

THE SPECTRUM OF ULTRAPRODUCTS OF FINITE  
CARDINALS FOR AN ULTRAFILTER  
SH1026

SAHARON SHELAH

ABSTRACT. We complete the characterization of the possible spectrum of regular ultrafilters  $D$  on a set  $I$ , where the spectrum is the set of ultraproducts of (finite) cardinals modulo  $D$  which are infinite.

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## § 0. INTRODUCTION

## § 0(A). Background, questions and results.

Ultraproducts were very central in model theory in the sixties, usually for regular ultrafilters. The question of ultraproducts of infinite cardinals had been resolved (see [CK73]): letting  $D$  be a regular ultrafilter on a set  $I$ , (for transparency we ignore the case of a filter)

(\*)<sub>1</sub> if  $\bar{\lambda} = \langle \lambda_s : s \in I \rangle$  and  $\lambda_s \geq \aleph_0$  for  $s \in I$  then  $\prod_{s \in I} \lambda_s / D = \mu^{|I|}$  when  $\mu = \limsup_D(\bar{\lambda}) := \sup\{\chi : \text{the cardinal } \chi \text{ satisfies } \{s \in I : \lambda_s \geq \chi\} \in D\}$ .

What about the ultraproducts of finite cardinals? Of course, under naive interpretation, if  $\{\lambda_s : \lambda_s = 0\} \neq \emptyset$  the result is zero, so for notational simplicity we always assume  $s \in I \Rightarrow \lambda_s \geq 1$ . Also for every  $n \geq 1$ , letting  $\lambda_s = n$  for  $s \in I$  we have  $\prod_s \lambda_s / D = n$  so the question was

{y4}

*Question 0.1.* Given an infinite set  $I$

- (a) [the singleton problem] what infinite cardinals  $\mu$  belong to  $\mathcal{C}_I = \mathcal{C}_I^{\text{car}}$ , i.e. can be represented as  $\{\prod_{s \in I} \lambda_s / D : D \text{ a regular ultrafilter on } I, 1 \leq \lambda_s < \aleph_0\} \setminus \{\lambda : 1 \leq \lambda < \aleph_0\}$
- (b) [the spectrum problem] moreover what are the possible spectra, i.e. which sets of cardinals belong to  $\mathbf{C}_I$  which is the family of sets  $\mathcal{C}$  such that for some  $D$ , a regular ultrafilter on  $I$  we have  $\mathcal{C} = \text{upf}(D)$  where  $\text{upf}(D) = \{\prod_{s \in I} \lambda_s / D : 1 \leq \lambda_s < \aleph_0 \text{ for } s \in I\} \setminus \{\lambda : 1 \leq \lambda < \aleph_0\}$

{y4}

Keisler [Kei67] asks and has started on 0.1: (assuming GCH was prevalent at the time as the situation was opaque otherwise)

- (\*)<sub>2</sub> assume GCH, a sufficient condition for  $\mathcal{C} \in \mathbf{C}_I$  is:
  - (a)  $\mathcal{C}$  is a set of successor (infinite) cardinals
  - (b)  $\max(\mathcal{C}) = |I|^+$
  - (c) if  $\mu = \sup\{\chi < \mu : \chi \in \mathcal{C}\}$  then  $\mu^+ \in \mathcal{C}$
  - (d) if  $\mu^+ \in \mathcal{C}$  then  $\mu \cap \mathcal{C}$  has cardinality  $< \mu$ .

Keisler used products and  $D$ -sums of ultrafilters. Concerning the problem for singletons a conjecture of Keisler [Kei67, bottom of pg.49] was resolved in [Sh:7]:

(\*)<sub>3</sub>  $\mu = \mu^{\aleph_0}$  when  $\mu \in \mathcal{C}_I$ , i.e. when  $\mu = \prod_{s \in I} \lambda_s / D$  is infinite,  $D$  an ultrafilter on  $I$ , each  $\lambda_s$  finite non-zero

The proof uses coding enough “set theory” on the  $n$ ’s and using the model theory of the ultra-product. This gives a necessary condition (for the singleton version), but is it sufficient? This problem was settled in [Sh:a, Ch.V,§3] = [Sh:c, Ch.VI,§3] proving that this is also a sufficient condition (+ the obvious condition  $\mu \leq 2^{|I|}$ ), that is

(\*)<sub>4</sub>  $\mu \in \mathcal{C}_I := \cup\{\mathcal{C} : \mathcal{C} \in \mathbf{C}_I\}$  iff  $\mu = \mu^{\aleph_0} \leq 2^{|I|}$ .

The constructions in [Sh:a, Ch.VI,§3] = [Sh:c, Ch.VI,§3], use a family  $\mathcal{F}$  of functions with domain  $I$  and a filter  $D$  on  $I$  such that  $\mathcal{F}$  is independent over  $D$  (earlier Kunen used such family  $\mathcal{F} \subseteq {}^\lambda \lambda$  for constructing a good ultrafilter on  $\lambda$  in ZFC; eliminating the use of an instance of GCH in the proof of Keisler; earlier Engelking-Karłowicz proved the existence of such  $\mathcal{F}$ ). In particular in the construction in [Sh:a, Ch.VI,§3] of maximal such filters and the Boolean Algebra  $\mathbb{B} = \mathcal{P}(I)/D$  are central. We decrease the family and increase  $D$  during the construction; specifically we construct  $\mathcal{F}_\ell (\ell \leq n)$  decreasing with  $\ell, D_\ell$  a filter on  $I$  increasing with  $\ell, D_\ell$  a maximal filter such that  $\mathcal{F}_\ell$  is independent mod  $D_\ell$ ; so if  $\mathcal{F}_n = \emptyset$  then  $D_0$  is an ultrafilter and we have  $\mathbb{B}_\ell = \mathcal{P}(I)/D_\ell$  is essentially  $\leftarrow$ -decreasing and in the ultrapowers  $\mathbb{N}^I/D_\ell$  the part which  $\mathbb{B}_\ell$  induces for  $\ell \leq n$ , is a sequence of initial segments of  $\mathbb{N}^{\mathbb{B}}/D_0$  decreasing with  $\ell$ .

In [Sh:a, Ch.VI,Exercise 3.35] = [Sh:c, pg.370] this is formalized:

- (\*)<sub>5</sub> if  $D_0$  is a filter on  $I, \mathbb{B}_0 = \mathcal{P}(I)/D_0, D_1 \supseteq D_0$  an ultrafilter,  $D = \{A/D_0 : A \in D_1\}$  so  $D \in \text{uf}(\mathbb{B}_0)$  then  $\mathbb{N}^{\mathbb{B}_0}/D_0^+$  is an initial segment of  $\mathbb{N}^I/D$ ; (also  $\mathbb{B}$  satisfies the c.c.c., but this is just to ensure  $\mathbb{B}$  is complete, anyhow this holds in all relevant cases here).

It follows that we can replace  $\mathcal{P}(I)$  by a Boolean Algebra  $\mathbb{B}_1$  extending  $\mathbb{B}_0$ . The Boolean Algebra related to  $\mathcal{F}$  is the completion of the Boolean Algebra generated by  $\{x_{f,a} : f \in \mathcal{F}, a \in \text{Rang}(f)\}$  freely except  $x_{f,a} \cap x_{f,b} = 0$  for  $a \neq b \in \text{Rang}(f)$  and  $f \in \mathcal{F}$ . So if  $\text{Rang}(f)$  is countable for every  $f \in \mathcal{F}$ , the Boolean Algebra satisfies the  $\aleph_1$ -c.c. (in fact, is free), this was used there to deal with  $\text{lcf}(\kappa, D)$  for  $\kappa = \aleph_0$  (for  $\kappa > \aleph_0$  we need  $\text{Rang}(f) = \kappa$ ) and is continued lately in works of Malliaris-Shelah. But for  $\text{upf}(D)$  only the case of  $f$ 's with countable range is used.

The problem of the spectrum (i.e. 0.1(b)) was not needed in [Sh:a, Ch.VI,§3] for the model theoretic problems which were the aim of [Sh:a, Ch.VI], still the case of finite spectrum was resolved there (also cofinality, i.e.  $\text{lcf}(\kappa, D)$  was addressed). {y4}

This was continued by Koppelberg [Kop80] using a possibly infinite  $\leftarrow$ -increasing chains of complete Boolean Algebras; also she uses a system of projections instead of maximal filters but this is a reformulation as this is equivalent, see 0.11 below. {x23}

Koppelberg [Kop80] returns to the full spectrum problem proving:

- (\*)<sub>6</sub>  $\mathcal{C} \in \mathbf{C}_I$  when  $\mathcal{C}$  satisfies:
  - (a)  $\mathcal{C} \subseteq \text{Card}$
  - (b)  $\max(\mathcal{C}) = 2^{|I|}$
  - (c)  $\mu = \mu^{\aleph_0}$  if  $\mu \in \mathcal{C}$
  - (d) if  $\mu_n \in \mathcal{C}$  for  $n < \omega$  then  $\prod_n \mu_n \in \mathcal{C}$ .

Central in the proof is (\*)<sub>5</sub> above ([Sh:a, Ch.VI,Ex3.35,pg.370]). The result of Koppelberg is very strong, still the full characterization is not obtained; also Kanamori in his math review of her work asked about it.

Here we give a complete answer to the spectrum problem 0.1(b), that is, Theorem 1.20 gives a full ZFC answer to 0.1, that is. {y4}

**Theorem 0.2.** For any infinite set  $I, \mathcal{C} \in \mathbf{C}_I$  iff  $\mathcal{C}$  is a set of cardinals such that  $\mu \in \mathcal{C} \Rightarrow \mu = \mu^{\aleph_0} \leq 2^{|I|}$  and  $2^{|I|} \in \mathcal{C}$ . {y6}  
{y8}

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We now comment on some further questions on ultra-powers.

The problem of cofinalities was central in [Sh:a, Ch.VI,§3] in particular,  $\text{lcf}(\aleph_0, \lambda)$  (see 0.6 below). [Why? E.g. if  $\text{Th}(M)$ , the complete first order theory of the model  $M$  is unstable then  $M^I/D$  is not  $\text{lcf}(\aleph_0, \lambda)^+$ -saturated.] Another question was raised by the author [Sh:14, pg.97] and independently by Eklof [Ekl73]:

{y9} *Question 0.3.* Assume  $f_n \in {}^I\mathbb{N}$ ,  $f_{n+1} <_D f_n$  and  $\mu \leq \prod_{s \in I} f_n(s)/D$  for every  $n$  then is there  $f \in {}^I\mathbb{N}$  such that  $f <_D f_n$  for every  $n$  and  $\mu \leq \prod_{s \in I} f(s)/D$ ?

The point in [Sh:14, pg.75] was investigating saturation of ultrapowers (and ultraproducts) and Keisler order on first order theories. The point in [Ekl73] was ultraproduct of Abelian groups.

{y12} To explain the cofinalities problem, see 0.4. We can consider the following: for  $D$  a regular ultrafilter on  $I$  we consider  $M = \mathbb{N}^\lambda/D$ ; for  $a \in M$  let  $\lambda_a = |\{b : b <_M a\}|$  and define  $E_M = \{(a, b) : a, b \in M \text{ and } \lambda_a = \lambda_b \geq \aleph_0\}$ . So  $E_M$  is a convex equivalence relation, and the equivalence classes are naturally linearly ordered and let  $A_{D, \lambda} = \{a \in M : \lambda_a = \lambda\}$ . So  $\text{upf}(D) = \{\lambda_a : A_{D, \lambda} \neq \emptyset\}$  and Question 0.3 asks: can the co-initiality of some  $A_{D, \lambda}$  be  $\aleph_0$ . As  $M$  is  $\aleph_1$ -saturated, in this case the cofinality of  $M \upharpoonright \{c : \lambda_c < \lambda_a \text{ (hence } c <_M a)\}$  is  $\text{lcf}(\aleph_0, D)$  which is the co-initiality of  $A_{D, \min(\text{upf}(D))}$ .

So a natural question is

{y12} *Question 0.4.* What are the possible  $\text{spec}_1(D) = \{(\lambda, \theta, \partial) : \lambda \in \text{upf}(D), \partial \text{ the cofinality of } A_{D, \lambda} \text{ and } \theta \text{ the co-initiality of } A_{D, \lambda}\}$  for  $D$  a regular ultrafilter on  $I$ ?

A further question is:

{y15} *Question 0.5.* Assume  $\kappa = \text{cf}(\kappa) < \lambda_1 = \aleph_1^{\aleph_0} < \lambda_2 = \aleph_2^{\aleph_0}, \lambda_1^{<\kappa>\text{tr}} \leq 2^\lambda$ ; see 0.7(4). Is there a regular ultrafilter  $D$  on  $\lambda$  such that for  $n_i \in \mathbb{N}$  for  $i < \lambda$  we have  $\prod_i n_i/D = \lambda_1$  and  $\prod_i 2^{n_i}/D = \lambda_2$ ?

{y9} This work was presented in the May 2013 Eilat Conference honoring Mati Rubin's retirement. In a work in preparation [Sh:F1427], we try to build a counterexample to question 0.3.

## § 0(B). Preliminaries.

We define  $\text{lcf}(\kappa, D)$  and  $M^\mathbb{B}/D$ , when  $\mathbb{B}$  is a Boolean Algebra and more.

{x16} **Definition 0.6.** For  $D$  an ultrafilter on  $I$ ,  $\kappa$  a regular cardinal let  $\mu = \text{lcf}(\kappa, D)$  be the co-initiality of the linear order  $(\kappa^I/D) \upharpoonright \{f/D : f \in {}^I\kappa \text{ is not } D\text{-bounded by any } \varepsilon < \kappa\}$ .

{x18} *Notation 0.7.* 1)  $\mathbb{B}$  denotes a Boolean Algebra, usually complete; let  $\text{comp}(\mathbb{B})$  be the completion of  $\mathbb{B}$ .

2)  $\text{uf}(\mathbb{B})$  is the set of ultrafilters on  $\mathbb{B}$ .

3) Let  $\mathbb{B}^+ = \mathbb{B} \setminus \{0_\mathbb{B}\}$ .

4) Let  $\text{cc}(\mathbb{B}) = \min\{\kappa : \mathbb{B} \text{ satisfies the } \kappa\text{-c.c.}\}$ , necessarily a regular cardinal.

5) For  $\lambda \geq \kappa = \text{cf}(\kappa)$  let  $\lambda^{(\kappa)} = \text{trp}_\kappa(\lambda) = \sup\{|\lim_\kappa(\mathcal{T})| : \mathcal{T} \subseteq {}^\kappa \lambda \text{ is a subtree of cardinality } \leq \lambda\}$  where  $\lim_\kappa(\mathcal{T}) = \{\eta \upharpoonright i \in \mathcal{T} \text{ for every } i < \kappa\}$ .

6) For a Boolean algebra  $\mathbb{B}$  let  $\text{comp}(\mathbb{B})$  be its completion.

{x20}

**Definition 0.8.** For a Boolean Algebra  $\mathbb{B}$  and a model or a set  $M$ .

1) Let  $M^{\mathbb{B}}$  be the set of partial functions  $f$  from  $\mathbb{B}^+$  into  $M$  such that for some maximal antichain  $\langle a_i : i < i(*) \rangle$  of  $\mathbb{B}$ ,  $\text{Dom}(f)$  includes  $\{a_i : i < i(*)\}$  and is included in<sup>1</sup>  $\{a \in \mathbb{B}^+ : (\exists i)(a \leq a_i)\}$  and  $f \upharpoonright \{a \in \text{Dom}(f) : a \leq a_i\}$  is constant for each  $i$ .

1A) Naturally for  $f_1, f_2 \in M^{\mathbb{B}}$  we say  $f_1, f_2$  are  $D$ -equivalent, or  $f_1 = f_2 \pmod D$  when for some  $b \in D$  we have  $a_1 \in \text{Dom}(f_1) \wedge a_2 \in \text{Dom}(f_2) \wedge a_1 \cap a_2 \cap b > 0_{\mathbb{B}} \Rightarrow f_1(a_1) = f_2(a_2)$ .

1B) Abusing notation, not only  $M^{\mathbb{B}_1} \subseteq M^{\mathbb{B}_2}$  but  $M^{\mathbb{B}_1}/D_1 \subseteq M^{\mathbb{B}_2}/D_2$  when  $\mathbb{B}_1 \leq \mathbb{B}_2, D_\ell \in \text{uf}(\mathbb{B}_\ell)$  for  $\ell = 1, 2$  and  $D_1 \subseteq D_2$ , that is, for  $f \in M^{\mathbb{B}_1}$  we identify  $f/D_1$  and  $f/D_2$ .

2) For  $D$  an ultrafilter on the completion of the Boolean Algebra  $\mathbb{B}$  we define  $M^{\mathbb{B}}/D$  naturally, as well as  $\text{TV}(\varphi(f_0, \dots, f_{n-1})) \in \text{comp}(\mathbb{B})$  when  $\varphi(x_0, \dots, x_{n-1}) \in \mathbb{L}(\tau_M)$  and  $f_0, \dots, f_{n-1} \in M^{\mathbb{B}}$  where  $\text{TV}$  stands for truth value and  $M^{\mathbb{B}}/D \models \varphi[f_0/D, \dots, f_{n-1}/D]$  iff  $\text{TV}_M(\varphi(f_0, \dots, f_{n-1})) \in D$ .

3) We say  $\langle a_n : n < \omega \rangle$   $D$ -represents  $f \in \mathbb{N}^{\mathbb{B}}$  when  $\langle a_n : n < \omega \rangle$  is a maximal antichain of  $\mathbb{B}$  (allowing  $a_n = 0_{\mathbb{B}}$ ) and for some  $f' \in \mathbb{N}^{\mathbb{B}}$  which is  $D$ -equivalent to  $f$  (see 0.8(1A)) we have  $f'(a_n) = n$ . We may omit  $D$  if  $D = \{1_{\mathbb{B}}\}$  and say just  $\langle a_n : n < \omega \rangle$  represents  $f$ . {x20}

4) We say  $\langle (a_n, k_n) : n < \omega \rangle$  represents  $f \in \mathbb{N}^{\mathbb{B}}$  when:

- (a) the  $k_n$  are natural numbers with no repetition
- (b)  $\langle a_n : n < \omega \rangle$  is a maximal antichain of  $\mathbb{B}$
- (c)  $f(a_n) = k_n$ .

The proofs in [Sh;a, Ch.VI,§3] use downward induction on the cardinals.

**Observation 0.9.** 1) If  $\mathbb{B}$  is a complete Boolean Algebra and  $f \in \mathbb{N}^{\mathbb{B}}$  then some sequence  $\langle a_n : n < \omega \rangle$  represents  $f$ . {x21}

1A) If  $\mathbb{B}$  is a c.c.c. Boolean Algebra and  $f \in \mathbb{N}^{\mathbb{B}}$ , then some sequence  $\langle a_n, k_n : n < \omega \rangle$  represents  $f$ .

2) For a model  $M$  and Boolean Algebra  $\mathbb{B}_1$  and ultrafilter  $D$  on its completion  $\mathbb{B}_2$  we have  $M^{\mathbb{B}_1}/D = M^{\mathbb{B}_2}/D$ .

**Fact 0.10.** 1) If  $\mathbb{B}_1 \subseteq \mathbb{B}_2$  are Boolean Algebras,  $\mathbb{B}$  is a complete Boolean Algebra and  $\pi_1$  is a homomorphism from  $\mathbb{B}_1$  into  $\mathbb{B}$  then there is a homomorphism  $\pi_2$  from  $\mathbb{B}_2$  into  $\mathbb{B}$  extending  $\pi_1$ . {x30}

2) There is a homomorphism  $\pi_3$  from  $\mathbb{B}_3$  into  $\mathbb{B}$  extending  $\pi_\ell$  for  $\ell = 0, 1, 2$  when:

- (a)  $\mathbb{B}_0 \subseteq \mathbb{B}_\ell \subseteq \mathbb{B}_3$  are Boolean Algebras for  $\ell = 1, 2$
- (b)  $\mathbb{B}_1, \mathbb{B}_2$  are freely amalgamated over  $\mathbb{B}_0$  inside  $\mathbb{B}_3$
- (c)  $\mathbb{B}$  is a complete Boolean Algebra
- (d)  $\pi_\ell$  is a homomorphism from  $\mathbb{B}_\ell$  into  $\mathbb{B}$  for  $\ell = 0, 1, 2$
- (e)  $\pi_0 \subseteq \pi_1$  and  $\pi_0 \subseteq \pi_2$ .

*Proof.* 1) Well known.

2) Straightforward. □<sub>0.10</sub>

<sup>1</sup>for the  $D_\ell \in \text{uf}(\mathbb{B}_\ell)$  ultra-product, without loss of generality  $\mathbb{B}$  is complete, then without loss of generality  $f \upharpoonright \{a_i : i < i(*)\}$  is one to one.

{x23}

**Observation 0.11.** Assume  $\mathbb{B}_1 \triangleleft \mathbb{B}_2$  are Boolean Algebras and  $\mathbb{B}_1$  is complete.

1) The following properties of  $D$  are equivalent:

- (a)  $D$  is a maximal filter on  $\mathbb{B}_2$  (among those) disjoint to  $\mathbb{B}_1 \setminus \{1_{\mathbb{B}_1}\}$
- (b) there is a projection  $\pi$  of  $\mathbb{B}_2$  onto  $\mathbb{B}_1$  such that  $D = \{a \in \mathbb{B}_2 : \pi(a) = 1_{\mathbb{B}_1}\}$ .

1A) Moreover  $D$  determines  $\pi$  uniquely and vice versa, in particular  $\pi(c)$  is the unique  $c' \in \mathbb{B}_1$  such that  $c = c' \bmod D$ .

2) If  $D$  satisfies (1)(a) and  $D_1$  is an ultrafilter of  $\mathbb{B}_1$ , then there is a one and only one ultrafilter  $D_2 \in \text{uf}(\mathbb{B}_2)$  extending  $D_1 \cup D$ .

*Proof.* 1) Clause (a) implies clause (b):

As  $D$  is a filter on  $\mathbb{B}_2$  clearly for some Boolean Algebra  $\mathbb{B}'_2$ , there is a homomorphism  $\mathbf{j}_0 : \mathbb{B}_2 \rightarrow \mathbb{B}'_2$  which is onto, such that  $a \in D \Leftrightarrow \mathbf{j}_0(a) = 1_{\mathbb{B}'_2}$ . As  $D \cap \mathbb{B}_1 = \{1_{\mathbb{B}_1}\}$  necessarily  $\mathbf{j}_0 \upharpoonright \mathbb{B}_1$  is one-to-one. Let  $\mathbb{B}'_1 = \mathbf{j}_0(\mathbb{B}_1)$  so  $\mathbf{j}_1 := (\mathbf{j}_0 \upharpoonright \mathbb{B}_1)^{-1}$  is an isomorphism from  $\mathbb{B}'_1$  onto  $\mathbb{B}_1$  hence by 0.10(1) and the assumption that  $\mathbb{B}_1$  is complete there is a homomorphism  $\mathbf{j}_2$  from  $\mathbb{B}'_2$  onto  $\mathbb{B}_1$  extending  $\mathbf{j}_1$ . Hence  $\mathbf{j}_3 = \mathbf{j}_2 \circ \mathbf{j}_0$  is a homomorphism from  $\mathbb{B}_2$  onto  $\mathbb{B}_1$  extending  $\text{id}_{\mathbb{B}_1}$ , so it is a projection.

Lastly,  $\mathbf{j}_3^{-1}\{1_{\mathbb{B}_1}\}$  is a filter extending  $D$  and disjoint to  $\mathbb{B}_1 \setminus \{1_{\mathbb{B}_1}\}$ . By the maximality of  $D$  we have equality.

An alternative proof is:

Let  $\mathbb{B}'_2$  be the sub-algebra of  $\mathbb{B}_2$  generated by  $\mathbb{B}_1 \cup D$ . Clearly every member of  $\mathbb{B}'_2$  can be represented as  $(a \cap b) \cup ((1-a) \cap \sigma(\bar{a}, \bar{b}))$  with  $a, a_m \in D$  for  $m < n = \ell g(\bar{a})$  and  $b \in \mathbb{B}_1, b_k \in \mathbb{B}_1$  for  $k < \ell g(\bar{b}), \sigma$  a Boolean term such that  $\bigwedge_{k < n} a \leq a_k$ , equivalently

$\bigwedge_{k < n} a \cap (1 - a_k) = 0$ . We try to define a function  $\pi$  from  $\mathbb{B}'_2$  into  $\mathbb{B}_1$  by:

$$\oplus \pi((a \cap b) \cup ((1 - a) \cap \sigma(\bar{a}, \bar{b}))) = b \text{ for } a, \bar{a}, b, \bar{b} \text{ as above.}$$

We have to prove that  $\pi$  is as promised.

(\*)<sub>1</sub>  $\pi$  is a well defined (function from  $\mathbb{B}'_2$  into  $\mathbb{B}_1$ ).

Why? Obviously for every  $c \in \mathbb{B}'_2$  there are  $a, \bar{a}, b, \bar{b}, \sigma$  as above, so  $\pi(c)$  has at least one definition, still we have to prove that any two such definitions agree. So assume  $c = (a_\ell \cap b_\ell) \cup ((1 - a_\ell) \cap \sigma_\ell(\bar{a}_\ell, \bar{b}_\ell))$  for  $\ell = 1, 2$  as above so with  $a_1, a_2, a_{1,k}, a_{2,m} \in D$  and  $b_1, b_2, \bar{b}_1, \bar{b}_2 \in \mathbb{B}_1$  such that  $a_1 \leq a_{1,k}$  for every  $k < \ell g(\bar{a}_1), a_2 \leq a_{2,m}$  for every  $m < \ell g(\bar{a}_2)$ . We should prove that  $b_1 = b_2$ , if not without loss of generality  $b_1 \not\leq b_2$  hence  $b := b_1 - b_2 > 0$ . Clearly  $a := a_1 \cap a_2 \in D$  and computing  $c \cap b \cap a$  in two ways we get  $a \cap b \cap b_1 = a \cap b \cap b_2$  hence  $a \cap b = a \cap b \cap b_1 = a \cap b \cap b_2 = a \cap 0 = 0$  recalling  $b = b_1 - b_2$ , hence  $a \leq 1 - b$  so as  $a \in D$  necessarily  $1 - b \in D$ . But  $b \in \mathbb{B}_1^+$  so  $1 - b \in \mathbb{B}_1 \setminus \{1_{\mathbb{B}_1}\}$ , contradiction to the assumption on  $D$ .

(\*)<sub>2</sub>  $\pi$  commutes with “ $x \cap y$ ”.

Why? Assume that for  $\ell = 1, 2$  we have  $c_\ell = (a_\ell \cap b_\ell) \cup ((1 - a_\ell) \cap \sigma_\ell(\bar{a}_\ell, \bar{b}_\ell))$  with  $a_\ell, b_\ell, \bar{a}_\ell, \bar{b}_\ell, \sigma_\ell$  as above.

So  $\pi(c_\ell) = b_\ell$  and letting  $a = a_1 \cap a_2 \in D$  we have  $c := c_1 \cap c_2 = (a \cap (b_1 \cap b_2)) \cup ((1 - a) \cap \sigma(\bar{a}, \bar{b}))$  where  $\bar{a} = \bar{a}_1 \wedge \langle a_1 \rangle \wedge \bar{a}_2 \wedge \langle a_2 \rangle, \bar{b} = \bar{b}_1 \wedge \langle b_1 \rangle \wedge \bar{b}_2 \wedge \langle b_2 \rangle$  for some suitable term  $\sigma$ .

As  $a \in D$ , clearly  $\pi(c) = b_1 \cap b_2 = \pi(c_1) \cap \pi(c_2)$ , as required.

(\*)<sub>3</sub>  $\pi$  commutes with “ $1 - x$ ”.

Why? Let  $c = (a \cap b) \cup ((1-a) \cap \sigma(\bar{a}, \bar{b}))$  hence  $1-c = (a \cap (1-b)) \cup ((1-a) \cap (1-\sigma(\bar{a}, \bar{b})))$  hence  $\pi(1-c) = 1-b = 1-\pi(c)$  so we are done.

(\*)<sub>4</sub>  $\pi$  is a projection onto  $\mathbb{B}_1$ .

[Why? By (\*<sub>1</sub>), (\*<sub>2</sub>), (\*<sub>3</sub>) clearly  $\pi$  is a homomorphism from  $\mathbb{B}'_2$  into  $\mathbb{B}_1$ . So its range is  $\subseteq \mathbb{B}_1$  and if  $c \in \mathbb{B}_1$  let  $b = c, a = 1_{\mathbb{B}_1}, \bar{a} = \langle \rangle = \bar{b}$  and  $\sigma(\bar{a}, \bar{b}) = 0_{\mathbb{B}_1}$  so  $c = (a \cap b) \cup ((1-a) \cap \sigma(\bar{a}, \bar{b}))$  and  $a, b, \bar{a}, \bar{b}, \sigma$  are as required so  $\pi((a \cap b) \cap ((1-a) \cap \sigma(\bar{a}, \bar{b}))) = b$  which means  $\pi(c) = b = c$ .]

Now we can finish: as  $\mathbb{B}_1 \subseteq \mathbb{B}'_2 \subseteq \mathbb{B}_2$  and  $\pi$  is a homomorphism from  $\mathbb{B}'_2$  into  $\mathbb{B}_1$  which is a complete Boolean Algebra, we can extend  $\pi$  to  $\pi^+$ , a homomorphism from  $\mathbb{B}_2$  into  $\mathbb{B}_1$ , see 0.10. But  $\pi$  is a projection hence so is  $\pi^+$ . Clearly  $(\pi^+)^{-1}\{1_{\mathbb{B}_1}\}$  includes  $D$  and equality holds by the assumption on the maximality of  $D$  and we have proved the implication. {x30}

Clause (b) implies clause (a):

First, clearly  $D$  is a filter of  $\mathbb{B}_2$ ; also  $a \in \mathbb{B}_1 \setminus \{1_{\mathbb{B}_1}\} \Rightarrow \pi(a) = a \neq 1_{\mathbb{B}_1} \Rightarrow a \notin D$ .

Toward contradiction assume  $D_2$  is a filter on  $\mathbb{B}_2, D \subsetneq D_2$  and  $D_2 \cap \mathbb{B}_1 = \{1_{\mathbb{B}_1}\}$ . Choose  $c_2 \in D_2 \setminus D$  and let  $c_1 = \pi(c_2)$ , consider the symmetric difference,  $c_1 \Delta c_2$  it is mapped by  $\pi$  to  $c_1 \Delta c_1 = 0_{\mathbb{B}_2}$  hence  $\pi(1_{\mathbb{B}_2} - (c_1 \Delta c_2)) = 1_{\mathbb{B}_2} - \pi(c_1 \Delta c_2) = 1_{\mathbb{B}_2} - 0_{\mathbb{B}_2} = 1_{\mathbb{B}_2}$ , so  $1_{\mathbb{B}_2} - (c_1 \Delta c_2) \in D$  so  $c_1 = c_2 \pmod{D}$ , hence (recalling  $D_1 \subseteq D_2$ ) we have  $c_1 = c_2 \pmod{D_2}$  but  $c_2 \in D_2$  hence  $c_1 \in D_2$ . But

- $c_1 \in \mathbb{B}_1$  being  $\pi(c_2)$
- $c_1 \neq 1_{\mathbb{B}_1}$  as  $\pi(c_2) = c_1$  and  $c_2 \notin D$

and recall

- $c_1 \in D_2$

so  $c_1$  contradicts  $D_2 \cap \mathbb{B}_1 = \{1_{\mathbb{B}_1}\}$ . We comment that for this direction we do not use the completeness of  $\mathbb{B}_1$ .

1A) Now  $\pi$  determines  $D$  in the statement (b). Also  $D$  determines  $\pi$  because if  $\pi_1, \pi_2$  are projections from  $\mathbb{B}_2$  onto  $\mathbb{B}_1$  such that  $D = \{a \in \mathbb{B}_2 : \pi_\ell(a) = 1_{\mathbb{B}_1}\}$  for  $\ell = 1, 2$  and  $\pi_1 \neq \pi_2$  let  $a \in \mathbb{B}_2$  be such that  $\pi_1(a) \neq \pi_2(a)$ ; then as in (b)  $\Rightarrow$  (a) in the proof of part (1),  $\pi_\ell(a) = a \pmod{D}$  for  $\ell = 1, 2$  hence  $\pi_1(a) = \pi_2(a) \pmod{D}$ , but  $\pi_1(a), \pi_2(a) \in \mathbb{B}_1$  and  $D_2 \cap \mathbb{B}_1 = 1_{\mathbb{B}_1}$  and  $D \cap \mathbb{B}_1 = 1_{\mathbb{B}_1}$  hence  $\pi_1(a) = \pi_2(a)$ , contradiction.

2) Straightforward, but we elaborate; clearly  $a \in D_1 \wedge b \in D \Rightarrow \pi(a \cap b) = \pi(a) \cap \pi(b) = a \cap 0_{\mathbb{B}_1} = a \geq 0_{\mathbb{B}_1}$  hence  $a \in D_1 \wedge b \in D \Rightarrow a \cap b > 0_{\mathbb{B}_2}$ , hence least one such  $D_2 \in \text{uf}(\mathbb{B}_2)$ . For uniqueness toward contradiction assume  $\mathcal{E}_1, \mathcal{E}_2$  are from  $\text{uf}(\mathbb{B}_2)$  and extend  $D_1 \cup D$ . So necessarily there is  $a \in \mathcal{E}_1 \setminus \mathcal{E}_2$  so as above  $a = \pi(a) \pmod{D}$  but  $D \subseteq \mathcal{E}_1 \cap \mathcal{E}_2$  hence for  $\ell = 1, 2$  we have  $a = \pi(a) \pmod{\mathcal{E}_\ell}$  so  $a \in \mathcal{E}_\ell \Leftrightarrow \pi(a) \in \mathcal{E}_\ell$ . But  $\mathcal{E}_1 \cap \mathbb{B}_1 = D_1 = \mathcal{E}_2 \cap \mathbb{B}_1$  and  $\pi(a) \in \mathbb{B}_1$  has  $\pi(a) \in \mathcal{E}_1 \Leftrightarrow \pi(a) \in \mathcal{E}_2$ . By the last two sentences  $a \in \mathcal{E}_1 \Leftrightarrow a \in \mathcal{E}_2$  contradicting the choice of  $a$ . □<sub>0.11</sub>

**Fact 0.12.** Assume  $\mathbb{B}_1 < \mathbb{B}_2$  are complete Boolean Algebras,  $D_\ell \in \text{uf}(\mathbb{B}_\ell)$  for  $\ell = 1, 2$ . If  $D$  is a maximal filter on  $\mathbb{B}_2$  disjoint to  $\mathbb{B}_1 \setminus \{1_{\mathbb{B}_1}\}$  and  $D \cup D_1 \subseteq D_2$  then  $\mathbb{N}^{\mathbb{B}_1}/D_1$  is an initial segment of  $\mathbb{N}^{\mathbb{B}_2}/D_2$ . {x33}

{x36}

*Remark 0.13.* 1) This is [Sh:a, Ch.VI,Ex3.35].

2) We can prove: if the homomorphism  $\mathbf{j} : \mathbb{B}_2 \rightarrow_{\text{onto}} \mathbb{B}_1$  maps  $D_2 \in \text{uf}(\mathbb{B}_2)$  onto  $D_1 \in \text{uf}(\mathbb{B}_1)$  then  $\mathbb{N}^{\mathbb{B}_1}/D_1$  is canonically isomorphic to an initial segment of  $\mathbb{N}^{\mathbb{B}_2}/D_2$  as in 0.12.

*Proof.* The desired conclusion will follow by  $(*)_3$  below:

$(*)_1$  If  $\mathcal{I}$  is a maximal antichain of  $\mathbb{B}_1$  then  $\{a/D : a \in \mathcal{I}\}$  is a maximal antichain of  $\mathbb{B}_2/D$ .

[Why? First,

- $a \in \mathcal{I} \Rightarrow a \in \mathbb{B}_1^+ \Rightarrow a/D \in (\mathbb{B}_2/D)^+$
- if  $a \neq b \in \mathcal{I}$  then  $\mathbb{B}_2 \models "a \cap b = 0_{\mathbb{B}_1}"$  hence  $\mathbb{B}_2/D \models "(a/D) \cap (b/D) = 0_{\mathbb{B}_2/D}"$ .

Hence, obviously  $\mathcal{I}^* := \{a/D : a \in \mathcal{I}\}$  is an antichain of  $\mathbb{B}_2/D$ . Toward contradiction assume  $\mathcal{I}^*$  is not maximal and let  $c/D$  witness it. By 0.11 there is  $c' \in \mathbb{B}_1$  such that  $c = c' \text{ mod } D$  and so without loss of generality  $c \in \mathbb{B}_1$ .

As  $c/D \neq 0/D$  necessarily  $c \in \mathbb{B}_1^+$  and if  $b \in \mathcal{I}$  then  $(b/D) \cap (c/D) = 0/D$  hence  $b \cap c = 0 \text{ mod } D$  but  $b, c \in \mathbb{B}_1$  hence  $b \cap c = 0$ , so  $c$  contradicts " $\mathcal{I}$  is a maximal antichain of  $\mathbb{B}_1$ ".]

$(*)_2$  If  $f \in \mathbb{N}^{\mathbb{B}_2}, c \in \mathbb{B}_1 \setminus D_1$  and  $\text{TV}(f > n) \cup c \in D$  for every  $n$  then  $g \in \mathbb{N}^{\mathbb{B}_1} \Rightarrow g/D_2 < f/D_2$ .

[Why? If  $g$  is a counter-example, then  $\text{TV}(f \leq g)$  belongs to  $D_2$  but  $1-c \in D_1 \subseteq D_2$  so  $\text{TV}(f \leq g) - c$  belongs to  $D_2$  hence to  $D^+ := \{a \in \mathbb{B}_2 : 1-a \notin D\}$  since  $D \subseteq D_2$ . Let  $\langle b_n : n < \omega \rangle$  represent  $g$  as a member of  $\mathbb{N}^{\mathbb{B}_1}$ , then by  $(*)_1$ ,  $\langle b_n/D : n < \omega \rangle$  is a maximal anti-chain of  $\mathbb{B}_2/D$  hence for some  $n$ ,  $\text{TV}(f \leq g) \cap b_n - c \in D^+$  but  $\text{TV}(f \leq n) - c \geq \text{TV}(f \leq g) \cap b_n - c$  hence  $\text{TV}(f \leq n) - c \in D^+$ , contradiction to an assumption of  $(*)_2$ ; so  $(*)_2$  holds indeed.]

$(*)_3$  If  $f \in \mathbb{N}^{\mathbb{B}_2}, g \in \mathbb{N}^{\mathbb{B}_1}$  and  $f/D_2 \leq g/D_2$  then for some  $g' \in \mathbb{N}^{\mathbb{B}_1}/D$  we have  $f/D_2 = g'/D_2$ .

[Why? Let  $\langle a_n : n < \omega \rangle$  represent  $f$  and let  $a_{\geq n} = \bigcup_{k \geq n} a_k \in \mathbb{B}_2$ . If for some  $b \in D_2$ , we have  $n < \omega \Rightarrow a_{\geq n} \cup (1-b) \in D$  then there is  $f' \in \mathbb{N}^{\mathbb{B}_2}$  such that  $f'/D_2 = f/D_2$  and  $n < \omega \Rightarrow \text{TV}(f' \geq n) \in D$ . Now we apply  $(*)_2$  with  $f', 0_{\mathbb{B}_1}$  here standing for  $f, c$  there and we get contradiction to " $f/D_2 \leq g/D_2$ ". So we can assume there is no such  $b$ .

Let  $a'_n \in \mathbb{B}_1$  be such that  $a_n = a'_n \text{ mod } D$  so possibly  $a'_n = 0_{\mathbb{B}_1}$ , such  $a'_n$  exists by 0.11(1A). Clearly  $\langle a'_n/D : n < \omega \rangle$  is an antichain of  $\mathbb{B}_2/D$ , so as  $D \cap \mathbb{B}_1 = \{1_{\mathbb{B}_1}\}$  clearly  $\langle a'_n : n < \omega \rangle$  is an antichain of  $\mathbb{B}_1$ .

Case 1:  $c := \bigcup_n a'_n \notin D_1$ .

First, we argue that for every  $n \in \omega$ ,  $\text{TV}(f > n) \cup c \in D$  as otherwise there is some  $n \in \omega$  such that  $\text{TV}(f \leq n) - c \in D^+$  hence for some  $\ell \leq n$ ,  $a'_\ell - c \in D^+$  hence (by the choice of  $a'_\ell$ ) we have  $a'_\ell - c \in D^+$ , contradiction to the choice of  $c$ . As  $c \in \mathbb{B}_1$  we see by the assumption of this case that  $c \in \mathbb{B}_1 \setminus D_1$ , hence by  $(*)_2$  we get a contradiction to the assumption " $f/D_2 \leq g/D_2$ " of  $(*)_3$ .



Case 2:  $c := \bigcup_n a'_n \in D_1$  and  $d = \bigcup_n (a_n \Delta a'_n) \notin D_2$ .

As  $D_2$  is an ultrafilter of  $\mathbb{B}_2$  extending  $D_1$ , clearly  $c' := c - d \in D_2$ . We define  $g' \in \mathbb{N}^{\mathbb{B}_1}$  as the function represented by  $\langle a'_n : n < \omega \rangle$  and  $g'' \in \mathbb{N}^{\mathbb{B}_2}$  as the function represented by  $\langle a''_n : n < \omega \rangle$ , where  $a''_n$  is  $a'_n \cap c'$  if  $n > 0$  and  $a'_n \cup (1 - c')$  if  $n = 0$ . Easily  $f/D_2 = g''/D_2$  because  $f, g''$  “agree” on  $c'$  which belongs to  $D_2$  and is disjoint to  $d$  and the choice of  $d$ ; also  $g''/D_2 = g'/D_2$  because  $c' \in D_2$ . Together we are done.

Case 3:  $c := \bigcup_n a'_n \in D_1$  and  $d = \bigcup_n (a_n \Delta a'_n) \in D_2$ .

Let  $d' \in \mathbb{B}_1$  be such that  $d'/D = d/D$ . Let  $d_1 := \bigcup_n (a_n - a'_n)$  and  $d_2 := \bigcup_n (a'_n - a_n)$  hence  $d = d_1 \cup d_2$ . Let  $k < \omega$ , now modulo  $D$  we have  $d' \cap \bigcup_{n \leq k} a'_n = d \cap \bigcup_{n \leq k} a'_n =$

$\bigcup_{\ell=1}^2 (d_\ell \cap \bigcup_{n \leq k} a'_n)$  and we shall deal separately with each term.

First,  $d_1 \cap \bigcup_{n \leq k} a'_n = \bigcup_{\ell \leq k} ((a_\ell - a'_\ell) \cap \bigcup_{n \leq k} a'_n) \cup (\bigcup_{\ell > k} (a_\ell - a'_\ell) \cap \bigcup_{n \leq k} a'_n)$ . Now the first term  $\bigcup_{\ell \leq k} ((a_\ell - a'_\ell) \cap \bigcup_{n \leq k} a'_n)$  is equal mod  $D$  to  $(\bigcup_{n \leq k} 0) \cap \bigcup_{n \leq k} a'_n = 0_{\mathbb{B}_1}$ , by the choice of the  $a'_\ell$ . Next, the second term in the union,  $(\bigcup_{\ell > k} (a_\ell - a'_\ell) \cap \bigcup_{n \leq k} a'_n)$  is modulo  $D$  again by the choice of the  $a'_\ell$ , equal to  $(\bigcup_{\ell > k} (a_\ell - a'_\ell)) \cap \bigcup_{n \leq k} a'_n$  which is zero as  $\langle a_n : n < \omega \rangle$  is an antichain; together by the previous sentences  $d_1 \cap \bigcup_{n \leq k} a'_n = 0_{\mathbb{B}_2}$  mod  $D$ .

Similarly  $d_2 \cap \bigcup_{n \leq k} a'_n = 0_{\mathbb{B}_2}$  mod  $D$  noting that  $\langle a'_n : n < \omega \rangle$  is necessarily an antichain of  $\mathbb{B}_1$ . Hence  $d' \cap \bigcup_{n \leq k} a'_n = d \cap \bigcup_{n \leq k} a'_n = \bigcup_{\ell=1}^2 (d_\ell \cap \bigcup_{n \leq k} a'_n) = 0_{\mathbb{B}_2} \cup 0_{\mathbb{B}_2} = 0_{\mathbb{B}_2}$  mod  $D$ . But  $d' \in \mathbb{B}_1$  and  $a'_n \in \mathbb{B}_1$  for every  $n$  (and  $D \cap \mathbb{B}_1 = \{1_{\mathbb{B}_1}\}$ , of course), hence  $d' \cap \bigcup_{n \leq k} a'_n = 0_{\mathbb{B}_1}$ . However, as this holds for every  $k$  and the choice of  $c$  it follows that  $d' \cap c = 0$ , but by the case first assumption of the present case  $c \in D_1 \subseteq D_2$  so  $d' \notin D_2$ , but by the case assumption  $d'/D = d/D$  and  $d \in D_2$  contradiction.  $\square_{0.12}$

## § 1. SPECTRUM OF THE ULTRAPRODUCTS OF FINITE CARDINALS

{k2}

**Definition 1.1.** Assume  $D$  is an ultra-filter on  $I$ .1) Let  $\text{upf}(D)$  be the spectrum of ultra-products mod  $D$  of finite cardinals, that is;  $\{\prod_{i \in I} n_i/D : n_i \in \mathbb{N} \text{ for } i \in I \text{ and } \prod_{i \in I} n_i/D \text{ is infinite}\}$ .2) For  $\lambda \in \text{upf}(D)$  let  $A_{D,\lambda} = \{a : a \in \mathbb{N}^I/D \text{ and the set } \{b \in \mathbb{N}^I/D : \mathbb{N}^I/D \models "b < a"\} \text{ has cardinality } \lambda\}$ ; we consider it as a linearly ordered set by the order inherited from  $\mathbb{N}^I/D$ .3) Let  $\text{spec}_1(D) = \{(\lambda, \theta, \partial) : \lambda \in \text{upf}(D) \text{ and } A_{D,\lambda} \text{ has cofinality } \partial \text{ and co-initiality } \theta\}$ .4) Let  $\text{spec}_2(D)$  be the set of triples  $(\lambda, \theta, \partial)$  such that:

- (a)  $\lambda \in \text{upf}(D)$
- (b)  $(\alpha)$  if  $\lambda < \max(\text{upf}(D))$  then  $A_{D,\lambda}$  has cofinality  $\partial$
- (β) if  $\lambda = \max(\text{upf}(D))$  then  $\partial = 0$  (or  $*$ )
- (c)  $\theta$  is the co-initiality of  $A_{D,\lambda}$ .

5) For  $D$  an ultrafilter on a complete Boolean Algebra  $\mathbb{B}$  we define the above similarly considering  $\mathbb{N}^{\mathbb{B}}/D$  instead  $\mathbb{N}^J/D$  but in clause (b),  $\partial$  is the cofinality of  $A_{D,\lambda}$  in all cases.

{f2}

**Definition 1.2.** Let  $K_\alpha$  be the class of objects  $\mathbf{k}$  consisting of:

- (a)  $\mathbb{B}_\beta$  is a Boolean Algebra for  $\beta \leq \alpha$
- (b)  $\langle \mathbb{B}_\beta : \beta \leq \alpha \rangle$  is increasing
- (c)  $\mathbb{B}_\beta$  is complete for  $\beta < \alpha$ ,  $\mathbb{B}_0$  is trivial
- (d)  $\mathbb{B}_\beta \triangleleft \mathbb{B}_\gamma$  if  $\beta < \gamma \leq \alpha$  and  $\cup\{\mathbb{B}_{\beta_1} : \beta_1 < \gamma\} \triangleleft \mathbb{B}_\gamma$  for limit  $\gamma \leq \alpha$
- (e)  $D_\beta$  is a filter on  $\mathbb{B}_\alpha$  such that  $\mathbb{B}_\beta \cap D_\beta = \{1_{\mathbb{B}_\beta}\}$
- (f)  $D_\beta$  is maximal under clause (e), so  $D_0$  is an ultrafilter and  $D_\alpha = \{1_{\mathbb{B}_\alpha}\}$
- (g)  $\langle D_\beta : \beta \leq \alpha \rangle$  is  $\subseteq$ -decreasing.

{f4}

**Definition 1.3.** 1) Above let  $\mathbb{B}[\mathbf{k}] = \mathbb{B}_{\mathbf{k}} = \mathbb{B}_\alpha, \mathbb{B}[\mathbf{k}, \beta] = \mathbb{B}_{\mathbf{k},\beta} = \mathbb{B}_\beta, \bar{\mathbb{B}}_{\mathbf{k}} = \langle \mathbb{B}_{\mathbf{k},\beta} : \beta \leq \alpha \rangle, D_{\mathbf{k},\beta} = D_\beta, D_{\mathbf{k}} = D_{\mathbf{k},0}, \ell g(\mathbf{k}) = \alpha_{\mathbf{k}} = \alpha(\mathbf{k}) = \alpha$ .1A) Let  $K_\alpha^{\text{com}}$  be the class of  $\mathbf{k} \in K_\alpha$  such that  $\mathbb{B}_{\mathbf{k}}$  is a complete Boolean Algebra.2) Assume  $\kappa > \aleph_0$  is regular. Let  $K_\alpha^{\text{cc}(\kappa),1}$  be the class of  $\mathbf{k} \in K_\alpha$  such that  $\mathbb{B}_\alpha$  satisfies the  $\kappa$ -c.c.3) Let  $K_\alpha^{\text{cc}(\kappa),2}$  be the class of  $\mathbf{k} \in K_\alpha^{\text{cc}(\kappa),1}$  such that:

- $\mathbb{B}_{\mathbf{k}}$  is complete; recall that for every  $\beta < \alpha, \mathbb{B}_\beta$  is complete
- if  $\delta \leq \alpha$  has cofinality  $\geq \kappa$  then  $\mathbb{B}_{\mathbf{k},\delta} = \bigcup_{\beta < \delta} \mathbb{B}_{\mathbf{k},\beta}$
- if  $\delta \leq \alpha$  is limit of cofinality  $< \kappa$ , then  $\mathbb{B}_{\mathbf{k},\delta}$  is the completion of  $\bigcup_{\beta < \delta} \mathbb{B}_{\mathbf{k},\beta}$ .

3A) We may omit  $\kappa$  when  $\kappa = \aleph_1$  so  $K_\alpha^{\text{cc},\iota} = K_\alpha^{\text{cc}(\aleph_1),\iota}$ ; if we omit  $\iota$  we mean 1.4) Let  $K = \bigcup_\alpha K_\alpha$  and  $K^{\text{cc}(\kappa),\iota} = \cup\{K_\alpha^{\text{cc}(\kappa),\iota} : \alpha \text{ is an ordinal}\}$  so  $K^{\text{cc}} = \bigcup_\alpha K_\alpha^{\text{cc}}$ .5) We say  $\mathbb{B}$  is above  $\bar{\mathbb{B}}_{\mathbf{k}}$  when  $\mathbb{B}_{\mathbf{k}} \subseteq \mathbb{B}$  and  $\mathbb{B}_{\mathbf{k},\beta} \triangleleft \mathbb{B}$  for  $\beta < \alpha_{\mathbf{k}}$ .6)  $K_\alpha^{\text{fr}(\kappa)}$  is the class of  $\mathbf{f}$  consisting of:

- {f2}
- (a)  $\mathbf{k}_f = (\bar{\mathbb{B}}, \bar{D})$  as in 1.2
  - (b)  $\bar{\xi} = \langle \xi_\gamma : \gamma \leq \alpha \rangle$  and  $\bar{x} = \langle x_{\beta, \zeta, i} : i < \kappa, \beta < \alpha, \zeta < \xi_\beta \rangle, x_{\beta, \zeta, i} \in \mathbb{B}_k$  are such that  $\bar{x}$  is free except that  $\beta < \xi_\alpha \wedge i < j < \kappa \Rightarrow x_{\beta, \zeta, i} \cap x_{\beta, \zeta, j} = 0$
  - (c) the sub-algebra which  $\langle x_{\beta, \zeta, i} : \zeta < \xi_\gamma, i < \kappa \rangle$  generates is dense in  $\mathbb{B}_{k, \gamma}$
  - (d) so  $\bar{\xi}_f = \bar{\xi}, \bar{x}_f = \bar{x}, \bar{\mathbb{B}}_f = \mathbb{B}_k$ , etc.

7) Let  $*K_\alpha$  be defined like  $K_\alpha$  in 1.2 omitting clause (d) of 1.2, and define  $*K$ , as above; not really needed here but we may comment. {f2}

**Definition 1.4.** 1) If  $\beta \leq \gamma$  and  $\mathbf{m} \in K_\gamma$  then  $\mathbf{k} = \mathbf{m} \upharpoonright \beta$  is the unique  $\mathbf{k} \in K_\beta$  such that  $\mathbb{B}_k = \mathbb{B}_{\mathbf{m}, \beta}, \mathbb{B}_{k, \alpha} = \mathbb{B}_{\mathbf{m}, \alpha}, D_{k, \alpha} = D_{\mathbf{m}, \alpha} \cap \mathbb{B}_k$  for  $\alpha \leq \beta$ . {f8}

1A) If  $\mathbf{k} \in K_\alpha$  and  $\beta < \alpha$  then  $\pi_{k, \beta}$  is the unique projection from  $\mathbb{B}_k$  onto  $\mathbb{B}_{k, \beta}$  such that  $\pi_{k, \beta}^{-1} \{1_{\mathbb{B}_{k, \beta}}\} = D_{k, \beta}$  recalling 0.11; let  $\pi_{k, \alpha} = \text{id}_{\mathbb{B}_{k, \alpha}}$  and if  $\gamma \leq \beta \leq \alpha$  then  $\pi_{k, \beta, \gamma} = \pi_{k, \gamma} \upharpoonright \mathbb{B}_{k, \beta}$ . {x23}

2) We define the following two-place relations on  $K$ :

(A)  $\mathbf{k} \leq_K^{\text{at}} \mathbf{m}$  where at stands for atomic iff:

- (a)  $\alpha_k = \alpha_m$
- (b)  $\mathbb{B}_{k, \beta} = \mathbb{B}_{m, \beta}$  for  $\beta < \alpha_k$
- (c)  $\mathbb{B}_{k, \alpha(k)} < \mathbb{B}_{m, \alpha(m)}$
- (d)  $D_{k, \beta} \subseteq D_{m, \beta}$  for  $\beta \leq \alpha_k$ .

(B)  $\mathbf{k} \leq_K^{\text{ver}} \mathbf{m}$ , where ver stands for vertical iff

- (a)  $\alpha_k \leq \alpha_m$
- (b)  $\mathbf{k} \leq_K^{\text{at}} (\mathbf{m} \upharpoonright \alpha_k)$

(C)  $\mathbf{k} \leq_K^{\text{hor}} \mathbf{m}$ , where hor stands for horizontal iff

- (a)  $\alpha_k = \alpha_m = \alpha$
- (b)  $\mathbb{B}_{k, \beta} < \mathbb{B}_{m, \beta}$  for  $\beta \leq \alpha$
- (c)  $D_{k, \beta} \subseteq D_{m, \beta}$  for  $\beta \leq \alpha$

(D) (a)  $\mathbf{f}_1 \leq_K^{\text{fr}(\kappa)} \mathbf{f}_2$  iff  $\mathbf{f}_\ell \in K_{\alpha_\ell}^{\text{fr}(\kappa)}$  and  $\mathbf{k}_{f_1} \leq_K^{\text{ver}} \mathbf{k}_{f_2}$  and  $\bar{x}_{f_1} \leq \bar{x}_{f_2}$  which means:

$$\bullet \beta < \alpha_1 \Rightarrow \bar{x}_{f_1, \beta} = \bar{x}_{f_2, \beta} \text{ and } \beta = \alpha_1 \Rightarrow \xi_{f_1, \beta} \leq \xi_{f_2, \beta} \wedge \bar{x}_{f_1, \beta} = \bar{x}_{f_2, \beta} \upharpoonright \xi_{f_2, \beta}$$

(b)  $\leq_K^{\text{at-fr}(\kappa)}$  is defined similarly

(E)  $\mathbf{k} \leq_K^{\text{wa}} \mathbf{m}$  where wa stands for weakly atomic iff

- (a), (b), (d) as in Clause (A)
- (c)  $\mathbb{B}_{k, \alpha(k)} \subseteq \mathbb{B}_{m, \alpha(m)}$ .

*Remark 1.5.* Note that for the present work it is not a loss to use exclusively c.c.c. Boolean Algebras; moreover ones which have a dense subalgebra which is free. So using only free Boolean Algebras or their completion, i.e.  $(K_\alpha^{\text{fr}(\kappa_1)}, \leq_K^{\text{fr}(\kappa)})$ ; so we are giving for  $\mathbb{B}$  a set of generators (and the orders respect this). {f3}

**Observation 1.6.** *The relations  $\leq_K^{\text{at}}, \leq_K^{\text{wa}}$  and  $\leq_K^{\text{ver}}$  and  $\leq_K^{\text{hor}}$  (the last one is not used) are partial orders on  $K$ .* {f10}

We need various claims on extending members of  $K$ , existence of upper bounds to an increasing sequence and amalgamation.

{f13}

**Claim 1.7.** *Let  $\delta$  be a limit ordinal.*

1) *If  $\langle \mathbf{k}_i : i < \delta \rangle$  is  $\leq_K^{\text{at}}$ -increasing then it has a  $\leq_K^{\text{at}}$ -lub  $\mathbf{k}_\delta$ , the union naturally defined so  $|\mathbb{B}_{\mathbf{k}_\delta}| \leq \Sigma\{|\mathbb{B}_{\mathbf{k}_i}| : i < \delta\}$ .*

1A) *Like part (1) for  $\leq_K^{\text{wa}}$ .*

2) *If  $\langle \mathbf{k}_i : i < \delta \rangle$  is a  $\leq_K^{\text{ver}}$ -increasing sequence, then it has a  $\leq_K^{\text{ver}}$ -upper bound  $\mathbf{k} = \mathbf{k}_\delta$  which is the union which means:*

- (a)  $lg(\mathbf{k}) = \cup\{lg(\mathbf{k}_i) : i < \delta\}$  call it  $\alpha$
- (b) if  $\beta < \alpha$  then  $\mathbb{B}_{\mathbf{k},\beta} = \mathbb{B}_{\mathbf{k}_i,\beta}$  for every large enough  $i$
- (c) (α) if  $\langle lg(\mathbf{k}_i) : i < \delta \rangle$  is eventually constant (so  $lg(\mathbf{k}_i) = \alpha$  for every  $i < \delta$  large enough) then
  - $\mathbb{B}_{\mathbf{k},\alpha} = \cup\{\mathbb{B}_{\mathbf{k}_i,\alpha} : i < \delta \text{ is such that } lg(\mathbf{k}_i) = \alpha\}$
  - $D_{\mathbf{k},\alpha} = \{1_{\mathbb{B}_{\mathbf{k},\alpha}}\}$ , redundant
- (β) if  $\langle lg(\mathbf{k}_i) : i < \delta \rangle$  is not eventually constant then
  - $\mathbb{B}_{\mathbf{k},\alpha} = \cup\{\mathbb{B}_{\mathbf{k}_i,lg(\mathbf{k}_i)} : i < \delta\}$
  - $D_{\mathbf{k},\alpha} = \{1_{\mathbb{B}_{\mathbf{k},\alpha}}\}$ , redundant
- (d) if  $\beta < \alpha$  then  $D_{\mathbf{k},\beta} = \cup\{D_{\mathbf{k}_i,\beta} : i < \delta \text{ is such that } \beta \leq lg(\mathbf{k}_i)\}$ .

3) *In part (2) if  $\mathbf{k}_i \in K^{\text{cc}(\kappa),2}$ ,  $\text{cf}(\delta) \geq \kappa$  and the sequence  $\langle lg(\mathbf{k}_i) : i < \delta \rangle$  is not eventually constant then  $\mathbb{B}_{\mathbf{k}}$  is complete and  $\text{upf}(D_{\mathbf{k}}) = \cup\{\text{upf}(D_{\mathbf{k}_i}) : i < \delta\}$ .*

4) *Similarly for the  $*K$  version.*

*Proof.* Straightforward (concerning part (3) note that recalling  $\text{cf}(\delta) \geq \kappa$  we have  $\mathbb{N}^{\mathbb{B}(\mathbf{k})} = \cup\{\mathbb{N}^{\mathbb{B}(\mathbf{k}_i)} : i < \delta\}$ . □<sub>1.7</sub>

{f16}

**Claim 1.8.** 1) *If  $\mathbf{k} \in K_\alpha$  and  $\mathbb{B}$  is above  $\bar{\mathbb{B}}_{\mathbf{k}}$  (i.e.  $\mathbb{B}_{\mathbf{k},\alpha} \subseteq \mathbb{B}$  and  $\beta < \alpha \Rightarrow \mathbb{B}_{\mathbf{k},\beta} < \mathbb{B}$ ) then there is  $\mathbf{m} \in K_\alpha$  such that  $\mathbf{k} \leq_K^{\text{wa}} \mathbf{m}$  and  $\mathbb{B}_{\mathbf{m},\alpha} = \mathbb{B}$ .*

2) *If  $\mathbf{k} \in K_\alpha$  and  $\mathbb{B} \subseteq \mathbb{B}_{\mathbf{k}}$  and  $\beta < \alpha \Rightarrow \mathbb{B}_{\mathbf{k},\beta} \subseteq \mathbb{B}$  (hence  $\mathbb{B}_{\mathbf{k},\beta} < \mathbb{B}$ ) then there is  $\mathbf{m} \in K_\alpha$  such that  $\mathbf{m} \leq_K^{\text{wa}} \mathbf{k}$  and  $\mathbb{B}_{\mathbf{m}} = \mathbb{B}$ .*

3) *If  $\mathbf{k} \in K_\alpha$ ,  $\mathbb{B}_{\mathbf{k}} < \mathbb{B}$  then for some  $\mathbf{m} \in K_\alpha$ ,  $\mathbf{k} \leq_K^{\text{at}} \mathbf{m}$ ,  $\mathbb{B}_{\mathbf{m}} = \mathbb{B}$*

4) *In part (2) if  $\mathbb{B} < \mathbb{B}_{\mathbf{k}}$  then we can add  $\mathbf{m} \leq_K^{\text{at}} \mathbf{k}$ .*

5) *If  $\mathbf{k} \leq_K^{\text{wa}} \mathbf{m}$  and  $\mathbb{B}_{\mathbf{k}} < \mathbb{B}_{\mathbf{m}}$  then  $\mathbf{k} \leq_K^{\text{at}} \mathbf{m}$ .*

*Proof.* 1) For simplicity we concentrate on the main case: all the  $\mathbb{B}_\beta$  and  $\mathbb{B}$  satisfies the c.c.c. By 1.7(1A) without loss of generality  $\mathbb{B}$  is generated by  $\mathbb{B}_{\mathbf{k},\alpha} \cup \{a\}$  where  $a \notin \mathbb{B}_{\mathbf{k},\alpha}$ . So  $\mathbb{B}_{\mathbf{m}}$  is uniquely defined and as required in Definition 1.2, but we have to define the  $D_{\mathbf{m},\beta}$ 's and, of course, let  $D_{\mathbf{m},\alpha} = \{1_{\mathbb{B}_{\mathbf{m}}}\}$ .

{f13}

{f2}

Case 1:  $\alpha = 0$

This is trivial.

Case 2:  $\alpha = \beta + 1$

As  $\mathbb{B}_{\mathbf{k},\beta} < \mathbb{B}$  and  $\pi_{\mathbf{k},\beta}$  is a projection from  $\mathbb{B}_{\mathbf{k},\alpha}$  onto  $\mathbb{B}_{\mathbf{k},\beta}$  and  $[\mathbb{B}_{\mathbf{k},\alpha} \subseteq \mathbb{B}$  and  $\mathbb{B}_{\mathbf{k},\beta}$  is complete by 0.10(1) there is a projection  $\pi$  from  $\mathbb{B}$  onto  $\mathbb{B}_{\mathbf{k},\beta}$  extending  $\pi_{\mathbf{k},\beta}$ . Now for  $\gamma < \alpha$  let  $D_{\mathbf{m},\gamma}$  be the filter on  $\mathbb{B}$  generated by  $D_{\mathbf{k},\gamma} \cup \{a \Delta \pi(a)\}$ .

{x30}

Case 3:  $\alpha$  is a limit ordinal,  $\mathbb{B}$  is c.c.c. (the main case) and  $\alpha$  is of cofinality  $> \aleph_0$

In this case,  $\mathbb{B}' = \bigcup_{\gamma < \alpha} \mathbb{B}_{\mathbf{k},\gamma}$  is complete and  $< \mathbb{B}$  so we can continue as in Case 2 using  $\mathbb{B}'$  instead  $\mathbb{B}_{\mathbf{k},\beta}$ .

Case 4:  $\text{cf}(\alpha) = \aleph_0$

Let  $\alpha = \bigcup_n \alpha_n, \alpha_n < \alpha_{n+1}$ . For  $\beta < \alpha$  let  $\pi_\beta$  be the projection of  $\mathbb{B}_k$  onto  $\mathbb{B}_{k,\beta}$  which maps  $D_{k,\beta}$  onto  $1_{\mathbb{B}_{k,\beta}}$ . Let  $\Pi_\beta$  be the set of homomorphisms from  $\mathbb{B}$  into  $\mathbb{B}_{k,\beta}$  extending  $\pi_\beta = \pi_{k,\beta}$ , so not empty hence (recalling  $\mathbb{B}_{k,\beta}$  is complete) there are  $b_\beta \leq c_\beta$  from  $\mathbb{B}_{k,\beta}$  such that  $\{\pi(a) : \pi \in \Pi_\beta\}$  is  $\{a' \in \mathbb{B}_{k,\beta} : b_\beta \leq a' \leq c_\beta\}$ . Clearly  $\gamma < \beta < \alpha \Rightarrow b_\gamma \leq b_\beta \leq c_\beta \leq c_\gamma$  in  $\mathbb{B}_{k,\beta}$ .

Now by induction on  $\zeta < (\|\mathbb{B}\|^{|\alpha|})^+$  we define  $\langle (b_{\beta,\zeta}, c_{\beta,\zeta}) : \beta < \alpha \rangle$  such that:

- (\*) $_\zeta$  (a)  $\mathbb{B}_{k,\beta} \models "b_{\beta,\zeta} \leq c_{\beta,\zeta}"$  for  $\beta < \alpha$
- (b) if  $\gamma < \beta < \alpha$  then  $\mathbb{B}_{k,\beta} \models "b_{\gamma,\zeta} \leq b_{\beta,\zeta} \leq c_{\beta,\zeta} \leq c_{\gamma,\zeta}"$
- (c) if  $\varepsilon < \zeta$  and  $\beta < \alpha$  then  $\mathbb{B}_{k,\beta} \models "b_{\beta,\varepsilon} \leq b_{\beta,\zeta} \leq c_{\beta,\zeta} \leq c_{\beta,\varepsilon}"$ .

Subcase 4A: For  $\zeta = 0$  let  $(b_{\beta,\zeta}, c_{\beta,\zeta}) = (b_\beta, c_\beta)$ , so clause (a),(b) holds as said above and clause (c) is empty.

Subcase 4B:  $\zeta$  is a limit ordinal

Let for  $\beta < \alpha$

- $b_{\beta,\zeta} = \cup\{b_{\gamma,\varepsilon} : \varepsilon < \zeta\}$  in  $\mathbb{B}_{k,\beta}$
- $c_{\beta,\zeta} = \cap\{c_{\gamma,\varepsilon} : \varepsilon < \zeta\}$ .

They are well defined because  $\mathbb{B}_{k,\beta}$  is a complete Boolean Algebra and it is easy to check that (a),(b),(c) hold.

Subcase 4C:  $\zeta = \varepsilon + 1$

Let  $b_{\beta,\zeta} = \cup\{\pi_{k,\gamma,\beta}(b_{\gamma,\varepsilon}) : \gamma \in (\beta, \alpha)\}$ ,  $c_{\beta,\zeta} = \cap\{\pi_{k,\gamma,\beta}(c_{\gamma,\varepsilon}) : \gamma \in (\beta, \alpha)\}$ .

Now check.

Having carried the induction, by (\*) $_\zeta$  for  $\zeta < (\|\mathbb{B}\|^{|\alpha|})^+$  for some  $\zeta_* < (\|\mathbb{B}\|^{|\alpha|})^+$ ,  $\langle (b_{\beta,\zeta}, c_{\beta,\zeta}) : \beta < \alpha \rangle$  is the same for all  $\zeta \geq \zeta_*$  and let  $a_\beta = b_{\beta,\zeta_*}$  for  $\beta < \alpha$ .

Easily  $\gamma < \beta < \alpha \Rightarrow \pi_\beta(a_\beta) = a_\gamma$  and let  $D_{\mathbf{m},\beta}$  be the filter of  $\mathbb{B}$  generated by  $D_{k,\beta} \cup \{(a \Delta a_\beta)\}$ .

2) Easy.

3) By (1).

4),5) Should be easy. □<sub>1.8</sub>

**Claim 1.9.** 1) If  $\mathbf{k} \in K_\alpha, \langle \beta_i : i \leq i(*) \rangle$  is increasing with  $\beta_{i(*)} = \alpha$  then there is one and only one  $\mathbf{m} \in K_{i(*)}$  such that  $(\mathbb{B}_{\mathbf{m},i}, D_{\mathbf{m},i}) = (\mathbb{B}_{k,\beta_i}, D_{\mathbf{m},\beta_i})$  for  $i \leq i(*)$ . {f17}

2) Above if  $\mathbf{m} \leq_K^{\text{at}} \mathbf{m}_1$  then there is  $\mathbf{k}_1$  such that  $\mathbf{k} \leq_K^{\text{at}} \mathbf{k}_1$  and  $\mathbb{B}_{\mathbf{k}_1} = \mathbb{B}_{\mathbf{m}_1}$  and  $D_{\mathbf{k}_1,\beta_i} = D_{\mathbf{m}_1,i}$  for  $i \leq i(*)$ .

2A) Similarly for  $\leq_K^{\text{wa}}$ .

3) Above if  $\mathbf{m} \leq_K^{\text{ver}} \mathbf{m}_1 \in K_{i(*)+j(*)}$  then there is  $\mathbf{k}_1 \in K_{\alpha+j(*)}$  such that  $\mathbf{k} \leq_K^{\text{ver}} \mathbf{k}_1, \mathbb{B}_{\mathbf{k}_1} = \mathbb{B}_{\mathbf{m}_1}, \mathbb{B}_{\mathbf{k}_1,\alpha+j} = \mathbb{B}_{\mathbf{m}_1,i(*)+j}, D_{\mathbf{k}_1,\alpha+j} = D_{\mathbf{m}_1,i(*)+j}, D_{\mathbf{k}_1,\beta_i} = D_{\mathbf{m}_1,i}$  for  $j < j(*), i \leq i(*)$ .

4) If  $\mathbf{k} \in K_\alpha$  and  $\beta_0 \leq \beta_1 \leq \beta_2 \leq \alpha$  then  $\pi_{\mathbf{k},\beta_2,\beta_0} = \pi_{\mathbf{k},\beta_1,\beta_0} \circ \pi_{\mathbf{k},\beta_2,\beta_1}$ .

*Proof.* Straightforward. □<sub>1.9</sub>

**Conclusion 1.10.** 1) If  $\mathbf{k} \in K_\alpha$  then there is  $\mathbf{m}$  such that  $\mathbf{k} \leq_K^{\text{at}} \mathbf{m}$  and  $\mathbb{B}_{\mathbf{m},\alpha}$  is the completion of  $\mathbb{B}_{k,\alpha}$  so  $\mathbf{m} \in K_\alpha^{\text{com}}$  and  $\mathbf{k} \leq_K^{\text{ver}} \mathbf{m}$ ; so if  $\mathbf{k} \in K_\alpha^{\text{cc}(\kappa)}$  then  $\mathbf{m} \in K_\alpha^{\text{cc}(\kappa)} \cap K^{\text{com}}$ . {f19}

2) If  $\alpha < \beta, \mathbf{k} \in K_\alpha, \mathbf{n} \in K_\beta$  and  $\mathbf{k} \leq_K^{\text{ver}} \mathbf{n}$ , then for some  $\mathbf{m}$  we have  $\mathbf{k} \leq_K^{\text{ver}} \mathbf{m} \leq_K^{\text{ver}} \mathbf{n}$  and  $\mathbb{B}_{\mathbf{m}}$  is the completion of  $\mathbb{B}_k$  inside  $\mathbb{B}_{\mathbf{n}}$ ; so if  $\mathbf{n} \in K^{\text{cc}(\kappa)}$  then  $\mathbf{m} \in K^{\text{cc}(\kappa)}$ .

*Proof.* 1) By 1.8.

2) Check the definitions. □<sub>1.10</sub>

{f22}

{f16}

**Claim 1.11.** *There is  $\mathbf{m}$  such that  $\mathbf{k} \leq_K^{\text{wa}} \mathbf{m}$ ,  $\mathbb{B}_{\mathbf{m}} = \mathbb{B}$  and  $Y \subseteq D_{\mathbf{m}}$  and  $\mathbb{B}_{\mathbf{k}} \triangleleft \mathbb{B} \Rightarrow \mathbf{k} \leq_K^{\text{at}} \mathbf{m}$  when:*

- (a)  $\mathbf{k} \in K_{\alpha}$
- (b)  $\mathbb{B}$  is a Boolean Algebra,  $Y \subseteq \mathbb{B}$
- (c) (α)  $\mathbb{B}_{\mathbf{k}} \subseteq \mathbb{B}$   
 (β)  $\mathbb{B}_{\mathbf{k},\beta} \triangleleft \mathbb{B}$  for  $\beta < \alpha_{\mathbf{k}}$
- (d) if  $\beta < \alpha_{\mathbf{k}}$  then for some  $X_{\beta}$  we have:
  - (α)  $X_{\beta} \subseteq Y$
  - (β)  $X_{\beta}$  is a downward directed subset of  $\mathbb{B}$
  - (γ) if  $x \in X_{\beta}$  and  $b \in D_{\mathbf{k},\beta}$  then  $x \cap b$  is not disjoint to any  $a \in \mathbb{B}_{\mathbf{k},\beta}^+$
  - (δ) if  $y \in Y$  then for some  $b \in D_{\mathbf{k},\beta}$  and  $x \in X_{\beta}$  we<sup>2</sup> have  $b \cap x \leq y$ .

{f16}

*Proof.* By 1.8 without loss of generality  $\mathbb{B}$  is generated by  $\mathbb{B}_{\mathbf{k}} \cup Y$  and let  $\alpha = \alpha_{\mathbf{k}}$  and let  $X_{\beta}$  be as in clause (d) in the claim for  $\beta < \alpha$  and define  $\mathbf{m} \in K_{\alpha}$  as follows:

- $D_{\mathbf{m},\alpha} = \{1_{\mathbb{B}}\}$ ,  $\mathbb{B}_{\mathbf{m},\alpha} = \mathbb{B}$
- for  $\beta < \alpha$  let  $\mathbb{B}_{\mathbf{m},\beta} = \mathbb{B}_{\mathbf{k},\beta}$  and  $D_{\mathbf{m},\beta}$  be the filter on  $\mathbb{B}_{\mathbf{m},\alpha}$  generated by  $D_{\mathbf{k},\beta} \cup X_{\beta}$ .

The point is to check  $\mathbf{m} \in K_{\alpha}$  as then  $\mathbf{k} \leq_K^{\text{at}} \mathbf{m}$  and  $Y \subseteq D_{\mathbf{m}}$  are obvious, also  $\bar{\mathbb{B}}_{\mathbf{m}}$  is as required and  $D_{\mathbf{m},\beta}$  a filter on  $\mathbb{B}_{\mathbf{m},\alpha}$  including  $D_{\mathbf{k},\beta}$  are obvious.

So proving  $(*)_1, (*)_2, (*)_3$  below will suffice

$(*)_1$  if  $\beta < \gamma < \alpha$  then  $D_{\mathbf{m},\gamma} \subseteq D_{\mathbf{m},\beta}$ .

[Why? If  $a \in D_{\mathbf{m},\gamma}$  then by the choice of  $D_{\mathbf{m},\gamma}$  (recalling  $D_{\mathbf{k},\gamma}$  is downward directed being a filter and  $X_{\gamma}$  is downward directed by its choice (i.e. Clause (d)(β) of the claim) for some  $b \in D_{\mathbf{k},\gamma}$  and  $x \in X_{\gamma}$  we have  $b \cap x \leq a$ . So by (d)(α) applied to  $\gamma$  we have  $x \in Y$  hence by (d)(δ) applied to  $\beta$  for some  $b_1 \in D_{\mathbf{k},\beta}$  and  $x_1 \in X_{\beta}$  we have  $b_1 \cap x_1 \leq x$  hence  $(b \cap b_1) \cap x_1 = b \cap (b_1 \cap x_1) \leq b \cap x \leq a$  but  $b \in D_{\mathbf{k},\gamma} \subseteq D_{\mathbf{k},\beta}$ ,  $b_1 \in D_{\mathbf{k},\beta}$  hence  $b \cap b_1 \in D_{\mathbf{k},\beta}$  and  $x_1 \in X_{\beta}$  hence  $a \in D_{\mathbf{m},\beta}$  by the choice of  $D_{\mathbf{m},\beta}$ .]

$(*)_2$   $D_{\mathbf{m},\beta}$  is a filter on  $\mathbb{B}_{\mathbf{m},\alpha} = \mathbb{B}$  disjoint to  $\mathbb{B}_{\mathbf{m},\beta} \setminus \{1_{\mathbb{B}_{\mathbf{k}}}\} = \mathbb{B}_{\mathbf{k},\beta} \setminus \{1_{\mathbb{B}_{\mathbf{k}}}\}$ .

[Why? By the definition of  $D_{\mathbf{m},\beta}$  and clause (d)(γ).]

$(*)_3$  if  $\beta < \alpha$  then  $D_{\mathbf{m},\beta}$  is a maximal filter of  $\mathbb{B}_{\mathbf{m}}$  disjoint to  $\mathbb{B}_{\mathbf{k},\beta} \setminus \{1_{\mathbf{m},\beta}\} = \mathbb{B}_{\mathbf{m},\beta} \setminus \{1_{\mathbb{B}_{\mathbf{m},\beta}}\}$ .

Why? If  $b \in \mathbb{B} = \mathbb{B}_{\mathbf{m},\alpha}$  then for some Boolean terms  $\sigma(y_0, y_1, \dots, z_0, z_1, \dots)$  and  $a_0, a_1, \dots \in \mathbb{B}_{\mathbf{k}}$  and  $x_0, x_1, \dots \in Y$  we have  $b = \sigma(a_0, a_1, \dots, x_0, x_1, \dots)$  hence modulo the filter  $D_{\mathbf{m},\beta}$ ,  $b$  is equal to  $\sigma(a_0, a_1, \dots, 1_{\mathbb{B}_{\mathbf{k},\beta}}, 1_{\mathbb{B}_{\mathbf{k},\beta}}, \dots)$ . But for each  $a_{\ell}$  there is  $c_{\ell} \in \mathbb{B}_{\mathbf{k},\beta}$  such that  $a_{\ell} = c_{\ell} \text{ mod } D_{\mathbf{k},\beta}$  hence  $b$  is equal to  $\sigma(c_0, c_1, \dots, 1_{\mathbb{B}_{\mathbf{k},\beta}}, 1_{\mathbb{B}_{\mathbf{k},\beta}}, \dots)$  which belongs to  $\mathbb{B}_{\mathbf{k},\beta}$ .

<sup>2</sup>Does this contradict (d)(γ)? No, as  $D_{\mathbf{k},\beta}$  is disjoint to  $\mathbb{B}_{\mathbf{k},\beta} \setminus \{1_{\mathbb{B}_{\mathbf{k},\beta}}\}$ .

{f24} As this holds for any  $b \in \mathbb{B}$  we are easily done. □<sub>1.11</sub>

**Definition 1.12.** 1) We say  $\mathbf{k}$  is reasonable in  $\alpha$  when  $\alpha + 1 \leq \alpha_{\mathbf{k}}$  (so  $\mathbb{B}_{\mathbf{k},\alpha}$  is complete) and there is a maximal antichain of  $\mathbb{B}_{\alpha+1}$  included in  $\{a \in \mathbb{B}_{\mathbf{k},\alpha+1} : \pi_{\mathbf{k},\alpha+1,\alpha}(a) = 0_{\mathbb{B}_{\mathbf{k},\alpha}}\}$ .

2) We say  $\mathbf{k}$  is reasonable when it is reasonable in  $\alpha$  whenever  $\alpha + 1 \leq \alpha_{\mathbf{k}}$ .

3) Let

- $A_{\mathbf{k},\alpha}^1 = \{f \in \mathbb{N}^{\mathbb{B}[\mathbf{k}]} : f \in \mathbb{N}^{\mathbb{B}[\mathbf{k},\alpha]} \text{ and if } \beta < \alpha \text{ then } f/D_{\mathbf{k}} \notin \mathbb{N}^{\mathbb{B}[\mathbf{k},\beta]}/D_{\mathbf{k}}\}$
- $A_{\mathbf{k},\alpha}^2 = \{f/D_{\mathbf{k}} : f \in A_{\mathbf{k},\alpha}^1\}$
- $A_{\mathbf{k},<\alpha}^1 = \bigcup_{\beta < \alpha} A_{\mathbf{k},\beta}^1$  and  $A_{\mathbf{k},<\alpha}^2 = \bigcup_{\beta < \alpha} A_{\mathbf{k},\beta}^2$ , etc.

4) We say  $f$  is reasonable<sup>3</sup> in  $(\mathbf{k}, \alpha, \beta)$  when  $\alpha < \beta < \alpha_{\mathbf{k}}$  and  $f \in \mathbb{N}^{\mathbb{B}[\mathbf{k},\beta]}$  and for some  $f' \in \mathbb{N}^{\mathbb{B}[\mathbf{k},\beta]}$ , we have  $f'/D_{\mathbf{k}} = f/D_{\mathbf{k}}$  and  $f'$  is represented by  $\langle a_n : n < \omega \rangle$  and  $\pi_{\mathbf{k},\alpha}(a_n) = 0$  for every  $n$  large enough and  $a_n \notin D_{\mathbf{k}}$  for every  $n$ . If  $\beta = \alpha + 1$  we may omit it.

5) We say  $f$  is reasonable in  $(\mathbf{k}, < \alpha)$  when it is reasonable in  $(\mathbf{k}, \beta, \gamma)$  for some  $\beta + 1 = \gamma < \alpha$ . {m0}

**Observation 1.13.** If  $\beta < \alpha_{\mathbf{k}}$ ,  $\mathbf{k} \in K^{\text{com}}$  and  $f \in \mathbb{N}^{\mathbb{B}[\mathbf{k}]}$  is represented by  $\langle a_n : n < \omega \rangle$ , then  $f \in A_{\mathbf{k},\leq\beta}^1$  iff  $\bigcup_n (a_n \triangle \pi_{\mathbf{k},\beta}(a_n)) \notin D_{\mathbf{k}}$ . {x33}

*Proof.* By the proof of 0.12. □<sub>1.13</sub>

**Claim 1.14.** 1) If  $\mathbf{k} \in K_{\alpha+1}^{\text{cc}}$  then there is  $\mathbf{m} \in K_{\alpha+1}^{\text{cc}}$  such that  $\mathbf{k} \leq_K^{\text{at}} \mathbf{m}$ ,  $\mathbb{B}_{\mathbf{m}}$  is complete and  $\mathbf{m}$  is reasonable in  $\alpha$  and  $\|\mathbb{B}_{\mathbf{m}}\| = \|\mathbb{B}_{\mathbf{k}}\|^{\aleph_0}$ . {f28}

2) If  $\mathbf{k} \in K_{\alpha+1}^{\text{cc}}$  is reasonable in  $\alpha$  and  $\mathbf{k} \leq_K^{\text{at}} \mathbf{m}$  or  $\mathbf{k} \leq_K^{\text{ver}} \mathbf{m}$  then  $\mathbf{m}$  is reasonable in  $\alpha$ .

3) If  $\langle \mathbf{k}_i : i < \delta \rangle$  is  $\leq_K^{\text{ver}}$ -increasing in  $K^{\text{cc}}$  and each  $\mathbf{k}_i$  is reasonable then there is a  $\leq_K^{\text{ver}}$ -upper bound  $\mathbf{k}$  of cardinality  $(\sum_i \|\mathbb{B}_{\mathbf{k}_i}\|)^{\aleph_0}$  which is reasonable.

4) If  $f$  is reasonable in  $(\mathbf{k}, \alpha)$  then it is reasonable in  $(\mathbf{k}, < \alpha + 1)$ .

5) If  $f \in A_{\mathbf{k},\alpha}^1$  then  $f$  is reasonable in  $(\mathbf{k}, \alpha)$ .

6) In 1.7,(2),(3) if  $\mathbf{k}_i$  is reasonable for every  $i < \delta$  then so is  $\mathbf{k}$ . {f13}

*Proof.* Straightforward; e.g.:

2) Because  $\mathbb{B}_{\mathbf{k}} < \mathbb{B}_{\mathbf{m},\alpha(\mathbf{k})}$ , see Definition 1.4(2) and read Definition 1.12(1). {f24}

5) Let  $\langle a_n : n < \omega \rangle$  represent  $f$ .

Let  $a'_n = \pi_{\mathbf{k},\alpha+1}(a_n)$ , so  $\pi_{\mathbf{k},\alpha+1,\alpha}(a_n - a'_n) = \pi_{\mathbf{k},\alpha+1,\alpha}(a_n) - \pi_{\mathbf{k},\alpha+1,\alpha}(a'_n) = a'_n - a'_n = 0$ . Now  $\langle a'_n : n < \omega \rangle$  is an antichain (using  $\pi_{\mathbf{k},\alpha}$ ) and let  $g \in \mathbb{N}^{\mathbb{B}_{\alpha}}$  be such that  $g(a'_n) = n$  and  $g(1_{\mathbb{B}_{\mathbf{k},\alpha}} - \bigcup_n a'_n) = 0$ . So  $g \in \mathbb{N}^{\mathbb{B}_{\mathbf{k}}}$  and by “ $f \in A_{\mathbf{k},\alpha}^1$ ”, necessarily  $\text{TV}(f \neq g) \in D_{\mathbf{k}}$  and clearly  $\text{TV}(f \neq g) \leq \bigcup_n (a_n - a'_n) \cup \text{TV}(f = 0)$

but  $\text{TV}(f \neq 0) \in D_{\mathbf{k}}$ , i.e.  $a_0 \notin D_{\mathbf{k}}$  by the assumption on  $f$ , hence necessarily  $b := \bigcup_n (a_n - a'_n) \in D_{\mathbf{k}}$ . Now define  $f'' \in \mathbb{N}^{\mathbb{B}_{\mathbf{k}}}$  represented by  $\langle b_n : n < \omega \rangle$  where  $b_n = a_n - a'_n$  for  $n \geq 1$  and  $b_n = 1 - \bigcup_{m \geq 1} b_m$  for  $n = 0$ . So  $\pi_{\mathbf{k},\alpha}(b_n) = 0_{\mathbb{B}_{\mathbf{k},n}}$  for  $n \geq 1$  and  $b_n \notin D_{\mathbf{k}}$  for  $n \geq 0$ . Clearly  $f''$  is reasonable in  $(\mathbf{k}, \alpha)$ . □<sub>1.14</sub>

<sup>3</sup>Note that “some  $f$  is reasonable in  $(\mathbf{k}, \alpha)$ ” is close to but not equivalent to “ $\mathbf{k}$  is reasonable in  $\mathbf{h}$ ”.

{f27}

**Claim 1.15.** *If (A) then (B) where:*

- (A) (a)  $\mathbf{k} \in K_\alpha^{\text{cc}}$   
 (b)  $\beta_n < \beta_{n+1} < \beta = \bigcup_k \beta_k \leq \alpha$   
 (c)  $\mathbf{k}$  is reasonable in  $\beta_n$  for every  $n \in \omega$
- (B) if  $f_1 \in A_{\mathbf{k},\beta}^1$ , i.e.  $f_1 \in \mathbb{N}^{\mathbb{B}[\mathbf{k},\beta]}$ ,  $f_1/D_{\mathbf{k}} \notin \cup\{\mathbb{N}^{\mathbb{B}[\mathbf{k},\gamma]}/D_{\mathbf{k}} : \gamma < \beta\}$  then there is  $f_2$  such that:
- (a)  $f_2 \in \mathbb{N}^{\mathbb{B}[\mathbf{k},\beta]}$   
 (b)  $f_2/D_{\mathbf{k}} \notin \cup\{\mathbb{N}^{\mathbb{B}[\mathbf{k},\gamma]}/D_{\mathbf{k}} : \gamma < \beta\}$   
 (c)  $f_2/D_{\mathbf{k}} < f_1/D_{\mathbf{k}}$   
 (d) there is  $\langle (a_i, k_i) : i < \omega \rangle$  representing  $f_2$  such that:  
 (α) for each  $i$ , letting  $k(i) = k_i$  we have  $a_i \in \mathbb{B}_{\mathbf{k},\beta_{k(i)+1}}$  and  $\pi_{\mathbf{k},\beta_{k(i)+1},\beta_{k(i)}}(a_i) = 0$   
 (β) for each  $\ell$  the set  $\{i : k_i < \ell\}$  is finite.

*Proof.* For each  $n$  let  $\langle a_{n,\ell} : \ell < \omega \rangle$  be a maximal antichain of  $\mathbb{B}_{\beta_{n+1}}$  such that  $\pi_{\mathbf{k},\beta_{n+1},\beta_n}(a_{n,\ell}) = 0$  for  $\ell < \omega$ , exists as  $\mathbf{k}$  is reasonable in  $\beta_n$  for every  $n \in \omega$ , see Definition 1.12(2).  
 {x24}

Let

- (\*)<sub>0</sub> (a)  $\mathcal{T}_n = \{\eta : \eta \in {}^n\omega \text{ and } \bigcap_{k < n} a_{k,\eta(k)} > 0\}$   
 (b)  $\mathcal{T} = \bigcup_n \mathcal{T}_n$ .

Hence

- (\*)<sub>1</sub> (a)  $\langle a_\eta : \eta \in \mathcal{T}_n \rangle$  is a maximal antichain of  $\mathbb{B}_{\beta_{n+1}}$  on which  $\pi_{\mathbf{k},\beta_n}$  is zero  
 (b)  $\mathcal{T}$  is a subtree of  ${}^\omega\omega$ .

Now choose a sequence  $\bar{k}$  of natural numbers such that:

- (\*)<sub>2</sub> (a)  $\bar{k} = \langle k_s : s \in \mathcal{T} \rangle$   
 (b) if  $\nu \triangleleft \eta$  then  $k_\nu < k_\eta$   
 (c) if  $k_\eta = k_\nu$  then  $\eta = \nu$
- {x20} (\*)<sub>3</sub> let  $g_n \in \mathbb{N}^{\mathbb{B}[\mathbf{k},\beta(n)+1]}$  be represented by  $\langle (a_\eta, k_\eta) : \eta \in \mathcal{T}_n \rangle$ , see Definition 0.8(4).

[Why? Well defined by (\*)<sub>1</sub>(a).]

- (\*)<sub>4</sub> let  $\mathbf{M}_{\mathbf{k}} = \{\bar{c} : \bar{c} = \langle c_\ell : \ell < \omega \rangle \text{ is a maximal antichain of } \mathbb{B}_{\beta_0+1} \text{ disjoint to } D_{\mathbf{k}}\}$ .

What is the point of  $\mathbf{M}_{\mathbf{k}}$ ?  $g_n \in A_{\mathbf{k},\beta(n)}^1$  hence  $\langle g_n/D_{\mathbf{k}} : n < \omega \rangle$  is increasing and cofinal in  $\cup\{\mathbb{N}^{\mathbb{B}[\mathbf{k},\beta(n)]}/D_{\mathbf{k}} : n < \omega\}$  hence if in  $\mathbb{N}^{\mathbb{B}[\mathbf{k}]} / D_{\mathbf{k}}$  we have a definable sequence, the  $n$ -th try being  $g_n/D_{\mathbf{k}}$ , in “non-standard places” we have the  $g_{\bar{c}}$ 's defined below members of  $A_{\mathbf{k},\alpha}^2$  and those are co-initial in it.

- (\*)<sub>5</sub> for each  $\bar{c} \in \mathbf{M}_{\mathbf{k}}$  let  
 (a)  $S_{\bar{c}} = \{(\ell, \eta) : \ell < \omega, \eta \in \mathcal{T}_\ell \text{ and } c_\ell \cap a_\eta > 0\}$



(b) for  $(\ell, \eta) \in S_{\bar{c}}$  let  $a_{(\ell, \eta)} = c_\ell \cap a_\eta$

(c)  $g_{\bar{c}} \in \mathbb{N}^{\mathbb{B}[\mathbf{k}, \beta]}$  be represented by  $\langle (a_{(\ell, \eta)}, k_\eta) : (\ell, \eta) \in S_{\bar{c}} \rangle$

(\*)<sub>6</sub>  $g_n$  is  $(\mathbf{k}, \beta_n)$ -reasonable.

[Why? By the Definition 1.12.]

{f24}

(\*)<sub>7</sub>  $g_n \in A_{\mathbf{k}, \beta_n+1}^1$ .

[Why? Follows from the definition of  $g_n$  in (\*)<sub>3</sub> and the choice of the  $a_\eta$ 's and  $k_\nu$ 's in (\*)<sub>1</sub> and (\*)<sub>2</sub>.]

(\*)<sub>8</sub>  $g_{\bar{c}} \in A_{\mathbf{k}, \beta}^1$  for  $\bar{c} \in \mathbf{M}_{\mathbf{k}}$ .

[Why? As  $\bar{k}$  is with no repetition and the definition.]

(\*)<sub>9</sub> there is  $\bar{c} \in \mathbf{M}_{\mathbf{k}}$  such that  $g_{\bar{c}}/D_{\mathbf{k}} < f_1/D_{\mathbf{k}}$ .

[Why? See explanation after (\*)<sub>4</sub> and 1.14(5).]

□<sub>1.15</sub> {f26}  
{f39}

**Claim 1.16.** *If (A) then (B) where:*

- (A) (a)  $\mathbf{k} \in K_\alpha^{\text{cc}}$  is  $\mathbb{B}_{\mathbf{k}, \alpha}$  infinite  
 (b)  $\beta_n = \beta(n) < \alpha$  is increasing with limit  $\alpha$   
 (c)  $\mathbf{k}$  is reasonable in  $\beta_n$   
 (d)  $f \in A_{\mathbf{k}, \alpha}^1$
- (B) *there are  $\mathbf{m}, g$  such that:*  
 (a)  $\mathbf{k} \leq_K^{\text{at}} \mathbf{m}$  and  $\|\mathbb{B}_{\mathbf{m}}\| = (\|\mathbb{B}_{\mathbf{k}}\|^{\aleph_0})^+$   
 (b)  $g \in A_{\mathbf{m}, \alpha}^1$   
 (c)  $g/D_{\mathbf{m}} < f/D_{\mathbf{m}}$   
 (d)  $g/D_{\mathbf{m}} \notin \mathbb{N}^{\mathbb{B}[\mathbf{k}]} / D_{\mathbf{m}}$ .

*Proof.* Without loss of generality  $\mathbb{B}_{\mathbf{k}}$  is complete of cardinality  $\lambda$ .

Let  $f_2, \langle (a_n, k_n) : n < \omega \rangle$  be as in 1.15 for  $f_1 = f$  and let  $u_n = \{\ell : a_\ell \in \mathbb{B}_{\mathbf{k}, \beta_n}\}$ , by 1.15 clearly  $u_n$  is finite. {f27}

Let  $\mathbb{B}^0$  be the Boolean algebra extending  $\mathbb{B}_{\mathbf{k}}$  generated by  $\mathbb{B}_{\mathbf{k}} \cup \{x_{\varepsilon, n, \ell} : \ell \leq n \text{ and } \varepsilon < \lambda^+\}$  freely except the equation  $x_{\varepsilon, n, \ell} \leq a_n, x_{\varepsilon, n, \ell_1} \cap x_{\varepsilon, n, \ell_2} = 0$  and  $\bigcup_{\ell \leq n} x_{\varepsilon, n, \ell} = a_n$  for  $\varepsilon < \lambda^+, \ell \leq n, \ell_1 < \ell_2 \leq n$  and let  $\mathbb{B}$  be the completion on  $\mathbb{B}^0$ . Let  $g_\varepsilon \in \mathbb{N}^{\mathbb{B}}$  be represented by  $\langle x_{g_\varepsilon, \ell} := \bigcup \{x_{\varepsilon, n, \ell} : n \text{ satisfies } \ell \leq n\} : \ell < \omega \rangle$ , clearly

(\*)<sub>1</sub>  $g_\varepsilon/D \leq f_2/D$  for any  $D \in \text{uf}(\mathbb{B})$ .

For  $\varepsilon \neq \zeta < \lambda^+$  let  $c_{\varepsilon, \zeta} = \bigcup_{\ell} (x_{g_\varepsilon, \ell} \Delta x_{g_\zeta, \ell})$ .

Now

(\*)<sub>2</sub>  $c_{\varepsilon, \zeta} = \bigcup_n \bigcup_{\ell \leq n} (x_{\varepsilon, n, \ell} \Delta x_{\zeta, n, \ell})$  and  $c_{\varepsilon, \zeta} = c_{\zeta, \varepsilon}$ .

[Why? As  $\langle x_{\varepsilon, n, \ell} : \ell \leq n \rangle$  is a partition of  $a_n$  and  $\langle a_n : n < \omega \rangle$  is an antichain of  $\mathbb{B}$ .]

Let  $\mathbb{B}'$  be the sub-algebra of  $\mathbb{B}$  generated by  $\mathbb{B}_{\mathbf{k}} \cup Y$  where  $Y := \{c_{\varepsilon, \zeta} : \varepsilon < \zeta < \lambda^+\}$

(\*)<sub>3</sub> we define  $\pi_n^1 : \mathbb{B}_{\mathbf{k}} \cup Y \rightarrow \mathbb{B}_{\beta_n}$  by:

- $\pi_n^1 \upharpoonright \mathbb{B}_{\mathbf{k}} = \pi_{\mathbf{k}, \alpha(\mathbf{k}), \beta(n)}$
- $\pi_n^1(c_{\varepsilon, \zeta}) = 1_{\mathbb{B}_{\beta(n)}}$  for  $\varepsilon \neq \zeta < \lambda^+$

(\*)<sub>4</sub>  $\pi_n^1$  has an extension  $\pi_n^2 \in \text{Hom}(\mathbb{B}', \mathbb{B}_{\beta(n)})$ , necessarily unique

[Why? It is enough to show that if  $d_0, \dots, d_{m-1} \in \mathbb{B}_{\mathbf{k}}$  and  $\varepsilon_\ell < \zeta_\ell < \lambda^+$  for  $\ell < k$  and  $\sigma(y_0, \dots, y_{m-1}, x_0, \dots, x_{k-1})$  is a Boolean term and

$$\mathbb{B} \models \sigma(d_0, \dots, d_{m-1}, c_{\varepsilon_0, \zeta_0}, \dots, c_{\varepsilon_{k-1}, \zeta_{k-1}}) = 0$$

then  $\mathbb{B}_{\mathbf{k}, \beta(n)} \models \sigma(\pi_n^1(d_0), \dots, \pi_n^1(d_{m-1}), \pi_n^1(c_{\varepsilon_0, \zeta_0}, \dots)) = 0$ . As  $d_0, \dots, d_{m-1} \in \mathbb{B}_{\mathbf{k}}$  and  $\pi_n^1(c_{\varepsilon_\ell, \zeta_\ell}) = 1_{\mathbb{B}_{\mathbf{k}, \beta(n)}}$  it is sufficient to prove: if  $d \in \mathbb{B}_{\mathbf{k}}$  and  $\mathbb{B} \models "d \cap \bigcap_{\ell < k} c_{\varepsilon_\ell, \zeta_\ell} =$

$0"$  then  $\mathbb{B}_{\mathbf{k}, \beta(n)} \models "(\pi_{\mathbf{k}, \alpha(\mathbf{k}), \beta(n)}(d)) = 0"$ .

Now if for some  $\ell \geq 1, \ell \notin u_n, d \cap a_\ell > 0$  the assumption does not hold and otherwise, necessarily  $d \leq \bigcup_{\ell \in u_n} a_\ell$  hence the conclusion holds. So indeed  $\pi_n^2$  is well defined.]

(\*)<sub>5</sub>  $\pi_n^2 = \pi_{\mathbf{k}, \beta(n+1), \beta(n)} \circ \pi_{n+1}^2$ .

{f17} [Why? See the definition of  $\pi_m^1$  recalling 1.9(4).]

(\*)<sub>6</sub> there is  $\mathbf{n}$  such that  $\mathbf{k} \leq_K^{\text{at}} \mathbf{n}$  and  $\mathbb{B}_{\mathbf{n}} = \mathbb{B}'$ ,  $\pi_{\mathbf{n}, \alpha(\mathbf{k}), \beta(n)} = \pi_n^2$  hence  $D_{\mathbf{n}} \supseteq Y$ .

[Why? Check the definitions.]

(\*)<sub>7</sub> there is  $\mathbf{m}$  such that  $\mathbf{n} \leq_K^{\text{wa}} \mathbf{m}$ ,  $\mathbb{B}_{\mathbf{m}} = \mathbb{B}$  hence  $\mathbf{k} \leq_K^{\text{wa}} \mathbf{m}$

{f16} [Why? By claim 1.8(1),(3), the "hence" by 1.8(5) recalling that  $\mathbb{B}_{\mathbf{k}} < \mathbb{B}$  by the choice of  $\mathbb{B}$  and  $\mathbb{B}_{\mathbf{k}} \subseteq \mathbb{B}' \subseteq \mathbb{B}$  by the choice of  $\mathbb{B}'$ .]

(\*)<sub>8</sub> there is  $\varepsilon < \lambda^+$  such that  $g_\varepsilon \in A_{\mathbf{n}, \alpha(\mathbf{k})}^1$ .

[Why? As  $A_{\mathbf{n}, < \alpha(\mathbf{k})}^1$  has cardinality  $\leq \lambda$ .]

(\*)<sub>9</sub>  $\mathbf{m}$  is as required.

{f16} [By (\*)<sub>7</sub> + (\*)<sub>8</sub> and 1.8(2).] □<sub>1.16</sub>

{f40} **Observation 1.17.** *In claim 1.16 we can demand  $\|\mathbb{B}_{\mathbf{m}}\| = \|\mathbb{B}_{\mathbf{k}}\|^{\aleph_0}$ .*

{f39} *Proof.* By the Löwenheim-Skolem-Tarski argument. □<sub>1.17</sub>

{f42} **Claim 1.18.** *Assume  $\mathbf{k} \in K_\alpha^{\text{cc}}$  and  $\text{cf}(\alpha) > \aleph_0$ .*

*If  $f \in \mathbb{N}^{\mathbb{B}[\mathbf{k}]}$  and  $f/D \notin \mathbb{N}^{\mathbb{B}[\mathbf{k}, \beta]}/D_{\mathbf{k}}$  for  $\beta < \alpha$ , then for some  $\mathbf{m}$  and  $g$  we have:*

- (\*) (a)  $\mathbf{k} \leq_K^{\text{at}} \mathbf{m}$
- (b)  $g \in \mathbb{N}^{\mathbb{B}[\mathbf{m}]}$
- (c)  $g/D_{\mathbf{m}} < f/D_{\mathbf{m}}$
- (d)  $h/D_{\mathbf{m}} < g/D_{\mathbf{m}}$  when  $h \in \mathbb{N}^{\mathbb{B}[\mathbf{k}, \beta]}$  for some  $\beta < \alpha$
- (e)  $|\mathbb{B}_{\mathbf{m}}| \leq |\mathbb{B}_{\mathbf{k}}|$ .

{f39} *Proof.* Like the proof of 1.16 only simpler and shorter. Let  $\bar{a} = \langle a_n : n < \omega \rangle$   
 {f26} represent  $f$ ,  $\lambda = \|\mathbb{B}_{\mathbf{k}}\|$ . By 1.14(4), without loss of generality  $f$  is reasonable in  
 $(\mathbf{k}, \alpha)$ ; let  $\{x_{\varepsilon, n, \ell} : \varepsilon < \lambda^+, \ell \leq n\}$ ,  $\mathbb{B}^0, \mathbb{B}, Y, \mathbb{B}'$  be as there and define  $\pi_1 : \mathbb{B}_{\mathbf{k}} \cup Y \rightarrow$   
 $\mathbb{B}_{\mathbf{k}, \alpha}$  as there.  $\pi_1 \upharpoonright \mathbb{B}_{\mathbf{k}, \alpha} = \pi_{\mathbf{k}, \alpha+1, \alpha}$ ,  $\pi_1(x_{\varepsilon, \zeta}) = 1_{\mathbb{B}_{\mathbf{k}, \alpha}}$  for  $\varepsilon < \zeta < \lambda^+$ .

The rest is as there. □<sub>1.18</sub>

**Claim 1.19.** *If  $\mathbf{k} \in K_{\alpha}^{\text{com}}$ ,  $\lambda \geq \|\mathbb{B}_{\mathbf{k}}\| + 2^{\aleph_0}$  and  $p(x)$  is a type in the model  $\mathbb{N}^{\mathbb{B}[\mathbf{k}]} / D_{\mathbf{k}}$   
 then for some  $\mathbf{m} \in K_{\alpha+1}$  we have  $\mathbf{k} \leq_K^{\text{ver}} \mathbf{m}$  and  $p(x)$  is realized in  $\mathbb{N}^{\mathbb{B}[\mathbf{m}]} / D_{\mathbf{m}}$ .* {f34}

*Proof.* Easy. □<sub>1.19</sub>

Having established all these statements, we can prove now the main result of this  
 paper:

**Theorem 1.20.** *For any infinite cardinal  $\lambda$ , for some regular ultrafilter  $D$  on  $\lambda$  we  
 have  $\text{upf}(D) = \mathcal{C}$  iff:* {k6}

- (\*) (a)  $\mathcal{C}$  is a set of cardinals  $\leq 2^\lambda$
- (b)  $\mu = \mu^{\aleph_0}$  whenever  $\mu \in \mathcal{C}$
- (c)  $2^\lambda \in \mathcal{C}$  hence is the maximal member of  $\mathcal{C}$ .

*Proof.* The implication  $\Rightarrow$ , we already know, so we shall deal with the  $\Leftarrow$  implica-  
 tion; the proof relies on earlier definitions and claims so the reader can return to a  
 second reading.

Let  $\langle \lambda_\alpha : \alpha \leq \alpha(*) \rangle$  list  $\mathcal{C}$  in increasing order. Let  $S = \{\alpha : \alpha \leq \alpha(*) + 1$  and  
 $\text{cf}(\alpha) \neq \aleph_0\}$ . We choose  $\mathbf{k}_\alpha$  by induction on  $\alpha \in S \cap (\alpha(*) + 2)$  such that:

- ⊞ (a)  $\mathbf{k}_\alpha \in K_{\alpha}^{\text{com}} \cap K_{\alpha}^{\text{cc}}$ , see Definition 1.2, 1.3(1A) {f2}
- (b)  $\mathbf{k}_\beta \leq_K^{\text{ver}} \mathbf{k}_\alpha$  for  $\beta \in \alpha \cap S$ , see 1.4(2)(B) {f8}
- (c) if  $f \in A_{\mathbf{k}, \beta}^1$ ,  $\beta < \alpha$  then  $\lambda_\beta = |\{g/D_{\mathbf{k}} : g \in \mathbb{N}^{\mathbb{B}}, g/D < f(D)\}|$
- (d) if  $\text{cf}(\alpha) > \aleph_0$  then  $\mathbb{B}_{\mathbf{k}_\alpha} = \cup\{\mathbb{B}_{\mathbf{k}_\beta} : \beta < \alpha\}$
- (e)  $\mathbf{k}_\alpha$  is reasonable (see Definition 1.12) {f24}
- (f) the set  $A_{\mathbf{k}, \beta}^1$  has cardinality  $\lambda_\beta$ .

Case 1: For  $\alpha = 0$

$\mathbb{B}_{\mathbf{k}_0}$  is the trivial Boolean Algebra, so really there is nothing to prove.

Case 2:  $\text{cf}(\alpha) > \aleph_0$

Use 1.7(2),(3) to find  $\mathbf{k}_\alpha$  satisfying clauses (a),(b),(c),(d). Now  $\mathbf{k}_\alpha$  satisfies clause {f13}  
 (e) by 1.14(6). {f26}

Case 3:  $\alpha = \beta + 1$

We choose  $\mathbf{k}_{\beta, i}$  by induction for  $i \leq \lambda_\beta$  such that

- (\*) (a) (α) if  $\beta \in S$  then  $\mathbf{k}_\beta \leq_K^{\text{ver}} \mathbf{k}_{\beta, i} \in K_{\alpha}^{\text{com}} \cap K_{\alpha}^{\text{cc}}$
- (β) if  $\beta \notin S$  then  $\gamma \in \beta \cap S \Rightarrow \mathbf{k}_\gamma \leq_K^{\text{ver}} \mathbf{k}_{\beta, i} \in K_{\alpha}^{\text{com}} \cap K_{\alpha}^{\text{cc}}$
- (γ) if  $i = 0$  then there is  $g \in \mathbb{N}^{\mathbb{B}[\mathbf{k}_{\beta, i}]}$  such that  $g/D_{\mathbf{k}_{\beta, i}} \notin \{f/D_{\mathbf{k}_{\beta, i}} : f \in \mathbb{N}^{\mathbb{B}[\mathbf{k}_\gamma]}\}$  for some  $\gamma \in \alpha \cap S$
- (δ)  $\mathbb{B}_{\mathbf{k}_{\beta, i}}$  is infinite
- (b)  $\langle \mathbf{k}_{\beta, j} : j \leq i \rangle$  is  $\leq_K^{\text{at}}$ -increasing continuous
- (c)  $\mathbb{B}_{\mathbf{k}_{\beta, i}}$  has cardinality  $\leq \lambda_\beta$

- (d) if  $i = j + 1$ :
- ( $\alpha$ ) bookkeeping gives us  $g_{\beta,j} \in \mathbb{N}^{\mathbb{B}[\mathbf{k}_{\beta,i}]}$  such that  $g_{\beta,j}/D_{\mathbf{k}_{\beta,j}} \notin \cup\{\mathbb{N}^{\mathbb{B}[\gamma,\mathbf{k}_{\beta}]} : \gamma \in \alpha \cap S\}$
  - ( $\beta$ ) there is  $f_{\beta,j} \in \mathbb{N}^{\mathbb{B}[\mathbf{k}_{\beta,i}]}$  such that  $f_{\beta,j}/D_{\mathbf{k}_{\beta,i}} < g_{\beta,j}/D_{\mathbf{k}_{\beta,i}}$  and  $f_{\beta,j}/D_{\mathbf{k}_{\beta,i}} \notin \cup\{\mathbb{N}^{\mathbb{B}[\gamma,\mathbf{k}_{\beta}]} : \gamma \in \alpha \cap S\}$
  - (e) if  $i < \lambda_{\beta}$  and  $g$  satisfies (d)( $\alpha$ ) then for some  $i_1 \in [i, \lambda_{\beta}]$ ,  $g_{\beta,i_1} = g$
  - (f) if  $i = j + 1$  then  $\mathbb{B}_{\mathbf{k}_{\beta,i}}$  is complete and reasonable.

{f26} Note that by 1.14(1) we can take care of clause (f), so we shall ignore it.

{f28} For  $i = 0$  we use 1.14(1) if  $\beta \in S$  and we use 1.7 if  $\beta \notin S$ .

{f13} For  $i$  limit use 1.7(1).

{f42} For  $i = j + 1$ ,  $\text{cf}(j) > \aleph_0$  use the claim 1.18.

{f39} If  $i = j + 1$ ,  $\text{cf}(j) = \aleph_0$  use the claim 1.16.

{f26} For  $i = j + 1$ ,  $\text{cf}(j) = 1$  we use 1.14(1).

Having carried the induction on  $i \leq \lambda_{\beta}$  let  $\mathbf{k}_{\alpha} = \mathbf{k}_{\beta,\lambda_{\beta}}$ . In particular  $\mathbb{B}_{\mathbf{k}_{\beta,\lambda_{\beta}}}$  is complete as  $\lambda_{\beta} = \sup\{i < \lambda_{\beta} : \mathbb{B}_{\mathbf{k}_{\beta,i}} \text{ is complete}\}$  by clause (f) and  $\text{cf}(\lambda_{\beta}) > \aleph_0$  as  $\lambda_{\beta} = \lambda_{\beta}^{\aleph_0}$ .

Having carried the induction on  $\alpha \leq \alpha(*) + 1$  clearly the pair  $(\mathbb{B}_{\mathbf{k}_{\alpha(*)+1}}, D_{\mathbf{k}_{\alpha(*)+1}})$  is almost as required. That is, (see [Sh:c, Ch.VI,§3]) we know that for some regular filter  $D_*$  on  $\mathcal{P}(I)$ , there is a homomorphism  $\mathbf{j}$  from the Boolean Algebra  $\mathcal{P}(I)$  onto  $\mathbb{B}_{\mathbf{k}_{\alpha(*)+1}}$  and let  $D = \{A \subseteq \lambda : \mathbf{j}(A) \in D_{\mathbf{k}_{\alpha(*)+1}}\}$ . □<sub>1.20</sub>

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EINSTEIN INSTITUTE OF MATHEMATICS, EDMOND J. SAFRA CAMPUS, GIVAT RAM, THE HEBREW UNIVERSITY OF JERUSALEM, JERUSALEM, 91904, ISRAEL, AND, DEPARTMENT OF MATHEMATICS, HILL CENTER - BUSCH CAMPUS, RUTGERS, THE STATE UNIVERSITY OF NEW JERSEY, 110 FRELINGHUYSEN ROAD, PISCATAWAY, NJ 08854-8019 USA

*Email address:* [shelah@math.huji.ac.il](mailto:shelah@math.huji.ac.il)

*URL:* <http://shelah.logic.at>