ON THE NUMBER OF DEDEKIND CUTS AND TWO-CARDINAL MODELS OF DEPENDENT THEORIES

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Abstract. For an infinite cardinal κ, let $\operatorname{ded}\kappa$ denote the supremum of the number of Dedekind cuts in linear orders of size κ. It is known that $\kappa < \operatorname{ded}\kappa \leq 2^\kappa$ for all κ and that $\operatorname{ded}\kappa < 2^\kappa$ is consistent for any κ of uncountable cofinality. We prove however that $2^\kappa \leq \operatorname{ded}(\operatorname{ded}(\operatorname{ded}(\operatorname{ded}\kappa)))$ always holds. Using this result we calculate the Hanf numbers for the existence of two-cardinal models with arbitrarily large gaps and for the existence of arbitrarily large models omitting a type in the class of countable dependent first-order theories. Specifically, we show that these bounds are as large as in the class of all countable theories.

1. Introduction

For an infinite cardinal κ, let

$$\operatorname{ded}\kappa = \sup\{|I| : I \text{ is a linear order with a dense subset of size } \leq \kappa\}.$$

In general the supremum need not be attained. Let I be a linear order and let $\mathfrak{c} = (I_1, I_2)$ be a cut of I (i.e. $I = I_1 \cup I_2$, $I_1 \cap I_2 = \emptyset$ and $i_1 < i_2$ for all $i_1 \in I_1, i_2 \in I_2$). By *cofinality of* $\mathfrak{c}$ *from the left* (respectively, *from the right*) we mean the cofinality of the linear order induced on I_1 (resp. the cofinality of I_2^*, that is I_2 with the order reversed).

Fact 1.1. *The following cardinalities are the same, see e.g.* [CKS12, Proposition 6.5]*:*

(1) $\operatorname{ded}\kappa$,

(2) $\sup\{\lambda :$ *exists a linear order* I *of size* $\leq \kappa$ *with* λ *cuts*$\}$,

(3) $\sup\{\lambda :$ *exists a regular* μ *and a linear order of size* $\leq \kappa$ *with* λ *cuts of cofinality* μ *both from the left and from the right*$\}$,

(4) $\sup\{\lambda :$ *exists a regular* μ *and a tree* T *of size* $\leq \kappa$ *with* λ *branches of length* $\mu\}$.

It is well-known that $\kappa < \operatorname{ded}\kappa \leq (\operatorname{ded}\kappa)^{\aleph_0} \leq 2^\kappa$ (for the first inequality, let μ be minimal such that $2^\mu > \kappa$, and consider the tree $2^{<\mu}$) and that $\operatorname{ded}\aleph_0 = 2^{\aleph_0}$ (as $\mathbb{Q} \subseteq \mathbb{R}$ is dense). Thus

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$\operatorname{ded}\kappa = (\operatorname{ded}\kappa)^{\aleph_0} = 2^\kappa$ for all κ in a model with GCH. Moreover, Baumgartner [Bau76] had shown that if $2^\kappa = \kappa^{+n}$ (i.e. the nth successor of κ) for some $n \in \omega$, then $\operatorname{ded}\kappa = 2^\kappa$. On the other hand, for any κ of uncountable cofinality Mitchell [Mit73] had proven that consistently $\operatorname{ded}(\kappa) < 2^\kappa$. Besides, in [CKS12, Section 6] it is demonstrated that for some κ it is consistent that $\operatorname{ded}\kappa < (\operatorname{ded}\kappa)^{\aleph_0}$ (but it is still open if both inequalities $\operatorname{ded}\kappa \leq \left(\operatorname{ded}\kappa^{\aleph_0}\right) \leq 2^\kappa$ can be strict simultaneously). The importance of the function $\operatorname{ded}\kappa$ from the model-theoretic point of view is largely due to the following fact:

Fact 1.2. [Kei76, She90] *Let T be a complete first-order theory in a countable language L. For a model M of T, $S_1(M)$ denotes the space of 1-types over M (i.e. the space of ultrafilters on the Boolean algebra of definable subsets of M). Define $f_T(\kappa) = \sup\{|S_T(M)| : M \models T, |M| = \kappa\}$. Then for any countable T, f_T is one of the following functions: κ, $\kappa + 2^{\aleph_0}$, $\kappa^{\aleph_0}$, $\operatorname{ded}\kappa$, $(\operatorname{ded}\kappa)^{\aleph_0}$ or 2^κ (and each of these functions occurs for some T).*

In the first part of the paper we prove that $2^\kappa \leq \operatorname{ded}(\operatorname{ded}(\operatorname{ded}(\operatorname{ded}\kappa)))$ holds for any κ. Our proof uses results from the PCF theory of the second author. Optimality of this bound remains open. Moreover, with two extra iterations we can ensure that the supremums are attained. I.e., for any cardinal κ there are linear orders $I_0, \ldots, I_6$ such that $|I_0| \leq \kappa, 2^\kappa \leq |I_6|$ and for every $i < 6$, the number of Dedekind cuts in I_i is at least $|I_{i+1}|$.

In the second part of the paper we apply these results to questions about cardinal transfer. Fix a complete first-order theory T in a countable language L, with a distinguished predicate $P(x)$ from L. Given two cardinals $\kappa \geq \lambda \geq \aleph_0$ we say that $M \models T$ is a (κ, λ)-model if $|M| = \kappa$ and $|P(M)| = \lambda$. A classical question in model theory is to determine implications between existence of two-cardinal models for different pairs of cardinals. It was studied by Vaught, Chang, Morley, Shelah and others.

Fact 1.3. *(Vaught) Assume that for some κ, T admits a $(\beth_n(\kappa), \kappa)$-model for all $n \in \omega$. Then T admits a (κ', λ')-model for any $\kappa' \geq \lambda'$.*

Vaught's theorem is optimal:

Example 1.4. Fix $n \in \omega$, and consider a structure M in the language $L = \{P_0(x), \ldots, P_n(x), \in_0, \ldots, \in_{n-1}\}$ in which $P_0(M) = \omega$, $P_{i+1}(M)$ is the set of subsets of $P_i(M)$, and $\in_i \subseteq P_i \times P_{i+1}$ is the membership relation. Let $T = \operatorname{Th}(M)$. Then M is a $(\beth_n, \aleph_0)$-model of T, but it is easy to see by "extensionality" that for any $M' \models T$ we have $|M'| \leq \beth_n(|P_0(M')|)$.

However, the theory in the example is wild from the model theoretic point of view, and stronger transfer principles hold for tame classes of theories.

Fact 1.5. *(1)* [Lac72] *If T is stable and admits a (κ, λ)-model for some $\kappa > \lambda$, then it admits a (κ', λ')-model for any $\kappa' \geq \lambda'$.*

(2) [Bay98] *If T is o-minimal and admits a (κ, λ)-model for some $\kappa > \lambda$, then it admits a (κ', λ')-model for any $\kappa' \geq \lambda'$.*

For further two-cardinal results for stable theories see [She90, Ch. V, §6] and also [BS06].

An important class of theories containing both the stable and the o-minimal theories is the class of *dependent* theories (also called NIP theories in the literature) introduced by the second author [She90]. In the countable case, dependent theories can be defined as those theories for which $f_T(\kappa) \leq (\operatorname{ded} \kappa)^{\aleph_0}$ (see Fact 1.2, and see Section 3 for a combinatorial definition). Recently dependent theories have attracted a lot of attention both in purely model theoretic work on generalizing the machinery of stable theories (e.g. [She09, She07, She12, CS13, CS]), and due to the analysis of some important algebraic examples [HP11, HHM08].

It is easy to see that the theory in Example 1.4 is not dependent, but also that a complete analogue of Fact 1.5 cannot hold for dependent theories: consider the theory of $(\mathbb{R}, <)$ expanded by a predicate naming $\mathbb{Q}$. In Section 3 we show that in fact the situation for dependent theories is not better than for arbitrary theories, in contrast to the stable and o-minimal cases. Namely, for every $n < \omega$ we construct a *dependent* theory T_n which has a $(\beth_m, \aleph_0)$-model for all $m < n$, but does not have a $(\beth_\omega, \aleph_0)$-model. In Section 4 we elaborate on this example and show that the Hanf number for omitting a type is again the same for countable dependent theories as for arbitrary theories — unlike in the stable [HS91] and in the *o*-minimal [Mar86] cases. Examples which we construct add to the list of dependent theories [KS10b, KS10a] demonstrating that the principle "dependent = stable + linear order" has only limited applicability.

2. On the number of Dedekind cuts

2.1. On $\mathrm{pp}_\kappa(\lambda)$.

We summarize some facts from the PCF theory of the second author (see also [HSW99, Chapter 9] for an exposition).

Definition 2.1. Given a set of cardinals A and a cardinal λ, we will write $\sup^+(A) = \min\{\mu : \forall \nu \in A, \nu < \mu\}$ and $\lambda \leq^+ \sup(A)$ if either $\lambda < \sup(A)$, or $\lambda = \sup(A)$ and $\lambda \in A$.

Definition 2.2. [She94, II.§1] For $\operatorname{cf} \lambda \leq \kappa < \lambda$ let

$$A = \left\{ \operatorname{cf}\left(\prod a/\mathcal{F}\right) : a \subset \mathrm{Reg} \wedge \sup(a) = \lambda \wedge |a| \leq \kappa \wedge \mathcal{F} \text{ is an ultrafilter on } a \wedge \mathcal{F} \cap I_b(a) = \emptyset \right\},$$

where Reg is the class of regular cardinals, and for a set B of ordinals with $\sup(B) \notin B$, $I_b(B) = \{X \subseteq B : \exists \beta \in B\, X \subseteq \beta\}$ denotes the ideal of bounded subsets of B. Then we define $\mathrm{pp}_\kappa(\lambda) = \sup(A)$ and $\mathrm{pp}^+_\kappa(\lambda) = \sup^+(A)$ (where "pp" stands for "pseudo-power").

Equivalently (see e.g [HSW99, Lemma 9.1.1]), for $\mathrm{cf}\,\lambda \leq \kappa < \lambda$ one has

$$\mathrm{pp}_\kappa(\lambda) = \sup\left\{\mathrm{tcf}\left(\prod_{i<\kappa}\lambda_i/I, <_I\right) : \lambda_i = \mathrm{cf}\,\lambda_i < \lambda = \sup_{i<\kappa}\lambda_i \wedge I \text{ is an ideal on } \kappa \wedge I_b(\kappa) \subseteq I\right\},$$

where $<_I$ is the lexicographic ordering modulo I and for a partial order P, $\mathrm{tcf}(P) = \kappa$ when there are $\langle p_i : i < \kappa \rangle$ in P such that $\kappa = \mathrm{cf}\,\kappa$ and $\bigwedge_{i<j}(p_i < p_j)$ and $\forall p \in P\left(\bigvee_{i<\kappa} p \leq p_i\right)$ (true cofinality may not exist). We recall that $\Gamma(\theta, \sigma) = \{I : \text{for some cardinal } \theta_I < \theta,\ I \text{ is a } \sigma\text{-complete ideal on } \theta_I\}$ and $\Gamma(\theta) = \Gamma(\theta^+, \theta)$. Then $\mathrm{pp}_{\Gamma(\theta,\sigma)}(\lambda)$ is defined in the same way as $\mathrm{pp}_\kappa(\lambda)$ but the supremum is taken only over ideals from $\Gamma(\theta, \sigma)$.

Fact 2.3. *See e.g.* [HSW99, Chapter 9]*:*

(1) $\lambda < \mathrm{pp}_\kappa(\lambda) \leq \lambda^\kappa$ *and if* $\mathrm{cf}\,\lambda = \kappa > \aleph_0$ *and* λ *is* κ*-strong (i.e.* $\rho^\kappa < \lambda$ *for all* $\rho < \lambda$*), then* $\mathrm{pp}_\kappa(\lambda) = \lambda^\kappa$*. In particular* $\mathrm{pp}_\kappa(\lambda) = \lambda^\kappa$ *holds for any strong limit* λ *with uncountable cofinality* κ*.*

(2) For any θ *we have* $\mathrm{pp}_{\Gamma(\theta)}(\lambda) \leq \mathrm{pp}_\theta(\lambda)$ *and* $\mathrm{pp}_{\Gamma(\theta^+,2)}(\lambda) = \mathrm{pp}_\theta(\lambda)$*.*

Fact 2.4. *(1)* [She93, 4.3] *Assume:*

- λ *is regular, uncountable,*
- $\kappa < \lambda$ *implies* $2^\kappa < 2^\lambda$*,*
- *for some regular* $\chi \leq 2^\lambda$ *there is no tree of cardinality* λ *with* $\geq \chi$*-many branches of length* λ*.*

Then $2^{<\lambda} < 2^{\leq\lambda}$*, and for some* $\mu \in \left(\lambda, 2^{<\lambda}\right]$ *with* $\mathrm{cf}\,\mu = \lambda$*:*

(a) for every regular χ *in* $\left(2^{<\lambda}, 2^\lambda\right]$ *there is a linear order of cardinality* χ *with a dense subset of cardinality* μ *(the linear order is* $\left(T_\chi, <_{lx}\right)$*, where* $T_\chi \subseteq 2^{<\mu}$ *has* $\leq \mu$ *nodes and* $\geq \chi$*-many branches of length* λ*),*

(b) $\mathrm{pp}_{\Gamma(\lambda)}(\mu) = 2^\lambda$*,*

(c) μ *is* $(\lambda, \lambda^+, 2)$*-inaccessible, i.e. (see* [She93, 3.2]*) for any* μ' *such that* $\lambda < \mu' < \mu \wedge \mathrm{cf}\mu' \leq \lambda$ *we have* $\mathrm{pp}_{\Gamma(\lambda^+,2)}(\mu') < \mu$*, which in view of Fact 2.3 implies* $\mathrm{pp}_\lambda(\mu') < \mu$*.*

(2) [She96, Claim 3.4] *Assume that* $\theta_{n+1} = \min\left\{\theta : 2^\theta > 2^{\theta_n}\right\}$ *for* $n < \omega$ *and* $\sum_{n<\omega}\theta_n < 2^{\theta_0}$ *(so* θ_{n+1} *is regular,* $\theta_{n+1} > \theta_n$*). Then for infinitely many* $n < \omega$*, for some* $\mu_n \in [\theta_n, \theta_{n+1})$ *(so* $2^{\mu_n} = 2^{\theta_n}$*) we have: for every regular* $\chi \leq 2^{\theta_n}$ *there is a tree of cardinality* μ_n *with* $\geq \chi$*-many branches of length* θ_n*.*

(3) [She94, II.2.3(2)] *If* $\lambda < \mu$ *are singulars of cofinality* $\leq \kappa$ *(and* $\kappa < \lambda$*) and* $\mathrm{pp}_\kappa(\lambda) \geq \mu$ *then* $\mathrm{pp}_\kappa(\mu) \leq^+ \mathrm{pp}_\kappa(\lambda)$*.*

Remark 2.5. See [GS89] concerning optimality of these results.

2.2. **Bounding exponent by iterated** ded.

Definition 2.6. By induction on the ordinal α we define a strictly increasing sequence of ordinals $\beth_\alpha$ such that:

- If $\alpha = 0$, then $\beth_\alpha = \aleph_0$.
- If $\alpha = \beta + 1$, then $\beth_\alpha = \min\left\{\beth : 2^\beth > 2^{\beth_\beta}\right\}$.
- If α is limit, then $\beth_\alpha = \sum\{\beth_\beta : \beta < \alpha\}$.

Lemma 2.7. *For any ordinal α, $2^{\beth_{\alpha+1}} \leq^+ \operatorname{ded}\left(2^{\beth_\alpha}\right)$.*

Proof. $2^{<\beth_{\alpha+1}}$ is a tree with $2^{\beth_{\alpha+1}}$ branches and $\leq \sum\left\{2^{|\beta|} : \beta < \beth_{\alpha+1}\right\}$ nodes. But if $\beta < \beth_{\alpha+1}$, then $2^\beta \leq 2^{\beth_\alpha}$ and $\beth_{\alpha+1} \leq 2^{\beth_\alpha}$ by the definition of $\beth$'s, so the number of nodes is bounded by $2^{\beth_\alpha}$. ☐

Proposition 2.8. *Assume that $\beth_{\alpha+k} \leq 2^{\beth_\alpha}$ for some $k \in \omega$. Then for some $m \leq k$:*

- $\operatorname{ded}\left(2^{\beth_\alpha}\right) \geq 2^{\beth_{\alpha+m}}$,
- $\operatorname{ded}\left(2^{\beth_{\alpha+m}}\right) \geq 2^{\beth_{\alpha+k}}$.

Proof. We follow the proof of [She96, Claim 3.4]. Let $\theta_n = \beth_{\alpha+n}$ for $n \leq k$. Note that θ_{n+1} is regular and $\theta_{n+1} > \theta_n$. We define:

$(*)_{\theta_n}$ for every regular $\chi \leq 2^{\theta_n}$ there is a tree of cardinality θ_n with $\geq \chi$-many branches of length θ_n.

Let $S_0 = \left\{0 < n \leq k : (*)_{\theta_n} \text{ fails}\right\}$.

By Fact 2.4(1) with $\lambda = \theta_n$ and the definitions of S_0 and of the $\beth$'s it follows that for each $n \in S_0$ there is μ_n such that:

$(\alpha)_n$ $\theta_n = \operatorname{cf}\mu_n < \mu_n \leq 2^{<\theta_n} = 2^{\theta_{n-1}}$(as $2^{<\theta_n} \leq \theta_n \times 2^{\theta_{n-1}} \leq 2^{\theta_0} \times 2^{\theta_{n-1}} \leq 2^{\theta_{n-1}}$).

$(\beta)_n$ $\operatorname{pp}_{\theta_n}(\mu_n) = \operatorname{pp}_{\Gamma(\theta_n)}(\mu_n) = 2^{\theta_n}$ (as $\operatorname{pp}_{\Gamma(\theta_n)}(\mu_n) = 2^{\theta_n}$ by Fact 2.4(1)(b), and $\operatorname{pp}_{\Gamma(\theta_n)}(\mu_n) \leq \operatorname{pp}_{\theta_n}(\mu_n) \leq \mu_n^{\theta_n} \leq \left(2^{\theta_{n-1}}\right)^{\theta_n} \leq 2^{\theta_n}$ by Fact 2.3).

$(\gamma)_n$ For any μ' we have that $\theta_n < \mu' < \mu_n \wedge \operatorname{cf}\mu' \leq \theta_n$ implies $\operatorname{pp}_{\Gamma(\lambda^+,2)}(\mu') < \mu_n$ (by Fact 2.4(1)(c)).

$(\delta)_n$ $\operatorname{ded}(\mu_n) \geq 2^{\theta_n}$ (as for any regular $\chi \leq 2^{\theta_n}$ there is linear order of cardinality $\geq \chi$ with a dense subset of size μ_n by Fact 2.4(1)(a)).

Let $S_1 = \left\{n \in S_0 : \mu_n \geq 2^{\beth_\alpha}\right\}$. Then we have the following claims.

$(*)_1$ If $n \leq k$ and $n \notin S_0$ then $\operatorname{ded}\left(2^{\beth_\alpha}\right) \geq 2^{\beth_{\alpha+n}}$.

Proof. By the definition of S_0 and of θ_n it follows that $\operatorname{ded}(\theta_n) \geq 2^{\beth_{\alpha+n}}$ (taking supremum over trees corresponding to regular χ's less or equal to 2^{θ_n}), and $\theta_n \leq 2^{\beth_\alpha}$ by assumption. Thus $\operatorname{ded}\left(2^{\beth_\alpha}\right) \geq 2^{\beth_{\alpha+n}}$ as wanted.

$(*)_2$ If $n \le k$ and $n \in S_0 \setminus S_1$ then $\operatorname{ded}\left(2^{\beth_\alpha}\right) \ge 2^{\beth_{\alpha+n}}$.

Proof. By the definition of S_1 we have $\mu_n < 2^{\beth_\alpha}$. On the other hand, as $n \in S_0$, we have $\operatorname{ded}(\mu_n) \ge 2^{\theta_n}$ by $(\delta)_n$. Combining we get $\operatorname{ded}\left(2^{\beth_\alpha}\right) \ge 2^{\beth_{\alpha+n}}$.

$(*)_3$ If n and $n+1$ are from S_1 then $\mu_n > \mu_{n+1}$.

Proof. By the assumption $\mu_n \ge 2^{\beth_\alpha} \ge \theta_{n+1} = \operatorname{cf}\theta_{n+1}$, and in fact $\mu_n > \theta_{n+1}$ as they are of different cofinality.

Assume that $\mu_n < \mu_{n+1}$. Then by Fact 2.4(3) with $\lambda = \mu_n$, $\mu = \mu_{n+1}$ and $\kappa = \theta_{n+1}$ (as $\max\{\operatorname{cf}\mu_n, \operatorname{cf}\mu_{n+1}\} = \max\{\theta_n, \theta_{n+1}\} < \min\{\mu_n, \mu_{n+1}\}$ by $(\alpha)_n$ and $(\alpha)_{n+1}$, and $\operatorname{pp}_{\theta_{n+1}}(\mu_n) \ge \operatorname{pp}_{\Gamma(\theta_n)}(\mu_n) = 2^{\theta_n} \ge \mu_{n+1}$) we would get $\operatorname{pp}_{\theta_{n+1}}(\mu_{n+1}) \le^+ \operatorname{pp}_{\theta_{n+1}}(\mu_n)$.

On the other hand by $(\gamma)_{n+1}$ we would get that $\theta_{n+1} < \mu_n < \mu_{n+1} \wedge \operatorname{cf}\mu_n \le \theta_{n+1}$ implies $\operatorname{pp}_{\theta_{n+1}}(\mu_n) < \mu_{n+1} \le 2^{\theta_{n+1}} = \operatorname{pp}_{\theta_{n+1}}(\mu_{n+1})$ — a contradiction. Thus we conclude that $\mu_n \ge \mu_{n+1}$, and in fact $\mu_n > \mu_{n+1}$ as they are of different cofinalities.

We try to define $m = \max\{0 < n \le k : n \notin S_1\}$.

Case 1. m not defined. So $S_1 = \{1, \dots, k\}$ (and we may assume that $k \ge 2$), hence $\mu_1 > \dots > \mu_k$ by $(*)_3$, hence $\mu_k < \mu_1 \le 2^{\theta_0}$. But by the definition of S_1 actually $\mu_k \ge 2^{\theta_0}$ — a contradiction.

Case 2. m is well-defined. So $\{m+1, \dots, k\} \subseteq S_1$ hence as in Case 1 we have $\mu_k < \mu_{m+1} \le 2^{\theta_m}$ hence $\operatorname{ded}\left(2^{\beth_{\alpha+m}}\right) \ge \operatorname{ded}(\mu_k) \ge 2^{\beth_{\alpha+k}}$ by $(\delta)_k$. Besides, $\operatorname{ded}\left(2^{\beth_\alpha}\right) \ge 2^{\beth_{\alpha+m}}$ (by $(*)_1$ if $m \notin S_0$ and by $(*)_2$ if $m \in S_1 \setminus S_0$) — so we are done.

□

Proposition 2.9. *Assume that $\beth_{\alpha+k} \le 2^{\beth_\alpha}$ for some $k \in \omega$. Then for some $m \le k$:*

- $2^{\beth_{\alpha+k}} \le^+ \operatorname{ded}\left(2^{\beth_{\alpha+k-1}}\right)$,
- $2^{\beth_{\alpha+k-1}} \le^+ \operatorname{ded}\left(2^{\beth_{\alpha+m}}\right)$,
- $2^{\beth_{\alpha+m}} \le^+ \operatorname{ded}\left(2^{\beth_{\alpha+m-1}}\right)$,
- $2^{\beth_{\alpha+m-1}} \le^+ \operatorname{ded}\left(2^{\beth_\alpha}\right)$.

Proof. We modify the proof of Proposition 2.8. We have:

$(*)_1^+$ If $n+1 \le k$ and $n+1 \notin S_0$ then $\operatorname{ded}\left(2^{\beth_\alpha}\right)^+ \ge 2^{\beth_{\alpha+n}}$.

Proof. As $\left(2^{\beth_{\alpha+n}}\right)^+$ is regular, $\left(2^{\beth_{\alpha+n}}\right)^+ \le 2^{\beth_{\alpha+n+1}}$ and $(*)_{\theta_{n+1}}$ holds by the definition of S_0, it follows that $\operatorname{ded}(\theta_{n+1})^+ \ge 2^{\beth_{\alpha+n}}$, and $\theta_{n+1} \le 2^{\beth_\alpha}$ by assumption. Thus $\operatorname{ded}\left(2^{\beth_\alpha}\right)^+ \ge 2^{\beth_{\alpha+n}}$ as wanted.

$(*)_2^+$ If $n+1 \le k$ and $n+1 \in S_0 \setminus S_1$ then $\operatorname{ded}\left(2^{\beth_\alpha}\right) \ge 2^{\beth_{\alpha+n}}$.

Proof. If $n+1 \in S_0 \setminus S_1$ then $\mu_{n+1} < 2^{\beth_\alpha}$ and $\operatorname{ded}(\mu_{n+1})^+ \geq 2^{\theta_n}$ by $(\delta)_{n+1}$.

Now in Case 1 we get a contradiction in the same way as before, so we may assume that m is well defined, i.e. $\{m+1, \ldots, k\} \subseteq S_1$. As before we get $\mu_k < \mu_{m+1} \leq 2^{\theta_m}$, hence $\operatorname{ded}\left(2^{\beth_{\alpha+m}}\right) \geq \operatorname{ded}(\mu_k)^+ \geq 2^{\beth_{\alpha+k-1}}$ by $(\delta)_k$. Besides, $\operatorname{ded}\left(2^{\beth_\alpha}\right)^+ \geq 2^{\beth_{\alpha+m-1}}$ (by $(*)_1^+$ if $m \notin S_0$ and by $(*)_2^+$ if $m \in S_1 \setminus S_0$). We can conclude by Lemma 2.7. □

Although, as it was already mentioned, it is consistent for κ of uncountable cofinality that $\operatorname{ded}\kappa < 2^\kappa$, we prove (in ZFC) that these values are not so far apart and that four iterations of ded are sufficient to get the exponent.

Theorem 2.10. *Let μ be an arbitrary cardinal. Then there are $\lambda_0, \ldots, \lambda_4$ such that:*

(1) $\lambda_0 \leq \mu$,
(2) $\lambda_{i+1} \leq \operatorname{ded}(\lambda_i)$ for $i < 4$,
(3) $2^\mu \leq \lambda_4$.

Proof. As the sequence of the $\beth$'s is increasing, for some α we have $\beth_\alpha \leq \mu < \beth_{\alpha+1}$, so also $\alpha \leq \mu$.

First of all, for any ordinal β with $\beta + \omega \leq \alpha$ and $2^{\beth_\beta} > \beth_{\beta+\omega}$ we have (by Fact 2.4(2) taking $\theta_0 = \beth_\beta$ and $\theta_n = \beth_{\beta+n}$):

$\odot_1$ For infinitely many $\gamma \in [\beta, \beta+\omega)$ and arbitrary regular $\beth \leq 2^{\beth_\gamma}$, there is a tree T with $|T| \in [\beth_\gamma, \beth_{\gamma+1})$ and at least $\beth$-many branches of length $\beth_\gamma$.

Let δ_* be the largest non-successor ordinal $\leq \alpha$, so $\alpha = \delta_* + n_*$ for some $n_* < \omega$. We have:

$\odot_2$ There is a linear order I of cardinality $\leq \mu$ with $\geq \sum\left\{2^{\beth_\beta} : \beta < \delta_*\right\}$ Dedekind cuts.

(Indeed, if $\beth_{\delta_*}$ is a strong limit cardinal then $\sum\left\{2^{\beth_\beta} : \beta < \delta_*\right\} \leq \mu$ and this is trivial. Otherwise, the demand $\beth_{\beta+\omega} \leq 2^{\beth_\beta} < 2^{\beth_{\beta+1}}$ holds for every large enough $\beta < \delta_*$, so by $\odot_1$ and Fact 1.1 we can conclude by taking the sum of the corresponding linear orders and noting that $\delta_* \leq \mu$).

Let $\lambda_0 = \mu$, $\lambda_1 = \sum\left\{2^{\beth_\beta} : \beta < \delta_*\right\}$ and $\lambda_{2+n} = 2^{\beth_{\delta_*+n}}$ for $n \in \{0, \ldots, n_*\}$. Note that $\lambda_{2+n_*} = 2^{\beth_\alpha} = 2^\mu$.

We have:

- $\lambda_1 \leq^+ \operatorname{ded}\lambda_0$ (by $\odot_2$).
- $\lambda_2 \leq^+ \operatorname{ded}\lambda_1$ (as $2^{<\beth_{\delta_*}}$ is a tree with $\sum\{2^\kappa : \kappa < \beth_{\delta_*}\} = \sum\left\{2^{\beth_\beta} : \beta < \delta_*\right\} = \lambda_1$ nodes and $2^{\beth_{\delta_*}} = \lambda_2$ branches).
- $\lambda_{2+n+1} \leq^+ \operatorname{ded}(\lambda_{2+n})$ for $n < n_*$ (by Lemma 2.7).

If $\delta_* = \alpha$ then we are done as $\lambda_2 = 2^{\beth_\alpha} = 2^\mu$ (as $\mu < \beth_{\alpha+1}$ and $\beth_{\alpha+1}$ is smallest with $2^{\beth_\alpha} < 2^{\beth_{\alpha+1}}$), so assume $\delta_* = \alpha_* + n_*$ and $n_* > 0$.

If $\beth_{\delta_*+n_*} \leq 2^{\beth_{\delta_*}}$, then by Proposition 2.8 there is some $m \leq n_*$ such that $\lambda_3' = \operatorname{ded}\left(2^{\beth_{\delta_*}}\right) \geq 2^{\beth_{\delta_*+m}}$ and $\lambda_4' = \operatorname{ded}\left(2^{\beth_{\delta_*+m}}\right) \geq 2^{\beth_{\delta_*+n_*}} = 2^{\beth_\alpha} = 2^\mu$. It then follows that $\lambda_0, \lambda_1, \lambda_2, \lambda_3', \lambda_4'$ are as wanted.

Otherwise $\beth_{\delta_*+n_*} > 2^{\beth_{\delta_*}}$, and let n be the biggest such that $\beth_{\delta_*+n_*} > 2^{\beth_{\delta_*+n}}$, it follows that $n \leq n_* - 1$. Then $\beth_{\delta_*+n_*} \leq 2^{\beth_{\delta_*+n+1}}$ and again by Proposition 2.8 we get some m such that:

- $\lambda_0'' = 2^{\beth_{\delta_*+n}} < \beth_{\delta_*+n_*} \leq \mu$,
- $\lambda_1'' = 2^{\beth_{\delta_*+n+1}} \leq^+ \operatorname{ded}\left(2^{\beth_{\delta_*+n}}\right)$ (by Lemma 2.7),
- $\lambda_2'' = 2^{\beth_{\delta_*+m}} \leq \operatorname{ded}\left(2^{\beth_{\delta_*+n+1}}\right)$,
- $2^\mu = 2^{\beth_{\delta_*+n_*}} \leq \lambda_3'' = \operatorname{ded}\left(2^{\beth_{\delta_*+m}}\right)$.

But then $\langle \lambda_i'' \rangle_{i \leq 3}$ are as wanted. □

Similarly we have:

Corollary 2.11. *Let μ be an arbitrary cardinal. Then there are $\lambda_0, \ldots, \lambda_6$ such that:*

(1) $\lambda_0 \leq \mu$,
(2) $\lambda_{i+1} \leq^+ \operatorname{ded}(\lambda_i)$ *for all* $i < 6$,
(3) $2^\mu \leq \lambda_6$.

Proof. Follows from the proof of Theorem 2.10 using Proposition 2.9 instead of Proposition 2.8. □

Problem 2.12. What is the smallest $1 < n \leq 4$ for which Theorem 2.10 remains true? Can the bound be improved at least for certain classes of cardinals? Also, how might the required number of iterations vary in different models of ZFC?

Corollary 2.13. *For every cardinal μ and $k < \omega$ there is some $n < \omega$ and a sequence $\langle \lambda_m : m \leq n \rangle$ such that:*

- $\lambda_0 \leq \mu$,
- $\lambda_0 < \ldots < \lambda_n$ *and* $\operatorname{ded}(\lambda_m)^+ \geq \lambda_{m+1}$,
- $\lambda_n \geq \beth_k(\mu)$.

Proof. Follows by iterating Corollary 2.11. □

3. On 2-cardinal models for dependent T

We recall that a formula $\varphi(x, y) \in L$ is said to have the independence property (or IP) with respect to a theory T if in some model of T there are elements $\langle a_i : i \in \omega \rangle$ and $\langle b_s : s \subseteq \omega \rangle$ such that $\varphi(a_i, b_s)$ holds if and only if $i \in s$. A complete first-order theory is called dependent (or NIP) if no formula has the independence property. The class of dependent theories contains both the stable and the o-minimal theories, but also for example the theory of algebraically closed valued fields.

Fact 3.1. [She90, Theorem II.4.11] *A countable theory T is dependent if and only if $|S_1(M)| \leq (\operatorname{ded}|M|)^{\aleph_0}$ for all $M \models T$.*

In this section we show that when considering the two-cardinal transfer to arbitrarily large gaps between the cardinals, the situation for dependent theories is not better than for arbitrary theories. Namely, for every $n < \omega$ we construct a dependent theory T which has a $(\beth_m, \aleph_0)$-model for all $m < n$, but does not have any $(\beth_\omega, \aleph_0)$-models.

Definition 3.2. For any $n \in \mathbb{N}$, let L_n be the language consisting of:

(1) P_m, Q_m are unary predicates for $m < n$.
(2) f_m is a unary function for $m + 1 < n$.
(3) $<_m$ is a binary relation for $m < n$.

Definition 3.3. We define a universal theory $T_n^\forall$ in the language L_n saying:

(1) $\langle Q_m : m < n \rangle$ is a partition of the universe.
(2) $<_m$ is a linear order on Q_m.
(3) P_m is a subset of Q_m.
(4) f_m is a unary function such that:
   (a) It is 1-to-1 from P_{m+1} into $Q_m \setminus P_m$.
   (b) It is 1-to-1 from $Q_m \setminus P_m$ into P_{m+1}.
   (c) $f(f(x)) = x$.
   (d) It is the identity on $\{x : x \notin P_{m+1} \cup (Q_m \setminus P_m)\}$.

Claim 3.4. (1) $T_n^\forall$ is a consistent universal theory.
(2) $T_n^\forall$ has JEP and AP.
(3) If $M \models T_n^\forall$ and $A \subseteq M$ is finite, then the substructure generated by A is finite, and in fact of size at most $2 \times |A|$.
(4) $T_n^\forall$ has a model completion T_n which is $\aleph_0$-categorical and eliminates quantifiers.

Proof. (1), (2) and (3) are easy to see, and (4) follows by e.g. [Hod93, Theorem 7.4.1]. □

Claim 3.5. In fact, T_n is axiomatized by:

(1) $T_n^\forall$
(2) $<_m$ is a dense linear order without end-points.
(3) P_m is both dense and co-dense in Q_m.
(4) f_m is a 1-to-1 function from P_{m+1} onto $Q_m \setminus P_m$.
(5) If $a_1 <_m c_1$ and $a_2 <_{m+1} c_2$, then there are $b_1 \in Q_m \setminus P_m$ and $b_2 \in P_{m+1}$ such that: $a_1 <_m b_1 <_m c_1$, $a_2 <_{m+1} b_2 <_{m+1} c_2$ and $f_m(b_2) = b_1$.

Proposition 3.6. *T_n is dependent.*

Proof. Let $M \models T_n$. Let $p(x) \in S_1(M)$ be a non-algebraic type. By quantifier elimination it is determined by:

- $Q_m(x)$ for the corresponding $m < n$.
- Fixing the corresponding cut of x over M in the order $<_m$.
- Saying if $P_m(x)$ holds or not.
- If it doesn't hold, fixing the cut of $f_m(x)$ over M in the order $<_{m+1}$.
- If it holds, fixing the cut $f_m(x)$ over M in the order $<_{m-1}$.

Then clearly $|S_1(M)| \leq \operatorname{ded}|M|$, so T_n is dependent. □

Remark 3.7. In fact it is easy to check that T_n is strongly dependent (see [She05]).

Proposition 3.8. *(1) If $M \models T_n$ and $\left|P_0^M\right| = \lambda$, then $|M| \leq \beth_n(\lambda)$.*

(2) Moreover: $\left|P_{m+1}^M\right| = \left|Q_m^M \setminus P_m^M\right| \leq \left|Q_m^M\right|$ and $\left|Q_m^M\right| \leq^+ \operatorname{ded}\left|P_m^M\right|$.

Claim 3.9. Assume that $\lambda_0 < \ldots < \lambda_n$ and $\lambda_{m+1} \leq^+ \operatorname{ded}\lambda_m$. Then T_n has a model M such that $\left|P_0^M\right| = \lambda_0$ and :

(1) $\left|P_m^M\right| = \lambda_m$.
(2) $\left|Q_m^M\right| = \lambda_{m+1}$.

Proof. By assumption, for every $m < n$ we can find a linear order J_m of cardinality λ_{m+1} with a dense subset I_m of cardinality λ_m. We may also assume that:

(1) For every $a < b$ in J_m, $|(a, b)| = \lambda_{m+1}$ and $|(a, b) \cap I_m| = \lambda_m$ (so in particular I_m is also co-dense in J_m).
(2) I_m and J_m are dense without end-points.

Indeed, given an arbitrary infinite linear order I and a dense subset J, let $I_* = I \times \mathbb{Q}$, $J_* = J \times \mathbb{Q}$ and let I_{**} be the lexicographic order on $I_*^{<\omega}$, $J_{**} = J_*^{<\omega}$. It is easy to see that $|I_{**}| = |I|$, $|J_{**}| = |J|$, J_{**} is dense in I_{**}, both orders are dense without end-points, and that for any $a < b$ in J_{**}, $|(a, b)| = |I|$ and $|(a, b) \cap J_{**}| = |J|$.

We define M by taking $Q_m^M = J_m$, $P_m^M = I_m$ and $<_m^M = <_{J_m}$. We may choose f_m satisfying 3.5(4) by transfinite induction as all the relevant intervals have "full cardinality" by the assumption. By Claim 3.5, $M \models T_n$. □

Theorem 3.10. *For every $n < \omega$ there is a dependent countable theory T which has a $(\beth_m, \aleph_0)$-model for all $m < n$, but does not have any $(\beth_\omega, \aleph_0)$-models.*

Proof. Follows by combining Propositions 3.6, 3.8, Claim 3.9 and Corollary 2.13. □

4. Hanf number for omitting types

Now we elaborate on the previous example, and for every countable ordinal $\beta < \omega_1$ we find a countable ordinal $\alpha_* < \omega_1$, a countable theory T_{α_*} and a partial type $p(x)$ such that:

- there is a model of T_{α_*} omitting $p(x)$ and of size $\geq \beth_\beta$,
- any model of T_{α_*} omitting $p(x)$ is of size at most $\beth_{\alpha_*}$.

Definition 4.1. Fix an ordinal $\alpha_* < \omega_1$. We describe our theory T_{α_*}.

(1) $\langle Q_\alpha(x) : \alpha \leq \alpha_*\rangle$ are pairwise disjoint infinite unary predicates.

(2) $<_\alpha$ is a dense linear order without end-points on $Q_\alpha(x)$.

(3) $P_\alpha(x)$ is a dense co-dense subset of $Q_\alpha(x)$.

(4) $R(x)$ is a unary predicate disjoint from all Q_α's.

(5) $\langle c_n : n \in \omega\rangle$ are constants and $R(c_n)$ for all $n \in \omega$.

(6) $<_R$ is a linear order on $R(x)$, and $(R, <_R, \langle c_n : n \in \omega\rangle)$ is a model of $\mathrm{Th}(\mathbb{N}, <, \langle n : n \in \mathbb{N}\rangle)$.

(7) $s_R(x), s_R^{-1}(x)$ are the successor and the predecessor functions on $R(x)$.

(8) $\langle d_r : r \in \mathbb{Q}\rangle$ are constants and $P_0(d_r)$ for all $r \in \mathbb{Q}$.

(9) For every successor ordinal $\delta + 1 \leq \alpha_*$:
   (a) f_δ is a bijection from $P_{\delta+1}$ onto $Q_\delta \setminus P_\delta$, identity on $\{x : x \notin P_{\delta+1} \cup (Q_\delta \setminus P_\delta)\}$ and such that $f_\delta(f_\delta(x)) = x$.
   (b) If $a_1 <_\delta c_1$ and $a_2 <_{\delta+1} c_2$ for some $a_1, c_1 \in Q_\delta \setminus P_\delta$ and $a_2, c_2 \in P_{\delta+1}$, then there are $b_1 \in Q_\delta \setminus P_\delta$ and $b_2 \in P_{\delta+1}$ such that: $a_1 <_\delta b_1 <_\delta c_1$, $a_2 <_{\delta+1} b_2 <_{\delta+1} c_2$ and $f_\delta(b_2) = b_1$.

(10) For every limit ordinal $\delta \leq \alpha_*$:
   (a) We fix some listing $\langle \alpha_{\delta,n} : n < \omega\rangle$ with $\sum_{n<\omega} \alpha_{\delta,n} = \delta$, where for every n we have that $\alpha_{\delta,n}$ is a *successor* ordinal larger than the successor of $\alpha_{\delta,n-1}$ and larger than any $\alpha_{\delta',m}$ from a similar listing for a smaller limit ordinal δ'.
   (b) We have a function $G_\delta(x)$ such that:
      (i) G_δ is the identity on $\{x : x \notin P_\delta\}$.
      (ii) $G_\delta : P_\delta(x) \to R(x)$ is onto.
      (iii) for every $y \in R(x)$, $G_\delta^{-1}(y)$ is a dense linear order without end-points.
      (iv) If $y_1 <_R y_2$, then $G_\delta^{-1}(y_1)$ is co-dense in $G_\delta^{-1}(y_2)$, and every cut of $G_\delta^{-1}(y_1)$ realized by some $a \in P_\delta$ is realized by some $a' \in G_\delta^{-1}(y_2)$.
   (c) We have a relation $E_\delta(x_1, x_2, y)$ which holds if and only if x_1 and x_2 are from $P_\delta \setminus G_\delta^{-1}(y)$ and realize the same cut over $G_\delta^{-1}(y)$.
   (d) For each $n \in \omega$ we have a function $F_{\delta,n}$ such that:
      (i) It is a bijection from $G_\delta^{-1}(c_n) \setminus G_\delta^{-1}(c_{n-1})$ onto $P_{\alpha_{\delta,n}}(x)$, the identity on $\{x : x \notin P_{\alpha_{\delta,n}} \cup G_\delta^{-1}(c_n)\}$ and such that $F_{\delta,n}(F_{\delta,n}(x)) = x$.
      (ii) For any $n \in \omega$, if $a_1 <_{\alpha_{\delta,n}} b_1$ with $a_1, b_1 \in P_{\alpha_{\delta,n}}$ and $a_2 <_\delta d <_\delta b_2$ with $a_2, b_2 \in G_\delta^{-1}(c_n)$, then there are $e_1 \in P_{\alpha_{\delta,n}}$ and $e_2 \in G_\delta^{-1}(c_n) \setminus G_\delta^{-1}(c_{n-1})$

such that: $a_1 <_\delta e_1 <_\delta b_1$, $a_2 <_\delta e_2 <_\delta b_2$, $F_{\delta,n}(e_2) = e_1$ and $E_\delta(d, e_2, \alpha)$ for all $\alpha < c_n$.

Claim 4.2. T_{α_*} is a complete dependent theory.

Proof. It it easy to check by back-and-forth that T is a complete theory eliminating quantifiers.

Let $M \models T_{\alpha_*}$ and let $p(x) \in S_1(M)$ be a non-algebraic type. We have the following options:

(1) $p(x) \vdash Q_\alpha(x)$ for some successor $\alpha < \alpha_*$. Then $p(x)$ is determined by:
   - (a) Fixing the cut of x over M in the order $<_\alpha$.
   - (b) If $p(x) \vdash \neg P_\alpha(x)$:
     - (i) Fixing the cut of $f_\alpha(x)$ over M in the order $<_{\alpha+1}$.
     - (ii) If $\alpha+1$ occurs as $\alpha_{\delta,n}$ for some limit $\delta < \alpha_*$, then fixing the cut of $F_{\delta,n}(f_\alpha(x))$ over M in the order $<_\delta$, and fixing the cut of $G_\delta(F_{\delta,n}(f_\alpha(x)))$ in $<_R$ over M.
   - (c) If $p(x) \vdash P_\alpha(x)$:
     - (i) fixing the cut $f_{\alpha-1}(x)$ over M in the order $<_{\alpha-1}$.
     - (ii) If α occurs as $\alpha_{\delta,n}$ for some limit $\delta < \alpha_*$, then fixing the cut of $F_{\delta,n}(x)$ over M in the order $<_\delta$, and fixing the cut of $G_\delta(F_{\delta,n}(x))$ in $<_R$ over M.

(2) $p(x) \vdash Q_\delta(x)$ for some limit δ. Then $p(x)$ is determined by:
   - (a) Fixing the cut of x over M in the order $<_\delta$.
   - (b) If $P_\delta(x)$ does not hold, then similar to 2(b).
   - (c) If $P_\delta(x)$ holds:
     - (i) Fixing the cut of $G_\delta(x)$ over M in $<_R$.
     - (ii) If $G_\delta(x) = c_n$ for some $n \in \omega$ also fixing the cut of $F_{\delta,n}(x)$ over M in $<_{\alpha_{\delta,n}}$.

(3) If $p(x) \vdash R(x)$, then fixing the cut of x in $<_R$ over M.

(4) $p(x) \vdash \{\neg Q_\alpha(x) : \alpha < \alpha_*\} \cup \{\neg R(x)\}$. Then $p(x)$ is a complete type.

Altogether it follows that $|S_1(M)| \leq (\mathrm{ded}\,|M|)^{\aleph_0}$, thus T is dependent by Fact 3.1. □

Consider the type $p_*(x) = \{\neg P_\alpha(x) : 0 < \alpha \leq \alpha_*\} \cup \{x \neq c_n : n \in \omega\} \cup \{x \neq d_r : r \in \mathbb{Q}\}$.

Claim 4.3. Let M be a model of T_{α_*} omitting $p_*(x)$. Then $|M| \leq \beth_{\alpha_*}$.

Proof. First of all, if M omits p_* then $\left|P_0^M\right| = \aleph_0$ and $\left|R^M\right| = \aleph_0$. We show by induction for $\delta \leq \alpha_*$ that $\left|P_\delta^M\right| \leq \beth_\delta$. If $\delta = \alpha+1$ is a successor, then clearly $\left|P_{\delta+1}^M\right| \leq^+ \mathrm{ded}\left|P_\delta^M\right|$, thus $\leq \beth_{\delta+1}$ by induction. If δ is a limit, then by construction $\left|P_\delta^M\right| \leq \sum_{n<\omega}\left(\left|P_{\alpha_{\delta,n}}^M\right|\right) \leq \sum_{n<\omega} \beth_{\alpha_{\delta,n}} = \beth_\delta$. The claim follows. □

Claim 4.4. For every $\beta < \omega_1$ there is $\alpha_* < \omega_1$ such that T_{α_*} has a model omitting $p_*(x)$ of size $\geq \beth_\beta$.

Proof. By Corollary 2.13 and induction there is $\alpha_* < \beta + \omega$ such that we can choose a strictly increasing sequence of cardinals $(\lambda_\alpha)_{\alpha<\alpha_*}$ satisfying:

- $\lambda_0 = \aleph_0$.
- $\lambda_{\alpha+1} \leq^+ \operatorname{ded} \lambda_\alpha$.
- For a limit α, $\lambda_\alpha = \sum_{\alpha'<\alpha} \lambda_{\alpha'}$.
- $\lambda_{\alpha_*} \geq \beth_\beta$.

We define a model of T_{α_*} omitting p_* and such that $\left|P_\alpha^M\right| = \lambda_\alpha$ by induction on α.

(1) Let $R^M = (\omega, <)$ with c_n naming n. Let $Q_0^M = (\mathbb{R}, <)$ and let $P_0^M = \mathbb{Q}$, with d_r naming r.

(2) For a successor $\delta = \alpha+1$: Similarly to Claim 3.9, we can find a linear order J of cardinality λ_δ with a dense subset I of cardinality λ_α. We may also assume that for every $a < b$ in J, $|(a,b)| = \lambda_\delta$ and $|(a,b) \cap I| = \lambda_\alpha$. We let $Q_\delta^M = J$, $P_\delta^M = I$ and $<_\delta^M = <_J$. We may choose f_δ satisfying Definition 4.1 by transfinite induction as all the relevant intervals have "full cardinality" by construction and the inductive assumption.

(3) For a limit $\delta \leq \alpha_*$:
   (a) First we construct orders I_n, J_n by induction on $n < \omega$:
      (i) Let $I_0 \subseteq J_0$ be dense linear orders without end-points and such that I_0 is dense-codense in J_0, $|I_0| = \lambda_{\alpha_{\delta,0}}$, $|J_0| = \lambda_{\alpha_{\delta,0}+1}$, and such that for every $a < b$ in J_0, $|(a,b)| = \lambda_{\alpha_{\delta,0}+1}$ and $|(a,b) \cap I_0| = \lambda_{\alpha_{\delta,0}}$ (can be chosen by assumption on λ_α as in the proof of Claim 3.9).
      (ii) Let I'_{n+1}, J'_{n+1} be dense linear orders without end-points and such that I'_{n+1} is dense-codense in J'_{n+1}, $\left|I'_{n+1}\right| = \lambda_{\alpha_{\delta,n+1}}$, $\left|J'_{n+1}\right| = \lambda_{\alpha_{\delta,n+1}+1}$, and such that for every $a < b$ in J'_{n+1}, $|(a,b)| = \lambda_{\alpha_{\delta,n+1}+1}$ and $\left|(a,b) \cap I'_{n+1}\right| = \lambda_{\alpha_{\delta,n+1}}$ (again can be chosen by assumption on λ_α as in the proof of Claim 3.9). Let I_{n+1} extend I_n with a copy of I'_{n+1} added in every cut, and similarly let J_{n+1} extend J_n with a copy of J'_{n+1} added in every cut. It follows that $\lambda_{\delta,n+1} \leq |I_{n+1}| \leq \lambda_{\alpha_{\delta,n}+1} \times \lambda_{\alpha_{\delta,n+1}} \leq \lambda_{\alpha_{\delta,n+1}}$ and $|J_{n+1}| \leq \lambda_{\alpha_{\delta,n}+2} \times \lambda_{\alpha_{\delta,n+1}+1} \leq \lambda_{\alpha_{\delta,n+1}+1}$, and that I_{n+1} is a dense-codense subset of J_{n+1}.
      (iii) Finally, let $I = \bigcup_{n<\omega} I_n$ and $J = \bigcup_{n<\omega} J_n$. In particular I is dense-codense in J and both I, J are of size λ_δ.
   (b) We let $P_\delta^M = I, Q_\delta^M = J$ and define G_δ^M by sending I_n to c_n. By construction of I_n and $P_{\alpha_{\delta,n}}^M$ and transfinite induction we can find bijections $F_{\delta,n}^M$ between $G_\delta^M(c_n) \setminus G_\delta^M(c_{n-1}) = I_n \setminus I_{n-1}$ and $P_{\alpha_{\delta,n}}^M$ satisfying the axioms of T_{α_*}. We let $E(x,y,c_n)$ hold for x, y in $I_n \setminus I_{n-1}$ realizing the same cut over I_{n-1}.

□

Theorem 4.5. *For every countable ordinal $\beta < \omega_1$ there is a complete countable dependent theory T and a partial type $p(x)$ such that:*

- *T has a model omitting p of size $\geq \beth_\beta$.*
- *Any model of T omitting p is of size $< \beth_{\omega_1}$.*

Proof. Combining Claims 4.2, 4.3 and 4.4. □

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