A} , so we may write $\nu R_\varepsilon \eta$
- (d) \mathcal{A} is not free (for \mathbf{a}), see $(*)_1$ below or just not strongly free, see $(*)_2$ below
- (e) $\mathbf{a} = (\mathcal{A}, \bar{R})$ is λ -free which means $\mathcal{B} \subseteq \mathcal{A} \wedge |\mathcal{B}| < \lambda \Rightarrow \mathcal{B}$ is \mathbf{a} -free

where

- $(*)_1$ if $\mathcal{B} \subseteq \mathcal{A}$ then \mathcal{B} is \mathbf{a} -free means that there is a witness $(h, <_*)$ which means
 - (α) $<_*$ a well ordering of \mathcal{B}
 - (β) h is a function from \mathcal{B} to κ
 - (γ) if $h(\eta) = h(\nu)$ and $\nu R_\zeta \eta$ for some ζ then $\nu <_* \eta$ (so really only $<_* \upharpoonright \{\eta \in \mathcal{B} : h(\eta) = \varepsilon\}$ for $\varepsilon < \kappa$ count); so it is reasonable to assume each R_ε is irreflexive
 - (δ) for any $\eta \in \mathcal{B}$ the set¹ $\text{exp}(\eta, h, <_*)$ has cardinality $< \kappa$ where (recall that $\mathcal{B} = \text{Dom}(h)$)
 - $\text{exp}(\eta, h, <_*) = \text{exp}(\eta, h, <_*, \mathbf{a}) = \{\zeta < \kappa : \text{there is } \nu <_* \eta \text{ such that } \nu R_\zeta \eta \text{ and } h(\nu) = h(\eta)\}$
- $(*)_2$ if $\mathcal{B} \subseteq \mathcal{A}$ then \mathcal{B} is strongly \mathbf{a} -free means that for every well ordering $<_*$ of \mathcal{B} there is a function $h : \mathcal{B} \rightarrow \kappa$ such that $(h, <_* \upharpoonright \mathcal{B})$ witness \mathcal{B} is \mathbf{a} -free
- $(*)_3$ if $\mathcal{B} \subseteq \mathcal{A}$ then \mathcal{B} is weakly free means that there is a witness h which means
 - (α) h is a function from \mathcal{B} to κ
 - (β) for every $\eta \in \mathcal{B}$ the set $\text{exp}(\eta, h)$ has cardinality $< \kappa$ where
 - $\text{exp}(\eta, h) = \text{exp}(\eta, h, \mathbf{a}) = \{\zeta < \kappa : \text{there is } \nu \in \mathcal{B} \text{ such that } \nu R_\zeta \eta \text{ and } h(\nu) = h(\eta)\}$.

2) Let $\text{Inc}(\mu, \lambda, \kappa)$ mean that we can find $(\mathcal{A}, \bar{\mathcal{A}}, \bar{R})$ witnessing it which means that:

- (a) – (d) as above
- (e)' $\bar{\mathcal{A}} = \langle \mathcal{A}_\alpha : \alpha < \lambda \rangle$ is a partition² of union \mathcal{A} such that for each $u \in [\lambda]^{< \lambda}$ the set $\cup \{\mathcal{A}_\alpha : \alpha \in u\}$ is free (i.e. for (\mathcal{A}, \bar{R})).
- 3) We call $\mathbf{a} = (\mathcal{A}, \bar{R})$ a pre-witness for $[\mu, \lambda, \kappa]$ or $[\mu, \kappa]$ when it satisfies clauses (a),(b),(c) of part (1). For such \mathbf{a} let $G_{\mathbf{a}}$ be the graph with set of nodes \mathcal{A} and set of edges $\{\{\eta, \nu\} : \eta R_\varepsilon \nu \text{ for some } \varepsilon < \kappa\}$.

¹exp stands for exceptional

²If λ is regular we can use $\langle \cup_{\alpha < \beta} \mathcal{A}_\alpha : \beta < \lambda \rangle$, so an increasing sequence of length λ with union

\mathcal{A} each set is free.

Claim 1.2. We have $\text{INC}_{\text{chr}}(\mu, \lambda, \kappa)$ or $\text{INC}_{\text{chr}}[\mu, \lambda, \kappa]$, see Definition 0.1(4) when: {r6}
{y8}

- ⊞ (a) $\text{Inc}(\chi, \lambda, \kappa)$ or $\text{Inc}[\chi, \lambda, \kappa]$ respectively
 (b) $\chi \leq \mu = \mu^\kappa$.

Proof. Fix $\mathbf{a} = (\mathcal{A}, \bar{\mathcal{A}}, \bar{R})$ or $\mathbf{a} = (\mathcal{A}, \bar{R})$ witnessing $\text{Inc}(\mu, \lambda, \kappa)$ or $\text{Inc}[\mu, \lambda, \kappa]$ respectively. Now we define $\tau_{\mathcal{A}}$ as the vocabulary $\{P_\eta : \eta \in \mathcal{A}\} \cup \{F_\varepsilon : \varepsilon < \kappa\}$ where P_η is a unary predicate, F_ε a unary function (but it may be interpreted as a partial function).

We further let $K_{\mathbf{a}}$ be the class of structures M such that:

- ⊞₁ (a) $M = (|M|, F_\varepsilon^M, P_\eta^M)_{\varepsilon < \kappa, \eta \in \mathcal{A}}$
 (b) $\langle P_\eta^M : \eta \in \mathcal{A} \rangle$ is a partition of $|M|$, so for $a \in M$ let $\eta[a] = \eta_a^M$ be the unique $\eta \in \mathcal{A}$ such that $a \in P_\eta^M$
 (c) if $a_\ell \in P_{\eta_\ell}^M$ for $\ell = 1, 2$ and $F_\zeta^M(a_2) = a_1$ then $\eta_1 R_\zeta \eta_2$.

Let $K_{\mathbf{a}}^*$ be the class of M such that:

- ⊞₂ (a) $M \in K_{\mathbf{a}}$
 (b) $\|M\| = \mu$
 (c) if $\eta \in \mathcal{A}, u \subseteq \kappa$ and for $\zeta \in u$ we have $\nu_\zeta \in \mathcal{A}, \nu_\zeta R_\zeta \eta$ and $a_\zeta \in P_{\nu_\zeta}^M$ then for some $a \in P_\eta^M$ we have $\zeta \in u \Rightarrow F_\zeta^M(a) = a_\zeta$ and $\zeta \in \kappa \setminus u \Rightarrow F_\zeta^M(a)$ not defined.

Clearly

- ⊞₃ there is $M \in K_{\mathbf{a}}^*$.

[Why? Obvious as we are assuming $|\mathcal{A}| = \chi \leq \mu = \mu^\kappa$.]

⊞₄ for $M \in K_{\mathbf{a}}$ let G_M be the graph with:

- set of nodes $|M|$
- set of edges $\{\{a, F_\varepsilon^M(a)\} : a \in |M|, \varepsilon < \kappa \text{ when } F_\varepsilon^M(a) \text{ is defined}\}$.

{y8} We shall show that the graph G_M is as required in Definition 0.1(1) or 0.1(4)
 {y8} (recalling κ^+ here stands for χ there, see 0.1(7)). Clearly G_M is a graph with μ
 {r2} nodes so recalling Definition 1.1(2) or 1.1(1) it suffices to prove ⊞₅ and ⊞₇ below.

⊞₅ if $\mathcal{B} \subseteq \mathcal{A}$ is free, and $M \in K_{\mathbf{a}}$ then $G_{M, \mathcal{B}} := G_M \setminus (\cup \{P_\eta^M : \eta \in \mathcal{B}\})$ has chromatic number $\leq \kappa$.

{r2} [Why? Let the pair $(h, <_*)$ witness that \mathcal{B} is free (for $\mathbf{a} = (\mathcal{A}, \bar{R})$, see 1.1(1)(*)₁) so $h : \mathcal{B} \rightarrow \kappa$ and let $\mathcal{B}_\varepsilon = \{\eta \in \mathcal{B} : h(\eta) = \varepsilon\}$ for $\varepsilon < \kappa$.

Clearly

⊞_{5.1} it suffices for each $\varepsilon < \kappa$ to prove that $G_{M, \mathcal{B}_\varepsilon}$ has chromatic number $\leq \kappa$.

Let $\langle \eta_\alpha : \alpha < \alpha(*) \rangle$ list \mathcal{B} in $<_*$ -increasing order. We define $\mathbf{c}_\varepsilon : G_{M, \mathcal{B}_\varepsilon} \rightarrow \kappa$ by defining a colouring $\mathbf{c}_{\varepsilon, \alpha} : G_{M, \{\eta_\beta : \beta < \alpha\} \cap \mathcal{B}_\varepsilon} \rightarrow \kappa$ by induction on $\alpha \leq \alpha(*)$ such that $\mathbf{c}_{\varepsilon, \alpha}$ is increasing continuous with α . For $\alpha = 0$, let $\mathbf{c}_{\varepsilon, \alpha} = \emptyset$, and for α limit take union. If $\alpha = \beta + 1$ and $\eta_\beta \notin \mathcal{B}_\varepsilon$ then we let $\mathbf{c}_\alpha = \mathbf{c}_\beta$.

Lastly, assume $\alpha = \beta + 1, \eta_\beta \in \mathcal{B}_\varepsilon$ then note that the set $u_{\varepsilon, \beta} = \{\zeta < \kappa : \text{there is } \nu <_* \eta_\beta \text{ such that } \nu \in \mathcal{B}_\varepsilon \text{ and } \nu R_\zeta \eta\}$ has cardinality $< \kappa$ because the pair $(<_*, h)$ witness “ \mathcal{B} is free”. Hence, recalling $M \in K_{\mathbf{a}}$, for each $a \in P_{\eta_\beta}^M$, the set $u_{\varepsilon, \beta, a} := \{\zeta < \kappa_\varepsilon : F_\zeta^M(a) \in \{P_\nu^M : \nu <_* \eta_\beta \text{ and } \nu \in \mathcal{B}_\varepsilon\}\}$ is $\subseteq u_{\varepsilon, \beta}$ hence has cardinality $\leq |u_{\varepsilon, \beta}| < \kappa$. But by $(*)_1(\gamma)$ of 1.1 and the definition of $K_{\mathbf{a}}, A_a := \{b \in G_{M, \{\eta_\gamma : \gamma < \beta \cap \mathcal{B}_\varepsilon\}} : \{b, a\} \text{ is an edge of } G_M\}$ is $\subseteq \{F_\zeta^M(a) : \zeta \in u_{\varepsilon, \beta, a}\}$ hence the set A_a has cardinality $\leq |u_{\varepsilon, \beta, a}| < \kappa$. So define $\mathbf{c}_{\varepsilon, \alpha}$ extending $\mathbf{c}_{\varepsilon, \beta}$ by, for $a \in P_{\eta_\beta}^M$ letting $\mathbf{c}_{\varepsilon, \alpha}(a) = \min(\kappa \setminus \{\mathbf{c}_{\varepsilon, \beta}(b) : b \in P_\nu^M \text{ for some } \nu <_* \eta_\beta \text{ from } \mathcal{B}_\varepsilon \text{ and } \{b, a\} \text{ is an edge of } G_M\})$. Recalling there is no edge $\subseteq P_{\eta_\beta}$ this is a colouring. So we can carry the induction. So indeed \boxplus_5 holds.]

\boxplus_6 if $\mathcal{B} \subseteq \mathcal{A}$ is free and $M \in K_{\mathbf{a}}$ then $G_{M, \mathcal{B}}$ is the union of $\leq \kappa$ sets each with colouring number $\leq \kappa$ hence also chromatic number $\leq \kappa$.

[Why? By the proof of \boxplus_5 .]

\boxplus_7 $\text{chr}(G_M) > \kappa$ if $M \in K_{\mathbf{a}}^*$.

Why? Toward contradiction assume $\mathbf{c} : G_M \rightarrow \kappa$ is a colouring and let $<_*$ be a well ordering of \mathcal{A} . For each $\eta \in \mathcal{A}$ and $\varepsilon, \zeta < \kappa$ let $\Lambda_{\eta, \varepsilon, \zeta} = \{\nu : \nu \in \mathcal{A}, \nu <_* \eta, \nu R_\zeta \eta \text{ and } \varepsilon \in \mathcal{H}_\nu\}$ where for $\nu \in \mathcal{A}$ we define $\mathcal{H}_\nu = \{\varepsilon : \text{for some } a \in P_\nu^M \text{ we have } \mathbf{c}(a) = \varepsilon\}$.

Case 1: There is $\eta \in \mathcal{A}$ such that $(\forall \varepsilon \in \mathcal{H}_\eta)(\exists^\kappa \zeta < \kappa)[\Lambda_{\eta, \varepsilon, \zeta} \neq \emptyset]$.

So we can find a one-to-one function $g : \mathcal{H}_\eta \rightarrow \kappa$ such that $\Lambda_{\eta, \varepsilon, g(\varepsilon)} \neq \emptyset$ for every $\varepsilon \in \mathcal{H}_\eta \subseteq \kappa$. For each $\varepsilon \in \mathcal{H}_\eta \subseteq \kappa$ choose $\nu_\varepsilon \in \Lambda_{\eta, \varepsilon, g(\varepsilon)}$; possible as $\Lambda_{\eta, \varepsilon, g(\varepsilon)} \neq \emptyset$ by the choice of the function g . By the definition of “ $\nu_\varepsilon \in \Lambda_{\eta, \varepsilon, g(\varepsilon)}$ ” there is $a_\varepsilon \in P_{\nu_\varepsilon}^M$ such that $\mathbf{c}(\nu_\varepsilon) = \varepsilon$; recalling $\nu_\varepsilon \in \Lambda_{\eta, \varepsilon, \zeta}$ we have $\nu_\varepsilon R_\zeta \eta$ holds. So as $M \in K_{\mathbf{a}}^*$ there is $a \in P_\eta^M$ such that $\varepsilon \in \mathcal{H}_\eta \subseteq \kappa \Rightarrow F_{g(\varepsilon)}^M(a) = a_\varepsilon$, but then $\{a, a_\varepsilon\} \in \text{edge}(G_M)$ hence $\mathbf{c}(a) \neq \mathbf{c}(a_\varepsilon) = \varepsilon$ for every $\varepsilon \in \mathcal{H}_\eta \subseteq \kappa$, contradiction to the definition of \mathcal{H}_η .

Case 2: Not Case 1

So for every $\eta \in \mathcal{A}$ there is $\varepsilon \in \mathcal{H}_\eta \subseteq \kappa$ such that there are $< \kappa$ ordinals $\zeta < \kappa$ such that $\Lambda_{\eta, \varepsilon, \zeta} \neq \emptyset$. This means that there is $h : \mathcal{A} \rightarrow \kappa$ such that:

- ₁ $\eta \in \mathcal{A} \Rightarrow h(\eta) \in \mathcal{H}_\eta$ and
- ₂ $\eta \in \mathcal{A} \Rightarrow \kappa > |\{\zeta < \kappa : \Lambda_{\eta, h(\eta), \zeta} \neq \emptyset\}|$.

This implies that:

- ₃ $\eta \in \mathcal{A} \Rightarrow \kappa > |\text{exp}(\eta, h, \mathbf{a}, <_*)|$

because (we have •₂ and):

- ₄ if $\eta \in \mathcal{A}$ and $\varepsilon = h(\eta)$ then $\text{exp}(\eta, h, <_*, \mathbf{a}) \subseteq \{\zeta < \kappa : \Lambda_{\eta, \varepsilon, \zeta} \neq \emptyset\}$.

[Why? As $h : \mathcal{A} \rightarrow \kappa$ and if $\zeta \in \exp(\eta, h, <_*, \mathbf{a})$ let ν exemplify this, that is, $\nu <_* \eta, \nu R_\zeta \eta$ and $h(\nu) = h(\eta) = \varepsilon$ and recall $h(\nu) = \varepsilon$ implies $\varepsilon \in \mathcal{H}_\nu$ by \bullet_1 . But this means that $\nu \in \Lambda_{\eta, \varepsilon, \zeta}$ hence $\Lambda_{\eta, \varepsilon, \eta} \neq \emptyset$ as required.]

{r2} As $<_*$ was any well ordering of \mathcal{A} , this means, see 1.1(*)₂ holds, that \mathcal{A} is
{r2} strongly free, contradiction to 1.1(d). □_{1.2}

We can now reprove a result from [Sh:1006].

{r7}

Conclusion 1.3. 1) We have $\text{Inc}(\mu, \lambda, \kappa)$ when

(*) for some \mathcal{F} and natural number $\mathbf{k} > 0$ we have

(a) $\mathcal{F} \subseteq {}^\kappa \mu$ has cardinality μ and is tree like (i.e. $f_1(\bar{d}) = f_2(j) \wedge \{f_1, f_2\} \subseteq \mathcal{F} \Rightarrow f_1 \upharpoonright i = f_2 \upharpoonright j$)

(b) \mathcal{F} is not free where

- $\mathcal{F}' \subseteq \mathcal{F}$ is free means:
- there is a sequence $\langle \mathcal{F}'_i : i < \kappa \rangle$ such that $\mathcal{F}' = \cup \{ \mathcal{F}'_i : i < \kappa \}$ and for each i, \mathcal{F}'_i has a transversal which means that $\{ \text{Rang}(\eta) : \eta \in \mathcal{F}'_i \}$ has a transversal (= one-to-one choice function)

(c) \mathcal{F} is the increasing union of $\langle \mathcal{F}_\alpha : \alpha < \lambda \rangle$ such that each \mathcal{F}_α is free.

2) We have $\text{Inc}[\mu, \lambda, \kappa]$ when

(*) as above but replacing clause (c) by:

(c)' every $\mathcal{F}' \subseteq \mathcal{F}$ of cardinality $< \lambda$ has a transversal.

Proof. 1), 2) We define \mathbf{a} by choosing (for our \mathcal{F}):

- $\mathcal{A}_\mathbf{a} = \mathcal{F}$
- $<_\mathcal{A}$ any well ordering of \mathcal{F} ; not part of \mathbf{a}
- R_ε is defined by: $f R_\varepsilon g$ iff $f <_\mathcal{A} g \wedge f(\varepsilon) = g(\varepsilon)$
- for part (1) let $\vec{\mathcal{A}}$ be a sequence witnessing clause (c).

So it suffices to prove $\text{Inc}(\mu, \lambda, \kappa)$ or $\text{Inc}[\mu, \lambda, \kappa]$; hence it suffices to prove that \mathbf{a} witness it.

{r2} Now in Definition 1.1, clauses (a),(b),(c) are obvious. For clause (e), assume
{r7} $\mathcal{F}_2 \subseteq \mathcal{F}$ is free in the sense of 1.3(1)(b), and we shall prove that \mathcal{F}_2 is \mathbf{a} -free, this suffices for clause (e). By the assumption on \mathcal{F}_2 , clearly \mathcal{F}_2 is the union of $\langle \mathcal{F}_{2, \zeta} : \zeta < \kappa \rangle$, $\mathcal{F}_{2, \zeta}$ has a transversal \mathbf{h}_ζ . Now we define $h : \mathcal{F}_2 \rightarrow \kappa$ by: $h(f) = \text{pr}(\zeta, \varepsilon)$ where $\zeta = \min\{\xi : f \in \mathcal{F}_{2, \xi}\}$ and ε is minimal such that $\mathbf{h}_\zeta(\text{Rang}(f)) = f(\varepsilon)$, now the pairs $(h, <_\mathcal{A} \upharpoonright \mathcal{F}_2)$ witness that \mathcal{F}_2 is free (for \mathbf{a}).

For clause (d) toward contradiction assume that $h : \mathcal{F} \rightarrow \kappa$ and well ordering $<_*$ of \mathcal{A} witness \mathcal{F} is free for \mathbf{a} , hence $\vec{\mathcal{B}} = \langle \mathcal{B}_\varepsilon : \varepsilon < \kappa \rangle$ is a partition of \mathcal{F} when we let $\mathcal{B}_\varepsilon = \{f \in \mathcal{F} : h(f) = \varepsilon\}$.

{r2} By Definition 1.1, for each $\varepsilon < \kappa$ and $f \in \mathcal{B}_\varepsilon$ the set $u_f = \{\zeta < \kappa : \text{for some } g \in \mathcal{B}_\varepsilon \text{ we have } g R_\zeta f\}$ has cardinality $< \kappa$ and let $\zeta_f \in \kappa \setminus u_f$. For $\varepsilon, \zeta < \kappa$ let $\mathcal{B}_{\varepsilon, \zeta} = \{f \in \mathcal{B}_\varepsilon : \zeta_f = \zeta\}$ so $\langle \mathcal{B}_{\varepsilon, \zeta} : \varepsilon, \zeta < \kappa \rangle$ is a partition of \mathcal{A} . Now for each $\varepsilon, \zeta < \kappa$, if $f \neq g \in \mathcal{B}_{\varepsilon, \zeta}$ then $f(\zeta) \neq g(\zeta)$. Why? By symmetry we can assume $g <_\mathcal{A} f$ now $\zeta = \zeta_f \in \kappa \setminus u_f$, so g cannot witness $\zeta \in u_f$. So $\langle \mathcal{B}_{\varepsilon, \zeta} : \varepsilon, \zeta < \kappa \rangle$ contradicts clause (b) of the claim's assumption. □_{1.3}

{r18}

Claim 1.4. *If $\text{INC}[\mu, \lambda, \kappa]$ or $\text{INC}(\mu, \lambda, \kappa)$ then $\text{Inc}[\mu, \lambda, \kappa]$ or $\text{Inc}(\mu, \lambda, \kappa)$ respectively.*

Proof. As the two cases are similar we do the $\text{INC}(\mu, \lambda, \kappa)$ case, so let $G, \langle G_i : i < \lambda \rangle$ witness it.

Let $<_*$ be a well ordering of the set of nodes of G . Define $\mathbf{a} = (\mathcal{A}, \bar{\mathcal{A}}, \bar{R})$ by:

- \mathcal{A} is the set of nodes of G
- $\bar{\mathcal{A}} = \langle \mathcal{A}_i : i < \lambda \rangle$ with \mathcal{A}_i the set of nodes of G_i
- $R_\varepsilon = \{(\nu, \eta) : \{\nu, \eta\} \text{ an edge of } G \text{ and } \nu <_* \eta\}$.

Now check, noting when checking, that e.g. in $(*)_1$ of Definition 1.1, $\text{exp}(\eta, \alpha, <_*)$ is equal to κ or to \emptyset as $\bigwedge_\varepsilon R_\varepsilon = R_0$. {r2} $\square_{1.4}$

Private Appendix

§ 2

{b1}

Definition 2.1. For a linear order I let the graph $G_{I,k}$ be defined by: set of nodes $[I]^k$

$$\begin{aligned} \text{set of edges : } \{ \{u_1, u_2\} : & u_1, u_2 \in [\mathcal{U}]^k \text{ and } |u_1 \triangleleft u_2| = 2 \\ & \text{for some } t_0 <_I \dots <_I t_k \text{ we have} \\ & u_1 = \{t_0, \dots, t_{k-1}\}, u_2 = \{t_1, \dots, t_k\} \}. \end{aligned}$$

Toward resolving \aleph_ω : warm up.

We delay the problems of “ $\beth_k(\kappa)$ is singular”; not only $\beth_k(\kappa), S \notin \check{I}_\theta[\lambda]$ (not not the C_α 's not exactly a tree, each is union on θ set over an initial segment.

{b3}

Theorem 2.2. We have $\text{INC}(2^{<\lambda}, \lambda, \kappa)$ when:

⊞ for some θ

- (a) $\theta = \beth_k(\kappa)$ and $k \geq 1$
- (b) $S \subseteq S_\theta^\lambda$ is stationary, so λ is regular $> \theta$
- (c) (α) S is non-reflecting and $S \in \check{I}_\theta[\lambda]$ or just
- (β) $\bar{C} = \langle C_\delta : \alpha < \lambda \rangle, C_\alpha \subseteq \delta = \text{otp}(C_\alpha) \leq \theta, C_{\alpha_1} = C_{\alpha_2} \cap \alpha_1$ and $\delta \in S \Rightarrow \text{sup}(C_\delta)$ and $\bar{C} \upharpoonright S$ is λ -free.

Proof.

(*)₁ let $\mathcal{T} = \cup \{ \mathcal{T}_\alpha : \alpha < \lambda \}$ where \mathcal{T}_α be the set η such that:

- (a) $\eta = \langle a_\beta : \beta \leq \alpha \rangle = \langle a_{\eta, \beta} : \beta < \alpha \rangle$
- (b) $a_\beta \subseteq \beta$ has order type $< \theta^+$
- (c) if $\text{cf}(\delta) = \theta$ and $\delta = \text{sup}(a_\beta \cap \delta) < \text{sup}(a_\beta)$ then $\delta \in a_\beta \cap S$
- (d) if $\gamma \in C_\beta$ then $a_\gamma = a_\beta \cap \gamma$

{b1}

(*)₂ (a) let G be the graph with set of nodes $\{u: \text{for some } \alpha < \lambda \text{ and } \eta \in \mathcal{T}_{\alpha_{k-1}} \text{ we have } u \subseteq a_{\eta, \alpha_{k-1}} \cup \{\alpha_{k-1}\}\}$ set of edges as in 2.1

(*)₃ for $\alpha < \lambda$ let $G_\alpha = G \upharpoonright \{u \in G : u \subseteq \mathcal{T}_{\leq \alpha}\}$.

Clearly $\langle G_\alpha : \alpha < \lambda \rangle$ is \subseteq -increasing with union G and G has $2^{<\lambda}$ models hence it is enough to prove (*)₇ + (*)₈ below.

For the free part we first define:

(*)₄ choose I satisfying:

- (a) I is a linear order of cardinality λ
- (b) $\text{cf}(I) = \theta$
- (c) every interval is isomorphic to I

(*)₅ for $\alpha < \lambda$ we let f solve $\mathcal{T}_{<\alpha}$ when:

- (a) f is a function from $\mathcal{T}_{<\alpha}$ into I
- (b) if $\eta \in \mathcal{T}_\alpha$ then $f \upharpoonright [\{\eta \upharpoonright (\gamma+1) : \gamma+1 < \alpha\}]^k$ is one to one and into some proper initial segment of I and then prove

(*)₆ if (A) then (B) where:

- (A) (a) $\alpha \in \lambda \setminus S$ and $\alpha < \beta < \lambda$

- (b) f solves \mathcal{T}_α
- (c) $\bar{\mathcal{U}} = \langle \mathcal{U}_\eta : \eta \in \mathcal{T}_\alpha \rangle$
- (d) if $\eta \in \mathcal{T}_\alpha$ then $\mathcal{U}_\eta \subseteq I$ is convex bounded stationary subset of θ disjoint to $\{f(u) : u \in [\{\eta \upharpoonright (\gamma + 1) : \gamma + 1 < \alpha\}]^k\}$
- (B) there is g such that
 - (a) g solves \mathcal{T}_β
 - (b) g extends f
 - (c) if $\eta \in \mathcal{T}_\beta$ and $u \in [\{\eta \upharpoonright (\gamma + 1) : \gamma + 1 < \alpha\}]^k, u \notin \text{Dom}(f)$ then $g(u) \in \mathcal{U}_{\eta \upharpoonright (\alpha+1)}$.

[Why? We prove this by induction on β .]

Case 1: β is a successor ordinal.

So without loss of generality $\beta = \alpha + 1$ and easy.

Case 2: $\text{cf}(\beta) \text{ in } [\aleph_0, \theta]$

We choose an increasing continuous sequence $\langle \alpha_\varepsilon : \varepsilon \leq \text{cf}(\beta) \rangle$ with $\alpha_0 = \alpha, \alpha_{\text{cf}(\beta)} = \beta$. For each $\eta \in \mathcal{T}_\alpha$ let $\langle \mathcal{U}_{\eta, \varepsilon} : \varepsilon < \text{cf}(\beta) \rangle$ be a partition of \mathcal{U}_η each $\mathcal{U}_{\eta, \varepsilon}$ of cardinality θ .

Now we choose f_ε by induction on $\varepsilon \leq \text{cf}(\beta)$ such that:

- f_ε solves $\mathcal{T}_{<\alpha_\varepsilon}$
- $f_\varepsilon = f$ if $\varepsilon = 0$
- $\langle f_\zeta : z\eta \leq \varepsilon \rangle$ is \subseteq -increasing continuous
- if $\eta \in \mathcal{T}_{\alpha_\varepsilon}$ and $u \in [\{\eta \upharpoonright (\gamma + 1) : \gamma + 1 < \alpha_\varepsilon\}]^k$ and $u \notin \text{dom}(f)$ then $f_\varepsilon(u) \in \cup \{\mathcal{U}_{\eta \upharpoonright \alpha, \zeta} : \zeta < \varepsilon\}$.

For $\varepsilon = 0$ we let $f_\varepsilon = f$ for ε successor say $\varepsilon = \zeta + 1$ use the induction hypothesis with $f_\zeta, \alpha_\zeta, \alpha_\varepsilon$ here standing for f, α, β there

- for ε limit ordinal let $f_\varepsilon = \cup \{f_\zeta : \zeta < \varepsilon\}$.

This is obviously possible and $f_{\text{cf}(\beta)}$ is as promised, so $(*)_6$ holds.

- $(*)_7$ $G_\alpha = G \upharpoonright \mathcal{T}_{<\alpha}$ has chromatic number $\leq \kappa$.

[Why? Let $\mathbf{c} : [\theta]^k \rightarrow \kappa$ be a colouring of the EH graphs.

As trivially there is a solution f_0 of \mathcal{T}_1 , by $(*)$ there is a solution f_α of $\mathcal{T}_{<\alpha}$. Let $\mathbf{c}_\alpha : G_\alpha \rightarrow \kappa$ be $\mathbf{c}_\alpha(u) = \text{boldc}(\{f(\nu) : \nu \in u\})$ for u a role of G_α . Now check.]

- $(*)_8$ G has no colouring with κ colours.

[Why? Toward contradiction assume $\mathbf{c} : G \rightarrow \kappa$ is a colouring. For $\alpha < \lambda$ let $\mathcal{A}_{<\alpha} = \{a : a \text{ a subset of order types } < \theta^+ \text{ such that if } \text{cf}(\delta) = \theta \wedge \delta < \sup(a) \wedge \delta = \sup(a \cap \delta) \text{ then } \delta \in a \wedge \delta \in S\}$. Let $\mathcal{P}_\alpha \subseteq [\alpha]^{<\theta}$ be of cardinality $< \theta$ such that $\langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$ witness $S \in \check{I}_\theta[\lambda]$.

For $\alpha < \lambda$ and $a \in \mathcal{A}_{<\alpha}$ let $\text{hrtg}_a = \{p : p \text{ is a set of cardinality } \leq \beth_{k-1}(\kappa) \text{ equation of the form } c(\{\alpha_0, \dots, \alpha_{k-2}, x\}) = i \text{ for some } \alpha_0 < \dots < \alpha_{k-2} \text{ are from } a, i < \kappa\}$.

- $(*)_8.1$ let \mathbf{Y} be the set of (η, \mathcal{S}, g) such that

- $(*)$ (a) $\eta \in \mathcal{T}$

- (b)(α) $\mathcal{X} \subseteq \ell g(\eta)$ is of cardinality $< \theta$
 (β) if $\beta \in \mathcal{X}$, $\beta \cap \mathcal{S}$ is non-empty with no last member then $X \cap \beta$ is unbounded in a_β
 (c) g is a function with domain \mathcal{X}
 (d) if $\beta \in \text{Dom}(g)$ then
 (α) $g(\beta)$ is of the form $\langle p_{g(\beta), \varepsilon} : \varepsilon \in w_g \rangle$
 (β) $w_{g, \beta}$ is a non-stationary subset of θ
 (γ) $p_{g, \varepsilon} \in \mathcal{P}_{a_{\eta, \eta}}$
 (e) if $\beta_1 \in a_{\eta, \beta_2}$ so $\beta_1 < \beta_2 < \ell g(\eta)$ then $w_{g, \beta_1} \subseteq w_{g, \beta_2}$; moreover $g(\beta_1) = g(\beta_2) \upharpoonright w_{g, \beta_2}$
- (*)_{8.2} we define the two-place realtion \leq_η on $\mathcal{Y} : (\eta_1, \mathcal{X}_1, g_1) \leq_{\bar{x}} (\eta_2, \mathcal{X}_2, g_2)$ iff
 (a) both belong to Y
 (b) $\eta_1 \leq \eta_2$
 (c) $\mathcal{X}_1 = \mathcal{X}_2 \cap \ell g(\eta_1)$
 (d) $g_1 = g_2 \upharpoonright X_1$
- (*)_{8.3} (a) for $(\eta_1, \mathcal{X}_1, g_1) \in \mathbf{Y}$ and $\beta \in \mathcal{X}_1$ let $(\xi, \beta, \eta, x, g) = \min\{\varepsilon : \varepsilon = \theta \text{ or } \varepsilon \in w_g \text{ for some } (\eta_2, \beta_2, X_3, g_2) \text{ which is } <_* \text{ above } (\eta_1, \mathcal{X}_1, g_2) \text{ and } \beta_a \in \mathcal{X}_2 \text{ we have } \beta_1 \in a_{\eta_2, \beta_2} \text{ and } \beta_1 \text{ realizes } p_{g(\beta_1), \varepsilon} \text{ but no } \alpha < \ell g(\eta) \text{ realize the type } p_{g(\eta), \varepsilon}, \text{ see below}\}$
 (b) we say $\alpha < \ell g(\eta)$ realizes the type p in η when: if $c(\{\alpha_0, \dots, \alpha_{k-1}, x\}) = i$ then $\mathbf{c}(\{\alpha_0, \dots, \alpha_{k-1}, \alpha\}) = i \text{edge} \alpha_{k-1}$ and so necessarily $\{\alpha_0, \dots, \alpha_{k-1}\} \subseteq a_{\eta, \alpha}$.
- (*)_{8.4} we define a two-place realtion \leq_2 on \mathbf{Y} by; $(\eta_1, \mathcal{X}_1, g_1) <_2 (\eta_2, \mathcal{X}_2, g_2)$ iff $(\eta_1, \mathcal{X}_1, g_1) <_1 (\eta_2, \mathcal{X}_2, g_2)$ and
 (e) $p_{g(\beta_1), \xi}$ is realized by some $\gamma \in a_{\beta_2}$ in η_2 when
 (α) $\beta_2 \in \mathcal{X}_2$ hence $< \ell g(\eta_2)$ but $\beta_2 \geq \ell g(\eta_1)$
 (β) $\beta_1 \in a_{\eta_2, \beta_2} \cap \mathcal{X}_1$
 (γ) $\xi = \xi(\beta_1, \eta_1, \mathcal{X}_1, g_1) < \theta$
 (δ) if $\text{otp}(a_{\beta_2} \cap \mathcal{X}_1)$ is a limit ordinal then $|\mathcal{X}_2 \setminus \mathcal{X}_1| > 1$
 (ε) there is no $\beta'_1 \in a_{\eta_2, \beta_2} \cap \mathcal{X}_1$ such that $\xi(\beta'_1, \eta_1, \mathcal{X}_1, g_1) < \xi$
- (*)_{8.5} (a) \leq_1 partially order \mathbf{Y}
 (b) any \leq_1 -increasing sequence of length $< \lambda$ in \mathbf{Y} has an upper bound
 (c) \leq_2 partially order \mathbf{Y}
 (d) for any \leq_2 -increasing sequence of length $< \lambda$ in \mathbf{Y} has an upper bound
- (*)_{8.6} if $\delta < \theta$ is a limit ordinal, $(\eta_1, \mathcal{X}_1, g_1) \in \mathbf{Y}$, $\beta_\varepsilon \in \mathcal{X}_1$ for $\varepsilon < \delta$, $\varepsilon < \zeta < \delta \Rightarrow \beta_\varepsilon \in a_{\eta_1, \beta_\zeta}$ then for some $(\eta_2, \mathcal{X}_2, g_2) \in Y$ which is \leq_2 -above $(\eta_1, \mathcal{X}_1, g_2)$ and $\beta \in \mathcal{X}_2$ we have $\varepsilon < \delta \rightarrow \beta_\varepsilon \in a_{\eta_2, \beta_\zeta}$.

[Why? Easily.]

- (*)_{8.7} if $(\eta_1, \mathcal{X}_1, g_1) \in \mathbf{Y}$ and $\beta \in \mathcal{X}_1$ then there is $(\eta_2, \mathcal{X}_2, g_2) \in \mathbf{Y}$ which is \leq_2 -above $(\eta_1, \mathcal{X}_1, g_2)$ and for some $\beta_2 \in [\ell g(\eta_1), \ell g(\eta_2))$ from \mathcal{X}_2 we have $a_{\eta_2, \beta_2} = a_{\eta_1, \beta_1} \cup \{\beta_1\}$

(*)_{8.8} there is a \leq_2 -increasing sequence $\{\mathbf{y}_\alpha = \langle (\eta_\alpha, \mathcal{X}_\alpha, g_\alpha) : \alpha < \lambda \rangle$ of members of \mathbf{Y} which is generic enough, i.e.

- (a) let $\chi > 2^\lambda$
- (b) $N_\alpha \prec (\mathcal{H}(\chi), \in)$ is of cardinality $< \lambda$ for $\alpha < \lambda$ be such that $\bar{N} = \langle N_\beta : \beta < \lambda \rangle$ is \prec -increasing continuous and $\bar{H} \upharpoonright (\alpha + 1) \in N_{\alpha+1}$ for $\alpha < \lambda$
- (c) $\bar{\mathcal{P}} \in N_0$ and $\{S, \bar{a}\} \subseteq N_0$
- (d) $\langle (\eta_\beta, \mathcal{X}_\beta, g_\beta) : \beta < \alpha \rangle \in N_{\alpha+1}$.

□

Discussion 2.3. We have two drawbacks in 2.2:

{b3}

(A) the demand “ θ is regular” and moreover “ $\theta = \beth_k$ ”.

For this it is desirable that instead of using the EH graph (2.1) on $[\theta]^k$, really $[I]^\theta$, we use a graph that exists for more θ 's, so it seems helpful to point out a sufficient condition for such graphs.

{b1}

It will help to have three successive cardinals $\theta^+, \theta^{+2}, \theta^{+3}$ and if θ is singular, $\theta^{+2}, \theta^{+3}, \theta^{+4}$. An easy solution is Definition 2.4 below, more profound is considering appropriate games, but it is not clear if this really helps, i.e. enable us to induct

{b12}

(B) eliminating “ $S \in \check{I}_\theta[\lambda]$ ”, this is helpful as then we have existence theorems, we intend to weaken the way the “memory for the a_i 's”.

We intend to allow for

- $a \triangleleft c_i$ for $i < \theta, i < j \Rightarrow \sup(a_i) < \min(a_2 \setminus a), \delta = \sup(\bigcup_i a_i), \langle \alpha_{\delta, j} : j < \theta \rangle$ list C_δ (guessing clubs) and $a_i a \subseteq [\alpha_{\delta, i}, \alpha_{\delta, i+1})$.

But we have to allow some edges between the $\langle a_{\delta, j} \setminus a : j < \theta \rangle$. Is it helpful to have $|a_\delta \setminus a| = \sigma, \sigma \geq \theta$. Note that here the presentation via k -tuples is not helpful.

{b12}

Definition 2.4. 1) $G_\theta^{k,m}$ is the graph such that:

- (a) the set of nodes is $[\theta]^{k+m}$
- (b) the set of edges is the set of pairs $\{u_1, u_2\}$ such that
 - (α) $u_\ell \in [\theta]^{k+m}$ let $\alpha_{\ell,0} < \dots < \alpha_{\ell,k+m}$ list it in increasing order
 - (β) $\alpha_{1,i+1} = \alpha_{2,i}$ for $i = 0, \dots, n-1$
 - (γ) $\alpha_{2,m} < \alpha_{1,n+1} < \alpha_{2,n+1} < \alpha_{1,n+2} < \dots$

2) Probably better to use the tree indiscernibility for (\aleph_n, \aleph_0)

(C) maybe letting $\sigma = \beth_{k-2}(\kappa), \theta = \beth_k(\kappa)$ if $\theta > \sigma^{+3}$ or so, for $\ell(*) \in \{1, 2, 3\}, S \subseteq S_{\sigma+\ell(\delta)}^\lambda$ stationary not reflecting $2^\mu = \lambda = \mu^+, \mu^{<\sigma} = \mu$.

So we have “ $C_\delta \subseteq \delta = \sup(C_\delta)$ ”, $\delta \in S \subseteq S_\theta^\lambda, \alpha_\delta$ and earlier “ $\alpha \in C_\delta \Rightarrow C_\delta \cap \alpha \in \mathcal{P}_\alpha$ or at least $\mathcal{P}_{<\delta}$ ” and now $\{u \cap C_\delta : u \in \mathcal{P}_{<\delta}, |u| \leq \sigma\}$ is equal to $[C_\delta]^{\leq \sigma}$ or just cofinal for \subseteq and $\text{cf}(\mu)^{+\ell(*)+1(?)} = \theta^{+\ell(*)}$.

(D) So if we fix $\sigma = \text{cf}(\sigma) < \beth_\omega$ and is $\lambda = \aleph_\sigma$

(E) we forgot to deal with $\mathcal{F} \subseteq {}^\theta\mu$ of cardinality μ^+ , $\text{cf}(\mu) = \theta$, \mathcal{F} is μ^+ -free. This calls for using $S \subseteq S^{\mu^+}$ and without loss of generality $\langle C_\delta : \delta \in S \rangle$ is μ^+ -free even if S reflects.

There is $\bar{C} = \langle C_\alpha : \alpha \in S \rangle$, $\text{otp}(C_\alpha) \leq \theta^+$, C_δ is θ -closed. Now \bar{C} forms a tree and the proof of 2.2 should deal with this “memory”.

{b3} This is an important part; it is better to combine it with ?? but have not elaborated (use convergence along with some C_δ of θ sets defined from a sequence of “lower” bound subsets of δ , etc...)

{k3} We only have $[C_\delta]^{<\sigma}$ in the memory, more complicated.

Let N_* be $(N_\lambda, \{\mathbf{y}_\alpha : \alpha < \lambda\})$, where $N_\lambda = \bigcup_{\alpha < \lambda} N_\alpha$. As $\mathcal{S} \subseteq \lambda$ is stationary, necessary for some $\delta \in S$ we have $N_\delta^* := (N_\delta, \{\mathbf{y}_\alpha : \alpha < \lambda\}) \prec N_*$. Now let $\mathbf{y}' = \bigcup_{\alpha < \delta} \mathbf{y}_\alpha$ and using $\bar{\mathcal{P}}_*$ we easily get contradiction.

§ 3

Below the case θ singular is on the one hand opaque; on the other hand we arrive to θ singular in the inductive construction. The aim of defining \mathbf{M} is to use $\mathbf{m} \in \mathbf{M}$ to build examples for $\text{INC}(\lambda, \kappa)$ for regular λ with some stationary $S \subseteq S_\theta^\lambda$ which does not reflect. It would be better if the property for θ implies the property for θ^+ , for 2^θ or $(2^\theta)^+$ and for \aleph_θ it implies $\text{INC}(\theta, \kappa)$.

Definition 3.1. Assume θ, σ are regular cardinals, $\mu \geq \theta \geq \sigma \geq \kappa$.

{c3}

Let $\mathbf{M} = M_{\mu, \theta, \sigma, \kappa}$ be the class of objects \mathbf{m} consisting of (so $\tau = \tau_{\mathbf{m}}, \dots$ omitting μ means for some μ) the cardinals $\mu, \theta, \sigma, \kappa$ and:

- (a) (α) a vocabulary τ but the function symbols will be interpreted as partial
- (β) τ has cardinality $\leq \mu$
- (γ) $R \in T$ a two-place predicate intended for a graph
- (b) (α) a normal tree \mathcal{T} with root $\text{rt}(\mathcal{T})$
- (β) \mathcal{T} has $\leq \theta^+$ levels, is ($< \theta$)-complete and has no θ^+ -branch
- (γ) $\text{lev}(\eta) = \text{lev}_{\mathcal{T}}(\eta) = \text{lev}_{\mathbf{m}}(\eta)$ is the level of η
- (δ) for $\varepsilon < \text{lev}(\eta)$ let $\eta|_\varepsilon$ be the unique $\nu <_{\mathcal{T}} \eta, \text{lev}(\nu) = \varepsilon$
- (ε) $\mathcal{T}_\varepsilon = \{\eta \in \mathcal{T} : \text{lev}(\eta) = \varepsilon\}$
- (ζ) for $\eta \in \mathcal{T}$ let $\text{pre}(\eta)$ be its predecessor if $\text{lev}(\eta)$ is a successor ordinal and is η otherwise
- (η) $\langle \mathcal{S}_\ell : \ell < 4(?) \rangle$ is a partition of $\mathcal{S}_* = \cup \{\mathcal{T}_{\varepsilon+1} : \varepsilon < e^+\}$
- (c) (α) for $\eta \in \mathcal{T}, M_\eta$ a τ -model
- (β) M_η is increasing continuous with η
- (γ) M_η is generated by P^{M_η}
- (δ) $G_\eta = (|M_\eta|, R^{M_\eta})$ is a graph
- (ε) option: \mathbf{c}_η is a colouring of G_η (or $\bigcup_{\nu <_{\mathcal{T}} \eta} G_\nu$) is increasing with η
- [helpful? omit?]
- (d) (α) $u_\eta \subseteq v_\eta$ are subsets of P^{M_η}
- (β) • u_η is of cardinality $< \sigma$
- v_η is of cardinality $< \theta$
- if $v_\eta \neq \emptyset$ then $\text{lev}(\eta)$ is a successor ordinal
- (γ) $P^{M_\eta} = \cup \{u_\nu : \nu \leq_{\mathcal{T}} \eta\}$
- (e) (α) \bar{a}_η lists u_η with no repetitions
- (β) $A_\eta \subseteq M_{\text{pre}(\eta)}$
- (γ) if $\eta \in \mathcal{S}_\eta$ then A_η has cardinality $< \sigma$
- (δ) let $N_\eta^- = M_\eta \upharpoonright \text{cl}_{M_\eta}(A_\eta)$ and $N_\eta = M_\eta \upharpoonright \text{cl}_{M_\eta}(A_\eta \cup v_\eta)$
- (ε) if $A_\eta \neq \emptyset$ then $\text{lg}(\eta)$ is a successor ordinal
- (η) [optional, used?] if $\eta \in \text{suc}_{\mathcal{T}}(\nu)$ then M_η is the union of $N_\eta^+, M_{\text{pre}(\eta)}$ over N_η^- , so in particular for every function symbol $F \in \tau, F^{M_\eta} = F^{N_\eta^+} \cup F^{M_{\text{pre}(\eta)}}$
- (f) (α) $<_{\mathbf{m}}$ is a (linear) well ordering of \mathcal{T} such that $\eta <_{\mathcal{T}} \nu \Rightarrow \eta <_{\mathbf{m}} \nu$

(β) if $\eta \in \mathcal{S}_1$ and $\nu = \text{pre}(\eta)$ then for some ρ we have $(\eta, \nu, \varrho, \rho) \in \mathbf{Q}_m$ which implies (means (?))

- ₁ $\nu <_{\mathcal{T}} \rho$ and $\eta \not<_{\mathcal{T}} \rho$ and $\rho <_m \eta$
 - ₂ $f = f_{\eta, \rho}$ is an embedding of M_ρ into M_η
 - ₃ f is the identity on N_ν^-
 - ₄ if $x \in M_\rho$ then $\mathbf{c}_\eta(f(x)) = \mathbf{c}_\rho(x)$
 - ₅ more? put the colouring as in Definition 3.2?
- {c6}

* * *

{c6}

Definition 3.2. We say $\mathbf{m} \in \mathbf{M}_{\mu, \theta, \sigma, \kappa}$ is colourable when for every $\lambda > \theta$ in the following game $\mathcal{D} = \mathcal{D}_{\mathbf{m}, \lambda}$ the colouring player has a winning strategy:

- (A) (a) a play lasts up to θ moves
 (b) if a player has no legal choice, it is a loss
 (B) before the ε -th move a pair $(\eta_\varepsilon, \mathbf{c}_\varepsilon)$ is given or has been produced such that

- (a) $\eta_\varepsilon \in \mathcal{T}_m$ and $\varepsilon = 0 \Rightarrow \eta_\varepsilon = \text{tr}(\mathcal{T}_m)$
 (b) \mathbf{c}_ε is a κ -colouring of the graph G_{η_ε} (no connection to $\mathbf{c}_{\eta_\varepsilon}$ is required!)

- (c) if $\zeta < \varepsilon$ then $\mathbf{c}_\zeta, \mathbf{c}_\varepsilon$ agree on $M_{\eta, \eta_\varepsilon}$
 (d) if $\varepsilon < \theta$ is a limit ordinal then

(α) if for some unbounded subset S of ε , $\langle \eta_\zeta : \zeta \in S \rangle$ is $<_I$ -increasing then $\eta_\varepsilon = \bigcup_{\zeta \in S} \eta_\zeta, \mathbf{c}_\varepsilon = \bigcup_{\zeta \in S} \mathbf{c}_\zeta$

(β) if there is no such S then by (A)(b) the play stops and the colouring plays

- (e) if $\varepsilon = \zeta + 1$ then $(\eta_\varepsilon, \mathbf{c}_\varepsilon)$ was chosen during the ζ -th move

- (C) in the ε -th move the anti-colouring player has some possibilities

Pos(a): (a) choose $\eta_{\varepsilon+1} \in \mathcal{T}$ such that $\eta_\varepsilon <_{\mathcal{T}} \eta_{\varepsilon+1}$ (but not necessary $\eta_\varepsilon = \text{pre}(\eta_{\varepsilon+1})!$)

(b) then the colouring player has to choose a legal $\mathbf{c}_{\varepsilon+1}$

Pos(b): choose $\nu_\varepsilon <_{\mathcal{T}} \eta_\varepsilon, \eta'_{\varepsilon+1} \in \text{suc}_{\mathcal{T}}(\nu_\varepsilon)$ and $\eta_{\varepsilon+1}$ such that

- $\eta_\varepsilon <_m \eta_{\varepsilon+1}$
- $x \in M_{\eta_\varepsilon} \wedge f_{\eta_{\varepsilon+1}, \eta_\varepsilon}(x) \in M_{\nu_\varepsilon} \Rightarrow \mathbf{c}_\varepsilon(f_{\eta_{\varepsilon+1}, \eta_\varepsilon}(x)) = \mathbf{c}_\varepsilon(x)$.

Then the colouring player has to choose a colouring $\mathbf{c}_{\varepsilon+1}$ of $G_{\eta_{\varepsilon+1}}$ extending $\mathbf{c}_\varepsilon \upharpoonright G_{\nu_\varepsilon}$ and such that $x \in M_{\eta_\varepsilon} \Rightarrow \mathbf{c}_{\varepsilon+1}(f_{\eta_{\varepsilon+1}, \eta_\varepsilon}(x)) = \mathbf{c}_\varepsilon(x)$.

Discussion 3.3. For θ singular maybe consider.

{c6} **Observation 3.4.** (How a play goes) In 3.2(B)(d), if we have two such S 's they give the same result (or waive the normality of \mathcal{T}_m).

Proof. FILL! □

* * *

Of course, it is easy to find $\mathbf{m} \in \mathbf{M}_{\mu, \theta, \sigma, \kappa}$ which is colourable, e.g. if all the graphs $G_{\mathbf{m}, \eta}$ are empty. So we like to have a property in the direction.

Definition 3.5. We say $\mathbf{m} \in \mathbf{M}_{\mu, \theta, \sigma, \kappa}$ is full when: the anti-colouring player has a winning strategy in the game $\mathfrak{D} = \mathfrak{D}_{\mathbf{m}}^c$ where

{c10}

- (A) a play of the game last θ moves
- (B) (a) just before the ε move a pair $\eta_\varepsilon(\eta_\varepsilon, \mathbf{c}_\varepsilon)$ is chosen
 (b) $\eta_\varepsilon \in \mathcal{T}_\varepsilon$ is $<_T$ -increasing continuous with ε
 (c) \mathbf{c}_ε is a κ -colouring of increasing continuous with ε
 (d) if $\varepsilon = 0$ then $\eta_\varepsilon = \text{rt}(\mathcal{T})$, so M_{η_ε} is empty and \mathbf{c}_ε the empty function

(C) in the ε -th move:

- (a) the anti-colouring player chooses a set \mathcal{X}_ε such that
 (α) $\mathcal{X}_\varepsilon \subseteq \{(\eta, \mathbf{c}) : \eta_\varepsilon \leq_{\mathcal{T}} \eta \text{ and } \mathbf{c} \text{ is a } \kappa\text{-colouring of } G_\eta \text{ extending } \mathbf{c}_{\eta_\varepsilon}\}$
 (β) \mathcal{X}_ε is closed under isomorphisms: if $(\mathbf{c}) \in \mathcal{X}_\varepsilon, \nu \in \mathcal{T}_{\text{lev}}$ a κ -colouring and there is an isomorphism from G_ρ onto G_ν over G_{η_ε} mapping \mathbf{c} to \mathbf{d} then $(\nu, \mathbf{d}) \in \mathcal{X}_\varepsilon$

(b) then the colouring player chooses $\eta_{\varepsilon+1} \in \text{suc}_{\mathcal{T}_\varepsilon}(\eta_\varepsilon)$ such that one of the following occurs:

- (α) there is no pair $(\eta, \mathbf{c}) \in \mathcal{X}_\varepsilon$ satisfying $\eta_{\varepsilon+1} \leq_{\mathcal{T}} \eta$
 (β) there is a pair $(\eta, \mathbf{c}) \in c\mathcal{X}_\varepsilon$ such that $(\eta_{\varepsilon+1}, \eta_\varepsilon, \eta, (\mathbf{c}))$ are as in 3.2(f) and $\mathbf{c}_{\eta_{\varepsilon+1}}(f_{\eta_{\varepsilon+1}, \eta}(x)) = \mathbf{c}(x)$

{c6}

(D) in the end of the play, the anti-colouring palyer wins when letting $\eta_\theta = \bigcup_{\varepsilon < \theta} \eta_\varepsilon, \mathbf{c}_\theta = \bigcup_{\varepsilon < \theta} \mathbf{c}_\varepsilon$ there is $\eta \in \text{suc}(\eta_\theta)$ such that no κ -colouring \mathbf{c} of G_η extending \mathbf{c}_θ such that: for some...?

Discussion 3.6. So we like to prove:

- ₁ if $\mathbf{M}_{\theta, \sigma, \kappa} \neq \emptyset$ then $\text{INC}[\theta^+, \sigma, \kappa]$
- ₂ if $\mathbf{M}_{\theta, \sigma, \kappa} \neq \emptyset$ then $\mathbf{M}_{\theta^+, \sigma, \kappa}$
 [at least if θ is regular this is similar to the \aleph_θ case]
- ₃ $\mathbf{M}_{\mu^+, \sigma, \kappa} \neq \emptyset$ when $\mu > \text{cf}(\mu) = \theta^+, \mathbf{M}_{\theta, \lambda, \kappa}$ and
 - (a)₁ $\text{pp}(\mu) > \theta^+$ or at least
 - (a)₂ there is $\bar{\lambda} = \langle \lambda_\varepsilon : \varepsilon < \theta^+ \rangle$ is an increasing sequence of regulars with limit $\mu, \mu^+ = \text{tcf}(\prod_{\varepsilon} \lambda_\varepsilon, <_{J_{\theta^+}^{\text{bd}}})$ and for some \bar{f} witnessing this, $\text{bad}(\bar{f}) = \emptyset$ or at least
 - (a)₃ there is $S \subseteq S_\theta^{\mu^+}$ and $\bar{C} = \langle C_\delta : \delta \in S \rangle$ a tree-like strict club system which is free and $S \in \check{I}_\theta[\mu^+]$
- ₄ assume $\mathbf{M}_{\theta, \sigma, \kappa} \neq \emptyset, \theta$ regular, $\lambda = \text{cf}(\lambda) > \theta, S \subseteq S_\theta^\lambda$ stationary not reflecting and $\alpha < \lambda \Rightarrow |\alpha|^{< \sigma} < \lambda$ then $\mathbf{M}_{\lambda, \sigma, \kappa} \neq \emptyset$.

Even more if

- ₅ if $\mathbf{M}_{\theta,\sigma,\kappa}$ and ∂ is the first regular cardinality $\geq 2^\theta$ then $\mathbf{M}_{\partial,\sigma,\kappa}$ or $M_{\partial,\theta,\kappa}$.

§ 4. PRIVATE APPENDIX

Moved 4/2013 from §1, pg.6:

Conclusion 4.1. *For any κ , for arbitrarily large regular $\lambda < \min\{\theta : \theta = \aleph_\theta > \kappa\}$ we have $\text{INC}^+(\lambda, \lambda, \kappa)$.*

{r8}

Remark 4.2. Note the variant of transversal we use for our purpose is equivalent by [Sh:161].

{r12}

Proof. If $\kappa = \aleph_0$, by Magidor-Shelah [MgSh:204]. Generally similar.

□_{4.1}

§ 5. 10 OLDER MATERIAL

Discussion 5.1. Do we have $\text{IC}_{\text{chr}}(\lambda, \aleph_\omega, \aleph_0)$ for some λ ? Assume not.

- 1) If $\mu \in \mathbf{C}_{\aleph_0}$, then necessarily $\text{pp}(\mu) = \mu^+$ hence $2^\mu = \mu^+$.
- 2) Let $\mu = \sum_n \lambda_n, \lambda_n = \text{cf}(\lambda_n) < \lambda_{n+1}, \bar{f} = \langle f_\alpha : \alpha < \mu^+ \rangle$ witness $\mu^+ = \text{tcf}(\prod \lambda_n, <_{J_{\aleph_0}^{\text{bd}}})$.

If $S_* = \text{bad}_{\leq \aleph_\omega}(\bar{f}) = \text{bad}(\bar{f} \cap S_{< \aleph_\omega}^{\mu^+})$ non-stationary, we are done. Otherwise for some $\ell \in \{1, 2, 3\}$ there is $S \subseteq S_{\aleph_\ell}^{\mu^+}$ stationary not reflecting in any $\delta \in S_{< \aleph_\omega}^{\mu^+}$, so we have \diamond_S .

- 3) Recall for every $n, \text{IC}_{\text{chr}}(\beth_n^+(\kappa), \beth_n^+(\kappa), \kappa)$; so we can assume $\bigwedge_n \beth_n < \aleph_\omega$, so $\beth_\omega = \aleph_\omega$.

4) Still we can prove: for some $\ell \in \{0, 1, 2, 3\}, (\forall \lambda), \text{IC}_{\text{chr}}(\lambda, \aleph_\omega, \aleph_\ell)$. Can we use $X \subseteq [\mu]^n$. Use [Sh:620]?

5) Can we use an example of part (3) on some $\kappa = \text{cf}(\kappa) < \beth_u$ and use...

6) Can we take stationary $S \subseteq S_\theta^\lambda$ not reflecting $S \notin \check{I}_\theta[\lambda]$, but on a partial square of it.

We use [Sh:1006, 0.3=y8].

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