THE COFINALITY OF THE SYMMETRIC GROUP AND THE COFINALITY OF ULTRAPOWERS

HEIKE MILDENBERGER AND SAHARON SHELAH

ABSTRACT. We prove that $\mathfrak{mcf} < \mathrm{cf}(\mathrm{Sym}(\omega)))$ and $\mathfrak{mcf} > \mathrm{cf}(\mathrm{Sym}(\omega)) = \mathfrak{b}$ are both consistent relative to ZFC. This answers a question by Banakh, Repovš and Zdomskyy and a question from [12].

1. Introduction

The two cardinals $\mathfrak{mcf}$, the minimal cofinality $\mathrm{cf}((\omega,<)^\omega/D)$ where D is a non-principal ultrafilter on ω, and the cofinality of the symmetric group on ω, $\mathrm{cf}(\mathrm{Sym}(\omega))$, are closely related: Both are cofinalities and hence regular. In ZFC, both cardinals have value in the interval $[\mathfrak{g}, \mathfrak{d}]$, namely Blass and Mildenberger [4] showed $\mathfrak{mcf} \geq \mathfrak{g}$, Brendle and Losada [7] showed $\mathrm{cf}(\mathrm{Sym}(\omega)) \geq \mathfrak{g}$, and Simon Thomas [21] showed $\mathrm{cf}(\mathrm{Sym}(\omega)) \leq \mathfrak{d}$. In their relations to $\mathfrak{b}$ the two cardinals behave differently: Obviously $\mathfrak{b} \leq \mathfrak{mcf}$, whereas Sharp and Thomas [16, Theorem 1.6] showed that $\mathrm{cf}(\mathrm{Sym}(\omega)) < \mathfrak{b}$ is consistent relative to ZFC. Before our research, in all investigated forcing extensions we have had $\mathrm{cf}(\mathrm{Sym}(\omega)) \leq \mathfrak{mcf}$ and in the forcing extensions in which both $\mathrm{cf}(\mathrm{Sym}(\omega)) \geq \mathfrak{b}$ and $\mathfrak{mcf} \geq \mathfrak{b}$, the two cardinal characteristics $\mathrm{cf}(\mathrm{Sym}(\omega))$ and $\mathfrak{mcf}$ coincide. The inequality $\mathrm{cf}(\mathrm{Sym}(\omega)) \leq \mathfrak{mcf}$ is partially due to a mathematical reason: Banakh, Repovš and Zdomskyy showed [1, Theorem 1.3]: If D is not nearly coherent to a Q-point then $\mathrm{cf}(\mathrm{Sym}(\omega)) \leq \mathrm{cf}((\omega,<)^\omega/D)$. In particular if there is no Q-point then $\mathrm{cf}(\mathrm{Sym}(\omega)) \leq \mathfrak{mcf}$.

Here we show that indeed an extra assumption is necessary. Our first forcing shows the relative consistency of $\aleph_1 = \mathfrak{mcf} < \aleph_2 = \mathrm{cf}(\mathrm{Sym}(\omega))$.

In our second forcing we show how to separate the two cardinals in the second direction above $\mathfrak{b}$: $\aleph_1 = \mathfrak{b} = \mathrm{cf}(\mathrm{Sym}(\omega)) < \mathfrak{mcf}$ is consistent. We use versions of the oracle-c.c. in the $\aleph_1$-$\aleph_2$-scenario.

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There are some known forcings establishing the relative consistency of $\mathfrak{b} < \mathfrak{mcf}$: Three interesting forcings for $\aleph_1 = \mathfrak{b} < \mathfrak{mcf}$ are given in [19, 20]. Since $\mathfrak{b} \leq \mathfrak{u}$ [15] and since NCF is equivalent to $\mathfrak{u} < \mathfrak{mcf}$ [11] the NCF-models show the relative consistency of $\mathfrak{b} < \mathfrak{mcf}$. In [12] we showed that also $\mathfrak{b}^+ < \mathfrak{mcf}$ is possible. In the second forcing extension of that work we arranged $\mathfrak{b}^+ < \mathfrak{mcf} = \mathrm{cf}(\mathrm{Sym}(\omega))$. In the other forcing extensions for $\mathfrak{b} < \mathfrak{mcf}$ the value of $\mathrm{cf}(\mathrm{Sym}(\omega))$ has not yet been computed or is possibly not determined by the forcing or by NCF.

We recall the definitions: We denote by ${}^\omega\omega$ the set of functions from $\omega \to \omega$. For $f, g \in {}^\omega\omega$ we write $f \leq^* g$ and say g eventually dominates f if $(\exists n)(\forall k \geq n)(f(k) \leq g(k))$. A set $B \subseteq {}^\omega\omega$ is called *unbounded* if there is no g that dominates all members of B. The *bounding number* $\mathfrak{b}$ is the minimal cardinality of an unbounded set.

Definition 1.1. *Let D be a non-principal ultrafilter over ω. By ultrapower we mean the usual modeltheoretic ultrapower: $(\omega, <)^\omega/D$ is the structure with domain $\{[f]_D : f \in {}^\omega\omega\}$ where $[f]_D = \{g \in {}^\omega\omega : \{n : f(n) = g(n)\} \in D\}$ and $[f]_D \leq_D [g]_D$ iff $\{n : f(n) \leq g(n)\} \in D$. The* minimal cofinality of an ultrapower of ω, $\mathfrak{mcf}$, *is defined as the*

$$\mathfrak{mcf} = \min\{\mathrm{cf}((\omega, <)^\omega/D) : D \text{ non-principal ultrafilter over } \omega\}.$$

We use the quasiorder $\leq_D$ also on the space ${}^\omega\omega$ by letting $f \leq_D g$ iff $\{n : f(n) \leq g(n)\} \in D$.

Definition 1.2. $\mathrm{Sym}(\omega)$ *is the group of permutations of ω. If $\mathrm{Sym}(\omega) = \bigcup_{i<\kappa} G_i$ and $\kappa = \mathrm{cf}(\kappa) > \aleph_0$, $\langle G_i : i < \kappa\rangle$ is strictly increasing, G_i is a proper subgroup of* $\mathrm{Sym}(\omega)$, *we call $\langle G_i : i < \kappa\rangle$ an* increasing decomposition. *We call the minimal κ such that an increasing decomposition of length κ exists the* cofinality of the symmetric group, *and denote it* $\mathrm{cf}(\mathrm{Sym}(\omega))$.

Definition 1.3. *A subset $\mathcal{G}$ of $[\omega]^\omega$ is called* groupwise dense *if*

(1) $(\forall X \in \mathcal{G})(\forall Y \subseteq^* X)(Y \text{ infinite } \to Y \in \mathcal{G})$, *and*

(2) for every partition of ω into finite intervals $\Pi = \{[\pi_i, \pi_{i+1}) : i \in \omega\}$ there is an infinite set A such that $\bigcup\{[\pi_i, \pi_{i+1}) : i \in A\} \in \mathcal{G}$.

The groupwise density number, $\mathfrak{g}$, *is the smallest number of groupwise dense families with empty intersection.*

An ultrafilter U over ω is called a *Q-point*, if given any strictly increasing function $f\colon \omega \to \omega$ there is an $X \in U$ such that $\forall n$, $X \cap [f(n), f(n+1))$ has just one element. The existence of a Q-point is independent of ZFC, see, e.g., [8] for existence and [14] for non-existence. An ultrafilter D is *nearly coherent to an ultrafilter U* if there is a finite-to-one function $f\colon \omega \to \omega$ such that $f(D) = f(U)$. Here $f(D) = \{E : f^{-1}[E] \in D\}$. Throughout we write $g[X]$ for the set $\{g(x) : x \in X\}$ and $g^{-1}[Y] = \{x : g(x) \in Y\}$. The principle NCF says that any two non-principal ultrafilters over ω are nearly coherent. Its consistency is established in [5, 6, 3]. *A base* for an ultrafilter is a subset $\mathcal{B}$ of $\mathscr{U}$ such that $(\forall Y \in \mathscr{U})(\exists X \in \mathcal{B})(X \subseteq Y)$. The character of an ultrafilter is the smallest size of a base. The *ultrafilter characteristic* $\mathfrak{u}$ is the smallest character of a non-principal ultrafilter.

In forcing the stronger condition is the larger one. For a forcing order $\mathbb{P}$ and a formula φ, we say $\mathbb{P}$ forces φ if the weakest condition in $\mathbb{P}$ forces φ.

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2. $\mathrm{Con}(\mathfrak{b} = \mathrm{cf}(\omega^\omega/D) < \mathrm{cf}(\mathrm{Sym}(\omega)))$

In this section we prove:

Theorem 2.1. *The constellation $\aleph_1 = \mathfrak{b} = \mathfrak{mcf} < \mathrm{cf}(\mathrm{Sym}(\omega))$ is consistent relative to* ZFC.

We essentially use oracle c.c. [18, Ch. 4] but we carry on a name for an ultrafilter $\underset{\sim}{D}$ and did not see to fit it in exactly. We establish a notion of forcing $\mathbb{P}$ such that for a $\mathbb{P}$-generic filter $\mathbf{G}$, $\underset{\sim}{D}[\mathbf{G}]$ will be an ultrafilter witnessing $\mathfrak{mcf} = \aleph_1$.

As in [18, Chapter IV] we fix some background for the oracles. Let S be a stationary subset of ω_1. We fix S throughout this section. A set $\mathscr{D} \subset \mathcal{P}(S)$ is called a filter over S if $\emptyset \notin \mathscr{D}$, $S \in \mathscr{D}$, $\mathscr{D}$ is closed under finite intersections and closed under supersets. A filter $\mathscr{D}$ over S is called *normal* if it contains all cobounded subsets of S and is closed under diagonal intersections, that is, given D_δ, $\delta \in \omega_1$, with $D_\delta \in \mathscr{D}$, their diagonal intersection $\triangle_{\delta\in\omega} D_\delta = \{\gamma \in S \,:\, \gamma \in \bigcap_{\delta\in\gamma} D_\delta\} \in \mathscr{D}$. For a filter $\mathscr{D}$ over ω_1 and $X, Y \subseteq \omega_1$ we let $X = Y \mod \mathscr{D}$ iff $(X \cap Y) \cup ((\omega_1 \smallsetminus X) \cap (\omega_1 \smallsetminus Y)) \in \mathscr{D}$, and $X \subseteq Y \mod \mathscr{D}$ if $X \smallsetminus Y = \emptyset \mod \mathscr{D}$.

We recall the notion of a $\diamondsuit_S^-$-sequence. A sequence $\bar{P} = \langle P_\delta \,:\, \delta \in S\rangle$ is called a $\diamondsuit_S^-$-sequence iff $P_\delta \subseteq \mathcal{P}(\delta)$ is countable and for any $X \subseteq \aleph_1$

$$\{\delta \in S \,:\, X \cap \delta \in P_\delta\} \text{ is a stationary subset of } S.$$

It is well known that $\diamondsuit_S^-$ and $\diamondsuit_S$ are equivalent (see [10, Ch. III]).

We fix a sufficiently large regular cardinal χ ($\chi \geq (2^{\aleph_2})^+$ suffices). We fix a well-order $<_\chi$ on $H(\chi)$.

Definition 2.2. *We assume that S is stationary and $\diamondsuit_S$.*

(1) (See [18, IV, Def 1.1]*) An S-oracle is a sequence $\bar{M} = \langle M_\delta \,:\, \delta \in S\rangle$ such that*
   - *(a) M_δ is countable and transitive and $\delta + 1 \subseteq M_\delta$,*
   - *(b) $i_\delta \colon M_\delta \hookrightarrow_{\mathrm{elem}} (H(\chi), \in, <_\chi)$ is elementary,*
   - *(c) $M_\delta \models \delta$ is countable,*
   - *(d) for $\delta < \varepsilon \in S$ the structures $(M_\delta, \in, <_{\chi,\delta})$ is an algebraic substructure of $(M_\varepsilon, \in, <_{\chi,\varepsilon})$ and i_ε extends i_δ, and*
   - *(e) for any $A \subseteq \omega_1$ the set $\{\delta \in S \,:\, A \cap \delta \in M_\delta\}$ is stationary in ω_1.*

*(2) We say that $\langle \bar{M}, \bar{N}, \bar{\eta}\rangle$ is an S-*oracle triple *if*
   - *(a) $\bar{M} = \langle M_\delta \,:\, \delta \in S\rangle$ is an S-oracle,*

(b) $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$,

(c) for $\delta \in S$, η_δ *is Cohen over* M_δ,

(d) $\bar{N} = \langle N_\delta : \delta \in S \rangle$,

(e) $N_\delta = M_\delta[\eta_\delta]$.

(3) $\mathscr{D}_{\bar{M}}$ *is the filter over* S *that is generated by the sets* $I_{\bar{M}}(A)$, $A \subseteq \omega_1$, *where*

$$I_{\bar{M}}(A) = \{\alpha \in S : A \cap \alpha \in M_\alpha\}.$$

From now on until the end of the section let $S \subseteq \omega_1$ be stationary and assume $\diamondsuit_S$. For L-structures $\mathcal{A}$, $\mathcal{M}$, we write $\mathcal{A} \prec \mathcal{M}$ if $\mathcal{A}$ is an elementary substructure of $\mathcal{M}$. Since for for L-structures $\mathcal{A}, \mathcal{B}, \mathcal{M}$ with $\mathcal{A}, \mathcal{B} \prec \mathcal{M}$ and $\mathcal{A} \subseteq \mathcal{B}$ also $\mathcal{A} \prec \mathcal{B}$ holds, we have that the structures on any oracle sequence are $\prec$-increasing.

If $f \colon A \to B$ is a function and $C \subseteq A$, then we write $f''C$ for $\{f(c) : c \in C\}$. We recall the following important properties of $\mathscr{D}_{\bar{M}}$.

Lemma 2.3. *(*[18, IV, Claim 1.4]*) The set* $\{I_{\bar{M}}(A) : A \subseteq \omega_1\}$ *is closed under finite intersections. The filter* $\mathscr{D}_{\bar{M}}$ *contains every closed unbounded set of limit ordinals in* ω_1*, it is normal and contains any club subset of* S*, and for every* $A \subseteq H(\aleph_1)$, $I_{\bar{M}}(A) \in \mathscr{D}_{\bar{M}}$.

Proof. We prove only the very last statement, the others are proved in the book. By $\diamondsuit_S$, $|H(\omega_1)| = \omega_1$. Let $f \colon H(\omega_1) \to \omega_1$ be the $<_\chi$-least bijection. Let $C = \{\delta \in \omega_1 : \delta \text{ limit and } (\forall \alpha < \delta)(f''M_\alpha \subseteq \delta)\}$. The set $\operatorname{acc}(C)$ of accumulation points of C is club in ω_1 and hence $\operatorname{acc}(C) \cap S \in \mathscr{D}_{\bar{M}}$. By its definition, $I_{\bar{M}}(f''A) \in \mathscr{D}_{\bar{M}}$. Now if $f''A \cap \delta \in M_\delta$ and $\delta \in \operatorname{acc}(C) \cap S$ then $M_\delta \ni (i_\delta^{-1}(f^{-1}))''(f''A \cap \delta) = \bigcup_{\alpha<\delta}(f^{-1} \restriction f''M_\alpha)''(f''A \cap \alpha) = \bigcup_{\alpha<\delta} A \cap \alpha = A \cap \delta$. Thus we have $I_{\bar{M}}(A) \supseteq I_{\bar{M}}(f''A) \cap \operatorname{acc}(C)$ and hence $I_{\bar{M}}(A) \in \mathscr{D}_{\bar{M}}$. $\dashv$

We recall when a notion of forcing $\mathbb{P}$ has the $\bar{M}$-c.c.

Definition 2.4. *(*[18, Ch. IV, Def. 1.5]*) Let* $\bar{M}$ *be an* S*-oracle sequence and let* $\mathbb{P}$ *be a notion of forcing. We define when* $\mathbb{P}$ *satisfies the* $\bar{M}$*-c.c. by cases:*

(a) If $|\mathbb{P}| \leq \aleph_0$, *always.*

(b) If $|\mathbb{P}| = \aleph_1$ *and if for every injective* $\pi \colon \mathbb{P} \to \omega_1$ *the set*

$$\big\{\delta \in S : (\forall A \in M_\delta \cap \mathcal{P}(\delta))\big(((\pi^{-1})''A \text{ is predense in } (\pi^{-1})''\delta) \\ \to ((\pi^{-1})''A \text{ is predense in } \mathbb{P})\big)\big\}$$

is an element of $\mathscr{D}_{\bar{M}}$.

(c) If $|\mathbb{P}| > \aleph_1$ *and for every* $\mathbb{P}^\dagger \subseteq \mathbb{P}$ *if* $|\mathbb{P}^\dagger| \leq \aleph_1$ *then here are* $\mathbb{P}''$ *such that* $|\mathbb{P}''| = \aleph_1$ *and* $\mathbb{P}^\dagger \subseteq \mathbb{P}'' \subseteq_{\mathrm{ic}} \mathbb{P}$ *and* $\pi \colon \mathbb{P}'' \to \omega_1$ *as in (b).* $\mathbb{P}'' \subseteq_{\mathrm{ic}} \mathbb{P}$ *means that* $\mathbb{P}''$ *is an incompatibility preserving suborder of* $\mathbb{P}$*, i.e., for any* $p, q \in \mathbb{P}''$, $p \leq_{\mathbb{P}''} q$ *iff* $p \leq_{\mathbb{P}} q$ *and* $p \perp_{\mathbb{P}''} q$ *iff* $p \perp_{\mathbb{P}} q$.

Oracle sequences are not continuous. The requirement $\delta \in M_\delta$ precludes continuity.

Lemma 2.5. *Assume S is stationary and $\diamondsuit_S$.*

(1) There is an oracle triple.

(2) Let $\langle \bar{M}, \bar{N}, \bar{\eta}\rangle$ be an oracle triple. Then $\{\delta \in S : \{(\varepsilon, \eta_\varepsilon) : \varepsilon < \delta\} \in M_\delta\} \in \mathscr{D}_{\bar{M}}$.

Proof. (1) Let $\langle P_\delta : \delta \in S\rangle$ be a $\diamondsuit_S^-$-sequence. Recall, we fixed $f\colon H(\omega_1) \to \omega_1$, the $<_\chi$-least bijection. We choose M_δ, i_δ by induction on δ. Suppose that M_γ, i_γ, $\gamma < \delta$ has been chosen. Let $M'_\delta \prec (H(\chi), \in, <_\chi)$ be a countable elementary substructure with $\langle M_\gamma, i_\gamma : \gamma < \delta\rangle, \delta, P_\delta \in M'_\delta$ and $\delta + 1 \subseteq M'_\delta$. Then we let M_δ be the Mostowski collapse of M'_δ. The Mostowski collapse maps P_δ to itself. Moreover, since P_δ is countable, $P_\delta \subseteq M_\delta$, and hence $X \cap \delta \in P_\delta$ implies $X \cap \delta \in M_\delta$. By now, we have taken care of Def. 2.2.(2) (a). For being definite, we let the Cohen forcing $\mathbb{C}$ be the set of finite partial functions from ω to ω, ordered by extension. By the Rasiowa-Sikorski theorem there is a Cohen-generic filter G_δ over M_δ. Then the function $\eta_\delta = \bigcup\{p : p \in G_\delta\} \in {}^\omega\omega$ is a Cohen real over M_δ. (2) The set $A = \{(\varepsilon, \eta_\varepsilon) : \varepsilon \in S\} \subseteq H(\omega_1)$. We fix a club C such for $\delta \in C$, $A \cap \delta = \{(\varepsilon, \eta_\varepsilon) : \varepsilon < \delta\}$. By Lemma 2.3 we have $I_{\bar{M}}(A) \in \mathscr{D}_{\bar{M}}$. By normality $C \cap I_{\bar{M}}(A) \in \mathscr{D}_{\bar{M}}$. Moreover, $C \cap I_{\bar{M}}(A) \subseteq \{\delta : \{(\varepsilon, \eta_\varepsilon) : \varepsilon < \delta\} \in M_\delta\}$. $\dashv$

From now until the end of the section we fix an S-oracle triple $(\bar{M}, \bar{N}, \bar{\eta})$. We let

$$C' = \{\delta \in S : (\forall \alpha < \delta)(M_\alpha[\eta_\alpha] \in M_\delta)\}. \tag{2.1}$$

Since C' contains a club subset of S, we have $C' \cap S \in \mathscr{D}_{\bar{M}}$. Since $M_\delta \subseteq N_\delta$, the thinned out sequence $\langle N_\delta : \delta \in C' \cap S\rangle$ is increasing and hence it is an S-oracle sequence. From now on we replace S by $S \cap C'$.

Oracle triples allow for the application of the "Omitting Types Theorem" for the types "η_ε is not Cohen any more over $M_\varepsilon[\mathbb{P}]$". Now we recall the omitting types theorem and explain its use.

Lemma 2.6. *(The Omitting Types Theorem, see* [18, Ch. IV, Lemma 2.1]*) Assume $\diamondsuit_S$. Suppose the $\psi_i(x)$, $i < \omega_1$, are Π^1_2 formulas on reals with a real parameter possibly. Suppose further that there is no solution to $\bigwedge_{i<\omega_1} \psi_i(x)$ in $\mathbf{V}$ and moreover even if we add a Cohen real to $\mathbf{V}$ there will be none. Then there is an S-oracle $\bar{M}'$ such that for any forcing $\mathbb{P}$,*

$$\text{if } \mathbb{P} \text{ has the } \bar{M}' - \text{c.c then in } \mathbf{V}^{\mathbb{P}} \text{ there is no solution to } \bigwedge_i \psi_i(x).$$

We apply this to $\psi_i(h, y, \eta_i)$, $i \in S$, where ψ_i says:

$$\begin{aligned} &y \in {}^\omega 2 \text{ and } h \in {}^\omega\omega \text{ is increasing and} \\ &(\forall^\infty n)(\eta_i \restriction [h(n), h(n+1)) \neq y \restriction [h(n), h(n+1))). \end{aligned} \tag{2.2}$$

If $(h, y) \in M_i[\mathbb{P}]$, the arithmetic formula $\psi_i(h, y, \eta_i)$ says "(h, y) codes a (new) meagre set that shows that η_i is not Cohen over $M_i[\mathbb{P}]$ any more", see

[2, Ch. 2]. Thus the omission of the type $\bigwedge_i \psi_i$ says that $\{\eta_i : i \in S\}$ is not meagre in $\mathbf{V}^{\mathbb{P}}$.

Actually we apply (in $\oplus_{3,i}$ below) the proof of the omitting types theorem. We use the following fact.

Lemma 2.7. *If η is a Cohen real over M, then there is no $p \in \mathbb{C}$ such that p forces in Cohen forcing over M that η is not Cohen over $M[\mathbb{C}]$.*

Proof. It suffices to show that there is no name $(\underset{\sim}{h}, \underset{\sim}{y}) \in M$ and no condition such that $p \Vdash \psi(h, y, \eta)$. Any name $(\underset{\sim}{h}, \underset{\sim}{y})$ of an increasing function h and $y \colon \omega \to 2$ has an equivalent one of the following canonical form: We think of

$$\mathbb{C} = \{p : p = (p_1, p_2) \colon n \to \{\{m\} \times 2^m, m \in \omega \setminus \{0\}\}, n \in \omega\}$$

such that $\emptyset \Vdash \underset{\sim}{h}(-1) = 0$, $p \Vdash \underset{\sim}{h}(m) = \sum_{k \le m} p_1(k)$ and $p \Vdash \underset{\sim}{y} \restriction [\underset{\sim}{h}(m-1), \underset{\sim}{h}(m)) = p_2(m)$. Then given η and $m \in \omega$ and p there is a $q \geq_{\mathbb{C}} p$ and an $n \geq m$ that forces $\underset{\sim}{y} \restriction [\underset{\sim}{h}(n-1), \underset{\sim}{h}(n)) = \eta \restriction [\underset{\sim}{h}(n-1), \underset{\sim}{h}(n))$. ⊣

Applied to this lemma, the proof of the omitting types theorem yields:

Lemma 2.8. *For each $\mathbb{P}$ with the $\bar{N}$-c.c. there is a set $Y \in \mathscr{D}_{\bar{N}}$ such that for any $i \in Y$, η_i is Cohen over $M_i[\mathbb{P}]$.*

For building up a name for an ultrafilter witnessing $\mathfrak{mcf} = \aleph_1$ in our planned forcing extension we introduce some notions for handling names.

Definition 2.9. *(1) A* canonical $\mathbb{P}$-name for a subset of ω *is a name of the form $\tau = \{\langle \check{n}, p\rangle : p \in A_n\rangle\}$, where the $A_n \subseteq \mathbb{P}$ are antichains.*

(2) A canonical $\mathbb{P}$-name for a subset of $\mathcal{P}(\omega)$ *is a name of the form $\underset{\sim}{K} = \{\langle \tau, q\rangle : q \in A_\tau, \tau \in X\}$, where X is a set of canonical $\mathbb{P}$-names τ for subsets of ω, for maps π as in (3), and for each $\tau \in X$, the set A_τ is an antichain in $\mathbb{P}$.*

(3) Let $\pi \colon \mathbb{P} \to \omega_1$ be injective. We let $\pi''\mathbb{P} = \mathbb{P}'$ and define a partial order (or a quasi order) on $\mathbb{P}'$ such that π is an isomorphism from $\mathbb{P}$ to $\mathbb{P}'$. Then we lift π to a map $\bar{\pi}$ of $\mathbb{P}$-names by letting $\bar{\pi}(\tau) = \{\langle \bar{\pi}(\sigma), \pi(p)\rangle : \langle \sigma, p\rangle \in \tau\}$.

For canonical names τ, $\underset{\sim}{K}$ as above, $\bar{\pi}(\tau) \in H(\omega_1)$, $\bar{\pi}(\underset{\sim}{K}) \subseteq H(\omega_1)$. Thus according to Lemma 2.3, $I_{\bar{M}}(\bar{\pi}(K)) \in \mathscr{D}_{\bar{M}}$. The names $\bar{\pi}(\underset{\sim}{K})$ and $\bar{\pi}(\tau)$ are canonical.

Definition 2.10. *Let $f \colon H(\omega_1) \to \omega_1$ be as above, the $<_\chi$-least bijection. The function $f \in \mathbf{V}$ will be fixed throughout the construction.*

(1) We let τ be a canonical $\mathbb{P}'$-name of a subset of ω. We let for $\delta \in \omega_1$,

$$\tau \restriction \delta = \begin{cases} \tau \text{ if} & \tau \text{ is a } \mathbb{P}' \cap \delta\text{-name, and } \tau \in M_\delta \\ \text{undefined} & \text{otherwise.} \end{cases}$$

(2) *For a canonical* $\mathbb{P}'$*-name* $\underset{\sim}{K} = \{(\tau, q) : q \in A_\tau, \tau \in X\}$ *for a subset of* $\mathcal{P}(\omega)$ *and* $\delta < \omega_1$ *we define the* δ*-part as follows:*

$$\begin{aligned}\underset{\sim}{K} \restriction \delta = &\{(\tau, q) : (\tau, q) \in \underset{\sim}{K}, q \in \mathbb{P}' \cap \delta, \tau \text{ is a } \mathbb{P}' \cap \delta\text{-name}, \\ &\tau \in M_\delta, A_\tau \subseteq \mathbb{P}' \cap \delta, A_\tau \in M_\delta\}.\end{aligned}$$

Now we are ready to define the set K^1 of pairs that serve as conditions in the first iterand of our final two-step forcing.

Definition 2.11. *(1) For an* S*-oracle triple* $(\bar{M}, \bar{N}, \bar{\eta})$ *as above we let* K^1 *be the set of* $(\mathbb{P}, \underset{\sim}{D})$ *with the following properties:*

(a) $\mathbb{P}$ *is a c.c.c. forcing with a nonstationary domain* $\mathbb{P} \subseteq \omega_1$*.*

(b) $\underset{\sim}{D}$ *is a canonical* $\mathbb{P}$*-name of a non-principal ultrafilter over* ω*.*

(c) $Y(\mathbb{P}, \underset{\sim}{D}) \in \mathscr{D}_{\bar{N}}$*, where* $Y(\mathbb{P}, \underset{\sim}{D})$ *is the set of* $\delta \in S$ *such that items* (α) *to* (ε) *hold:*

(α) $\mathbb{P} \cap \delta \in M_\delta$.

(β) *If* $I \subseteq \mathbb{P} \cap \delta$ *and* $I \in N_\delta$ *and* I *is predense in* $\mathbb{P} \cap \delta$ *then* I *is predense in* $\mathbb{P}$ *(so we have that* $\mathbb{P}$ *has the* $\bar{N}$*-oracle-c.c.).*

(γ) $\underset{\sim}{D} \restriction \delta \in M_\delta$ *and* $M_\delta \models$ *"*$\underset{\sim}{D} \restriction \delta$ *is a canonical* $\mathbb{P} \cap \delta$*-name of an ultrafilter over* ω*".*

(δ) $N_\delta \models \Vdash_{\mathbb{P}\cap\delta}$ *"*η_δ *is Cohen-generic over* $M_\delta[\underset{\sim}{\mathbf{G}}_{\mathbb{P}\cap\delta}]$*".*

(ε) $\underset{\sim}{D} \cap H(\omega_1)^{N_\delta} \in N_\delta$ *and*

$$N_\delta \models \big(\underset{\sim}{D} \cap H(\omega_1)^{N_\delta} \text{ is a canonical } \mathbb{P}\text{-name of an ultrafilter over } \omega\big), \text{ and}$$

$$\Vdash_{\mathbb{P}\cap\delta} (\forall f \in M_\delta[\underset{\sim}{\mathbf{G}}_{\mathbb{P}\cap\delta}] \cap {}^\omega\omega)(f \leq_{\underset{\sim}{D}} \eta_\delta).$$

(2) *For an oracle triple* $(\bar{M}, \bar{N}, \bar{\eta})$ *we let* K^2 *be the set of* $(\mathbb{P}, \underset{\sim}{D}) \in H(\aleph_2)$ *such that there are a non-stationary* $\mathbb{P}' \subseteq \omega_1$ *and a one-to-one* $\pi \colon \mathbb{P}' \to \mathbb{P}$ *and* $(\mathbb{P}', \underset{\sim}{D}') \in K^1$*,* π *is an isomorphism from* $\mathbb{P}'$ *onto* $\mathbb{P}$ *with* $\bar{\pi}(\underset{\sim}{D}') = \underset{\sim}{D}$*.*

Remark 2.12. (1) If for a set of δ's in $\mathscr{D}_{\bar{N}}$ we have $N'_\delta \subseteq N_\delta$ and $\bar{N}'$ is an oracle sequence, then the $\bar{N}$-oracle c.c. implies the $\bar{N}'$-oracle c.c.

(2) Since we do not add new types that have to be omitted in the course of the iteration, one fixed oracle $\bar{N} \in \mathbf{V}$ is sufficient.

(3) We recall the the successor step for oracle-c.c. Lemma [18, IV 3.2]: If $\mathbb{Q}$ has the $\bar{M}$-c.c. and $\mathbb{Q}$ forces that $\underset{\sim}{\mathbb{Q}}'$ has the $\langle M_\delta[\mathbb{Q}] : \delta \in S\rangle$-c.c. then $\mathbb{Q} * \underset{\sim}{\mathbb{Q}}'$ has the $\bar{M}$-c.c.

Lemma 2.13. *Assume*

(a) $(\mathbb{P}, \underset{\sim}{D}) \in H(\aleph_2)$*,* $\mathbb{P}$ *is a forcing notion,* $\mathbb{P} \in H(\omega_2)$ *and* $\underset{\sim}{D} \in H(\omega_2)$ *is a canonical* $\mathbb{P}$*-name of an ultrafilter over* ω*.*

(b) $\mathbb{P}'_\ell \subseteq \omega_1$ *is a non-stationary set for* $\ell = 1, 2$*,* π_ℓ *is an injective function from* $\mathbb{P}'_\ell$ *onto* $\mathbb{P}$ *for* $\ell = 1, 2$*.*

(c) $\mathbb{P}'_\ell$ *is a notion of forcing, such that* π_ℓ *is an isomorphism from* $\mathbb{P}'_\ell$ *onto* $\mathbb{P}$ *for* $\ell = 1, 2$.

(d) $\underset{\sim}{D}'_\ell$ *is a* $\mathbb{P}'_\ell$*-name of a subset of* $\mathcal{P}(\omega)$ *such that* π_ℓ *maps* $\underset{\sim}{D}'_\ell$ *onto* $\underset{\sim}{D}$.

Then $(\mathbb{P}'_1, \underset{\sim}{D}'_1) \in K^1$ *iff* $(\mathbb{P}'_2, \underset{\sim}{D}'_2) \in K^1$.

Proof. The map $\pi = \pi_2^{-1} \circ \pi_1$ is an isomrophism from $\mathbb{P}'_1$ onto $\mathbb{P}'_2$, and its lifting $\bar{\pi}$ maps $\underset{\sim}{D}'_1$ to $\underset{\sim}{D}'_2$. So the set

$$Z = \{\delta \in \omega_1 : \pi \restriction \delta \text{ is a one-to-one mapping from } \mathbb{P}'_1 \cap \delta \text{ to } \mathbb{P}'_2 \cap \delta \text{ and } \pi \restriction \delta \in M_\delta\}$$

is a club subset of ω_1 and hence $Z \cap S$ belongs to $\mathscr{D}$. If $\delta \in Z \cap S$ then $\delta \in Y(\mathbb{P}'_1, \underset{\sim}{D}'_1)$ iff $\delta \in Y(\mathbb{P}'_2, \underset{\sim}{D}'_2)$, since $M_\delta \models \mathsf{ZFC}^-$, that is ZFC without the power set axiom. ⊣

This shows that in Definition 2.11(2) the following is true: If the demand holds for some pair $(\mathbb{P}', \pi)$ then it holds for every such pair. The primed partial orders in Lemma 2.13 that shall ensure that the domain is a non-stationary subset of ω_1 and that the names the oracles shall anticipate are subsets of $H(\omega_1)$. According to the lemma, they are invariant under bijections of ω_1. Since any property of the forcing is named modulo $\mathscr{D}_{\bar{N}}$ the particular choice of the injections does not matter. For the actual construction of forcing posets it is convenient to use non-stationary domains for the $\mathbb{P}' \in K^1$, since non-stationarity is preserved by countable unions and by diagonal unions.

The property in Def. 2.11(1)(c)(ε) will be crucial to ensure that $\underset{\sim}{D}$ will be forced to be an ultrafilter such that the weakest condition in the two-step forcing forces $\mathrm{cf}(\omega^\omega/\underset{\sim}{D}) = \aleph_1$, as witnessed by $\langle \eta_\delta : \delta \in S\rangle$. Technically it is more convenient to carry on the property (δ) together with (ε). In the case of an $\leq^*$-increasing sequence $\langle \eta_\delta : \delta < S\rangle$ unboundedness is preserved in limits of finite support iterations if each initial segments preserves it [2, Ch. 6, §4]. So it might be possible to carry (ε) and the contrary of (δ). We have not investigated this issue.

Concerning the preservation of (δ), we will frequently use [2, Chapter 6 Section 4]:

Lemma 2.14. *Let* $\mathbb{P}_n \lessdot \mathbb{P}_{n+1}$ *for* $n \in \omega$ *and let* $\mathbb{P}$ *be the direct limit of* $\langle \mathbb{P}_n : n \in \omega\rangle$. *If* $\mathbb{P}_n \Vdash$ *"*η_δ *is Cohen generic over* $M_\delta[G_{\mathbb{P}_n}]$*" for all* n*, then* $\mathbb{P} \Vdash$ *"*η_δ *is Cohen generic over* $M_\delta[G_{\mathbb{P}}]$*."*

For $\mathbb{P}$ from our classes by properties 2.11(1)(c)(β) and (δ) we get: There are $\mathscr{D}_{\bar{M}}$-many δ such that ($\mathbb{P} \Vdash$ "η_δ is Cohen generic over $M_\delta[G_{\mathbb{P}}]$" iff $\mathbb{P} \cap \delta \Vdash$ "η_δ is Cohen generic over $M_\delta[G_{\mathbb{P}\cap\delta}]$"). We will use this fact frequently in our inductive construction of elements of K^2 below. We extend the forcing orders gradually in $\lessdot$ and $\leq_{AP}$ (see Def. 2.16) in order to get $(\mathbb{P}, \underset{\sim}{D}) \in K^2$.

Let unif$(\mathcal{M})$ denote the smallest cardinality of a non-meagre set. The following proposition is not used in the proof. It gives the additional information that unif$(\mathcal{M}) = \aleph_1$ in our forcing extensions, as witnessed by $\{\eta_\delta : \delta \in S\}$.

Proposition 2.15. *If $(\mathbb{P}, \underset{\sim}{D}) \in K^2$ then $\mathbb{P}$ forces that $\{\eta_\delta : \delta \in S\}$ is a non-meagre subset of ${}^\omega\omega$.*

Proof. Let $p \in \mathbb{P}$ force that $\{\eta_\delta : \delta \in S\}$ is meagre. Let τ be a name for a meagre F_σ-set. By the c.c.c., there is a $\delta \in Y(\mathbb{P}, \underset{\sim}{D})$ such that $\tau, p \in M_\delta$, $p \in \mathbb{P} \cap \delta$, τ is a $\mathbb{P} \cap \delta$-name, and $p \Vdash \{\eta_\delta : \delta \in S\} \subseteq \tau$. Then $p \Vdash_{\mathbb{P}} \eta_\delta \in \tau$. By the oracle-c.c, that is item (β) in the definition of $Y(\mathbb{P}, \underset{\sim}{D})$, also $p \Vdash_{\mathbb{P}\cap\delta} \eta_\delta \in \tau$, in contradiction to item (1)(c)(ε) of the definition of $Y(\mathbb{P}, \underset{\sim}{D})$. $\dashv$

Proposition 2.15 has a sort of an inverse direction for the class of Suslin forcings. A forcing $\mathbb{Q} \subseteq \omega^\omega$ is called Suslin if $\mathbb{Q}$ is an analytic subset of ω^ω and the relations $\leq_{\mathbb{Q}}$ and $\perp_{\mathbb{Q}}$ are analytic sets in $\omega^\omega \times \omega^\omega$. For Suslin proper forcings, not making the ground model meager is equivalent to preserving the genericity of a Cohen real over a countable model [9, 6.21, 6.22].

Any forcing forcing over a fround model of CH that increases the continuum to $\aleph_2$ has size at least $\aleph_2$. We find such a forcing after a preliminary forcing. We extend $\mathbf{V}$ in two steps via

$$AP * \underset{\sim}{\mathbb{Q}}, \text{ and } \underset{\sim}{\mathbb{Q}} = \bigcup \{\mathbb{P} : \exists \underset{\sim}{D} (\mathbb{P}, \underset{\sim}{D}) \in G_{AP}\}.$$

In order to destroy any short sequence $\langle G_i : i < \omega \rangle$ that would witness cf(Sym$(\omega)) \leq \omega_1$ we take recourse to dense sets of a suitable approximation forcing AP. Any generic filter of the approximation forcing will be a forcing with the desired properties. Here is the definition of the approximation forcing.

Definition 2.16. *We let K^2 be as above.*

(A) Let $\mathbf{p} = (\mathbb{P}_p, \underset{\sim}{D}_p), \mathbf{q} = (\mathbb{P}_q, \underset{\sim}{D}_q) \in K^2$. We define $\mathbf{p} \leq_{AP} \mathbf{q}$ iff
  - *(a) $\mathbb{P}_p \lessdot \mathbb{P}_q$,*
  - *(b) $\Vdash_{\mathbb{P}_q} \underset{\sim}{D}_p \subseteq \underset{\sim}{D}_q$.*

(B) For $i = 1, 2$, we let forcing order of approximations be $AP^i = (K^i, \leq_{AP})$. We let $AP = AP^2$.

The following is the parallel of the basic claim on oracle c.c. forcing, Claim 3.2 from [18, Ch. IV]. The forcing $\mathbb{P}_i$ does not mean iteration up to stage i. The variable i, ranging over $\omega + 1$ or $\omega_1 + 1$ or ω_2, is just an index for $\mathbb{P}_i$ being a component of $(\mathbb{P}_i, \underset{\sim}{D}_i) \in K^2$. $\mathbb{P}_i$ is an $\bar{N}$-oracle c.c. forcing and $|\mathbb{P}_i| \leq \aleph_1$.

Lemma 2.17. *(A) The structure $(K^2, \leq_{AP})$ is a partial order of cardinality $|H(\aleph_2)|$.*

(B) $K^2 \neq \emptyset$.

(C) *If* $\mathbf{p}_n = (\mathbb{P}_n, \underset{\sim}{D}_n) \in K^2$ *for* $n \in \omega$ *and* $\mathbf{p}_n \leq_{AP} \mathbf{p}_{n+1}$ *then the set has an upper bound* $\mathbf{p}_\omega = (\mathbb{P}_\omega, \underset{\sim}{D}_\omega)$ *with* $\mathbb{P}_\omega = \bigcup\{\mathbb{P}_n : n \in \omega\}$.

(D) $(K^2, \leq_{AP})$ *is* $(\omega_1 + 1)$*-strategically closed, that is, for every* $\mathbf{p} \in AP$ *the protagonist has a winning strategy in the following game* $\Game(\mathbf{p})$*: A play lasts* $\omega_1 + 1$ *moves. During the play the player COM, the protagonist, chooses for* $i \leq \omega_1$, $\mathbf{p}_i = (\mathbb{P}_i, \underset{\sim}{D}_i) \in K^2$, *and INC, the antagonist, chooses* $\mathbf{q}_i \in K^2$ *such that*

(a) $\mathbf{p}_i \leq_{AP} \mathbf{q}_i$,

(b) $(\forall j < i)(\mathbf{q}_j \leq_{AP} \mathbf{p}_i)$,

(c) $\mathbf{p}_0 = \mathbf{p}$.

The protagonist COM wins the game if he can always move. The hard case is the choice of $\mathbf{p}_{\omega_1}$.

Proof. (A) and (B) are obvious.

(C) Let $\mathbf{p}_n = (\mathbb{P}_n, \underset{\sim}{D}_n)$ and let $\langle \mathbf{p}_n : n \in \omega\rangle$ be $\leq_{AP}$-increasing. We choose $(\mathbb{P}'_n, \pi_n, \mathbb{P}'_n, \underset{\sim}{D}'_n)$ by induction on n as in the definition of K^2 such that $\mathbb{P}'_n$ is not stationary and $\pi_n \colon \mathbb{P}'_n \to \mathbb{P}_n$ is an isomorphism, $\pi^{-1}(\underset{\sim}{D}_n) = \underset{\sim}{D}'_n$, $\pi_n \subseteq \pi_{n+1}$ and $(\mathbb{P}'_n, \underset{\sim}{D}'_n) \in K^1$. Then we let $\mathbb{P}'_\omega = \bigcup_{n\in\omega} \mathbb{P}'_n$, and the latter is not stationary. Moreover we let $\pi_\omega = \bigcup_{n\in\omega} \pi_n$. $\mathbb{P}'_\omega$ has the $\bar{N}$-oracle c.c. according to [18, Ch. IV, Claim 3.2]. We fix for $n \in \omega$ a reduction $r_{\mathbb{P}'_\omega, \mathbb{P}'_n} \colon \mathbb{P}'_\omega \to \mathbb{P}'_n$ and we set $C = \{\delta \in S : \delta \text{ limit of } S \text{ and } (\forall n) r''_{\mathbb{P}'_\omega, \mathbb{P}'_n}(\mathbb{P}'_\omega \cap \delta) \subseteq \delta\}$. Of course C is club in ω_1. We let

$$Y = \bigcap_{k\in\omega} Y(\mathbb{P}'_k, \underset{\sim}{D}'_k) \cap C.$$

Since $\mathscr{D}_{\bar{N}}$ is normal and hence $< \aleph_1$-complete, we have $Y \in \mathscr{D}_{\bar{N}}$. We show that there is $\underset{\sim}{D}'_\omega$ such that $(\mathbb{P}'_\omega, \underset{\sim}{D}'_\omega)$ is an upper bound of $\langle \mathbf{p}'_n : n < \omega\rangle$ in $\leq_{AP}$.

As in the proof of [18, Ch. IV, Claim 3.2] in the case of countable cofinality it is shown: The set Y witnesses that $\mathbb{P}'_\omega$ has the $\bar{N}$-oracle c.c., since for $\delta \in Y$ for any $I \in N_\delta$ that is a predense subset of $\mathbb{P}'_\omega \cap \delta$ the set I is also predense in $\mathbb{P}'_\omega$. So $\mathbb{P}'_\omega$ satisfies clause (c)(β) of Def. 2.11. By Lemma 2.14 the set Y is also a witness to clause (c)(δ) for $\mathbb{P}'_\omega \in K^1$.

Now we define an $\mathbb{P}'_\omega$-name $\underset{\sim}{D}'_\omega$ for an ultrafilter such that $\mathbf{p}_\omega = (\mathbb{P}'_\omega, \underset{\sim}{D}'_\omega) \in K^1$ and we define Y_* such that $Y_* \in \mathscr{D}_{\bar{N}}$ and $Y_* \subseteq Y(\mathbb{P}'_\omega, \underset{\sim}{D}'_\omega)$. We let

$$\mathbb{P}'_\omega \Vdash \underset{\sim}{E}' = \bigcup_{k\in\omega} \underset{\sim}{D}'_k.$$

Since $\mathbb{P}'_k$ is a complete suborder of $\mathbb{P}'_\omega$ the $\underset{\sim}{D}'_k$ are names for filters and $0_{\mathbb{P}'_{k+1}} \Vdash \underset{\sim}{D}'_k \subseteq \underset{\sim}{D}'_{k+1}$ the weakest element of $\mathbb{P}'_\omega$ forces that $\underset{\sim}{E}'$ is a $\mathbb{P}'_\omega$-name for a filter.

As in Equation (2.1) we let $C' = \{\delta \in \omega_1 : \delta \text{ limit and } (\forall \alpha \in \delta)(N_\alpha \in M_\delta)\}$. The set C' is club and hence and $Y_* := Y \cap C' \in \mathscr{D}_{\bar{N}}$. We write

$\mathrm{next}(Y_*,\varepsilon)$ for the next element in Y_* after ε, i.e., $\mathrm{next}(Y_*,\varepsilon) = \min\{\delta > \varepsilon \,:\, \delta \in Y_*\}$. By induction on $\delta \in Y_*$, we define a canonical $\mathbb{P}'_\omega \cap \delta$-name $\underset{\sim}{D}'_\omega(\delta) \in M_\delta$ such that

$$\begin{aligned} \mathbb{P}'_\omega \cap \delta \Vdash \text{``}\underset{\sim}{D}'_\omega(\delta) \supseteq \bigcup\{\underset{\sim}{D}'_\omega(\gamma) \,:\, \gamma \in Y_* \cap \delta\} \\ \text{and } \underset{\sim}{D}'_\omega(\delta) \text{ is an ultrafilter in } M_\delta,\text{''} \end{aligned}$$

and

$$\begin{aligned} \mathbb{P}'_\omega \cap \mathrm{next}(Y_*,\delta) \Vdash \text{``}(\forall f \in M_\delta[\mathbb{P}'_\omega])(\eta_\delta \geq_{\underset{\sim}{D}'_\omega(\mathrm{next}(Y_*,\delta))} f) \\ \text{and } \underset{\sim}{D}'_\omega(\mathrm{next}(Y_*,\delta)) \cap \mathcal{P}(\omega)^{N_\varepsilon} \text{ is an ultrafilter in } N_\varepsilon.\text{''} \end{aligned}$$

In the end we let $\mathbb{P}'_\omega \Vdash \underset{\sim}{D}'_\omega = \bigcup\{\underset{\sim}{D}'_\omega(\delta) \,:\, \delta \in Y_*\}$. The restriction of names was defined in Definition 2.10(2), and there is the following connection:

$$\{\delta \,:\, \underset{\sim}{D}'_\omega(\delta) = \underset{\sim}{D}'_\omega \restriction \delta\} \in \mathscr{D}_{\bar{N}}.$$

Assume that $\langle \underset{\sim}{D}'_\omega(\gamma) \,:\, \gamma \in Y_* \cap \delta\rangle$ has been defined. By the induction hypothesis on $(\mathbf{p}'_k, \pi_k)$, the $\mathbb{P}'_k$-names for ultrafilters $\underset{\sim}{D}'_k$ are defined and increasing in k.

We first consider the limit steps in the induction. Let $\delta \in Y_*$ be a limit of Y_*. First case: $\langle \underset{\sim}{D}'_\omega(\gamma) \,:\, \gamma < Y_* \cap \delta\rangle \notin M_\delta$. Then we let

$$1_{\mathbb{P}\cap\delta} \Vdash \underset{\sim}{D}'_\omega(\delta) = \bigcup\{\underset{\sim}{D}'_\omega(\gamma) \,:\, \gamma \in Y_* \cap \delta\}.$$

Second case: $\langle \underset{\sim}{D}'_\omega(\gamma) \,:\, \gamma \in Y_* \cap \delta\rangle \in M_\delta$. We first show

$$1 \Vdash_{\mathbb{P}'_\omega \cap \delta} \underset{\sim}{F}'(\delta) := \underset{\sim}{E}' \restriction \delta \cup \bigcup\{\underset{\sim}{D}'_\omega(\gamma) \,:\, \gamma \in Y_* \cap \delta\} \text{ is a filter base.''}$$

We assume, for a contradiction, that there are a condition $p \in \mathbb{P}'_\omega$, $k \in \omega$, and a $\gamma \in Y_* \cap \delta$ and there are names X, Y, such that p forces that $X \in \underset{\sim}{D}'_k \restriction \delta$ and $Y \in \underset{\sim}{E}' \restriction \delta$, $\gamma \in Y_* \cap \delta$ such that $X \cap Y$ is empty. Then $p \restriction \mathbb{P}'_k \Vdash X \in \underset{\sim}{D}'_k \restriction \delta$. Let $\mathbf{G}_k$ be $\mathbb{P}'_k$-generic over N_δ with $p \restriction \mathbb{P}'_k \in \mathbf{G}_k$. We let $Z[\mathbf{G}_k] = \{n \,:\, (\exists \tilde{q} \in \mathbb{P}'_\omega \cap \delta/\mathbf{G}_k)(\tilde{q} \geq p[\mathbf{G}_k] \wedge \tilde{q} \Vdash n \in Y[\mathbf{G}_k] \cap X)\}$. Since $\mathbf{p}_k$ is a condition the name $\underset{\sim}{D}'_\omega(\gamma) \restriction \delta$ is an ultrafilter compatible with $\underset{\sim}{D}'_k(\gamma)$. Therefore we have that $p \restriction \mathbb{P}'_k \Vdash_{\mathbb{P}'_k}$ "$Z[\mathbf{G}_k]$ is infinite." Now we take $n \in \omega$, $\tilde{q}$ as in the definition of $Z[\mathbf{G}_k]$, so that $\tilde{q} \Vdash n \in X \cap Y$. So we have a contradiction. Hence for any $\gamma \in Y_* \cap \delta$, the weakest condition forces that $\underset{\sim}{E}' \restriction \delta \cup \underset{\sim}{D}'_\omega(\gamma)$ is a filter basis. Since the names $\underset{\sim}{D}'_\omega(\gamma)$ are forced to be increasing with $\gamma \in Y_* \cap \delta$, also their union, $\underset{\sim}{F}'(\delta)$, is forced to be a filter basis. Now we choose a name $\underset{\sim}{D}'_\omega(\delta) \in M_\delta$ for an ultrafilter that extends $\underset{\sim}{F}'(\delta)$.

Now we consider the case of a δ that is a successor in Y_*, $\delta = \mathrm{next}(Y_*,\gamma)$. Then $N_\gamma \in M_\delta$. We extend $\underset{\sim}{D}'_\omega(\gamma)$ to $\underset{\sim}{D}'_\omega(\delta) \in M_\delta$ so that $\underset{\sim}{D}'_\omega(\delta)$ is a $\mathbb{P}' \cap \delta$-name for an ultrafilter such that

$$\begin{aligned} &1_{\mathbb{P}\cap\delta} \Vdash \underset{\sim}{D}'_\omega(\delta) \supseteq \underset{\sim}{F}(\delta) := (\underset{\sim}{E}' \restriction \delta) \cup \underset{\sim}{D}'_\omega(\gamma) \\ &\cup \big\{\{n \in \omega \,:\, \eta_\gamma(n) \geq \underset{\sim}{f}(n)\} \,:\, \underset{\sim}{f} \in M_\gamma \text{ a } \mathbb{P}'_\omega \cap \delta\text{-name for a function}\big\}. \end{aligned}$$

Since $\gamma \in Y_*$, we can restrict the considerations to $\mathbb{P}'_\omega \cap \gamma$ names $\underset{\sim}{f}$. Again we show that the weakest condition forces that $\underset{\sim}{F}(\delta)$ has the finite intersection property. Let $q_0 \in \mathbb{P}'_\omega \cap \delta$ be given. Let q_0 force that $\underset{\sim}{A}_1$ be a name of a member of $\underset{\sim}{D}'_k \restriction \delta$ and $q_0 \Vdash \underset{\sim}{A}_2 \in \underset{\sim}{D}'_\omega(\delta)$ and $A_3 = \{n : \eta_\gamma(n) > \underset{\sim}{f}(n)\}$. Now in M_δ we define a $(\mathbb{P}'_k \cap \delta)$-name $\underset{\sim}{A}_{23}$ as follows: if $\mathbf{G}_k \subseteq \mathbb{P}'_{\mathbf{p}_k}$, $q_0 \restriction \mathbb{P}'_k \in G_k$ is $\mathbb{P}'_k$-generic over M_δ we let

$$\begin{aligned}\underset{\sim}{A}_{23}[\mathbf{G}_k] =\{n \,:\, &(\exists \hat{q} \in \mathbb{P}'_\omega \cap \delta / \mathbf{G}_k)\\ &(\hat{q} \geq q_0[\mathbf{G}_k] \wedge \hat{q} \Vdash (n \in \underset{\sim}{A}_2[\mathbf{G}_k] \wedge \eta_\gamma(n) \geq \underset{\sim}{f}[\mathbf{G}_{\mathbf{p}_k}](n)))\}.\end{aligned}$$

Then $q_0 \restriction \mathbb{P}'_k \Vdash_{\mathbb{P}'_k} \underset{\sim}{A}_1 \cap \underset{\sim}{A}_{23}[\mathbf{G}_k]$ is infinite, since for $\mathbb{P}'_k$ is already an approximation and hence η_γ is Cohen generic also over $M_\gamma[\mathbb{P}'_k]$ and not $\leq_{\underset{\sim}{D}'_k} f$. We take $\hat{q}$, n as in the definition of $A_{23}[\mathbf{G}_k]$. Since $q_0 \restriction \mathbb{P}'_k$ is $\mathbb{P}'_k$-generic over M_δ, we may assume that $\hat{q} \in \mathbb{P}'_\omega$, $\hat{q} \restriction \mathbb{P}'_k \geq q_0$ and $\hat{q} \Vdash$ "$n \in \underset{\sim}{A}_1 \cap \underset{\sim}{A}_{23}$." Hence in M_δ there is a name for an ultrafilter $\underset{\sim}{D}'_\omega(\delta)$ containing $\underset{\sim}{F}(\delta)$, and we choose and fix the $<_\chi$-least one and call it $\underset{\sim}{D}'_\omega(\delta)$. Since $N_\gamma \subseteq \mathbb{M}_\delta$ and $N_\gamma \in M_\delta$, $\underset{\sim}{D}'_\omega(\delta) \cap \mathcal{P}(\omega)^{N_\gamma}$ is an ultrafilter in N_γ. Now the induction on $\delta \in Y_*$ is carried out. We let $\mathbb{P}'_\omega \Vdash \underset{\sim}{D}'_\omega = \bigcup\{\underset{\sim}{D}'_\omega(\delta) : \delta \in Y_*\}$. We mirror the construction back to the class K^2 by letting $\underset{\sim}{D}_\omega = \pi(\underset{\sim}{D}'_\omega)$ with $\pi = \bigcup_{n\in\omega} \pi_n$, where π means also the canonical extension of the map π from conditions to names.

(D) The proof is to a large extent contained in the proof of Lemma 2.23, item $\oplus_{3,i}$ on page 15, but we elaborate[1]. So let $\mathbf{p} \in K^2$ be given. We write $\mathbf{p}_i = (\mathbb{P}_i, \underset{\sim}{D}_i)$, $i < \omega_1$. The strategy of the protagonist is to choose in addition to $\mathbf{p}_i \geq_{AP} \mathbf{q}_j$ for $j < i$, on the side also $\mathbf{p}'_i = (\mathbb{P}'_i, \underset{\sim}{D}'_i)$ and $\pi_i \colon \mathbb{P}'_i \to \mathbb{P}_i$ and $\xi_i \in \omega_1$ with the following properties:

(a) $\langle \xi_i : i < \omega_1 \rangle$ is continuously increasing and $\xi_i \in C'$ (where C' is from (2.1))

(b) $(\mathbb{P}'_i, \underset{\sim}{D}'_i) \in K^1$, $\mathbb{P}'_i \setminus \bigcup\{\mathbb{P}'_j : j < i\} \subseteq [\xi_i + 1, \omega_1)$.

(c) π_i is a isomorphism from $\mathbb{P}'_i$ onto $\mathbb{P}_i$ mapping $\underset{\sim}{D}'_i$ onto $\underset{\sim}{D}_i$.

(d) for $j < i$, $\pi_j \subseteq \pi_i$, (so the $\mathbb{P}'_i$ are $\subseteq$-increasing in ω_1),

(e) for $j < i$, $(\mathbb{P}'_j, \underset{\sim}{D}'_j) \leq_{AP^1} (\mathbb{P}'_i, \underset{\sim}{D}'_i)$ and $(\mathbb{P}_j, \underset{\sim}{D}_j) \leq_{AP} (\mathbb{P}_i, \underset{\sim}{D}_i)$.

(f) If $i < j \leq k$, $p \in \mathbb{P}'_i$ and $q \in \mathbb{P}_j \cap \xi_k$ and p and q are compatible in $\mathbb{P}_j$, then they are compatible with a witness in $\mathbb{P}_j \cap \xi_k$. (Then the proof of [18, Claim 3.2] for showing that also each limit $\mathbb{P}_k$ has the $\bar{N}$-c.c. works)

(g) If $i = j + 1 < \omega_1$ is a successor ordinal, then COM chooses $\mathbf{p}_i = \mathbf{q}_j$.

(h) If $i < \omega_1$ is a limit ordinal and $\xi_i = i$ and if there is $j(*) < i$ such that

$$H = \bigcap\{Y(\mathbb{P}'_j, \underset{\sim}{D}'_j) : j \in [j(*), i)\} \in \mathscr{D}_{\bar{N}}.$$

[1]Here in the proof of item(D), player COM just has to cope with any steps of INC. In the mentioned Lemma we let INC play such that the cofinality of the symmetric group is increased by his steps. We know that COM can cope and we work on INC's part there.

then player COM takes for $\mathbf{p}_i$ the limit of a countable cofinal sequence of $\mathbf{q}_j$'s in the manner described in (C). Thus

$$H \subseteq Y(\mathbb{P}'_i, \underset{\sim}{D}'_i). \tag{2.3}$$

Now if $\mathbf{p}'_i$, $i < \omega_1$, are defined, in the ω_1-limit COM chooses $\mathbb{P}'_{\omega_1}$ as the direct limit. Then equation (2.3) implies that

$$Y(\mathbb{P}'_{\omega_1}, \underset{\sim}{D}'_{\omega_1}) \supseteq \triangle_{i\in\omega_1} Y(\mathbb{P}'_i, \underset{\sim}{D}'_i) \cap \{i : \xi_i = i\},$$

and hence $Y(\mathbb{P}'_{\omega_1}, \underset{\sim}{D}'_{\omega_1}) \in \mathscr{D}_{\bar{N}}$. Hence $1_{\mathbb{P}'}$ that $\underset{\sim}{D}'_{\omega_1} = \bigcup_{i<\omega_1} \underset{\sim}{D}'_i$ is an ultrafilter extending $\underset{\sim}{D}'_i$, $i < \omega_1$. We mirror the primed objects via $\tilde{\bigcup}_{j<\omega_1} \pi_j$ back to K^2 and thus we get a forcing $\mathbb{P}_{\omega_1} = \bigcup\{\mathbb{P}_i : i < \omega\}$ and a $\mathbb{P}_{\omega_1}$-name $\underset{\sim}{D}_{\omega_1}$ for an ultrafilter over ω. Hence the protagonist COM has a winning strategy. $\dashv$

Lemma 2.18. *The AP-names*

$$\underset{\sim}{\mathbb{Q}} = \bigcup\{\mathbb{P} : (\exists \underset{\sim}{D})\ (\mathbb{P}, \underset{\sim}{D}) \in \underset{\sim}{\mathbf{G}}_{AP}\}$$

and

$$\underset{\sim}{E} = \bigcup\{\underset{\sim}{D} : (\exists \mathbb{P})\ (\mathbb{P}, \underset{\sim}{D}) \in \underset{\sim}{\mathbf{G}}_{AP}\}$$

satisfy

(a) $\Vdash_{AP} \underset{\sim}{\mathbb{Q}}$ *is a c.c.c. forcing of cardinality* $\aleph_2$*,*

(b) $\Vdash_{AP} \underset{\sim}{E}$ *is* $\underset{\sim}{\mathbb{Q}}$*-name of a non-principal ultrafilter,*

(c) if $(\mathbb{P}, \underset{\sim}{D}) \in AP$ *then* $(\mathbb{P}, \underset{\sim}{D}) \Vdash_{AP} \Vdash_{\underset{\sim}{\mathbb{Q}}} \langle \eta_\delta : \delta \in S\rangle$ *is a* $\leq_{\underset{\sim}{E}}$*-increasing sequence and cofinal in* $\omega^\omega/\underset{\sim}{E}$*.*

Proof (a): See [18, Ch. IV, Claim 1.6].

Proof (b): By the c.c.c. and the construction with direct limits, for every $AP * \underset{\sim}{\mathbb{Q}}$-name τ for a real there are a pair $\mathbf{p} = (\mathbb{P}, \underset{\sim}{D}) \in AP$ and a condition $p \in \mathbb{P}$, and a $\mathbb{P}$-name real τ' for such that $(\mathbf{p}, p) \Vdash_{AP*\underset{\sim}{\mathbb{Q}}} \tau' = \tau$.

Proof (c): We go over to the approximation forcing AP^1, the one working with the posets whose domain is a subset of ω_1. Suppose for a contradiction that $(\mathbb{P}', \underset{\sim}{D}') \geq_{AP^1} (\mathbb{P}, \underset{\sim}{D})$ and $(\mathbb{P}', \underset{\sim}{D}') \Vdash_{AP^1} \Vdash_{\underset{\sim}{\mathbb{Q}}} (\exists f \in {}^\omega\omega)(f \geq_{\underset{\sim}{E}} \langle \eta_\delta : \delta \in S\rangle)$. Then there is $(\mathbb{P}'', \underset{\sim}{D}'') \geq_{AP^1} (\mathbb{P}', \underset{\sim}{D}')$ and there is a canonical $\mathbb{P}''$-name $\underset{\sim}{h}$ such that

$$((\mathbb{P}'', \underset{\sim}{D}''), p) \Vdash_{AP^1 * \underset{\sim}{\mathbb{Q}}} \underset{\sim}{h} \geq_{\underset{\sim}{E}} \langle \eta_\delta : \delta \in S\rangle. \tag{2.4}$$

Since $\underset{\sim}{h}$ is a name of a real in the c.c.c. forcing $\mathbb{P}''$, there are some for some $\delta_0 < \omega_1$, $\underset{\sim}{h}' \in M_{\delta_0}$ and $\underset{\sim}{h}$ is a $\mathbb{P}'' \cap \delta_0$-name such that $((\mathbb{P}'', \underset{\sim}{D}''), p) \Vdash_{AP^1 * \underset{\sim}{\mathbb{Q}}} \underset{\sim}{h} = \underset{\sim}{h}'$. We fix such a δ_0, $\underset{\sim}{h}'$. Since $(\mathbb{P}'', \underset{\sim}{D}'') \in K^1$, by Lemma 2.8 there is $\delta \geq \delta_0$ such that $N_\delta \models (\forall h \in M_\delta[G_{\mathbb{P}'' \cap \delta}])(h \not\geq_{\underset{\sim}{D}''[G_{\mathbb{P}'' \cap \delta}]} \eta_\delta)$. We take a condition $q \in \mathbb{P}'' \cap \delta$, $q \geq_{\mathbb{P}''} p$, forcing $\forall h \in M_\delta[G_{\mathbb{P}''}] h \not\geq_{\underset{\sim}{D}''} \eta_\delta$. Thus $((\mathbb{P}'', \underset{\sim}{D}''), q) \geq ((\mathbb{P}'', \underset{\sim}{D}''), p)$ and this is a contradiction to Equation (2.4). $\dashv$

Now we show that the union of the generic filter of the approximation forcing, i.e., the $\underset{\sim}{\mathbb{Q}}$ as given in Lemma 2.18, fulfils $\Vdash_{AP*\underset{\sim}{\mathbb{Q}}} \mathrm{cf}(\mathrm{Sym}(\omega)) = \aleph_2$. The conditions of the form $((\mathbb{P}_*, \underset{\sim}{D}_*), p)$ with $p \in \mathbb{P}_*$ are dense in $AP * \underset{\sim}{\mathbb{Q}}$.

A forcing destroying a given increasing cofinal chain of subgroups $\langle \tilde{G}_i : i < \omega_1\rangle$ of $\mathrm{Sym}(\omega)$ is written down in [12]. Such a forcing adds one particular real, a new permutation g that simultaneously conjugates certain $f_i \in G_{i+1} \setminus G_i$ for cofinally many $i < \omega_1$. Thus in the extension we have $g \in \mathrm{Sym}(\omega) \setminus \bigcup\{G_i : i < \omega_1\}$. We modify this situation slightly:

Lemma 2.19. *Suppose that chain of subgroups $\langle G_i : i < \omega_1\rangle$ is an increasing chain of subgroups of $\mathrm{Sym}(\omega)$ such that all permutations that move only finitely many elements are elements of G_0. Suppose that $U \subseteq \omega_1$ is uncountable and there are*

$$\langle \zeta_i^0, \zeta_i^1, \zeta_i^2, f_i^1, f_i^2 : i \in U\rangle \text{ and } g$$

with the following properties:

(1) for $i < j \in U$, $i \le \zeta_i^0 < \zeta_i^1 < \zeta_i^2 < \zeta_j^0$,

(2) for $i \in U$, $f_i^1 \in G_{\zeta_i^1}$ and $f_i^2 \in G_{\zeta_i^2} \setminus G_{\zeta_i^1}$, and

(3) for $i \in U$, $(\forall^\infty n)((g \circ f_i^1)(n) = (f_i^2 \circ g)(n))$.

Then $g \in \mathrm{Sym}(\omega) \setminus \bigcup\{G_i : i \in \omega_1\}$.

Proof. If $g \in G_{\zeta_i^1}$ for some $i \in U$, then by (3) also $f_i^2 \in G_{\zeta_i^1}$, contradiction. ⊣

Now we show that, given a condition $(\mathbf{p}, p)$ with $\mathbf{p} = (\mathbb{P}, \underset{\sim}{D})$, $p \in \mathbb{P}$ and a $\mathbb{P}$-name of a sequence $(G_i : i < \omega_1)$ there is $(\mathbf{q}, q) \ge (\mathbf{p}, p)$, $\mathbf{q} = (\mathbb{P} * \underset{\sim}{\mathbb{R}}, \underset{\sim}{D}')$, $q \in \mathbb{P} * \underset{\sim}{\mathbb{R}}$ and there are $\mathbb{P} * \underset{\sim}{\mathbb{R}}$-names such that $(\mathbf{q}, q)$ forces that $\langle \zeta_i^0, \zeta_i^1, \zeta_i^2, f_i^1, f_i^2 : i \in \omega\rangle$ as in (1) and (2) with $U = \omega_1$, and at the same time there is a $\mathbb{P} * \underset{\sim}{\mathbb{R}}$-name (g, U) as in (3). So $(\mathbf{q}, q)$ forces that $(G_i : \underset{\sim}{i} < \omega_1)$ is not cofinal in $\mathrm{Sym}(\omega)$.

For carrying this out we use some notions describing permutation groups.

Definition 2.20. *Let $f : \omega \to \omega$.* $\mathrm{supp}(f) = \{n : f(n) \neq n\}$.

Observation 2.21. *If $f \in \mathrm{Sym}(\omega)$, then $f[\mathrm{supp}(f)] = \mathrm{supp}(f)$.*

For $f \in \mathrm{Sym}(\omega)$, we say f has order 2 if $f \circ f$ is the identity.

The point in the search for $\mathbf{q}$ is that the supports of the f_i^k, $k = 1, 2$, $i < \omega_1$, that will be called w_i^k, are almost disjoint Cohen reals that are "sufficiently disconnected from the η_δ and the $\underset{\sim}{D}_{\mathbf{p}}$" so that the forcing adding the new conjugating permutation preserves the unboundedness of the $\langle \eta_\delta : \eta < \omega_1\rangle$ in the order with respect to the ultrafilter belonging to $\underset{\sim}{D}_{\mathbf{q}}$. The phrase "sufficiently disconnected" is made precise in item $\oplus_{3,i}$ properties (j) and (k) in the proof of Lemma 2.23.

For arguing with given supports, we use:

Lemma 2.22. *(*[12, Lemma 3.3]*) If $\langle G_i : i < \omega_1\rangle$ is an increasing sequence of proper subgroups of* $\mathrm{Sym}(\omega)$ *with union* $\mathrm{Sym}(\omega)$*, and G_0 contains all permutations with finite support, then for any $W \in [\omega]^{\aleph_0}$ the sequence*

$$\langle G_i \cap \{f \in \mathrm{Sym}(\omega) \,:\, \mathrm{supp}(f) \subseteq W \wedge f \textit{ is of order } 2\} \,:\, i < \omega_1\rangle$$

is not eventually constant.

Now we return to forcing. For establishing item $\oplus_{3,i}(k)$ in the next lemma, we use the omitting types theorem for oracles:

Lemma 2.23. $\Vdash_{AP*\underset{\sim}{\mathbb{Q}}}$ *"*$\mathrm{cf}(\mathrm{Sym}(\omega)) = \aleph_2$*".*

Proof. Assume towards a contradiction:

$\oplus_1$ $((\mathbb{P}_*, \underset{\sim}{D}_*), p_*) \Vdash_{AP*\underset{\sim}{\mathbb{Q}}}$ "$\langle \underset{\sim}{G}_i : i < \omega_1\rangle$ is an increasing sequence of proper subgroups of $\mathrm{Sym}(\omega)$ with union $\mathrm{Sym}(\omega)$, and $\underset{\sim}{G}_0$ contains all permutations with finite support".

$\oplus_2$ By Lemma 2.22, $\oplus_1$ implies: $((\mathbb{P}_*, \underset{\sim}{D}_*), p_*) \Vdash_{AP*\underset{\sim}{\mathbb{Q}}}$ "if $W \in [\omega]^{\aleph_0}$ then $\langle \underset{\sim}{G}_i \cap \{f \in \mathrm{Sym}(\omega) : \mathrm{supp}(f) \subseteq W \wedge f$ is of order $2\} : i < \omega_1\rangle$ is not eventually constant".

$\oplus_3$ We let $\langle m_\eta : \eta \in {}^{\omega>}\omega\rangle$ be a sequence of natural numbers without repetitions. For $\eta \in {}^{\omega}\omega$ we let $W(\eta) = \{m_{\eta\restriction n} : n \in \omega\}$. Then for $\eta \neq \eta'$ and $k = \min\{n : \eta(n) \neq \eta'(n)\}$ we have $W(\eta) \cap W(\eta') = \{m_{\eta\restriction n} : n < k\}$.

We now choose $\mathbf{p}_i = (\mathbb{P}_i, \underset{\sim}{D}_i) \in AP$, π_i, $\mathbf{p}'_i \in AP^1$, $\xi_i \in \omega_1$, and $(\zeta^0_i, \zeta^1_i, \zeta^2_i, \underset{\sim}{f}^1_i, \underset{\sim}{f}^2_i, \underset{\sim}{\mathbb{R}}'_i)$ by induction on $i < \omega_1$ such that

$\oplus_{3,i}$ (a) $\mathbf{p}_0 = \mathbf{p}_*$,

(b) $\mathbf{p}_i = (\mathbb{P}_i, \underset{\sim}{D}_i) \in AP$ and $j < i \to \mathbf{p}_j \leq_{AP} \mathbf{p}_i$.

(c) $\mathbf{p}'_i = (\mathbb{P}'_i, \underset{\sim}{D}'_i) \in AP^1$ satisfies

(α) $\mathbb{P}'_0 \cap C = \emptyset$, the set of members of $\mathbb{P}'_i \setminus \bigcup\{\mathbb{P}'_j : j < i\} \subseteq [\xi_i + 1, \omega_1)$, hence $\mathbb{P}'_i \cap \xi_i = \mathbb{P}'_j \cap \xi_i$ for any $j \geq i$,

(β) $\pi_i \colon \mathbb{P}'_i \to \omega_1$ is a one-to-one function mapping $\mathbb{P}'_i$ onto $\mathbb{P}_i$ and mapping $\underset{\sim}{D}'_i$ onto $\underset{\sim}{D}_i$,

(γ) if $j < i$, then $\pi_j \subseteq \pi_i$,

δ $\langle \xi_i : i < \omega_1\rangle$ has the properties (a) to (d) of the proof of Lemma 2.17 (D) with respect to the sequence $\langle \mathbf{p}'_i, \pi_i : i < \omega_1\rangle$.

(d) At double successor steps of limit ordinals we add a new Cohen real: If $i = \omega j + 1$ then $\mathbb{P}'_{i+1} = \mathbb{P}'_i * ({}^{\omega>}\omega, \triangleleft)$, we let $\underset{\sim}{\nu}_i$ be a name for $({}^{\omega>}\omega, \triangleleft)$-generic real. So $\underset{\sim}{\nu}_i$ is a Cohen real over $\mathbf{V}^{\mathbb{P}'_{\omega\cdot j}}$.

(e) If $i = j + 2$ then $\langle \underset{\sim}{G}_\ell \cap \mathcal{P}(\omega)^{\mathbb{P}'_j} : \ell < \omega_1\rangle$ and even $\langle \underset{\sim}{G}_\ell \cap \mathcal{P}(\omega)^{\mathbb{P}'_i} : \ell < \omega_1\rangle$ is a $\mathbb{P}'_i$-name.

(f) At triple successors to limit ordinals we fix witnessing functions with the new Cohen ν_i as information in their support, i.e., if $i = \omega \cdot j + 2$ then $(\zeta^0_i, \zeta^1_i, \zeta^2_i, \underset{\sim}{f}^1_i, \underset{\sim}{f}^2_i)$ satisfies

(α) $\zeta_i^0 = i < \zeta_i^1 < \zeta_i^2$,

(β) for $\ell = 1, 2$, $\mathbf{p}'_{i+1}$ forces that $\underset{\sim}{f}_i^2 \in G_{\underset{\sim}{\zeta}_i^2} \setminus G_{\underset{\sim}{\zeta}_i^1}$, $\underset{\sim}{f}_i^1 \in G_{\underset{\sim}{\zeta}_i^1}$ is a $\mathbb{P}'_{i+1}$-name of a member of $\mathrm{Sym}(\omega)$ of order 2 such that

$$\mathbb{P}'_{i+1} \Vdash \mathrm{supp}(f_i^\ell) = \underset{\sim}{w}_i^\ell = W(\langle \ell \rangle \underset{\sim}{\frown} \nu_i).$$

Here $\langle \ell \rangle ^\frown \nu$ is the concatenation of the singleton $\langle \ell \rangle$ and ν i.e. $(\langle \ell \rangle ^\frown \nu)(k) = \ell$ if $k = 0$, and $= \nu(k-1)$ else.

By Lemma 2.20, countable ordinals ζ_i^1, ζ_i^2 and names $\underset{\sim}{f}_i^1$, $\underset{\sim}{f}_i^2$ exist.

(g) Now finally we explain the successors to limit ordinals. If i is a limit ordinal, $j < i$, and $H = \bigcap \{Y(\mathbb{P}'_\varepsilon, \underset{\sim}{D}'_\varepsilon) : \varepsilon \in [j, i)\} \neq \emptyset \in \mathscr{D}_{\bar{N}}$, then $H \cap C \subseteq Y(\mathbb{P}'_i, \underset{\sim}{D}'_i)$. Remark: This is like the crucial clause of the proof of Lemma 2.17(D)(f). At limit steps we take the direct limit the and a name for an ultrafilter as descibed in Lemma 2.17. For limit ordinals $i < \omega_1$, we let ξ_i be as follows

$$\begin{aligned} \xi_i = \min\Big\{\delta \in Y(\mathbb{P}'_i, \underset{\sim}{D}'_i) \; : \; & (\forall j < i)(\delta > \xi_j) \wedge (\forall j_1 \in \delta) \\ & \big((\zeta_{j_1}^0, \zeta_{j_1}^1, \zeta_{j_1}^2, \underset{\sim}{f}_{j_1}^1, \underset{\sim}{f}_{j_1}^2) \in M_\delta \; \wedge N_{j_1} \in M_\delta \wedge \\ & \zeta_{j_1}^0, \zeta_{j_1}^1, \zeta_{j_1}^2, \underset{\sim}{f}_{j_1}^1, \underset{\sim}{f}_{j_1}^2 \text{ are } \mathbb{P}'_i \cap \delta\text{-names}\big)\Big\}. \end{aligned} \tag{2.5}$$

The set of relevant δ's is in $\mathscr{D}_{\bar{N}}$, hence it is not empty, and ξ_i is well-defined.

(i) Now we define $\underset{\sim}{\mathbb{R}}'_i \in M_\xi$: $\underset{\sim}{\mathbb{R}}'_i$ is a $\mathbb{P}'_i \cap \xi$-name of a c.c.c. forcing notion. A member of $\mathbb{R}'_i$ has the form (u, g) such that

(α) $u \subseteq \{\omega \cdot j + 1 : \omega \cdot j + 1 \in \xi_i\}$ is finite, g a finite partial permutation of order two, $\mathrm{dom}(g) \subseteq \bigcup_{\varepsilon \in u} w_\varepsilon^2$, such that $\varepsilon \in u$ implies $\mathrm{range}(g) \subseteq w_\varepsilon^1$.

(β) The sets $\mathrm{dom}(g)$ and $\mathrm{range}(g)$ are sufficiently large in the following sense:

- if $\delta \neq \varepsilon \in u$ then we fix n, such that $\nu_\delta \restriction n \neq \nu_\varepsilon \restriction n$ and then require that for $k = 1, 2$ the set $\{m_{\langle k \rangle ^\frown \nu_\delta \restriction \ell} : \ell < n\} \subseteq \mathrm{dom}(g) \cap \mathrm{range}(g)$,
- $\forall \varepsilon \in \mathrm{dom}(p)$, if ε is Cohen coordinate and $p(\varepsilon) \in 2^n$, $\ell \leq n$, $k = 1, 2$, then $m_{k ^\frown p(\varepsilon) \restriction \ell} \in \mathrm{dom}(g) \cap \mathrm{range}(g)$.

(γ) If $\varepsilon \in u$ then $\mathrm{dom}(g) \cap w_\varepsilon^2$ is closed under f_ε^1 and $\mathrm{range}(g) \cap w_\varepsilon^1$ is closed under f_ε^2.

(δ) For $(u_1, g_1), (u_2, g_2) \in \mathbb{R}'_i$ we let $(u_1, g_1) \leq (u_2, g_2)$ iff

(i) $u_1 \subseteq u_2$,

(ii) $g_1 \subseteq g_2$,

(iii) $(\forall \varepsilon \in u_1)(\forall n \in w_\varepsilon^2 \cap (\mathrm{dom}(g_2) \setminus \mathrm{dom}(g_1))\big(g_2(n) \in w_\varepsilon^1 \wedge f_\varepsilon^2(g_2(n)) = g_2(f_\varepsilon^1(n))\big)$.

We let $\mathbb{P}'_{i+1} = \mathbb{P}'_i * \mathbb{R}'_i$.

(j) We prove below: If $i_1 < i$ then $\underset{\sim}{\mathbb{R}}'_{i_1} \subseteq_{ic} \underset{\sim}{\mathbb{R}}'_i$ and if $I \in N_{i_1}$ is a predense subset of $\mathbb{P}'_{i_1} \cap \xi_{i_1} * \underset{\sim}{\mathbb{R}}'_{i_1}$ then I is predense in $\mathbb{P}'_i \cap \xi_i * \underset{\sim}{\mathbb{R}}'_i$.

(k) Together with $\mathbb{P}'_i$ we choose $\underset{\sim}{D}'_i$ such that $(\mathbb{P}'_i, \underset{\sim}{D}'_i) \in K^1$. In the limit steps this is done as in the proof of Lemma 2.17 (C). In the successor steps, since for a set $Y \in D_{\bar{N}}$ for $\varepsilon \in Y$, the Cohen real η_ε stays Cohen, hence unbounded, over $M_\varepsilon[\mathbb{P}'_{i+1}]$, there is such a $\underset{\sim}{D}'_i$.

We recall [12, Lemma 3.4]: Any two permutations with almost disjoint supports are almost conjugated and if the permutations both have order 2 then there is a conjugator of order 2.

We prove item (j): $\mathbb{P}'_{\xi_i} \Vdash \underset{\sim}{\mathbb{R}}'_{i_1} \subseteq_{ic} \underset{\sim}{\mathbb{R}}'_i$ follows from the definition of the orders $\mathbb{R}'_j$.

Assume that $D_0 \in N_{i_1}$ is an open dense subset of $\mathbb{P}'_{i_1} \cap \xi_{i_1} * \mathbb{R}_{i_1}$, and $p = (p \restriction \xi_{i_1}, p(\xi_{i_1})) \in (\mathbb{P}'_i \cap i * \mathbb{R}'_i)$. We have to find a condition in $q \in D_0$ that is compatible with p. Assume that $p \cap \xi_{i_1} \Vdash \mathbb{P}'_{\xi_{i_1}} p(i_1) = (u, g)$ and u, g are pinned down in $\mathbf{V}$, not names. After possibly strengthening p and g we can assume that g is so strong that it fulfils:

$$\begin{aligned}\mathrm{dom}(g) \supseteq & \{m_{p(\beta)\restriction k} \,:\, \beta \in \mathrm{supp}(p), \beta \text{ successor ordinal },\\ & \beta \in u, k \leq |p(\beta)| \wedge \mathbb{P}'_\beta = \mathbb{P}'_{\beta-1} * ({}^{\omega>}\omega, \lhd)\}\\ \mathrm{range}(g) \supseteq & \{(f^1_\beta)(m_{p(\beta)}) \,:\, \beta \in \mathrm{supp}(p), \beta \text{ successor ordinal }, \beta \in u,\\ & k \leq |p(\beta)| \wedge \mathbb{P}'_\beta = \mathbb{P}'_{\beta-1} * ({}^{\omega>}\omega, \lhd)\}\end{aligned}$$

After possibly further strengthening p we can assume that $p \restriction \xi_{i_1}$ determines ζ^j_β for $j = 0, 1, 2$ and determines $\underset{\sim}{f}^2_\beta$ restricted to the set on the right-hand side of the first eqution, and determines $\underset{\sim}{f}^1_\beta$ on the right-hand side of the second equation for any $\beta \in u$. We assume the analogous strength of p' for all triples $(p', (u', g'))$ appearing later in the proof. We assume that $\mathrm{dom}(g) \in \omega$ and that $\mathrm{dom}(g)$ is larger than any $W^2_\varepsilon \cap W^2_\zeta$ for $\varepsilon \neq \zeta \in u$ and that $\mathrm{range}(g)$ is a superset of $W^1_\varepsilon \cap W^1_\zeta$ for $\varepsilon \neq \zeta \in u$.

Now we choose $p_0 = (p \restriction \xi_{i_1}, u \cap \xi_{i_1}, g)) \in M_{\xi_{i_1}}$ We choose $q_0 \geq p_0$, $q_0 \in D \cap \xi_{i_1} \cap M_{\xi_{i_1}}$. Then q_0 does not determine more of the ν_ε than p_0 does. Then we take $q_0 \leq q_1$ such that

$$\begin{aligned}& q_1 = (q_0 \restriction \xi_{i_1} \cup \{(\varepsilon, q_1(\varepsilon)) \,:\, \varepsilon \in u \setminus \xi_{i_1}\}, (u \cup u_{q_1}, g_{q_0}))\\ & \quad \text{where for each } \varepsilon \in u \setminus \xi_{i_1},\\ & q_1(\varepsilon) \Vdash W(0^\frown \underset{\sim}{\nu}_\varepsilon) \cap (\mathrm{dom}(g_{q_0}) \setminus \mathrm{dom}(g)) = \emptyset \wedge\\ & \qquad W(1^\frown \underset{\sim}{\nu}_\varepsilon) \cap (\mathrm{range}(g_{q_0}) \setminus \mathrm{range}(g)) = \emptyset.\end{aligned}$$

This special point (not in [18, Ch. VI],[17]) is that the $\underset{\sim}{\nu}_i$, η_i are really Cohen: Defining relevant generic objects that have a Cohen real as domain allows us to carry on the oracle-c.c. and thus to preserve the Cohenness of the η_i. This main trick is also used in the next section.

Now q_1 is compatible with p. $\dashv$

So the oracle-c.c., i.e. item (j) at N_i, is proved. Hence by the omitting types theorem, η_i stays Cohen generic over M_i also in the extension by $\mathbb{P}'_{i+1}$.

$\oplus_4$ Once the induction is performed, we define $\mathbf{p}_{\omega_1} = (\mathbb{P}_{\omega_1}, \underset{\sim}{D}_{\omega_1})$ and $\mathbf{p}'_{\omega_1} \in K^1$ and $\pi = \bigcup_{i<\omega_1} \pi_i$ which maps $\mathbf{p}'_{\omega_1}$ onto $\mathbf{p}_{\omega_1}$ as follows:

(a) $\mathbb{P}'_{\omega_1} = \bigcup\{\mathbb{P}'_i \cap \xi_i * \mathbb{R}'_i \,:\, i < \omega_1\}$,

(b) $\mathbb{P}'_{\omega_1} \Vdash \underset{\sim}{D}'_{\omega_1} = \bigcup\{\underset{\sim}{D}'_i \,:\, i < \omega_1\}$,

(c) $\pi = \bigcup_{i<\omega_1} \pi_i$ is a isomorphism from $\mathbb{P}'_{\omega_1}$ onto $\mathbb{P}_{\omega_1}$ mapping $\underset{\sim}{D}'_{\omega_1}$ to $\underset{\sim}{D}_{\omega_1}$.

(d) $\bigwedge_{i<\omega_1} \mathbf{p}_i \le \mathbf{p}_{\omega_1} \in K^2$, $\bigwedge_{i<\omega_1} \mathbf{p}'_i \le \mathbf{p}'_{\omega_1} \in K^1$.

This finishes the construction of a stronger member in in AP-forcing.

$\oplus_5$ Let

$$\underset{\sim}{g} = \bigcup_{i\in S''} \{g \,:\, \exists p \exists u (p, (u,g)) \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}'_{\omega_1}}\}$$

$$\underset{\sim}{U} = \bigcup_{i\in S''} \{u \,:\, \exists p \exists g (p, (u,g)) \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}'_{\omega_1}}\}$$

We claim:

$$\mathbb{P}'_{\omega_1} \Vdash |\underset{\sim}{U} = \aleph_1| \wedge \text{“}g \notin \bigcup\{\underset{\sim}{G}_i \,:\, i < \omega_1\}\text{”}.$$

Proof: By the construction of $\mathbb{P}'_{\omega_1}$ we have

$$(\forall i < j \in S \cap C)\big(\underset{\sim}{f}^\ell_i \in M_j \wedge \underset{\sim}{f}^\ell_i \text{ is a } \mathbb{P}'_{\omega_1} \cap j\text{-name}\big).$$

The forcing $\mathbb{P}'_{\omega_1} \le_{K^1} \mathbb{Q}$, and hence $\underset{\sim}{\mathbb{Q}}$ adds a g: $\bigcup_{\varepsilon\in U} w_\varepsilon \to \bigcup_{\varepsilon\in U} w_\varepsilon$ that conjugates for $i \in \underset{\sim}{U}$, $f^1_i \in G_{\zeta^1_i}$ and $f^2_i \in G_{\zeta^2_i} \smallsetminus G_{\zeta^1_i}$, namely for some j such that $\mathbf{t}_j =$ true there is $i \in u$, with $(u,g) \in \mathbf{G}_{\mathbb{R}'_j}$, $(u,g) \Vdash i \in \underset{\sim}{U}$. Hence $i \in X$ and g conjugates f^1_i and f^2_i up to a finite mistake, by $\oplus_{3,j}$ item (i)(δ)(iii). Since $\mathrm{dom}(f^\ell_i) = w^\ell_i = W_{\langle \ell\rangle ^\frown \nu_i}$ this means full conjugation: $g \circ f^1_i \circ g = f^2_i$ up to finitely many arguments. But g is in some subgroup G_ζ. So for $i > \zeta$, $i \in X$, $f^2_i \in G_{\zeta^1_i}$, contradiction.

$\dashv$

End of proof of Theorem 2.1:

We assume that $S \subseteq \omega_1$ is stationary and $\mathbf{V} \models \diamondsuit^-_S$. We extend $\mathbf{V}$ with the forcing poset $AP * \underset{\sim}{\mathbb{Q}}$. By Lemma 2.18, $\mathfrak{mcf} = \aleph_1$ in the extension, and by Lemma 2.23, $\mathrm{cf}(\mathrm{Sym}(\omega)) = \aleph_2$. $\dashv$

3. On $\mathrm{Con}(\mathfrak{b} = \mathrm{cf}(\mathrm{Sym}(\omega)) < \mathfrak{mcf})$

Now we show that $\mathfrak{b} = \mathrm{cf}(\mathrm{Sym}(\omega)) < \mathfrak{mcf}$ is consistent relative to ZFC. In [13] we established that it is consistent relative to ZFC that $\aleph_1 = \mathfrak{b} = \mathfrak{g} < \aleph_2 = \mathfrak{mcf}$. Brendle and Losada showed that $\mathfrak{g} \leq \mathrm{cf}(\mathrm{Sym}(\omega))$ in ZFC, see [7]. So the following theorem gives another consistency proof for $\aleph_1 = \mathfrak{b} = \mathfrak{g} < \aleph_2 = \mathfrak{mcf}$.

Theorem 3.1. *It is consistent relative to* ZFC *that* $\mathfrak{b} = \mathrm{cf}(\mathrm{Sym}(\omega)) < \aleph_2 = \mathfrak{mcf}$.

For the proof we will again work with oracle c.c.-forcing. Let $D \subseteq [\omega]^\omega$ be a filter over ω. Then we write D^+ for the D-positive sets, i.e., $X \in D^+$ iff $X \cap Y$ is infinite for any $Y \in D$.

Lemma 3.2. *Let* $\kappa \geq \aleph_2$ *be a cardinal in* $\mathbf{V}$. *The* $(A)_\kappa$ *implies* $(B)_\kappa$.

$(A)_\kappa$ *For every filter* $D \subseteq [\omega]^\omega$ *over* ω *such that* $\mathcal{P}(\omega)/D$ *has the c.c.c. (that is: for every* A_i, $i < \omega_1$, *such that* $A_i \in D^+$ *there are* $i \neq j$ *such that* $A_i \cap A_j \in D^+$*) for every regular* $\kappa_* < \kappa$, *for every sequence* $\langle f_i : i < \kappa_* \rangle$ *of functions* $f_i \in {}^\omega\omega$ *there is* $g \in {}^\omega\omega$ *such that for unboundedly many* $i < \kappa_*$, $\neg g \leq_D f_i$.

$(B)_\kappa$ *After forcing with the forcing* $\mathbb{Q}$ *for adding* $\aleph_1$ *random reals in the extension* $\mathbf{V}^{\mathbb{Q}}$ *for every non-principal ultrafilter* D *on* ω, $\mathrm{cf}({}^\omega\omega/D) \geq \kappa$, *and* $\mathfrak{b}^{\mathbf{V}} = \mathfrak{b}^{\mathbf{V}^{\mathbb{Q}}}$.

Proof. Assume A_κ and that $\kappa_* < \kappa$ and $q_0 \in \mathbb{Q}$ forces "$\underset{\sim}{D}$ is an ultrafilter over ω and $\langle \underset{\sim}{f}_\alpha : \alpha < \kappa_* \rangle$ is increasing modulo $\underset{\sim}{D}$". So κ_* is regular and uncountable in $\mathbf{V}^{\mathbb{Q}}$ and hence regular and uncountable in $\mathbf{V}$. We shall show that there is $q_* \geq q_0$,

$$(\boxdot) \qquad q_* \Vdash \exists f \in ({}^\omega\omega) \bigwedge_{\alpha < \kappa_*} \underset{\sim}{f}_\alpha <_{\underset{\sim}{D}} f,$$

and thus we will have established $(B)_\kappa$.

Since $\mathbb{Q}$ is ${}^\omega\omega$-bounding, we can take $g_\alpha \in \mathbf{V}$ for $\alpha \in \kappa_*$ such that $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{f}_\alpha \leq^* g_\alpha$".

We let

$$E = \{A \in \mathcal{P}(\omega)^{\mathbf{V}} : (\exists q \in \mathbb{Q}) q \geq q_0 \wedge q \Vdash \check{A} \in \underset{\sim}{D}\}$$

and we let

$$D' = \{A \in \mathcal{P}(\omega)^{\mathbf{V}} : q_0 \Vdash \check{A} \in \underset{\sim}{D}\}.$$

Then we have $E, D' \in \mathbf{V}$ and the following holds:

(1) D' is a filter over ω.

(2) $E \subseteq (D')^+$. Let $A \in E$, say $q \Vdash A \in \underset{\sim}{D}$, $q \geq q_0$ and let $B \in D'$. Then $q \Vdash A \in \underset{\sim}{D} \wedge B \in \underset{\sim}{D}$, so $q \Vdash$ '$A \cap B$ is infinite." Since $A, B \in \mathbf{V}$, $A \cap B$ is infinite. Since this holds for every $B \in D'$, item (2) is proved.

(3) $(D')^+ \subseteq E$. Suppose that $X \notin E$. Then $\forall q \in \mathbb{Q}$, $q \geq q_0$ implies that $q \not\Vdash X \in \underset{\sim}{D}$, so $q_0 \Vdash X \notin \underset{\sim}{D}$. Since $\underset{\sim}{D}$ is a name of an ultrafilter $q_0 \Vdash X^c \in \underset{\sim}{D}$. So $X^c \in D'$ and $X \notin (D')^+$.

(4) So together: $(D')^+ = E$.

(5) D' is a c.c.c. filter. Proof: Let $A_\alpha \in (D')^+ = E$ for $\alpha \in \omega_1$, via q_α. Since $\mathbb{Q}$ is c.c.c there are $\alpha \neq \beta$ such that $q_\alpha \not\perp q_\beta$. Then there is $r \in \mathbb{Q}$, $r \Vdash A_\alpha \in \underset{\sim}{D}, A_\beta \in \underset{\sim}{D}$, and hence $r \Vdash A_\alpha \cap A_\beta \in \underset{\sim}{D}$ since $\underset{\sim}{D}$ is forced to be a filter. So $A_\alpha \cap A_\beta \in D'^+$.

Let g be as in the condition $(A)_\kappa$, applied to D' and $\langle g_\alpha : \alpha < \kappa \rangle$, so for some cofinal set $u \sup \kappa_*$ we have for $\alpha \in u \subseteq \kappa_*$, $\neg g \leq_{D'} g_\alpha$. Hence for $\alpha \in u$, $q_0 \not\Vdash \{n : g(n) \leq g_\alpha(n)\} \in \underset{\sim}{D}$ and there is $\tilde{q}_\alpha \geq q_0$, $\tilde{q}_\alpha \Vdash \{n : g(n) \leq g_\alpha(n)\} \notin \underset{\sim}{D}$. Thus $\tilde{q}_\alpha \Vdash \{n : g(n) > g_\alpha(n)\} \in \underset{\sim}{D}$ and the choice of g_α implies $\tilde{q}_\alpha \Vdash \{n : g(n) > \underset{\sim}{f}_\alpha(n)\} \in \underset{\sim}{D}$. Since $\mathbb{Q}$ has the c.c.c., we have $\mathrm{cf}(\kappa_*) > \omega$. Therefore κ_*-many of the $\tilde{q}_\alpha$ are in the generic filter. So for any $\mathbb{Q}$-generic filter G with $q_0 \in G$ we have $\underset{\sim}{f}_\alpha[G] \leq_{\underset{\sim}{D}[G]} g$ for cofinally many $\alpha \in u$. Hence a condition $q_* \geq q_0$ forces this. Since the sequence $\langle \underset{\sim}{f}_\alpha : \alpha < \kappa_* \rangle$ is $\leq_{\underset{\sim}{D}}$-increasing, we get $q_* \Vdash$ "$(\forall \alpha < \kappa_*)(\underset{\sim}{f}_\alpha \leq_{\underset{\sim}{D}} g)$." Thus Equation ($\boxdot$) and the first statement of $(\mathrm{B})_\kappa$ are proved.

Since the forcing adding $\aleph_1$ random reals is ${}^\omega\omega$-bounding, we have $\mathfrak{b}^{\mathbf{V}} = \mathfrak{b}^{\mathbf{V}^{\mathbb{Q}}}$. ⊣

In the extension $\mathbf{V}^{\mathbb{Q}}$ of Lemma 3.2 we have $\mathrm{cf}(\mathrm{Sym}(\omega)) = \aleph_1$ by [16, Theorem 1.6]. So if we succeed to establish the condition $(A)_\kappa$ of the lemma together with $\mathfrak{b} < \kappa$ for some $\kappa \geq \aleph_2$, we are done. We fix a stationary $S \subseteq \omega_1$ and take $\kappa = \aleph_2$ and we work again with oracle-c.c. forcings in order to establish the consistency of $(A)_{\aleph_2}$ and $\mathfrak{b} = \aleph_1$.

Lemma 3.3. *We assume that in $\mathbf{V}$, the set S is stationary in ω_1 and the two diamond principles $\diamondsuit_S$ and $\diamondsuit_{\{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_1\}}$ hold. Then there is an oracle c.c. forcing notion $\mathbb{P}$ such that in $\mathbf{V}^{\mathbb{P}}$ we have $(A)_{\aleph_2}$ of the previous lemma, and $\mathfrak{b} = \omega_1$.*

Proof. We fix in $\mathbf{V}$ a $\leq^*$-increasing sequence $\langle g_\delta : \delta < \omega_1 \rangle$ that is $\leq^*$-unbounded. We fix an oracle $\bar{M} = \langle M_\varepsilon : \varepsilon \in S \rangle$ such that the $\bar{M}$-c.c. ensures that the type $\bigwedge_{\delta < \omega_1} x \geq^* g_\delta$ is omitted. Indeed, $\langle g_\delta : \delta \in \omega_1 \rangle \in M_0' \prec H(\chi)$ and M_0 being the Mostowski collapse of M_0' suffices for this. In addition we take care that $\langle T_\alpha : \alpha \in \omega_2, \mathrm{cf}(\alpha) = \aleph_1 \rangle \in M_0'$ is a sequence of sets $T_\alpha \subseteq \alpha$ witnessing $\diamondsuit_{\{\alpha < \aleph_2 : \mathrm{cf}(\alpha) = \aleph_1\}}$.

We fix more notation: α, α' will range over ω_2, $i, j, \varepsilon, \zeta, \xi$ over ω_1, and the letters β, γ, δ will denote particular functions with values in ω_2, ω_1, ω_1. We fix a bijection $b \colon 2^{<\omega} \to \omega$ and another bijection $b_2 \colon \aleph_2 \to (\mathcal{P}(H(\omega_1)))^2$.

The forcing $\mathbb{P}$ is constructed by induction on $\alpha \leq \omega_2$ as the finite support iteration $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \omega_2, \alpha \leq \omega_2 \rangle$.

For an *odd stage* $\alpha \in \omega_2$ we force via $\mathbb{Q}_\alpha = \mathbb{C}$, and we conceive Cohen forcing in the form

$$\{p : p \text{ is a partial function from } 2^{<\omega} \text{ to } 2, |p| < \omega\}$$

and fix for $\eta \in 2^\omega \cap \mathbf{V}$ sets $A_{\alpha,\eta} = \{b((p(\eta \restriction 0), \dots, p(\eta \restriction n-1))) : n \in \omega, p \in G\} \subseteq \omega$ in the extension, where b is the bijection from above. Note that for $\eta \neq \eta'$, $A_{\alpha,\eta} \cap A_{\alpha,\eta'}$ is finite. We also take a bijection $c \colon 2^\omega \cap \mathbf{V} \to \omega_1$ and write $A'_{\alpha,\varepsilon} = A_{\alpha,c^{-1}(\varepsilon)}$.

For even $\alpha < \omega_2$ we define $\mathbb{Q}_\alpha$ as follows: If $\mathrm{cf}(\alpha) < \omega_1$, we let $\mathbb{Q}_\alpha$ be the trivial forcing, i.e. $\mathbb{Q}_\alpha = \{0\}$. Now suppose that $\alpha < \omega_2$, $\mathrm{cf}(\alpha) = \aleph_1$.

Assume that $\mathbb{P}_\alpha$ is defined and $|\mathbb{P}_\alpha| \leq \aleph_1$, and to simplify notation we assume that $\mathbb{P}_\alpha \subseteq \omega_1$. Then every canonical $\mathbb{P}_\alpha$-name $\underset{\sim}{D}$ for a subset of $\mathcal{P}(\omega)$ is a subset of $H(\omega_1)$. We say that $T_\alpha \subseteq \alpha$ codes $(\underset{\sim}{D}, \langle \underset{\sim}{f}_i : i < \omega_1 \rangle)$ if $b''_2 T_\alpha = (\underset{\sim}{D}, \langle \underset{\sim}{f}_i : i < \omega_1 \rangle)$.

If $\mathrm{cf}(\alpha) = \omega_1$ and T_α is a canonical $\mathbb{P}_\alpha$-name of a pair $(\underset{\sim}{D}, \langle \underset{\sim}{f}_{\alpha,i} : i < \omega_1 \rangle)$ such that $\underset{\sim}{D}$ is subset of $\mathcal{P}(\omega)$ that contains the cofinite sets and $\mathbb{P}_\alpha \Vdash \mathcal{P}(\omega)/\underset{\sim}{D}$ is c.c.c.", and $\langle \underset{\sim}{f}_{\alpha,i} : i < \omega_1 \rangle$ is a canonical $\mathbb{P}_\alpha$-name of a sequence in ω^ω then we first fix in the ground model an increasing sequence $\langle \beta(\alpha, i) : i < \omega_1 \rangle$ that converges to α such that each $\beta(\alpha, i)$ is an odd member of ω_2.

Next we define by induction on $i < \omega$ countable ordinals as follows:

$$\begin{aligned} &\gamma(\alpha, 0) = \min\{\varepsilon < \omega_1 : \underset{\sim}{f}_{\alpha,0} \in \mathbf{V}^{\mathbb{P}_{\beta(\alpha,\varepsilon)}}\} \\ &\gamma(\alpha, i) = \min\{\varepsilon < \omega_1 : \underset{\sim}{f}_{\alpha,i} \in V^{\mathbb{P}_{\beta(\alpha,\varepsilon)}} \wedge (\forall j < i)(\varepsilon > \gamma(\alpha, j))\} \end{aligned} \tag{3.1}$$

Later it will be important that the $\gamma(\alpha, i)$, $i < \omega_1$, are pairwise different.

Then for each $i < \omega_1$ we choose with the maximum principle a name $\delta(\alpha, i) \in \omega_1$ such that

$$\mathbb{P}_\alpha \Vdash A^c_{\beta(\alpha,\gamma(\alpha,i)),\delta(\alpha,i)} \in \underset{\sim}{D}. \tag{3.2}$$

We do not write the tildes under the names of the δ. For the existence of such $\delta(\alpha, i)$ we use the following claim.

Claim: For any $i < \omega_1$ there are coboundedly many ε such that

$$\mathbb{P}_\alpha \Vdash A^c_{\beta(\alpha,\gamma(\alpha,i)),\varepsilon} \in \underset{\sim}{D}.$$

Proof: Assume for a contradiction that $i < \omega_1$ is a couterexample to the claim. Then there are unboundedly many $\varepsilon \in \omega_1$ such that there is $p_\varepsilon \in \mathbb{P}_\alpha$ such that $p_\varepsilon \Vdash A_{\beta(\alpha,\gamma(\alpha,i)),\varepsilon)} \in D^+$. Since $\mathbb{P}_\alpha$ has the c.c.c. there is a $\mathbb{P}_\alpha$-generic G that that contains $\aleph_1$ many p_ε as above. Call this uncountable set of ε's X. However for $\varepsilon \neq \varepsilon' \in X$, $\mathbb{P}_\alpha \Vdash A_{\beta(\alpha,\gamma(\alpha,i)),\varepsilon} \cap A_{\beta(\alpha,\gamma(\alpha,i)),\varepsilon'}$ is finite. This contradicts the fact that $\mathbb{P}_\alpha \Vdash \mathcal{P}(\omega)/\underset{\sim}{D}$ is c.c.c., and thus the claim is proved.

We use only one $\delta(\alpha, i)$ and its value in ω_1 is not important. However, for the $\gamma(\alpha, i)$, the pairwise inequality $\beta(\alpha, \gamma(\alpha, i)) \neq \beta(\alpha, \gamma(\alpha, j))$ for $i \neq j$ is important, so that there are no conflicts between the various instances of condition (6) below.

Once the $\langle\gamma(\alpha,i),\delta(\alpha,i) : i<\omega_1\rangle$ is chosen, we define in $\mathbf{V}^{\mathbb{P}_\alpha}$ the forcing $\mathbb{Q}_\alpha$ as follows: $p\in\mathbb{Q}_\alpha$ iff

(1) $p=(u_p,h_p)$,

(2) $h_p\in{}^{\omega>}\omega$,

(3) $u_p\subseteq\omega_1$ is finite.

$\mathbb{Q}_\alpha\models p\le q$ if

(4) $u_p\subseteq u_q$ and

(5) $h_p\trianglelefteq h_q$ and

(6) if $|h_p|\le m<|h_q|$, $\xi\in u_p$ and $m\in(\omega\setminus A_{\beta(\alpha,\gamma(\alpha,\xi)),\delta(\alpha,\xi)})$ then $f_{\alpha,\xi}(m)<h_q(m)$.

We show that by induction on α that $\mathbb{P}_\alpha$ has the $\bar{M}$-c.c. Since we take direct limits, the limit steps are covered by [18, Ch. IV, 3.2]. The start of the induction is trivial. Now we look at the successor steps $\mathbb{P}_{\alpha+1}=\mathbb{P}_\alpha*\underset{\sim}{\mathbb{Q}}_\alpha$.

Odd α: $\mathbb{Q}_\alpha$ is the Cohen forcing. Any countable forcing has the $\bar{M}[\mathbb{P}_\alpha]$-c.c. Putting this together with the induction hypothesis, $\mathbb{P}_{\alpha+1}$ has the $\bar{M}$-c.c.

Even α*:* To simplify notation, we assume that $\mathbb{P}_\alpha\subseteq\omega_1$ and we assume $\mathbb{P}_\alpha\Vdash\mathbb{Q}_\alpha\cap\varepsilon=\{(u,p)\in\mathbb{Q}_\alpha : u\subseteq\varepsilon\}$. We fix a witness $Y(\mathbb{P}_\alpha)\in\mathscr{D}_{\bar{M}}$ for the $\bar{M}$-c.c. of $\mathbb{P}_\alpha$, i.e., for every $\varepsilon\in Y(\mathbb{P}_\alpha)$ for every $I\in M_\varepsilon$ that is a dense subset of $\mathbb{P}_\alpha\cap\varepsilon$, I is dense in $\mathbb{P}_\alpha$.

We intersect $Y(\mathbb{P}_\alpha)$ with the club $C\subseteq\omega_1$ of countable limit ordinals that are closed under the functions $\gamma(\alpha,\cdot)$ and $\delta(\alpha,\cdot)$ that are defined as in equations (3.1), (3.2). Since $\mathbb{P}_\alpha$ is c.c.c. such a club can be found in the ground model although $\delta(\alpha,\cdot)$ is a name.

Next we prove that $Y(\mathbb{P}_\alpha)\cap C$ witnesses that $\mathbb{P}_{\alpha+1}$ has the $\bar{M}$-c.c. Let $\varepsilon\in Y(\mathbb{P}_\alpha)\cap C$, $D\in M_\varepsilon$ be an open and dense subset of $\mathbb{P}_{\alpha+1}\cap\varepsilon$. Let $p\in\mathbb{P}_{\alpha+1}$. We have to show that there is $q\in D$ that is compatible with p.

We write $p=(p\restriction\alpha,(u_{p(\alpha)},h_{p(\alpha)}))$ and we assume that $p\restriction\alpha$ determines the finite sets $u_{p(\alpha)}$ and $h_{p(\alpha)}$ so that they to elements of $[\omega_1]^{<\omega}$ and ${}^{\omega>}\omega$ and that it also determines $\gamma(\alpha,\xi)$ and $\delta(\alpha,\xi)$ for any $\xi\in u_{p(\alpha)}$.

The search for q proceeds in four steps:

First step: We apply the induction hypothesis. We let $D'=D\cap\mathbb{P}_\alpha$. $D'\in M_\varepsilon$ is dense and open in $\mathbb{P}_\alpha\cap\varepsilon$. Since $\mathbb{P}_\alpha$ has the $\bar{M}$-c.c. and $\varepsilon\in Y(\mathbb{P}_\alpha)$ there is $q'\in D'\cap M_\varepsilon$ that is compatible with $p\restriction\alpha$. We fix a witness $r'\in\mathbb{P}_\alpha$ for compatibility.

Second step: We carefully prolong $h_{p(\alpha)}$ to h' to take a record of r' on its finitely many Cohen coordinates as follows. We do not increase $u_{p(\alpha)}$. We choose $n\in\omega$ so large such that

$$\begin{aligned}&(\forall m)(\forall\xi\in u_{p(\alpha)})(\forall\beta=\beta(\alpha,\gamma(\alpha,\xi))\in\mathrm{supp}(r'))\\&\big((r'\Vdash(m\notin A_{\beta(\alpha,\gamma(\alpha,\xi)),\delta(\alpha,\xi)}))\to m<n\big).\end{aligned}\tag{3.3}$$

Now on $n\setminus\mathrm{dom}(h_{p(\alpha)})$ we choose $h'(k)\ge f_{\alpha,\xi}(k)$ for all $\xi\in u_{p(\alpha)}$.

Third step: We go again into $D \cap M_\varepsilon$. We choose $q(\alpha) \in M_\varepsilon$ such that $q' \Vdash \underset{\sim}{q(\alpha)} \geq_{\mathbb{Q}_\alpha} (u_{p(\alpha)} \cap \varepsilon, h') \wedge \underset{\sim}{q(\alpha)} \in D_\alpha[\mathbb{P}_\alpha]$ and let $q = (q', \underset{\sim}{q(\alpha)})$. Then $q = (q', \underset{\sim}{q(\alpha)}) \in M_\varepsilon \cap D$.

Fourth step: We let

$$r = \Big(r' \cup \{(\beta(\alpha,\gamma(\alpha,\xi)), q_1(\beta(\alpha,\gamma(\alpha,\xi))))\ :\ \xi \in u_{\underset{\sim}{p(\alpha)}} \setminus \varepsilon\}, \\ (u_{\underset{\sim}{p(\alpha)}} \cup u_{\underset{\sim}{q(\alpha)}}, h_{\underset{\sim}{q(\alpha)}})\Big),$$

where for any $\xi \in u_{p(\alpha)} \setminus \varepsilon$

$$q_1(\beta(\alpha,\gamma(\alpha,\xi))) \Vdash_{\mathbb{Q}_{\beta(\alpha,\gamma(\alpha,\xi))}} (\forall n \in \mathrm{dom}(h_{\underset{\sim}{q(\alpha)}} \setminus \mathrm{dom}(h'))) \\ (n \in A_{\beta(\alpha,\gamma(\alpha,\xi)),\delta(\alpha,\xi)}). \tag{3.4}$$

The condition r is well defined, since for any $\xi \in u_{p(\alpha} \setminus \varepsilon$, the condition $q_1(\beta(\alpha,\gamma(\alpha,\xi)) \in \mathbb{P}_\alpha$ can be chosen to be compatible with $r'(\beta(\alpha,\gamma(\alpha,\xi))$, by the choice of n as in Equation (3.3).

Next we show $r \geq p, q$. First $r' = r \restriction \alpha \geq p \restriction \alpha, q'$ and $q' = q \restriction \alpha$. Finally we show $r \restriction \alpha \Vdash_{\mathbb{P}_\alpha} (u_{\underset{\sim}{p(\alpha)}} \cup u_{\underset{\sim}{q(\alpha)}}, h_{\underset{\sim}{q(\alpha)}}) \geq_{\mathbb{Q}_\alpha} (u_{\underset{\sim}{p(\alpha)}}, h'), (u_{\underset{\sim}{q(\alpha)}}, h_{\underset{\sim}{q(\alpha)}})$. The latter is trivial. For the former, let $\xi \in u_{\underset{\sim}{p(\alpha)}}$. First case: $\xi \in M_\delta$. We chose (after Equation (3.3)) the function $h_{\underset{\sim}{q(\alpha)}}(k)$ such that it dominates $f_{\alpha,\xi}(k)$ on any coordinate k not in $\mathrm{dom}(h_{p(\alpha)})$ such that $r' \Vdash k \notin A_{\beta(\alpha,\gamma(\alpha,\xi)),\delta(\alpha,\xi)}$. Thus r' forces the relevant instances of clause (6) of $r(\alpha) \geq p(\alpha)$.

Second case: $\xi \in u_{\underset{\sim}{p(\alpha)}} \setminus \varepsilon$. Equation (3.4) implies $r \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \underset{\sim}{r(\alpha)} \geq \underset{\sim}{q(\alpha)}$. ⊣

Remark: We work with the assumption $\diamondsuit_{\{\delta<\aleph_2\,:\,\mathrm{cf}(\delta)=\aleph_1\}}$. Alternatively, we could force as in the previous section by approximations of size $\aleph_1$ in a first step and thereafter force with the generic filter of the first forcing. The diamond $\diamondsuit_{\{\delta<\aleph_2\,:\,\mathrm{cf}(\delta)=\aleph_1\}}$ hands downs at stage α a possible $\mathbb{P}_\alpha$-name for objects D, $\langle g_i\ :\ i < \aleph_1\rangle$ as in property $(\mathrm{A})_{\aleph_2}$ of Lemma 3.2 and thus allows to construct a finite support iteration up to stage ω_2 instead of using an approximation forcing in a first forcing step. So our $\mathbb{P}$ in this proof corresponds in the sense of the outline of the forcing construction to the generic $\underset{\sim}{\mathbb{Q}}$ of the approximation forcing from the previous section.

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Abteilung für Mathematische Logik, Mathematisches Institut, Universität Freiburg, Eckerstr. 1, 79104 Freiburg im Breisgau, Germany

Einstein Institute of Mathematics, The Hebrew University, Edmond Safra Campus Givat Ram, Jerusalem 91904, Israel

E-mail address: heike.mildenberger@math.uni-freiburg.de, shelah@math.huji.ac.il