ON PARTIAL ORDERINGS HAVING PRECALIBRE-$\aleph_1$ AND FRAGMENTS OF MARTIN’S AXIOM

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Abstract. We define a countable antichain condition (ccc) property for partial orderings, weaker than precalibre-$\aleph_1$, and show that Martin’s axiom restricted to the class of partial orderings that have the property does not imply Martin’s axiom for $\sigma$-linked partial orderings. This yields a new solution to an old question of the first author about the relative strength of Martin’s axiom for $\sigma$-centered partial orderings together with the assertion that every Aronszajn tree is special. We also answer a question of J. Steprans and S. Watson (1988) by showing that, by a forcing that preserves cardinals, one can destroy the precalibre-$\aleph_1$ property of a partial ordering while preserving its ccc-ness.

A question asked in [1] is if $MA(\sigma$-centered) plus “Every Aronszajn tree is special” implies $MA(\sigma$-linked). The interest in this question originated in the result of Harrington-Shelah [5] showing that if $\aleph_1$ is accessible to reals, i.e., there exists a real number $x$ such that the cardinal $\aleph_1$ in the model $L[x]$ is equal to the real $\aleph_1$, then $MA$ implies that there exists a $\Delta^3_1(x)$ set of real numbers that does not have the Baire property. The hypothesis that $\aleph_1$ is accessible to reals is necessary, for if $\aleph_1$ is inaccessible to reals and $MA$ holds, then $\aleph_1$ is actually weakly-compact in $L$ ([5]), and K. Kunen showed that starting form a weakly compact cardinal one can get a model where $MA$ holds and every projective set of reals has the Baire property. In [1], using Todorčević’s $\rho$-functions ([12]), it was shown that $MA(\sigma$-centered) plus “Every Aronszajn tree is special” is sufficient to produce a $\Delta^3_1(x)$ of real numbers without the Baire property, assuming $\aleph_1 = \aleph_1^{L[x]}$. Thus, it was natural to ask how weak is $MA(\sigma$-centered) plus “Every Aronszajn tree is special” as compared to the full $MA$, and in particular if it implies $MA(\sigma$-linked). The answer is negative, as it has been observed by D. Chodounský and J. Zapletal that a finite-support iteration of $\sigma$-centered posets combined with the forcing that specializes Aronszajn trees has the Y-c.c. property, and therefore does not add random reals (see [2]). In the first part of the paper we give a new and stronger negative answer to the question, namely we show that a fragment of $MA$ that includes $MA(\sigma$-centered), and even

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MA(3-Knaster), and implies “Every Aronszajn tree is special”, does not imply MA(σ-linked). A partial ordering with the precalibre-$\aleph_1$ property plays the key role in the construction of the model.

In the second part of the paper we answer a question of Steprans-Watson [9]. They ask if it is possible to destroy the precalibre-$\aleph_1$ property of a partial ordering, while preserving its ccc-ness, in a forcing extension of the set-theoretic universe $V$ that preserves cardinals. This is a natural question considering that, as shown in [9], on the one hand, assuming MA plus the Covering Lemma, every precalibre-$\aleph_1$ partial ordering has precalibre-$\aleph_1$ in every forcing extension of $V$ that preserves cardinals; and on the other hand the ccc property of a partial ordering having precalibre-$\aleph_1$ can always be destroyed while preserving $\aleph_1$, and consistently even preserving all cardinals.

We answer the Steprans-Watson question positively, and in a very strong sense. Namely, we show that it is consistent, modulo ZFC, that the Continuum Hypothesis holds and there exist a forcing notion $T$ of cardinality $\aleph_1$ that preserves $\aleph_1$ (and therefore it preserves all cardinals, cofinalities, and the cardinal arithmetic), and two precalibre-$\aleph_1$ partial orderings, such that forcing with $T$ preserves their ccc-ness, but it also forces that their product is not ccc and therefore they don’t have precalibre-$\aleph_1$.

1. Preliminaries

Recall that a partially ordered set (or poset) $P$ is ccc if every antichain of $P$ is countable; it is productive-ccc if the product of $P$ with any ccc poset is also ccc; it is Knaster (or has property-K) if every uncountable subset of $P$ contains an uncountable subset consisting of pairwise compatible elements. More generally, for $k \geq 2$, $P$ is $k$-Knaster if every uncountable subset of $P$ contains an uncountable subset such that any $k$-many of its elements have a common lower bound. Thus, Knaster is the same as 2-Knaster. $P$ has precalibre-$\aleph_1$ if every uncountable subset of $P$ has an uncountable subset such that any finite set of its elements has a common lower bound; it is $\sigma$-linked (or $\sigma$-2-linked) if it can be partitioned into countably-many pieces so that each piece is pairwise compatible. More generally, for $k \geq 2$, $P$ is $\sigma$-$k$-linked if it can be partitioned into countably-many pieces so that any $k$-many elements in the same piece have a common lower bound. Finally, $P$ is $\sigma$-centered if it can be partitioned into countably-many pieces so that any finite number of elements in the same piece have a common lower bound. We have the following implications, for every $k \geq 2$:

$$\sigma\text{-centered} \Rightarrow \sigma\text{-}k\text{-linked} \Rightarrow k\text{-Knaster} \Rightarrow \text{productive-ccc} \Rightarrow \text{ccc},$$

and

$$\sigma\text{-centered} \Rightarrow \text{precalibre-$\aleph_1$} \Rightarrow k\text{-Knaster}.$$ 

These are the only implications that can be proved in ZFC.

For any property $\Gamma$ of posets that implies the ccc, and an infinite cardinal $\kappa$, Martin’s Axiom for $\Gamma$ and for families of $\kappa$-many dense open sets, denoted by $MA_{\kappa}(\Gamma)$, asserts: for every $P$ that satisfies the property $\Gamma$ and every family $\{D_\alpha : \alpha < \kappa\}$ of dense open subsets of $P$, there exists a filter $G \subseteq P$ that is generic for the family, that is, $G \cap D_\alpha \neq \emptyset$ for every $\alpha < \kappa$. 


When $\kappa = \aleph_1$ we omit the subscript and write $MA(\Gamma)$ for $MA_{\aleph_1}(\Gamma)$. Also, for an infinite cardinal $\theta$, the notation $MA_{< \theta}(\Gamma)$ means: $MA_{\kappa}(\Gamma)$ for all $\kappa < \theta$. The axiom $MA_{\aleph_0}(\Gamma)$ is provable in ZFC; and it is consistent, modulo ZFC, that the Continuum Hypothesis fails and $MA_{\aleph_0}(\text{ccc})$ holds (see [7], or [6]). Martin’s axiom, denoted by $MA$, is $MA(\text{ccc})$.

Thus, we have the following implications, for every $k \geq 2$:

$$MA_{\kappa}(\text{ccc}) \Rightarrow MA_{\kappa}(\text{productive-ccc}) \Rightarrow$$

$$\Rightarrow MA_{\kappa}(k\text{-Knaster}) \Rightarrow MA_{\kappa}(\sigma\text{-k-linked}) \Rightarrow MA_{\kappa}(\sigma\text{-centered}),$$

and

$$MA_{\kappa}(k\text{-Knaster}) \Rightarrow MA_{\kappa}(\text{precalibre}^-\aleph_1) \Rightarrow MA_{\kappa}(\sigma\text{-centered}).$$

Again, the arrows cannot be reversed (see [13], [10] for even finer distinctions, and also [11] for Borel examples).

For all the facts mentioned in the rest of the paper without a proof, as well as for all undefined notions and notations, see [6].

2. THE PROPERTY $Pr_k$

Let us consider the following property of partial orderings, weaker than the $k$-Knaster property.

**Definition 1.** For $k \geq 2$, let $Pr_k(Q)$ mean that $Q$ is a forcing notion such that if $p_e \in Q$, for all $e < \aleph_1$, then we can find $\bar{u}$ such that:

(a) $\bar{u} = \langle u_\xi : \xi < \aleph_1 \rangle$.
(b) $u_\xi$ is a finite subset of $\aleph_1$.
(c) $\bigcup_{\xi_0} u_\xi \cap u_{\xi_1} = \emptyset$, whenever $\xi_0 \neq \xi_1$.
(d) If $\xi_0 < \ldots < \xi_{k-1}$, then we can find $\varepsilon_l \in u_{\xi_l}$, for $l < k$, such that

$$\{p_{\varepsilon_l} : l < k\} \text{ have a common lower bound}.$$ 

Notice that $Pr_k(Q)$ implies that $Q$ is ccc, and that $Pr_{k+1}(Q)$ implies $Pr_k(Q)$. Also note that if $Q$ is $k$-Knaster, then $Pr_k(Q)$. For given a subset $\{p_e : e < \aleph_1\}$ of $Q$, there exists an uncountable $X \subseteq \aleph_1$ such that $\{p_{\varepsilon_l} : l < k\}$ has a common lower bound, for every $\varepsilon_0 < \ldots < \varepsilon_{k-1}$ in $X$, so we can take $u_\xi$ to be the singleton that contains the $\xi$-th element of $X$. Finally, observe that if $Q$ has precalibre-$\aleph_1$, then $Pr_k(Q)$ holds for every $k \geq 2$.

Recall that if $T$ is an Aronszajn tree on $\omega_1$, then the forcing that specializes $T$ consists of finite functions $p$ from $\omega_1$ into $\omega$ such that if $\alpha \neq \beta$ are in the domain of $p$ and are comparable in the tree ordering, then $p(\alpha) \neq p(\beta)$. The ordering is the reversed inclusion. It is consistent, modulo ZFC, that the specializing forcing is not productive-ccc, an example being the case when $T$ is a Suslin tree. However, we have the following:

**Lemma 2.** If $T$ is an Aronszajn tree and $Q = QT$ is the forcing that specializes $T$ with finite conditions, then $Pr_k(Q)$ holds, for every $k \geq 2$.

**Proof.** Without loss of generality, $T = (\omega_1, < T)$. Let $p_\alpha \in Q$, for $\alpha < \aleph_1$. By a $\Delta$-system argument we may assume that $\{\text{dom}(p_\alpha) : \alpha < \aleph_1\}$ forms a $\Delta$-system, with root $r$. Moreover, we may assume that for some fixed $n$, $|\text{dom}(p_\alpha) \setminus r| = n$, for all $\alpha < \omega_1$. Let $\langle \alpha_1, \ldots, \alpha_n \rangle$ be an enumeration of $\text{dom}(p_\alpha) \setminus r$. We may also assume that if $\alpha < \beta$, then the highest level of $T$
that contains some $\alpha_i$ ($1 \leq i \leq n$) is strictly lower than the lowest level of $T$ that contains some $\beta_j$ ($1 \leq j \leq n$).

Fix a uniform ultrafilter $D$ over $\omega_1$. For each $\alpha < \omega_1$ and $1 \leq i, j \leq n$, let

$$D_{\alpha,i,j} := \{ \beta > \alpha : \alpha_i <_T \beta_j \}$$

and let

$$D_{\alpha,i,0} := \{ \beta > \alpha : \alpha_i <_T \beta_j, \text{ all } j \}.$$

For every $\alpha$ and every $i$, there exists $j_{\alpha,i} \leq n$ such that $D_{\alpha,i,j_{\alpha,i}} \in D$. Moreover, for every $1 \leq i \leq n$, there exists $E_i \in D$ such that $j_{\alpha,i}$ is fixed, say with value $j_i$, for all $\alpha \in E_i$. We claim that $j_i = 0$, for all $1 \leq i \leq n$. For suppose $i$ is so that $j_i \neq 0$. Pick $\alpha < \beta < \gamma$ in $E_i \cap D_{\alpha,i,j_i} \cap D_{\beta,i,j_i}$. Then $\alpha_i, \beta_i <_T \gamma_j$, hence $\alpha_i <_T \beta_i$. This yields an $\omega_1$-chain in $T$, which is impossible. Now let $E := \bigcap_{1 \leq i \leq n} E_i \in D$.

We claim that for every $m$ and every $\alpha$ we can find $u \in [\omega_1 \setminus \alpha]^m$ such that if $\beta < \gamma$ are in $u$, then $\beta_i <_T \gamma_j$, for every $1 \leq i, j \leq n$. Indeed, given $m$ and $\alpha$, choose any $\beta^0 \in E \setminus \alpha$. Now given $\beta^0, \ldots, \beta^m$, all in $E$, let $\beta^{m+1} \in E \cap \bigcap_{1 \leq i \leq m} \bigcap_{1 \leq j \leq n} D_{\beta^i,j,i,0}$. Then the set $u := \{ \beta^0, \ldots, \beta^{m-1} \}$ is as required.

We can now choose ($u_\xi : \xi \in \aleph_1$) pairwise-disjoint, with $|u_\alpha| > k \cdot n$, so that if $\xi_1 < \xi_2$, then $\sup(u_{\xi_1}) < \min(u_{\xi_2})$, and each $u_\xi$ is as above, i.e., if $\beta < \gamma$ are in $u_\xi$, then $\beta_i <_T \gamma_j$, for every $1 \leq i, j \leq n$. We claim that ($u_\xi : \xi \in \aleph_1$) is as required. So, suppose $\xi_0 < \ldots \xi_{k-1}$. We choose $\alpha^\ell \in u_{\xi_\ell}$ by downward induction on $\ell \in \{0, \ldots, k-1\}$ so that $\{p_\alpha : \ell < k\}$ has a common lower bound. Let $\alpha^{k-1}$ be any element of $u_{\xi_{k-1}}$. Now suppose $\alpha^{k+1}, \ldots, \alpha^{k-1}$ have been already chosen and we shall choose $\alpha^\ell$. We may assume that for each $\beta \in u_{\xi_\ell}$, $p_\beta$ is incompatible with $p_{\alpha^\ell}$, some $\ell' \in \{\ell+1, \ldots, k-1\}$, for otherwise we could take as our $\alpha^\ell$ any $\beta \in u_{\xi_\ell}$ with $p_\beta$ compatible with all $p_{\alpha^{\ell'}}$, $\ell' \in \{\ell+1, \ldots, k-1\}$. Thus, for each $\beta \in u_{\xi_\ell}$, there exist $\ell' \in \{\ell+1, \ldots, k-1\}$ and $1 \leq i, j \leq n$ such that $\beta_i <_T \alpha^{\ell'}_j$. So, since $|u_{\xi_\ell}| > k \cdot n$, there must exist $\beta, \beta' \in u_{\xi_\ell}$ and $\ell'$ such that $\beta_i, \beta'_i <_T \alpha^{\ell'}_j$, for some $1 \leq i, j \leq n$ with $\beta_i \neq \beta'_i$. But this implies that $\beta_i$ and $\beta'_i$ are $<_T$-comparable, contradicting our choice of $u_{\xi_\ell}$. \qed

We show next that the property $Pr_k$ for forcing notions is preserved under iterations with finite support, of any length.

**Lemma 3.** For any $k \geq 2$, the property $Pr_k$ is preserved under finite-support forcing iterations. That is, if

$$\langle P_\alpha, Q_\beta ; \alpha \leq \lambda , \beta < \lambda \rangle$$

is a finite-support iteration of forcing notions such that $Pr_k(P_0)$ and $\Vdash_{P_\beta}$ "$Pr_k(Q_\beta)$", for every $\beta < \lambda$, then $Pr_k(P_\lambda)$.

**Proof.** By induction on $\alpha \leq \lambda$. For $\alpha = 0$ it is trivial. If $\alpha$ is a limit ordinal with $cf(\alpha) \neq \aleph_1$, and $p_\vee \in P_\alpha$, for all $\vee < \aleph_1$, then either uncountably many $p_\vee$ have the same support (in the case $cf(\alpha) = \omega$) or the support of all $p_\vee$ is bounded by some $\alpha' < \alpha$. In either case $Pr_k(P_\alpha)$ follows easily from the induction hypothesis.
If \( cf(\alpha) = \aleph_1 \), then we may use a \( \Delta \)-system argument, as in the usual proof of the preservation of the ccc.

So, suppose \( \alpha = \beta + 1 \). Let \( p_\varepsilon \in \mathbb{P}_\alpha \), for all \( \varepsilon < \aleph_1 \). Without loss of generality, we may assume that \( \beta \in dom(\varepsilon) \), for all \( \varepsilon < \aleph_1 \).

Since \( \mathbb{P}_\beta \) is ccc, there is \( q \in \mathbb{P}_\beta \) such that
\[
q \Vdash \sett{\varepsilon : p_\varepsilon \upharpoonright \beta \in \langle \mathcal{G}_\beta \rangle} = \aleph_1.
\]

Let \( G \subseteq \mathbb{P}_\beta \) be generic over \( V \) and with \( q \in G \). In \( V[G] \) we have that \( p_\varepsilon(\beta)[G] \in \mathcal{Q}_\beta[G] \), and \( Pr_k(\mathcal{Q}_\beta[G]) \) holds. So, there is \( \langle u^{0}_{\xi} : \xi < \aleph_1 \rangle \) as in Definition 1 for the sequence \( \langle p_\varepsilon(\beta) : p_\varepsilon \upharpoonright \beta \in \mathcal{G}_\beta \rangle \).

For each \( \xi \), let \( (q_\xi, u^1_\xi) \) be such that
\[
q_\xi \in \mathbb{P}_\beta \text{ and } q_\xi \leq q,
q_\xi \Vdash \sett{u^0_{\xi} = u^1_\xi} \text{, so } u^1_\xi \text{ is finite.}
q_\xi \leq p_\varepsilon \upharpoonright \beta \text{, for every } \varepsilon \in u^1_\xi. \text{ (This can be ensured because if } \varepsilon \in u^1_\xi \text{, then } q_\xi \Vdash \sett{p_\varepsilon \upharpoonright \beta \in \mathcal{G}_\beta} \text{, so we may as well take } q_\xi \leq p_\varepsilon \upharpoonright \beta.\)
\]

Now apply the induction hypothesis for \( \mathbb{P}_\beta \) to obtain \( \langle u^{2}_{\xi} : \xi < \aleph_1 \rangle \) as in the definition of \( Pr_k \) for the sequence \( \langle q_\xi : \xi < \aleph_1 \rangle \). We may assume, by refining the sequence if necessary, that \( \max(u^{2}_{\xi}) < \min(u^{2}_{\xi}) \) whenever \( \xi < \zeta' \).

Let \( u^*_\xi := \bigcup \sett{u^{2}_{\zeta} : \xi \in u^{2}_{\zeta}} \). We claim that \( u^*_\xi = \langle u^{2}_{\zeta} : \zeta < \aleph_1 \rangle \) is as in the definition, for the sequence \( \langle p_\varepsilon : \varepsilon < \aleph_1 \rangle \). Clearly, the \( u^*_\xi \) are finite and pairwise-disjoint. Moreover, given \( \xi_0 < \ldots < \xi_{k-1} \), we can find \( \xi_0 \in u^{2}_{\xi_0}, \ldots, \xi_{k-1} \in u^{2}_{\xi_{k-1}} \) such that in \( \mathbb{P}_\beta \) there is a common lower bound \( q_* \) to \( \langle q_{\xi_0}, \ldots, q_{\xi_{k-1}} \rangle \). Since \( q_* \leq q_{\xi_0}, \ldots, q_{\xi_{k-1}} \leq q \), there are some \( q_{**} \leq q_* \) and \( \varepsilon_1 \in u^{2}_{\xi_0} \), for each \( l < k \), such that for some \( \mathbb{P}_\beta \)-name \( p \),
\[
q_{**} \Vdash \sett{p \leq q_\xi, p_{\xi_0}(\beta), \ldots, p_{\xi_{k-1}}(\beta)}.
\]

Then the condition \( q_{**} * p \) is a common lower bound for the conditions \( p_{\xi_0}, \ldots, p_{\xi_{k-1}} \).

\( \square \)

3. ON FRAGMENTS OF MA

We shall now prove that \( MA(Pr_{k+1}) \) does not imply \( MA(\sigma\text{-}k\text{-}linked) \), which yields a negative answer to the first question stated in the Introduction. The following is the main lemma.

**Lemma 4.** For \( k \geq 2 \), there is a forcing notion \( \mathbb{P}_* = \mathbb{P}_*^k \) and \( \mathbb{P}_* \)-names \( \mathcal{A} \) and \( \mathcal{Q}_* = \mathcal{Q}_*^k \) such that

1. \( \mathbb{P}_* \) has precalibre-\( \aleph_1 \) and is of cardinality \( \aleph_1 \).
2. \( \Vdash \mathcal{A} \subseteq \aleph_1^{k+1} \)
3. \( \Vdash \mathcal{Q}_* = \sett{v \in \aleph_1^{\aleph_0} : \forall \xi^{k+1} \cap \mathcal{A} = \emptyset} \) is \( \sigma\text{-}k\text{-}linked \).
4. \( \Vdash \mathcal{I}_\alpha := \sett{v \in \mathcal{Q}_* : v \not\subseteq \alpha} \) is dense, all \( \alpha < \aleph_1 \).
(5) $\|\mathbb{P}_{\ast}\| = \aleph_0$. "If $v_\alpha \in \mathcal{Q}_\mathbb{A}$ is such that $v_\alpha \not\in \alpha$, for $\alpha < \aleph_1$; and $u_\xi \in [\aleph_1]^{<\aleph_0}$, for $\xi < \aleph_1$, are non-empty and pairwise disjoint, then there exist $\xi_0 < \ldots < \xi_k$ such that for every $\langle \alpha_\ell : \ell \leq k \rangle \in \prod_{\ell \leq k} u_\alpha$, the set $\bigcup_{\ell \leq k} v_\alpha$ does not belong to $\mathcal{Q}_\mathbb{A}$."

Proof. We define $\mathbb{P}_{\ast}$ by: $p \in \mathbb{P}_{\ast}$ if and only if $p$ has the form $(u, A, h_\forall, A_\forall, h_\forall, \ldots, h_\forall, A_\forall, h_\forall)$, where

(a) $u \in [\aleph_1]^{<\aleph_0}$,
(b) $A \subseteq [u]^{k+1}$, and
(c) $h : \varphi_p \to \omega$, where $\varphi_p := \{v \subseteq u : [v]^{k+1} \cap A = \emptyset\}$, is such that $w_0, \ldots, w_{k-1} \in \varphi_p$ and $h$ is constant on $\{w_0, \ldots, w_{k-1}\}$, then $w_0 \cup \ldots \cup w_{k-1} \in \varphi_p$.

The order is given by: $p \leq q$ if and only if $u_q \subseteq u_p$, $A_q = A_p \cap [u_q]^{k+1}$, and $h_q \subseteq h_p$ (hence $\varphi_q = \varphi_p \cap \mathcal{P}(u_q)$ and $h_p \upharpoonright \varphi_q = h_q$).

(1): Clearly, $\mathbb{P}_{\ast}$ has cardinality $\aleph_1$, so let us show that it has precalibre-$\aleph_1$. Given $\{q_\xi = (u_\xi, A_\xi, h_\xi) : \xi < \aleph_1\} \subseteq \mathbb{P}_{\ast}$, and writing $\varphi_\xi$ instead of the more cumbersome $\varphi_{q_\xi}$, we can find an uncountable $W \subseteq \aleph_1$ such that:

(i) The set $\{u_\xi : \xi \in W\}$ forms a $\Delta$-system with heart $u_*$.
(ii) The sets $[u_\xi]^{k+1} \cap \mathcal{A}_\xi$, for $\xi \in W$, are all the same. Hence the sets $\varphi_\xi \cap \mathcal{P}(u_\xi)$, for $\xi \in W$, are also all the same.
(iii) The functions $h_\xi \upharpoonright \{\varphi_\xi \cap \mathcal{P}(u_\xi)\}$, for $\xi \in W$, are all the same.
(iv) The ranges of $h_\xi$, for $\xi \in W$, are all the same, say $R$. So, $R$ is finite.
(v) For each $i \in R$, the sets $\{w \cap u_\xi : h_\xi(w) = i\}$, for $\xi \in W$, are the same.

We will show that every finite subset of $\{q_\xi : \xi \in W\}$ has a common lower bound. Given $\xi_0, \ldots, \xi_m \in W$, let $q = (u_q, A_q, h_q)$ be such that

- $u_q = \bigcup_{\xi \leq m} u_\xi$
- $A_q = \bigcup_{\xi \leq m} A_\xi$. Note that this implies that the $\varphi_\xi$ are contained in $\varphi_q = \{v \subseteq u_q : [v]^{k+1} \cap A_q = \emptyset\}$. Indeed, if, say, $w \in \varphi_q$, then $[w]^{k+1} \cap \mathcal{A}_q = \emptyset$, and we claim that there is also $[w]^{k+1} \cap \mathcal{A}_j = \emptyset$, for $j \leq m$. For if $v \subseteq [w]^{k+1} \cap \mathcal{A}_j$, with $j \neq \ell$, then $v \subseteq u_\xi$, and therefore $v \subseteq [u_\xi]^{k+1} \cap \mathcal{A}_j = [u_\xi]^{k+1} \cap \mathcal{A}_\ell$. Hence, $v \subseteq [w]^{k+1} \cap \mathcal{A}_\xi$, which is impossible because $[w]^{k+1} \cap \mathcal{A}_\ell$ is empty.
- $h_q : \varphi_q \to \omega$ is such that $h_q(v) = h_\xi(v)$ for all $v \subseteq \varphi_\xi$, and the $h_q(v)$ are all distinct and greater than $\sup\{h_q(v) : v \in \bigcup_{\xi \leq m} \varphi_\xi\}$, for $v \not\subseteq \bigcup_{\xi \leq m} \varphi_\xi$. Notice that $h_q$ is well-defined because the restrictions $h_\xi \upharpoonright \{\varphi_\xi \cap \mathcal{P}(u_\xi)\}$, for $\xi \leq m$, are all the same.

We claim that $q \in \mathbb{P}_{\ast}$. For this, we only need to show that if $\{w_0, \ldots, w_{k-1}\} \subseteq \varphi_q$ and $h_q$ is constant on $\{w_0, \ldots, w_{k-1}\}$, then $\bigcup_{\xi \leq m} \varphi_q$ and suppose $h_q$ is constant on it, say with constant value $i$. By definition of $h_q$ we must have $\{w_0, \ldots, w_{k-1}\} \subseteq \bigcup_{\xi \leq m} \varphi_\xi$. Now suppose, towards a contradiction, that $v \in \bigcup_{\xi \leq m} \varphi_\xi$, some $\ell \leq m$. Let $\ell = \{w_j : j < k\} \cap \varphi_\xi$, and let $t = \{w_j : j < k\} \setminus s$. Thus, $v \subseteq \bigcup_s \bigcup_t \{\ell \cap u_*\}$, for $\alpha \in v \cup \mathcal{P}(u_\xi)$, and $\alpha \in \bigcup_{\ell' \neq \ell} \varphi_{\xi'\xi}$, for some $\ell' \neq \ell$, hence $\alpha \in u_\xi \cap u_{\xi'} = u_*$. 

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By (v),
\[ \{ w \cap u_\xi : h_\xi(w) = i \} = \{ w \cap u_\xi : h_\xi'(w) = i \} \]
for every \( \ell' \leq m \). So, for every \( w_j \in t \), there exists \( w_j' \in \wp_\xi \) such that \( w_j \cap u_\xi = w_j' \cap u_\xi \) and \( h_\xi(w_j') = i \). Let \( t' = s \cup \{ w_j' : w_j \in t \} \). Note that \( t' \subseteq \wp_\xi \) and \( t' \subseteq \{ w : h_\xi(w) = i \} \). So,
\[ v \subseteq \bigcup t' \subseteq \bigcup \{ w : h_\xi(w) = i \}. \]
Thus, \( v \in \bigcup \{ w : h_\xi(w) = i \} \). But this is impossible because \( \bigcup \{ w : h_\xi(w) = i \} \in \wp_\xi \) (since \( h_\xi \) satisfies property (c) above) and therefore
\[ \bigcup \{ w : h_\xi(w) = i \} \subseteq A_\xi = \emptyset. \]

Now one can easily check that \( q \leq q_{\xi_0}, \ldots, q_{\xi_m} \). And this shows that the set \( \{ q_\xi : \xi \in W \} \) is finite-wise compatible.

(2): Let
\[ \mathcal{A} = \{ (\dot{v}, p) : v \in A_p, p \in \mathbb{P}_s \}. \]
Thus, \( \mathcal{A} \) is a name for the set \( \bigcup \{ A_p : p \in G \} \), where \( G \) is the \( \mathbb{P}_s \)-generic filter. Clearly, (2) holds.

(3): Let
\[ \mathcal{Q}_\mathcal{A} = \{ (\dot{v}, p) : v \in \wp_p, p \in \mathbb{P}_s \}. \]
Thus, \( \mathcal{Q}_\mathcal{A} \) is a name for the set \( \bigcup \{ \wp_p : p \in G \} \), where \( G \) is the \( \mathbb{P}_s \)-generic filter. Clearly, \( \Vdash_{\mathbb{P}_s} \) "\( \mathcal{Q}_\mathcal{A} = \{ v \in [\aleph_1]^{<\aleph_0} : [v]^{k+1} \cap \mathcal{A} = \emptyset \} \)". Moreover, if \( G \) is \( \mathbb{P}_s \)-generic over \( V \), then, by (c), the function \( \bigcup \{ h_p : p \in G \} \) witnesses that the interpretation \( i_G(\mathcal{Q}_\mathcal{A}) \), ordered by \( \supseteq \), is \( \sigma \)-\( k \)-linked.

(4): Clear.

(5): Suppose that \( p \in \mathbb{P}_s \) forces \( \dot{v}_\alpha \in \mathcal{Q}_\mathcal{A} \) is such that \( \dot{v}_\alpha \not\subseteq \alpha \), all \( \alpha < \aleph_1 \); and it also forces \( \dot{u}_\xi \in [\aleph_1]^{<\aleph_0} \), all \( \xi < \aleph_1 \), are non-empty and pairwise disjoint.

For each \( \xi < \aleph_1 \), let \( q_\xi = (u_\xi, A_\xi, h_\xi) \leq p \) and let \( u_\xi^* \in [\aleph_1]^{<\aleph_0} \) and \( \dot{v}_\alpha^* = (v_\alpha^*, \alpha : \alpha \in u_\xi^*) \), with \( v_\alpha^* \in [\aleph_1]^{<\aleph_0} \), be such that
\[ q_\xi \Vdash_{\mathbb{P}_s} \" \dot{u}_\xi = u_\xi^* \text{ and } \dot{v}_\alpha = v_\alpha^*, \text{ for } \alpha \in u_\xi^* \". \]

We may assume, by extending \( q_\xi \) if necessary, that \( u_\xi^* \cup \bigcup_{\alpha \in u_\xi^*} v_\alpha^* \subseteq u_\xi^* \).

As in (1), we can find an uncountable \( W \subseteq \aleph_1 \) such that (i)-(v) hold for the set of conditions \( \{ q_\xi : \xi \in W \} \). Hence \( \{ q_\xi : \xi \in W \} \) is pairwise compatible (in fact, finite-wise compatible), from which it follows that the set \( \{ u_\xi^* : \xi \in W \} \) is pairwise disjoint. Now choose \( \xi_0 < \ldots < \xi_k \) from \( W \) so that
- The heart \( u_\xi \) of the \( \Delta \)-system \( \{ u_\xi : \xi \in W \} \) is an initial segment of \( u_\xi', \text{ all } \ell \leq k \);
- \( \sup (u_\xi) < \inf (u_{\xi+1} \setminus u_\xi) \), for all \( \ell < k \), and
- \( u_{\xi+1} \subseteq u_\xi \setminus u_\xi \), for all \( \ell \leq k \).
For each $\sigma = (\alpha_\ell : \ell \leq k) \in \prod_{\ell \leq k} u_\ell^*$, pick $w_\sigma \in \bigcup_{\ell \leq k} u_{\ell, \alpha_\ell}^*$ such that $|w_\sigma \cap (v_{\ell, \alpha_\ell}^* \setminus \alpha_\ell)| = 1$, for all $\ell \leq k$. This is possible because $v_{\ell, \alpha_\ell}^* \not\subseteq \alpha_\ell$.

Claim 5. $w_\sigma \not\subseteq u_\xi$, hence $w_\sigma \not\in A_\xi$, for all $\sigma \in \prod_{\ell \leq k} u_{\ell, \alpha_\ell}^*$ and all $\ell \leq k$.

Proof of Claim. Fix $\sigma = (\alpha_\ell : \ell \leq k)$ and $\ell \leq k$, and suppose, for a contradiction, that $w_\sigma \subseteq u_\xi$. Then $w_\sigma \subseteq u_\xi \setminus u_*$. If $\ell < k$, then since $\sup(w_{\ell+1}) < \inf(u_{\ell+1}) \leq \inf(u_{\ell+1}) = \alpha_{\ell+1}$, we would have $w_\sigma \setminus \alpha_{\ell+1} = \emptyset$, which contradicts our choice of $w_\sigma$. But if $\ell = k$, then since $\sup(w_{k-1}) \leq \sup(w_{k-1}) < \inf(u_k \setminus u_*)$, we would have $w_\sigma \cap u_{\xi-1}, \alpha_{\ell-1} = \emptyset$, which contradicts again our choice of $w_\sigma$. \hfill $\Box$

Now define $q = (u_q, A_q, h_q)$ as follows:

- $u_q = \bigcup_{\ell \leq k} u_\xi$
- $A_q = (\bigcup_{\ell \leq k} A_\xi) \cup \{ w_\sigma : \sigma \in \prod_{\ell \leq k} u_{\ell, \alpha_\ell}^* \}$. Note that since $w_\sigma \not\subseteq u_\xi$ (Claim 5), we have that $w_\sigma \not\in \varphi_{\xi}^\ell$, for all $\sigma \in \prod_{\ell \leq k} u_{\ell, \alpha_\ell}^*$ and $\ell \leq k$. Hence, $\varphi_{\xi}^\ell \subseteq \varphi_q$, all $\ell \leq k$.
- $h_q : \varphi_q \rightarrow \omega$ is such that $h_q(v) = h_{\xi}^\ell(v)$ for $v \in \varphi_{\xi}^\ell$, for all $\ell \leq k$, and the $h_q(v)$ are all distinct and greater than $\sup\{ h_q(v) : v \in \bigcup_{\ell \leq k} \varphi_{\xi}^\ell \}$, for $v \not\in \bigcup_{\ell \leq k} \varphi_{\xi}^\ell$.

As in (1), we can now check that $q \in P_s$. Moreover, by Claim 5, $A_\ell = A_q \cap [u_\ell^*]^{k+1}$. Hence, $q \leq q_{\ell^*}$, all $\ell \leq k$, and so

$q \Vdash \text{"} u_\xi^* = u_\ell^* \text{ and } v_\alpha = v_{\ell, \alpha}^* \text{ for } \alpha \in u_\ell^* \text{."}$

And since $w_\sigma \in \bigcup_{\ell \leq k} u_{\ell, \alpha_\ell}^*$, for every $\sigma \in \prod_{\ell \leq k} u_{\ell, \alpha_\ell}^*$, we have that

$q \Vdash \text{"} \bigcup_{\ell \leq k} v_{\alpha_\ell} \not\subseteq Q_\alpha \text{ for all } \langle \alpha_\ell : \ell \leq k \rangle \in \prod_{\ell \leq k} u_{\ell, \alpha_\ell}^* \text{."}$

\hfill $\Box$

Lemma 6. Let $k \geq 2$ and let $P_s$ be as in Lemma 4. Suppose $Q$ is a $P_s$-name for a forcing notion that satisfies $Pr_{k+1}$. Then,

$q \Vdash Q \text{"} \text{There is no directed } G \subseteq Q_\alpha \text{ such that } I_\alpha \cap G \neq \emptyset, \text{ all } \alpha < \aleph_1 \text{."}$

where $I_\alpha$ is a name for the dense open set $\{ v \in Q_\alpha : v \not\subseteq \alpha \}$.

Proof. Suppose, for a contradiction, that $p * q \in P_s * Q$ and $p * q \Vdash Q \text{"} \text{There exists } G \subseteq Q_\alpha \text{ directed, with } I_\alpha \cap G \neq \emptyset, \text{ all } \alpha < \aleph_1 \text{."}$

Suppose $G_0 \subseteq P_s$ is a filter generic over $V$, with $p \in G_0$. So, in $V[G_0]$, letting $q = i_{G_0}(q)$ and $Q = i_{G_0}(Q)$, we have that for some $Q$-name $G$

$q \Vdash Q \text{"} G \subseteq Q_\alpha \text{ is directed and } I_\alpha \cap G \neq \emptyset, \text{ all } \alpha < \aleph_1 \text{."}$

For each $\alpha < \aleph_1$, let $q_\alpha \leq q$, and let $v_\alpha \in [\aleph_1]^{< \aleph_0}$ be such that

$q_\alpha \Vdash Q \text{"} \text{"} v_\alpha \subseteq I_\alpha \cap G \text{."}$

Thus, $v_\alpha \not\subseteq \alpha$, for all $\alpha < \aleph_1$.

Since $Q$ satisfies $Pr_{k+1}$, there exists $\bar{u} = \langle u_\xi : \xi < \aleph_1 \rangle$ such that

(a) $u_\xi$ is a finite subset of $\aleph_1$, all $\xi < \aleph_1$, ...
(b) \( u_{\xi_0} \cap u_{\xi_1} = \emptyset \) whenever \( \xi_0 \neq \xi_1 \), and 
(c) if \( \xi_0 < \ldots < \xi_k \), then we can find \( \alpha_\ell \in u_{\xi_\ell} \), for \( \ell \leq k \), such that 
\[ \{ q_\alpha_\ell : \ell \leq k \} \] have a common lower bound.

By Lemma 4, we can find \( \xi_0 < \ldots < \xi_k \) such that for every \( \{ \alpha_\ell : \ell \leq k \} \subseteq \prod_{\ell \leq k} u_{\xi_\ell} \) the set \( \bigcup_{\ell \leq k} v_{\alpha_\ell} \) does not belong to \( Q_A \).

By (c), let \( \alpha_\ell \in u_{\xi_\ell} \), for \( \ell \leq k \), be such that \( \{ q_\alpha_\ell : \ell \leq k \} \) have a common lower bound, call it \( r \). Then \( r \) forces that \( \{ \check{v}_{\alpha_\ell} : \ell \leq k \} \subseteq G \). And since \( r \) forces that \( \tilde{G} \) is directed, it also forces that \( \bigcup_{\ell \leq k} v_{\alpha_\ell} \in Q_A \) a contradiction.

All elements are now in place to prove the main result of this section.

**Theorem 7.** Let \( k \geq 2 \). Assume \( \lambda = \lambda^{<\theta} \), where \( \theta = cf(\theta) > \aleph_1 \). Then there is a finite-support iteration 
\[ \bar{P} = \langle P_\alpha, Q_\beta ; \alpha \leq \lambda, \beta < \lambda \rangle \]
where

1. \( P_0 \) is the forcing \( P^* \) from Lemma 4.
2. \( \Vdash_{P_0} \text{"} P_{r_{k+1}}(Q_\beta) \text{"} \), for every \( 0 < \beta < \lambda \).
3. In \( V^{\bar{P}_0} \) the axiom \( MA_{\leq \theta}(P_{r_{k+1}}) \) holds, hence in particular (Lemma 2) every Aronszajn tree on \( \omega_1 \) is special.
4. \( Q_A \) witnesses that \( MA(\sigma- \text{k-linked}) \) fails in \( V^{\bar{P}_0} \).

**Proof.** To obtain (3), we proceed in the standard way as in all iterations forcing (some fragment of) \( MA \), that is, we iterate all posets with the \( P_{r_{k+1}} \) property and having cardinality \( < \theta \), which are given by some fixed bookkeeping function (see [6] or [7] for details).

Since after forcing with \( P_0 \) the rest of the iteration \( \bar{P} \) has the property \( P_{r_{k+1}} \) (Lemma 3), (4) follows immediately from Lemma 6.

**Corollary 8.** For every \( k \geq 2 \), ZFC plus \( MA(P_{r_{k+1}}) \) does not imply \( MA(\sigma- \text{k-linked}) \).

Thus, since \( MA(P_{r_{k+1}}) \) implies both \( MA(\sigma-\text{centered}) \) and “Every Aronszajn tree is special”, the corollary answers in the negative and in a strong way the question from [1]: Does \( MA(\sigma-\text{centered}) \) plus “Every Aronszajn tree is special” imply \( MA(\sigma-\text{linked}) \)?

### 4. ON DESTROYING PRECALIBRE-\( \aleph_1 \) WHILE PRESERVING THE CCC

We turn now to the second question stated in the Introduction (Steprans-Watson [9]): Is it consistent that there exists a precalibre-\( \aleph_1 \) poset which is ccc but does not have precalibre-\( \aleph_1 \) in some forcing extension that preserves cardinals?

Note that the forcing extension cannot be ccc, since ccc forcing preserves the precalibre-\( \aleph_1 \) property. Also, as shown in [9], assuming \( MA \) plus the Covering Lemma, every forcing that preserves cardinals also preserves the precalibre-\( \aleph_1 \) property. Moreover, the examples provided in [9] of cardinal-preserving forcing notions that destroy the precalibre-\( \aleph_1 \) they do so by actually destroying the ccc property.
A positive answer to Question 1 is provided by the following theorem. But first, let us recall a strong form of Jensen’s diamond principle, *diamond-star relativized to a stationary set S*, which is also due to Jensen. For S a stationary subset of \( \omega_1 \), let

\[ \diamondsuit^*_S: \text{There exists a sequence } \langle S_\alpha : \alpha \in S \rangle, \text{ where } S_\alpha \text{ is a countable set of subsets of } \alpha, \text{ such that for every } X \subseteq \omega_1 \text{ there is a club } C \subseteq \omega_1 \text{ with } X \cap \alpha \in S_\alpha, \text{ for every } \alpha \in C \cap S. \]

The principle \( \diamondsuit^*_S \) holds in the constructible universe \( L \), for every stationary \( S \subseteq \omega_1 \) (see \([3]\), 3.5, for a proof in the case \( S = \omega_1 \), which can be easily adapted to any stationary \( S \)). Also, \( \diamondsuit^*_S \) can be forced by a \( \sigma \)-closed forcing notion (see \([7]\), Chapter VII, Exercises H18 and H20, where it is shown how to force the even stronger form of diamond known as \( \diamondsuit^*_S \)).

**Theorem 9.** It is consistent, modulo ZFC, that the CH holds and there exist

1. A forcing notion \( T \) of cardinality \( \aleph_1 \) that preserves cardinals.
2. Two posets \( P_0 \) and \( P_1 \) of cardinality \( \aleph_1 \) that have precalibre-\( \aleph_1 \) and such that

\[ \models_T " P_0, P_1 \text{ are ccc, but } P_0 \times P_1 \text{ is not ccc}. " \]

Hence \( \models_T " P_0 \text{ and } P_1 \text{ don’t have precalibre-} \aleph_1". \)

**Proof.** Let \( \{ S_1, S_2 \} \) be a partition of \( \Omega := \{ \delta < \omega_1 : \delta \text{ a limit} \} \) into two stationary sets. By a preliminary forcing, we may assume that \( \diamondsuit^*_S \) holds. So, there exists \( \langle S_\alpha : \alpha \in S_1 \rangle \), where \( S_\alpha \) is a countable set of subsets of \( \alpha \), such that for every \( X \subseteq \omega_1 \) there is a club \( C \subseteq \omega_1 \) with \( X \cap \alpha \in S_\alpha \), for every \( \alpha \in C \cap S_1 \). In particular, the CH holds. Using \( \diamondsuit^*_S \), we can build an \( S_1 \)-oracle, i.e., an \( \subseteq \)-increasing sequence \( M = \langle M_\delta : \delta \in S_1 \rangle \), with \( M_\delta \) countable and transitive, \( \delta \in M_\delta \), \( M_\delta \models " ZFC^- + \delta \text{ is countable} " , \) and such that for every \( A \subseteq \omega_1 \) there is a club \( C_A \subseteq \omega_1 \) such that \( A \cap \delta \in M_\delta \), for every \( \delta \in C_A \cap S_1 \). (For the latter, one simply needs to require that \( S_5 \subseteq M_\delta \) for all \( \delta \in S_1 \).) Moreover, we can build \( M \) so that it has the following additional property:

(\( \ast \)) For every regular uncountable cardinal \( \chi \) and a well ordering \( \triangleleft^*_\chi \) of \( H(\chi) \), the set of all (universes of) countable \( N \preceq H(\chi), \in, \triangleleft^*_\chi \) such that the Mostowski collapse of \( N \) belongs to \( M_\delta \), where \( \delta := N \cap \omega_1 \), is stationary in \( [H(\chi)]^{\aleph_0} \). The property (\( \ast \)) will be needed to prove that the tree partial ordering \( T \) (defined below) has many branches, and also to prove that the product partial ordering \( \mathbb{Q} \times T \) (defined below) is \( S_1 \)-proper (Claim 10), and so it does not collapse \( \aleph_1 \).

To ensure (\( \ast \)), take a big-enough regular cardinal \( \lambda \) and define the sequence \( M \) so that, for every \( \delta \in S_1 \), \( M_\delta \) is the Mostowski collapse of a countable elementary substructure \( X \) of \( H(\lambda) \) that contains \( M \upharpoonright \delta \), all ordinals \( \leq \delta \), and all elements of \( S_\delta \). To see that (\( \ast \)) holds, fix a regular uncountable cardinal \( \chi \), a well ordering \( \triangleleft^*_\chi \) of \( H(\chi) \), and a club \( E \subseteq [H(\chi)]^{\aleph_0} \). Let \( \tilde{N} = \langle N_\alpha : \alpha < \aleph_1 \rangle \) be an \( \subseteq \)-increasing and \( \in \)-increasing continuous chain of elementary substructures of \( \langle H(\chi), \in, \triangleleft^*_\chi \rangle \) with the universe of \( N_\alpha \) in \( E \)
for all \( \alpha < \aleph_1 \). We shall find \( \delta \in S_1 \) such that the transitive collapse of \( N_\delta \)
stands to \( M_\delta \), where \( \delta = N_\delta \cap \omega_1 \).

Fix a bijection \( h : \aleph_1 \to \bigcup_{\alpha < \aleph_1} \aleph_\alpha \), and let \( \Gamma : \aleph_1 \times \aleph_1 \to \aleph_1 \) be the standard pairing function (cf. [6], 3). Observe that the set

\[
D := \{ \delta < \aleph_1 : \delta \text{ is closed under } \Gamma \text{ and } h \text{ maps } \delta \text{ onto } N_\delta \}
\]

is a club. Now let

\[
X_1 := \{ \Gamma(i, j) : h(i) \in h(j) \}
\]

\[
X_2 := \{ \Gamma(\alpha, i) : h(\alpha) \in N_\alpha \}
\]

\[
X_3 := \{ \Gamma(i, j) : h(i) \subseteq h(j) \}
\]

\[
X := \{ 3j + i : j \in X_1 \text{ and } i \in \{1, 2, 3\} \}.
\]

The set \( S'_1 := \{ \delta \in S_1 : X \cap \delta \in M_\delta \} \) is stationary. Thus, since the set \( C := \{ \delta < \aleph_1 : \delta = N_\delta \cap \omega_1 \} \) is a club, we can pick \( \delta \in C \cap D \cap S'_1 \). Since \( \delta \in D \), the structure

\[
Y := \{ X_2 \cap \delta, \{ (i, j) : \Gamma(i, j) \in X_1 \cap \delta \}, \{ (i, j) : \Gamma(i, j) \in X_3 \cap \delta \} \}
\]

is isomorphic to \( N_\delta \), and therefore \( Y \) and \( N_\delta \) have the same transitive collapse. And \( Y \) belongs to \( M_\delta \), because \( \delta \in S'_1 \). Hence, since \( M_\delta \models ZFC^- \), the transitive collapse of \( Y \) belongs to \( M_\delta \). Finally, since \( \delta \in C \), \( \delta = N_\delta \cap \omega_1 \).

We shall define now the forcing \( T \). Let us write \( \aleph_1^{<\aleph_1} \) for the set of all countable sequences of countable ordinals. Let

\[
T := \{ \eta \in \aleph_1^{<\aleph_1} : \text{Range}(\eta) \subset S_1, \eta \text{ is increasing and continuous, of successor length, and if } \varepsilon < lh(\eta), \text{ then } \eta \upharpoonright \varepsilon \in M_{\eta(\varepsilon)} \}.
\]

Let \( \leq_T \) be the partial order on \( T \) given by end-extension. Thus, \( (T, \leq_T) \) is a tree. Note that, since \( \delta \in M_\delta \) for every \( \delta \in S_1 \), if \( \eta \in T \), then \( \eta \in M_{\text{supRange}(\eta)} \). Also notice that if \( \eta \in T \), then \( \eta^{-}\langle \delta \rangle \in T \), for every \( \delta \in S_1 \) greater than \( \text{supRange}(\eta) \). In particular, every node of \( T \) of finite length has \( \aleph_1 \)-many extensions of any bigger finite length. Now suppose \( \alpha < \omega_1 \) is a limit, and suppose, inductively, that for every successor \( \beta < \alpha \), every node of \( T \) of length \( \beta \) has \( \aleph_1 \)-many extensions of every higher successor length below \( \alpha \). We claim that every \( \eta \in T \) of length less than \( \alpha \) has \( \aleph_1 \)-many extensions in \( T \) of length \( \alpha + 1 \) (and in fact, the set of their suprema is stationary). For every \( \delta < \omega_1 \), let \( T_\delta := \{ \eta \in T : \text{supRange}(\eta) < \delta \} \). Notice that \( T_\delta \) is countable: otherwise, uncountably-many \( \eta \in T_\delta \) would have the same \( \text{supRange}(\eta) \), and therefore they would all belong to the model \( M_{\text{supRange}(\eta)} \), which is impossible because it is countable. Now fix a node \( \eta \in T \) of length less than \( \alpha \), and let \( B := \{ b_\gamma : \gamma < \omega_1 \} \) be an enumeration of all the branches (i.e., linearly-ordered subsets of \( T \) closed under predecessors) \( b \) of \( T \) that contain \( \eta \) and have length \( \alpha \) (i.e., \( \bigcup \{ \text{dom}(\eta') : \eta' \in b \} = \alpha \)). For a club \( C \) of the set \( \{ b_\gamma : \gamma < \delta \} \) belongs to \( M_\delta \). We shall next build a sequence \( B^* := \langle b^*_\xi : \xi < \omega_1 \rangle \) of branches from \( B \) so that the set \( \text{sup}B^* := \langle \text{supRange}(\bigcup b^*_\xi) : \xi < \omega_1 \rangle \) is the increasing enumeration of a club. To this end, start by fixing an increasing sequence \( \langle \alpha_n : n < \omega \rangle \) of successor ordinals converging to \( \alpha \), with \( \alpha_0 \) greater than the length of \( \eta \). Then let \( b^*_0 \) := \( b_0 \). Given \( b^*_\xi \), let \( \gamma \) be the least ordinal such that \( \bigcup b_\gamma(\alpha_0) > \text{supRange}(\bigcup b^*_\xi) \), and let \( b^*_\xi+1 := b_\gamma \). Finally, given \( b^*_\xi \) for all \( \xi < \delta \), where \( \delta < \omega_1 \) is a limit ordinal, pick an increasing sequence \( \langle \xi_n : n < \omega \rangle \) converging to \( \delta \).
By construction, the sequence \( \langle \text{supRange}(\bigcup b^*_n) : n < \omega \rangle \) is increasing. Now let \( f : \alpha \rightarrow \aleph_1 \) be such that \( f| [0, \alpha_0] = \bigcup b^*_0 \upharpoonright [0, \alpha_0], \) and \( f| (\alpha_n, \alpha_{n+1}] = \bigcup b^*_{\delta_n+1} \upharpoonright (\alpha_n, \alpha_{n+1}], \) for all \( n < \omega. \) Then set \( b^*_\delta := \{ f| \beta : \beta < \alpha \) is a successor \} \). One can easily check that \( b^*_\delta \) is a branch of \( T \) of length \( \alpha \) with \( \text{supRange}(\bigcup b^*_\delta) = \text{sup}\{\text{supRange}(\bigcup b^*_\xi) : \xi < \zeta \}. \) Finally, notice that if \( \delta \in S_1 \cap C \) is greater than \( \alpha \) and belongs to the club enumerated by \( \text{sup}B^* \), then since \( M_\delta \models \text{"}\delta \text{ is countable"}, \) we can pick the sequences \( \langle \alpha_n : n < \omega \rangle \) and \( \langle \xi_n : n < \omega \rangle \) in \( M_\delta. \) Then the sequence \( \langle b^*_\delta : n < \omega \rangle \) belongs to \( M_\delta, \) and therefore \( (\bigcup b^*_\delta)^- (\delta) \in T. \)

By (\( \ast \)) the set of all countable \( N \subseteq \langle H(\aleph_2), \in, <^*_k \rangle \) that contain \( B^* \) and \( \langle \alpha_n : n < \omega \rangle, \) with \( \alpha \subseteq N, \) and such that the Mostowski collapse of \( N \) belongs to \( M_\delta, \) where \( \delta := N \cap \omega_1, \) is stationary in \( [H(\chi)]_{\aleph_0}. \) So, since the set \( \text{Lim}(\text{sup}B^*) \) of limit points of \( \text{sup}B^* \) is a club, there is such an \( N \) with \( \delta := N \cap \omega_1 \in \text{Lim}(\text{sup}B^*). \) If \( N \) is the transitive collapse of \( N, \) we have that \( B^* \upharpoonright \delta \in N \in M_\delta, \) and so in \( M_\delta \) we can build, as above, the branch \( b^*_\delta. \) Therefore, since \( \delta = \text{supRange}(\bigcup b^*_\delta), \) we have that \( \bigcup b^*_\delta \cup \{ (\alpha, \delta) \} \in T \) and extends \( \eta. \) We have thus shown that \( \eta \) has \( \aleph_1 \)-many extensions in \( T \) of length \( \alpha + 1. \) Even more, the set \( \{ \text{supRange}(\bigcup b) : b \text{ is a branch of length } \alpha + 1 \text{ that extends } \eta \} \) is stationary.

Note however that since the complement of \( S_1 \) is stationary, \( T \) has no branch of length \( \omega_1, \) because the range of such a branch would be a club contained in \( S_1. \) But since every \( \eta \in T \) has extensions of length \( \alpha + 1, \) for every \( \alpha \) greater than or equal to the length of \( \eta, \) forcing with \( (T, \geq_T) \) yields a branch of \( T \) of length \( \omega_1. \)

In order to obtain the forcing notions \( \mathbb{P}_0 \) and \( \mathbb{P}_1 \) claimed by the theorem, we need first to force with the forcing \( \mathbb{Q}, \) which we define as follows. For \( u \) a subset of \( T, \) let \( [u]_{\mathbb{P}}^2 \) be the set of all pairs \( \{ \eta, \nu \} \subseteq u \) such that \( \eta \neq \nu \) and \( \eta \) and \( \nu \) are \( <_T \)-comparable. Let \( \mathbb{Q} := \{ p : [u]_{\mathbb{P}}^2 \rightarrow \{0, 1\} : u \text{ is a finite subset of } T \}, \) ordered by reversed inclusion.

It is easily seen that \( \mathbb{Q} \) is ccc, and it has cardinality \( \aleph_1, \) so forcing with \( \mathbb{Q} \) does not collapse cardinals, does not change cofinalities, and preserves cardinal arithmetic. (In fact, \( \mathbb{Q} \) is equivalent, as a forcing notion, to the poset for adding \( \aleph_1 \) Cohen reals, which is \( \sigma \)-centered, but we shall not make use of this fact.)

Notice that if \( G \subseteq \mathbb{Q} \) is a generic filter over \( V, \) then \( \bigcup G : [T]_{\mathbb{P}}^2 \rightarrow \{0, 1\}. \)

Recall that, for \( S \subseteq \aleph_1 \) stationary, a forcing notion \( \mathbb{P} \) is called \( S \)-proper if for all (some) large-enough regular cardinals \( \chi \) and all (stationary-many) countable \( \langle N, \in \rangle \preceq \langle H(\chi), \in \rangle \) that contain \( \mathbb{P} \) and such that \( N \cap \aleph_1 \in S, \) and all \( p \in \mathbb{P} \cap N, \) there is a condition \( q \leq p \) that is \( (N, \mathbb{P}) \)-generic. If \( \mathbb{P} \) is \( S \)-proper, then it does not collapse \( \aleph_1. \) (See [8], or [4] for details.)

**Claim 10.** The forcing \( \mathbb{Q} \times T \) is \( S_1 \)-proper, hence it does not collapse \( \aleph_1. \)

**Proof of the claim.** Let \( \chi \) be a large-enough regular cardinal, and let \( \prec^*_\chi \) be a well-ordering of \( H(\chi). \) Let \( N \preceq \langle H(\chi), \in, \prec^*_\chi \rangle \) be countable and such that \( \mathbb{Q} \times T \) belongs to \( N, \) \( \delta := N \cap \aleph_1 \in S_1, \) and the Mostowski collapse of \( N \)
that belongs to $M_\delta$. Fix $(q_0, \eta_0) \in (Q \times T) \cap N$. It will be sufficient to find a condition $\eta_* \in T$ such that $\eta_0 \leq_T \eta_*$ and $(q_0, \eta_*)$ is $(N, Q \times T)$-generic.

Let

$$Q_\delta := \{ p \in Q : \{ \eta, \nu \} \in \text{dom}(p), \text{then } \eta, \nu \in T_\delta \}.$$ 

Thus, $Q_\delta$ is countable. Moreover, notice that $T_\delta = T \cap N$, and therefore $Q_\delta = Q \cap N$. Hence, $T_\delta$ and $Q_\delta$ are the Mostowski collapses of $T$ and $Q$, respectively, and so they belong to $M_\delta$.

In $M_\delta$, let $\langle (p_n, D_n) : n < \omega \rangle$ list all pairs $(p, D)$ such that $p \in Q_\delta$, and $D$ is a dense open subset of $Q_\delta \times T_\delta$ that belongs to the Mostowski collapse of $N$. That is, $D$ is the Mostowski collapse of a dense open subset of $Q \times T$ that belongs to $N$.

Also in $M_\delta$, fix an increasing sequence $\langle \delta_n : n < \omega \rangle$ converging to $\delta$, and let

$$D'_n := \{ (p, \nu) \in D_n : lh(\nu) > \delta_n \}.$$ 

Clearly, $D'_n$ is dense open.

Note that, as the Mostowski collapse of $N$ belongs to $M_\delta$, we have that $\langle <^* \rangle (Q_\delta \times T_\delta) = (\langle <^* \rangle (Q \times T)) \cap N \in M_\delta$.

Now, still in $M_\delta$, and starting with $(q_0, \eta_0)$, we inductively choose a sequence $\langle (q_n, \eta_n) : n < \omega \rangle$, with $q_n \in Q_\delta$ and $\eta_n \in T_\delta$, and such that if $n = m + 1$, then:

(a) $p_n \geq q_n$ and $\eta_n <_T \eta_n$.

(b) $(q_n, \eta_n) \in D'_n$.

(c) $(q_n, \eta_n)$ is the $<_^*$-least such that (a) and (b) hold.

Then, $\eta_* := \bigcup \{ \eta_n \cup \{ \delta, \delta \} \} \in T$, and $\eta_* \in M_\delta$, hence $(q_0, \eta_*) \in Q \times T$. Clearly, $(q_0, \eta_0) \leq (q_0, \eta_0)$. So, we only need to check that $(q_0, \eta_*)$ is $(N, Q \times T)$-generic.

Fix an open dense $E \subseteq Q \times T$ that belongs to $N$. We need to see that $E \cap N$ is predense below $(q_0, \eta_*)$. So, fix $(r, \nu) \leq (q_0, \eta_*)$. Since $Q_\delta$ is ccc, $q_0$ is $(N, Q)$-generic, so we can find $r' \in \{ p : (p, \eta) \in E \}$, some $\eta \cap N$ that is compatible with $r$. Let $n$ be such that $p_n = r'$ and $D_n$ is the Mostowski collapse of $E$. Then $(p_n, \eta_n)$ belongs to the transitive collapse of $E$, hence to $E \cap N$, and is compatible with $(r, \nu)$, as $(p_n, \eta_n) \leq (p_n, \eta_n)$.

We thus conclude that if $G \subseteq Q$ is a filter generic over $V$, then in $V[G]$ the forcing $T$ does not collapse $\aleph_1$, and therefore, being of cardinality $\aleph_1$, it preserves cardinals, cofinalities, and the cardinal arithmetic.

We shall now define the $Q$-names for the forcing notions $P_\ell$, for $\ell \in \{0, 1\}$, as follows: in $V^Q$, let $\beta = \bigcup \mathcal{G}$, where $\mathcal{G}$ is the standard $Q$-name for the $Q$-generic filter over $V$. Then let

$$\mathbb{P}_\ell := \{ (w, c) : w \subseteq T \text{ is finite, } c \text{ is a function from } w \text{ into } \omega \text{ such that if } \{ \eta, \nu \} \in [w]_T^2 \text{ and } \beta(\{ \eta, \nu \}) = \ell, \text{ then } c(\eta) \neq c(\nu) \}.$$ 

A condition $(w, c)$ is stronger than a condition $(v, d)$ if and only if $w \supseteq v$ and $c \supseteq d$.

We shall show that if $G$ is $Q$-generic over $V$, then in the extension $V[G]$, the partial orderings $\mathbb{P}_\ell = \mathbb{P}_{\ell}[G]$, for $\ell \in \{0, 1\}$, and $T$ are as required.

**Claim 11.** In $V[G]$, $\mathbb{P}_\ell$ has precalibre-$\aleph_1$. 

Proof of the claim. Assume \( p_\alpha = (w_\alpha, c_\alpha) \in \mathbb{P}_\ell \), for \( \alpha < \omega_1 \). We shall find an uncountable \( S \subseteq \mathbb{R}_1 \) such that \( \{p_\alpha : \alpha \in S\} \) is finite-wise compatible. For each \( \delta \in S_2 \), let

\[ s_\delta := \{ \eta | (\gamma + 1) : \eta \in w_\delta, \text{ and } \gamma \text{ is maximal such that } \gamma < lh(\eta) \wedge \eta(\gamma) < \delta \}. \]

As \( \eta \) is an increasing and continuous sequence of ordinals from \( S_1 \), hence disjoint from \( S_2 \), the set \( s_\delta \) is well-defined. Notice that \( s_\delta \) is a finite subset of \( T_\delta := \{ \eta | T : supRange(\eta) < \delta \} \), which is countable.

Let \( s_\delta^1 \subseteq s_\delta \). Note that \( s_\delta^1 \subseteq s_\delta \).

Let \( f : S_2 \rightarrow \omega_1 \) be given by \( f(\delta) = max\{supRange(\eta) : \eta \in s_\delta \} \). Thus, \( f \) is regressive, hence constant on a stationary \( S_3 \subseteq S_2 \). Let \( \delta_0 \) be the constant value of \( f \) on \( S_3 \). Then, \( s_\delta \in T_{\delta_0} \), for every \( \delta \in S_3 \). So, since \( T_{\delta_0} \) is countable, there exist \( S_1 \subseteq S_3 \) stationary and \( s_\alpha \) such that \( s_{\delta_0} = s_\alpha, \) for every \( \delta \in S_4 \).

Further, there is a stationary \( S_5 \subseteq S_3 \) and \( s_{\delta_0} \) such that for all \( \delta \in S_5 \),

\[ s_{\delta_0}^1 = s_{\alpha}, \quad c_\delta \upharpoonright s_{\alpha}^1 = c_\alpha, \quad \text{and } \forall \alpha < \delta(w_\alpha \subseteq T_{\delta_0}). \]

Hence, if \( \delta_1 < \delta_0 \) are from \( S_5 \), then not only \( w_{\delta_1} \cap w_{\delta_0} = s_{\alpha}^1 \), but also if \( \eta_1 \in w_{\delta_1} - w_{\delta_0} \) and \( \eta_2 \in w_{\delta_0} - w_{\delta_1} \), then \( \eta_1 \) and \( \eta_2 \) are \( \tau \)-incomparable: for suppose otherwise, say \( \eta_1 < \tau \eta_2 \). If \( \gamma + 1 = lh(\eta_1) \), then \( \eta_2 \upharpoonright (\gamma + 1) = \eta_1 < \tau \eta_2 \), and \( \eta_2(\gamma) = \eta_1(\gamma) < \delta_2 \), by choice of \( S_5 \). Hence, by the definition of \( s_{\delta_0} \), \( \eta_2 \upharpoonright (\gamma + 1) = \eta_1 \) is an initial segment of some pair of \( s_{\delta_0} = s_{\alpha} \), and so it belongs to \( T_{\delta_1} \), hence \( \eta_1 \in s_{\alpha}^1 \), contradicting the assumption that \( \eta_1 \notin s_{\alpha}^1 \).

So, \( \{p_\delta : \delta \in S_5 \} \) is as required.

It only remains to show that forcing with \( T \) over \( V[G] \) preserves the ccc-ness of \( \mathbb{P}_0 \) and \( \mathbb{P}_\ell \), but makes their product not ccc.

**Claim 12.** If \( G_T \) is \( T \)-generic over \( V[G] \), then in the generic extension \( V[G][G_T] \), the forcing \( \mathbb{P}_\ell \) is ccc.

**Proof of the claim.** First notice that, by the Product Lemma (see \([6], 15.9\)), \( G \) is \( \mathbb{Q} \)-generic over \( V[G_T] \), and \( V[G][G_T] = V[G_T][G] \). Now suppose \( A = \{(w_\alpha, c_\alpha) : \alpha < \omega_1 \} \in V[G_T] \) is a \( \mathbb{Q} \)-name for an uncountable subset of \( \mathbb{P}_\ell \). For each \( \alpha < \omega_1 \), let \( p_\alpha \in \mathbb{Q} \) and \( (w_\alpha, c_\alpha) \) be such that \( p_\alpha \Vdash \text{"}(w_\alpha, c_\alpha) = (w_\alpha, c_\alpha)\" \). Let \( u_\alpha \) be such that \( dom(p_\alpha) = [u_\alpha]^2 \). By extending \( p_\alpha \), if necessary, we may assume that \( w_\alpha \subseteq u_\alpha \), for all \( \alpha < \omega_1 \). We shall find \( \alpha \neq \beta \) and a condition \( p \) that extends both \( p_\alpha \) and \( p_\beta \) and forces that \( (w_\alpha, c_\alpha) \) and \( (w_\beta, c_\beta) \) are compatible. For this, first extend \( (w_\alpha, c_\alpha) \) to \( (u_\alpha, d_\alpha) \) by letting \( d_\alpha \) give different values in \( \omega \setminus Range(c_\alpha) \) to all \( \eta \in u_\alpha \setminus w_\alpha \). We may assume that the set \( \{u_\alpha : \alpha < \omega_1\} \) forms a \( \Delta \)-system with root \( r \). Moreover, we may assume that \( p_\alpha \) restricted to \( [r]^2 \), is the same for all \( \alpha < \omega_1 \), and also that \( d_\alpha \), restricted to \( r \) is the same for all \( \alpha < \omega_1 \). Now pick \( \alpha \neq \beta \) and let \( p : [u_\alpha \cup u_\beta]^2_T \rightarrow \{0, 1\} \) be such that \( p \upharpoonright [u_\alpha]^2_T = p_\alpha, \ p \upharpoonright [u_\beta]^2_T = p_\beta, \) and \( p(\{\eta, \nu\}) \neq \ell, \) for all other pairs in \( [u_\alpha \cup u_\beta]^2_T \). Then, \( p \) extends both \( p_\alpha \) and \( p_\beta \), and forces that \( (u_\alpha, d_\alpha) \) and \( (u_\beta, d_\beta) \) are compatible, hence it forces that \( (w_\alpha, c_\alpha) \) and \( (w_\beta, c_\beta) \) are compatible.

But in \( V[G][G_T] \), the product \( \mathbb{P}_0 \times \mathbb{P}_\ell \) is not ccc. For let \( \eta^* = \bigcup G_T \). For every \( \alpha < \omega_1 \), let \( p_\alpha^\ell := \{\eta^* \upharpoonright (\alpha + 1), c_\alpha^\ell \} \in \mathbb{P}_\ell \), where \( c_\alpha^\ell (\eta^* \upharpoonright (\alpha + 1)) = 0 \). Then the set \( \{p_\alpha^0, p_\alpha^\ell : \alpha < \omega_1\} \) is an uncountable antichain.
ON PARTIAL ORDERINGS HAVING PRECALIBRE-$\aleph_1$ ...  15

References


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