ON BOREL HULL OPERATIONS

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Abstract. We show that some set-theoretic assumptions (for example Martin's Axiom) imply that there is no translation invariant Borel hull operation on the family of Lebesgue null sets and on the family of meager sets (in $\mathbb{R}^n$). We also prove that if the meager ideal admits a monotone Borel hull operation, then there is also a monotone Borel hull operation on the σ–algebra of sets with the property of Baire.

1. Introduction

Sometimes a property of subsets of the real line $\mathbb{R}$ is introduced by means of a cover or a representation of the given set in terms of other sets. For instance,

(i) a set $A \subseteq \mathbb{R}$ is meager if $A \subseteq \bigcup\limits_{n<\omega} A_n$ for some closed nowhere dense sets $A_0, A_1, A_2, \ldots \subseteq \mathbb{R}$;

(ii) a set $A \subseteq \mathbb{R}$ is said to be Σ^0_ξ if $A = \bigcup\limits_{n<\omega} A_n$ for some $A_0, A_1, A_2, \ldots \in \bigcup\limits_{\zeta<\xi} \Pi^0_\zeta$;

(iii) a set $A \subseteq \mathbb{R}$ is Lebesgue measurable if $A \subseteq B$ for some Borel set $B \subseteq \mathbb{R}$ such that $B \setminus A$ is Lebesgue negligible;

(iv) a set $A \subseteq \mathbb{R}$ has the Baire property if for some Borel set $B \subseteq \mathbb{R}$ we have $A \subseteq B$ and $B \setminus A$ is meager, etc.

It is natural to ask if the "witnesses" in the above definitions can be chosen in a somewhat canonical or uniform way. For instance, we may wonder if they can depend monotonically on the input set A or if they can be translation invariant. Thus, in the relation to definition (i), we may ask if there are mappings $\varphi_0, \varphi_1, \varphi_2, \ldots$ defined on the ideal of meager subsets of $\mathbb{R}$, with values in the family of all closed nowhere dense subsets of $\mathbb{R}$ such that

($\circledast$) $A \subseteq \bigcup_{i<\omega} \varphi_i(A)$ (for each meager $A \subseteq \mathbb{R}$)

and with one of the following properties $(\circledast)^{\rm trans}$ or $(\circledast)^{\rm monot}$.

$(\circledast)^{\rm trans}$ For every meager set A and a real number r we have $\varphi_i(A+r) = \varphi_i(A)+r$ for all i.

$(\circledast)^{\rm monot}$ For all meager sets A, B such that $A \subseteq B$ we have $\varphi_i(A) \subseteq \varphi_i(B)$ for all i.

Easily, neither of these is possible. Suppose towards contradiction that there are mappings φ_i (for $i < \omega$) satisfying $(\circledast) + (\circledast)^{\rm trans}$. Then for each rational number $q \in \mathbb{Q}$ we have $\varphi_i(\mathbb{Q}) = \varphi_i(\mathbb{Q}+q) = \varphi_i(\mathbb{Q}) + q$ and hence each $\varphi_i(\mathbb{Q})$ is a closed nowhere dense set invariant under rational translations and this is impossible. Let us argue that the mappings φ_i (for $i < \omega$) cannot be monotone. So suppose they

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satisfy $(\circledast)+(\circledast)^{\rm monot}$. By induction on $\alpha<\omega_1$ construct a sequence $\langle A_\alpha:\alpha<\omega_1\rangle$ of meager sets so that

$$\bigcup_{\beta<\alpha}\bigcup_{i<\omega}\varphi_i(A_\beta)\subsetneq A_\alpha \quad \text{for all } \alpha<\omega_1.$$

Then for some n the sequence $\langle\varphi_n(A_\alpha):\alpha<\omega_1\rangle$ has a strictly increasing (cofinal) subsequence, a contradiction (as all $\varphi_n(A_\alpha)$ are closed).

A similar question associated with definition (ii) also has the negative answer: Mátrai and Zelený [8] showed that it is not possible to choose monotone presentations for Σ^0_ξ sets. However, problems concerning the monotonicity of the choice of witnesses for (iii) and (iv) cannot be decided within the standard set theory. A function ψ choosing such witnesses will be called a *Borel hull operation*, see Definition 1.1 below. Elekes and Máthé [7] proved that the existence of monotone Borel hulls for measurable sets is independent from ZFC, and parallel results for the Baire property were given by Balcerzak and Filipczak [2].

Notation and basic definitions. In the current note $\mathbf{X}$ is a Polish space, $\mathtt{Borel}$ denotes the family of all Borel subsets of $\mathbf{X}$, $\mathcal{M}$ is the σ–ideal of all meager subsets of $\mathbf{X}$, and $\mathcal{N}$ is the σ–ideal of all Lebesgue negligible subsets of $\mathbb{R}^n$. The same notation $\mathcal{M},\mathtt{Borel}$ will be used in $\mathbb{R}^n$, too.

Let $\mathcal{I}$ be a σ–ideal of subsets of $\mathbf{X}$. We say that a family $\mathcal{D}\subseteq\mathcal{I}$ is a *base* of $\mathcal{I}$ if

$$\big(\forall A\in\mathcal{I}\big)\big(\exists B\in\mathcal{D}\big)\big(A\subseteq B\big).$$

We say that $\mathcal{I}$ has a *Borel base* if every set from $\mathcal{I}$ can be covered by a Borel set from the ideal $\mathcal{I}$, i.e., $\mathtt{Borel}\cap\mathcal{I}$ is a base of $\mathcal{I}$. For a σ–ideal $\mathcal{I}$ with a Borel base, let $\mathcal{S}_\mathcal{I}$ denote the σ–algebra of subsets of $\mathbf{X}$ generated by $\mathrm{Borel}\cup\mathcal{I}$. Thus, in particular, the σ–algebra of all sets with the Baire property is $\mathtt{Baire}=\mathcal{S}_\mathcal{M}$.

Let $\mathcal{I}\subseteq\mathcal{P}(\mathbf{X})$ be a proper σ–ideal with a Borel base and containing all finite subsets of $\mathbf{X}$. We defne the following cardinal coefficients of $\mathcal{I}$:

$$\begin{aligned}
\mathrm{add}\,(\mathcal{I}) &:= \min\Big\{|\mathcal{F}|:\mathcal{F}\subseteq\mathcal{I},\ \bigcup\mathcal{F}\notin\mathcal{I}\Big\},\\
\mathrm{cov}\,(\mathcal{I}) &:= \min\Big\{|\mathcal{F}|:\mathcal{F}\subseteq\mathcal{I},\ \bigcup\mathcal{F}=\mathbf{X}\Big\},\\
\mathrm{non}\,(\mathcal{I}) &:= \min\{|A|:A\subseteq X,\ A\notin\mathcal{I}\},\\
\mathrm{cof}\,(\mathcal{I}) &:= \min\{|\mathcal{F}|:\mathcal{F}\subseteq\mathcal{I},\ (\forall A\in\mathcal{I})\,(\exists B\in\mathcal{F})\,(A\subseteq B)\}.
\end{aligned}$$

If $\mathbf{X}=\mathbb{R}^n$, $r\in\mathbb{R}^n$ and $A,B\subseteq\mathbb{R}^n$, then we define $A+r=\{a+r:a\in A\}$ and $A+B=\{a+b:a\in A,\ b\in B\}$. A family $\mathcal{F}$ of subsets of $\mathbb{R}^n$ is *translation invariant* if $A+r\in\mathcal{F}$ for all $A\in\mathcal{F}$ and $r\in\mathbb{R}^n$. For a translation invariant ideal $\mathcal{I}\subseteq\mathcal{P}(\mathbb{R}^n)$ we define the *transitive covering number* of $\mathcal{I}$ as

$$\mathrm{cov}^*\,(\mathcal{I}):=\min\{|A|:A\subseteq\mathbb{R}^n,\ (\exists B\in\mathcal{I})(A+B=\mathbb{R}^n)\}.$$

For systematic study of the cardinal invariants mentioned above for the case of $\mathcal{N}$ and $\mathcal{M}$ we refer the reader to Bartoszyński and Judah [4].

Definition 1.1. Let $\mathcal{I}$ be a σ–ideal on $\mathbf{X}$ with a Borel base and let $\mathcal{F}\subseteq\mathcal{S}_\mathcal{I}$.

1. A *Borel hull operation* on $\mathcal{F}$ with respect to $\mathcal{I}$ is a mapping $\psi:\mathcal{F}\longrightarrow\mathtt{Borel}$ such that $A\subseteq\psi(A)$ and $\psi(A)\setminus A\in\mathcal{I}$ for all $A\in\mathcal{F}$.
2. If the range of a Borel hull operation ψ consists of sets of some Borel class Γ, then we say that ψ is a Γ *hull operation.*

(3) A Borel hull operation ψ on $\mathcal{F}$ is *monotone* if $\psi(A_1) \subseteq \psi(A_2)$ whenever $A_1 \subseteq A_2$ are from $\mathcal{F}$.
(4) Assume $\mathbf{X} = \mathbb{R}^n$ and both $\mathcal{I}$ and $\mathcal{F}$ are translation invariant. If a Borel hull operation ψ on $\mathcal{F}$ satisfies $\psi(A + x) = \psi(A) + x$ for all $A \in \mathcal{F}$ and $x \in \mathbb{R}^n$, then ψ is called *a translation invariant hull operation.*

By [7, 2], under CH there exist monotone Borel hull operations on $\mathcal{S}_{\mathcal{I}}$ where $\mathcal{I}$ denotes either the ideal of Lebesgue negligible sets or the meager ideal. Adding many random or Cohen reals to a model of CH gives a model with no monotone Borel hull operations for $\mathcal{I}$ (where $\mathcal{I}$ is either the null or the meager ideal, respectively). More examples of universes with and without monotone Borel hulls for the null and meager ideals were given in Rosłanowski and Shelah [11].

The content of the paper. In [7, Question 4.2], the authors ask if it is possible to define a Borel hull operation on $\mathcal{N}$ which is monotone and translation invariant. In the second section we show that some set-theoretic assumptions (for example Martin's Axiom) imply that there is no translation invariant Borel hull operation on $\mathcal{N}$ and on $\mathcal{M}$ (even without the requirement of monotonicity).

The non-existence of monotone Borel hull operations on $\mathcal{I}$ implies non-existence of such operations on $\mathcal{S}_{\mathcal{I}}$ but not much had been known about the converse implication. In particular, Balcerzak and Filipczak [2, Question 2.23] asked if it is possible that there exists a monotone Borel hull operation on $\mathcal{I}$ (with respect to $\mathcal{I}$) but there is no such hull operation on $\mathcal{S}_{\mathcal{I}}$ for a ccc ideal $\mathcal{I}$. In the third section we give a negative answer for the case of the meager ideal. We show that the existence of a monotone Borel hull operation on $\mathcal{M}$ (with respect to $\mathcal{M}$) is equivalent with the existence of such hull operation on $\mathcal{S}_{\mathcal{M}}$.

2. No translation invariant Borel hulls operations

It is known that pairs $(\mathcal{S}_{\mathcal{N}}, \mathcal{N})$ and $(\mathcal{S}_{\mathcal{M}}, \mathcal{M})$ have the *Extended Steinhaus Property,* i.e., for any $A, B \in \mathcal{S}_{\mathcal{N}} \setminus \mathcal{N}$ ($\mathcal{S}_{\mathcal{M}} \setminus \mathcal{M}$, respectively) the set $A + B = \{a + b : a \in A,\ b \in B\}$ has an interior point. Many variants of Steinhaus Property have been investigated in the literature (see Bartoszewicz, Filipczak and Natkaniec [3]). We need a generalization in which only one of the sets A, B has to belong to the σ–algebra. In [9] and [1] such properties were proved for topological groups or locally compact groups with complete Haar measure. We formulate them for $\mathbb{R}^n$.

Theorem 2.1. (1) [9, Cor. 4] *If $A, B \subseteq \mathbb{R}^n$ are non-meager sets and A has the Baire property, then $A + B$ has an interior point.*
(2) [1, Thm 1] *If $A, B \subseteq \mathbb{R}^n$ are not Lebesgue null sets and A is measurable, then $A + B$ has an interior point.*

To prove that there is no translation invariant Borel hull operation on $\mathcal{N}$ (on $\mathcal{M}$, respectively) it is enough to show that the additive group $\mathbb{R}^n$ has a subgroup which is a non-meager null set (a meager set of positive outer measure, respectively).

Theorem 2.2. (1) *If $\mathbb{R}^n$ has a subgroup $G \in \mathcal{N} \setminus \mathcal{M}$, then there is no translation invariant Borel hull operation on $\mathcal{N}$.*
(2) *If $\mathbb{R}^n$ has a subgroup $H \in \mathcal{M} \setminus \mathcal{N}$, then there is no translation invariant Borel hull operation on $\mathcal{M}$.*

Proof. (1) Suppose, contrary to our claim, that there is a translation invariant hull operation $\varphi : \mathcal{N} \longrightarrow \mathcal{N} \cap \mathtt{Borel}$. For every x from G we have $G + x = G$, which gives $\varphi(G) + x = \varphi(G + x) = \varphi(G)$, and consequently $\varphi(G) + G = \varphi(G)$. Since $G \notin \mathcal{M}$ and $\varphi(G) \in \mathtt{Borel} \setminus \mathcal{M}$, Theorem 2.1 implies that $\varphi(G) + G$ has an interior point, contrary to $\varphi(G) \in \mathcal{N}$. The proof of (2) is similar. □

We will show that under some set-theoretic assumptions one can find a linear subspace of $\mathbb{R}^n$ (considered over the field $\mathbb{Q}$ of rational numbers) which belongs to $\mathcal{N} \setminus \mathcal{M}$ ($\mathcal{M} \setminus \mathcal{N}$, respectively).

Let us consider inequalities $\mathrm{non}(\mathcal{N}) > \mathrm{non}(\mathcal{M})$ and $\mathrm{non}(\mathcal{M}) > \mathrm{non}(\mathcal{N})$. It is well known that each of them is independent of ZFC.

Theorem 2.3 (S. Głąb)**.** (1) *If* $\mathrm{non}(\mathcal{N}) > \mathrm{non}(\mathcal{M})$*, then there exists a linear subspace of* $\mathbb{R}^n$ *which belongs to* $\mathcal{N} \setminus \mathcal{M}$*.*
(2) *If* $\mathrm{non}(\mathcal{M}) > \mathrm{non}(\mathcal{N})$*, then there exists a linear subspace of* $\mathbb{R}^n$ *which belongs to* $\mathcal{M} \setminus \mathcal{N}$*.*

Proof. (1) Let $A \subseteq \mathbb{R}^n$ be a set such that $|A| = \mathrm{non}(\mathcal{M})$ and $A \notin \mathcal{M}$. Let $\mathtt{span}(A)$ be the linear subspace of $\mathbb{R}^n$ generated by A (over $\mathbb{Q}$). Since $|\mathtt{span}(A)| = |A| < \mathrm{non}(\mathcal{N})$, we obtain $\mathtt{span}(A) \in \mathcal{N} \setminus \mathcal{M}$. The proof of (2) is similar. □

A *κ–Luzin set* for an ideal $\mathcal{I}$ on $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ of cardinality $\geq \kappa$, and such that its intersection with any set from $\mathcal{I}$ has cardinality less than κ (compare Bukovský [5, Section 8.2] or Cichoń [6]). If $\mathcal{I} = \mathcal{N}$, then κ–Luzin sets for $\mathcal{I}$ are also called κ–Sierpiński sets, and κ–Luzin sets for the meager ideal are called just κ–Luzin. If $\kappa = \aleph_1$ then we may omit it.

Of course, a κ–Luzin set for $\mathcal{I}$ does not belong to $\mathcal{I}$. If there exists a κ–Luzin set for $\mathcal{I}$, then $\mathrm{non}(\mathcal{I}) \leq \kappa$ and $\mathrm{cf}(\kappa) \leq \mathrm{cov}(\mathcal{I})$. It is known that if $\mathrm{cov}(\mathcal{I}) = \mathrm{cof}(\mathcal{I}) = \kappa$, then there exists a κ–Luzin set for $\mathcal{I}$ (see [5, Theorem 8.26]). Since $\mathbb{R}^n$ can be decomposed into a null set and a meager set, it follows that for any regular κ, every κ–Luzin set has measure zero and every κ–Sierpiński set is meager.

Theorem 2.4 ([5, Theorem 8.28])**.** (1) *If* $\kappa \leq \mathrm{non}(\mathcal{N})$ *and A is a κ–Luzin set, then $A \in \mathcal{N} \setminus \mathcal{M}$. In particular, if κ is regular and A is a κ–Luzin set, then $A \in \mathcal{N} \setminus \mathcal{M}$.*
(2) *If* $\kappa \leq \mathrm{non}(\mathcal{M})$ *and A is a κ–Sierpiński set, then $A \in \mathcal{M} \setminus \mathcal{N}$. In particular, if κ is regular and A is a κ–Sierpiński set, then $A \in \mathcal{M} \setminus \mathcal{N}$.*

Smítal proved that if the Continuum Hypothesis holds then there exists a linear subspace of $\mathbb{R}^n$, which is a Luzin set (see [12]). One can construct a linear subspace which belongs to $\mathcal{N} \setminus \mathcal{M}$ ($\mathcal{M} \setminus \mathcal{N}$) assuming a condition weaker than CH. The proof is a small modification of the proof of [12, Lemma 1].

Theorem 2.5. *Let* $\{\mathcal{I}, \mathcal{J}\} = \{\mathcal{N}, \mathcal{M}\}$*.*
(1) *If* $\mathrm{cov}^*(\mathcal{I}) \geq \mathrm{cof}(\mathcal{I})$*, then there exists a linear subspace H of* $\mathbb{R}^n$ *such that* $H \in \mathcal{J} \setminus \mathcal{I}$*.*
(2) [Bukovský [5, Exercise 8.7(b)]] *If* $\mathrm{cov}(\mathcal{I}) = \mathrm{cof}(\mathcal{I}) = \kappa$*, then there exists a linear subspace H of* $\mathbb{R}^n$*, which is a κ–Luzin set for $\mathcal{I}$.*

Proof. (1) Let $\kappa = \mathrm{cof}(\mathcal{I}) \leq \mathrm{cov}^*(\mathcal{I})$. Let $B \in \mathcal{I}$ be such that $\mathbb{R}^n \setminus B \in \mathcal{J}$ and let $\{B_\alpha : \alpha < \kappa\}$ be a base for the ideal $\mathcal{I}$ such that $B \subseteq B_\alpha$ for all $\alpha < \kappa$. By induction on α we choose a sequence $\langle x_\alpha : \alpha < \kappa\rangle$. Suppose that $\langle x_\beta : \beta < \alpha\rangle$ has been defined and let $Z_\alpha := \mathtt{span}(\{x_\beta : \beta < \alpha\})$, i.e., it is the linear subspace of $\mathbb{R}^n$ generated by $\{x_\beta : \beta < \alpha\}$ over $\mathbb{Q}$. The set $\mathbb{Q}B_\alpha := \{qx : x \in B_\alpha,\ q \in \mathbb{Q}\}$ belongs to $\mathcal{I}$ and $|Z_\alpha| < \mathrm{cov}^*(\mathcal{I})$, so we may choose

$(*)_\alpha$ $x_\alpha \in \mathbb{R}^n \setminus \bigcup\limits_{y \in Z_\alpha} (\mathbb{Q}B_\alpha + y)$.

Then, after the construction is carried out, we set

$$H := \mathtt{span}(\{x_\alpha : \alpha < \kappa\}).$$

Since $x_\alpha \notin B_\alpha$, we have $H \notin \mathcal{I}$. Also, $H \setminus \{0\} \subseteq \mathbb{R}^n \setminus B$. Indeed, suppose towards contradiction that $x \in H \cap B \setminus \{0\}$ and let $x = q_1 x_{\alpha_1} + \ldots + q_m x_{\alpha_m}$, where $\alpha_1 < \alpha_2 < \ldots < \alpha_m < \kappa$, $q_1, \ldots, q_m \in \mathbb{Q}$ and $q_m \neq 0$. Let $y = -\frac{1}{q_m}(q_1 x_{\alpha_1} + \ldots + q_{m-1} x_{\alpha_{m-1}})$. Clearly, $y \in Z_{\alpha_m}$. Since $x_{\alpha_m} \notin \mathbb{Q}B_{\alpha_m} + y$ and $x \in B \subseteq B_{\alpha_m}$, we conclude

$$x_{\alpha_m} \neq \frac{1}{q_m} x - \frac{1}{q_m}(q_1 x_{\alpha_1} + \ldots + q_{m-1} x_{\alpha_{m-1}}) = x_{\alpha_m},$$

a contradiction. Now we easily see that $H \in \mathcal{J} \setminus \mathcal{I}$.

(2) The arguments are essentially the same as in (1). Let $\kappa = \mathrm{cov}(\mathcal{I}) = \mathrm{cof}(\mathcal{I})$ and let $\{B_\alpha : \alpha < \kappa\}$ be a base for $\mathcal{I}$. Choose a sequence $\langle x_\alpha : \alpha < \kappa \rangle$ so that

$(*)^+_\alpha$ $x_\alpha \in \mathbb{R}^n \setminus \bigcup \{\mathbb{Q}B_\beta + y : \beta \leq \alpha \ \&\ y \in Z_\alpha\}$,

where $Z_\alpha := \mathtt{span}(\{x_\beta : \beta < \alpha\})$. Finally put $H := \mathtt{span}(\{x_\alpha : \alpha < \kappa\})$. Let us argue that $|H \cap B_\alpha| < \kappa$ for each $\alpha < \kappa$. Suppose $x \in H \cap B_\alpha$, $x \neq 0$. Then $x = q_1 x_{\alpha_1} + \ldots + q_m x_{\alpha_m}$ for some $q_1, \ldots, q_m \in \mathbb{Q}$, $q_m \neq 0$ and $\alpha_1 < \ldots < \alpha_m$. So,

$$y_0 := x_{\alpha_m} - \frac{x}{q_m} = -(\frac{q_1}{q_m} x_{\alpha_1} + \ldots + \frac{q_{m-1}}{q_m} x_{\alpha_{m-1}}) \in \mathtt{span}(\{x_\beta : \beta < \alpha_m\}) = Z_{\alpha_m}$$

and

$$x_{\alpha_m} = \frac{1}{q_m} x + y_0 \in \mathbb{Q}B_\alpha + Z_{\alpha_m}.$$

Since, by $(*)^+_\alpha$, $x_{\alpha_m} \notin \mathbb{Q}B_\beta + Z_{\alpha_m}$ for $\beta \leq \alpha_m$ we conclude $\alpha > \alpha_m$. Thus $H \cap B_\alpha \subseteq Z_\alpha$, and consequently $|H \cap B_\alpha| < \kappa$. □

Remark 2.6. Concerning the assumptions in Theorem 2.5(1), note that $\mathrm{cov}^*(\mathcal{N}) \leq \mathrm{non}(\mathcal{M}) \leq \mathrm{cof}(\mathcal{N})$, so in the case of the null ideal the assumption here is actually $\mathrm{cov}^*(\mathcal{N}) = \mathrm{cof}(\mathcal{N})$. However, for the meager ideal we can only say that $\mathrm{cov}^*(\mathcal{M}) \leq \mathrm{non}(\mathcal{N})$ and it is consistent that $\mathrm{cov}^*(\mathcal{M}) > \mathrm{cof}(\mathcal{M})$. For further discussion of $\mathrm{cov}^*(\mathcal{M})$ we refer the reader to Bartoszyński and Judah [4, Section 2.7] or Miller and Steprāns [10].

In Theorem 2.5(2) note that $\mathrm{cov}(\mathcal{I}) \leq \mathrm{non}(\mathcal{J})$ and therefore the subspace H as there satisfies $H \in \mathcal{J} \setminus \mathcal{I}$ (by Theorem 2.4).

It is well known that for $\mathcal{I} \in \{\mathcal{N}, \mathcal{M}\}$, the Martin Axiom MA implies $\mathrm{add}(\mathcal{I}) = \mathrm{cov}(\mathcal{I}) = \mathrm{cov}^*(\mathcal{I}) = \mathrm{non}(\mathcal{I}) = \mathrm{cof}(\mathcal{I}) = 2^{\aleph_0}$.

The following corollary sums up our previous considerations.

Corollary 2.7. (1) *There is no translation invariant Borel hull operation on $\mathcal{N}$ if any of the following conditions holds:*
$\mathrm{non}(\mathcal{N}) > \mathrm{non}(\mathcal{M})$ *or* $\mathrm{cov}^*(\mathcal{M}) \geq \mathrm{cof}(\mathcal{M})$ *or MA.*

(2) *There is no translation invariant Borel hull operation on $\mathcal{M}$ if any of the following conditions holds:*
$\mathrm{non}(\mathcal{M}) > \mathrm{non}(\mathcal{N})$ *or* $\mathrm{cov}^*(\mathcal{N}) = \mathrm{cof}(\mathcal{N})$ *or MA.*

3. Monotone Borel hulls operations on Baire

Let us fix a countable base $\mathcal{B}$ of our Polish space $\mathbf{X}$. We also require that $\mathcal{B}$ is *closed under intersections* and $\mathbf{X} \in \mathcal{B}$.

Definition 3.1. A monotone Borel hull operation $\varphi : \mathcal{M} \longrightarrow \mathtt{Borel} \cap \mathcal{M}$ is $\mathcal{B}$–*regular* whenever

$$\varphi(A) \cap U \subseteq \varphi(A \cap U)$$

for all $A \in \mathcal{M}$ and $U \in \mathcal{B}$.

Theorem 3.2. *The following conditions are equivalent:*

(i) *There is a monotone Borel hull operation on $\mathcal{M}$ with respect to $\mathcal{M}$.*
(ii) *There is a $\mathcal{B}$–regular monotone Borel hull operation on $\mathcal{M}$ with respect to $\mathcal{M}$.*
(iii) *There is a monotone Borel hull operation on* Baire *with respect to $\mathcal{M}$.*

Proof. (i) $\Rightarrow$ (ii) Let $\varphi : \mathcal{M} \longrightarrow \mathtt{Borel} \cap \mathcal{M}$ be a monotone Borel hull operation on $\mathcal{M}$. For $A \in \mathcal{M}$ let

$$\psi(A) = \bigcap \big\{ (\mathbf{X} \setminus U) \cup \varphi(A \cap U) : U \in \mathcal{B} \big\}.$$

Clearly, $\psi(A)$ is a Borel subset of $\mathbf{X}$ (as a countable intersection of Borel sets) and $A \subseteq \psi(A) \subseteq \varphi(A)$ (as $\mathbf{X} \in \mathcal{B}$). Also, if $A \subseteq B \in \mathcal{M}$ and $U \in \mathcal{B}$, then $\varphi(A \cap U) \subseteq \varphi(B \cap U)$ (as φ is monotone) and hence $(\mathbf{X} \setminus U) \cup \varphi(A \cap U) \subseteq (\mathbf{X} \setminus U) \cup \varphi(B \cap U)$. Consequently, if $A \subseteq B \in \mathcal{M}$, then $\psi(A) \subseteq \psi(B)$. Therefore $\psi : \mathcal{M} \longrightarrow \mathtt{Borel} \cap \mathcal{M}$ is a monotone Borel hull on $\mathcal{M}$. To show that it is $\mathcal{B}$–regular suppose $A \in \mathcal{M}$ and $U \in \mathcal{B}$.

Fix $V \in \mathcal{B}$ for a moment and let $W = U \cap V$. Then $W \in \mathcal{B}$ and

$$\psi(A) \cap U \subseteq \big((\mathbf{X} \setminus W) \cup \varphi(A \cap W) \big) \cap U = \big((\mathbf{X} \setminus (U \cap V)) \cup \varphi(A \cap U \cap V) \big) \cap U \subseteq (U \setminus V) \cup \varphi(A \cap U \cap V) \subseteq (\mathbf{X} \setminus V) \cup \varphi((A \cap U) \cap V).$$

Thus $\psi(A) \cap U \subseteq (\mathbf{X} \setminus V) \cup \varphi((A \cap U) \cap V)$ for all $V \in \mathcal{B}$ and therefore $\psi(A) \cap U \subseteq \psi(A \cap U)$.

(ii) $\Rightarrow$ (iii) Suppose that $\varphi : \mathcal{M} \longrightarrow \mathtt{Borel} \cap \mathcal{M}$ is a $\mathcal{B}$–regular monotone Borel hull operation on $\mathcal{M}$. For a set $Z \subseteq \mathbf{X}$ let

$$\mathbf{K}(Z) = \mathbf{X} \setminus \bigcup \big\{ U \in \mathcal{B} : U \cap Z \in \mathcal{M} \big\}.$$

Clearly, $\mathbf{K}(Z)$ is a closed subset of $\mathbf{X}$ and

$(*)_1$ $Z \subseteq Y \subseteq \mathbf{X}$ implies $\mathbf{K}(Z) \subseteq \mathbf{K}(Y)$,
$(*)_2$ $Z \setminus \mathbf{K}(Z) \in \mathcal{M}$, and
$(*)_3$ if $Z \subseteq \mathbf{X}$ has the Baire property, then $\mathbf{K}(Z) \setminus Z \in \mathcal{M}$.

For $Z \in \mathtt{Baire}$ let

$$\psi(Z) = \mathbf{K}(Z) \cup \varphi(Z \setminus \mathbf{K}(Z)).$$

Then $\psi : \mathtt{Baire} \longrightarrow \mathtt{Borel}$ and for $Z \in \mathtt{Baire}$:

$(*)_4$ $Z \subseteq \mathbf{K}(Z) \cup (Z \setminus \mathbf{K}(Z)) \subseteq \mathbf{K}(Z) \cup \varphi(Z \setminus \mathbf{K}(Z)) = \psi(Z)$, and
$(*)_5$ $\psi(Z) \setminus Z \subseteq \big(\mathbf{K}(Z) \setminus Z\big) \cup \varphi(Z \setminus \mathbf{K}(Z)) \in \mathcal{M}$.

Thus ψ is a Borel hull operation on Baire. Let us argue that ψ is monotone, i.e.,

$(*)_6$ if $Z \subseteq Y \subseteq \mathbf{X}$, $Z, Y \in \mathtt{Baire}$, then $\psi(Z) \subseteq \psi(Y)$.

Suppose that sets $Z \subseteq Y$ have the Baire property and let us argue that $\psi(Z) \subseteq \psi(Y)$. Assume $x \in \psi(Z)$. If $x \in \mathbf{K}(Y)$, then $x \in \psi(Y)$ by the definition of ψ. So suppose that $x \notin \mathbf{K}(Y)$. Then also $x \notin \mathbf{K}(Z)$ (remember $(*)_1$) so $x \in \varphi(Z \setminus \mathbf{K}(Z))$. Let $U \in \mathcal{B}$ be such that $x \in U \subseteq \mathbf{X} \setminus \mathbf{K}(Y)$. Then $(Z \setminus \mathbf{K}(Z)) \cap U \subseteq Y \setminus \mathbf{K}(Y)$ and, since φ is $\mathcal{B}$–regular,

$$x \in \varphi(Z \setminus \mathbf{K}(Z)) \cap U \subseteq \varphi((Z \setminus \mathbf{K}(Z)) \cap U) \subseteq \varphi(Y \setminus \mathbf{K}(Y)) \subseteq \psi(Y).$$

(iii) $\Rightarrow$ (i) Straightforward. □

The proof of Theorem 3.2 also shows the following.

Corollary 3.3. *If there is a monotone Σ^0_ξ (Π^0_ξ, respectively) hull operation on $\mathcal{M}$ with respect to $\mathcal{M}$, $2 \le \xi < \omega_1$, then there exists a monotone $\Pi^0_{\xi+1}$ (Π^0_ξ, respectively) hull operation on* `Baire` *with respect to $\mathcal{M}$.*

Assuming CH, or more generally $\mathrm{add}(\mathcal{M}) = \mathrm{cof}(\mathcal{M})$, the σ–ideal of meager sets has a monotone Σ^0_2 hull operation and hence, under the same assumption, there is a monotone Π^0_3 hull operation on `Baire`, see [2].

Definition 3.4 (See [11, Definition 3.4])**.** Let $\mathcal{I}$ be an ideal of subsets of $\mathbf{X}$ and let α^*, β^* be limit ordinals. *An $\alpha^* \times \beta^*$–base for $\mathcal{I}$* is a sequence $\langle B_{\alpha,\beta} : \alpha < \alpha^* \ \&\ \beta < \beta^* \rangle$ of Borel sets from $\mathcal{I}$ such that

(a) the family $\{B_{\alpha,\beta} : \alpha < \alpha^*,\ \beta < \beta^*\}$ is a base for $\mathcal{I}$, and
(b) for each $\alpha_0, \alpha_1 < \alpha^*$, $\beta_0, \beta_1 < \beta^*$ we have

$$B_{\alpha_0,\beta_0} \subseteq B_{\alpha_1,\beta_1} \quad \Leftrightarrow \quad \alpha_0 \le \alpha_1 \ \&\ \beta_0 \le \beta_1.$$

Note that if an ideal $\mathcal{I}$ has an $\alpha^* \times \beta^*$–base, then $\mathrm{add}(\mathcal{I}) = \min\{\mathrm{cf}(\alpha^*), \mathrm{cf}(\beta^*)\}$ and $\mathrm{cof}(\mathcal{I}) = \max\{\mathrm{cf}(\alpha^*), \mathrm{cf}(\beta^*)\}$.

Theorem 3.5 (See [11, Proposition 3.6])**.** *Assume that α^*, β^* are limit ordinals. If an ideal $\mathcal{I}$ has an $\alpha^* \times \beta^*$–base consisting of Π^0_ξ sets, $\xi < \omega_1$, then there exists a monotone Π^0_ξ hull operation on $\mathcal{I}$ with respect to $\mathcal{I}$.*

Theorem 3.6 (See [11, Theorem 3.7])**.** *Let κ, λ be cardinals of uncountable cofinality, $\kappa \le \lambda$. There is a ccc forcing notion $\mathbb{Q}^{\kappa,\lambda}$ of size $\lambda^{\aleph_0}$ such that*

$$\Vdash_{\mathbb{Q}^{\kappa,\lambda}} \text{“ the meager ideal } \mathcal{M} \text{ has a } \kappa \times \lambda\text{–base consisting of } \Sigma^0_2 \text{ sets ”.}$$

Putting the results quoted above together with Corollary 3.3 we obtain the following.

Corollary 3.7. *Let κ, λ be uncountable regular cardinals, $\kappa \le \lambda$. There is a ccc forcing notion $\mathbb{Q}^{\kappa,\lambda}$ of size $\lambda^{\aleph_0}$ such that*

$$\Vdash_{\mathbb{Q}^{\kappa,\lambda}} \text{“ there is a monotone } \Pi^0_3 \text{ hull operation on } \texttt{Baire} \text{ and } \mathrm{add}(\mathcal{M}) = \kappa \text{ and } \mathrm{cof}(\mathcal{M}) = \lambda \text{ ”.}$$

4. Open problems

Problem 4.1. (1) Can we find, in ZFC, a subgroup of $\mathbb{R}^n$ which belongs to $\mathcal{N} \setminus \mathcal{M}$ ($\mathcal{M} \setminus \mathcal{N}$, respectively) ?
(2) When do such subgroups exist in a locally compact group (with complete Haar measure)?

If the answer to Problem 4.1(1) is positive, then in ZFC there is no translation invariant Borel hull on $\mathcal{N}$ ($\mathcal{M}$, respectively). Should the existence of such subgroups be independent from ZFC, we still may suspect that there are no translation invariant Borel hulls, or at least that there are no translation invariant monotone Borel hulls.

Problem 4.2. Is it consistent that there are translation invariant Borel hulls on $\mathcal{N}$ ($\mathcal{M}$, respectively)? If yes, can we additionally have that this hull operation is monotone?

The following problem is motivated by Theorem 2.5(2).

Problem 4.3. Let $\mathcal{I} \in \{\mathcal{N}, \mathcal{M}\}$. Assume that there exists a κ–Luzin set for $\mathcal{I}$. Does there exists a subgroup of $\mathbb{R}^n$ which is also a κ–Luzin set for $\mathcal{I}$?

Every set with Baire property has a Σ^0_2 hull, so one may wonder if in Corollary 3.7 we may claim the existence of monotone Σ^0_2 hulls. Or even:

Problem 4.4. Is it consistent that there is a monotone Π^0_3 hull operation on `Baire` but there is no monotone Σ^0_2 hull operation on `Baire` (with respect to $\mathcal{M}$) ?

We do not know if an analogue of Theorem 3.2 holds for the null ideal.

Problem 4.5 (Cf. Balcerzak and Filipczak [2, Question 2.23])**.** Is it consistent that there exists a monotone Borel hull on the ideal $\mathcal{N}$ of Lebesgue negligible subsets of $\mathbb{R}$ (with respect to $\mathcal{N}$) but there is no such hull on the algebra $\mathcal{S}_\mathcal{N}$ of Lebesgue measurable sets? In particular, is it consistent that $\mathrm{add}(\mathcal{N}) = \mathrm{cof}(\mathcal{N})$ but there is no monotone Borel hull operation on $\mathcal{S}_\mathcal{N}$?

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