MODEL THEORY FOR A COMPACT CARDINAL
SH1019

SAHARON SHELAH

Abstract. We would like to develop classification theory for T, a complete theory in $\mathbb{L}_{\theta,\theta}(\tau)$ when θ is a compact cardinal. We already have bare bones stability theory and it seemed we can go no further. Dealing with ultrapowers (and ultraproducts) naturally we restrict ourselves to "D a θ-complete ultrafilter on I, probably (I,θ)-regular". The basic theorems of model theory work and can be generalized (like Łos theorem), but can we generalize deeper parts of model theory?

The first section is trying to sort out what occurs to the notion of "stable T" for complete $\mathbb{L}_{\theta,\theta}$-theories T. We generalize several properties of complete first order T, equivalent to being stable (see [Sh:c]) and find out which implications hold and which fail.

In particular, can we generalize stability enough to generalize [Sh:c, Ch.VI]? Let us concentrate on saturation in the local sense (types consisting of instances of one formula). We prove that at least we can characterize the T's (of cardinality $\le \theta$ for simplicity) which are minimal for appropriate cardinal $\lambda \ge 2^\kappa + |T|$ in each of the following two senses. One is generalizing Keisler order $\triangleleft$ which measures how saturated are ultrapowers. Another generalizes the results on $\triangleleft^*$, that is, we ask: Is there an $\mathbb{L}_{\theta,\theta}$-theory $T_1 \supseteq T$ of cardinality $|T| + 2^\theta$ such that for every model M_1 of T_1 of cardinality $> \lambda$, the $\tau(T)$-reduct M of M_1 is λ^+-saturated. Moreover, the two versions of stable used in the characterization are different.

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The author would like to thank the Israel Science Foundation for partial support of this research (Grant No. 1053/11). References like [Sh:797, 2.11=La18] means we cite from [Sh:797], Claim 2.11 which has label a18, this to help if [Sh:797] will be revised. The author thanks Alice Leonhardt for the beautiful typing. First typed May 10, 2012.

Annotated Content

§ 0. Introduction

{0A}

§ 0(A). **Background and results.** In Winter 2012, I have tried to explain in a model theory class, a position I held for long: model theory can extensively deal with $\mathbb{L}_{\lambda^+,\aleph_0}$-classes and a.e.c. however while we can generalize basic model theory to $\mathbb{L}_{\lambda,\kappa}$-classes, $\lambda \geq \kappa > \aleph_0$, see [Dic85], we cannot do considerably more. The latter logics are known to have downward LST theorems and various connections to large cardinals and consistency results, and only rudimentary stability theory (see [Sh:300a]). Note that, e.g. there is $\psi \in \mathbb{L}_{\aleph_1,\aleph_1}$ such that $M \models \psi$ iff M is isomorphic to $(\mathbf{L}_\alpha, \in)$ for some ordinal α such that $\beta < \alpha \Rightarrow [\mathbf{L}_\beta]^{\leq\aleph_0} \cap \mathbf{L} \subseteq \mathbf{L}_\alpha$. Hence, assuming $\mathbf{V} = \mathbf{L}$ if $\mu > \mathrm{cf}(\mu) = \aleph_0$ then ψ has a model of cardinality μ and every model M of ψ of cardinality μ is isomorphic to $(\mathbf{L}_\mu, \in)$. It folows that, e.g. for every second order sentence φ, there is $\psi \in \mathbb{L}_{\aleph_1,\aleph_1}$ which is categorical in the cardinal λ iff $(\exists\mu)(\mathbf{L}_\mu \models \varphi$ and $\lambda = \mu^{+\omega})$; so the categoricity spectrum is not so nice. Similar results hold if, e.g. $0^\#$ does not exist, noting that: if $\geq \theta^\#$ and $\mu > \mathrm{cf}(\mu) = \aleph_0$ then for some real r, in $\mathbf{L}[r], \mu$ has cofinality $\aleph_0$. Such views have been quite general - see Väänänen's book [Vää11].

This work is dedicated to starting to try to disprove this for the logic $\mathbb{L}_{\theta,\theta}$ for $\theta > \aleph_0$ a compact cardinal. Still Łos theorem on ultra-products was known to generalize so let us review the background in this direction.

In the sixties, ultra-products were very central in model theory, see e.g. the books [BS69] and [CK73].

Concerning isomorphisms of ultrapowers see Keisler [Kei61] and then Shelah [Sh:13]; later for infinitary logics see Hodges-Shelah [HoSh:109].

In [Sh:797], the logic $\mathbb{L}^1_\theta$ is introduced. By [Sh:1101], elementary equivalence for $\mathbb{L}^1_\theta$ is characterized by isomorphic ultra-limits; this was originally part of the present paper (it was called §3).

Here we deal with the logic $\mathbb{L}_{\theta,\theta}$ itself. We are mainly interested in generalizations of [Sh:c, Ch.VI], on Keisler order $\triangleleft$ and saturation of ultra-powers and the order $\triangleleft^*$ from [Sh:500]. See history there, in [Sh:c] and recent works with Malliaris ([MiSh:996], [MiSh:997], [MiSh:998]) dealing with unstable T's and lately [MiSh:1050], [MiSh:1051], [MiSh:1069], [MiSh:1070].
In particular after [Sh:c, Ch.VI] the picture was:

{v1}

Theorem 0.1. *Assume T is a complete countable first order theory.*
1) The following conditions are equivalent, for any $\lambda \geq 2^{\aleph_0}$:

- $(a)''$ *if D is a regular ultrafilter on λ and M is a model of T then M^λ/D is λ^+-saturated*[1]
- $(b)''$ *there is a first order theory $T_1 \supseteq T$ such that: $M_1 \models T_1 \Rightarrow M_1{\restriction}\tau(T)$ is locally saturated (i.e. for types $\subseteq \{\varphi(\bar{x},\bar{a}) : a \in {}^{\ell g(\bar{y})}(M_1)\}$ for some $\varphi = \varphi(\bar{x},\bar{y})$)*
- $(c)''$ *T is stable*[2] *without the* f.c.p.
- $(d)''$ *like $(b)''$ but $|T_1| = \aleph_0$.*

[1] We can use "2^λ-saturated".

[2] For first order T, stability follows from "without the f.c.p."

2) The following conditions are equivalent:

(a) *if* $\mathbf{x} = \langle D_\alpha : \alpha < \delta \rangle$, *where* δ *is a limit ordinal and for each* $\alpha < \delta, D_\alpha$ *is a regular ultra-filter on a cardinal* λ_α, *<u>then</u> for any (equivalently some) model* M *of* T, M_δ *is* $\sup\{2^{\lambda_\alpha} : \alpha < \delta\}$*-saturated where* M_δ *is ultra-limit of* M *by* $\mathbf{x}$ *(i.e.* $M_\alpha(\alpha \le \delta)$ *is* $\prec$*-increasing continuous,* $M_0 = M, M_{\alpha+1} = M_\alpha^{\lambda_\alpha}/D_\alpha$*)*

(b) *there is a first order theory* $T_1 \supseteq T$ *such that:* $M_1 \models T_1 \Rightarrow M_1 {\restriction} \tau(T)$ *is saturated*

(c) T *is superstable without the* f.c.p.

(d) *like (b) but* $|T_1| = 2^{\aleph_0}$.

3) The following conditions are equivalent:

$(b)'$ *like (b) but* $|T_1| = \aleph_0$

$(c)'$ T *is* $\aleph_0$*-stable without the* f.c.p.

See more in [BGSh:570] *and* [Sh:500].

{v1} Our main topic is generalizing results like 0.1 replacing first order logic with $\mathbb{L}_{\theta,\theta}$, so "countable" is replaced by "of cardinality $\le \theta$". More specifically, one aim is to characterize the complete $\mathbb{L}_{\theta,\theta}$-theories T such that for some $\mathbb{L}_{\theta,\theta}$-theory T_1 extending T, for every model M_1 of T_1, the $\tau(T)$-reduct of the model M_1 is (locally) {a33g} saturated, such T will be called (locally) minimal. The main conclusions are 3.19, {a54} 3.20, 4.9.

{v1} Note that $(a)'' \Leftrightarrow (c)''$ of Theorem 0.1(1) characterizes when T is $\triangleleft_\lambda$-minimal and even $\triangleleft$-minimal (but not $\triangleleft_\lambda$-minimal in the case $\aleph_0 < \lambda < 2^{\aleph_0}$, on it see [Sh:c]). There is much more to be said on this order.

{v1} Parallelly, $(b) \Leftrightarrow (c)$ of Theorem 0.1 is related to the partial orders $\triangleleft^*, \triangleleft^*_\lambda$ implicitly investigated in [Sh:c, Ch.VI] but introduced in [Sh:500], see more on them in Dzamonja-Shelah [DjSh:692], Shelah-Usvyatsov [ShUs:844] and lately Malliaris-Shelah ([MiSh:1051]); related is Baldwin-Grossberg-Shelah [BGSh:570].

{v1} But in our context trying to generalize Theorem 0.1, i.e. the minimal case was hard enough. In fact, there is a problem already in generalizing the notion of being stable. In §1 we suggest some reasonable definitions and try to map their relations. Note that those generalizations are really very different in the present context (though equivalent for the first order case). For some versions, some "unstable" T's are categorical in all relevant λ's; while other "unstable" versions imply maximal number of models up to isomorphism in relevant cardinalities, and some "stable T's" have an intermediate behaviour (i.e. $\dot{I}(\lambda, T) = \lambda^+$).

To get sufficient conditions on T for having many models we may consider the tree ${}^{\theta \ge}\lambda$ and try to combine it with the identities for $(\aleph_1, \aleph_0)$ (see [Sh:74]) which is a kind of the relevant indiscernibility, we hope to deal with this in [Sh:F1396].

Originally we were interested in generalizing the characterization of the minimal theories in Keisler order $(\triangleleft, \triangleleft_\lambda)$, where T is bigger for $\triangleleft_\lambda$ if for fewer regular ultrafilters D on the cardinal $\lambda, M^\lambda/D$ is λ^+-saturated for some (equivalent any) model of T.

Earlier version was flawed but we succeed in characterizing the $\blacktriangleleft^*_{\lambda,\theta}$-minimal ones, see §3. Later we get also the characterization of the $\blacktriangleleft_{\lambda,\theta}$-minimal ones where $\blacktriangleleft_{\lambda,\theta}$ is defined below but we use a different version of stable.

Of course, before all this we have to define saturation and local saturation. This is straightforward, but "unfortunately" two wonderful properties true in the first order case are missing: existence and uniqueness.

The main achievements are in §3,§4: first (in 3.19), a characterization of the (locally) minimal theories as stable with θ-n.c.p. under reasonable definitions (see Definition 2.9). But unlike the first order case, some stable theories (even just theories of one equivalence relation) are maximal. In fact we get two characterizations: one for the local version (dealing with types containing formulas $\varphi(\bar{x}_{[\varepsilon]}, \bar{a})$ only for one φ, various $\bar{a}$'s) and another for the global one (naturally for theories $T, |T| = \theta$). Second (in 3.20), we characterize the $\blacktriangleleft_{\lambda,\theta}$-minimal T as definably stable with the θ-n.c.p. {a33} {a9} {a33g}

We may hope this will help us to resolve the categoricity spectrum. It is natural to try to first prove: having long linear orders implies many models. But this is not so - see 1.12; so the situation has a marked difference from the first order case. We hope to continue this in [Sh:F1396] and see the related [Sh:1064]; note that criterions for "there is no universal model of T in λ" help to prove non-categoricity of $T \subseteq \mathbb{L}_{\theta,\theta}$ in λ. See survey [Mir05] and the recent [Sh:F1808]. {y45}

This work was presented in a lecture in MAMLS meeting, Fall 2012 and in courses in The Hebrew University, Spring 2012 and 2013.

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* * *

{v2}

Discussion 0.2. 1) We may wonder, for $\theta > \aleph_0$ a compact cardinal what about $\mathbb{L}_{\theta,\aleph_0}$-theories?
2) Recall the logic from [HoSh:271, §2], that is, given two compact cardinals $\kappa > \theta > \aleph_0$, a logic $\mathbb{L}_{\kappa/\theta,\kappa/\theta}$ is defined and proved to be "nice", e.g. it is λ-compact for $\lambda < \theta$, has interpolation, has downward LST property down to κ and the upward LST property for models of cardinality $\geq \lambda$ but is not θ^+-compact.
3) On the classical results on $\mathbb{L}_{\lambda,\kappa}$ see e.g. [Dic85]; on "when for given M_1, M_2 there are I and $D \in \mathrm{uf}_\theta(I)$ such that $M_1^I/D \cong M_2^I/D$", see Hodges-Shelah [HoSh:109].
4) Recently close works are Malliaris-Shelah [MiSh:999] which deals with κ-complete ultrafilters (on sets and relevant Boolean algebras) on the way to understanding the amount of saturation of ultra-powers by regular ultra-filters. On reduced power, see [Sh:1064].
5) Concerning dependent (non-elementary) classes, see also Kaplan-Lavi-Shelah [KpLaSh:1055].
6) Is the lack of uniqueness of saturation a sign this is a bad choice? It does not seem so to me.
7) If we insist on "union of $<_{\mathscr{L}}$-increasing countable chain" is an $<_{\mathscr{L}}$-extension, we can restrict ourselves to $\mathbb{L}^1_\theta$, but what about unions of length $\kappa \in \mathrm{Reg} \cap (\aleph_0, \theta)$? If

we restrict our logic as in $\mathbb{L}^1_\theta$ for all those $\kappa < \theta$ maybe we get close to a.e.c., or get an interesting new logic with EM models (as indicated in [Sh:797], [Sh:893]).
8) Presently, our intention here is to show $\mathbb{L}_{\theta,\theta}$ has a model theory, in particular classification theory. At this point having found significant dissimilarities to the first order case on the one hand, and solving the parallel of serious theorems on the other hand, there is no reason to abandon this direction.

We may wonder

{v4}

Question 0.3. Characterize the (first order complete) T such that M^λ/D is not λ^+-saturated whenever M is a λ-saturated model of $T, \lambda \geq \theta > \aleph_0, D$ a (λ,θ)-regular θ-complete ultrafilter on λ.

{v7}

Question 0.4. Can we prove nice things on the following logics:

- (A) let $\mathbb{L}^*_\kappa$ be $\{\psi$: for every $\mu < \kappa$ large enough we have $\psi \in \mathbb{L}_{\mu^+,\mu^+}$ and if $\langle M_s : s \in I\rangle$ is $\prec_{\mathbb{L}_{\mu^+,\mu^+}}$-increasing, I a directed partial order then $\bigcup\limits_s M_s \models \psi$ iff $\bigwedge\limits_s M_s \models \psi\}$. How close is $\mathbb{L}^*_\kappa$ to a.e.c. when κ is a compact cardinal?
- (B) As above but I is linearly ordered.

{0B}

§ 0(B). Preliminaries.

{w0}

Hypothesis 0.5. θ is a compact uncountable cardinal (of course, we use only restricted versions of this).

{w2}

Notation 0.6. 1) Let $\varphi(\bar{x})$ mean: φ is a formula of $\mathbb{L}_{\theta,\theta}, \bar{x}$ is a sequence of variables with no repetitions including the variables occuring freely in φ, also $\ell g(\bar{x}) < \theta$ if not said otherwise. We use φ, ψ, ϑ to denote formulas and for a logical statement $\{\mathbf{st}\}$ let $\varphi^{\mathbf{st}}$ or $\varphi^{[\mathbf{st}]}$ or $\varphi^{\mathrm{if}(\mathbf{st})}$ be φ if $\mathbf{st}$ is true or 1 and be $\neg\varphi$ if $\mathbf{st}$ is false or 0.
2) For a set u, usually of ordinals, let $\bar{x}_{[u]} = \langle x_\varepsilon : \varepsilon \in u\rangle$, now u may be an ordinal but, e.g. if $u = [\alpha,\beta)$ we may write $\bar{x}_{[\alpha,\beta)}$; similarly for $\bar{y}_{[u]}, \bar{z}_{[u]}$; let $\ell g(\bar{x}_{[u]}) = u$.
3) τ denotes a vocabulary, i.e. a set of predicates and function symbols each with $< \theta$ places.
4) T denotes a theory in $\mathbb{L}_{\theta,\theta}$; usually complete in the vocabulary τ_T and with a model of cardinality $\geq \theta$ if not said otherwise.
5) Let Mod_T be the class of models of T.
6) For a model M let its vocabulary be τ_M.

{w4}

Notation 0.7. 1) ε, ζ, ξ are ordinals $< \theta$.
2) For a linear order I let $\mathrm{comp}(I)$ be its completion.

{w8}

Definition 0.8. 1) Let $\mathrm{uf}_\theta(I)$ be the set of θ-complete ultrafilters on I, non-principal if not said otherwise. Let $\mathrm{fil}_\theta(I)$ be the set of θ-complete filters on I; mainly we use (θ,θ)-regular ones (see below).
2) The filter $D \in \mathrm{fil}_\theta(I)$ is called (λ,θ)-regular <u>when</u> there is a witness $\bar{w} = \langle w_t : t \in I\rangle$ which means: $w_t \in [\lambda]^{<\theta}$ for $t \in I$ and $\alpha < \lambda \Rightarrow \{t : \alpha \in w_t\} \in D$.
3) Let $\mathrm{ruf}_{\lambda,\theta}(I)$ be the set of (λ,θ)-regular $D \in \mathrm{uf}_\theta(I)$; let $\mathrm{rfil}_{\lambda,\theta}(I)$ be the set of (λ,θ)-regular $D \in \mathrm{fil}_\theta(I)$; when $\lambda = |I|$ we may omit λ; so necessarily $\lambda \leq \theta$.

4) For $S \subseteq \mathrm{Card} \cap \theta$ with $\sup(S) = \theta$ and $D \in \mathrm{uf}_\theta(I)$ which is not θ^+-complete let $\mathrm{lcr}(S,D) = \min\{\mu : \mu \geq \theta$ and for some $f \in {}^I S$ we have $\mu = |\prod_{s\in I} f(s)/D|\}$ and let $\mathrm{Cr}(S,D) = \{\mu$: for some $f \in {}^I S$ the cardinality of $\prod_{s\in I} f(s)/D$ is $\mu\}$.

Note that

{w10}

Observation 0.9. *If $S = \mathrm{Card} \cap \theta$ and $D \in \mathrm{uf}_\theta(I)$ and μ is the cardinal θ^I/D __then__ $\mathrm{lcr}(S,D)$ is θ and $\mathrm{Cr}(S,D)$ is $\mathrm{Card} \cap \mu^+$ or $\mathrm{Card} \cap \mu$. Moreover, if D is (λ,θ)-regular then $\mathrm{Cr}(S,D) \nsubseteq 2^\lambda$ hence $|I| = \lambda \Rightarrow 2^\lambda \in \mathrm{Cr}(S,D)$; and so $|I| = \lambda \Rightarrow 2^\lambda = \max(\mathrm{Cr}(S,D))$.*

Proof. E.g., concerning the second sentence assume that D is (λ,θ)-regular and choose $\bar{w} = \langle w_s : s \in I\rangle$ witnessing it, i.e. $w_s \in [\lambda]^{<\theta}$ and $\alpha < \lambda \Rightarrow A_\alpha := \{s \in I : \alpha \in w_s\}$ belongs to D. We define $f \in I_S$ by $f(m) = \min(S \backslash 2^{|w_s|})$, hence $f(s) \in S$ and let $\langle u_{s,i} : i < 2^{|w_s|}\rangle$ list $\mathscr{P}(w_s)$.

Now for every $u \subseteq \lambda$ let $f_u \in \prod_{s\in I} f(s)$ be defined by: $f_u(s)$ is the $i < 2^{|w_s|} < f(s)$ such that $u \cap w_t = u_{s,i}$.

So

(a) $\{f_u/D : u \subseteq \lambda\}$ is a subset of $\prod_s f(s)/D$ and

(b) if $u_1 \neq u_2 \subseteq \lambda$ then $f_{u_1}/D \neq f_{u_)2}/D$.

[Why? Choose $\alpha \in u_1 \triangle u_2$, hence $\{s \in I : f_{u_1}(s) \neq f_{u_2}(s)\} \supseteq \{s : \alpha \in w_s\} \in D$.]

Together we are done proving $\mathrm{Cr}(S,D) \nsubseteq 2^\lambda$. Lastly, if $I = \lambda$ then $g \in {}^I S \Rightarrow |\prod_{s\in I} g(s)/D| \leq |\prod_s g(s)| \leq \theta^{|I|} = \theta^\lambda = 2^\lambda$ well assuming $0 \notin S$ for transparency. $\square_{0.9}$

{x0}

Notation 0.10. 1) A vocabulary τ means with $\mathrm{arity}(\tau) \leq \theta$ if not said otherwise, where $\mathrm{arity}(\tau) = \aleph_0 + \sup\{|\mathrm{arity}(P)|^+ : P$ is a predicate (or function symbol) from $\tau\}$, of course, where $\mathrm{arity}(P)$ is the number of places of P.
2) If $A \subseteq N, \bar{a} \in {}^\varepsilon N$ and $\Delta \subseteq \mathbb{L}_{\theta,\theta}(\tau_M)$ __then__ $\mathrm{tp}_\Delta(\bar{a},A,N) = \{\varphi(\bar{x}_{[\varepsilon]},\bar{b}) : \varphi(\bar{x}_{[\varepsilon]},\bar{y}) \in \Delta, N \models$ "$\varphi[\bar{a},\bar{b}]$" and $\bar{b} \in {}^{\ell g(\bar{y})}M\}$.
3) $\mathbf{S}^\varepsilon_\Delta(A,M) = \{\mathrm{tp}_\Delta(\bar{a},A,N)$: for some $N, M \prec_{\mathbb{L}_{\theta,\theta}} N$ and $\bar{a} \in {}^\varepsilon N\}$.
4) If $\Delta = \mathbb{L}_{\theta,\theta}$ then we may omit Δ.
4A) If Δ is the set of quantifier free formulas from $\mathbb{L}(\tau_N)$, we may write $\mathrm{tp}_{\mathrm{qf}}$ instead of tp_Δ.

{x1}

Definition 0.11. 1) $\mathbb{L}_{\theta,\theta}(\tau)$ is the set of formulas of $\mathbb{L}_{\theta,\theta}$ in the vocabulary τ.
2) For τ-models M,N let $M \prec_{\mathbb{L}_{\theta,\theta}} N$ mean: if $\varphi(\bar{x}) \in \mathbb{L}_{\theta,\theta}(\tau_M)$ and $\bar{a} \in {}^{\ell g(\bar{x})}M$ then $M \models \varphi[\bar{a}] \Leftrightarrow N \models \varphi[\bar{a}]$.

{x2}

Definition 0.12. For a set v of ordinals, a sequence $\bar{u} = \langle u_\alpha : \alpha \in v\rangle$ and models M_1, M_2 of the same vocabulary τ and $\Delta \subseteq \mathbb{L}_{\theta,\theta}(\tau)$ a set of formulas we define a game $\eth = \eth_{\Delta,\bar{u}}(M_1,M_2)$ but when $(\forall \alpha \in v)(u_\alpha = u)$ we may write $\eth_{\Delta,u,v}(M_1,M_2)$:

(a) a play lasts some finite number of moves not known in advance

(b) in the n-th move the antagonist chooses

- $\alpha_n \in v$ such that $m < n \Rightarrow \alpha_n < \alpha_m$
- sequence $\langle a_{n,i,\ell} : (n,i,\ell) \in I \rangle$ where
- $I = \{(n,i,\ell_{n,i}) : i \in u_{\alpha_n}\}$
- $\ell_{n,i} = \ell(n,i) \in \{1,2\}$
- $a_{n,i,\ell(n,i)} \in M_{\ell_{n,i}}$

(c) in the n-th move (after the antagonist's move) the protagonist chooses $a_{n,i,3-\ell(n,i)} \in M_{3-\ell(n,i)}$ for $i \in u_{\alpha_n}$

(d) the play ends when the antagonist cannot choose α_n

(e) the protagonist wins a play when:

- the set $\{(a_{m,i,1}, a_{m,i,2}) : i \in u_{\alpha_m}$ and the m-th move was done$\}$ is a function and even
- is a partial one-to-one function from M_1 into M_2 and moreover
- it preserves satisfaction of Δ-formulas and their negations.

We know (see, e.g. [Dic85])

{x4}

Fact 0.13. The τ-models M_1, M_2 are $\mathbb{L}_{\theta,\theta}$-equivalent iff for every set Δ of $< \theta$ atomic formulas and $\alpha, \beta < \theta$ the protagonist wins in the game $\partial_{\Delta,\alpha,\beta}(M_1, M_2)$.

And, of course

{x6}

Fact 0.14. For a complete $T \subseteq \mathbb{L}_{\theta,\theta}(\tau)$.
1) $(\mathrm{Mod}_T, \prec_{\mathbb{L}_{\theta,\theta}})$ has amalgamation and the joint embedding property (JEP), that is:

(a) amalgamation: if $M_0 \prec_{\mathbb{L}_{\theta,\theta}} M_\ell$ for $\ell = 1,2$ then there are $M_3, f_1, f_2, M_1', M_2'$ such that

- $M_0 \prec_{\mathbb{L}_{\theta,\theta}} M_3$
- for $\ell = 1,2, f_\ell$ is a $\prec_{\mathbb{L}_{\theta,\theta}}$-embedding of M_ℓ into M_3 over M_0, that is, for some τ_T-models M_ℓ' for $\ell = 1,2$ we have $M_\ell' \prec_{\mathbb{L}_{\theta,\theta}} M_3$ and f_ℓ is an isomorphism from M_ℓ onto M_ℓ' over M_0;

(b) JEP: if M_1, M_2 are $\mathbb{L}_{\theta,\theta}$-equivalent τ-models then there is a τ-model M_3 and $\prec_{\mathbb{L}_{\theta,\theta}}$-embedding f_ℓ of M_ℓ into M_3 for $\ell = 1,2$.

2) Types are well defined, see [Sh:300b], i.e. the orbital type **tp** and the types as a set of formula $\mathrm{tp}_{\mathbb{L}_{\theta,\theta}}$ are essentially equivalent, that is:

$(*)$ if $M_0 \prec_{\mathbb{L}_{\theta,\theta}} M_\ell, \zeta < \theta, \bar{a}_\ell \in {}^{\zeta}(|M_\ell|)$ for $\ell = 1,2$ and so $\tau = \tau(M_\ell)$ for $\ell = 0,1,2$ then the following conditions are equivalent:

(a) the set of formulas (= type) $\mathrm{tp}_{\mathbb{L}_{\theta,\theta}}(\bar{a}_1, M_0, M_1)$ is equal to $\mathrm{tp}_{\mathbb{L}_{\theta,\theta}}(\bar{a}_2, M_0, M_2)$,
{x0} see 0.10(2), that is, if $\xi < \theta, \bar{b} \in {}^{\xi}(M_0)$ and $\varphi(\bar{x}_{[\zeta]}, \bar{y}_{[\xi]}) \in \mathbb{L}_{\theta,\theta}(\tau)$ then $M_1 \models \varphi[\bar{a}_1, \bar{b}] \Leftrightarrow M_2 \models \varphi[\bar{a}_2, \bar{b}]$

{x6} (b) (orbital types) there are M_3, f_1, f_2 as in 0.14(1)(a) such that $f_1(\bar{a}_1) = f_2(\bar{a}_2)$.

The well known generalization of Łos theorem (see e.g. [Jec03] or [HoSh:109]) is:

{x7}

Theorem 0.15. *If $\varphi(\bar{x}_{[\zeta]}) \in \mathbb{L}_{\theta,\theta}(\tau), D \in \mathrm{uf}_\theta(I)$ and M_s is a τ-model for $s \in I$ and $f_\varepsilon \in \prod\limits_{s\in I} M_s$ for $\varepsilon < \zeta$ then $M \models \varphi[\ldots, f_\varepsilon/D, \ldots]_{\varepsilon<\zeta}$ iff the set $\{s \in I : M_s \models \varphi[\ldots, f_\varepsilon(s), \ldots]_{\varepsilon<\zeta}\}$ belongs to D.*

Recall {x8}

Fact 0.16. Assume $D \in \mathrm{uf}_\theta(I)$ is not θ^+-complete and $\mathfrak{B} = (\mathscr{H}(\chi), \in, \theta)^I/D$.
1) If $\mathrm{cf}(\chi) \geq \theta$ and $a_\alpha \in \mathfrak{B}$ for $\alpha < \theta$ then there is $\bar{b} \in \mathfrak{B}$ such that $\mathfrak{B} \models$ "$\bar{b}$ is a sequence of length $< \theta$ with the α-th element being a_α" for[3] every $\alpha < \theta$.
2) If $\mathrm{cf}(\chi) > \lambda$ and D is (λ, θ)-regular and $a_\alpha \in \mathfrak{B}$ for $\alpha < \lambda$ then there is $w \in \mathfrak{B}$ such that $\alpha < \lambda \Rightarrow \mathfrak{B} \models$ "$|w| < \theta$ and $a_\alpha \in w$", (in fact, also the inverse holds).
3) For some function h from I onto $\theta, D/h = \{u \subseteq \theta : h^{-1}(u) \in D\}$ is a normal ultrafilter on θ.

Proof. 1) Let $a_\alpha = f_\alpha/D$ where $f_\alpha \in {}^I\mathscr{H}(\chi)$. Let $F : I \to \theta$ be such that $\alpha < \theta \Rightarrow \{s : \alpha \leq F(s)\} \in D$, such a function F exists by the assumption on D. We define $g : I \to \mathscr{H}(\chi)$ by:

- $g(s) = \langle f_\alpha(s) : \alpha < F(s)\rangle$.

Now g/D is as required, check.
2) Similarly using $\bar{w} = \langle w_s : s \in I\rangle$ from 0.8, so {w8}

- $g(s) = \{f_\alpha(s) : \alpha \in w_s\}$.

3) See, e.g. [Jec03]. $\square_{0.16}$

Recall (see history [Sh:950, §1] in the literature usually we say "strongly convergent" instead of "convergent" to distinguish from other versions; but here this is not needed).

{x11}

Definition 0.17. Assume $\Delta \subseteq \mathbb{L}_{\theta,\theta}(\tau_M)$ and I is a linear order and $\mathbf{\bar{a}} = \langle \bar{a}_t : t \in I\rangle$ and $t \in I \Rightarrow \bar{a}_t \in {}^uM$ and $\bar{\theta} = \langle \theta_\varphi = \theta_{\varphi(\bar{x}_{[u]},\bar{y})} : \varphi = \varphi(\bar{x}_{[u]}, \bar{y}) \in \Delta\rangle$ where θ_φ is a cardinal $\leq \theta$; if $\bigwedge\limits_{\varphi\in\Delta} \theta_\varphi = \sigma$ we may write σ; if $\sigma = \theta$ we may omit it.

1) We say $\mathbf{\bar{a}}$ is a $(\Delta, \bar{\theta})$-convergent sequence in M when for every $\varphi(\bar{x}_{[u]}, \bar{y}) \in \Delta$ and $\bar{b} \in {}^{\ell g(\bar{y})}M$ there is $J \subseteq \mathrm{comp}(I)$ of cardinality $< \sigma$ or $< \theta_{\varphi(\bar{x}_{[u]},\bar{y})} < \theta$ respectively, such that:

- if $s, t \in I$ and $\mathrm{tp}_{\mathrm{qf}}(s, J, \mathrm{comp}(I)) = \mathrm{tp}_{\mathrm{qf}}(t, J, \mathrm{comp}(I))$ then $M \models$ "$\varphi[\bar{a}_s, \bar{b}] \equiv \varphi[\bar{a}_t, \bar{b}]$".

1A) We say $\mathbf{\bar{a}}$ is a middle (Δ, σ)-convergent sequence when $\mathbf{\bar{a}}$ is $(\Delta, \bar{\theta})$-convergent for some $\bar{\theta} = \langle \theta_\varphi : \varphi \in \Delta\rangle$ satisfying $\varphi \in \Delta \Rightarrow \theta_\varphi < \sigma$. If $\sigma = \theta$ then we may omit it.
2) We say "strictly $(\Delta, \bar{\theta})$-convergent" when we demand "J$\subseteq I$"; similarly in the other variant.

[3]We are identifying elements of $\mathscr{H}(\chi)$ with ones of $\mathfrak{B}$ naturally, see 0.22(2). Alternatively, expand $\mathfrak{A} = (\mathscr{H}(\chi), \in, \theta)$ by having $c_\alpha^{\mathfrak{A}^+} = \alpha$, so $c_\alpha \in \tau(\mathfrak{A}^+)$ is an individual constant for $\alpha < \lambda$, so $\mathfrak{B}^+ = (\mathfrak{A}^+)^I/D$ is an expansion of $\mathfrak{B}$ and $\mathfrak{B}^+ \models$ "a_α is the c_α-th element of the sequence b". {x18}

{x13}

Definition 0.18. For a linear order I.
1) I^* is its inverse, $\mathrm{cf}(I)$ is the cofinality of I (so $0, 1$ or a regular cardinal) and $\mathrm{co}-\mathrm{in}(I)$ is the co-initiality of I which means the cofinality of its inverse.
2) A cut is a pair (C_1, C_2) such that C_1 is an initial segment of I and $C_2 = I \backslash C_1$.
3) The cofinality (κ_1, κ_2) of the cut (C_1, C_2) is the pair (κ_1, κ_2) of regular cardinals (or 0 or 1) such that $\kappa_1 = \mathrm{cf}(I \restriction C_1), \kappa_2 = \text{co-in}(I \restriction C_2)$.
4) We say (C_1, C_2) is a pre-cut of I [of cofinality (κ_1, κ_2)] <u>when</u> $C_1, C_2 \subseteq I$ and $(\{s \in I : (\exists t \in C_1)(s \le_I t), \{s \in I : (\exists t \in C_2)(t \le_I s)\}$ is a cut of I [of cofinality (κ_1, κ_2)].

{x17}
Definition 0.19. 0) We say X respects E <u>when</u> for some set I, E is an equivalence relation on I and $X \subseteq I$ and $sEt \Rightarrow (s \in X \leftrightarrow t \in X)$.
1) We say $\mathbf{x} = (I, D, \mathscr{E})$ is a $(\kappa, \sigma)-$l.u.f.t. (limit-ultra-filter-iteration triple) <u>when</u>:

- (a) D is a filter on the set I
- (b) $\mathscr{E}$ is a family of equivalence relations on I
- (c) $(\mathscr{E}, \supseteq)$ is σ-directed, i.e. if $\alpha(*) < \sigma$ and $E_i \in \mathscr{E}$ for $i < \alpha(*)$ <u>then</u> there is $E \in \mathscr{E}$ refining E_i for every $i < \alpha(*)$
- (d) if $E \in \mathscr{E}$ then D/E is a κ-complete ultrafilter on I/E where $D/E := \{X/E : X \in D \text{ and } X \text{ respects } E\}$.

1A) We say $\mathbf{x}$ is a $(\kappa, \sigma)-$l.f.t. <u>when</u> above we weaken clause (d) to:

- $(d)'$ if $E \in \mathscr{E}$ then D/E is a κ-complete filter.

2) Omitting "(κ, σ)" means $(\theta, \aleph_0)$, recalling θ is our fixed compact cardinal.
3) Let $(I_1, D_1, \mathscr{E}_1) \le^1_h (I_2, D_2, \mathscr{E}_2)$ mean that:

- (a) h is a function from I_2 onto I_1
- (b) if $E \in \mathscr{E}_1$ <u>then</u> $h^{-1} \circ E \in \mathscr{E}_2$ where $h^{-1} \circ E = \{(s, t) : s, t \in I_2 \text{ and } h(s)Eh(t)\}$
- (c) if $E_1 \in \mathscr{E}_1$ and $E_2 = h^{-1} \circ E_1$ <u>then</u> $D_1/E_1 = h(D_2/E_2)$.

{x17f}
{x17} *Remark* 0.20. Note that in 0.19(3), if $h = \mathrm{id}_{I_2}$ then $I_1 = I_2$.

{x16}
Definition 0.21. Assume $\mathbf{x} = (I, D, \mathscr{E})$ is a (κ, σ)-l.u.f.t.
1) For a function f let $\mathrm{eq}(f) = \{(s_1, s_2) : f(s_1) = f(s_2)\}$. If $\bar{f} = \langle f_i : i < i_* \rangle$ and $i < i_* \Rightarrow \mathrm{dom}(f_i) = I$ then $\mathrm{eq}(\bar{f}) = \cap\{\mathrm{eq}(f_i) : i < i_*\}$.
2) For a set U let $U^I | \mathscr{E} = \{f \in {}^I U : \mathrm{eq}(f) \text{ is refined by some } E \in \mathscr{E}\}$.
3) For a model M let $\mathrm{l.r.p.}_{\mathbf{x}}(M) = M^I_D | \mathscr{E} = (M^I/D) \restriction \{f/D : f \in {}^I M \text{ and } \mathrm{eq}(f)$ is refined by some $E \in \mathscr{E}\}$, pedantically (as $\mathrm{arity}(\tau_M)$ may be $> \aleph_0$), $M^I_D | \mathscr{E} = \cup\{M^I_D \restriction E : E \in \mathscr{E}\}$; l.r.p. stands for limit reduced power.
4) If $\mathbf{x}$ is l.u.f.t. we may in (3) write $\mathrm{l.u.p.}_{\mathbf{x}}(M)$.

We now give the generalization of Keisler [Kei63]; Hodges-Shelah [HoSh:109, Lemma 1,pg.80] is the case $\kappa = \partial$.

{x18}
Theorem 0.22. *1) If $(I, D, \mathscr{E})$ is $(\kappa, \partial)-$l.u.f.t., $\varphi = \varphi(\bar{x}_{[\zeta]}) \in \mathbb{L}_{\kappa, \partial}(\tau)$ so $\zeta < \partial, f_\varepsilon \in M^I | \mathscr{E}$ for $\varepsilon < \zeta$ <u>then</u> $M^I_D | \mathscr{E} \models \varphi[\ldots, f_\varepsilon/D, \ldots]$ iff $\{s \in I : M \models \varphi[\ldots, f_\varepsilon(s), \ldots]_{\varepsilon < \zeta}\} \in D$.*

2) Moreover $M \prec_{\mathbb{L}_{\kappa,\partial}} M^I_D/\mathscr{E}$*, pedantically* $\mathbf{j} = \mathbf{j}_{M,\mathbf{x}}$ *is a* $\prec_{\mathbb{L}_{\kappa,\partial}}$*-elementary embedding of* M *into* $M^I_D/\mathscr{E}$ *where* $\mathbf{j}(a) = \langle a : s \in I\rangle/D$.
3) We define $(\prod\limits_{s\in I} M_s)^I_D|\mathscr{E}$ *similarly* <u>*when*</u> *the equivalence relation* $\{(s,t) \in I \times I : M_s = M_t\}$ *is refined by some* $E \in \mathscr{E}$.

{x21}

Convention 0.23. *1) Abusing notation in* $\prod\limits_{s\in I} M_s/D$ *we allow* f/D *for* $f \in \prod\limits_{s\in S} M_s$ *when* $S \in D$.
2) For $\bar{c} \in {}^{\gamma}(\prod\limits_{s\in I} M_s/D)$ *we can choose* $\langle \bar{c}_s : s \in I\rangle$ *such that* $\bar{c}_s \in {}^{\gamma}(M_s)$ *and* $\bar{c} = \langle \bar{c}_s : s \in I\rangle/D$ *which means: if* $i < \ell g(\bar{c})$ <u>*then*</u> $c_{s,i} \in M_s$ *and* $c_i = \langle c_{s,i} : s \in I\rangle/D$.

{x24}
{x16}

Remark 0.24. 1) Why the "pedantically" in 0.21(3)? Otherwise if $\mathbf{x}$ is a $(\theta,\sigma)-$ l.u.f.t., $(\mathscr{E}_{\mathbf{x}}, \supseteq)$ is not κ^+-directed, $\kappa < \text{arity}(\tau)$ <u>then</u> defining l.u.p.$_{\mathbf{x}}(M)$, we have freedom: if $R \in \tau, \text{arity}_\tau(R) \geq \kappa$, i.e. on $R^N {\restriction} \{\bar{a} : \bar{a} \in {}^{\text{arity}(P)}N$ and no $E \in \mathscr{E}$ refines $\text{eq}(\bar{a})\}$ so we have no restrictions.
2) So, e.g. for categoricity we better restrict ourselves to vocabularies τ such that $\text{arity}(\tau) = \aleph_0$.

{x28}

Definition 0.25. We say that M is a θ-complete model <u>when</u> for every $\varepsilon < \theta, R_* \subseteq {}^{\varepsilon}M$ and $F_* : {}^{\varepsilon}M \to M$ there are $R, F \in \tau_M$ such that $R^M = R_* \wedge F^M = F_*$.

{x31}

Observation 0.26. *1) If* M *is a* τ*-model of cardinality* λ <u>*then*</u> *there is a* θ*-complete expansion* M^+ *of* M *so* $\tau(M^+) \supseteq \tau(M)$ *and* $\tau(M^+)$ *has cardinality* $|\tau_M| + 2^{(\|M\|^{<\theta})}$.
2) For models $M \prec_{\mathbb{L}_{\theta,\theta}} N$ *and* M^+ *as above the following conditions are equivalent:*

- (a) $N = \text{l.u.p.}_{\mathbf{x}}(M)$ *up to isomorphisms over* M *identifying* $a \in M$ *with* $\mathbf{j}_{\mathbf{x}}(a) \in N$*, for some* $(\theta,\theta) - \text{l.u.f.t.}\ \mathbf{x}$
- (b) *there is* N^+ *such that* $M^+ \prec_{\mathbb{L}_{\theta,\theta}} N^+$ *and* $N^+{\restriction}\tau_M$ *is isomorphic to* N *over* M.

3) For a model M*, if* $(P^M, <^M)$ *is a* θ*-directed partial order and* $\chi = \text{cf}(\chi) \geq \theta$ *and* $\lambda = \lambda^{\|M\|} + \chi$ <u>*then*</u> *for some* $(\theta,\theta) - \text{l.u.f.t.}\ \mathbf{x}$*, the model* $N := \text{l.u.p.}_{\mathbf{x}}(M)$ *satisfies* $(P^N, <^N)$ *has a cofinal increasing sequence of length* χ *and* $|P^N| = \lambda$.

Proof. Easy, e.g.
3) Let M^+ be as in part (1). Note that M^+ has Skolem functions for formulas $\varphi(\bar{x},\bar{y}) \in \mathbb{L}_{\theta,\theta}(\tau_{M^+})$ and let $T' := Th_{\mathbb{L}_{\theta,\theta}}(M^+) \cup \{P(\sigma(x_{\varepsilon_0},\ldots,x_{\varepsilon_i},\ldots)_{i<i(*)}) \to \sigma(x_{\varepsilon_0},\ldots,x_{\varepsilon_i},\ldots)_{i<i(*)} < x_\varepsilon : \sigma$ is a $\tau(M^+)$-term so $i(*) < \theta$ and $i < i(*) \Rightarrow \varepsilon_i < \varepsilon < \lambda \cdot \chi\}$.

Clearly

- $(*)$ T' is $(<\theta)$-satisfiable in M^+.

[Why? Because if $T'' \subseteq T'$ has cardinality $< \theta$ then the set $u = \{\varepsilon < \lambda \cdot \chi : x_\varepsilon$ appears in $T''\}$ has cardinality $< \theta$ and let $i(*) = \text{otp}(u)$; now for each $\varepsilon \in u$ the set $\Gamma_\varepsilon = T' \cap \{P(\sigma(x_{\varepsilon_0},\ldots)) \to \sigma(x_{\varepsilon_0},\ldots,x_{\varepsilon_i},\ldots)_{i<i(*)} < x_\varepsilon : i(*) < \theta$ and $\varepsilon_i < \varepsilon$ for $i < i(*)\}$ has cardinality $< \theta$. Now we choose $c_\varepsilon \in M$ by induction on $\varepsilon \in u$ such that the assignment $x_\zeta \mapsto c_\zeta$ for $\zeta \in \varepsilon \cap u$ in M^+ satisfies Γ_ε, possible because

$|\Gamma_\varepsilon| < \theta$ and $(P^M, <^M)$ is θ-directed. So the model M^+ with the assignment $x_\varepsilon \mapsto c_\varepsilon$ for $\varepsilon \in u$ is a model of T'', so T' is $(< \theta)$-satisfiable indeed.]

Recalling that $|M| = \{c^{M^+} : c \in \tau(M^+)$ an individual constant$\}$, T' is realized in some $\prec_{\mathbb{L}_{\theta,\theta}}$-elementary extension N^+ of M^+ by the assignment $x_\varepsilon \mapsto a_\varepsilon$ (for $\varepsilon < \lambda \cdot \chi$). Without loss of generality N^+ is the Skolem hull of $\{a_\varepsilon : \varepsilon < \lambda \cdot \chi\}$, so $N := N^+ \restriction \tau(M)$ is as required. Now $\mathbf{x}$ as required exists by part (2). $\square_{0.26}$

{x32} **Observation 0.27.** *1) If* $\mathbf{x}$ *is a non-trivial* $(\theta, \theta) -$ l.u.f.t. *and* $\chi = \mathrm{cf}(\text{l.u.p.}(\theta <))$ *<u>then</u>* $\chi = \chi^{<\theta}$.
2) Also $\mu = \mu^{<\theta}$ *when* μ *is the cardinality of* l.u.p.$(\theta, <)$.

Proof. 1) By the choice of $\mathbf{x}$ clearly $\chi = \mathrm{cf}(\chi) \geq \theta$. As χ is regular $\geq \theta$ by a theorem of Solovay [Sol74] we have $\chi^{<\theta} = \chi$.
{a27c} 2) See the statement and the proof of 3.11. $\square_{0.27}$

§ 1. Basic stability

{1}

For a complete first order T, being stable has many equivalent definitions; see [Sh:c]. We define the parallel properties for a complete $\mathbb{L}_{\theta,\theta}$-theory and try to sort out the implications.

A difference with the first order case which may be confusing is that the existence of long orders is not so strong and does not imply other versions of unstability, see in particular 1.12. {y45}

In Definition 1.1 below, defining the notions "ι-unstable" generally demand more {y2} when ι increase; it seems reasonable that we shall order the parts of 1.1 in increasing {y2} order by ι, but we deviate putting "4-unstable" just after "1-unstable" as it is more easy to define than 2/3-unstable.

{y2}

Definition 1.1. Let $T \subseteq \mathbb{L}_{\theta,\theta}$, not necessarily complete; below "T is ι-stable" is the negation of "T is ι-unstable"; below if $\Delta = \mathbb{L}_{\theta,\theta}(\tau_T)$ then we may omit Δ except in parts (4),(4A).

1) T is 1-unstable <u>iff</u> for some $\varepsilon, \zeta < \theta$ and formula $\varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\zeta]}) \in \mathbb{L}_{\theta,\theta}$ there is a model M of T and $\bar{a}_\alpha \in {}^\varepsilon M, \bar{b}_\alpha \in {}^\zeta M$ for $\alpha < \theta$ such that $M \models \varphi[\bar{a}_\alpha, \bar{b}_\beta]^{\mathrm{if}(\alpha<\beta)}$ for $\alpha, \beta < \theta$.

2) We say T is 4-unstable <u>when</u> there are $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\theta,\theta}$ and a model M of T and $\bar{b}_\eta \in {}^{\ell g(\bar{y})}M$ for $\eta \in {}^{\theta>}2$ such that $p_\eta(\bar{x}) = \{\varphi(\bar{x}, \bar{b}_{\eta\restriction\alpha})^{\mathrm{if}(\eta(\alpha))} : \alpha < \theta\}$ is a type in M for every $\eta \in {}^\theta 2$, i.e. every subset of cardinality $< \theta$ is realized.

3) For a class $\mathbf{I}$ of linear orders we say T is $\mathbf{I}$-unstable <u>when</u> for some $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\theta,\theta}$ for every $I \in \mathbf{I}$ there are M and $\langle(\bar{a}_s, \bar{b}_s) : s \in I\rangle$ is as in part (1). If $\mathbf{I} = \{I\}$ we may write I-unstable. We say T is $(\Delta, \mathbf{I})$-unstable when above $\varphi(\bar{x}, \bar{y}) \in \Delta$.

4) We say T is strongly $(\Delta, \mathbf{I})$-unstable <u>when</u>[4] for some $\varphi(\bar{x}, \bar{y}) \in \Delta$ satisfying $\ell g(\bar{x}) = \ell g(\bar{y})$ for every linear order $I \in \mathbf{I}$ there are $M \models T$ and sequence $\langle \bar{a}_s \hat{\ } \bar{b}_s : s \in I\rangle$ in M such that:

- (a) $M \models \varphi[\bar{a}_s, \bar{b}_t]^{\mathrm{if}(s<t)}$ for $s, t \in I$,
- (b) $\langle \bar{a}_s : s \in I\rangle$ is strictly $\varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\zeta]})$-convergent where $\ell g(\bar{a}_s) = \varepsilon$
- (c) $\langle \bar{b}_s : s \in I\rangle$ strictly $\psi(\bar{x}_{[\zeta]}, \bar{y}_{[\varepsilon]})$-convergent where $\ell g(\bar{b}_s) = \zeta$ and $\psi(\bar{x}_{[\zeta]}, \bar{y}_{[\varepsilon]}) = \varphi(\bar{y}_{[\varepsilon]}, \bar{x}_{[\zeta]})$ also called $\varphi^+(\bar{x}_{[\zeta]}, \bar{y}_{[\varepsilon]})$

recalling Definition 0.17(1),(2). Let the default value of Δ be $\{\varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\zeta]}), \psi(\bar{x}_{[\zeta]}, \bar{y}_{[\varepsilon]})\}$. {x11}

4A) We say T is middle Δ-unstable <u>when</u> in part (4) we replace "strictly Δ-convergent" by "strictly middle Δ-convergent", see Definition 0.17(1),(2). The {x11} default value of Δ is as in part (4).

5) We say T is 3-unstable <u>when</u> it is strongly $\mathbf{I}_2$-unstable where $\mathbf{I}_2 = \{\sum\limits_{i<i(*)} I_i : i(*)$ an ordinal and for each i, I_i is anti-isomorphic to some ordinal $\delta_i, \mathrm{cf}(\delta_i) \geq \theta\}$.

6) We say T is 2-unstable <u>iff</u> it is $\mathbf{I}_2$-unstable.

7) We say T is 5-unstable if it is $({}^{\theta>}2, <_{\mathrm{lex}})$-unstable.

{y2f}

Remark 1.2. We shall clarify all implications between "ι-unstable" and definably stable which is defined below; this is summed up in 1.15. {y62}

[4]The difference between 1.1(3) and 1.1(4) is the "convergent". In part (5) for the applications {y2} we have in mind it is enough to restrict ourselves to the case $\mathbf{I}_2 = \{\sum\limits_{i<(*)} \delta_i$: <u>where</u> $\delta_i \in \{\theta, \theta^+\}, i(*)$ an ordinal$\}$.

{y3}

Definition 1.3. Let T be as in 1.1. {y2}
1) T is definably stable (definably unstable is the negation) <u>when</u>: if $\varphi = \varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\zeta]}) \in \mathbb{L}_{\theta,\theta}$ <u>then</u> there is $\psi(\bar{y}_{[\zeta]}, \bar{z}_{[\xi]}) \in \mathbb{L}_{\theta,\theta}$ such that:

(∗) if $M \prec_{\mathbb{L}_{\theta,\theta}} N$ are models of T and $\bar{a} \in {}^{\varepsilon}N$ <u>then</u> there is $\bar{c} \in {}^{\xi}M$ satisfying: $\psi(\bar{y}_{[\zeta]}, \bar{c})$ defines $\mathrm{tp}_{\varphi}(\bar{a}, M, N)$, that is:

- if $\bar{b} \in {}^{\zeta}M$ <u>then</u> $N \models \varphi[\bar{a}, \bar{b}]$ iff $M \models \psi[\bar{b}, \bar{c}]$.

{y2} 2) We say $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\theta,\theta}(\tau_T)$ is 1-stable (for T) <u>when</u> 1.1(1) fails for φ (and T). Similarly for the other versions. We say $\varphi(\bar{x}, \bar{y})$ is symmetrically 1-stable (for T) <u>when</u> it is 1-stable and also $\varphi^{\perp}(\bar{y}, \bar{x})$ is 1-stable where $\varphi^{\perp}(\bar{y}, \bar{x}) = \varphi(\bar{x}, \bar{y})$ is called the dual of $\varphi(\bar{x}, \bar{y})$.
3) We say T is (λ, Δ)-stable <u>when</u> $\Delta \subseteq \mathbb{L}_{\theta,\theta}(\tau_T)$ and for every model M of T and $A \subseteq M$ of cardinality $\leq \lambda$ and $\varphi = \varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\zeta]}) \in \Delta$ the set $\mathbf{S}^{\varepsilon}_{\varphi}(A, M)$ has cardinality $\leq \lambda$ <u>where</u> $\mathbf{S}^{\varepsilon}_{\Delta}(A, M) = \{\mathrm{tp}_{\Delta}(\bar{a}, A, N) : N, \bar{a}$ satisfy $M \prec_{\mathbb{L}_{\theta,\theta}} N, \bar{a} \in {}^{\varepsilon}N\}$.
4) We say T is Δ-stable <u>when</u> T is (λ, Δ)-stable for every $\lambda = \lambda^{<\theta} + \lambda^{|T|}$.
4A) In part 3) and 4) omitting Δ means $\Delta = \mathbb{L}_{\theta,\theta}(\tau_T)$.

{y5}

Claim 1.4. *Let $T \subseteq \mathbb{L}_{\theta,\theta}$ (not necessarily complete), $\tau = \tau(T)$ and let $\partial = (\theta + |T|)^{<\theta}$.*
1) We have $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (x) \Rightarrow (f) \Rightarrow (g) \Rightarrow (h) \Rightarrow (i) \Leftrightarrow (j)$ for $x = d, e$ where:

{y2} (a) *T is 5-unstable, see 1.1(7)*
{y2} (b) *T is 4-unstable, see 1.1(2)*
(c) *for some $\varepsilon < \theta$ for every $\lambda \geq \theta$ there are $A \subseteq M \models T, |A| = \lambda$ such that $\mathbf{S}^{\varepsilon}(A, M) = \{\mathrm{tp}_{\mathbb{L}_{\theta,\theta}}(\bar{a}, A, N) : M \prec_{\mathbb{L}_{\theta,\theta}} N, \bar{a} \in {}^{\varepsilon}N\}$ has cardinality $> \lambda$*
(d) *for some $\varepsilon < \theta$, for every $\lambda = \lambda^{\partial}$ for[5] some $\varphi = \varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\zeta]}) \in \mathbb{L}_{\theta,\theta}$ there are $A \subseteq M \models T, |A| = \lambda$ such that $\mathbf{S}^{\varepsilon}_{\varphi}(A, M)$ has cardinality $> \lambda$*
(e) *like (c) but for some $\lambda = \lambda^{\partial}$*
(f) *like (d) but for some $\lambda = \lambda^{\partial}$*
(g) *T is definably unstable*
(h) *there are $\varepsilon < \theta, M \models T, \varphi = \varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\zeta]}) \in \mathbb{L}_{\theta,\theta}(\tau_T)$ and $\langle(\bar{b}_{\alpha,0}, \bar{b}_{\alpha,1}, \bar{c}_{\alpha}) : \alpha < \theta\rangle$ such that:*
- *$\bar{b}_{\alpha,0}, \bar{b}_{\alpha,1} \in {}^{\zeta}M$ and $\bar{c}_{\alpha} \in {}^{\varepsilon}M$*
- *$\mathrm{tp}(\bar{b}_{\alpha,0}, \cup\{\bar{b}_{\beta,0}, \bar{b}_{\beta,1}, \bar{c}_{\beta} : \beta < \alpha\}, M) = \mathrm{tp}(\bar{b}_{\alpha,1}, \cup\{\bar{b}_{\beta,0}, \bar{b}_{\beta,1}, \bar{c}_{\beta} : \beta < \alpha\}, M)$*
- *$\{\varphi(\bar{x}_{\varepsilon}, \bar{b}_{\beta,1}), \neg\varphi(\bar{x}_{\varepsilon}, \bar{b}_{\beta,0}) : \beta < \alpha\}$ is realized by $\bar{c}_{\alpha}$ in M*

{y2} (i) *T is 2-unstable, see 1.1(6)*
{y2} (j) *T is 1-unstable, see 1.1(1).*

[5] What if we ask for a fixed φ, not depending on λ? This makes $(c) \Rightarrow (d)$ problematic.

2) T is 3-unstable $\Rightarrow T$ is definably unstable.
3) T is 1-unstable iff *T is $\{(\lambda, <)\}$-unstable for every (equivalently some) $\lambda \geq \theta$.*
4) T is 5-unstable iff *T is $\{I\}$-unstable for every linearly ordered I.*
5) T is 2-unstable iff *for every $\varepsilon, \zeta < \theta$ it is $\varepsilon \times \zeta^*$-unstable.*
6) In Definition 1.1(1), we can use $\bar{a}_\alpha = \bar{b}_\alpha$ so $\varepsilon = \zeta$. {y2}

Proof. 1) $(a) \Rightarrow (b)$

Obvious; by clause (a) there is $\varphi = \varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\theta,\theta}(\tau_T)$ which witnesses T is $({}^{\theta>}2, <_{\text{lex}})$-unstable, so there is a model M of T and $\bar{a}_\eta \in {}^{\ell g(\eta)}M$ for $\eta \in {}^{\theta>}2$ such that $M \models$ "$\varphi[\bar{a}_\eta, \bar{a}_\nu]$" iff $(\eta <_{\text{lex}} \nu)$ for every $\eta, \nu \in {}^{\theta>}2$. Let $\bar{y} = \bar{y}_{[\zeta]}, \bar{y}' = \bar{y}_{[\zeta+\zeta]}$ and let $\varphi' = \varphi'(\bar{x}, \bar{y}')$ be $(\varphi(\bar{x}, \bar{y}' \restriction [0, \zeta)) \equiv \varphi(\bar{x}, \bar{y}' \restriction [\zeta, \zeta + \zeta))$, easily φ' witnesses T is 4-unstable as witnessed by $\langle \bar{b}_\eta : \eta \in {}^{\theta>}2 \rangle$ where $\bar{b}_\eta = \bar{a}_{\eta\char`^\langle 0 \rangle}{}^\frown \bar{a}_{\eta\char`^\langle 1 \rangle}$.

$(b) \Rightarrow (c)$

Let $\varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\zeta]})$ be as in 1.1(2). Note that {y2}

$(*)$ in Definition 1.1(2), without loss of generality there are $\bar{c}_\eta \in {}^\varepsilon M$ for $\eta \in {}^\theta 2$ realizing $p_\eta(\bar{x}_{[\varepsilon]})$. {y2}

[Why? There is a θ-complete uniform ultrafilter D on θ hence in M^θ/D there are such $\bar{c}_\eta$'s.]

So by compactness for $\mathbb{L}_{\theta,\theta}$, for every λ there are $M_\lambda \models T$ and $\bar{a}^\lambda_\nu \in {}^\zeta(M_\lambda)$ for $\nu \in {}^{\lambda>}2$ and $\bar{c}^\lambda_\eta \in {}^\varepsilon(M_\lambda)$ for $\eta \in {}^\lambda 2$ such that $M_\lambda \models \varphi[\bar{c}^\lambda_\eta, \bar{a}^\lambda_\nu]^{\text{if}(\eta(\ell g(\nu)))}$ when $\nu \triangleleft \eta \in {}^\lambda 2$.

For any cardinal λ let $\mu = \min\{\mu : 2^\mu > \lambda\}$ hence $\mu \leq \lambda \wedge (\forall \partial < \mu)(2^\partial \leq \lambda)$ and so $2^{<\mu} \leq \lambda$ hence $\mu \leq \lambda$, let $A = \cup\{\bar{a}^\mu_\nu : \nu \in {}^{\mu>}2\}$, so $A \subseteq M_\mu$ has cardinality $\leq 2^{<\mu} + \theta \leq \lambda$ and $\mathbf{S}^\varepsilon(A, M_\mu)$ has cardinality $\geq |\{\text{tp}(\bar{c}^\mu_\eta, A, M_\mu) : \eta \in {}^\mu 2\}| \geq 2^\mu > \lambda$.

$(c) \Rightarrow (d)$

It suffices to prove $\neg(d) \Rightarrow \neg(c)$. So assume that $\neg(d)$ holds and note that clearly the set $\{\varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\zeta]}) \in \mathbb{L}(\tau_T) : \varepsilon, \zeta < \theta\}$ has cardinality ∂ recalling $\partial = (|T|^{<\theta} + \theta)$.

Hence if $A \subseteq M \models T$ and $|A| \leq \lambda$ then

- $|\mathbf{S}^\varepsilon(A, M)| \leq \Pi\{|\mathbf{S}^\varepsilon_\varphi(A, M)| : \varphi = \varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\zeta]}) \in \mathbb{L}_{\theta,\theta}(\tau_T)\} \leq (\sup\{|\mathbf{S}^\varepsilon_\varphi(A)| : \varphi = \varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\zeta]}) \in \mathbb{L}_{\theta,\theta}(\tau_T)\})^\partial \leq \lambda^\partial = \lambda$.

[Why? First inequality by the definitions of $\mathbf{S}^\varepsilon(-), \mathbf{S}^\varepsilon_\varphi(-)$, second inequality because the number of relevant φ's is $\leq \partial$, third inequality by the present assumption $\neg(d)$; the last inequality by the meaning of $\neg(d)$. But the deduced inequality means $\neg(c)$.]

$(c) \Rightarrow (e)$

Easy as there are $\lambda = \lambda^\partial$.

$(d) \Rightarrow (f)$

As there are cardinals λ such that $\lambda = \lambda^\partial$.

$(e) \Rightarrow (f)$

As in $(c) \Rightarrow (d)$.

$\underline{(f) \Rightarrow (g)}$

Clearly $\neg(g) \Rightarrow \neg(f)$ holds by counting.

$\underline{(g) \Rightarrow (h)}$

So by compactness for $\mathbb{L}_{\theta,\theta}$ for some $\varepsilon < \theta$ and $M \models T$ and $p \in \mathbf{S}^{\varepsilon}(M)$ and {y3} $\varphi = \varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\zeta]})$ there are no $\psi(\bar{y}_{[\zeta]}, \bar{z}_{[\xi]})$ and $\bar{c} \in {}^{\xi}M$ as in Definition 1.3. Again by compactness for $\mathbb{L}_{\theta,\theta}$ without loss of generality $|\tau_T| < \theta$.

For each $\kappa < \theta$ we try by induction on $\alpha < \kappa$ to choose $\bar{b}^{\kappa}_{\alpha,0}, \bar{b}^{\kappa}_{\alpha,1}, \bar{c}^{\kappa}_{\alpha}$ such that {y2} (recalling 1.1(3)):

- $\bar{b}^{\kappa}_{\alpha,0}, \bar{b}^{\kappa}_{\alpha,1} \in {}^{\zeta}M$ realize the same $\{\varphi^{\perp}(\bar{x}_{[\zeta]}, \bar{y}_{[\varepsilon]})\}$-type over $A^{\kappa}_{\alpha} := \cup\{\bar{b}^{\kappa}_{\beta,0}, \bar{b}^{\kappa}_{\beta,1}, \bar{c}^{\kappa}_{\alpha} : \beta < \alpha\}$
- $\varphi(\bar{x}_{[\varepsilon]}, \bar{b}^{\kappa}_{\alpha,1}), \neg\varphi(\bar{x}_{[\varepsilon]}, \bar{b}^{\kappa}_{\alpha,0}) \in p$
- $\bar{c}^{\kappa}_{\alpha}$ realizes $\{\varphi(\bar{x}_{[\varepsilon]}, \bar{b}^{\kappa}_{\beta,1}), \neg\varphi(\bar{x}_{[\varepsilon]}, \bar{b}^{\kappa}_{\beta,0}) : \beta \le \alpha\}$.

<u>Case 1</u>: For every κ we succeed to carry the induction.

Let $\bar{c}^{\kappa} \in {}^{\varepsilon}M$ realize $\{\varphi(\bar{x}_{[\varepsilon]}, \bar{b}^{\kappa}_{\alpha,1}) \wedge \neg\varphi(\bar{x}_{[\varepsilon]}, \bar{b}^{\kappa}_{\alpha,0}) : \alpha < \kappa\}$. By compactness for $\mathbb{L}_{\theta,\theta}$ we can get clause (h).

<u>Case 2</u>: For some κ and $\alpha < \kappa$, we cannot choose $\bar{b}^{\kappa}_{\alpha,0}, b^{\kappa}_{\alpha,1}$ (but have chosen $\langle \bar{b}^{\kappa}_{\beta,\ell} : \beta < \alpha, \ell < 2\rangle$).

We can find ψ contradicting our choice of M, φ, p.

$\underline{(h) \Rightarrow (j)}$

Let $\varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\zeta]}), M, \bar{b}_{\alpha,0}, \bar{b}_{\alpha,1}, \bar{c}_{\alpha}(\alpha < \theta)$ be as in clause (h) and let φ' be as in the proof of $(a) \Rightarrow (b)$. Now $\varphi', \langle(\bar{c}_{\alpha}, \bar{b}_{\alpha,0}{}^{\frown}\bar{b}_{\alpha,1}) : \alpha < \theta\rangle$ are as required in clause (j) because for $\alpha, \beta < \theta$ we have $M \models$ "$\varphi[\bar{c}_{\alpha}, \bar{b}_{\beta,0}] \equiv \varphi[\bar{c}_{\alpha}, \bar{b}_{\beta,1}]$" iff $\beta > \alpha$.

$\underline{(j) \Rightarrow (i)}$

Let $I = \theta \times \theta^*$, i.e. $\{(\alpha,\beta) : \alpha, \beta < \theta\}$ ordered by $(\alpha_1, \beta_1) < (\alpha_2, \beta_2)$ iff $\alpha_1 < \alpha_2$ or $\alpha_1 = \alpha_2 \wedge \beta_1 > \beta_2$.

Let $\varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\zeta]})$ witness T is 1-unstable and $M, \langle(\bar{a}_{\alpha}, \bar{b}_{\alpha}) : \alpha < \theta\rangle$ exemplify this. Let $\bar{x}' = \bar{x}_{[\varepsilon+\varepsilon]}, \bar{y}' = \bar{y}_{[\zeta+\zeta+\varepsilon]}$ and for $\alpha, \beta < \theta$ let $\bar{a}'_{(\alpha,\beta)} = \bar{a}_{\alpha}{}^{\frown}\bar{a}_{\beta}, \bar{b}'_{(\alpha,\beta)} = \bar{b}_{\alpha}{}^{\frown}\bar{b}_{\beta+1}{}^{\frown}\bar{a}_{\alpha}$ and let $\varphi'(\bar{x}', \bar{y}')$ say $\varphi(\bar{x}' {\restriction} \varepsilon, \bar{y}' {\restriction} \zeta)$ or $(\bar{x}' {\restriction} \varepsilon = \bar{y}' {\restriction} [\zeta + \zeta, \zeta + \zeta + \varepsilon) \wedge \neg\varphi(\bar{x}' {\restriction} [\varepsilon, \varepsilon + \varepsilon), \bar{y}'[\zeta, \zeta + \zeta))$.

{y2} Now $\varphi', M, \langle(\bar{a}'_{\alpha}, \bar{b}'_{\alpha}) : \alpha < \theta\rangle$ are as required in Definition 1.1(3) by part (5) proved below.

$\underline{(i) \Rightarrow (j)}$

Trivially.

{x11} 2) Note that "3-unstable $\Rightarrow$ definably unstable" holds by recalling the Definitions {y3} 0.17(1), 1.1(5), 1.3(1).

3) Easy, too.

4) First, the implication $\Rightarrow$ holds by "θ is compact" because every linear order I is embeddable into $({}^{\alpha}2, <_{\text{lex}})$ for some ordinal α. Second, the implication $\Leftarrow$ is trivial.

5) First, the implication $\Rightarrow$ holds as θ is a compact cardinal. Second, the implication $\Leftarrow$ is trivial.

6) Easy, too, using enough dummy variables; i.e. let $\bar{a}' = \bar{a}_{\alpha}{}^{\frown}\bar{b}_{\alpha}$ and $\varphi''(\bar{x}_{[\varepsilon+\zeta]}, \bar{y}_{[\varepsilon+\zeta]}) := \varphi(\bar{x}_{[\varepsilon+\zeta]} {\restriction} [0, \varepsilon), \bar{y}_{[\varepsilon+\zeta]} {\restriction} [\varepsilon, \varepsilon + \zeta))$. $\square_{1.4}$

{y10} **Conclusion 1.5.** *1) Assume $T \subseteq \mathbb{L}_{\theta,\aleph_0}$ is (complete and) $(\varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\zeta]}, \{\theta\})$-unstable for some $\varepsilon < \theta$ and $\varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\varepsilon]}) \in \mathbb{L}_{\theta,\aleph_0}$.*

For every $\lambda \geq \theta^+ + |T|$, there are $M_\alpha \in \mathrm{Mod}_T$ for $\alpha < 2^\lambda$ which are pairwise non-isomorphic each of cardinality λ.
2) If $T \subseteq \mathbb{L}_{\theta,\theta}$ is strongly 3-unstable and $\lambda = \lambda^{<\theta} \geq \theta^+ + |T|$, then the conclusion of part (1) holds.

Proof. Follows by [Sh:E59, §3] (which improve [Sh:300, Ch.III]) but we explain the background. By [Sh:c, Ch.VIII], if $T \subseteq T_1$ are complete first order and $\lambda \geq |T_1| + \aleph_1$ and T unstable then there are models M_α of T of cardinality λ for $\alpha < 2^\lambda$, pairwise non-isomorphic each from $\mathrm{PC}(T_1, T)$, i.e. each M_α can be expanded to a model of T_1. This was done mainly using E.M. models, i.e. for some $T_2 \supseteq T_1$ of cardinality $\leq \lambda$ with Skolem functions each M_α can be expanded to a model N_α of T_2 which is generated by $\{\bar{a}^\alpha_t : t \in I_\alpha\}, I_\alpha$ a linear order $\bar{\mathbf{a}}_\alpha = \langle \bar{a}^\alpha_t : t \in I_\alpha \rangle$ is an indiscernible sequence in N_α and for some $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T), N_\alpha \models \varphi[\bar{a}^\alpha_s, \bar{a}^\alpha_s]$ iff $s <_{I_\alpha} t$.

Now [Sh:E59, §3] improve it by just requiring $\langle (N_\alpha, \bar{\mathbf{a}}_\alpha) : \alpha < 2^\lambda \rangle$ and $\varepsilon < \theta, \varphi = \varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\varepsilon]}) \in \mathbb{L}_{\theta,\aleph_0}(\tau_T)$ to have some of the properties of such E.M. models (called there "being κ-skeleton like").

This means here just (where λ is regular for transparency):

$(*)$ $(A) \Rightarrow (B)$ where:
- (A)
  - (a) $\bar{\mathbf{a}}_\alpha = \langle \bar{a}_{\alpha,s} : s \in I_\alpha \rangle, \bar{\mathbf{b}}_\alpha = \langle \bar{b}_{\alpha,s} : s \in I_\alpha \rangle, \bar{b}_{\alpha,s} = \bar{a}_{\alpha,s} \in {}^\varepsilon(M_\alpha), \zeta = \varepsilon, M_\alpha, \varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\varepsilon]}), \varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\varepsilon]})$ are as in Definition 1.1(4) {y2}
  - (b) $I_\alpha = \sum\limits_{i<\lambda} I_{\alpha,i}, S_\alpha \subseteq \lambda, I_{\alpha,\varepsilon}$ is isomorphic to $(\theta, >)$ if $\varepsilon \in S_\alpha$ and to $(\theta^+, >)$ if $\varepsilon \in \lambda \backslash S_\alpha$
- (B) $\{M_\alpha / \cong : \alpha < 2^\lambda\}$ has cardinality 2^λ.

Why there are models as in (A)? For part (2) by Definition 1.1(5) and see 1.1(4). {y2}
For part (1) by the definition on E.M. nodes. Note that in [Sh:E59, §3] we first deal with the case ε is finite, but we are assuming $\lambda = \lambda^{<\theta}$ hence allowing $\varepsilon \in [\omega, \theta)$ cause no problem, see [Sh:E59, Th.3.28,pg.48,L3c.16]. $\square_{1.5}$

{y12}

Question 1.6. 1) Can we add in 1.5 "pairwise not $\mathbb{L}_{\infty,\theta^+}$-equivalent"? {y10}
2) Does the logic $\mathscr{L}$ have interpolation when $\mathbb{L}_{\theta,\aleph_0} \subseteq \mathscr{L} \subseteq \mathbb{L}_{\theta,\theta}$ and $\mathscr{L}$ is defined by: $\psi \in \mathscr{L}(\tau)$ iff $\psi \in \mathbb{L}_{\theta,\theta}(\tau)$ and for $\mathbf{t} \in \{\mathrm{yes}, \mathrm{no}\}$ the class of models of $\psi^{\mathbf{t}}$ is closed under $M^I_D | \mathscr{E}$ when $(I, D, \mathscr{E})$ is $(\theta, \aleph_0)$-complete, see Definition 0.21. {x16}

Now recall stability implies the existence of convergence sub-sequences, specifically:

{y15}

Claim 1.7. *Assume $|T| \geq \lambda = \mathrm{cf}(\lambda)$ and $\mu < \lambda \Rightarrow (\mu^{|T|})^+ + \theta < \lambda, |T|^{<\theta} < \partial = \mathrm{cf}(\partial) < \lambda$. If T is 1-stable, $\varepsilon < \theta, M$ is a model of T and $\bar{a}_\alpha \in {}^\varepsilon M$ for $\alpha < \lambda$ then for some stationary $S \subseteq S^\lambda_\partial$ the sequence $\langle \bar{a}_\alpha : \alpha \in S \rangle$ is $(<\omega)$-indiscernible and strongly $\mathbb{L}_{\theta,\theta}$-convergent in M, see Definition 0.17(1).* {x11}

Proof. By [Sh:300a] but we explain the background. First, we may find a $\prec_{\mathbb{L}_{\theta,\theta}}$ increasing sequence $\langle M_\alpha : \alpha \leq \lambda \rangle$ such that $M_\alpha \prec_{\mathbb{L}_{\theta,\theta}} M, \|M_\alpha\| \leq |\alpha|^{<\theta} + \theta + |T|^{<\theta}$ and $\bar{a}_\alpha \subseteq M_{\alpha+1}$.

Second, for each $\alpha \in S_0 := \{\delta < \lambda : \mathrm{cf}(\delta) = \partial\}$ we can find $B_\alpha \subseteq M_\alpha$ of cardinality $\le |T|^{<\theta}$ such that $\mathrm{tp}_{\mathbb{L}_{\theta,\theta}}(\bar{a}_\alpha, M_\alpha, M)$ is definable over B_α.

Third, by Fodour lemma there is a stationary $S_1 \subseteq S$ such that $\langle B_\delta : \delta \in S_1\rangle$ is constantly B, and even the definition scheme is the same. We then prove $\langle \bar{a}_\alpha : \alpha \in S_1\rangle$ is n-indiscernible by induction on n (as there).

Lastly, for proving convergence, we fix $\bar{b} \in {}^{\theta>}M$ and use "$\mathrm{tp}_{\mathbb{L}_{\theta,\theta}}(\bar{b}, M_\lambda, M)$ is definable. $\square_{1.7}$

The experience with first order classes says categoricity even for PC-classes (see below) implies stability (also $\triangleleft_{\lambda,\theta}$-minimality) however this is not so here (where {a22} on $\blacktriangleleft_{\lambda,\theta}$, see Definition 2.9) hence we now consider some examples (see also 3.3). In the rest of this section we prove this and give other examples.

{y41}

Claim 1.8. *T being 1-unstable does not imply T being definably unstable, and does* {y5} *not imply satisfying 1.4(h).*

Proof. Let $M = (\theta, <)$ and $T = \mathrm{Th}_{\mathbb{L}_{\theta,\theta}}(M)$; clearly T is 1-unstable and is de- {y5} finably stable. As for 1.4(h), toward contradiction assume $N \models T$ and $\varphi =$ {y5} $\varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\zeta]}), \langle(\bar{a}_\alpha, \bar{b}_\alpha, \bar{c}_\alpha) : \alpha < \theta\rangle$ are as in clause (h) of 1.4. As θ is a compact cardinal without loss of generality $\langle \bar{a}_\alpha{}^\frown\bar{b}_\alpha{}^\frown\bar{c}_\alpha : \alpha < \theta\rangle$ is an indiscernible sequence in M, i.e. n-indiscernible for every n. Now check. $\square_{1.8}$

{y43}

Thesis 1.9. A big difference with the first order, that is the $\theta = \aleph_0$ case, is:

{y40}

- (a) long linear orders does not contradict categoricity, in particular see 1.10 below
- (b) consider interpreting for $\partial < \theta$, a group isomorphic to the Abelian group $(\{\eta \in {}^A2 : (\exists^{<\partial} a \in A)(\eta(a) = 1)\}, \triangle)$ where $\triangle$ is the symmetric difference; it appears "for free" (formally[6], if we allow equality for the group being just a congruence relation)
- (c) similarly for the group generated by $\{x_a : a \in A\}$ freely.

{y40}

Example 1.10. 1) There are T and T_1 such that:

- (a) $T \subseteq \mathbb{L}_{\theta,\theta}(\{<\})$ is complete
- (b) $T_1 \subseteq \mathbb{L}_{\theta,\theta}(\tau_1)$ is complete, τ_1 finite and $<$ belongs to τ_1
- (c) $T_1 \supseteq T$
- (d) models of T are dense linear orders
- (e) $\mathrm{PC}(T, T_1)$ is categorical in every $\lambda \ge \theta$, recalling
  - $\mathrm{PC}(T, T_1) = \{M_1 \restriction \tau_T : M_1 \in \mathrm{Mod}_{T_1}\}$
- (f) T is 1-unstable
- (g) T is definably stable.

---

[6]Why? E.g. for a model M let

- the set of elements in $\varphi(M)$ where $\varphi = \varphi(\bar{x}_{[\omega]})$ says: $\bigwedge_{n \ne m} (x_{2n} \ne x_{2n+1} \wedge x_{2m} \ne x_{2m+1} \to x_{2n} \ne x_{2m})$, let $\mathrm{Rang}^*(\bar{x}_{2n}) = \{x_{2n} : x_{2n} = x_{2n+1}\}$
- the congruence $\varphi_{\mathrm{eq}}(\bar{x}_{[\omega]}, \bar{y}_{[\omega]})$ says $\mathrm{Rang}^*(\bar{x}_{[\omega]}) = \mathrm{Rang}(\bar{y}_{[\omega]})$
- $\varphi_{\mathrm{mult}}(\bar{x}_{[\omega]}, \bar{y}_{[\omega]}, \bar{z}_{[\omega]}) = \mathrm{Rang}^*(\bar{x}_{[\omega]}) \triangle \mathrm{Rang}^*(\bar{y}_{[\omega]}) = \mathrm{Rang}^*(\bar{z}_{[\omega]})$.

{y43} For clause (c) of 1.9 - more cumbersome.

2) Moreover $T = \mathrm{Th}_{\mathbb{L}_{\theta,\theta}}(N)$ where:

(a) (α) N is a dense linear order
(β) N is of cardinality θ

(b) (α) N is the union of $\aleph_0$ well ordered sets
(β) N has cofinality $\aleph_0$, also its inverse

(c) (α) if σ is regular uncountable, any increasing sequence of length σ has no lub
(β) if $s \in N$ then $N_{<t} = N {\restriction} \{s : s <_I t\}$ has cofinality $\aleph_0$ and $N_{>t} = N {\restriction} \{s : t <_I s\}$ has co-initiality $\aleph_0$

(d) any two intervals of N are isomorphic (note: T cannot say this but T_1 can).

3) Moreover T_1 extends T and just says in addition only that every two intervals of N are isomorphic.

{y40g}

Remark 1.11. 1) See [Sh:E62, §2] as explained below.
2) Hausdorff has introduced and investigated the class of scattered linear orders. Galvin and Laver, see [Lav71] investigate the class $\mathscr{M}$ of linear orders which are a countable union of scattered linear orders. They were interested in linear orders up to embeddability inside the class $\mathscr{M} = \cup\{\mathscr{M}_{\lambda,\mu_1,\mu_2} : \mu_1, \mu_2$ are regular uncountable such that $\lambda^+ = \mu_1 + \mu_2\}$ where $\mathscr{M}_{\lambda,\mu_1,\mu_2}$ is the class of linear orders from $\mathscr{M}$ of cardinality λ with no increasing sequences of length μ_1 and no decreasing sequences of length μ_2. Galvin defined $\mathscr{M}_{\lambda,\mu_1,\mu_2}$ and prove existence of a universal member.

Laver, solving a long standing conjecture of Fraïsse, and using the theory of better quasi orders of Nash Williams prove the following. The class $\mathscr{M}$ is well quasi ordered and even better quasi order under embeddability; this answers affirmatively Fraïsse's conjecture which says that $\mathscr{M}_{\aleph_1,\aleph_1}$ = the class of countable linear orders, is well ordered. So categoricity (1.10(1)(e)) and clause (c) of 1.10(2) were irrelevant there, the latter is crucial here for categoricity. In [Sh:220, pp.308,309], this is continued being interested in uniqueness. We do more in [Sh:E62, §2]. {y40}
3) As requested we explain that in [Sh:E62, §2], we investigate classes of I^+ of the form: a linear order, I expanded by unary relations $P_s^{I^+}$ $(s \in S)$ such that $\langle P_s^{I^+} : s \in S\rangle$ is a partition of I and if, e.g. $\langle t_i : i < \kappa\rangle$ is increasing with lub $t_\kappa, \kappa = \mathrm{cf}(\kappa) > \aleph_0$ and $t_\kappa \in P_s^{I^+}$ then we know for a club of $\delta < \kappa$, what is the co-initiality of $\{s \in I : (\forall i < \delta)(t_i <_I s)\}$ and more. It is proved there that under such restritions we get uniqueness for those expanded linear orders.

Proof. We know (see [Sh:E62, §2] and 1.11 above) {y40g}

$(*)_1$ there is a linear order N satisfying Clauses (a)-(d) of part (2)

$(*)_2$ (a) choose N_* as in $(*)_1$
(b) let $T = \mathrm{Th}_{\mathbb{L}_{\theta,\theta}}(N_*)$
(c) let T_1 be $T \cup \{\psi\}$, where ψ says that: if $x_1 < y_1, x_2 < y_2$ then $z \mapsto F(z, x_1, y_1, x_2, y_2)$ is an isomorphism from the interval (x_1, y_1) onto the interval (x_2, y_2) for the linear order
(d) note that the theory T_1 is consistent as we can expand N_* to a model of T_1

$(*)_3$ (a) if N is a linear order failing sub-clause (α) of (b) of 1.10(2) <u>then</u> there is $N_1 \subseteq N$ of cardinality $< \theta$ failing it, hence N is not a model of T {y40}

(b) similarly for $(b)(\beta), (c)(\beta)$ and even $(c)(\alpha)$ for $\sigma < \theta$.

[Why? By θ being a compact cardinal.]

So easily

{y40} $(*)_4$ (a) if M is a model of T then M satisfies Clauses $(a)(\alpha), (b), (c)$ of 1.10(2)

(b) if $M \in \mathrm{PC}(T, T_1)$, i.e. $M = M_1 {\restriction} \{<\}$ where $M_1 \models T_1$ <u>then</u> M satisfies
{y40} Clauses $(a)(\alpha), (b), (c), (d)$ of 1.10(2).

{y40} [Why? Mainly by $(*)_3$, e.g. why M satisfies clause $(c)(\alpha)$ of 1.10(2)? let $\bar{a} = \langle a_\alpha : \alpha < \partial \rangle$ be increasing, ∂ regular uncountable and we shall prove it has no
{y40} lub. If $\partial < \theta$ this is said in T. If $\partial \geq \theta$ or just $\partial \geq \aleph_1$, then $\bar{a}$ is bounded (see 1.10(2)(b)(β)) so there is a decreasing $b = \langle b_\beta : \beta < \kappa \rangle$ such that $(\bar{a}, \bar{b})$ is a pre-cut
{y40} of M, see 0.18(4) and κ is 1 or a regular cardinal. Now by 1.10(2)$(b)(\alpha)$ necessarily
{y40} $\kappa = \aleph_0$ or $\kappa = 1$; but by $M \models T$ recalling 1.10$(c)(\beta), \kappa = 1$ is impossible.]

Also

$(*)_5$ $\mathrm{PC}(T, T_1)$ is categorical in every $\lambda \geq \theta$.

{y40g} [Why? By [Sh:E62, §2] and see 1.11(3).]

{y40} So T satisfies all the clauses of 1.10(1), e.g. we shall prove that T is definably stable; toward this assume

$(*)_{6.1}$ $M \prec_{\mathbb{L}_{\theta,\theta}} N$ are models of T and we should prove that for $\bar{a} \in {}^{\theta>}M, \mathrm{tp}_{\mathbb{L}_{\theta,\theta}}(\bar{a}, M, N)$ is definable (in M).

Toward this for $a \in N \backslash M$ clearly $M_{>a} := \{b \in M : a <^N b\}$ has co-initiality 1 or $\aleph_0$ so let $\bar{b}_{a,1}$ list a countable subset of $M_{>a}$ unbounded from below in $M_{>a}$.

Let $M_{<a} = \{b \in M : b <^N a\}$ and let $\bar{b}_{a,2}$ be a sequence of elements of $M_{<a}$ of length $< \theta$ which is unbounded in $N_{<a} \cap M$ if possible, empty otherwise. Letting $\bar{b} = \bar{b}_{a,1} {}^\frown \bar{b}_{a,2}$ clearly it is a sequence of elements of M of length $< \theta$ (but actually $\bar{b}_2$ is not necessary).

So clearly it suffices to prove:

$(*)_{6.2}$ if $\bar{a} \in {}^{\theta>}N$ and $\bar{b} \in {}^{\theta>}M$ includes $\bar{b}_{a_\varepsilon}$ (or just $\bar{b}_{a_\varepsilon,1}$) for every $\varepsilon < \ell g(\bar{a})$ then $\mathrm{tp}_{\mathbb{L}_{\theta,\theta}}(\bar{a}, M, N)$ is definable over $\bar{b}$.

For this it suffices to prove:

$(*)_{6.3}$ Assume $\partial \leq \theta$ is regular and e.g. inaccessible, $\varepsilon < \partial$ and $\bar{a}_1, \bar{a}_2 \in {}^{\varepsilon}N$. The following are equivalent:

(a) $\mathrm{tp}_{\mathbb{L}_{\theta,\tau}}(\bar{a}_1, M, N) = \mathrm{tp}_{\mathbb{L}_{\theta,\theta}}(\bar{a}_2, M, N)$

(b) (α) if $\xi, \zeta < \varepsilon$ then $a_{1,\xi} <_M a_{1,\zeta} \Leftrightarrow a_{2,\xi} < a_{2,\zeta}$ (in M)

(β) if $u \subseteq \varepsilon$ <u>then</u> the cofinalities of $\bigcap\limits_{\zeta \in u} M_{<a_{1,\zeta}}, \bigcap\limits_{\zeta \in u} M_{<a_{2,\zeta}}$ are equal or are both $\geq \partial$

(γ) if $u \subseteq \varepsilon$ <u>then</u> the co-initialities of $\bigcap\limits_{\zeta \in u} M_{>a_{1,\zeta}}, \bigcap\limits_{\zeta \in u} M_{>a_{2,\zeta}}$ are equal or are both $\geq \partial$.

This is easy to check. $\square_{1.10}$

{y45}

Example 1.12. 0) $\mathrm{Th}_{\mathbb{L}_{\theta,\theta}}(\theta,<)$ is 1-unstable, definably stable.
1) Let $T_2 = \mathrm{Th}(N), N$ is the linear order $\theta \times (\theta+1)^*$ ordered lexicographically expanded by $P^N = \theta \times \{\theta+1\}$.

Then:

- (a) T_2 is 2-unstable as exemplified by a formula $\varphi = \varphi(x,y)$ but T_2 is 3-stable and stable as well as 4-stable and 5-stable
- (b) M is a model of T_2 when M is $\sum\limits_{i<\delta} M_i, \delta$ an ordinal of cofinality $\geq \theta$ and each M_i is isomorphic to $\delta_i + 1, \delta_i$ an ordinal of cofinality $\geq \theta$.

2) Let $T_3 = \mathrm{Th}_{\mathbb{L}_{\theta,\theta}}(N), N$ is the linear order $\theta \times \theta^*$.

Then

- (a) T_3 is 3-unstable but stable hence 4-stable and 5-stable
- (b) like 1.12(1)(b) but $M_i \cong \delta_i$. {y45}

3) Let $T_4 = \mathrm{Th}_{\mathbb{L}_{\theta,\theta}}({}^{\theta>}2, \triangleleft)$

- (a) T_4 is 4-unstable but 5-stable and 3-stable
- (b) M is a model of T iff it is isomorphic to $(\mathscr{T}, \triangleleft)$ where for some ordinal α of cofinality $\geq \theta, \mathscr{T}$ is a subset of ${}^{\alpha>}2$, closed under initial segments, $\eta \in \mathscr{T} \Rightarrow \eta\hat{\ }\langle 0\rangle \in \mathscr{T} \wedge \eta\hat{\ }\langle 1\rangle \in \mathscr{T}$ and $\mathscr{T}$ is closed under increasing unions of length $< \theta$.

4) Let T_5 be the $\mathbb{L}_{\theta,\theta}$-theory of any dense linear order which is θ-saturated in the first order sense (so with neither first nor last element), can use also $\mathrm{Th}_{\mathbb{L}_{\theta,\theta}}({}^{\theta>}2, <_{\mathrm{lex}})$

- (a) T_5 is ι-unstable, for $\iota = 1,\ldots,5$.

5) Let $T_6 = \mathrm{Th}_{\mathbb{L}_{\theta,\theta}}(M)$ where $M = ({}^{\theta>}2, \triangleleft, P^M), P^M = \{\eta\hat{\ }\langle 1\rangle : \eta \in {}^{\theta>}2\}$ so $\tau_M = \{<, P\}$ so $<, P$ are two-place, one-place predicates respectively, <u>then</u> T_6 is 5-unstable but 3-stable.

Proof. This proof almost always uses only $\theta = \mathrm{cf}(\theta) > \aleph_0$; we shall mention when not.
0) See the proof of 1.8. {y41}
1) Note that

- $(*)_1$ (a) if (C_1, C_2) is a cut of $\theta \times (\theta+1)^*$, <u>then</u> the cofinality of (C_1,C_2) is one of the following: $(0,1),(1,\theta),(1,\partial),(1,1),(\partial,1),(\theta,0)$ with $\partial = \mathrm{cf}(\partial) < \theta$
  - (b) every one of those cofinalities appear.

[Why? By inspection.]

- $(*)_2$ if N is a model of T_2 and (C_1,C_2) is a cut of N <u>then</u> the cofinality of (C_1,C_2) is one of the following: $(0,1),(1,\lambda_1),(1,\partial),(1,1),(\partial,1),(\lambda_2,0)$ with $\partial = \mathrm{cf}(\partial) < \theta, \lambda_1 = \mathrm{cf}(\lambda_1) \geq \theta$ and $\lambda_2 = \mathrm{cf}(\lambda_2) \geq \theta$.

[Why? Follows from $(*)_3$ which is proved below.]

$(*)_3$ (a) let $\varphi_1(x,y)$ say: $x < y$ and there is no $z \in (x,y]$ such that $P(z)$

(b) let $\varphi_2(x,y) = \varphi_1(x,y) \vee \varphi_1(y,x) \vee x = y$

(c) if $N \models T_2$ then φ_2 defines an equivalence relation on N, each equivalence class A is $\mathbb{L}_{\theta,\theta}$-equivalent to $(\theta+1)^*$ ($\mathbb{L}_{\aleph_1,\aleph_1}$ suffice) hence $N{\restriction}A$ is anti-well (linearly) ordered, with a first element and last element and omitting the first element of co-initiality $\geq \theta$

(d) if $N \models T_2$ then the linear order φ_2^N is $\mathbb{L}_{\theta,\theta}$-elementarily equivalent to θ.

[Why? Should be clear.]

{y45} By $(*)_3$, Clause (b) of 1.12(1) holds. Now Clause (a) of 1.12(1) follows by {y2} checking Definition 1.1.

2) Similarly replacing $(\theta+1)^*$ by θ^*.

3) Let $\tau = \{<\}, M = ({}^{\theta>}2, \triangleleft)$ a τ-model so $<^M = \trianglelefteq {\restriction}{}^{\theta>}2$. Clause (b) should be clear and anyhow we use just $\Rightarrow$. For Clause (a), T_4 being 4-unstable holds for the {y2} formula $\varphi = \varphi(x,y) = (y < x)$ by the definition of 4-unstable in 1.1(2). As being "5-stable" is easier, we shall just prove "T_4 is 3-stable".

For this we prove the following, using θ is a compact cardinal; clearly this suffices; {y2} the φ, ψ below are not related to Definition 1.1(4):

⊞ Assume $M \models T_4$ and δ_1, δ_2 are ordinals of cofinality $\geq \theta$, but $\mathrm{cf}(\delta_1) \neq \mathrm{cf}(\delta_2)$ and $J = (\{1\} \times \delta_1) \cup (\{2\} \times \delta_2)$ ordered by $\alpha_1 < \beta_1 < \delta_1 \wedge \alpha_2 < \beta_2 < \delta_2 \Rightarrow (1,\alpha_1) < (1,\beta_1) < (2,\beta_2) < (2,\alpha_2)$ and $\varphi = \varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\zeta]}) \in \mathbb{L}_{\theta,\theta}(\tau_M), \bar{a}_s \in {}^{\varepsilon}M, \bar{b}_s \in {}^{\zeta}M$ for $s \in J$ and $M \models \varphi[\bar{a}_s, \bar{b}_t]^{\mathrm{if}(s<t)}$. Then for some $\psi(\bar{x}, \bar{z}) \in \mathbb{L}_{\theta,\theta}(\tau_M)$ and $\bar{c}$ from ${}^{\ell g(\bar{z})}M$ we have:

(a) $\delta_1 = \sup\{\alpha_1 < \delta_1 : M \models \text{“}\psi[\bar{a}_{(1,\alpha_1)}, \bar{c}]\text{”}\}$

(b) $\delta_2 = \sup\{\alpha_2 < \delta_2 : M \models \text{“}\neg\psi[\bar{a}_{(2,\alpha_2)}, \bar{c}]\text{”}\}$.

Why? For $\ell = 1,2$ let D_ℓ be a θ-complete ultrafilter on δ_ℓ such that $\alpha < \delta_\ell \Rightarrow$ {y45} $[\alpha, \delta_\ell) \in D_\ell$. As in 1.4(6), without loss of generality $\bar{a}_s = \bar{b}_s$ and by clause (b) of 1.12(3), $M = (\mathscr{T}, \triangleleft)$ where $\mathscr{T}, \alpha$ are as there.

Let $\mathscr{T}^+ = \mathscr{T} \cup \{\eta \in {}^{\alpha>}2 : \ell g(\eta)$ is a limit ordinal and $\beta < \ell g(\eta) \Rightarrow \eta{\restriction}\beta \in \mathscr{T}\}$, clearly $\eta \in \mathscr{T}^+ \backslash \mathscr{T} \Rightarrow \mathrm{cf}(\ell g(\eta)) \geq \theta$ using $T_3 = \mathrm{Th}_{\mathbb{L}_{\theta,\theta}}(M)$. For $s \in J$ let $\bar{a}_s = \langle a_{s,i} : i < \zeta \rangle$ and for each $i < \zeta$ we choose $\eta_i^1, \eta_i^2 \in \mathscr{T}^+$ such that:

- $\eta_i^\ell = \cup\{\nu \in \mathscr{T} : \{\alpha < \delta_\ell : \nu \trianglelefteq a_{(\ell,\alpha),i}\} \in D_\ell\}$.

Let $u_\ell = \{\varepsilon < \zeta : \{\alpha < \delta_\ell : a_{(\ell,\alpha),\varepsilon} = \eta_\varepsilon^\ell\} \in D_\ell\}$ clearly

$(*)_1$ $\varepsilon \in u_\ell \Rightarrow \eta_\varepsilon^\ell \in \mathscr{T}$

$(*)_2$ $u_\ell \neq \zeta$.

[Why? By $s,t \in J \Rightarrow M \models \varphi[\bar{a}_s, \bar{b}_t]^{\mathrm{if}(s<t)}$, see the statement of ⊞ hence $s \neq t \Rightarrow \bar{a}_s \neq \bar{a}_t$ but $u_\ell = \zeta \Rightarrow \bigwedge\limits_{\alpha,\beta<\delta_\ell} \bar{a}_{(\ell,\alpha)} = \bar{a}_{(\ell,\beta)}$.]

Now we prove ⊞ by cases.

Case 1: $\varepsilon \in u_1$ but $\varepsilon \notin u_2 \vee (\varepsilon \in u_2 \wedge \eta_\varepsilon^1 \neq \eta_\varepsilon^2)$

Let $\psi(\bar{x}_{[\zeta]}, \bar{c}) = (x_{[\varepsilon]} = \eta_\varepsilon^1)$ and check.

Case 2: $\varepsilon \in u_2$ but $\varepsilon \notin u_1 \vee (\varepsilon \in u_1 \wedge \eta^1_\varepsilon \neq \eta^2_\varepsilon)$

Let $\psi(\bar{x}_{[\zeta]}, \bar{c}) = (x_{[\varepsilon]} \neq \eta^2_\varepsilon)$ and check.

Case 3: $\varepsilon < \zeta, \varepsilon \notin u_1, \varepsilon \notin u_2$ but $\eta^1_\varepsilon \neq \eta^2_\varepsilon$

By symmetry without loss of generality $\ell g(\eta^1_\varepsilon) > \ell g(\eta^2_\varepsilon)$, let $\nu \in \mathscr{T}$ be such that $\nu \triangleleft \eta^1_\varepsilon$ but $\nu \ntrianglelefteq \eta^2_\varepsilon$, clearly exists and let $\psi(\bar{x}_\zeta, \bar{c}) = (\nu \triangleleft x_\varepsilon)$ and check.

Case 4: $\varepsilon < \zeta, \varepsilon \notin u_1 \cup u_2, \eta^1_\varepsilon = \eta^2_\varepsilon$ but for some $\nu \triangleleft \eta^1_\varepsilon$ we have $\delta_1 = \sup\{\alpha < \delta_1 : \nu \triangleleft a_{(1,\alpha),\varepsilon}\}$

Let $\psi(\bar{x}_\zeta, \bar{c}) = (\nu \ntrianglelefteq x_\varepsilon)$.

Case 5: Like Case 4 for δ_2

Similarly.

Now if none of the cases above holds, then by $(*)_2$ there is $\varepsilon < \zeta$ such that $\varepsilon \notin u_1$; by not case 2, $\varepsilon \notin u_2$, by not case 3, $\eta^1_\varepsilon = \eta^2_\varepsilon$, by not case 4, $\mathrm{cf}(\ell g(\eta^1_\varepsilon)) = \mathrm{cf}(\delta_1)$, and by not case 5, $\mathrm{cf}(\ell g(\eta^2_\varepsilon)) = \mathrm{cf}(\delta_2)$. Together necessarily $\mathrm{cf}(\delta_1) = \mathrm{cf}(\delta_2)$ contradicting an assumption.

So $\boxplus$ holds indeed. (We may wonder what we can do without assuming "θ a compact cardinal"; in short, if $\partial < \theta \wedge \alpha < \mathrm{cf}(\delta_\ell) \Rightarrow |\alpha|^\partial < \mathrm{cf}(\delta_\ell)$, we can use the Δ-system lemma; otherwise use [Sh:620, §7] which gives a weaker relative of the Δ-system lemma for, e.g. $\lambda = \mu^+, \mu > 2^{\mathrm{cf}(\mu)}$.)

4) Easy.

5) Like the proof of part (3), noting that $<_{\mathrm{lex}}$ is definable in M. $\square_{1.12}$

{y50} **Definition 1.13.** For a linear order I and $\sigma < \theta$ we define $M_{I,\sigma}$ as the following model:

- (a) universe $\{\eta : \eta$ a sequence of length $< \sigma, \eta(2i) \in \mathbb{Q}, \eta(2i+1) \in I\}$
- (b) $<^M$ is the natural lexicographic order.

{y56} **Example 1.14.** 1) There is a complete $T \subseteq \mathbb{L}_{\theta,\theta}(\{<\})$ which is definably unstable, 1-unstable but "3-stable and 4-stable".

2) We can add "T has $\theta -$ n.c.p.", see Definition 3.1 below. {a17}

Proof. 1) Let $\tau = \{<\}$ and for any cardinality λ we define a τ-model M_λ by:

- (A) $s \in M_\lambda$ iff for some $\alpha = \alpha(s) < \lambda, s$ is a function from α to $\{0,1\}$ such that the set $\{\beta < \alpha : s(\beta) = 1\}$ is finite}
- (B) $M_\lambda \models$ "$s < t$" iff $s \triangleleft t$.

Let $T = \mathrm{Th}_{\mathbb{L}_{\theta,\theta}}(M_\lambda)$.

Now

- $(*)$ if M is a model of T then for some cardinal λ and M' we have:
  - (a) M' is isomorphic to M
  - (b) $M' \subseteq M_\lambda$
  - (c) $|M'|$ is closed under initial segments
  - (d) if $\eta \in M'$ and $\gamma < \lambda$ then $\eta\hat{\ }\langle(0)_\gamma\rangle \in M'$.

The rest should be clear.

2) As above use the linear order of 1.10 instead of θ. $\square_{1.14}$ {y40}

We now sum up the implications among the generalizations of stable.

Conclusion 1.15. *1) For T a complete $\mathbb{L}_{\theta,\theta}$-theory the following implications hold:* {y62}

(a) *5-unstable $\Rightarrow$ 4-unstable $\Rightarrow$ T is unstable $\Rightarrow$ T is λ-unstable for some $\lambda = \lambda^{<\theta} + \theta + \lambda^{|T|}$ $\Rightarrow$ definably unstable $\Rightarrow$ 2-unstable $\Leftrightarrow$ 1-unstable*

(b) *3-unstable $\Rightarrow$ definably unstable $\Rightarrow$ 2-unstable $\Leftrightarrow$ 1-unstable.*

2) The results in part (1) are best possible, i.e. all implications not appearing there fail for some such T.

Proof. 1) Clause (a):

- $\bullet_1$ "T is 5-unstable implies T is 4-unstable".

{y5} [Why? By 1.4(1)(a) $\Rightarrow$ (b).]

- $\bullet_2$ "4-unstable implies T is unstable" .

{y5} [Why? By 1.4(1)(b) $\Rightarrow$ (c).]

- $\bullet_3$ "T implies λ-unstable for some $\lambda = \lambda^{<\theta} + \lambda^{|T|}$".

{y5} [Why? By 1.4(1)(c) $\Rightarrow$ (e).]

- $\bullet_4$ "λ-unstable for some $\lambda = \lambda^{<\theta} + \lambda^{|T|}$ implies definably unstable".

{y5} [Why? By 1.4(a)(b) $\Rightarrow$ (e).]

- $\bullet_5$ "definably unstable implies 2-unstable".

{y5} [Why? By 1.4(1)(g) $\Rightarrow$ (i).]

- $\bullet_6$ "2-unstable is equivalent to 1-unstable".

{y5} [Why? By 1.4(1)(i) $\Rightarrow$ (j).]

Clause (b):

- $\bullet_1$ "3-unstable implies definably unstable".

So we are done.

{y5} [Why? By 1.4(2), the second phrase. The other implications hold by clause (a).]

2) Note that:

- $\bullet_1$ "1-unstable does not imply definably unstable".

{y41} [Why? This holds by 1.8.]

- $\bullet_2$ "3-unstable does not imply stable.

{y41} [Why? This holds by 1.8(2).]

- $\bullet_3$ "4-unstable does not imply 3-unstable".

{y45} [Why? This holds by 1.12(3).]

- $\bullet_4$ "4-unstable does not imply definably 5-unstable".

{y45} [Why? This holds by 1.12(3).]

So we are done. $\square_{1.15}$

{2}

§ 2. Saturation of ultrapowers

We define versions of notions of saturation and deal with basic properties.

Note that unlike the first order case, two $(\lambda, \lambda, \mathbb{L}_{\theta,\theta})$-saturated models of cardinality λ are not necessarily isomorphic, see Definition 2.2 and examples in 2.3. {a6} We consider calling the notion in 2.2, compact instead of saturated, but the word compact has been over used. {a5}

{a0}

Context 2.1. θ a compact cardinal.

{a5}

Definition 2.2. 1) We say M is fully (λ, ∂, L)-saturated (may omit the fully; where $L \subseteq \mathscr{L}(\tau_M)$ and $\mathscr{L}$ is a logic; we may write $\mathscr{L}$ if $L = \mathscr{L}(\tau_M)$, the default value is $\mathscr{L} = \mathbb{L}_{\theta,\theta}$) <u>when</u>: if Γ is a set of $< \lambda$ formulas from L with parameters from M with $< 1 + \partial$ free variables, and Γ is $(< \theta)$-satisfiable in M, <u>then</u> Γ is realized in M.
2) We say "locally" <u>when</u> using one $\varphi = \varphi(\bar{x}, \bar{y}) \in \mathscr{L}$, i.e. all members of Γ have the form $\varphi(\bar{x}, \bar{b})$, that is:

- (a) if $\partial \le \theta$, then we consider a set of formulas of the form $\{\varphi(\bar{x}_{[\varepsilon]}, \bar{a}_\alpha) : \alpha < \alpha_*\}$ where $\varepsilon < \partial, \alpha_* < \lambda$ (so $\ell g(\bar{x}) = \varepsilon$)
- (b) if $\partial > \theta$ letting $j_* = \ell g(\bar{x})$, we consider a set of formulas of the form $\{\varphi(\langle x_{\varepsilon(i,\alpha)} : i < j_\alpha\rangle, \bar{a}_\alpha) : \alpha < \alpha_*\}$ where $\{\varepsilon(i,\alpha) : i < j_\alpha, \alpha < \alpha_*\} \subseteq j_*$.

3) In the full case omitting ∂ means $\partial = \lambda$ and in the local case omitting ∂ means $\partial = \theta$; writing "$\le \partial$" means ∂^+. Omitting L means $\mathbb{L}_{\theta,\theta}$ and omitting λ means $\lambda = \|M\|$.
4) Assume ε is an ordinal $< \theta$ and Δ is a set of formulas of the form $\varphi(\bar{x}_{[\varepsilon]}, \bar{y})$. We say M is (λ, Δ)-saturated when: Γ is realized in M whenever Γ is a set of $< \lambda$ formulas of the form $\varphi(\bar{x}_{[\varepsilon]}, \bar{a}), \bar{a} \subseteq M$, which is $(< \theta)$-satisfiable in M. May write $(\lambda, \theta, \Delta)$-saturated abusing notation.

As said above, this notion does not have the most desirable properties it has in the first order case as:

{a6}

Claim 2.3. *Let $\tau = \{<\}, <$ a two-place predicate.*
1) If $T = \mathrm{Th}_{\mathbb{L}_{\theta,\theta}}(\theta, <)$, <u>then</u> no model of T is $(\theta^+, 1, \mathbb{L}_{\theta,\theta}(\tau))$-saturated.
2) There is a complete $T \subseteq \mathbb{L}_{\theta,\theta}(\tau)$ such that: $\tau = \tau_T$ is finite and if $\mu = \mu^{<\kappa}, \kappa = \mathrm{cf}(\kappa) \ge \theta$, so possibly $\mu = \kappa$ <u>then</u> T has non-isomorphic $(\kappa, \kappa, \mathbb{L}_{\theta,\theta}(\tau))$-saturated models of cardinality μ (but a unique smooth one - see the proof).
3) In part (2), if μ is strong limit singular then:

- (A) *if μ is of cofinality $\ge \theta$ <u>then</u> T has non-isomorphic special models of cardinality μ; where:*
  - *M is called special when M is the union of the $\prec_{\mathbb{L}_{\theta,\theta}}$-increasing sequence $\bar{M} = \langle M_\alpha : \alpha < \mathrm{cf}(\mu)\rangle$ such that $\|M_\alpha\| < \mu$ and $M_{\alpha+1}$ is $(\|M_\alpha\|^+, \|M_\alpha\|^+, \mathbb{L}_{\theta,\theta}(\tau))$-saturated*
- (B) *if μ has cofinality $\in [\aleph_1, \theta)$ then T has $> \mu$ special models of cardinality μ pairwise non-isomorphic; but unique if we demand "M is smooth" (see in the proof)*
- (C) *if μ has cofinality $\aleph_0$ <u>then</u> T has a special model of cardinality μ and this model is unique up to isomorphism.*

{a7}

Remark 2.4. 1) The claim above tells us that saturation does not behave as in the first order case, neither concerning existence nor concerning uniqueness.
2) So in part 2.3(2), the counterexample is when $\mu = \kappa$; note that there are such μ's: any successor of strong limit singular cardinal which is $\geq \theta$ by [Sol74]. {a6}
3) Concerning 2.3(3) note that we regain uniqueness if we demand smoothness; see [Sh:88r, 2.15=L88r-2.10,2.17=L88r-2.11.1]. {a6}
4) Concerning 2.3(3)(c), recall that Chang proved that for such μ, if two models are $\mathbb{L}_{\mu^+,\mu}$-equivalent then they are isomorphic. {a6}
5) Let $L = \mathbb{L}_{\theta,\theta}(\tau_M)$. Why in first order logic in 2.2 we use only $\partial = 1$ and here not? If $(\forall \alpha < \lambda)(|\alpha|^{<\theta} < \lambda)$ then the cases $\partial = 1$ and $\partial = 2$ are equivalent <u>but</u> for $\partial = \aleph_1$, a type $p = p(\bar{x}_{[\omega]})$ may not be realized though the model is (λ, ∂, L)-saturated for every finite ∂, unlike first order logic. {a5}

Proof. 1) Any model of T is isomorphic to $M = (\delta, <)$ for some ordinal δ of cofinality $\geq \theta$. Hence it is enough for such δ to prove that $M = (\delta, <)$ satisfies the desired conclusion. If $\delta = \theta$ the model M omits the type $\{\alpha < x : \alpha < \theta\}$ and if $\delta > \theta$ then M omits $\{\alpha < x \wedge x < \theta : \alpha < \theta\}$.
2) Let $\tau = \{<\}, <$ a two-place predicate; toward defining a theory T we first let $\mathfrak{k} = (K, \leq_{\mathfrak{k}})$ be defined by:

$(*)_1$ (a) K is the class of τ-models M which are trees in the model theoretic sense, i.e. satisfies:

- $x < y \rightarrow x \neq y$
- $(x < y \wedge y < z) \rightarrow x < z$
- $(x < z \wedge y < z) \rightarrow (x < y \vee y < x \vee y = x)$

(b) $\leq_{\mathfrak{k}}$ is the following two-place relation on $K_1 : M \leq_{\mathfrak{k}} N$ <u>iff</u>
  (α) $M \subseteq N$
  (β) if $\langle a_n : n < \omega \rangle$ is increasing with no upper bound in M, <u>then</u> it has no upper bound in N.

Now observe

$(*)_2$ $\mathfrak{k}$ is a weak a.e.c., in the sense that:
(A) (a) K and $\leq_{\mathfrak{k}}$ are closed under isomorphisms
(b) $\leq_{\mathfrak{k}}$ is a partial order and $M \in K \Rightarrow M \leq_{\mathfrak{k}} M$
(c) if $\langle M_i : i < \delta \rangle$ is $\leq_{\mathfrak{k}}$-increasing then $M_\delta := \bigcup_{i<\delta} M_i \in K$ and $i < \delta \Rightarrow M_i \leq_{\mathfrak{k}} M_\delta$
(d) if $\langle M_i : i \leq \delta \rangle$ is $\leq_{\mathfrak{k}}$-increasing then $\bigcup_{i<\delta} M_i \leq_{\mathfrak{k}} M_\delta$ provided that $\mathrm{cf}(\delta) \neq \aleph_0$
(e) if $M_1 \subseteq M_2$ are $\leq_{\mathfrak{k}} N$ <u>then</u> $M_1 \leq_{\mathfrak{k}} M_2$
(f) LST: if $\lambda = \lambda^{\aleph_0}$ <u>then</u> the LST-property holds up to λ
(B) (a) $\mathfrak{k}$ satisfies the amalgamation property, in fact, essentially disjoint union suffice, i.e. if $M_0 \subseteq M_1, M_0 \subseteq M_2$ are from K and $M_1 \cap M_2 = M_0$, <u>then</u> $M_3 = M_1 \cup M_2$ does $\leq_{\mathfrak{k}}$-extend M_ℓ for $\ell = 0, 1, 2$. Note that to say $M_3 := M_1 \cup M_2$ means M_3 has universe $|M_1| \cup$

$|M_2|$ and $<^{M_*}$ is defined by $a_1 <^{M_3} a_2$ iff at least one of the following holds:

(α) $a_1 <^{M_1} a_2$

(β) $a_1 <^{M_2} a_2$

(γ) $a_1 \in M_1 \backslash M_0$ and $a_2 \in M_2 \backslash M_0$ and for some $b \in M_0$
- $a_1 \le^{M_1} b <^{M_2} a_2$

(δ) as in (γ) but we interchange M_1, M_2

(ε) $a_1 \in M_1 \backslash M_0$ and $a_2 \in M_1 \backslash M_0$ and the set $\{b \in M_0; a_1 \le^{M_1} b\}, \{b \in M_0 : a_2 \le^{M_1} b\}$ are equal and non-empty (recalling M_ℓ is a tree)

(b) similarly $\mathfrak{k}$ has the JEP, even as the disjoint union

(c) (skewed amalgamation) if $M_0 \subseteq M_1$ and $M_0 \le_{\mathfrak{k}} M_2$ all from K and $M_1 \cap M_2 = M_0$ then $M_3 = M_1 \cup M_2$ defined as in (B)(a) above satisfies $M_2 \subseteq M_3$ and $M_1 \le_{\mathfrak{k}} M_3$

(d) if $A \subseteq M \in K, A \neq \emptyset$ then $M{\restriction}A \in K$ (but possibly $M{\restriction}A \nleq_{\mathfrak{k}} M$).

[Why? For clause (B)(c), clearly $\ell \le 3 \Rightarrow M_\ell \in K$ and $\ell < 3 \Rightarrow M_\ell \subseteq M_3$. For proving $M_1 \le_{\mathfrak{k}} M_3$ let $\bar{a} = \langle a_n : n < \omega \rangle$ be $<^{M_1}$-increasing and $c \in M_3 \backslash M_1$ be an upper bound (for $<^{M_3}$) of $\{a_n : n < \omega\}$. So one of the five cases in (B)(a) holds for infinitely many pairs (a_n, c), so without loss of generality for all (a_n, c).

If clause (α) - then $c \in M_1$ and we are done, and if clause (β) then $a_n \in M_0$ and use $M_0 \le_{\mathfrak{k}} M_2$. If clause ($\gamma$), then there is $b_n \in M_0$ such that $a_n \le^{M_1} b_n \le^{M_2} c$, so $b_n \in M_0, \{b_n : n < \omega\}$ linearly ordered, by Ramsey theorem (as M_1 is a tree) without loss of generality $\bar{b} = \langle b_n : n < \omega \rangle$ is monotone. If $\bar{b}$ is increasing, then it is increasing in M_1 and clearly has no upper bound in M_1 (as it will be an upper bound of $\bar{a}$), hence in M_0 but it has one in M_2, contradicting $M_0 \le_{\mathfrak{k}} M_2$. If $\bar{b}$ is (monotone and) not increasing then it is $\le$-decreasing hence $b_0 \in M_0 \subseteq M_1$ is an upper bound of $\bar{a}$, contradiction.

Next, if we use Clause (δ), the proof is easier: $\bigwedge\limits_n a_n \in M_2$ hence $\bigwedge\limits_n a_n \in M_1 \cap M_2 = M_0$ and $c \in M_3 \backslash M_1 = M_2 \backslash M_0$ so use $M_0 \le_{\mathfrak{k}} M_1$.

Lastly, if clause (ε), then there is $b \in M_0$ above all the a_n's so we finish as earlier.

So we are done proving $(*)_2$.]

In particular

$(*)_3$ if $\langle M_i : i < \delta \rangle$ is $\le_{\mathfrak{k}}$-increasing then $\bigcup\limits_{i<\delta} M_i \in K$ does $\le_{\mathfrak{k}}$-extend M_i for $i < \delta$.

Next for $\kappa \ge \theta$ and let

$(*)_4$ $K^{\rm ec}_\kappa = \{M \in K$: if $M \le_{\mathfrak{k}} N, A \subseteq M$ has cardinality $< \kappa$ and $\bar{a} \in {}^{\kappa>}N$ then some $\bar{b} \in {}^{\ell g(\bar{a})}M$ realizes $\mathrm{tp}_{\rm qf}(\bar{a}, A, N)\}$.

Clearly

$(*)_5$ (a) if $M_1 \in K$ has cardinality $\le \mu = \mu^{<\kappa}$ then some $M_2 \in K^{\rm ec}_\kappa$ has cardinality μ and $\le_{\mathfrak{k}}$-extends M_1

(b) any $M \in K^{\rm ec}_\kappa$ has elimination of quantifiers in $\mathbb{L}_{\theta,\theta}$ up to $x < y, x = y$ and $\varphi_*(\bar{x}_{[w]}) = (\exists y)(\bigwedge\limits_n x_n < y)$; also M is $(\kappa,\kappa,\mathbb{L}_{\theta,\theta})$-saturated, recalling $\kappa \ge \theta$

(c) any $M_1, M_2 \in K^{\rm ec}_\kappa$ are $\mathbb{L}_{\theta,\theta}$-equivalent and even $\mathbb{L}_{\infty,\theta}$-equivalent

(d) $K^{\rm ec}_{\kappa_2} \subseteq K^{\rm ec}_{\kappa_1}$ when $\theta \le \kappa_1 \le \kappa_2$.

Hence we define T as (it is well defined by $(*)_5(c)$)

$(*)_6$ $T = {\rm Th}_{\mathbb{L}_{\theta,\theta}}(M)$ whenever $M \in K^{\rm ec}_\theta$.

So

$(*)_7$ T is a complete $\mathbb{L}_{\theta,\theta}$-theory, $\tau_T = \{<\}$ and if $\kappa \ge \theta, \mu = \mu^{<\kappa}$ <u>then</u> T has a $(\kappa,\kappa,\mathbb{L}_{\theta,\theta})$-saturated model of cardinality μ (even extending any pregiven $M \in {\rm Mod}_T$ of cardinality $\le \mu$).

Lastly

$(*)_8$ if $\mu = \mu^{<\kappa}, \kappa \ge \theta$ <u>then</u> there are $> \mu$ pairwise non-isomorphic $(\kappa,\kappa,\mathbb{L}_{\theta,\theta})$-saturated models of T of cardinality μ.

Why? First, <u>Case 1</u>: assume μ is regular uncountable

For $M \in K$ with universe λ let ${\rm smooth}_0(M) = \{\delta < \mu : {\rm cf}(\delta) = \aleph_0$ and $M{\restriction}\delta \le_{\mathfrak{k}} M\}$ and for any $M \in K$ of cardinality λ let ${\rm smooth}(M) = {\rm smooth}_0(N)/\mathscr{D}_\mu$ for any N isomorphic to M with universe λ recalling $\mathscr{D}_\mu$ is the club filter on μ.

This makes sense because:

- if $M_1, M_2 \in K$ have universe λ then ${\rm smooth}_0(M_1) = {\rm smooth}_0(M_2) \mod \mathscr{D}_\mu$.

We say such M is smooth when ${\rm smooth}(M) = \lambda/\mathscr{D}_\lambda$.

Easily for any $S \subseteq \{\delta < \lambda : {\rm cf}(\delta) = \lambda\}$ there is $M = M_S \in {\rm Mod}_T$ of cardinality μ such that ${\rm smooth}(M) = S/\mathscr{D}_\mu$ and even $M_S \in K^{\rm ec}_\kappa$. So if $S_1, S_2 \subseteq \lambda$ and $S_1 \backslash S_2$ is stationary then $M_{S_1} \not\cong M_{S_2}$, so by $(*)_5(c)$ we are done.

Note

$\boxplus_1$ If $\mu = \mu^{<\mu} > \aleph_0$ <u>then</u> up to isomorphism there is one and only one smooth $M \in K^{\rm ec}_\mu$ which is $(\mu,\mu,\mathbb{L}_{\theta,\theta})$-saturated of cardinality μ; where

$\boxplus_2$ $M \in K$ of cardinality $\mu = {\rm cf}(\mu)$ is smooth when ${\rm smooth}(M) = \emptyset/\mathscr{D}_\mu$.

Details on $\boxplus_2$ see $(*)_9 - (*)_{11}$ in the end of the proof.

Second, next <u>Case 2</u>: assume μ singular of cofinality $\ge \aleph_1$

For special models in our context the hope was to show that any two special model are $\mathbb{L}_{\infty,\theta}$-equivalent.

Let $\bar\kappa = \langle \kappa_i : i < {\rm cf}(\mu)\rangle$ be increasing with limit μ such that $\kappa_i > \theta, \lambda_i = 2^{\kappa_i} < \kappa_{i+1}$.

So we can consider:

$\boxplus_3$ $K^{\rm sep}_{\bar\kappa} = \{\cup\{M_i : i < {\rm cf}(\mu)\} : M_i \in K^{\rm ec}_{\kappa_i}$ be κ_i^+-saturated of cardinality $\lambda_i, \le_{\mathfrak{k}}$-increasing with $i\}$.

Now

$\oplus$ (a) any $M \in K^{\rm sep}_{\bar\kappa}$ is special and $K^{\rm sep}_{\bar\kappa} \neq \emptyset$, moreover, if $M_1 \in K$ has cardinality $\le \mu$ <u>then</u> there is $N \in K^{\rm sep}_{\bar\kappa}$ such that $M \le_{\mathfrak{k}} N$

(b) any two models from $K^{\rm sep}_{\bar\kappa}$ are $\mathbb{L}_{\infty,\theta}$-equivalent

(c) there are non-isomorphic $M_1, M_2 \in K^{\rm sep}_{\bar\kappa}$.

Why $\oplus$ holds?

<u>Clause (a)</u>: The existence of $N \in K^{\rm ec}_\mu$ as well as "any $M \in K^{\rm sep}_{\bar\kappa}$ is special" are obvious by the definitions. For the second demand, (density), assume $M \in K$ has cardinality $\le \mu$, without loss of generality of cardinality μ. Let $|M|$ be $\bigcup\limits_{i<\kappa} A_i, |A_i| = \lambda_i$.

We choose M_i by induction on $i \le \kappa$ such that:

$\oplus_1$ (a) $M_i \subseteq M$ has cardinality

(b) $\langle M_j : j \le i\rangle$ is $\le_{\mathfrak{k}}$-increasing

(c) $M_i \le_{\mathfrak{k}} M$

(d) if $i = j+1$ then $A_j \subseteq M_i$.

Next we choose N_i by induction on $i \le \kappa$ such that:

$\oplus_2$ (a) $N_i \in K$ is κ_i-saturated of cardinality λ_i

(b) $\langle N_j : j \le i\rangle$ is $\le_{\mathfrak{k}}$-increasing

(c) $M_i \le_{\mathfrak{k}} N_i$

(d) $N_i \cap M = M_i$.

Why can we carry the induction? For $i = 0$ obviously, by the JEP and the density of κ_i^+-saturated in cardinality λ_i. For $i = j+1$ recalling $\mathfrak{k}$ has amalgamation (LST and as above). For limit i of cofinality $> \aleph_0$ - similarly.

Lastly, for i of cofinality $\aleph_0$ the proof is as in $(*)_2(B)(c)$.

<u>Clause (b)</u>: Is obvious when $\mathrm{cf}(\mu) \ge \theta$.

But even without this assumption we can prove a stronger result:

$\oplus_3$ (b)$^+$ if $M_\ell \in K^{\rm sep}_{\bar\kappa}$ for $\ell = 1,2$ and $\kappa < \mu$ then M_1, M_2 are $\mathbb{L}_{\infty,\kappa}$-equivalent.

Why? Without loss of generality $\kappa = \lambda_0^+ \ge \mathrm{cf}(\mu)$ and $\bar M_\ell = \langle M_{\ell,i} : i < \mathrm{cf}(\mu)\rangle$ witness $M_\ell \in K^{\rm sep}_{\bar\kappa}$.

Let $\mathscr{A}_\ell$ be the set of A such that:

(α) $A \subseteq M_\ell, |A| \le \lambda_0$

(β) if $a \in A \backslash M_i, i < \mathrm{cf}(\mu)$ and $B^\ell_{a,i} = \{b \in M_{1,i} : b <_{M_1} a\}$ has cofinality $\le \lambda_0$ then $B^\ell_{a,i} \cap A$ is cofinal in M_ℓ

(γ) if $a_n \le^{M_\ell} a_{n+1} \le^{M_\ell} b$ and $a_n \in A \cap M_i$ for $n < \omega, b \in M_j$, <u>then</u> there is such b in $A \cap M_j$.

Let $\mathscr{F}_0$ be the set of f such that:

- for some $A_1 \in \mathscr{A}_1, A_2 \in \mathscr{A}_2, f$ is a isomorphism from $M_1 {\restriction} A_1$ onto $M_2 {\restriction} A_2$ preserving the property in (β) above.

Now $\mathscr{F}$ witness "M_1, M_2 are $\mathbb{L}_{\infty,\kappa}$-equivalent. We leave the checking to the reader.

What about Clause (c): "Two non-isomorphic ones"? We give three ways to do this.

First Way:

We can get 2^μ pairwise non-isomorphic $(\kappa, \kappa, \mathbb{L}_{\theta,\theta})$-equivalent models which are special and even in $K^{\rm sep}_{\bar{\kappa}}$ when μ is strong limit singular. A way to do it is to work as in [Sh:511] where we construct "complicated" sequences of subtrees of ${}^{\sigma\geq}\lambda$ and use them to construct, e.g. Boolean Algebras. We do not elaborate, but shall give details in the other ways.

A Second Way:

Giving in some respect a stronger version, when μ is strong limit of cofinality $\kappa > \aleph_0$ is as follows. Let $\langle \lambda_i : i < \kappa \rangle$ be increasing continuous with limit $\lambda_\kappa = \mu, \lambda_{i+1} = (\lambda_{i+1})^{\lambda_i}, \lambda_0 = (\lambda_0)^{\aleph_0}$ and $S_0, S_1 \subseteq S^\kappa_{\aleph_0}$ be stationary disjoint and $\varepsilon \in S_1 \Rightarrow \lambda_{\varepsilon+1} = 2^{\lambda_\varepsilon}$. We choose $\bar{M}_\varepsilon$ by induction on $\varepsilon \leq \kappa$ such that:

$(*)_{8.1}$
(a) $\bar{M}_\varepsilon = \langle M_\eta : \eta \in {}^{(\lambda_\varepsilon)}2 \rangle$
(b) $\langle M_{\eta \restriction \lambda_\zeta} : \zeta \leq \varepsilon \rangle$ is $\subseteq$-increasing continuous
(c) $M_\eta \in K$ has universe $\lambda_{\ell g(\eta)}$
(d) $M_\eta \in K^{\rm ec}_{\lambda_\varepsilon}$ if $\eta \in {}^{\lambda_{\varepsilon+1}}2$
(e) $M_{\eta \restriction (\zeta+1)} \leq_{\mathfrak{k}} M_\eta$ for $\eta \in {}^{\lambda_\varepsilon}2, \zeta < \varepsilon$
(f) if $\eta \neq \nu \in {}^{(\lambda_\varepsilon)}2, \varepsilon = \zeta + 1, \zeta \in S_1$ and $f \in \mathscr{F}_{\eta,\nu}$ (see below) then for some $\rho \in \lim(I_f)$ (see below) we have: there is $a \in M_\eta$ such that $(\forall n)(\rho(n) < a)$ but for no $b \in M_\nu$ do we have $M_\nu \models (\forall n)(f(\rho(n)) < b))$, where
 $\oplus$ $\mathscr{F}_{\eta,\nu}$ is the set of functions f such that
 - $\mathrm{dom}(f)$ is a subtree of ${}^{\omega>}(\lambda_\varepsilon)$ with $\lim(\mathrm{dom}(f))$ of cardinality 2^{λ_ε}
 - if $\rho \in \mathrm{dom}(f) \Rightarrow M_\eta \models$ "$\langle \rho(\ell) : \ell < \ell g(\rho) \rangle$ is increasing"
 - for every $\xi < \zeta$, all but $< \lambda_\varepsilon$ members ρ of $\mathrm{dom}(f), \mathrm{Rang}(\rho) \nsubseteq \lambda_\xi$
 - if $\rho\hat{\ }\langle \alpha \rangle, \varrho\hat{\ }\langle \beta \rangle \in \mathrm{Dom}(f)$ are $\triangleleft$-incomparable then $M_\eta \models$ "$\rho\hat{\ }\langle \alpha \rangle, \rho\hat{\ }\langle \beta \rangle$ are incomparable"
 - f is one to one.

Now

$(*)_{8.2}$ we can carry the induction.

[Why? For $\varepsilon = 0$ trivially and ε limit use union; for $\varepsilon = \zeta + 1, \zeta \notin S_1$ use $(*)_5(a)$ and for $\varepsilon = \zeta + 1, \zeta \in S_1$ by cardinality consideration we can take care of clause (f) and then use $(*)_5(a)$ to take care of clause (d).]

$(*)_{8.3}$ if $\eta \in {}^\mu 2$ then M_η is a special model of T.

[Why? By $(*)_{8.1}(b), (c), (d)$.]

$(*)_{8.4}$ if $\eta \neq \nu \in {}^\mu 2$ then $M_\eta \in K^{\rm ec}_\mu$ is not $\leq_{\mathfrak{k}}$-embeddable into M_ν.

[Why? By [Sh:136, Claim 2.4,pg.111], see more in Rubin-Shelah [RuSh:117] and [Sh:f, Ch.XI].]

Third Way: Giving μ^+ non-isomorphic models is by the simple black box of [Sh:309, §1,1.5=L4.5A,pg.3], but we elaborate[7] giving a self contained proof. Let $\langle M_i : i < \mu \rangle$ be a sequence of members of $K^{\rm ec}_\kappa$ so models of T, each of cardinality μ and we shall find a model from K of cardinality μ not $\le_{\mathfrak{k}}$-embeddable into any M_i, this clearly suffices by $\oplus(a)$, the density.

We define a model $M \in K$ as follows:

- (a) its set of elements is the set of η's such that
  - (α) η is a sequence of length $\le \omega$
  - (β) $\eta(0) \in \mu$ if $\ell g(\eta) > 0$
  - (γ) $\eta(1+n) \in M_{\eta(0)}$ when $1+n < \ell g(\eta)$
  - (δ) $M_{\eta(0)} \models$ "$\eta(1+n) < \eta(1+n+1)$" when $1+n+1 < \ell g(\eta)$
  - (ε) if $\ell g(\eta) = \omega$ then $M_{\eta(0)} \models$ "$\neg(\exists x)(\bigwedge\limits_n \eta(1+n) < x)$"
- (b) the order $<^M$ is $\triangleleft$, being an initial segment.

Let $N \in K^{\rm ec}_\kappa$ be such that $M \le_{\mathfrak{k}} N$ and N has cardinality μ. Now indeed $i < \mu = N$ is not $\le_{\mathfrak{k}}$-embeddable into M_i as in [Sh:309, §1,1.5=L4.5A]; in details toward contradiction assume f is an isomorphism from N onto M_i. Define $\eta_n \in N$ of length $n+1$ by induction on n as follows: if $n = 0$ then $\eta_n = \langle i \rangle \in N$ so $\eta_n(0) = i$ and if η_n has been defined then we let $\eta_{n+1} = \eta_n{}^\frown\langle f(a_n)\rangle$, it is well defined as $a_n \in N$ hence $f(\eta_n) \in M_i$ and clearly $\eta_n \triangleleft \eta_{n+1}$ hence $M_i \models f(\eta_n) < f(\eta_{n+1})$.

Now we ask: does the $<^{M_i}$-increasing sequence $\langle f(\eta_n) : n < \omega \rangle$ have an upper bound in M_i? If a is such an upper bound, $f^{-1}(a)$ is above $\{\eta_n : n < \omega\}$ so necessarily is the sequence $\bigcup\limits_n \eta_n$ which does not belong to N. If there is no such $a, \eta = \bigcup\limits_n \eta_n \in N$ and $f(\eta)$ satisfies the demand, contradiction, so we are done proving $(*)_8$.]

Why are we done proving part (3)? Clauses (A),(B) - the existence of 2^μ pairwise non-isomorphic special models from $K^{\rm ec}_\theta$ of cardinality λ is proved in "the second way" of the proof of $(*)_8$ in part (1). The uniqueness of the smooth special model is just like Lemma [Sh:88r, 2.18=L88r-2.11,pg.18] and see Definition [Sh:88r, 2.15=L88r-2.10], but see $(*)_{10}$ below.

Proof of $\boxplus_2$: Easy as above because here smoothness holds automatically as quoted above but we elaborate:

$(*)_9$ if $\lambda = \lambda^{<\lambda} > \aleph_0$ and $\alpha < \lambda \Rightarrow |\alpha|^{\aleph_0} < \lambda$ and M_1, M_2 are smooth $\le_{\mathfrak{k}}$-saturated λ-saturated models of cardinality λ, then M_1, M_2 are isomorphic.

Why? For $\ell = 1,2$ let $\langle M_{\ell,\alpha} : \alpha < \lambda \rangle$ be $\le_{\mathfrak{k}}$-increasing continuous with union M_ℓ such that $\alpha < \lambda \Rightarrow \|M_{\ell,\alpha}\| < \lambda$; possible because $\alpha < \lambda \Rightarrow |\alpha|^{\aleph_0} < \lambda$.]

Now we choose $f_\varepsilon, \alpha_{1,\varepsilon}, \alpha_{2,\varepsilon}, N_{1,\varepsilon}, N_{2,\varepsilon}$ by induction on $\varepsilon < \lambda$ such that:

$(*)_{10}$ (a) $N_{\ell,\varepsilon} \le_{\mathfrak{k}} M_\ell$ has cardinality $< \lambda$

[7] Can we get 2^μ ones? In this particular case, yes, but we shall not elaborate; we can use [Sh:309, 1.9=L4.6,pg.5].

(b) f_ε is an isomorphism from $N_{1,\varepsilon}$ onto $N_{2,\varepsilon}$
(c) $\alpha_{\ell,\varepsilon} = \alpha(\ell,\varepsilon)$ is increasing with ε for $\ell = 1,2$
(d) if $\zeta < \varepsilon, \ell = 1,2$ then $M_{\ell,\alpha(\ell,\zeta)} \subseteq N_{\ell,\varepsilon} \subseteq M_{\alpha(\ell,\varepsilon)}$.

The rest should be clear.

$(*)_{11}$ We have $M_1 \cong M_2$ when for $\ell = 1,2$:
(a) $M_\ell \in K$ is of cardinality μ
(b) $M_\ell = \bigcup\limits_{\ell<\kappa} M_{\ell,i}$
(c) $\langle M_{\ell,i} : i < \kappa\rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous
(d) $M_{\ell,i+1} \in K$ is $\|M_{\ell,i}\|^+$-saturated.

Why true? Similar to the proof above. Note that if $\kappa = \aleph_0$, then the "continuous" in clause (c) is redundant.
3) Clauses (A),(B) of 2.2(3) were proved inside the proof of part (2) and Clause (C) follows from $L_{\infty,\mu}$-equivalence. $\square_{2.3}$ {a5}

{a8}

Claim 2.5. *1) If $D \in \mathrm{uf}_\theta(I)$ is (λ,θ)-regular and M_1, M_2 are $\mathbb{L}_{\theta,\theta}$-equivalent and $\tau(M) = \tau$ has cardinality $\leq \lambda$ <u>then</u> $M_1^I/D, M_2^I/D$ are $\mathbb{L}_{\lambda^+,\lambda^+}$-equivalent, moreover $\mathbb{L}_{\infty,\lambda^+,\lambda^+}$-equivalent (so one is $(\lambda^+,\lambda^+,\mathbb{L}_{\theta,\theta})$-saturated iff the other is).*
2) Similarly for $D \in \mathrm{fil}_\theta(I)$ which is (λ,θ)-regular.

{a8d}

Remark 2.6. Recall that $\mathbb{L}_{\chi,\mu,\gamma}(\tau) = \{\varphi(\bar{x}) \in \mathbb{L}_{\chi,\mu}(\tau) : \varphi(\bar{x})$ has quantifier depth $< \gamma\}$ and $\mathbb{L}_{\infty,\lambda^+}(\tau) = \cup\{\mathbb{L}_{\chi,\lambda^+}(\tau) : \chi$ a cardinal $> \lambda\}$ and $\mathbb{L}_{\lambda^+,\lambda^+}(\tau) = \cup\{\mathbb{L}_{\lambda^+,\lambda^+,\gamma} : \gamma < \lambda^+\}$.

Note that unlike the first order case we cannot demand $\mathbb{L}_{\infty,\lambda^+}$-equivalence.

Proof. 1) Let $\gamma < \lambda^+$. As D is (λ,θ)-regular there is a sequence $\langle (u_s, v_s, \Delta_s) : s \in I\rangle$ such that $v_s \in [\gamma]^{<\theta}, u_s \in [\lambda]^{<\theta}, \Delta_s$ a set of $< \theta$ formulas of $\mathbb{L}_{\theta,\theta}(\tau_T)$ and $\alpha < \gamma \wedge \beta < \lambda \wedge \varphi(\bar{x}) \in \mathbb{L}_{\theta,\theta}(\tau_T) \Rightarrow \{s : \alpha \in v_s, \beta \in u_s$ and $\varphi(\bar{x}) \in \Delta_s\} \in D$. For $s \in I$ let $\Game_s$ be the game $\Game_{\Delta_s,u_s,v_s}(M_1,M_2)$, see Definition 0.12. As M_1, M_2 are {x2} $\mathbb{L}_{\theta,\theta}$-equivalent by 0.13 the protagonist wins this game $\Game_s$ which means that it has {x4} a winning strategy $\mathbf{st}_s$. Let $N_\ell = M_\ell^I/D$ and it suffices to find a strategy $\mathbf{st}$ for the protagonist in the game $\Game_{\mathbb{L}_{\theta,\theta},\lambda,\gamma}$. The strategy is obvious, see proof in [Sh:1101, 1.3=Ld11] but we give details.

We say $\mathbf{s}$ is a reasonable state when it consists of:

(a) $\gamma_{\mathbf{s}} < \gamma, n_{\mathbf{s}} < \omega$
(b) a member A of D
(c) a set J of cardinality $< \theta$
(d) $f_\alpha^\ell \in M_\ell^I$ for $\ell \in \{1,2\}, \alpha < \lambda$
(e) if $s \in A$, <u>then</u> $\gamma_{\mathbf{s}} \in v_s$ and $(n_{\mathbf{s}}, g_{\mathbf{s},s})$ is a winning state for the isomorphism player in the game $\Game_{\Delta_s,u_s,v_\alpha}$ where the partial function $g_{\mathbf{s},s}$ is $\{(f_\alpha^1(s), f_\alpha^2(s)) : \alpha \in u_s\}$, so necessarily of cardinality $\leq |u_s| < \theta$.

2) The same proof as part (1) using only Δ's which are sets of $< \theta$ atomic formulas of $\mathbb{L}_{\theta,\theta}(\tau_T)$. $\square_{2.5}$

{a10}

Definition 2.7. 1) Assume $\bar{\mu} = (\mu_1, \mu_2)$ but if $\mu_1 = \mu, \mu_2 = \theta$ we may write μ; and $\lambda \geq \mu_1 \geq \mu_2 \geq \theta$. We define a two-place relation $\blacktriangleleft_{\lambda,\bar{\mu},\theta}$ on the class of complete theories T (in $\mathbb{L}_{\theta,\theta}$, of course) of cardinality $\leq \lambda$. We say $T_1 \blacktriangleleft_{\lambda,\bar{\mu},\theta} T_2$ iff for every $D \in \mathrm{ruf}_\theta(\lambda)$ and models M_1, M_2 of T_1, T_2, respectively we have: if M_2^λ/D is locally $(\mu_1^+, \mu_2^+, \mathbb{L}_{\theta,\theta})$-saturated then so is M_1^λ/D.
2) We say fully or write $\blacktriangleleft^{\mathrm{ful}}_{\lambda,\bar{\mu},\theta}$, when we deal with full saturation. We may omit $\bar{\mu}$ when $\lambda = \mu_1, \mu_2 = \theta$. We define $\blacktriangleleft_{\lambda,\bar{\mu},\theta}$, $\blacktriangleleft^{\mathrm{ful}}_{\lambda,\bar{\mu},\theta}$ parallelly.

{a11}

Remark 2.8. 1) NOte that $\blacktriangleleft$ is a quasi-order and not a partial order, in particular, is not a strict order.
2) The relation of $\blacktriangleleft$ here to the classical one of Keisler is quite close. Keisler uses "D a regular ultrafilter on λ". The demand of regular is natural for several reasons. The most relevant is that using it Keisler proves that λ^+-saturation of M^λ/D depends only on the first order theory of M. By request we use a different symbol.

Naturally, we demand here (λ, θ)-regularity because to preserve the $\mathbb{L}_{\theta,\theta}$-theory we need the ultrafilter to be θ-complete, so the strongest possible regularity is for (λ, θ). Also the choice of saturation is natural.

We now turn to generalizing $\trianglelefteq^*$.

{a9}

Definition 2.9. Assume $\bar{\mu} = (\mu_1, \mu_2), \bar{\chi} = (\chi_1, \chi_2)$ and $\lambda \geq \theta, \mu_1 \geq \mu_2 \geq \theta$; if $\mu_1 = \mu, \mu_2 = \theta$ we may write μ instead of $\bar{\mu}$; similarly for $\bar{\chi}$; if $\bar{\chi} = (\mu, \theta)$ then we may omit $\bar{\chi}$.
1) We say T is locally/fully $(\lambda, \bar{\mu}, \theta)$-minimal when for every complete $T_0 \supseteq T$ with $\tau(T_0) \backslash \tau(T)$ of cardinality $\leq \lambda$, for some T_1 we have:

(a) $T_1 \supseteq T_0$ is a complete theory in $\mathbb{L}_{\theta,\theta}(\tau_{T_1})$
(b) T_1 has no model of cardinality $< \theta$
(c) $\tau(T_0) \subseteq \tau(T_1)$ and $|\tau(T_1) \backslash \tau(T_0)| \leq \lambda$
(d) if M_1 is a model of T_1 of cardinality $> \mu_2$ then $M_1 \restriction \tau_T$ is locally/fully $(\mu_1^+, \mu_2^+, \mathbb{L}_{\theta,\theta})$-saturated.

2) For complete T_1, T_2 with no model of cardinality $< \theta$, we say $T_1 \blacktriangleleft^*_{\lambda,\bar{\mu},\bar{\chi},\theta} T_2$ when for every complete $T_1^+ \supseteq T_1$ such that $|\tau(T_1^+) \backslash \tau(T_1)| \leq \lambda$ for some T_3, τ_2 we have:

(a) T_3 is a complete theory in $\mathbb{L}_{\theta,\theta}(\tau(T_3))$
(b) $|\tau(T_3) \backslash \tau(T_1^+)| \leq \lambda$ and $\tau(T_1) \subseteq \tau(T_1^+) \subseteq \tau(T_3)$
(c) $T_1^+ \subseteq T_3$
(d) $\tau_2 \subseteq \tau(T_3)$ and $T_3 \restriction \tau_2'$ is isomorphic to T_2 over $\tau(T_1)$, (if $\tau(T_1^+) \cap \tau(T_2) = \emptyset$ we can demand $T_1^+ \cup T_2 \subseteq T_3$; so the isomorphism above maps τ_2' onto $\tau(T_2), T_3 \restriction \tau_2$ onto T_2, preserving the number of places and being a predicate/function symbol) and is the identity on $\tau(T_1)$
(e) if M_3 is a model of T_3 and $M_3 \restriction \tau_2$ is locally (μ_1^+, μ_2^+)-saturated then $M_3 \restriction \tau(T_1)$ is locally (χ_1^+, χ_2^+)-saturated.

3) We define $T_1 \blacktriangleleft^{*,\mathrm{ful}}_{\lambda,\bar{\mu},\theta} T_2$ is as in part (2) omitting the "locally".

4) In part (2), if we omit $\bar{\mu}, \bar{\chi}$ we mean $\|M_3\|$, i.e. $T_1 \blacktriangleleft^*_{\lambda,\theta} T_2$ means as above but we replace clause (e) in part (2) by:

$(e)'$ if M_3 is a model of T_3 and $M_3 \restriction \tau_2'$ is locally $(\|M_3\|, \|M_2\|)$-saturated then $M_3 \restriction \tau_1'$ is locally $(\|M_3\|, \|M_3\|)$-saturated.

{a9d}

Remark 2.10. 0) We may note that $\trianglelefteq^*$ is defined similarly in the first order case.
1) Why the T_0 in 2.9(1) and T_1^+ in 2.9(2) in the definition? Because otherwise it is not clear why those relations are partial orders because $\mathbb{L}_{\theta,\theta}$ fail the Robinson lemma, i.e. if $T_\ell \subseteq \mathbb{L}_{\theta,\theta}(\tau_\ell)$ is complete for $\ell = 1,2$ and $\tau_0 = \tau_1 \cap \tau_2, T_1 \cap \mathbb{L}_{\theta,\theta}(\tau_0) = T_2 \cap \mathbb{L}_{\theta,\theta}(\tau_0)$ then $T_1 \cup T_2$ does not necessarily have a model); see [Be85]. {a9}
2) We may be worried that this will cause $\neg(T_1 \blacktriangleleft^*_{\lambda,\bar{\mu},\bar{\chi},\theta} T_2)$ because of trivial reasons, i.e. because for some $T_1^+ \supseteq T_2$ there is no T_3 satisfying clauses (a)-(d) of Definition 2.9(2). But this is not the case because {a9}

⊞ if $T_\ell \subseteq \mathbb{L}_{\theta,\theta}(\tau_\ell)$ has a model of cardinality $\geq \theta$ for $\ell = 1,2$ and $\tau_1 \cap \tau_2 = \emptyset$ then $T_1 \cup T_2$ has a model of cardinality $\geq \theta$.

[Why? Because by the compactness for $\mathbb{L}_{\theta,\theta}$ and the downward LST property if $\lambda = \lambda^{<\theta} + |T_\ell|$ then T_ℓ has a model of cardinality λ.]
3) For $\mathbb{L}^1_\kappa$ it holds; see §3.

{a12}

Conclusion 2.11. *1) $\blacktriangleleft^*_{\lambda,\bar{\mu},\theta}, \blacktriangleleft_{\lambda,\bar{\mu},\theta}$ are partial orders (as are the full versions).*
2) In Definition 2.7 the choice of M_1, M_2 does not matter. {a10}
3) If $T_1 \blacktriangleleft^*_{\lambda,\bar{\mu},\theta} T_2$ then $T_1 \blacktriangleleft_{\lambda,\bar{\mu},\theta} T_2$; also for the full versions.

Proof. 1) Easy.
2) By 2.5. {a8}
3) By part (2). $\square_{2.11}$

{a14}

Claim 2.12. *1) $\mathrm{Th}_{\mathbb{L}_{\theta,\theta}}((\theta,<))$ is a $\blacktriangleleft^*_{\lambda,\bar{\mu},\theta}$-maximal and a $\blacktriangleleft_{\lambda,\bar{\mu},\theta}$-maximal theory (so $\bar{\chi} = (\mu,\theta)$, see beginning of Definition 2.9).* {a9}
2) $\mathrm{Th}_{\mathbb{L}_{\theta,\theta}}(\theta,=)$ is a $\blacktriangleleft^*_{\lambda,\bar{\mu},\theta}$-minimal and $\triangleleft_{\lambda,\mu,\theta}$-minimal theory.
3) T is $(\lambda,\bar{\mu},\theta)$-minimal, (see Definition 2.9(1)) iff T is $\blacktriangleleft^*_{\lambda,\bar{\mu},\theta}$-minimal. {a9}

Proof. 1) Easy: we never get even local saturation, recalling 2.10(2). {a9d}
2) Easy: even the (full) $(\lambda^+, \lambda^+, \mathbb{L}_{\theta,\theta})$-saturated means just "of cardinality $\geq \lambda^+$".
3) Easy, too, just read the definitions. $\square_{2.12}$

§ 3. The n.c.p. and local minimality

{a13}

Definition 3.1. 1) We say T has the θ-n.c.p. <u>when</u> it fails the θ-c.p. We say T has the θ-c.p. <u>when</u>: some $\varphi = \varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\zeta]}) \in \mathbb{L}_{\theta,\theta}(\tau_T)$ so $\varepsilon, \zeta < \theta$ is a witness of θ-c.p., that is, for every $\partial < \theta$ there are a model M of T and Γ such that:

$(*)_{M,\Gamma,\theta,\partial}$
- $\Gamma \subseteq \{\varphi(\bar{x}_{[\varepsilon]}, \bar{b}) : \bar{b} \in {}^{\zeta}M\}$
- $|\Gamma| < \theta$
- Γ is $(< \partial)$-satisfiable in M
- Γ is not satisfiable in M.

2) For $\varepsilon < \theta$, if $\Delta \subseteq \Phi_{T,\varepsilon} := \{\varphi(\bar{x}_{[\varepsilon]}, \bar{y}_\varphi) : \varphi \in \mathbb{L}_{\theta,\theta}(\tau_T)\}$ is of cardinailty $< \theta$ we define the spec(Δ, T) as the set of cardinals $\partial < \theta$ such that $\partial \geq 2$ and for some model M of T and sequence $\langle \varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{y}_{\varphi_\alpha}) : \alpha < \partial \rangle$ of members of Δ and $\bar{a}_\alpha \subseteq M$ of length $\ell g(\bar{y}_{\varphi_\alpha})$ for $\alpha < \partial$, the set $\{\varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{a}_\alpha) : \alpha < \partial\}$ is not realized in M but any subset of cardinality $< \partial$ is realized.
3) For $\varphi = \varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\zeta]}) \in \Phi_{T,\varepsilon}$ let $\operatorname{spec}(\varphi, T) = \operatorname{spec}(\{\varphi\}, T)$.
4) We may replace Δ by a sequence listing its members (even with repetitions).

{a19}

Observation 3.2. *1) T has θ-c.p. <u>iff</u> for some φ, $\operatorname{spec}(\varphi, T)$ is unbounded in θ <u>iff</u> for some $\varepsilon < \theta$ and $\Delta \subseteq \Phi_{T,\varepsilon}$ of cardinality $< \theta$ the set $\operatorname{spec}(\Delta, T)$ is unbounded in θ.*
2) In the definition of "the theory T has the θ-c.p.", of "$S = \operatorname{spec}(\varphi, T)$" and of "$S = \operatorname{spec}(\Delta, T)$" see Definition 3.1, the model M does not matter; of course, for T a complete $\mathbb{L}_{\theta,\theta}$-theory. {a17}
3) If $\varepsilon < \theta$ and $\Delta \subseteq \Phi_{T,\varepsilon}$ has cardinality $< \theta$ <u>then</u> for some $\psi = \psi(\bar{x}_{[\varepsilon]}, \bar{y}_\psi)$ we have:

(a) $\operatorname{spec}(\Delta, T) \subseteq \operatorname{spec}(\psi, T)$; *<u>moreover</u> they are equal*
(b) *if $M \models T$ then $\{\emptyset\} \cup \{\varphi(M, \bar{a}) : \varphi(\bar{x}_{[\varepsilon]}, \bar{y}) \in \Delta$ and $\bar{a} \in {}^{\ell g(\bar{y})}M\} = \{\psi(M, \bar{a}) : \bar{a} \in {}^{\ell g(\bar{y})}M\}$; well assuming $\|M\| > 1$.*

Proof. 1) Obviously, the second assertion implies the first and the third trivially implies the first by part (3) so we are left with proving "the first implies the second".
{a17} For $\partial < \theta$, let M, Γ be as in 3.1(1) for ∂, so necessarily $|\Gamma| \geq \partial$, let $\Gamma_1 \subseteq \Gamma$ be of minimal cardinality such that Γ_1 is not realized in M. So $\partial \leq |\Gamma_1| \in \operatorname{spec}(\varphi, T)$.
{a17} 2) Read Definition 3.1.
3) Use definition by cases as in [Sh:c], (see [Sh:c, Ch.II,§(2.1),pg.29] and §2 here; it suffices to assume the theory T has no model with just one element). That is, let $\langle \varphi_i(\bar{x}_{[\varepsilon]}, \bar{y}_i) : i < i_* \rangle$ list $\Delta, \zeta = \sup\{\ell g(\bar{y}_i) : i < i_*\}$ so $\zeta < \theta$ and let $\psi = \psi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\zeta+i_*+1]}) = \bigwedge\limits_{i<i_*} ((y_{\zeta+i_*} = y_{\zeta+i} \wedge \bigwedge\limits_{j<i} y_{\zeta+i_*} \neq y_{\zeta+j}) \rightarrow \varphi(\bar{x}_{[\varepsilon]}, \bar{y} \restriction \zeta_i))$. Now check. $\square_{3.2}$

For first order T, $\aleph_0 -$ c.p. = f.c.p. follows from unstability (by [Sh:a, Ch.II,§2] = [Sh:c, Ch.II,§2]), but not so here. The most interesting part of 3.3 is 3.3(4) as we have many non-implications. {a22}

{a22}

Claim 3.3. *1) There is a 5-unstable T with $\operatorname{spec}(\mathbb{L}(\tau_T), T) = \aleph_0$ which is 3-unstable (see Definition 3.1(2); yes, here we use Δ = the set of first order formulas).* {a17}
2) There is a 1-unstable, definably stable T which has the $\theta -$ c.p..

3) Assume $M = (\lambda, E^M), E^M$ *an equivalence relation on* λ *and* $\lambda \geq \theta, T = \mathrm{Th}_{\mathbb{L}_{\theta,\theta}}(M)$, *<u>then</u>* T *is 1-stable; and* T *has the* θ*-c.p. iff* $\theta = \sup\{(a/E^M) : a \in M$ *and* $\theta > |a/E^M|\}$.

4) If T *is* θ*-n.c.p., and is 1-unstable, then it is definably stable.*

Proof. 1) Let T be the theory of I for any dense linear order I which is θ-saturated (in the first order sense) with neither first nor last member. This is the T_5 of 1.12(4). {y45}

2) $T_0 = \mathrm{Th}((\theta, <))$ which by 1.12(1) is 1-unstable, definably stable; by inspection spec$(\varphi, T) = \mathrm{Card} \cap \theta$ when $\varphi(x, y_0, y_1) = (x < y_1 \wedge x \neq y_0)$ so T_0 has the θ-c.p. {y45}

3) Easy, too.

4) So we are assuming T has the θ-n.c.p. and is 1-unstable. As T is 1-unstable there is $\varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\varepsilon]}) \in \mathbb{L}(\tau_T)$ witnessing it, hence we can choose:

$(*)_1$ (a) a model M of T and $\bar{a}_\alpha \in {}^\varepsilon M$ such that
(b) $M \models \varphi[\bar{a}_\alpha, \bar{a}_\beta]^{\mathrm{if}(\alpha<\beta)}$ for $\alpha < \beta < \theta$
(c) without loss of generality M and T has cardinality θ
(d) $\varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\varepsilon]} \vdash \neg\varphi(\bar{y}_{[\varepsilon]}, \bar{x}_{[\varepsilon]})$.

By θ being a compact cardinal and $M \in \mathrm{Mod}_T$, every $p \in \mathbf{S}_\varphi(M)$ being definable because T is definably stable, we can find:

$(*)_2$ $\psi = \psi(\bar{y}_{[\zeta]}, \bar{z}_{[\xi]}) \in \mathbb{L}(\tau_T)$ such that: if $M \models T$ and $p \in \mathbf{S}_\varphi(M)$ then for some $\bar{c} \in {}^\xi M$ we have: if $\bar{b} \in {}^\zeta M$ then $\varphi(\bar{x}_{[\varepsilon]}, \bar{b}) \in p$ <u>iff</u> $M \models \psi[\bar{b}, \bar{c}]$

$(*)_3$ (a) $\Delta = \{\varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\varepsilon]}), \varphi^\perp(\bar{x}_{[\varepsilon]}, \bar{y}_{[\varepsilon]})\}$ see Definition 1.3(2) {y3}
(b) let $\partial = \partial_\Delta$ be $< \theta$ but $> \sup[\mathrm{spec}(\Delta_\ell, \tau)]$ for $\ell = 1, 2$, see Definition 3.1(2). {a17}

Let

$(*)_4$ $\langle \bar{c}_\xi : \xi < \theta\rangle$ list ${}^\xi\mu$ each appearing θ-times
$(*)_5$ let $S = \{\delta < \theta : \mathrm{cf}(\delta) > \partial\}$.

Now fix $\delta \in S$ for a while, we choose $\bar{b}_{\delta,\alpha}$ by induction on $\alpha < \theta$ such that:

$(*)_6$ (a) $\bar{b}_{\delta,\alpha} \in {}^\varepsilon M$
(b) $M \models \varphi[\bar{b}_\beta, \bar{b}_{\delta,\alpha}]$ for $\beta < \delta$
(c) $M \models \varphi[\bar{b}_{\delta,\alpha}, \bar{b}_{\delta,\beta}]$ for $\beta < \alpha$
(d) if possible (under (a)+(b)+(c)) then we have $M \models \psi[\bar{b}_{\delta,\alpha}, \bar{c}^*_\alpha]$.

We can carry the induction, because for $\bar{b}$ to satisfy clauses (a),(b),(c) it has to realize a Δ-type $p_{\delta,\alpha}$ and every member is satisfied by $\bar{a}_\beta$ for $\beta < \alpha$ large enough, so recalling $\mathrm{cf}(\delta) > \partial$ and the choice of ∂, we can carry the induction indeed; where $p_{\delta,\theta} = \{\varphi(\bar{a}_\alpha, \bar{x}), \varphi(\bar{x}, a_{\delta,\beta}) : \alpha < \delta, \beta < \theta\}$ is a type in M. Hence there is $q_\delta \in \mathbf{S}(M)$ extending it.a

Now by the choice of ψ, there is $\bar{d}_\delta \in {}^\xi M$ such that:

- $\bar{b} \in {}^\zeta M \Rightarrow [M \models \psi[\bar{b}, \bar{c}_\delta]$ iff $\varphi(\bar{x}, \bar{b}) \in p_\delta]$.

Clearly there is $\alpha(\delta) < \theta$ such that $\bar{c}_{\alpha(\delta)} = \bar{d}_\delta$ hence

- $r_\delta = p_{\delta,\alpha(\delta)}(\bar{x}_{[\varepsilon]}) \cup \{\neg\psi(\bar{x}_{[\varepsilon]}, \bar{c}_{\alpha(\delta)})\}$ is contradictory, but of course
- every subset of r_δ with $< \mathrm{cf}(\delta)$ members is realized.

So r_δ contradicts "T has the θ-n.c.p. $\square_{3.3}$

More generally

{a23}

Claim 3.4. *Assume $T = \mathrm{Th}_{\mathbb{L}_{\theta,\theta}}(M), M$ a θ-saturated model (in the first order sense) with $\mathrm{Th}_{\mathbb{L}}(M)$, the first order theory of M, being unstable (e.g. random graph).*
1) T is 5-unstable.
2) T has $\theta -$ n.c.p. provided that $\theta = \sup\{\theta' : \theta' < \theta$ is a compact cardinal$\}$.
3) T has the $\theta -$ c.p. <u>when</u> (a) and $(b) \vee (b)' \vee (b)''$ where:

- (a) *the first order theory $\mathrm{Th}_{\mathbb{L}}(M)$ has the independence property (hence is unstable)*
- (b) *for some $\kappa < \theta$ we have $\theta = \sup\{\mu$: there is a graph G on μ such that $\mathrm{chr}(G) > \kappa$ but $A \in [\mu]^{<\mu} \Rightarrow \mathrm{chr}(G{\restriction}A) \le \kappa\}$*

(maybe $(b)', (b)''$ are more transparent)

- $(b)'$ *$\theta = \sup\{\mu : \mu = \mathrm{cf}(\mu) < \theta$ and some stationary $S \subseteq S^{\mu}_{\aleph_0}$ does not reflect$\}$ or just*
- $(b)''$ *like (b) replacing $\aleph_0$ by some regular $\kappa < \theta$.*

4) T has the $\theta -$ c.p. <u>when</u> (a) and (b) $\vee$ (b)$'$ where:

- (a) *the first order theory $\mathrm{Th}_{\mathbb{L}}(M)$ has the strict order property (hence is unstable)*
- (b) *for some regular $\kappa < \theta$ we have $\theta = \sup\{\mu^{<\kappa} : \mu = \mathrm{cf}(\mu)$ and I^κ/D has a (μ,μ)-cut for some ultrafilter D on κ and θ-saturated dense linear order $I\}$, we can fix D and I; see Golshani-Shelah* [GsSh:1075, Th.3.3]

(maybe more transparently)

- (b)$'$ *for some regular $\kappa < \theta$ we have $\theta = \sup\{\mu^{<\kappa} : \mu$ is a successor cardinal, $\mu = \mu^{<\mu} > \kappa^+$ and there are a stationary $S \subseteq S^{\mu}_{\kappa}$ and $\bar{C} = \langle C_\delta : \delta < \mu$ limit$\rangle$ such that C_δ is a closed unbounded subset of δ disjoint to S and $\delta_1 \in C_{\delta_2} \Rightarrow C_{\delta_1} = C_{\delta_2} \cap \delta_1\}$.*

5) T has the $\theta -$ n.c.p. if $\mathrm{Th}_{\mathbb{L}}(M)$ is stable.

{a23d}

Remark 3.5. 1) Recall that a first order T_0 is unstable iff it has the independence
{a23} property or the strict order property, hence part (3),(4),(5) of 3.4 covers all complete first order T.
{a23} 2) The statements in $3.4(3)(b)'$, $3.4(4)(b)'$ are consistent by a relative of Laver indestructability; see, e.g. [Sh:945, 1.3=La7].
{a23} Note that [GsSh:1075, Th.3.3] use conditions weaker than 3.4(4)(b)$'$, because by [Sh:922] the assumptions on μ and κ implies $\diamondsuit_S$.

Proof. 1) Let $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T)$ be a first order formula which has the order property for T. Easily it witnesses that T is 5-unstable.
2) Easy, but we shall elaborate.

So let $\varphi = \varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\theta,\theta}(\tau_T)$ be a formula and we shall prove that $\mathrm{spec}(\varphi, T)$ is bounded in θ. As θ is strongly inaccessible there is $\sigma < \theta$ such that $\varphi \in \mathbb{L}_{\sigma,\sigma}(\tau_T)$ so $\ell g(\bar{x}) + \ell g(\bar{y}) < \sigma$. By the assumption without loss of generality σ is a compact cardinal. Now for every cardinal $\partial \in [\sigma, \theta)$ and τ_M-model N consider the statement

$(*)^+_{N,\varphi,\partial}$ if $\bar{b}_i \in {}^{\ell g(\bar{y})}N$ for $i < \partial$ and every subset of $p(\bar{x}) := \{\varphi(\bar{x}, \bar{b}_i) : i < \partial\}$ of cardinality $< \partial$ is realized in N <u>then</u> $p(\bar{x})$ is realized in N.

Now first it suffices to prove $(*)^+_{M,\varphi,\partial}$ for every such ∂ because this statement can be phrased as a sentence $\psi_{\varphi,\partial}$ in $\mathbb{L}_{\theta,\theta}(\tau_T)$ and it means $\partial \notin \mathrm{spec}(\varphi, T)$.

Second, assume the antecedent of $(*)^+_{M,\varphi,\partial}$ so $\langle \bar{b}_i : i < \partial\rangle$ are as above, let $B = \cup\{\bar{b}_i : i < \partial\}$ hence p is a $(< \sigma)$-satisfiable $\ell g(\bar{x})$-type in M over B, $B \subseteq M$, $|B| = \partial$. Hence there is an $\mathbb{L}_{\sigma,\sigma}(\tau_T)$-complete type $q(\bar{x})$ in $\mathbf{S}^{\ell g(\bar{x})}_{\mathbb{L}_{\sigma,\sigma}(\tau_T)}(M)$ extending it; the existence of $q(\bar{x})$ is the point at which we use "σ is a compact cardinal".

Let $q'(\bar{x})$ be the set of first order formulas in $q(\bar{x})$ so clearly $q'(\bar{x}) \in \mathbf{S}^{\ell g(\bar{x})}_{\mathbb{L}}(M)$; as M is θ-saturated clearly some $\bar{a} \in {}^{\ell g(\bar{x})}M$ realizes $q'(\bar{x}) \restriction B$. We are done because in M every $\mathbb{L}_{\sigma,\sigma}(\tau_T)$ formula is equivalent to a Boolean combination of first order formulas. In other words, without loss of generality M has elimination of quantifiers for first order formulas; and it follows that it has elimination of quantifiers also for $\mathbb{L}_{\sigma,\sigma}(\tau_T)$; so we are done.
3) Trivially $(b)' \Rightarrow (b)''$ and by [Sh:1006, 1.2=La6] we have $(b)'' \Rightarrow (b)$ so we can assume (a) + (b).

Let $\varphi(\bar{x}_{[m]}, \bar{y}_{[n]}) \in \mathbb{L}(\tau_T)$ be a first-order formula with the independence property for $\mathrm{Th}_{\mathbb{L}}(M)$. Define $\psi(\bar{x}_{[\kappa]}, \bar{y}^0_{[n]}, \bar{y}^1_{[n]}) \in \mathbb{L}_{\kappa^+,\aleph_0}(\tau_T)$ or pedantically $\in \mathbb{L}_{\kappa^+,\kappa^+}(\tau_T)$ as saying:

$(*)_1$ for each $\ell \in \{0,1\}$ there is a unique $i_\ell < \kappa$ such that $\varphi(\bar{x}_{[mi_\ell, m(i_\ell+1))}, \bar{y}^\ell_{[n]})$ and moreover $i_0 \neq i_1$.

We claim $\sup(\mathrm{spec}_\psi(T)) = \theta$. By clause (b), for some unbounded $\Theta \subseteq \mathrm{Card} \cap \theta$ for every $\mu \in \Theta$ there is a graph G_μ with set of nodes μ such that $\mathrm{chr}(G_\mu) > \kappa$ but $u \in [\mu]^{<\mu}$ implies $\mathrm{chr}(G_\mu \restriction u) \leq \kappa$. Since φ has the independence property and M is (first-order) saturated, we can find $\langle \bar{b}_i : i < \mu\rangle$ with $\bar{b}_i \in {}^nM$ such that for every $\bar{\mathbf{t}} \in {}^\mu 2$ there is $\bar{a} \in {}^mM$ with $\bigwedge\limits_{i<\mu} \varphi^M[\bar{a}, \bar{b}_i]^{\mathrm{if}(\mathbf{t}(i))}$.

Now let:

$(*)_2$ $\Gamma_\mu = \{\psi(\bar{x}, \bar{b}_i, \bar{b}_j) : i < j < \mu \text{ and } (i,j) \in \mathrm{edge}(G_\mu)\}$.

Easily

$(*)_3$ Γ_μ demonstrates $\mu \in \mathrm{spec}_\psi(T)$.

Let I be as there and let D be a uniform ultrafilter on κ such that Θ is unbounded in θ where

$$\Theta = \{\mu : \mu = \mu^{<\kappa} \text{ and in } I^\kappa/D \text{ there is a } (\mu,\mu)\text{-cut}\}.$$

Let $\mu \in \Theta$; let the first order formula $\varphi = \varphi(\bar{x}_{[n]}, \bar{x}_{[m]})$ exemplify that $\mathrm{Th}_{\mathbb{L}}(M)$ has the strict order property. For notational simplicity assume $n = 1 = m$. We choose $a_s \in {}^m M$ for $s \in I$ such that $M \models (\forall x)(\varphi(x, a_s) \to \varphi(x, a_2))$ iff $s <_I t$.

By the choice of μ, there are $f^1_\alpha, f^2_\alpha \in {}^\kappa I$ such that in I^κ/D we have $\alpha < \beta < \mu \Rightarrow f^1_\alpha/D < f^1_\beta/D < f^2_\beta/D < f^2_\alpha/D$, but I^κ/D omits the type $p = \{f^1_\alpha/D < x < f^2_\alpha/D : \alpha < \mu\}$. By [GsSh:1075, Lemma 2.1] if J is the completion of I then also J^κ/D omits the type p.

Let $\psi(\bar{x}_{[\kappa]}, \bar{y}_{[\kappa]}, \bar{z}_{[\kappa]})$ be the formula $\bigvee\limits_{A \in D} \bigwedge\limits_{i \in A} (\varphi(x_i, z_i \wedge \neg\varphi(x_i, y_i)$.

We define $\bar{b}^\ell_\alpha = \langle b^\ell_{\alpha,\varepsilon} : \varepsilon < \kappa\rangle$ for $\alpha < \mu, \ell \in \{1, 2\}$ by $b^\ell_{\alpha,\varepsilon} = a_{f^\ell_\alpha(\varepsilon)} \in M$.

Now let $\Gamma_\mu = \{\psi(\bar{x}, \bar{b}^1_\alpha, \bar{b}^2_\alpha) : \alpha < \mu\}$ and the rest should be clear.

4) Clause (b)$'$ implies clause (b) is proved in Golshani-Shelah [GsSh:1075, Th.3.3]. So we can assume (a) + (b) and the proof is similar to the proof of part (2).

5) Without loss of generality $\tau(T)$ has cardinality $< \theta$. Assume $\varepsilon < \theta, \varphi(\bar{x}_{[\varepsilon]}, \bar{y}) \in \mathbb{L}_{\theta,\theta}(\tau_T)$, let $\zeta = \ell g(\bar{y})$ and $\Gamma = \{\varphi(\bar{x}_{[\varepsilon]}, \bar{a}_\alpha) : \alpha < \alpha_* < \theta\}$ is a set of $\mathbb{L}_{\theta,\theta}$-formulas with parameters from M. Without loss of generality $\langle \bar{a}_\alpha : \alpha < \alpha_*\rangle$ is with no repetitions, we let $\kappa = (|T| + |\zeta|)^{|T|+|\varepsilon|}$.

We shall use freely:

- $(*)$ if $\alpha < \alpha'_*$ and $\bar{b}', \bar{b}'' \in {}^\delta M$ realize the same first order type over $\bar{a}_\alpha$ then $M \models \varphi[\bar{b}', \bar{a}_\alpha] \equiv \varphi[\bar{b}'', \bar{a}_\alpha]$.

We shall assume Γ is $(\le 2^\kappa)$-satisfiable in M and prove that it is satisfiable in M; this easily suffices. Let $A = \cup\{\bar{a}_\alpha : \alpha < \alpha_*\}$ and we try by induction on $i < \kappa^+$ to choose $M_i \prec_{\mathbb{L}} M$ of cardinality $\le 2^\kappa$, increasing continuous with i such that: if $p(\bar{x}_{[\varepsilon]}) \in \mathbf{S}^\varepsilon_{\mathbb{L}}(M_i \cup A)$ does not fork over M_i then for some $\alpha < \alpha_*, \bar{a}_\alpha \subseteq M_{i+1}$ and $p(\bar{x}_{[\varepsilon]}) \nvdash \varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{a}_\alpha)$. If we are stuck in i, i.e. M_i is well defined but we cannot choose M_{i+1}, then as $[p_1, p_2 \in \mathbf{S}^\varepsilon_{\mathbb{L}}(M_i \cup A)$ does not fork over $M_i \Rightarrow (p_1 = p_2 \Leftrightarrow p_1 \restriction M_i = p_2 \restriction M_i)]$ and $\mathbf{S}^\varepsilon_{\mathbb{L}}(M_i)$ has cardinality $(\sup_n |\mathbf{S}^n_{\mathbb{L}}(M_i)|)^{|\varepsilon|} \le (2^\kappa)^{|\varepsilon|} = 2^\kappa$, clearly for some $p(\bar{x}) \in \mathbf{S}^\varepsilon_{\mathbb{L}}(M_i \cup A)$ not forking over M_i there is no such α, but $p(\bar{x})$ is realized in M hence so is Γ.

What if we succeed to carry the induction? Choose $\bar{b}$ which realizes $\Gamma' = \{\varphi(\bar{x}_{[\varepsilon]}, \bar{a}_\alpha) : \bar{a}_\alpha \subseteq M_i$ for some $i < \kappa^+\}$, now $\{\alpha < \alpha_* : \bar{a}_\alpha \subseteq M_{\kappa^+}\} \le \|M_{\kappa^+}\|^{|\zeta|} \le 2^\kappa$, hence Γ' indeed is realized in M say by $\bar{b} \in {}^\varepsilon M$ and let $q \in \mathbf{S}^\varepsilon_{\mathbb{L}}(M_{\kappa^+} \cup A)$ extend $\mathrm{tp}_{\mathbb{L}}(\bar{b}, M_{\kappa^+}, M)$ and does not fork over M_{κ^+}. Without loss of generality $\bar{b}$ realizes q in M using a partial automorphism of M.

Now for every $i < \kappa^+$, by the induction $\mathrm{tp}_{\mathbb{L}}(\bar{b}, M_\kappa \cup A)$ is not a non-forking extension of $\mathrm{tp}(\bar{b}, M_i) = p$ hence also $\mathrm{tp}(\bar{b}, M_\kappa)$ is not. Contradiction to "$\mathrm{Th}_{\mathbb{L}}(M)$ is stable". $\square_{3.4}$

{a25} **Claim 3.6.** *The model $N = M^I/D$ is not $(\chi^+, \theta, \mathbb{L}_{\theta,\theta})$-saturated (even locally, and even just for φ-types) when:*

- (a) $D \in \mathrm{uf}_\theta(I)$
- (b) $\varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\zeta]})$ *witnesses T has the θ-c.p.*
- {w8} (c) $\chi = \mathrm{lcr}_\theta(\mathrm{spec}(\varphi, T), D)$ *see 0.8(3), equivalently letting* $(J, <_J, P^J) = (\theta, <, \mathrm{spec}(\varphi, T))^I/D$ *we have* $\chi = \min\{|\{s : s <_J t\}| : t \in P^J$, *but* $(\exists^{\ge\theta} s)(s <_J t)\}$.

{a46} *Proof.* Straightforward or see the proof of 4.3 below. $\square_{3.6}$

Remark 3.7. In 3.8, 3.14 + more below the distinction T, T_1 is not necessary. But it is natural in the way we shall quote them; that is we consider properties of T and ask for $T_1 \supseteq T$ large enough such that "$M \models T_1 \Rightarrow M \restriction \tau_T$ satisfies ..." {a28}

{a26}

Definition 3.8. We say that $(\varphi, M, \bar{\mathbf{a}}, \bar{\mathbf{b}})$ strongly χ-witnesses or $(M, \bar{\mathbf{a}}, \bar{\mathbf{b}})$ strongly (χ, φ)-witness that T is 1-unstable <u>when</u> for some $T_1 \supseteq T$: (if $\chi = \theta$ we may omit it)

$\circledast_1$ (a) M is a model of T_1

(b) $\varphi = \varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\zeta]}) \in \mathbb{L}_{\theta,\theta}(\tau(T_1))$

(c) (α) $\bar{a}_\alpha \in {}^{\varepsilon}M, \bar{b}^1_\beta \in {}^{\zeta}M$ for $\alpha, \beta < \chi$ are such that $M \models \varphi[\bar{a}_\alpha, \bar{b}_\beta]^{\mathrm{if}(\alpha<\beta)}$

(β) $\bar{\mathbf{a}} = \langle \bar{a}_\alpha : \alpha < \chi \rangle$ and $\bar{\mathbf{b}} = \langle \bar{b}_\alpha : \alpha < \chi \rangle$

(d) for every $\bar{a} \in {}^{\varepsilon}M$ for some truth value $\mathbf{t}$ for every $\beta < \chi$ large enough we have $M \models \varphi[\bar{a}, \bar{b}_\beta]^{\mathrm{if}(\mathbf{t})}$

(e) for every $\bar{b} \in {}^{\zeta}M$ for some truth value $\mathbf{t}$ for every $\alpha < \chi$ large enough we have $M \models \varphi[\bar{a}_\alpha, \bar{b}]^{\mathrm{if}(\mathbf{t})}$.

{a26d}

Remark 3.9. Definition 3.8 is a case of "$\langle \bar{a}^1_\alpha {}^\frown \bar{b}^1_\alpha : \alpha < \chi \rangle$ is convergent", see [Sh:300a, §2,Def 2.1=L300a-2.1,pg.25]. {a26}

{a27}

Observation 3.10. *1) Assume the triple $(M, \bar{\mathbf{a}}, \bar{\mathbf{b}})$ strongly (χ, φ)-witnesses that T is 1-unstable and $\chi = \mathrm{cf}(\chi) \geq \theta$. If $\lambda = \lambda^{<\theta} + |\tau_T|$ and $\sigma = \mathrm{cf}(\sigma) \in [\theta, \lambda]$, <u>then</u> there is a triple $(M', \bar{\mathbf{a}}', \bar{\mathbf{b}}')$ which strongly (σ, φ)-witness T is 1-unstable and $\|M'\| = \lambda$. We can add $\|M\| \leq \lambda \Rightarrow M \prec_{\mathbb{L}_{\theta,\theta}} M'$ and $\chi > \lambda \Rightarrow M' \prec_{\mathbb{L}_{\theta,\theta}} M$.*
2) If for every $\tau' \subseteq \tau(T)$ of cardinality $< \theta$ such that $\varphi \in \mathbb{L}_{\theta,\theta}(\tau')$ there is a strong (χ, φ)-witness for $T \cap \mathbb{L}_{\theta,\theta}(\tau)$ being 1-unstable for some $\chi = \mathrm{cf}(\chi) \geq \theta$ <u>then</u> there is a strong (χ, φ)-witness for T being 1-unstable for every $\chi = \mathrm{cf}(\chi) \geq \theta$.

Proof. 1) First let $D \in \mathrm{ruf}_\theta(\lambda)$ and so by 0.26(3) for some $\chi_1 = \mathrm{cf}(\chi_1) \in [\lambda^+, 2^\lambda)$ and $\bar{\mathbf{a}}', \bar{\mathbf{b}}'$, we have $(M^I/D, \bar{\mathbf{a}}', \bar{\mathbf{b}}')$ strongly (χ_1, φ) witness T is 1-unstable. Now apply the downward LST argument. {x31}
2) Easy, too. $\square_{3.10}$

{a27c}

Observation 3.11. *For any model M satisfying $\|M\| = \|M\|^{<\theta}$ there is an expansion M_1^* by the new function symbols $F_\xi(\xi < \theta), F_\xi$ being ξ-place such that $M' \equiv_{\mathbb{L}_{\theta,\theta}} M \Rightarrow \|M'\| = \|M'\|^{<\theta}$.*

Proof. Choose $F_\xi^{M_2} : {}^{\xi}M_2 \to M$ which is one-to-one. $\square_{3.11}$

{a27g}

Claim 3.12. *Assume $T \subseteq \mathbb{L}_{\theta,\theta}(T_1)$ is complete 1-unstable theory as witnessed by $\varphi(\bar{x}, \bar{y})$.*

For any theory $T_1 \supseteq T$ and regular $\chi \geq \theta$ there are $M, \bar{\mathbf{a}}, \bar{\mathbf{b}}$ as in Definition 3.8 with $M \in \mathrm{Mod}_{T_1}$. {a26}

Proof. Let $\ell g(\bar{x}) = \varepsilon < \theta, \ell g(\bar{y}) = \zeta < \theta$.

Let $P, <$ be new predicates, i.e. $\notin \tau(T_1)$ with $\varepsilon + \zeta, \varepsilon + \zeta + \varepsilon + \zeta$ places respectively and let F_ξ be a new ξ-place function symbol.

Let T_2 be the set of $\mathbb{L}_{\theta,\theta}(\tau_{T_1} \cup \{P, <, F_\xi : \xi < \theta\})$-sentences such that for any $\tau(T)$-model M_2 we have: $M_2 \models T_2$ iff

$(*)_1$ (a) $M_2 \models T_1$

(b) $<^{M_2}$ linearly ordered P^{M_2}, of cofinality $\geq \theta_1$ for any $\theta_1 < \theta$

(c) if $\bar{a}_1 \hat{} \bar{b}_1 \in P^{M_2}, \bar{a}_2 \hat{} \bar{b}_2 \in P^{M_2}, \bar{a}_\ell \in {}^\varepsilon(M_2), \bar{b}_\ell \in {}^\zeta(M_2)$ for $\ell = 1, 2$ and $\bar{a}_1 \hat{} \bar{b}_1 <^{M_2} \bar{a}_2 \hat{} \bar{b}_2$ <u>then</u> $M_2 \models \varphi(\bar{a}_1, \bar{b}_2) \wedge \neg\varphi(\bar{a}_2, \bar{b}_1)$

(d) for every $\bar{a}' \in {}^\varepsilon(M_2)$ for some truth value $\mathbf{t}$, for every $\bar{a} \hat{} \bar{b} \in P^{M_2}$ which is $<^{M_2}$-large enough (and $(\ell g(\bar{a}), \ell g(\bar{b})) = (\varepsilon, \zeta)$, of course) we have $M_2 \models \varphi[\bar{a}', \bar{b}]^{\text{if}(\mathbf{t})}$

(e) for every $\bar{b}' \in {}^\zeta(M_2)$ for some truth value $\mathbf{t}$, for every $\bar{a} \hat{} \bar{b} \in P^{M_2}$ which is $<^{M_2}$-large enough, we have $M_2 \models \varphi[\bar{a}, \bar{b}']^{\text{if}(\mathbf{t})}$.

Now

$(*)_2$ T_2 is an $\mathbb{L}_{\theta,\theta}$-theory.

Why? For this it suffices to prove that every $T_2' \subseteq T_2$ of cardinality $< \theta$ has a model, so without loss of generality $|\tau_{T_1}| < \theta$ and let $M_1 \models T_1$. As T is complete 1-unstable as witnessed by φ for every $\gamma < \theta$ there are $\langle (\bar{a}_i^\gamma, \bar{b}_i^\gamma) : i < \gamma \rangle$ in M_1 as
{y2} in Definition 1.1(1), i.e. $M_1 \models \varphi[\bar{a}_i^\gamma, \bar{b}_j^\gamma]^{\text{if}(i<j)}$ for $i, j < \gamma$.

By compactness of $\mathbb{L}_{\theta,\theta}$ possibly changing M_1 we have $\langle (\bar{a}_i, \bar{b}_i) : i < \theta \rangle$ as above. By the LST argument without loss of generality $\|M_1\| = \theta$, hence $|{}^\varepsilon(M_1)| + |{}^\zeta(M_1)| = \theta$.

Let $\langle \bar{c}_\alpha : \alpha < \theta \rangle$ list ${}^\varepsilon|(M_1)|$ and $\langle \bar{d}_\alpha : \alpha < \theta \rangle$ list ${}^\zeta|(M_1)|$.

We define $f : [\theta]^3 \to \{0, 1\}$ by:

$(*)_3$ if $\alpha < \beta < \gamma < \theta$ then $f(\{\alpha, \beta, \gamma\}) = 1$ <u>iff</u> $j < \alpha \Rightarrow M_1 \models$ "$\varphi[\bar{c}_j, \bar{b}_\beta] \equiv \varphi[\bar{c}_j, \bar{b}_\gamma]$" and $j < \alpha \Rightarrow M_1 \models$ "$\varphi[\bar{a}_\beta, \bar{d}_j] \equiv \varphi[\bar{a}_\gamma, \bar{d}_j]$".

But θ is, of course, weakly compact so f is constant on $[\mathscr{U}]^3$ for some $\mathscr{U} \in [\theta]^\theta$; easily necessarily f is constantly 1.

We now define M_2 expanding M_1 by

$$P^{M_2} = \{\bar{a}_\alpha \hat{} \bar{b}_\alpha : \alpha \in \mathscr{U}\}$$

$$<^{M_2} = \{\bar{a}_\alpha \hat{} \bar{b}_\alpha \hat{} \bar{a}_\beta \hat{} \bar{b}_\beta : \alpha < \beta \text{ are from } \mathscr{U}\}.$$

Easily $M_2 \models T_2'$ hence we are done proving $(*)_2$.

$(*)_4$ for every λ there is a model M_2 of T_2 such that $\text{cf}(P^{M_2}, <^{M_2}) \geq \lambda^+$.

{x31} [Why? Let $M_2 \models T_2, D \in \text{ruf}_{\chi,\theta}(\lambda)$ then $(M_2)^\lambda / D$ is as required by 0.26(3).]

$(*)_5$ for every regular $\chi \geq \theta$ and $\lambda = \lambda^{<\theta} + |T_1| + \chi$ there is a model M_2 of T_2 of cardinality λ such that $\text{cf}(P^{M_2}, <^{M_2}) = \chi$.

[Why? By $(*)_4$ applied with $((\chi + \lambda + \theta)^{<\theta})^+$ here standing for λ there and then use the LST argument.]

To finish note that

$(*)_6$ if $M_2 \models T_2$ and $\langle (\bar a_\alpha {}^\frown \bar b_\alpha) : \alpha < \chi \rangle$ is $<^{M_2}$-increasing cofinal in P^{M_2} and $(\ell g(\bar a_\alpha), \ell g(\bar b_\alpha)) = (\varepsilon, \zeta)$ then $(\varphi, M_2, \langle \bar a_\alpha : \alpha < \chi \rangle, \langle \bar b_\alpha : \alpha < \chi \rangle)$ is as required in Definition 3.8. {a26}

[Why? Read the Definition of T_2.] $\square_{3.12}$ {a27k}

Remark 3.13. 1) We can strengthen the conclusion of 3.12 to {a27g}

$(*)$ for every $\bar d \in {}^{\theta >}\mu$ the sequence $\langle \mathrm{tp}_{\mathbb{L}_{\theta,\theta}(\tau)}(\bar a^1_\alpha {}^\frown \bar a^2_\alpha, \mathrm{Rang}(\bar d), M) : \alpha < \chi \rangle$ is eventually constant.

How? In $(*)_3$ we can change somewhat the demand:

$(*)'_3$ for $\alpha < \beta < \gamma < \theta$ then $f(\{\alpha, \beta, \gamma\}) = 1$ iff for every $j < \alpha$ and formula $\vartheta(\bar x_{[\varepsilon+\zeta]}, \bar y_{[\varepsilon+\zeta]})(\tau(T'_2))$ we have $M_1 \models \vartheta[a^1_\beta {}^\frown \bar a^2_\beta, \bar c_j] \Leftrightarrow M_1 \models \vartheta[\bar a^1_\gamma, \bar a^2_\gamma, \bar c_j]$.

We similarly change $(*)_1(c) + (d)$.
2) Clearly if $T \vdash$ "$(P, <)$ is a linear order of cofinality $\ge \partial$" for every $\partial < \theta$ and $\lambda = \lambda^{<\theta} + |T| \ge \kappa = \mathrm{cf}(\kappa) \ge \theta$, then T has a model N of cardinality λ such that $\mathrm{cf}(P^N, <^N) = \kappa$. This is proved inside the proof of 3.12 and holds by 0.26(3). {a27g} {a28}

Claim 3.14. *If (A) then (B) where:*

(A) (a) T *is a complete* $\mathbb{L}_{\theta,\theta}(\tau_T)$*-theory*
 (b) T *is 1-unstable as witnessed by* $\varphi(\bar x_{[\varepsilon]}, y_{[\zeta]})$ *and let* $\psi = \psi(\bar x_{[\zeta]}, \bar y_{[\varepsilon]}) = \varphi(\bar y_{[\varepsilon]}, \bar x_{[\zeta]})$
 (c) $T_1 \supseteq T$ *is a complete* $\mathbb{L}_{\theta,\theta}(\tau_1)$*-theory and* $|\tau(T_1) \backslash \tau(T)| \le \lambda$
 (d) $\mathbf{x}$ *is a non-trivial* $(\theta, \theta) -$ l.u.f.t.
 (e) $\chi = \mathrm{cf}(\mathrm{l.u.p.}_{\mathbf{x}}(\theta, <))$ *hence* $\chi = \chi^{<\theta}$*, see 0.19 - 0.22* {x18}

(B) *for some* $M_1 \models T_1$ *the model* $\mathrm{l.u.p.}_{\mathbf{x}}(M_1)$ *is not* $(\chi^+, \{\varphi\})$*-saturated or not* $(\chi^+, \{\psi\})$*-saturated, see Definition 2.2(4).* {a5}

Proof. Case 1: $|T_1| \le \theta$.

Choose $D_* \in \mathrm{ruf}_{\chi,\theta}(\chi)$ hence D_* is uniform. Let $(M, \langle \bar a^1_\alpha : \alpha < \theta \rangle, \langle \bar b^1_\alpha : \alpha < \theta \rangle)$ be a strong φ-witness for T being 1-unstable, see Definition 3.8, exists by Claim 3.12. {a26g}

Let $M^+ = (M, P^{M^+}, <^{M^+})$ where $P^{M^+} = \{\bar a^1_\alpha {}^\frown \bar b^1_\alpha : \alpha < \theta\}$ and $<^{M^+} = \{(\bar a^1_\alpha {}^\frown \bar b^1_\alpha, \bar a^1_\beta {}^\frown \bar b^1_\beta) : \alpha < \beta < \theta\}$ and let $N^+ = \mathrm{l.u.p.}_{\mathbf{x}}(M^+)$ hence clearly $N^+ = (\mathrm{l.u.p.}_{\mathbf{x}}(M), P^{N^+}, <^{N^+})$ and $N = \mathrm{l.u.p.}_{\mathbf{x}}(M)$. By clause (A)(e) of the claim, clearly $(P^{N^+}, <^{N^+})$ is a linear order of cofinality χ so we can choose an increasing cofinal sequence $\langle \bar a^3_\alpha {}^\frown \bar b^3_\alpha : \alpha < \chi \rangle$ in $(P^{N^+}, <^{N^+})$, and by 0.15 {x7}

$(*)_1$ if $\bar a \in {}^\varepsilon|N^+|$ and $\bar b \in {}^\zeta|N^+|$ then for some truth values $\mathbf{t}(1), \mathbf{t}(2)$ for every $\alpha < \chi$ large enough $N^+ \models$ "$\varphi[\bar a, \bar b^3_\alpha]^{\mathrm{if}(\mathbf{t}(1))} \wedge \varphi[\bar a^3_\alpha, \bar b]^{\mathrm{if}(\mathbf{t}(2))}$"; of course this is a property of N.

We try to choose $(N_\alpha, \bar b^4_\alpha)$ by induction on $\alpha < \chi$ such that:

$(*)^2_\alpha$ (a) $N_\alpha \prec_{\mathbb{L}_{\theta,\theta}} N$ has cardinality χ
 (b) if $\beta < \alpha$ then $\bar a^3_\beta {}^\frown \bar a^4_\beta \subseteq N_\alpha \subseteq N$

(c) if $\beta < \alpha$ then $N_\beta \cup \bar{b}^3_\beta {}^\frown \bar{b}^4_\beta \subseteq N_\alpha$

(d) $\bar{b}^4_\alpha \in {}^\zeta N$ is from N^+ satisfies:

- for every $\bar{a} \in {}^\varepsilon(N_\alpha + \bar{a}^4_\alpha)$ we have $N \models \varphi[\bar{a}, \bar{b}^4_\alpha]$ iff $\{\beta < \chi : N \models \varphi[\bar{a}, \bar{b}^3_\beta]\} \in D_*$, equivalently
- $\bar{b}^4_\alpha$ realizes $\{\varphi(\bar{a}, \bar{y}_{[\zeta]})^{\text{if}(\mathbf{t})} : \bar{a} \in {}^\varepsilon(N_\alpha + \bar{a}^4_\alpha)$ and $\{\beta < \chi : N \models \varphi(\bar{a}, \bar{b}^3_\beta)^{\text{if}(\mathbf{t})}\} \in D_*$ and $\mathbf{t} \in \{0,1\}\}$.

If we are stuck at α then obviously we can choose N_α as required in clauses (a),(b),(c) of $(*)^2_\alpha$ hence there is no $\bar{b}^4_\alpha$ as required in $(*)^2_\alpha(d)$ hence N is not $(\chi^+, \theta, \{\psi\})$-saturated, (as otherwise N_α easily exists). Now as $N = \text{l.u.p.}_{\mathbf{x}}(M)$ the desired conclusion (B) holds for $M_1 = M$. So we can assume that we succeed to carry the induction so $M_3 := \cup\{N_\alpha : \alpha < \chi\}$ is $\prec_{\mathbb{L}_{\theta,\theta}} N$. Now the pair
{x32} $(M_3, \langle(\bar{a}^3_\alpha, \bar{b}^3_\alpha, \bar{b}^4_\alpha) : \alpha < \chi\rangle)$, recalling that (by 0.27) necessarily $\chi = \chi^{<\theta}$, satisfies $\boxplus^\chi_{M_3, \langle(\bar{a}^3_\alpha, \bar{b}^3_\alpha, \bar{b}^4_\alpha):\alpha<\chi\rangle}$, where for a linear order I and model M_* we let

$\boxplus^I_{M_*, \langle(\bar{a}^3_s, \bar{b}^3_s, \bar{b}^4_s):s \in I\rangle}$

(a) M_* is a model of T_1

(b) $\bar{b}^3_s, \bar{b}^4_s \in {}^\zeta(M_*)$ and $\bar{a}^3_s \in {}^\varepsilon(M_*)$

(c) if $\bar{a} \in {}^\varepsilon(M_*)$ then for some truth value $\mathbf{t}$ we have for every $s \in I$ large enough $M_* \models \varphi[\bar{a}, \bar{b}^3_s]^{\text{if}(\mathbf{t})} \wedge \varphi[\bar{a}, \bar{b}^4_s]^{\text{if}(\mathbf{t})}$

(d) $M_* \models$ "$\varphi[\bar{a}^3_s, \bar{b}^4_t]$" for $s, t \in I$

(e) if $s, t < \chi$ then $M_* \models$ "$\varphi[\bar{a}^3_s, \bar{b}^3_t]$" iff $s < t$.

Why? For clause (c) let $\alpha < \chi$ be such that $\bar{a} \in {}^\varepsilon(N_\alpha)$. Now for all $\beta \in [\alpha, \chi)$ recall
{a26} clause $(*)^2_\beta(d)$ and $(*)_1$. For clause (d), by $\circledast_1(c)(\alpha)$ of 3.8 we have $\alpha_1 < \beta_1 \Rightarrow N \models \varphi[\bar{a}^1_{\alpha_1}, \bar{b}^1_{\beta_1}]$, hence by the choice of $\langle \bar{a}^3_\gamma {}^\frown \bar{b}^3_\gamma : \gamma < \chi\rangle$ we have $\gamma \in (\alpha, \chi) \Rightarrow N \models \varphi[\bar{a}^3_\alpha, \bar{a}^3_\gamma]$ so by $(*)^2_\alpha(d)$ we have $N \models \varphi[\bar{a}^3_\alpha, \bar{b}^4_\beta]$ as required in (d).

{a26} As for clause (e) by $\circledast_1(c)(\alpha)$ of 3.8 we have $\beta, \alpha < \chi \Rightarrow N \models \varphi[\bar{a}^1_\alpha, \bar{b}^1_\beta]^{\text{if}(\alpha<\beta)}$ hence by the choice of $\langle \bar{a}^3_\gamma {}^\frown \bar{b}^3_\gamma : \gamma < \chi\rangle$ we have $\alpha, \beta < \chi \Rightarrow N \models \varphi[\bar{a}^3_\alpha, \bar{b}^3_\beta]^{\text{if}(\alpha<\beta)}$. So the pair $(M_3, \langle(\bar{a}^3_\alpha, \bar{b}^3_\alpha, \bar{b}^4_\alpha) : \alpha < \chi\rangle$ is as promised.

As $|\tau_{T_1}| \le \theta$ by the case assumption, by the downward LST theorem there are $M_4 \prec_{\mathbb{L}_{\theta,\theta}} M_3$ of cardinality θ and an increasing sequence $\langle \alpha(i) : \varepsilon < \theta\rangle$ of ordinals $< \chi$ such that $(M_4, \langle(b^3_{\alpha(\varepsilon)}, \bar{a}^3_{\alpha(\varepsilon)}, \bar{b}^4_{\alpha(\varepsilon)}) : \varepsilon < \theta\rangle$ satisfies $\boxplus^\chi_{M_4, \langle(\bar{a}^3_{\alpha(\varepsilon)}, \bar{b}^3_{\alpha(\varepsilon)}, \bar{b}^4_{\alpha(\varepsilon)}):\varepsilon<\chi\rangle}$.

Now it is easy to see that $\text{l.u.p.}_{\mathbf{x}}(M_4)$ is not locally $(\chi^+, \theta, \{\varphi\})$-saturated, a detailed proof is included in the proof of Case 2.

Case 2: $|T_1| > \theta$

Let $\tau_2 = \tau(T_1) \cup \{P, <, F_i, G_j, H_j) : i < \varepsilon, j < \zeta\}$ where the union is disjoint, and $P, <$ are unary and binary predicates respectively and F_i, G_j, H_j are unary function symbols.

Let T_2 be the set of $\mathbb{L}_{\theta,\theta}(\tau_2)$-sentences such that

$(*)_3$ for a τ_2-model M_2 we have $M_2 \models T_2$ iff

(a) $M_2 \models T_1$

(b) $(P^{M_2}, <^{M_2})$ is a linear order of cofinality $> \partial$ for every $\partial < \theta$

(c) $I = (P^{M_2}, <^{M_2}), M_3' = M_2 \restriction \tau(T_1), \mathbf{\bar a} = \langle (\bar a^3_t, \bar b^3_t, \bar b^4_t) : t \in P^{M_2}\rangle$ satisfies $\boxplus^I_{M_2,\mathbf{\bar a}}$ where we let
- $\bar a^3_t = \langle F_i^{M_2}(t) : i < \varepsilon\rangle$
- $\bar b^3_t = \langle G_j^{M_2}(t) : j < \zeta\rangle$
- $\bar b^4_t = \langle H_j^{M_2}(t) : j < \zeta\rangle$.

By Case 1 applied to $T_1 \cap \mathbb{L}_{\theta,\theta}(\tau')$ for any $\tau' \subseteq \tau_T$ of cardinality $\le \theta$ such that $\varphi(\bar x, \bar y) \in \mathbb{L}_{\theta,\theta}(\tau')$, hence clearly T_2 is a theory.

By the proof of 3.12, for every $\lambda = \lambda^{<\theta} + |T_1| \ge \kappa = \mathrm{cf}(\kappa) \ge \theta$, the theory T_2 has a model $N = N_{\lambda,\kappa}$ of cardinality λ such that $\mathrm{cf}(P^N, <^N) = \kappa$, see 3.13(2), 0.26(3). Applying this to the case $\kappa = \theta$, consider $N^* = \mathrm{l.u.p.}_{\mathbf{x}}(N_{\lambda,\theta})$, so $(P^{N^*}, <^{N^*})$ has cofinality χ, so let $\langle t_\varepsilon = t(\varepsilon) : \varepsilon < \chi\rangle$ be increasing and cofinal in it and for $t \in P^{N_{\lambda,\theta}}$ let $\bar a^3_t = \langle F_i^{N_*}(t) : i < \varepsilon\rangle, \bar b^3_t = \langle G_j^{N_*}(t) : j < \zeta\rangle, \bar b^4_t = \langle H_j^{N_*}(t : j < \zeta)$, so the statement $\boxplus = \boxplus^\chi_{N_*,\mathbf{\bar a}_1}$ where $\mathbf{\bar a}_1 = \langle(\bar a^3_{t(\xi)}, \bar b^3_{t(\xi)}, \bar b^4_{t(\xi)}) : \xi < \chi\rangle$ clearly holds. {a27g} {a27k} {a31}

Now for every $\bar a \in {}^\varepsilon(N^*)$ by $(*)_3(c)$ and clause (c) of $\boxplus$ clearly for some ordinal $\varepsilon(\bar a) < \chi$ and truth value $\mathbf{t}(\bar a)$ we have

$(*)_5$ if $\varepsilon(\bar a) \le \xi < \chi$ then $N_* \models$ "$\varphi[a, \bar b^3_{t(\xi)}]^{\mathrm{if}(\mathbf{t}(\bar a))} \wedge \varphi[\bar a, \bar b^4_{t(\xi)}]^{\mathrm{if}(\mathbf{t}(\bar a))}$".

For $\alpha \le \chi$ let $p_\alpha = \{\varphi(\bar x, \bar b^4_{t(\xi)}), \neg\varphi(\bar x, \bar b^3_{t(\xi)}) : \xi < \alpha\}$. Now by $(*)_3(c)$ and clauses (d),(e) of $\boxplus$ the sequence $\bar a^3_{t(\alpha)}$ realizes p_α in N_* when $\alpha < \chi$ hence p_χ, the increasing union of $\langle p_\alpha : \alpha < \chi\rangle$ is $(< \chi)$-satisfiable in N_*. However, by $(*)_5$ no $\bar a \in {}^\varepsilon(N_*)$ realizes p_χ, so p_χ exemplifies $N_* = \mathrm{l.u.p.}(M_4)$ is not $(\chi^+, \varphi(\bar x, \bar y))$-saturated so we have gotten the desired conclusion. $\square_{3.14}$

Theorem 3.15. *Assume T is a complete theory (in $\mathbb{L}_{\theta,\theta}$), has θ-n.c.p. and is definably stable and $\lambda = \lambda^{<\theta}$.* {a30}
1) T is locally $\blacktriangleleft_{\lambda,\theta}$-minimal.
2) If $D \in \mathrm{ruf}_{\lambda,\theta}(I)$ and $M \models T$ __then__ M^I/D is locally $(\lambda^+, \theta, \mathbb{L}_{\theta,\theta})$-saturated.

Remark 3.16. Note Theorem 3.15 deals with local $\blacktriangleleft_\lambda$-minimality, whereas 3.17 below deals with local $\blacktriangleleft^*_\lambda$-minimality and Claim 3.14 deals with non-$\blacktriangleleft^*_\chi$-minimality. {a3k} {a32} {a30} {a28}

Proof. 1) By part (2).
2) Without loss of generality $|\tau_T| \le \theta$.

Let $\varphi(\bar x, \bar y) \in \mathbb{L}_{\theta,\theta}$ and $\partial = \partial_\varphi < \theta$ witness $\varphi(\bar x, \bar y)$ fail the θ-c.p. and let $\varepsilon = \ell g(\bar x), \zeta = \ell g(\bar y)$ and $N = M^I/D$, where $D \in \mathrm{ruf}_\theta(\lambda)$ and M is a model of T and $p(\bar x) = p_0(\bar x)$ is a positive φ-type in N of cardinality $\le \lambda$, so $p(\bar x) \subseteq \{\varphi(\bar x, \bar b) : \bar b \in {}^{\ell g(\bar y)}N\}$ is $(< \theta)$-satisfiable in N.

As θ is a compact cardinal there is $p_1(\bar x) \in \mathbf{S}^\varepsilon_\varphi(N)$ extending $p_0(x)$. By Definition 1.3 there are $\psi(\bar y, \bar z) \in \mathbb{L}_{\theta,\theta}(\tau_T)$ and $\bar c \in {}^{\ell g(\bar z)}N$ which define $p_1(\bar x)$. Let $\bar c_s \in {}^{\ell g(\bar z)}M$ for $s \in I$ be such that $\bar c = \langle \bar c_s : s \in I\rangle/D$ and for $s \in I$ let $\Gamma_s = \{\varphi(x, \bar b)^{\mathrm{if}(\mathbf{t})} : M \models$ "$\psi[\bar b, \bar c_s]^{\mathrm{if}(\mathbf{t})}$" and $\mathbf{t} \in \{0,1\}\}$. {y3}

Let $I_\partial = \{s \in I : \Gamma_s$ is $(< \partial)$-satisfiable in M_s, that is if $\bar b_\alpha \in {}^\zeta(M_s)$ and $M_s \models \psi[\bar b_\alpha, \bar c_s]^{\mathrm{if}(\mathbf{t}(\alpha))}$ for $\alpha < \partial$ then $M \models \exists \bar x \bigwedge\limits_{\alpha<\partial} \varphi(\bar x, \bar b_\alpha)^{\mathrm{if}(\mathbf{t}(\alpha))}\}$; so by 0.15 necessarily $I_\partial \in D$. {x7}

By the choice of ∂ and of I_∂ for every $s \in I_\partial$ the set $\Gamma^+_{\mathbf{s}} = \{\varphi(\bar{x}, \bar{b}) : M \models$ "$\psi[b, \bar{c}_s]$"$\}$ is $(< \theta)$-satisfiable in M_s.

Let χ be large enough such that $M \in \mathscr{H}(\chi)$ and let $\mathfrak{B} = (\mathscr{H}(\chi), \in, M)^I/D$. As $s \in I \Rightarrow \Gamma^+_s \in \mathscr{H}(\chi)$ we have $\Gamma^+ := \langle \Gamma^+_s : s \in I\rangle/D \in \mathfrak{B}$ and $\mathfrak{B} \models$ "Γ^+ is a $(< \mathbf{j}(\theta))$-satisfiable over M" where $\mathbf{j} : \mathscr{H}(\chi) \to \mathfrak{B}$ is the canonical embedding. Let $\Gamma' = \{\varphi(\bar{x}, \bar{a}) : \mathfrak{B} \models$ "$\varphi(\bar{x}, \bar{a}) \in \Gamma$"$\}$. Hence to prove $p_0(\bar{x})$ is realized it suffices to show

- there is $w \in \mathfrak{B}$ such that $\varphi(\bar{x}, \bar{b})^{\mathrm{if}(\mathbf{t})} \in p_0(x) \Rightarrow \mathfrak{B} \models$ "$\bar{b} \in w$ and $|w| < \mathbf{j}(\theta)$".

{x8} By 0.16(2) this holds. $\square_{3.15}$

{a32} **Theorem 3.17.** *Assume the complete $T \subseteq \mathbb{L}_{\theta,\theta}$ has $\theta-$n.c.p. and is 1-stable hence*
{y5} *(by 1.4) definably stable and $T_0 \supseteq T$ is a complete $\mathbb{L}_{\theta,\theta}$-theory. Then for some $\mathbb{L}_{\theta,\theta}$-theory $T_1 \supseteq T_0$ of cardinality $(|T| + \theta)^{<\theta}$, we have:*

- *if M_1 is a model of T_1, letting λ be its cardinality, then $M'{\restriction}\tau_T$ is locally $(\lambda, \theta, \mathbb{L}_{\theta,\theta})$-saturated and $\lambda = \lambda^{<\theta} \subseteq |T|$.*

{a32d} *Remark* 3.18. Instead of "T is 1-stable" to prove M_1 is locally $(\lambda, \theta, \Delta)$-saturated it is enough to assume

(*a*) $\Delta \subseteq \mathbb{L}_{\theta,\theta}(\tau_T)$ has cardinality $< \theta$
(*b*) if $\varphi_1(\bar{x}, \bar{y}) \in \Delta$ then some $\psi_{\varphi_1}(\bar{y}, \bar{z})$ is as in the definition of definably stable
(*c*) Δ is closed under redividing the variables and permuting variables
(*d*) each $\varphi_1(\bar{x}, \bar{y}) \in \Delta$ is 1-stable in T.

{y3} *Proof.* For any $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\theta,\theta}(\tau_T)$ let $\psi_\varphi(\bar{y}, \bar{z}_\varphi)$ be as in Definition 1.3 of definably
{y8} stable for φ and T, see Definition 1.3(1) recalling T is definably stable by 1.4(1). For $\gamma < \theta$ let $\vartheta_{\varphi,\gamma}(\bar{z}_\varphi)$ be the formula saying that $(\forall \ldots \bar{y}_i \ldots)_{i<\gamma}(\bigwedge_{i<\gamma} \psi(\bar{y}_i, \bar{z}) \to \exists x \bigwedge_{i<\gamma} \varphi(\bar{x}, \bar{y}))$ and let $\vartheta_\varphi(\bar{z}_\varphi) = \vartheta_{\varphi,\partial_\varphi}(\bar{z}_\varphi)$.

{a19} Let $\Delta_\varphi \subseteq \{\varphi, \neg\varphi\}$ and let $\varphi^{[*]}(\bar{x}, \bar{y}_*)$ be as in 3.2(3) for Δ and let $\theta_\varphi < \theta$ be large enough and for $\Delta \subseteq \mathbb{L}_{\theta,\theta}(\tau_T)$ be of cardinality $< \theta$, let $\theta_\Delta < \theta$ be large enough.

Now

$(*)_1$ let T_2 be the set of sentences in $\mathbb{L}_{\theta,\theta}(\tau_2)$ where τ_2 implicitly defined below such that $M_2 \models T_2$ iff:
   (a) $M_2 \models T_0$
   (b) $<^{M_2}$ is a well ordering of $|M_2|$ of cofinality $\geq \theta$
   (c) if $\varphi = \varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\theta,\theta}(\tau_T)$ and $\bar{c} \in \vartheta_\varphi(M_2)$ and $d \in M_2$ then $\bar{a}^{\varphi,M_2}_{\bar{c},d} := \langle F_{\varphi,i}(d, \bar{c}) : i < \ell g(\bar{z}_\varphi)\rangle$ realizes $p^{\varphi,M_2}_{\bar{c},d} := \{\varphi(x, \bar{b}) : \bar{b} \in {}^\zeta(M_2)$ and $i < \ell g(\bar{b}) \Rightarrow b_i < d$ and $M_2 \models \psi_\varphi[\bar{b}, \bar{c}]\}$
   (d) P^{M_2} is a closed unbounded set of d's such that: if $\Delta \subseteq \mathbb{L}_{\theta,\theta}(\tau_{T_2})$ has cardinality $< \theta$ and $\partial = \partial_\Delta < \theta$ is large enough and $\mathrm{cf}(\{d' : d' <^{M_1} d\}, <^{M_1}) \geq \theta_\Delta$ then $M_2^{<d} := M_2{\restriction}\{d' : d' < d^{M_2}\} \prec_\Delta M_2$

(e) $a \mapsto \langle G^{M_2}_\varepsilon(a) : \varepsilon < \zeta\rangle$ is a function from M_2 onto ${}^\zeta(M_2)$ for each $\zeta < \theta$.

Now

$(*)_2$ T_2 is a theory.

[Why? Choose $\chi = \chi^{<\theta} \ge |T_2|$, let $M_0 \models T_0$ be a $(\chi^+, \{\varphi\})$-saturated model (or just a locally $(\chi^+, \theta, \mathbb{L}_{\theta,\theta}(\tau_T))$-saturated model); exists by 3.15 + L.S.T. Choose $\langle M^2_\alpha : \alpha < \chi^+\rangle$ a $\prec_{\mathbb{L}_{\theta,\theta}}$-increasing sequence of $\prec_{\mathbb{L}_{\theta,\theta}}$-submodels of M_0, each of cardinality χ increasing fast enough, i.e. choose M^2_α by induction on α. The rest should be clear.] {a30}

$(*)_3$ let $\tau_3 = \tau_2 \cup \{Q, F\}, Q$ a unary predicate, F a unary function symbol and $T_3 \subseteq \mathbb{L}_{\theta,\theta}(\tau_3)$ is a set of sentences such that a τ_3-model M_3 satisfies T_3 iff:

(a) $M_3 \models T_2$

(b) $Q^{M_3} \subseteq P^{M_3}$ is $<^{M_3}$-unbounded

(c) F^{M_3} maps Q^{M_3} onto $|M_3|$ hence Q^{M_3} is of cardinality $\|M_3\|$

(d) if $d \in M_3$ and $\bar c \in {}^{\ell g(\bar z)}(M^{<d}_3)$ then $\langle e \in M_3 : e$ satisfies $M_3 \models$ "$d < e \wedge Q(e)$"$\rangle$ is 2-indiscernible (even n-indiscernible for every n) over $\bar c$ in $M_3 \restriction \tau_2$

$(*)_4$ T_3 is a theory.

[Why? Easy, e.g. it is enough to consider $(\Delta, 2)$-indiscernibility and for this imitate the proof of 3.12.] {a27g}

$(*)_5$ assuming $\varphi = \varphi(\bar x, \bar y) \in \mathbb{L}_{\theta,\theta}(\tau_T)$ for some cardinal $\partial^1_\varphi < \theta$, if $M_3 \models T_3, \bar c \in \vartheta_\varphi(M_3)$ and $\bar b \in {}^{\ell g(\bar y)}(M_3)$ then for some $A = A^{\varphi,M_3}_{\bar c,\bar b} \subseteq P^{M_3}_{\Delta_\varphi}$ of cardinality $< \partial^1_\Delta$ we have:

- if $d_1, d_2 \in P^M$ and $(\forall d \in A)(d_1 \le d \equiv d_2 \le d)$ then $M_3 \models$ "$\varphi[\bar a^{M_3,\varphi}_{\bar c,d_1}, \bar b] \equiv \varphi[\bar a^{M_3,\varphi}_{\bar c,d_2}, \bar b]$".

[Why? Straightforward because T is definably stable and $<^{M_3}$ is a linear well ordering but we give details. Let $\partial^1_\varphi < \theta$ be large enough.

Suppose $M_3 \models T_3$ hence $(|M_3|, <^{M_3})$ is a well ordering. Without loss of generality $|M_3|$ is an ordinal α_* and $<^{M_3}$ is the usual order so $\mathrm{cf}(\alpha_*) \ge \theta$. Suppose $\bar c \in \vartheta_\varphi(M_3)$ and $\bar b \in {}^{\ell g(\bar y)}(|M_3|)$ and we shall prove that there is $A = A^{\varphi,M_1}_{\bar c,\bar b} \subseteq P^{M_2}_{\Delta_\varphi}$ as required.

Toward this we choose by induction on n a set A_n such that:

$(*)_{5.1}$ (a) $A_n \subseteq P^{M_3}$ has cardinality $\le \partial^1_\varphi$

(b) $m < n \Rightarrow A_m \subseteq A_n$ and $A_0 = \{\min\{\alpha \in P^{M_3} : \bar b \subseteq M^{<\alpha}_3\}\}$

(c) if $\alpha \in A_n$ and $\mathrm{cf}(M^{<\alpha}_3 \cap P^{M_3}) \ge \theta_{\Delta_\varphi}$, then there are $\psi_*, \bar c_\alpha$ such that (letting $\psi_{\varphi[*]} = \psi(\bar y_{[*]}, \bar z_*)$: we have

(α) $\bar c_\alpha \in {}^{\ell g(\bar z_*)}(M^{<\alpha}_3)$

(β) if $\bar a \in (M^{<\alpha}_3)$ then $M_3 \models \varphi[\bar a, \bar b]$ iff $M_3 \models \psi_*[\bar a, \bar c_\alpha]$

(γ) $\bar c_\alpha \subseteq M^{<\beta}_3$ for some $\beta < \alpha$ which belongs to A_{n+1}

(d) if $\alpha \in A_n$ and $\mathrm{cf}(M_1^{<\alpha} \cap P^{M_3}_{\Delta_\varphi}, <^{M_3}) < \theta_{\Delta_\varphi}$ then $(A_{n+1} \cap M_3^{<\alpha} \cap P^{M_3})$ is cofinal in $(P^{M_3}, <^{M_3})$.

Recall $(P^{M_3}_{\Delta_\varphi}, <)$ is a well order of cofinality $\ge \theta$.

Now let $A = \bigcup\limits_n A_n$ and we shall prove $\bullet$ of $(*)_5$; suppose $d_1, d_2 \in P^{M_3} \backslash A$ and $(\forall d \in A)(d < d_1 \equiv d < d_2)$. If $\bar{b} \subseteq {}^{\min(d_1,d_2)>}(M_3)$ then d_1, d_2 are $<^{M_3}$-above the unique member of A_0, hence clearly $M_3 \models$ "$\varphi[\bar{a}^{M_3}_{\bar{c},d_1}, \bar{b}] \equiv \varphi[\bar{a}^{M_3}_{\bar{c},d_2}, \bar{b}]$" as required.

If not, let $d'' \in A \subseteq P^{M_3}$ be minimal such that $d_1 < d''$ (equivalently $d_2 < d''$). Now d'' cannot be the first, a successor or of cofinality $< \theta$ in $(P^{M_3}, <^{M_3})$ hence $(M_3^{<d''} \cap P^{M_3})$ has cofinality $\ge \theta_{\Delta_\varphi}$ (see $(*)_{5.1}(d)$ and use $(*)_{5.1}(c)$). Let $\alpha = d''$ and $\beta = \sup(A \cap \alpha)$, by $(*)_{5.1}(c)(\gamma)$ we have $\bar{c}_\alpha \subseteq M_3^{<\beta}$ so by $(*)_{5.1}(c)(\beta)$ again $M_3 \models$ "$\varphi[\bar{a}^{M_3}_{\bar{c},d_1}, \bar{b}] \equiv \varphi[\bar{a}^{M_3}_{\bar{c},d_2}, \bar{b}]$". So we are done proving $(*)_5$.]

$(*)_6$ if $\varphi = \varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\theta,\theta}(\tau_T)$, for $\partial^2_\varphi < \theta$ large enough, if $M_3 \models T_3, \bar{c} \in \vartheta_i(M_3), \bar{b} \in {}^{\ell g(\bar{y})}(M_3)$ <u>then</u> for some $B \subseteq Q^{M_3}$ of cardinality $< \partial^2_\varphi$ and for some truth value $\mathbf{t}$ we have

- if $\alpha \in Q^{M_3} \backslash B$ then $M_3 \models$ "$\varphi[\bar{a}^{M_3}_{\bar{c},d}, \bar{b}]^{\mathrm{if}(\mathbf{t})}$".

[Why? As otherwise we get contradiction to φ is 1-stable. In details, let $M_3, \bar{b}$ be a counterexample; let $\partial_2 < \theta$ be large enough and $\kappa = \mathrm{cf}(|M_3|, <^{M_3})$ let $\kappa \ge \theta$; and let $\langle d_i : i < \kappa \rangle$ be $<^{M_3}$-increasing cofinal and $d_i \in Q^{M_3}$.

Now $\bar{b} \in {}^\zeta(M_3)$ hence there is $d_* \in Q^M$ such that $\bar{b} \subseteq M_3^{<d_*}$; so for some truth value, $d_* \le^{M_3} d \Rightarrow M_3 \models$ "$\varphi[\bar{a}^{M_3}_{\bar{c},d}, \bar{b}]^{\mathrm{if}(\mathbf{t})}$".

Let $A^{\varphi,M_3}_{M_3,\bar{c},\bar{b}}$ be as in $(*)_5$ and $E = E_{M_3,\bar{c},\bar{b}} = \{(d_1, d_2) : d_1, d_2 \in Q^{M_3}$ and $(\forall d \in A^{\varphi,M_3}_{M_3,\bar{c},\bar{b}})(d < d_1 \equiv d < d_2 \wedge d = d_1 \equiv d = d_2)\}$ is an equivalence relation and let $A^+_{M_3,\bar{c},\bar{b}} = \{d \in Q^M : d/E_{M_3,\bar{c},\bar{b}}$ has $\le \partial_2$ members$\}$. Now if $d \in Q^{M_3} \backslash A^+_{M_3,\bar{c},\bar{b}} \Rightarrow M_3 \models$ "$\varphi[\bar{a}^{M_3}_{\bar{c},d}, \bar{b}]^{\mathrm{if}(\mathbf{t})}$", we are done, otherwise let d^* be a counterexample. Let $d^*_1 = \min(d^*/E)$ and $d^*_2 \in (A_{M_3,\bar{c},\bar{b}} \backslash M_3^{<d^*})$ and let $d^*_3 = d_*$.

Now M_3 satisfies

$(*)_{6.1}$ (a) $M_3 \models$ "$d^*_1 < d^*_2 < d^*_3 \wedge Q(d^*_1) \wedge Q(d^*_2) \wedge Q(d^*_3)$

(b) for some $\bar{b}' \in {}^\varepsilon(M_3)$ we have $M_3 \models (\forall t) \in [d^*_1 < t < d^*_2 \wedge P(t) \rightarrow \varphi(\langle F_i(t), \bar{c}) : i < \varepsilon, \bar{b}']^{\mathrm{if}(\neg\mathbf{t})}]$ and $M^*_3 \models (\forall t)[d^*_3 < t \wedge P(t) \rightarrow \varphi(\langle F_i(t), \bar{c}) : i < \varepsilon\rangle, \bar{b})^{\mathrm{if}(\mathbf{t})}]$.

By the demand on Q^{M_3}

- for every $d'_1 < d'_2 < d'_3$ from Q^{M_3} for some $\bar{b}' \in {}^\zeta(M_3)$ we have $M_3 \models (\forall t)[d'_1 < t < d'_2 \wedge P(t) \rightarrow \varphi(\langle F_i(t, \bar{c}) : i < \varepsilon\rangle, \bar{b}')^{\mathrm{if}(\neg\mathbf{t})}]$ and $M^*_3 \models (\forall t)[d'_3 < t \wedge P(t) \rightarrow \varphi(\langle F_i(t, \bar{c}) : i < \varepsilon), \bar{b}')^{\mathrm{if}(\mathbf{t})}]$.

From this clearly T has the order property, contradiction, so $(*)_6$ holds indeed.]

Now the required saturation follows. That is, assume $\bar{c} \in \vartheta(M_3), p_{\bar{c}} = \{\varphi(\bar{x}, \bar{b}) :$

{x32} $M \models \psi[\bar{b}, \bar{c}]\}$, so a type of cardinality $\le \|M\|^{|\ell g(\bar{x})|}$ but $\|M\| = \|M\|^{<\theta}$ by 0.27, and every $\varphi(\bar{x}, \bar{b}) \in p_{\bar{c}}$ is realized by every $\bar{a}^{M_3}_{\bar{c},d}$ for every $d \in Q^{M_3}$ except possibly $\le \partial_2$ many. As $|Q^M| = \|M\|$ by $(*)_5(c)$, we are done. $\square_{3.17}$

We can now sum up, giving full characterization of two versions of local minimality. Note that at last we state the main results 3.19, 3.20. {a33g} {a33}

Conclusion 3.19. *Assume T is a complete $\mathbb{L}(\tau_T)$-theory.*

Assume $\lambda = \lambda^{<\theta} \ge 2^\theta + |T|$, <u>then</u> T is locally (λ, θ)-minimal <u>iff</u> T is 1-stable with θ-n.c.p.

Proof. <u>Case 1</u>: T has the θ-c.p.

Let $T_1 \supseteq T$. Let $D_1 \in \mathrm{ruf}_\theta(\lambda)$ and D_2 be an e.g. normal ultrafilter on θ and so $D = D_1 \times D_2 \in \mathrm{ruf}_\theta(\lambda \times \theta)$. If $M \models T_1$ then $M^{\lambda \times \theta}/D \cong (M^\lambda/D_1)^\theta/D_2$; let $M_0 = M, M_1 = M_0^\lambda/D$ and $M_2 = M_1^\theta/D$, all models of T_1. So $M^{\lambda \times \theta}/D$ is isomorphic to M_1^θ/D and the latter is not locally $((2^\theta)^+, \theta, \mathbb{L}_{\theta,\theta}(\tau_T))$-saturated by 3.6, (hence not $(\lambda^+, \theta, \mathbb{L}_{\theta,\theta})$-saturated). {a25}

<u>Case 2</u>: T is 1-unstable.

Let $T_1 \supseteq T$ and $M \models T_1$ and M^+ be a θ-complete expansion of M.

Now apply Claim 3.14 to the theory T_1 so for some $M_1 \models T_1$, so for some {a28} $(\theta, \theta) -$ l.u.f.t. $\mathbf{x}$ we have $\theta = \mathrm{cf}(\text{l.u.p.}_{\mathbf{x}}(\theta, <))$, exists by 0.26(3), hence the model {x31} l.u.p.$_{\mathbf{x}}(M)$ is not locally $(\theta^+, \theta, \mathbb{L}_{\theta,\theta}(\tau_T))$-saturated so we are done.

<u>Case 3</u>: T is 1-stable with θ-n.c.p.

Use Theorem 3.17. $\square_{3.19}$ {a32} {a33g}

Conclusion 3.20. *Assume $\lambda = \lambda^{<\theta} \ge 2^\theta + |T|$ and T is a complete $\mathbb{L}_{\theta,\theta}(\tau)$-theory of cardinality $\le \lambda$. <u>Then</u> T is $\triangleleft_{\lambda,\theta}$-minimal <u>iff</u> T is definably stable with the $\theta -$ n.c.p. <u>iff</u> T is 1-stable with the θ-n.c.p.*

Proof. The third and second clauses are equivalent by 3.3(4). The proof splits to {a22} cases and is similar to the proof of 3.19. {a33}

<u>Case 1</u>: T has the $\theta -$ c.p.

Exactly as in the proof of 3.19. {a33}

<u>Case 2</u>: T is definably unstable

By Claim 1.4(1), T is 1-unstable. Again use 3.14 but now using $\mathbf{x}$ which is simply {a28} $D \in \mathrm{ruf}_\theta(\lambda)$; true 3.14 say "for some M_1" but recall 2.5. {a28}

<u>Case 3</u>: T is definably stable with the $\theta -$ n.c.p.

Use 3.15. $\square_{3.20}$ {a30} {a36}

Claim 3.21. *1) If the set $\mathrm{spec}(\varphi(\bar{x}, \bar{y}), T)$ includes every regular $\partial < \theta$ or just belongs to every normal ultrafilter on θ and $\lambda \ge \theta$ <u>then</u> T is $\blacktriangleleft_{\lambda,\theta}$-maximal.*

1A) Moreover, if $\mathrm{spec}(\varphi(\bar{x}, \bar{y}), T)$ belongs to every normal ultrafilter on θ and $\lambda \ge 2^\theta$ <u>then</u> for every theory $T_0 \supseteq T$ of cardinality $\le \lambda$ for some $\mathbb{L}_{\theta,\theta}$-theory T_1 extending T_0 of cardinality λ for every model M_1 of $T_1, M_1 \restriction \tau_T$ is not locally θ^+-saturated; so T is $\blacktriangleleft_{\lambda,\theta}$-maximal.

1B) In (1A) we can replace "$\lambda \ge 2^\theta$" by "$\lambda \ge \theta$ and $\theta \backslash \mathrm{spec}(\varphi, T)$ is not in the (λ, θ)-weakly compact ideal on θ (see in the proof)".

2) There is a model $M_* = (\theta, E^M), E^M$ *an equivalence relation such that* $T = \mathrm{Th}_{\mathbb{L}_{\theta,\theta}}(M)$ *satisfies* $\mathrm{spec}(xEy, T) = \theta \cap \mathrm{Card}$ *hence* T *is* $\blacktriangleleft_{\lambda,\theta}$*-maximal for every* λ *and even* $\triangleleft^*_{\lambda,\bar{\mu},\theta}$*-maximal.*
3) Assume κ *is supercompact with the Laver diamond. There is a sequence of models* $\langle M_A : A \subseteq \theta\rangle$ *such that:*

- (a) $M_A = (\theta, E_A)$ *for* $A \subseteq \theta, E_A$ *an equivalence relation on* θ *such that letting* $T_A = \mathrm{Th}(M_A)$ *we have*
- (b) *for* $\lambda = \lambda^{<\theta}, T_A \blacktriangleleft_{\lambda,\theta} T_B$ *iff* $A \subseteq B$ *iff* $T_A \trianglelefteq^*_{\lambda,\theta} T_B$

{a25} *Proof.* 1) By 3.6, because for θ-complete which is not θ^+-complete[8] ultrafilter on a
{x8} set I recalling 0.16(3) and "$\prod\limits_{\alpha<\theta} \alpha/D$ has cardinality θ" we know that $\theta \in \{\prod\limits_{s\in I} \theta_s/E : \theta_s \in \mathrm{spec}(\varphi(\bar{x},\bar{y}))a\}$.
1A) To make the rest of the proof be also a proof of part (1B), let $\mathbb{B}$ be the Boolean Algebra $\mathscr{P}(\theta)$ and let $\mathscr{F} = \{f : f \in {}^\theta\theta \text{ satisfies } f(\alpha) < 1+\alpha\}$. Also without loss of generality $|T| \le \theta$.

Let M_0 be a model of T_0 such that letting $M = M_0{\restriction}\tau$ we have $\mathscr{H}(\theta) \subseteq M, M{\restriction}\mathscr{H}(\theta) \prec_{\mathbb{L}_{\theta,\theta}} M$. Let M_1 be an expansion of M by $\le \lambda$ symbols including $P^{M_1} = \mathscr{H}(\theta), P^{M_1}_u = u$ for $u \in \mathbb{B}, F^M_f{\restriction}\theta = f$ for $f \in \mathscr{F}$ and the relations $R_1 = (\in {\restriction}\mathscr{H}(\theta))$ and $R^{M_1}_2 = \{(\beta,\partial)\char`^\bar{a}_{\partial,\beta} : \partial \in \mathrm{spec}(\varphi,T), \beta < \partial\}$, where $\{\varphi(\bar{x},\bar{a}_{\partial,\beta}) : \beta < \partial\}$ exemplified $\partial \in \mathrm{spec}(\varphi,T)$ in the model M.

Lastly, let $T_1 = \mathrm{Th}_{\mathbb{L}_{\theta,\theta}}(M_1) \cup \{P_\theta(c) \wedge (\exists^{\ge\partial} y)(y \in c) : \partial < \theta\}$ recalling $\theta \in \mathbb{B}$. The rest should be clear but we shall give details.

Let M_2 be a model of T_1, so $(P^{M_2}_\theta, \in^{M_2} {\restriction}P^{M_2}_\theta)$ is a linear order which is a well ordering, so without loss of generality $P^{M_2}_\theta = \alpha_*$ for some ordinal α_* and $\in^{M_2}{\restriction}P^{M_2}$ is the usual order and $c^{M_2} \in P^{M_2}_\theta = \alpha_*$ is necessarily $\ge \theta$, so $\theta \in P^{M_2}_\theta$.

Let $D = \{u \in \mathbb{B} : M_2 \models P_u(\theta)\}$ so this is an ultrafilter on the Boolean algebra $\mathbb{B}$ which is θ-complete and normal (for $\mathscr{F}$, i.e. $(\forall f \in \mathscr{F})(\exists A \in D)[f{\restriction}A$ is constant$]$). By the assumption of the claim, $u_* := \mathrm{spec}(\varphi,T) \in D$, so $M_2 \models$ "$P_{u_*}(\theta)$" and let $p_* = \{\varphi(\bar{x},\bar{a}) : \langle\beta,\theta\rangle\char`^\bar{a} \in R^{M_2}_2$ for some $\beta < \theta\}$.

Now

- $p_*(\bar{x})$ is not realized in M_2, i.e. $M_2{\restriction}\tau_T$.

[Why? Because M_1 satisfies the sentence saying this even replacing θ by any member of $P_{\mathrm{spec}(\varphi,T)}$ and $M_2 \models T_2$.]

- if $\partial < \theta$ then every subset of p_* of cardinality $\le \partial$ is satisfiable in $M_2{\restriction}\tau_T$.

[Why? Similarly.]
1B) The proof is as in (1A), but the demand

- $(*)$ there is $\mathbb{B} \subseteq \mathscr{P}(\theta)$ of cardinality λ, including $[\theta]^{<\theta}$ but we also have $\mathscr{F} \subseteq \{f \in {}^\theta\theta : (\forall\alpha < \theta)(f(\alpha) < 1+\alpha)\}$ of cardinality $\le \lambda$ satisfying $\alpha < \theta \wedge f \in \mathscr{F} \Rightarrow f^{-1}\{\alpha\} \in \mathbb{B}$ such that there is no uniform θ-complete ultrafilter D on $\mathbb{B}$ such that $f \in \mathscr{F} \Rightarrow (\exists\alpha)(f^{-1}\{\alpha\} \in D)$.

[8] being (λ,θ)-regular is a stronger condition

In the proof "the ultra-filter D is normal for $\mathscr{F}$" means $f \in \mathscr{F} \Rightarrow (\exists \alpha < \theta)(f^{-1}\{\alpha\} \in D)$. By the way this implies θ-complete when $\mathscr{F}$ is the set of all regressive $f \in {}^{\theta}\theta$. Why? If $A = \bigcup_{i<\partial} A_i$, let $f : \theta \to \theta$ be $f(\alpha)$ is 0 if $\alpha < \partial$ and if $\min\{i < \partial : \alpha \in A_i\}$ if $\alpha \geq \partial$.
2) E.g. $E^M = \{(\alpha, \beta) : \alpha + |\alpha| = \beta + |\beta|\}$ satisfies the first demand; the first "hence" follows by (1), the second hence by (1B).
3) Let $C = \{\mu : \mu < \theta$ is strong limit$\}$, let $\langle S_i : i < \theta \rangle$ be a partition of C to θ unbounded subsets of C such that for each i there is a normal ultrafilter D_i^* on θ to which S_i belongs; moreover, for every $\lambda \geq \theta$ for some normal ultrafilter D on $[\lambda]^{<\theta}$ the set $\{u \in [\lambda]^{<\theta} : u \cap \theta \in S_i\}$ belongs to D. Well known to exist, see Kanamori-Magidor [KM78]. For $A \subseteq \theta$, let E_A be an equivalence relation on θ such that $\{|(\alpha/E_A| : \alpha < \theta\} = \cup\{S_i : i \in A\}$. So the following claim 3.22 will suffice. {a38} $\square_{3.21}$

{a38}

Claim 3.22. *Assume $\theta < \lambda = \lambda^{<\theta}$ and $f_* : \theta \to \theta$ satisfies $\alpha < \theta \Rightarrow \alpha < f_*(\alpha) \in \mathrm{Card}$ and there is a transitive class $\mathbf{M} \supseteq {}^{\lambda}\mathbf{M}$, a model of ZFC including the ordinals and an elementary embedding $\mathbf{j}$ of $\mathbf{V}$ into $\mathbf{M}$ with critical point θ such that $(\mathbf{j}(f_*))(\theta) = \lambda$.*

Let E be a thin enough club of $\theta, S_1 = \mathrm{Rang}(f_* \restriction E)$ and let $S_2 = \{2^\mu : \mu \in S_1\}$.

Then there is $D \in \mathrm{ruf}_\theta(\lambda)$ such that we have:

(a) *if $f : \lambda \to S_1$ then the cardinal $\prod_{\alpha<\lambda} f(\alpha)/D$ is $< \theta$ or is $\geq \lambda$*

(b) *for some $f : \lambda \to S_1$ we have $\prod_{\alpha<\lambda} f(\alpha)/D$ is λ*

(c) *if $f : \lambda \to S_2$ then the cardinality $\prod_{\alpha<\lambda} f(\alpha)/D$ is $< \theta$ or is $\geq 2^\lambda$*

(d) *for some $f : \lambda \to S_2$ we have $\prod_{\alpha<\lambda} f(\alpha)/D$ is 2^λ.*

Proof. Let $E = \{\mu < \theta : \mu$ strong limit and $\mathrm{Rang}(f_* \restriction \mu) \subseteq \mu\}$, it is the club of θ, mentioned in the claim. Let $S_1 = \{f_*(\mu) : \mu \in E\}$ and $S_2 = \{2^{f_*(\mu)} : \mu \in S_1\}$.

Let D be the following normal ultrafilter on $I = [\lambda]^{<\theta}$

$$\{\mathscr{U} \subseteq I : \{\mathbf{j}(\alpha) : \alpha < \lambda\} \in \mathbf{j}(\mathscr{U})\}.$$

Hence the following set belongs to $D : \{s \in I : s \cap \theta \in E$ and $|s| = f_*(s \cap \theta)\}$.

Clearly D is a θ-complete (λ, θ)-regular ultrafilter on a set I, even normal and fine, and the set I has cardinality $\lambda^{<\theta} = \lambda$, so (by renaming) can serve as D in the claim.

Let $G_s : \mathscr{P}(s) \to |\mathscr{P}(s)|$ be one to one onto for each $s \in I$.

By the normality of D, in $(\theta, <)^I/D$, the θ-th element is f_0/D where $f_0 : I \to \theta$ is defined by $f_0(s) = \min(\theta \backslash s)$.

Now clause (b) holds for the function $f_* \circ f_0$, because $\prod_{s\in I}(f_* \circ f_0)(s), <)$ is isomorphic to $(\lambda, <)$ by the choice of D, hence $f_* \circ f_0/D$ is the λ-th member of $(\theta, <)^I/D$. As for clause (a) if $g/D \in \theta^I/D, \mathrm{Rang}(g) \subseteq S_1$ and $g <_D f_* \circ f_0$ then by the normality of $D, \prod_s g(s)/D$ has cardinality $< \theta$.

Note that $f_* \circ f_0(s) = \min\{\gamma \in S_1 : \gamma > \sup(s \cap \theta)\}$.

To prove clause (d) let $f_2 \in {}^I\theta$ be $f_2(s) = \min\{\gamma \in S_2 : \gamma > \sup(s \cap \theta)\}$, so $f_2(s) = 2^{f_*(s\cap\theta)}$ when $s \cap \theta \in E$ and easily $\prod\limits_{s\in I} f(s)/D$ is of cardinality $\le \theta^I = \theta^\lambda = 2^\lambda$. In fact, it is of cardinality 2^λ as exemplified by $\langle f_{\mathscr{U}}/D : \mathscr{U} \subseteq \lambda\rangle$ where for $\mathscr{U} \subseteq \lambda$ let $f_{\mathscr{U}} : I \to \theta$ be $f_{\mathscr{U}}(s) = G_s(\mathscr{U} \cap s)$. Also clause (c) follows, similarly to the proof of clause (a). $\square_{3.22}$

§ 4. Global c.p. and full minimality

{a43}

Definition 4.1. 1) Let $T \subseteq \mathbb{L}_{\theta,\theta}(\tau_T)$ be complete. We say T has the global θ-c.p. (negation: global θ- n.c.p.) when for some pair $(\bar{\varphi}, \bar{\partial})$ it has the global $(\bar{\varphi}, \bar{\partial})$-c.p., see below.
2) T has the global $(\bar{\varphi}, \bar{\partial})$-c.p. when for some S and ε:

- (a) $S \subseteq \theta$ belongs to some normal ultrafilter on θ and is a set of cardinals
- (b) $\varepsilon < \theta$ and $\bar{\varphi} = \langle \varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{y}_{\varphi_\alpha}) : \alpha < \theta \rangle$ where $\varphi_\alpha \in \mathbb{L}_{\theta,\theta}(\tau_T)$
- (c) $\bar{\partial} = \langle \partial_\alpha : \alpha \in S \rangle$ and ∂_α is a cardinal $\in [\alpha, \theta)$
- (d) if $\alpha \in S$ then $\partial_\alpha \in \mathrm{spec}(\bar{\varphi} \restriction \alpha, T)$, see Definition 3.1(3),(4). {a17}

{a44}

Observation 4.2. *If T has the $\theta-$c.p. then T has the global c.p..*

{a46}

Claim 4.3. *Assume D is a normal ultrafilter on θ and T has the global $(\bar{\varphi}, \bar{\partial})$-c.p., $S = \mathrm{Dom}(\bar{\partial}) \in D$ and M is a model of T and $\chi = \theta^\theta/D$ or just $\chi = \Pi\bar{\partial}/D$.*
1) $N = M^\theta/D$ is not fully $(\chi^+, \theta, \mathbb{L}_{\theta,\theta})$-saturated.
2) If $T_1 \supseteq T$ then for some model M_1 of T_1, the model $(M_1 \restriction \tau(T))^\theta/D$ is not fully $(\chi^+, \theta, \mathbb{L}_{\theta,\theta})$-saturated.

Proof. 1) Let $M \models T$ and for $i \in S$ let $\langle \varphi_{\xi(i,j)}(\bar{x}_{[\varepsilon]}, \bar{a}_{i,j}) : j < \partial_i \rangle$ witness $\partial_i \in \mathrm{spec}(\bar{\varphi} \restriction i, T)$ and $j < \partial_i \Rightarrow \xi(i,j) < i$. Let ∂'_ε be ∂_ε if $\varepsilon \in S$ and 1 if $\varepsilon \in \lambda \backslash S$. We can fix $\bar{f} = \langle f_\alpha : \alpha < \chi \rangle$ such that $f_\alpha \in \prod\limits_{\varepsilon<\theta} \partial'_\varepsilon$ and $\bar{f}$ is a set of representatives for $\prod\limits_{i<\theta} \partial'_i/D$. For each $\alpha < \chi$, as D is a normal ultrafilter on θ to which S belongs and $i \in S \Rightarrow \xi(i, f_\alpha(i)) < i$ clearly for some $\zeta(\alpha) < \theta$ we have $S_\alpha := \{i < \theta : i \in S$ and $\xi(i, f_\alpha(i)) = \zeta(\alpha)\} \in D$ and let $\bar{a}^*_\alpha \subseteq N$ be of length $\ell g(\bar{y}_{\varphi_{\zeta(\alpha)}})$ such that $\bar{a}_\alpha = \langle \bar{a}_{i,f_\alpha(i)} : i \in S_\alpha \rangle/D$ and let $\Gamma = \{\varphi_{\zeta(\alpha)}(\bar{x}_{[\varepsilon]}, \bar{a}_\alpha) : \alpha < \chi\}$.
Of course,

- $(*)_0$ Γ has cardinality $\le \chi$
- $(*)_1$ Γ is a set of $\mathbb{L}_{\theta,\theta}(\tau_T)$-formulas with parameters from N
- $(*)_2$ Γ is $(<\theta)$-satisfiable M.

[Why? Let $u \subseteq \chi$ have cardinality $< \theta$, hence $\zeta(*) = \sup\{\zeta(\alpha) : \alpha \in u\}$ is $< \theta$ and let $S_* = \{i \in S$: if $\alpha \in u$ then $f_\alpha(i) = \zeta(\alpha)$ and $|u| < i\}$. Clearly $S_* \in D$ and if $i \in S_*$ then $\{\varphi_{\zeta(\alpha)}(\bar{x}_{[\varepsilon]}, \bar{a}_{i,f_\alpha(i)}) : \alpha \in u\} \subseteq \{\varphi_{\xi(i,j)}(\bar{x}_{[\varepsilon]}, \bar{a}_{i,j}) : j < \partial_i\}$ and[9] has cardinality $< |i| < \partial_i$ hence is realized in M, so $M \models (\exists \bar{x}_{[\varepsilon]}) \bigwedge\limits_{\alpha\in u} \varphi_{\zeta(\alpha)}(\bar{x}_{[\varepsilon]}, \bar{a}_{i,f_\alpha(i)})$. Hence $N \models (\exists \bar{x}_{[\varepsilon]}) \bigwedge\limits_{\alpha\in u} \varphi_{\zeta(\alpha)}(\bar{x}_{[\varepsilon]}, \bar{a}_\alpha)$ so we are done.]

- $(*)_3$ Γ is not realized in N.

[Why? As in the proof of Case 2 of 3.14, without loss of generality $\theta \subseteq M$. Let $\tau^* = \tau_T \cup \{P_\zeta, Q, <, R, F : \zeta < \theta\}$ where P_ζ is a $(2 + \ell g(\bar{y}_{\varphi_\zeta}))$-place predicate, Q is unary, R is a $(1+\varepsilon)$ place predicate and F a unary function symbol. {a28}

For $i \in S$ let $M^+_i = (M, Q^{M^+_i}, P^{M^+_i}_\zeta, <^{M^+}, R^{M^+_i}, F^{M^+_i})_{\zeta<\theta}$ where

$(*)_{3.1}$ • $Q^{M^+_i} = \partial_i$

[9] The $\le \partial_i$ is for technical reasons, anyhow $\partial_i = |\partial_i + 1|$.

- $<^{M_i^+}$ the order on ∂_i
- $P_\zeta^{M_i^+} = \{\langle \zeta, j\rangle\hat{\ }\bar{a}_{i,j} : j < \partial_i \text{ and } \xi(i,j) = \zeta\}$
- $R^{M_i^+} = \{\langle j\rangle\hat{\ }\bar{b} : j < \partial_i$ and $\ell g(\bar{b}) = \varepsilon$ and $M \models \varphi_{\xi(i,j)}[\bar{b}, \bar{a}_{i,j}]\}$
- $F^{M_i^+}(j) = \xi(i,j) < i$.

Let $N^+ = \prod\limits_{i \in S} M_i^+/D$, so $N = N^+ \restriction \tau_T$, let $\mathbf{i} = \langle i : i \in S\rangle/D \in N^+$ and $\partial = \langle \partial_i : i \in S\rangle/D \in N^+$

$(*)_{3.2}$ in N^+ there is no $\bar{b} \in {}^\varepsilon(N^+)$ such that for every $j \in Q^{N^+}, N^+ \models$ "$j < \partial \rightarrow R[j, \bar{b}]$"

$(*)_{3.3}$ in N^+ if $j \in Q^{N^+}$ and $F^{N^+}(j) = \zeta < \theta$ then $N^+ \models (\forall \bar{x}_{[\varepsilon]})(\forall \bar{y})[P_\zeta(j, \zeta, \bar{y}) \rightarrow R(j, \bar{x}_{[\varepsilon]}) \equiv \varphi_\zeta(\bar{x}_{[\varepsilon]}, \bar{y})]$.

Let

$(*)_{3.4}$ $\Gamma = \{\varphi_\zeta(\bar{x}_{[\varepsilon]}, \bar{a})$: for some $j \in Q^{N^+}, \zeta = F^{N^+}(j)$ we have $N^+ \models$ "$P_\zeta(j, \zeta, \bar{a})$"$\}$.

Together

$(*)_{3.5}$ Γ is a set of χ formulas from $\mathbb{L}_{\theta,\theta}(\tau_T)$ with parameters from N which is $(<\theta)$-satisfiable in N but not realize in N so we are done.

2) Follows by (1). $\square_{4.3}$

{a49d}

{a35g} **Discussion 4.4.** Considering Theorem 3.20, 4.9 it is natural to wonder what are the implications between "T has the $\theta-$n.c.p." and "T has the global $\theta-$n.c.p.".

{a40} By 4.5 below the second does not imply the first and by 4.2, the first implies the second.

{a49} **Claim 4.5.** *There are a vocabulary $\tau, |\tau| \le \theta$ and a complete $T \subseteq \mathbb{L}_{\theta,\theta}(\tau)$ which have θ-n.c.p. but has the global c.p.*

Proof. For $i < \theta$ let ∂_i be an infinite cardinal $\in [i, \theta)$. Let $\tau = \{E, P_\zeta : \zeta < \theta\}, E$ a two-place predicate, P_ζ a unary predicate.

We choose a τ-model M as follows:

(a) its universe is $\theta \times \theta$

(b) $E^M = \{((i, j_1), (i, j_2)) : i < \theta \text{ and } j_1, j_2 < \theta)\}$, an equivalence relation

(c) $P_\zeta^M \subseteq |M|$ for $\zeta < \theta$

(d) for $i < \theta$, letting $a_i = (i, 0), A_i = a_i/E^M$, for every $\eta \in {}^i2$ the following are equivalent:

  (α) there are θ elements $a \in A_i$ such that $(\forall \zeta < i)(a \in P_\zeta^M \equiv \eta(\zeta) = 1)$

  (β) the set $\{a \in A_i$: if $\zeta < i$ then $a \in P_\zeta^M \equiv \eta(\zeta) = 1\}$ has cardinality $\neq \partial_i$

  (γ) the set $\{j < i : \eta(j) = 1\}$ has cardinality $< 1 + |i|$.

We shall check that $T := \mathrm{Th}_{\mathbb{L}_{\theta,\theta}(\tau)}(M)$ is as required.

Let $A'_i := \{a \in A_i$: if $\iota < i$ then $a \in P_\iota^M\}$; it is a subset of A_i of cardinality exactly ∂_i by clause (d)(α) above

$\boxplus_1$ T has global θ-c.p.

Why? Let $\varepsilon = 1, \bar{y} = \langle y_0, y_1 \rangle$ and $\varphi_i = \varphi_i(x, \bar{y}) = xEy_0 \wedge P_i(x) \wedge x \neq y_1$ for $i < \theta$ and let $\bar{\varphi} = \langle \varphi_i : i < \theta \rangle$.

For $i < \theta$ let $\Gamma_i = \{\varphi_j(x, \langle a_i, b \rangle) : b \in A'_i$ and $j < i\}$

- Γ_i is formally is as required for witnessing $\partial_i \in \text{spec}(\bar{\varphi} \restriction i, T)$ in particular $|\Gamma_i| = \partial_i$.

[Why? As $|A'_i| = \partial_i \geq i$.]

- Γ_i is not realized.

[Why? As $\{xEa_i \wedge x \neq b \wedge P_\zeta(x) : b \in A'_i$ and $\zeta < i\}$ is not realized.]

- if $\Gamma \subseteq \Gamma_i$ has cardinality $< \partial_i$ then Γ is realized.

[Why? As all but $< \partial_i$ members of A'_i realize Γ.]

So $\boxplus_1$ holds indeed.

$\boxplus_2$ T has the θ-n.c.p.

[Why? Let $\varphi = \varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\zeta]})$ and so for some $\kappa < \theta, \varphi$ belongs to $\mathbb{L}_{\theta,\theta}(\{E, P_\zeta : \zeta < \kappa\})$, hence M satisfies:

- if $a \in M, a \notin a_j/E^M$ for $j < \kappa^+$ then for any $\eta \in {}^\kappa 2$ the set $\{b : b \in a/E^M$ and $\zeta < \kappa \Rightarrow b \in P^M_\zeta \leftrightarrow \eta(\zeta) = 1\}$ has cardinality θ.

The rest should be clear.

$\boxplus_3$ T is 1-stable.

[Why? Obvious.]

Together we are done. $\square_{4.5}$

{a52}

Theorem 4.6. *Assume T is complete of cardinality θ and T is definably stable with global θ-n.c.p. and $\lambda = \lambda^{<\theta}$.*

1) T is $\triangleleft^{\rm ful}_{\lambda,\theta}$-minimal.

2) Moreover, if $D \in \text{ruf}_{\lambda,\theta}(I)$ and $\theta^I/D > \lambda$ and M is a model of T <u>then</u> M^I/D is fully $(\lambda^+, \theta, \mathbb{L}_{\theta,\theta})$-saturated.

Proof. 1) By part (2).

2) As T is definably stable we can use 1.7 and as T has $\theta -$ n.c.p. by 4.2, we can use 3.1, 3.2.

Let $M \models T$ and $N = M^I/D$, let $\varepsilon < \theta, A \subseteq N, |A| \leq \lambda$ and $p_0 \in \mathbf{S}^\varepsilon(A, N)$ and we shall prove that $p_0(\bar{x}_{[\varepsilon]})$ is realized; by 2.5 and 3.15 without loss of generality M is locally $(\lambda^+, \theta, \mathbb{L}_{\theta,\theta})$-saturated. Let $\{\varphi(\bar{x}_{[\varepsilon]}, \bar{y}_{[\zeta]}) : \varphi \in \mathbb{L}_{\theta,\theta}(\tau_T)$ and $\zeta < \theta\}$ be listed as $\langle \varphi_i(\bar{x}_{[\varepsilon]}, \bar{y}_{\zeta(i)}) : i < \theta \rangle$. Let $p_1(\bar{x}_{[\varepsilon]}) \in \mathbf{S}^\varepsilon(N)$ extends $p_0(\bar{x}_{[\varepsilon]})$ and for each $i < \theta$ let $\psi_i = \psi_i(\bar{y}_{\zeta(i)}, \bar{c}_i)$ be a formula from $\mathbb{L}_{\theta,\theta}(\tau_T)$ with parameters from N defining $p_1(\bar{x}_{[\varepsilon]}) \restriction \varphi_i$ and let $\bar{c}_\zeta = \langle \bar{c}_{\zeta,s} : s \in I \rangle / D$.

As D is a (λ, θ)-regular ultrafilter, by 0.16(2) there is $\bar{A} = \langle A_s : s \in I \rangle, A_s \in [M_s]^{<\theta}$ which is non-empty and $A = \{f_\alpha/D : \alpha < \lambda\}$ and $\alpha < \lambda \Rightarrow f_\alpha \in \prod\limits_{s \in I} A_s$ and for $i \leq \theta$ let $\Delta_i = \{\varphi_j(\bar{x}_{[\varepsilon]}, \bar{y}_{\zeta(j)}) : j < i\}$ and let $p_{s,i}(\bar{x}_{[\varepsilon]}) = \{\varphi_j(\bar{x}_{[\varepsilon]}, \bar{b}) : j < i, \bar{b} \in A_s, M \models \psi_j(\bar{b}, \bar{c}_{j,s})\}$. {x8}

For each $i < \theta$ let $\partial_i = \sup(\mathrm{spec}(\Delta_i, T))$, see 3.1(3) so $\partial_i < \theta$ and let $I_i = \{s \in I$: there is $p \in \mathbf{S}^{\varepsilon}_{\Delta_i}(A_s)$ such that $\psi_j(\bar{y}_{[\zeta(j)]}, \bar{c}_{j,s})$ defines $p \restriction \varphi_j$ for each $j < i\}$. {a17}

Now

$(*)$ $I_i \in D_i$.

[Why? Clear but we shall elaborate. Clearly for every $\gamma < \theta$, letting $\bar{y}_{j,\gamma}$ be of length $\ell g(\bar{y}_{\zeta(j)})$ the model N satisfies $\vartheta_{i,\partial}(\ldots, \bar{c}_j, \ldots)_{j<i}$ where

$$\begin{aligned}\vartheta_{i,j} = \vartheta_{i,\partial}(\ldots, \bar{z}^j, \ldots)_{j<i} \quad &:= (\forall \ldots \bar{y}_{j,\gamma} \ldots)_{j<i,\gamma<\partial}[\bigwedge_{j<i,\gamma<j} \psi_j(\bar{y}_{j,\gamma}, \bar{z}^j)^{\mathrm{if}(\gamma \text{is even})} \\ &\Rightarrow (\exists x_{[\varepsilon]})(\bigwedge_{j<i,\partial<j} \varphi_i(\bar{x}_{[\varepsilon]}, \bar{y}_{j,\gamma})^{\mathrm{if}(\gamma \text{is even})})].\end{aligned}$$

Hence $I_i \supseteq \{s \in I : M \models \vartheta_{i,\partial_i}(\ldots, \bar{c}_{j,s}, \ldots)_{j<i}\}$ and so $I_i \in D$.]

Clearly $I_i \in D$ is decreasing with i. Let $I'_\theta = \cap\{I_j : j < \theta\}$ and for $i < \theta$ let $I'_i = \cap\{I_j : j < i\} \backslash I_i$ for $i > 0$ and let $I'_0 = I \backslash I_0$ and $\langle I'_i : i < \theta\rangle$ is a partition of $I \backslash I'_\theta$ to θ sets $= \varnothing \mod D$.

If $I'_\theta \in D$, recall that M is $(\lambda^+, \theta, \mathbb{L}_{\theta,\theta})$-saturated, hence we can find $f \in {}^I M$ such that $s \in I'_\theta \Rightarrow f(s)$ realizes $p_{s,\theta}$, clearly f/D realizes p in N so we are done; hence without loss of generality $I'_\theta = \varnothing$.

Hence we can find $\mathbf{h} : I \to \theta$ such that $s \in I'_i \Rightarrow \mathbf{h}(s) = i$.

Let $\mathbf{h}_* \in {}^I\theta$ be such that $\mathbf{h}_*/D$ is the θ-th member of $(\theta, <)^I/D$ and without loss of generality $\mathbf{h}_* \le \mathbf{h}$.

Case 1: $\mathbf{h}_* <_D \mathbf{h}$.

In this case we can prove that $p_0(\bar{x}_{[\varepsilon]})$ is realized in N.

Case 2: Not Case 1.

In this case we can prove that T has global θ-c.p., contradicting an assumption. $\square_{4.6}$

{a53}

Theorem 4.7. *Assume T is complete of cardinality θ and T is 1-stable with the global $\theta -$ n.c.p. and $\lambda = \lambda^{<\theta}$. Then T is $\blacktriangleleft^{*,\mathrm{ful}}_{\lambda,\theta}$-minimal.*

{a52} *Question* 4.8. In the proof of 4.6 can we use "M is locally $(\lambda^+, \theta, \mathbb{L}_{\theta,\theta})$-saturated"?
{a52} We expect that we can prove this by combining the proofs of 4.6 and 3.17.

We now arrive to one of our main results.

{a54}

Conclusion 4.9. *Assume $\lambda \ge 2^\theta, T$ is a complete $\mathbb{L}_{\theta,\theta}(\tau_T)$-theory of cardinality θ. Then T is $\trianglelefteq^{\mathrm{ful}}_{\lambda,\theta}$-minimal iff T is definably stable and globally θ-n.c.p.*

{a5?} *Proof.* Like the proof of 3.20 by using 4.3, 4.6 instead of 3.14 and 3.15 respectively. $\square_{4.9}$

{a58}

Question 4.10. 0) What are the implications between "T has $\theta -$ n.c.p." and "T has the global $\theta -$ n.c.p.". Debt.

1) For which T, for every $T_1 \supseteq T$, for every large enough $\mu, \lambda = \lambda^\mu$ and $M_1 \neq T_2$ of cardinality λ, there is a $(\mu^+, \theta, \mathbb{L}_{\theta,\theta})$-saturated M_2 of cardinality λ such that $M_1 \prec_{\mathbb{L}_{\theta,\theta}} M_2$?

2) Can we characterize fully (λ, θ)-minimal T of cardinality θ? We have to generalize superstable, say: every $p \in \mathbf{S}^{\varepsilon}(M)$ is almost definable over some $A \in [M]^{<\theta}, \lambda = \lambda^{<\theta} \geq 2^{\theta} + |T|, T$ a complete $\mathbb{L}_{\theta,\theta}(\tau_T)$-theory

References

[Be85] Jon Barwise and Solomon Feferman (editors), *Model-theoretic logics*, Perspectives in Mathematical Logic, Springer Verlag, Heidelberg-New York, 1985.

[BS69] J. L. Bell and A. B. Slomson, *Models and ultraproducts: An introduction*, North-Holland Publishing Co., Amsterdam-London, 1969.

[CK73] Chen C. Chang and H. Jerome Keisler, *Model Theory*, Studies in Logic and the Foundation of Math., vol. 73, North–Holland Publishing Co., Amsterdam, 1973.

[Dic85] M. A. Dickman, *Larger infinitary languages*, Model Theoretic Logics (J. Barwise and S. Feferman, eds.), Perspectives in Mathematical Logic, Springer-Verlag, New York Berlin Heidelberg Tokyo, 1985, pp. 317–364.

[Jec03] Thomas Jech, *Set theory*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, The third millennium edition, revised and expanded.

[Jec03] ———, *Set theory*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, The third millennium edition, revised and expanded.

[Kei61] H. Jerome Keisler, *Ultraproducts and elementary classes*, Nederl. Akad. Wetensch. Proc. Ser. A **64** (1961), 477–495.

[Kei63] ———, *Limit ultrapowers*, Transactions of the American Mathematical Society **107** (1963), 382–408.

[KM78] Akihiro Kanamori and Menachem Magidor, *The evolution of large cardinal axioms in set theory*, Higher Set Theory, Lecture Notes in Mathematics, vol. 669, Springer – Verlag, 1978, pp. 99–275.

[Lav71] Richard Laver, *On Fraissé's order type conjecture*, Annals of Mathematics **93** (1971), 89–111.

[Mir05] Džamonja Mirna, *Club guessing and the universal models*, Notre Dame J. Formal Logic **46** (2005), 283–300.

[Sol74] Robert M. Solovay, *Strongly compact cardinals and the GCH*, Proceedings of the Tarski Symposium, Berkeley 1971, Proceedings of Symposia in Pure Mathematics, vol. 25, A.M.S., 1974, pp. 365–372.

[Vää11] Jouko Väänänen, *Models and games*, Cambridge Studies in Advanced Mathematics, vol. 132, Cambridge University Press, Cambridge, 2011.

[Sh:a] Saharon Shelah, *Classification theory and the number of nonisomorphic models*, Studies in Logic and the Foundations of Mathematics, vol. 92, North-Holland Publishing Co., Amsterdam-New York, xvi+544 pp, $62.25, 1978.

[Sh:c] ———, *Classification theory and the number of nonisomorphic models*, Studies in Logic and the Foundations of Mathematics, vol. 92, North-Holland Publishing Co., Amsterdam, xxxiv+705 pp, 1990.

[Sh:f] ———, *Proper and improper forcing*, Perspectives in Mathematical Logic, Springer, 1998.

[Sh:13] ———, *Every two elementarily equivalent models have isomorphic ultrapowers*, Israel Journal of Mathematics **10** (1971), 224–233.

[Sh:E59] ———, *General non-structure theory and constructing from linear orders*, arxiv:1011.3576.

[Sh:E62] ———, *Combinatorial background for Non-structure*, arxiv:math/1512.04767.

[Sh:74] ———, *Appendix to: "Models with second-order properties. II. Trees with no undefined branches" (Annals of Mathematical Logic 14(1978), no. 1, 73–87)*, Annals of Mathematical Logic **14** (1978), 223–226.

[Sh:88r] ———, *Abstract elementary classes near* $\aleph_1$, Chapter I. 0705.4137. arxiv:0705.4137.

[HoSh:109] Wilfrid Hodges and Saharon Shelah, *Infinite games and reduced products*, Annals of Mathematical Logic **20** (1981), 77–108.

[RuSh:117] Matatyahu Rubin and Saharon Shelah, *Combinatorial problems on trees: partitions, Δ-systems and large free subtrees*, Annals of Pure and Applied Logic **33** (1987), 43–81.

[Sh:136] Saharon Shelah, *Constructions of many complicated uncountable structures and Boolean algebras*, Israel Journal of Mathematics **45** (1983), 100–146.

[Sh:220] ———, *Existence of many* $L_{\infty,\lambda}$*-equivalent, nonisomorphic models of* T *of power* λ, Annals of Pure and Applied Logic **34** (1987), 291–310, Proceedings of the Model Theory Conference, Trento, June 1986.

[HoSh:271] Wilfrid Hodges and Saharon Shelah, *There are reasonably nice logics*, The Journal of Symbolic Logic **56** (1991), 300–322.

[Sh:300] Saharon Shelah, *Universal classes*, Classification theory (Chicago, IL, 1985), Lecture Notes in Mathematics, vol. 1292, Springer, Berlin, 1987, Proceedings of the USA–Israel Conference on Classification Theory, Chicago, December 1985; ed. Baldwin, J.T., pp. 264–418.

[Sh:300a] ———, *Stability theory for a model*, Chapter V (A), in series Studies in Logic, vol. 20, College Publications.

[Sh:300b] ———, *Universal Classes: Axiomatic Framework [Sh:h]*, Chapter V (B).

[Sh:309] ———, *Black Boxes*, 0812.0656. 0812.0656. arxiv:0812.0656.

[Sh:500] ———, *Toward classifying unstable theories*, Annals of Pure and Applied Logic **80** (1996), 229–255, arxiv:math/9508205.

[Sh:511] ———, *Building complicated index models and Boolean algebras*.

[BGSh:570] John Baldwin, Rami Grossberg, and Saharon Shelah, *Transfering saturation, the finite cover property, and stability*, Journal of Symbolic Logic **64** (2000), 678–684, arxiv:math/9511205.

[Sh:620] Saharon Shelah, *Special Subsets of* ${}^{\mathrm{cf}(\mu)}\mu$*, Boolean Algebras and Maharam measure Algebras*, Topology and its Applications **99** (1999), 135–235, 8th Prague Topological Symposium on General Topology and its Relations to Modern Analysis and Algebra, Part II (1996). arxiv:math/9804156.

[DjSh:692] Mirna Dzamonja and Saharon Shelah, *On* $\lhd^*$*-maximality*, Annals of Pure and Applied Logic **125** (2004), 119–158, arxiv:math/0009087.

[Sh:797] Saharon Shelah, *Nice infinitary logics*, Journal of the American Mathematical Society **25** (2012), 395–427, arxiv:1005.2806.

[ShUs:844] Saharon Shelah and Alex Usvyatsov, *More on* SOP_1 *and* SOP_2, Annals of Pure and Applied Logic **155** (2008), 16–31, arxiv:math/0404178.

[Sh:893] Saharon Shelah, *A.E.C. with not too many models*, Logic Without Borders (A. Hirvonen, M. Kesala, J. Kontinen, R. Kossak, and A. Villaveces, eds.), vol. Ontos Mathematical Logic, vol. 5, Berlin, Boston: DeGruyter [2015], 2015, arxiv:math/1302.4841 . arxiv:math/1302.4841, pp. 367–402.

[Sh:922] ———, *Diamonds*, Proceedings of the American Mathematical Society **138** (2010), 2151–2161, arxiv:0711.3030 . arxiv:0711.3030.

[Sh:945] ———, *On CON(*$\mathfrak{d}_\lambda > cov_\lambda$*(meagre))*, Transactions of the American Mathematical Society **submitted**, arxiv:0904.0817 . arxiv:0904.0817.

[Sh:950] ———, *Dependent dreams: recounting types*, arxiv:1202.5795.

[MiSh:996] Maryanthe Malliaris and Saharon Shelah, *Constructing regular ultrafilters from a model-theoretic point of view*, Transactions of the American Mathematical Society **367** (2015), 8139–8173, arxiv:1204.1481.

[MiSh:997] ———, *Model-theoretic properties of ultrafilters built by independent families of functions*, Journal of Symbolic Logic **79** (2014), 103–134.

[MiSh:998] ———, *Cofinality spectrum theorems in model theory, set theory and general topology*, Journal of the American Mathematical Society **29** (2016), 237–297, arxiv:1208.5424.

[MiSh:999] ———, *A dividing line within simple unstable theories*, Advances in Mathematics **249** (2013), 250–288.

[Sh:1006] Saharon Shelah, *On incompactness for chromatic number of graphs*, Acta Mathematica Hungarica **139(4)** (2013), 363–371, arxiv:math/1205.0064.

[MiSh:1050] Maryanthe Malliaris and Saharon Shelah, *Keisler's order has infinitely many classes*, Israel Journal of Mathematics **224** (2018), 189–230.

[MiSh:1051] ———, *Model-theoretic applications of cofinality spectrum problems*, Israel Journal of Mathematics **220** (2017), 947–1014.

[KpLaSh:1055] Itay Kaplan, Noa Lavi, and Saharon Shelah, *The generic pair conjecture for dependent finite diagrams*, Israel Journal of Mathematics **212** (2016), 259–287.

[Sh:1064] Saharon Shelah, *Atomic saturation of reduced powers*, Mathematical Logic Quarterly **submitted**, arxiv:math/1601.04824.

[MiSh:1069] Maryanthe Malliaris and Saharon Shelah, *Open problems on ultrafilters and some connections to the continuum*, Contemporary Mathematics **690** (2017), 145–159.

[MiSh:1070] ———, *Cofinality spectrum problems: the axiomatic approach*, Topology and its Applications **213** (2016), 50–79.

[GsSh:1075] Mohammad Golshani and Saharon Shelah, *On cuts in ultraproducts of linear orders I*, Journal of Mathematical Logic **16** (2016), 34 pps, arxiv:math/1510.06278.

[Sh:1101] Saharon Shelah, *Isomorphic limit ultrapowers for infinitary logic*, Israel Journal of Mathematics **submitted**, arxiv:math/1810.12729.
[Sh:F1396] ______, *Model theory for a compact cardinal II.*
[Sh:F1808] ______, *Divide and conquer dividing lines on univerality.*

Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem, 9190401, Israel, and, Department of Mathematics, Hill Center - Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019 USA
Email address: shelah@math.huji.ac.il
URL: http://shelah.logic.at