RIGIDITY OF CONTINUOUS QUOTIENTS

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ABSTRACT. We study countable saturation of metric reduced products and introduce continuous fields of metric structures indexed by locally compact, separable, completely metrizable spaces. Saturation of the reduced product depends both on the underlying index space and the model. By using the Gelfand–Naimark duality we conclude that the assertion "Stone–Čech remainder of the half-line has only trivial automorphisms" is independent from ZFC. Consistency of this statement follows from the Proper Forcing Axiom and this is the first known example of a connected space with this property.

The present paper has two largely independent parts moving in two opposite directions. The first part (§§1–4) uses model theory of metric structures and it is concerned with the degree of saturation of various reduced products. The second part (§5) uses set-theoretic methods and it is mostly concerned with rigidity of Stone–Čech[1] remainders of locally compact, Polish spaces. (A topological space is *Polish* if it is separable and completely metrizable.) The two parts are linked by the standard fact that saturated structures have many automorphisms (the continuous case of this fact is given in Theorem 3.1).

By βX we denote the Stone–Čech compactification of X and by X^* we denote its remainder (also called corona), $\beta X \setminus X$. A continuous map $\Phi\colon X^* \to Y^*$ is *trivial* if there are a compact subset K of X and a continuous map $f\colon X\setminus K \to Y$ such that $\Phi = \beta f \restriction X^*$, where $\beta f\colon \beta X \to \beta Y$ is the unique continuous extension of f. Continuum Hypothesis (CH) implies that all Stone–Čech remainders of locally compact, zero dimensional, non-compact Polish spaces are homeomorphic. This is a

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[1]Since in Czech alphabet letter 'č' precedes letter 's' some authors write Čech–Stone compactification instead of Stone–Čech compactification.

consequence of Parovičenko's theorem, see e.g., [31]. By using Stone duality, this follows from the fact that all atomless Boolean algebras are elementarily equivalent and the countable saturation of corresponding reduced products. The latter also hinges on the fortuitous fact that the theory of atomless Boolean algebras admits elimination of quantifiers. See also [8] where similar model-theoretic methods were applied to the lattice of closed subsets of a Stone–Čech remainder.

Gelfand–Naimark duality (see e.g., [3]) associates autohomeomorphisms of a compact Hausdorff space X to automorphisms of the C*-algebra $C(X)$ of continuous complex-valued functions on X. Logic of metric structures ([2]), or rather its version adapted to C*-algebras ([18]) is applied to analyze these algebras. The idea of defining autohomeomorphism of a compact Hausdorff space X indirectly via an automorphism of $C(X)$ dates back at least to the discussion in the introduction of [32].

The study of saturation properties of corona algebras was initiated in [16] where it was shown that all coronas of separable C*-algebras satisfy a restricted form of saturation, the so-called countable degree-1 saturation. Although it is not clear whether degree-1 saturation suffices to construct many automorphisms, even this restricted notion has interesting consequences ([16, Theorem 1], [5, Theorem 8], [37, §2]). We also note that most coronas are not $\aleph_2$-saturated provably in ZFC. For $C(\omega^*)$[2] this is a consequence of Hausdorff's construction of a gap in $\mathcal{P}(\omega)/\operatorname{Fin}$, and for the Calkin algebra this was proved in [39].

It was previously known that CH implies the existence of nontrivial autohomeomorphisms of the Stone–Čech remainder of $[0,1)$ (this is a result of Yu, see [24, §9]).

Definitions of types and countable saturation are reviewed in §1.

Theorem 1. *C*-algebra $C([0,1)^*)$ is countably saturated.*

This is a consequence of Theorem 2.5, where sufficient conditions for a quotient continuous field of models indexed by a Stone–Čech remainder of a locally compact Polish space to be countably saturated are provided. See also Proposition 2.1 where the necessity of some conditions of Theorem 2.5 was shown. We also prove countable saturation of reduced products of metric structures corresponding to the Fréchet ideal (Theorem 1.5) and so-called layered ideals (Theorem 2.7). More general metric reduced products are considered in §2.5 where a model-theoretic interpretation of a result of [27] is given.

[2]We use the notation commonly accepted in set theory and denote the least infinite ordinal (identified with the set of natural numbers including zero) by ω.

We note that Theorem 1 implies a strengthening of a result of Yu (see [24, §9]).

Corollary 2. *Continuum Hypothesis implies that* $C([0,1)^*)$ *has* $2^{\aleph_1}$ *automorphisms and that* $[0,1)^*$ *has* $2^{\aleph_1}$ *autohomeomorphisms. In particular it implies that* $[0,1)^*$ *has nontrivial automorphisms.*

Proof. By Theorem 1 and CH $C([0,1)^*)$ is saturated. By Theorem 3.1 it has $2^{\aleph_1}$ automorphisms. Gelfand–Naimark duality implies that $[0,1)^*$ has $2^{\aleph_1}$ automorphisms. Finally, CH implies that $2^{\aleph_0} < 2^{\aleph_1}$ and there are only $2^{\aleph_0}$ continuous functions from $[0,1)$ to itself. □

Let us now consider the situation in which quotient structures are maximally rigid. The first result in this direction was the second author's result that consistently with ZFC all autohomeomorphisms of ω^* are trivial ([33, §IV]). PFA implies that all homeomorphisms between Stone–Čech remainders of locally compact Polish spaces that are in addition countable or zero-dimensional are trivial ([11, §4.1], [11, Theorem 4.10.1], and [19]). The effect of the Proper Forcing Axiom (PFA) to quotient structures extends to the non-commutative context; see the discussion at the beginning of §2.1 as well as [15], [30] and [22]. All of these results, as well as the following theorem, appear to be instances of a hitherto unknown general result (see [14]).

Theorem 3 (PFA)**.** *Every autohomeomorphism of* $[0,1)^*$ *is trivial.*

We prove a more general result, Theorem 5.3, as a step towards proving that all Stone–Čech remainders of locally compact Polish spaces have only 'trivial' automorphisms assuming PFA. An inspection of its proof shows that it uses only consequences of PFA whose consistency does not require large cardinal axioms.

The proof of Theorem 3 introduces a novel technique. In all previously known cases rigidity of the remainder X^* was proved by representing it as an inverse limit of spaces homeomorphic to ω^* (see e.g., [11, §4]). This essentially applies even to the non-commutative case, where the algebras were always presented as direct limits of algebras with an abundance of projections. This approach clearly works only in the case when X is zero-dimensional (or, in the noncommutative case, when the C*-algebra has real rank zero) and our proof of Theorem 3 necessarily takes a different route.

Organization of the paper. In §1 we review conditions, types, and saturation of metric structures. In §1.2, , it is proved that reduced product with respect to the Fréchet ideal is always countably saturated. Proof of Theorem 1.5 proceeds via discretization of ranges of metric formulas given in §1.3 and is completed in §1.4. Models $C_b(X, A)$ and $C_0(X, A)$ are introduced in §2.1 and countable saturation of the corresponding quotients under additional assumptions is proved in Theorem 2.5, whose immediate consequence is Theorem 1. Proposition 2.1 provides some limiting examples. §5 is independent from the rest of the paper. In it we prove Theorem 3 by using set-theoretic methods. We conclude with some brief remarks in §6.

Notation. If $a \subseteq \mathrm{dom}(h)$ we write $h[a]$ for the pointwise image of a. An element a of a product $\prod_n A_n$ is always identified with the sequence $(a_n : n \in \omega)$; in particular indices in subscripts are usually used for this purpose. For a set A we denote its cardinality by $|A|$. In some of the literature (e.g., [8] or [24]) the half-line is denoted by $\mathbb{H}$. Since the same symbol is elsewhere used to denote the half-plane, we avoid using it. In our results about Stone–Čech remainders $[0, 1)$ can be everywhere replaced with $\mathbb{H}$. A subset D of a metric space is *ϵ-discrete* if $d(a, b) \geq \epsilon$ for all distinct a and b in D. We also follow [2] and write $x \dot{-} y$ for $\max(x - y, 0)$.

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1. Countable saturation

1.1. **Conditions, types and saturation.** A quick review of the necessary model-theoretic background is in order; see [2] and [23] for more details. Our motivation comes from study of saturation properties of C*-algebras ([18], [16]), but we prove novel results for general metric structures. Fix language $\mathcal{L}$ in the logic of metric structures whose variables are listed as $\{x_n : n \in \omega\}$. In the ensuing discussion we shall write $\bar{x}, \bar{y}, \bar{a}, \ldots$ to denote tuples of unspecified length and sort. In most interesting cases all entries of the tuple will belong to a single sort, such as the unit ball of the C*-algebra under the consideration, and we shall suppress discussion of sorts by assuming all variables are of the same sort.

For a metric formula $\phi(\bar{x})$, metric structure A of the same signature and tuple $\bar{a}$ in A of the appropriate sort, by $\phi(\bar{a})^A$ we denote the interpretation (i.e., evaluation) of $\phi(\bar{x})$ at $\bar{a}$ in structure A. A (closed) *condition* is an expression of the form $\phi(\bar{x}) = r$ for a formula $\phi(\bar{x})$ and a real number r. We consider conditions over a model A, in which case ϕ is allowed to have elements from A as parameters. Formally, we expand the language by adding constants for these elements; for details see [2] or [18, §2.4.1]. An *n-type* is a set of conditions all of whose free variables are included in the set $\{x_0, \dots, x_{n-1}\}$. We shall suppress n throughout and write x instead of x_0 if $n = 1$. In general, a *type* over a model A is a set of conditions with parameters from A. An n-type $\mathbf{t}(\bar{x})$ is *realized* in A if some n-tuple $\bar{a}$ in A we have that $\phi(\bar{a})^A = r$ for all conditions $\phi(\bar{x}) = r$ in $\mathbf{t}(\bar{x})$. A type is *consistent* (or *finitely approximately realizable* in the terminology of [16]) if every one of its finite subsets can be realized up to an arbitrarily small $\epsilon > 0$.

If κ is an infinite cardinal, we say that a model A is *κ-saturated* if every consistent type $\mathbf{t}$ over A with fewer than κ conditions is realized in A. If the density character of A is κ then A is *saturated*. Instead of *$\aleph_1$-saturated* we shall usually say *countably saturated*. Saturated models have remarkable properties. Every saturated model of density character κ has 2^κ automorphisms, and two saturated models of the same language and same character density are isomorphic if and only if they have the same theory (see any standard text on model theory, e.g., [4], [25] or [29]).

Following [16] one may consider restricted versions of saturation. If all consistent quantifier-free types of cardinality $< \kappa$ over a model are realized in it, the model is said to be *quantifier-free κ-saturated*. In case of C*-algebras a weaker notion of degree-1 countable saturation was considered in [16]. C*-algebra C is *countably degree-1 saturated* if every type consisting of conditions of the form $\|p\| = r$, where p is a sum of monomials of the form a, axb or ax^*b for a, b in C and variable x if and only if it is realizable in C. All coronas of separable C*-algebras have this property, and it is strong enough to imply many of the known properties of such coronas ([16]; see also [9]).

However, the existence of saturated models of unstable theories requires nontrivial assumptions on cardinal arithmetic, such as the Continuum Hypothesis (see [4, §6]). Nevertheless, in a situation where focus is on separable objects, countable saturation is sufficient. The fact that the ultrapowers as well as the relative commutants of separable subalgebras in ultrapowers are countably saturated (see e.g.,

[18]) is largely responsible for their usefulness in the study of separable C*-algebras. Saturation of all ultrapowers of a separable model of an unstable theory associated to a nonprincipal ultrafilter on ω is equivalent to the Continuum Hypothesis ([17]; see also [20] for a quantitative strenghtening).

Metric formula $\phi(\bar{y})$ is in *prenex normal form* if it is of the following form for some n and k ($\bar{x}$ stands for $(x(0), \dots, x(2n-1))$)

$$\mathrm{Q}_{x(0)}\,\mathrm{Q}_{x(1)} \dots \mathrm{Q}_{x(2n-2)}\,\mathrm{Q}_{x(2n-1)}\, f(\alpha_0(\bar{x},\bar{y}), \dots \alpha_{k-1}(\bar{x},\bar{y}))$$

where each Q stands for sup or inf, f is a continuous function and α_i for $i < k$ are atomic formulas.

Lemma 1.1. *Every type is equivalent to a type such that all formulas occurring in its conditions are in prenex normal form.*

Proof. This is an easy consequence of [2, Proposition 6.9], which states that every formula can be uniformly approximated by formulas in the prenex normal form and its proof. As pointed out in this proof, connectives $\dot{-}$ and $|\cdot|$ are monotonic in all of their arguments and therefore standard proof that a (discrete) formula is equivalent to one in prenex normal form applies to show that if ϕ and ψ are in prenex normal form then $\phi \dot{-} \psi$ and $|\psi|$ are equivalent to formulas in prenex normal form.

Fix a condition $\phi(\bar{x}) = r$. For every n fix a formula ϕ_n in prenex normal form such that

$$\sup_{\bar{x}} |\phi_n(\bar{x}) - \phi(\bar{x})| \leq 1/n.$$

The type consisting of conditions

$$|\psi(\bar{x}) \dot{-} r| \dot{-} 1/n = 0$$

for $n \geq 1$ is equivalent to condition $\phi(\bar{x}) = r$ and by the above formula $|\psi(\bar{x}) \dot{-} r| \dot{-} 1/n$ is equivalent to a formula in prenex normal form. □

1.2. **Reduced products over the Fréchet ideal.** Fix a language $\mathcal{L}$ in the logic of metric structures with a distinguished constant symbol 0. Assume A_n, for $n \in \omega$ are $\mathcal{L}$-structures. We form two $\mathcal{L}$-structures as follows (see also [28, §2] for more details)

$$\textstyle\prod_n A_n = \{\bar{x} : (\forall n) x_n \in A_n \text{ and all } x_n \text{ belong to the same domain}\},$$

$$\textstyle\bigoplus_n A_n = \{\bar{x} \in \prod_n A_n : (\forall n) x_n \in A_n \text{ and } \limsup_n d_n(x_n, 0^{A_n}) = 0\}.$$

We shall write

$$A_\infty := \textstyle\prod_i A_i.$$

Both structures are considered with respect to the sup metric, in which they are both complete. Interpretations of function symbols are defined in the natural way, pointwise. For a tuple $\bar{a} = (\bar{a}_i : i \in \omega)$ in one of these models and a function symbol f of the appropriate sort we have

$$f(\bar{a}) := (f(\bar{a}_n) : n \in \omega).$$

The interpretation of relational symbols is admittedly less canonical. If $\bar{a}$ is a tuple in one of the above models and R is a predicate symbol of the appropriate sort, then we define

$$R(\bar{a}) := \sup_n R(\bar{a}_n).$$

This agrees with our choice of the sup metric on the product space. It is also compatible with the convention adopted in the discrete logic, where $R(\bar{a})$ is true if and only if $R(\bar{a}_n)$ is true for all n.

Consider the quotient structure

$$\mathcal{A} = \prod_n A_n / \bigoplus_n A_n$$

(also denoted by $\prod_{\mathrm{Fin}} A_n$) with the quotient map

$$q\colon \prod_n A_n \to \prod_n A_n / \bigoplus_n A_n.$$

We shall use same notation for the natural extension of q to k-tuples for $k \geq 1$. The interpretation of an $\mathcal{L}$-sort in $\mathcal{A}$ is the q-image of the interpretation of this sort in $\prod_n A_n$. On this quotient structure define metric d via

$$d(q(\bar{a}), q(\bar{b})) = \limsup_n d(a_n, b_n).$$

For a predicate symbol R and $\bar{a}$ such that $q(\bar{a})$ is of the appropriate sort define

$$R(q(\bar{a})) = \limsup_n R(a_n),$$

and for a function symbol f and $\bar{a}$ such that $q(\bar{a})$ is of the appropriate sort define

$$f(q(\bar{a})) = q(f(\bar{a})).$$

Claim 1.2. *Interpretations of function and predicate symbols are well-defined and have the correct moduli of uniform continuity.*

Proof. Since all proofs are similar, we shall prove the claim only for function symbol f. Fix $\epsilon > 0$ and $\Delta(\epsilon) > 0$ such that each A_n satisfies

$$d(x, y) < \Delta(\epsilon) \Rightarrow d(f(x), f(y)) \leq \epsilon.$$

Fix $\bar{a}$ and $\bar{b}$ in $\prod_n A_n$ such that $d(q(\bar{a}), q(\bar{b})) < \Delta(\epsilon)$. Then the set

$$Z = \{n : d(a_n, b_n) \geq \Delta(\epsilon)\}$$

is finite and for all $n \in \omega \setminus Z$ we have $d(f(a_n), f(b_n)) \leq \epsilon$. Therefore $d(f(q(\bar{a}), q(\bar{b})) \leq \epsilon$. □

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We record two immediate consequences of definitions in order to furnish the intuition (see also [28, §2]).

Lemma 1.3. *If $\alpha(\bar{x})$ is an atomic formula and $\bar{a} \in \prod_n A_n$ is of the appropriate sort, then $\alpha(\bar{a})^{\prod_n A_n} = \sup_n \alpha(\bar{a}_n)^{A_n}$.* □

Lemma 1.4. *For a tuple $\bar{a}$ in A_∞ and a formula $\phi(\bar{x})$ of the appropriate sort we have*

$$\phi(q(\bar{a})) = \lim_m \lim_n \tilde{\phi}(\bar{a})^{A_{[m,n)}}.$$

(with $A_{[m,n)}$ denoting $\prod_{i=m}^{n-1} A_i$). □

In [6] the following result was alluded to as 'obviously true.' It may not be obvious, but at least it is true.

Theorem 1.5. *Every reduced product corresponding to the Fréchet ideal, $\prod_n A_n / \bigoplus_n A_n$, is countably saturated.*

The proof of Theorem 1.5 given in §1.4 is subtler than the proof of the corresponding result for the classical 'discrete' logic (see e.g., [25]), although it proceeds by 'discretizing' ranges of formulas (see §1.3).

Saturation of reduced products over ideals other than the Fréchet ideal will be considered in Theorem 2.7 below.

1.3. **Deconstructing formulas.** In this subsection we prepare the grounds for the proof of Theorem 1.5. For a bounded $D \subseteq \mathbb{R}$ and $n \geq 1$ let

$$F(D,n) = \{k2^{-n-1} : D \cap ((k-1)2^{-n-1}, (k+1)2^{-n-1}) \neq \emptyset\}.$$

Proof of the following lemma is straightforward.

Lemma 1.6. *Assume $D \subseteq \mathbb{R}$ is bounded and $n \in \omega$. Then*

(1) *$F(D,n)$ is finite*
(2) *every element of D is within 2^{-n} of an element of $F(D,n)$.*
(3) *$F(D,n+1)$ uniquely determines $F(D,n)$.*
(4) *If D is compact then*

$$\{E \subseteq \mathbb{R} : E \text{ is compact and } F(D,n) = F(E,n)\}$$

is an open neighbourhood of D in the Hausdorff metric. □

Let $\phi(\bar{y})$ be a formula in prenex normal form. By adding dummy variables if necessary, for some n and k we can represent $\phi(\bar{y})$ as

$$\text{(PNF)} \qquad \sup_{x(0)} \inf_{x(1)} \dots \sup_{x(2n-2)} \inf_{x(2n-1)} f(\alpha_0(\bar{x},\bar{y}), \dots \alpha_{k-1}(\bar{x},\bar{y}))$$

where f is a continuous function and α_i for $i < k$ are atomic formulas, possibly with parameters in a fixed model A.

If α is an atomic formula then we let $F(\alpha, n) := F(D, n)$ where D is the set of all possible values of α. Recall that D is always compact. This is true even if A is a C*-algebra and $\alpha(x)$ is a *-polynomial because the syntax requires variable x to range over a fixed bounded ball (see [18]).

If ϕ, n, k and α_i for $i < k$, are as in (PNF) and $\bar{a}$ is a tuple in A of the appropriate sort then we define *n-pattern of* $\phi(\bar{a})$ in A to be

$$\begin{aligned} P(\phi, \bar{a}, n)^A := \{\bar{r} \in \prod_{i=0}^{k-1} F(\alpha_i, n) : \\ (\forall x(0))(\exists x(1)) \dots (\forall x(2n-2))(\exists x(2n-1)) \\ \max_i |\alpha_i(\bar{x}, \bar{a})^A - r_i| < 2^{-n}\}. \end{aligned}$$

All $x(j)$ range over the relevant domain in A.

Assume $\phi_0(\bar{y}), \dots, \phi_{m-1}(\bar{y})$ are as in (PNF) with free variables $\bar{y}$ of the same sort and $\bar{a}$ is a tuple in A of the appropriate sort. Then we define the *n-pattern of $\phi_0(\bar{a}), \dots, \phi_{m-1}(\bar{a})$ in A* (or *the n-pattern of $\phi_0, \dots, \phi_{m-1}$ and $\bar{a}$ in A*) to be the set

$$\textstyle\prod_{i=0}^{m-1} P(\phi_i, \bar{a}, n)^A.$$

Lemma 1.7. *For all $n \geq 1$ and every language $\mathcal{L}$, a tuple of $\mathcal{L}$-formulas in prenex normal form has at most finitely many possible distinct n-patterns in all $\mathcal{L}$-structures.*

Proof. The range of every atomic formula is a compact subset of $\mathbb{R}$, and therefore every n-pattern of an m-tuple of formulas is a finite subset of

$$\prod_{i=0}^{m-1} \{k/n : |k| \leq K\}$$

for some fixed $K < \infty$. $\square$

Given a formula $\phi(\bar{x})$ as in (PNF), a tuple $\bar{a}$ of elements in $\prod_n A_n$ of the appropriate sort and $n \geq 1$, consider the set of all n-patterns of the form $P_i = P(\phi, \bar{a}_i, n)^{A_i}$. An n-pattern P in this set is *relevant* (for ϕ and $\bar{a}$ in $\prod_n A_n$) if it is equal to P_i for infinitely many i.

Lemma 1.8. *For every formula $\phi(\bar{x})$ in prenex normal form and every $\epsilon > 0$ there is m such that for every quotient structure of the form $\prod_n A_n / \bigoplus_n A_n$ and tuple $\bar{a} \in \prod_n A_n$ of the appropriate sort the value $\phi(q(\bar{a}))^{\mathcal{A}}$ is determined up to ϵ by the set of all relevant m-patterns for ϕ and $\bar{a}$ in $\prod_n A_n$.*

Proof. By adding dummy variables if necessary, we may assume that there exist natural numbers n and k and atomic formulas α_i, α_{k-1} such that $\phi'(\bar{x})$ is of the form

$$\sup_{x(0)} \inf_{x(1)} \dots \sup_{x(2n-2)} \inf_{x(2n-1)} f(\alpha_0(\bar{x},\bar{y}), \dots \alpha_{k-1}(\bar{x},\bar{y}))$$

Lemma 1.3 implies that

$$\alpha_n(\bar{x},\bar{a})^{A_\infty} = \sup_i \alpha_n(\bar{x}_i,\bar{a}_i)^{A_i}$$

and therefore $\phi'(\bar{a})^{A_\infty} \geq r$ if and only if for every $\delta > 0$ we have (writing $\bar{x} = (x(0), \dots, x(2n-1))$ and $\bar{x}_i = (x(0)_i, \dots, x(2n-1)_i)$ for $i \in \omega$)

$$\exists x(0) \forall x(1) \dots \exists x(2n-2) \forall x(2n-1)$$
$$f(\sup_i \alpha_0(\bar{x}_i,\bar{a}_i)^{A_i}, \dots \sup_i \alpha_{k-1}(\bar{x}_i,\bar{a}_i)^{A_i}) \geq r - \delta.$$

Hence $\phi'(q * \bar{a}))^{\mathcal{A}} \geq r$ if and only if for every $\delta > 0$ and every $l \in \omega$ we have

$$\exists x(0) \forall x(1) \dots \exists x(2n-2) \forall x(2n-1)$$
$$f(\sup_{i \geq l} \alpha_0(\bar{x}_i,\bar{a}_i)^{A_i}, \dots \sup_{i \geq l} \alpha_{k-1}(\bar{x}_i,\bar{a}_i)^{A_i}) \geq r - \delta.$$

Since ϕ is equipped with a fixed modulus of uniform continuity we can choose m such that changing the value of each α_i by no more than 2^{-m} does not affect the change of value of ϕ by more than $\epsilon/2$.

Therefore the value of $\phi(q(\bar{a}))^{\mathcal{A}}$ up to ϵ depends only on the set of relevant m-patterns for ϕ and $\bar{a}$ in $\prod_n A_n$. □

1.4. **Proof of Theorem 1.5.** Let $\mathbf{t} = \{\phi_i(\bar{y}) = r_i : i \in \omega\}$ be a type with parameters in $\mathcal{A}$ consistent with its theory. By Lemma 1.1 we may assume that all ϕ_i are given in prenex normal form.

By lifting parameters of $\phi_i(\bar{y})$ from the quotient $\mathcal{A}$ to A_∞ we obtain formulas $\tilde{\phi}_i(\bar{y})$, for $i \in \omega$. Since $\mathbf{t}$ is consistent, for every m we can choose $\bar{a}(m) \in A_\infty$ so that

$$|\phi_i(q(\bar{a}(m)))^{\mathcal{A}} - r_i| \leq 2^{-m}$$

for all $i \leq m$. Fix m for a moment and let

$$R(m,n) = \{P : P \text{ is a relevant } n\text{-pattern of } \tilde{\phi}_0, \dots, \tilde{\phi}_{m-1} \text{ and } \bar{a}(m) \text{ in } \textstyle\prod_i A_i\}.$$

Recursively choose increasing sequences $l(i)$ and $m(i)$ for $i \in \omega$ and infinite sets $\omega = X_0 \supseteq X_1 \supseteq \dots$ so that for every k and all m in X_k we have the following.

(1) $R(m,k) = R(m(k),k)$.

(2) The set of all patterns of $\tilde{\phi}_0(\bar{a}(m(k))_j), \dots, \tilde{\phi}_{n-1}(\bar{a}(m(k))_j)$ occurring in A_j for some $l(m) \leq j < l(m+1)$ is equal to $R(m,k)$.

Let us describe the construction of these objects. Assume that $k \in \omega$ is such that $l(i)$, $m(i)$ and X_i for $i \leq k$ are as required. Since there are only finitely many relevant patterns we can find an infinite $X_{k+1} \subseteq X_k$ and R such that $R(m, k+1) = R$ for all $m \in X_{k+1}$. Let $m(k+1) = \min(X_{k+1} \setminus k)$. Now increase $l(k)$ if necessary to assure that all patterns $P_j(m(k+1), k)$ for $j \geq l(k)$ are relevant. Finally, choose $l(k+1)$ large enough so that

$$\{P_j(m(k), k) : l(k) \leq j < l(k+1)\} = R(m(k), k).$$

(This is done with the understanding that $l(k+1)$ may have to be suitably increased once $m(k+2)$ is chosen and that this change is innocuous.)

Once all of these objects are chosen define $\bar{a} \in A_\infty$ via

$$\bar{a}_i = \bar{a}(m(k))_i, \text{ if } l(k) \leq i < l(k+1).$$

The salient property of $\bar{a}$ is that for every n the set of relevant n-patterns for $\tilde{\phi}_0(\bar{a}), \dots, \tilde{\phi}_{n-1}(\bar{a})$ is equal to $R(m(k), n)$ for all but finitely many k. We claim that $q(\bar{a})$ realizes type $\mathbf{t}$ in $\mathcal{A}$.

Fix i and $\epsilon > 0$. By Lemma 1.8 and the construction of $\bar{a}$, for all large enough m we have that $|\phi_i(q(\bar{a}))^{\mathcal{A}} - \phi_i(q(\bar{a}(m))^{\mathcal{A}}| < \epsilon$. Therefore for every i we have $\phi_i(q(\bar{a}))^{\mathcal{A}} = r_i$, and $q(\bar{a})$ indeed realizes $\mathbf{t}$.

2. Countable saturation of other reduced products

We extend Theorem 1.5 in two different directions. In §2.1 continuous fields of models are introduced with an eye on Theorem 1. In §2.5 we consider more traditional reduced products over arbitrary ideals on ω.

2.1. **Continuous reduced products.** We consider saturation of continuous reduced products of the form $\mathrm{C_b}(X, A)/C_0(X, A)$ (see below for the definitions), and the main result section is Theorem 2.5 that has Theorem 1 as a consequence.

By the Gelfand–Naimark duality (see e.g., [3]), the categories of compact Hausdorff spaces and unital abelian C*-algebras are equivalent. In particular, for a locally compact Polish space X there is a bijective correspondence between autohomeomorphisms of Stone–Čech remainder X^* and automorphisms of the quotient C*-algebra $\mathrm{C_b}(X)/C_0(X) \cong C(X^*)$. Here $\mathrm{C_b}(X)$ is the algebra of all bounded functions $f\colon X \to \mathbb{C}$ and $C_0(X)$ is its subalgebra of functions vanishing at infinity. Since $C_b(X)$ is naturally isomorphic to $C(\beta X)$, this quotient algebra is isomorphic to the *corona* (also called the *outer multiplier algebra*) of the algebra $C_0(X)$.

Hence results about autohomeomorphisms of Stone–Čech remainders X^* are special cases of results about automorphisms of coronas of separable, nonunital C*-algebras. A tentative and very inclusive definition of *trivial automorphisms* of the latter was given in [6, Definition 1.1]. In the present paper we shall be concerned with the abelian case only.

If A is a metric structure and X is a Hausdorff topological space consider the space

$$\begin{aligned} \mathrm{C_b}(X,A) = \{f\colon X \to A : f \text{ is continuous} \\ \text{and its range is included in a single domain of } A\}. \end{aligned}$$

Hence every $f \in \mathrm{C_b}(X,A)$ has a well-defined sort and domain (see [18, §2]) and we write $\mathcal{S}^{\mathrm{C_b}(X,A)} = \{f \in \mathrm{C_b}(X,A) : f \text{ is of sort } \mathcal{S}\}$. Equip $\mathrm{C_b}(X,A)$ with the metric

$$d(f,g) = \sup_{x\in X} d(f(x),g(x)),$$

and interpret predicate and function symbols as in $\prod_n A_n$ in §1.2. If the language of A has a distinguished constant $0_{\mathcal{S}}$ for every sort $\mathcal{S}$, consider a submodel of $\mathrm{C_b}(X,A)$ defined as follows ($\mathcal{S}$ ranges over all sorts in the language of A).

$$\begin{aligned} C_0(X,A) = \textstyle\bigcup_{\mathcal{S}}\{f \in \mathcal{S}^{\mathrm{C_b}(X,A)} :\ \text{function } x \mapsto d(f(x), 0^A_{\mathcal{S}}) \\ \text{vanishes at infinity}\}. \end{aligned}$$

When X is ω with the discrete topology, these models are isomorphic to $\prod_n A$ and $\bigoplus_n A$, respectively. In general $\mathrm{C_b}(X,A)$ is a submodel of $\prod_{t\in X} A$. In §2.1 we make a few remarks on general continuous fields of metric structures.

Consider the quotient structure $\mathrm{C_b}(X,A)/C_0(X,A)$ with the interpretations of predicate and function symbols as defined in §1.2. The question whether this quotient is countably saturated is quite sensitive to the choice of X and A. Before plunging into the main discussion we record a few relatively straightforward limiting facts.

Proposition 2.1. *Let X be a locally compact, non-compact Polish space and let A be a metric structure. Write $\mathcal{A} := \mathrm{C_b}(X,A)/C_0(X,A)$.*

(1) *If every connected component of X is compact then $\mathcal{A}$ is countably saturated.*
(2) *If X is connected and A is discrete then $\mathcal{A} \cong A$.*
(3) *If X is connected and A has a countably infinite definable discrete subset then $\mathcal{A}$ is not countably saturated.*

(4) *If X is connected and A is a separable, infinite-dimensional abelian C*-algebra with infinitely many projections then $\mathcal{A}$ is not countably saturated.*

Proof. (1) If $X = \bigoplus_n K_n$ and each K_n is compact, then with $A_n = \mathrm{C_b}(K_n, A)$ we have $\mathrm{C_b}(X, A)/C_0(X, A) \cong \prod_n A_n / \bigoplus_n A_n$ and the assertion follows from Theorem 1.5.

(2) If A is discrete and X is connected then $\mathrm{C_b}(X, A) \cong A$, $C_0(X, A)$ is a singleton, the quotient is isometrically isomorphic to A, and (3) is an immediate consequence of (2). In a C*-algebra the set of projections is definable by the formula $\|x^2 - x\| + \|x^* - x\|$. This is a consequence of the weak stability of these relations. Since two distinct commuting projections are at distance 1, this set is discrete. Hence (4) is a special case of (3). □

The following is a limiting example showing why one of the assumptions of Theorem 2.5 below is needed.

Theorem 2.2. *There exists a one-dimensional, connected space X such that $\mathcal{A}$ is not countably saturated for any unital C*-algebra A.*

Proof. With $Y = \{0\} \times [0, \infty)$ and $X_n = \{n\} \times [0, \infty)$ for $n \in \omega$ let $X = Y \cup \bigcup_n X_n$ with the subspace topology inherited from $\mathbb{R}^2$.

We give a proof in the case when $A = \mathbb{C}$. The general case is a straightforward extension of the argument by using continuous functional calculus.

Define $a_n \in \mathrm{C_b}(X, A)$ for $n \in \omega$ so that $a_n(x) = 1$ if $x \in X_n$, $a_n(x) = 0$ if $x \in X_j$ for $j \neq n$, $a_n(x) = 0$ on $\{0\} \times [n + 1, \infty)$, and on $\{0\} \times [0, n - 1]$ (with $[0, k] = \emptyset$ if $k < 0$), and linear on intervals $\{0\} \times [n - 1, n]$ and $\{0\} \times [n, n + 1]$.

Although X is connected and $\mathrm{C_b}(X, A)$ is projectionless we have that $q(a_n)$, for $n \in \omega$, are orthogonal projections in $\mathrm{C_b}(X, A)/C_0(X, A)$.

Claim 2.3. *For every projection p in $\mathrm{C_b}(X, A)/C_0(X, A)$ there exists a finite $F \subseteq \omega$ such that $p = \sum_{n\in F} q(a_n)$ or $p = 1 - \sum_{n\in F} q(a_n)$.*

Proof. Let $a \in \mathrm{C_b}(X, A)$ be such that $q(a) = p$. By replacing a with $(a^* + a)/2$ and truncating its spectrum by using continuous functional calculus we may assume a maps X into $[0, 1]$. Fix $\epsilon < 1/2$. Since $q(a)$ is a projection, the set

$$Z = \{x \in X : \epsilon < a(x) < 1 - \epsilon\}$$

is compact. Hence there exists a compact K so that on every $X_n \setminus K$ and on $Y \setminus K$ all values of a lie outside of Z. If $G_0 = a^{-1}([0, \epsilon])$ and $G_1 = a^{-1}([1 - \epsilon, 1])$ then by the continuity of a one of $Y \setminus G_0$ or $Y \setminus G_1$ is compact.

Assume for a moment that $Y \setminus G_0$ is compact. By the continuity of a there exists k such that $X_n \setminus G_0$ is compact for all $n \geq k$. With $F = \{n : X_n \setminus G_1 \text{ is compact}\}$ letting $\epsilon \to 0$ one easily checks that $p = \sum_{n \in F} q(a_n)$.

Otherwise, if $Y \setminus G_1$ is compact, a similar argument shows that $F = \{n : X_n \setminus G_0 \text{ is compact}\}$ is a finite set and $p = 1 - \sum_{n \in F} q(a_n)$. □

Consider type $\mathbf{t}(x)$ consisting of conditions $\|x^* - x\| = 0$, $\|x^2 - x\| = 0$, $\|x\| = 1$, and $xq(a_n) = 0$ for all n. Its realization would be a nonzero projection orthogonal to $q(a_n)$ for all n. By Claim, there is no such projection. □

Theorem 2.2 provides a countably degree-1 saturated abelian C*-algebra such that the Boolean algebra of its projections is isomorphic to the subalgebra of $\mathcal{P}(\omega)$ generated by finite subsets, and therefore not countably saturated.

In [16, Lemma 2.1] the following was proved for all $a < b$ in $\mathbb{R}$ and every countably degree-1 saturated algebra C. Every contable type such that each of its finite subsets is approximately satisfiable in C by a self-adjoint element with spectrum included in $[a, b]$ is realized in C by a self-adjoint element with spectrum included in $[a, b]$. The following implies that this cannot be extended to projections, answering a question raised on p. 53 of [16].

Corollary 2.4. *There are a countably degree-1 saturated C*-algebra C and a countable 1-type $\mathbf{t}$ over C such that every finite subset of $\mathbf{t}$ is realizable by a projection in C, but $\mathbf{t}$ is not realizable by a projection in C.*

Proof. Let C be $\mathrm{C_b}(X)/C_0(X)$, with X as in Theorem 2.2 and let $\mathbf{t}$ be as defined there. □

We should also remark that [16, Question 5.3], intended as a test for the question answered in Corollary 2.4, trivially has a negative answer in a projectionless algebra such as $C([0, 1)^*)$.

The Calkin algebra also fails to be quantifier-free countably saturated ([16, §4]), although all coronas of separable C*-algebras are countably degree-1 saturated ([16, Theorem 1], see also [37] and [9]). No example of an algebra which is quantifier-free countably saturated but not countably saturated is presently known.

Recall that the topological boundary of a subset K of a topological space X is $\partial K := \overline{K} \cap \overline{X \setminus K}$.

Theorem 2.5. *Assume X is a locally compact Polish space which can be written as an increasing union of compact subspaces, $X = \bigcup_n K_n$,* *such that*

$$\sup_n |\partial K_n| < \infty.$$

Then for any metric model A such that each domain of A is compact and locally connected the quotient $\mathrm{C_b}(X,A)/C_0(X,A)$ is countably saturated.

Proof of this theorem is given below in §2.3. Theorem 2.2 shows that the assumption on ∂K_n in Theorem 2.5 cannot be relaxed. The assumption that the compact sets K_n can be chosen to satisfy $\sup_n |\partial K_n| < \infty$ implies that for every $\epsilon > 0$ and every sort in $\mathcal{L}$ whose interpretation is a compact subset of A there exists N such that every ϵ-discrete subset of this sort in $A^{\partial K_n}$ has cardinality $\leq N$.

Proof of Theorem 1. Apply Theorem 2.5 to $\mathrm{C_b}([0,1),\mathbb{C})/C_0([0,1),\mathbb{C})$. □

2.2. **Relevant patterns.** The notion of relevant pattern from §1.3 is modified to present context in the natural way. For a tuple of formulas $\phi_0(\bar{x}),\dots,\phi_{k-1}(\bar{x})$ in prenex normal form and $\bar{a}$ in $\mathrm{C_b}(X,A)$ of the appropriate sort let P_t denote the pattern of $\phi_0(\bar{x}),\dots,\phi_{k-1}(\bar{x})$ and a_t in A (where a_t denotes the value of a at t). An n-pattern P is *relevant* (for $\tilde{\phi}_j(\bar{a})$, $0 \leq j < k$, in $\mathrm{C_b}(X,A)$) if closure of the set $\{t : P = P_t\}$ is not included in X. Equivalently, an n-pattern P is relevant (for $\tilde{\phi}_j(\bar{a})$, $0 \leq j < k$, in $\mathrm{C_b}(X,A)$) if for every k there exists $t \in X \setminus K_k$ such that $P = P_t(m,n)$.

The following is analogue of Lemma 1.8 for $\mathrm{C_b}(X,A)/C_0(X,A)$. Its proof is very similar to the proof of Lemma 1.8.

Lemma 2.6. *For every formula $\phi(\bar{x})$ and $\epsilon > 0$ there is m such that for every quotient $C_b(X,A)/C_0(X,A)$ and tuple $\bar{a} \in C_b(X,A)$ of the appropriate sort the value of $\phi(q(\bar{a}))$ in $C_b(X,A)/C_0(X,A)$ is determined up to ϵ by the set of all relevant m-patterns for $\phi(\bar{x})$ and $\bar{a}$ in* $\mathrm{C_b}(X,A)$. □

2.3. **Proof of Theorem 2.5.** We roughly follow the proof of Theorem 1.5.

Let $\mathbf{t} = \{\phi_i(\bar{y}) = r_i : i \in \omega\}$ be a type over $\mathrm{C_b}(X,A)/C_0(X,A)$ consistent with its theory. By Lemma 1.1 we may assume that all ϕ_i are given in prenex normal form. Lift parameters of $\phi_i(\bar{y})$ to $\mathrm{C_b}(X,A)$ to obtain formulas $\tilde{\phi}_i(\bar{y})$. For every m choose $\bar{a}(m) \in \mathrm{C_b}(X,A)$ so that

$$|\phi_i(q(\bar{a}(m)))^{\mathrm{C_b}(X,A)/C_0(X,A)} - r_i| \leq 2^{-m}$$

for all $i \leq m$.

Let $R(m,n)$ denote the set of relevant n-patterns for $\tilde{\phi}_j(\bar{a}(m))$, $0 \leq j < m$ in $\prod_n A_n$.

Main difference between the present proof and the proof of Theorem 1.5 is that when 'cutting and pasting' partial realizations of $\mathbf{t}$ we need to assure continuity of the resulting element $\bar{a}$.

For every k consider the set $A^{\partial K_k}$ with respect to the sup metric d. These sets are compact, and since cardinalities of ∂K_k, for $k \in \omega$, are uniformly bounded for every $\epsilon > 0$ there exists $N_\epsilon < \infty$ such that no $A^{\partial K_k}$ has an ϵ-discrete subset of cardinality $> N_\epsilon$. For the simplicity of notation for every k we introduce a pseudometric on $C_b(X,A)$ by

$$d_k(a,b) = d(a\restriction\partial K_k, b\restriction\partial K_k).$$

Recursively choose a sequence of infinite sets $\omega = Y_0 \supseteq Y_1 \supseteq \dots$, so that for every n and all m_1 and m_2 in Y_n we have the following.

(1) $R(m_1,n) = R(m_2,n)$.
(2) $\{k : d_k(a(m_1), a(m_2)) \leq 1/n\}$ is infinite.

Assume that Y_{n-1} has been chosen to satisfy the conditions. Consider the set of unordered pairs of elements of Y_{n-1},

$$[Y_{n-1}]^2 = \{\{m,m'\} \subseteq Y_i : m \neq m'\}$$

and define partition $c\colon [Y_{n-1}]^2 \to \{0,1\}$ by $c(\{m_1,m_2\}) = 0$ if (2) holds and $c(\{m_1,m_2\}) = 1$ otherwise.

Assume for a moment that there exists an infinite $Z \subseteq Y_{n-1}$ such that $c(\{m,m'\}) = 1$ for all $\{m,m'\} \in [Z]^2$. With $N_{1/n}$ for which there are no $1/n$-discrete subsets of $C_b(X,A)$ of cardinality $N_{1/n}$ for any d_k, consider the least $N_{1/n}$ elements of Z, listed as m_i, for $1 \leq i \leq N_{1/n}$. For all $i < j$ and every large enough k we have $d_k(a(m_i), a(m_j)) > 1/n$. Therefore $a(m_i)$, for $1 \leq i \leq N_{1/n}$, form a $1/n$-discrete set for some k, contradicting the choice of $N_{1/n}$.

By Ramsey's theorem there exists an infinite $Y \subseteq Y_{n-1}$ such that $c(\{m_1,m_2\}) = 0$ for all $\{m_1,m_2\} \in [Y]^2$. Therefore Y_{n-1} has an infinite subset for which (2) holds. Since there are only finitely many relevant patterns we can find an infinite subset Y_n of Y satisfying (1) and proceed.

This describes the recursive construction. Now we follow construction from the proof of Theorem 1.5 and recursively choose increasing sequences $k(i)$ and $m(i) \in Y_i$ for $i \in \omega$ so that for every n we have the following.

(3) The set of all patterns of $\tilde{\phi}_0(\bar{a}(m(n))_j), \dots, \tilde{\phi}_{n-1}(\bar{a}(m(n))_j)$ occurring in A_t for some $t \in K_{k(n+1)} \setminus K_{k(n)}$ is equal to $R(m(n),n)$.
(4) $d_{k(n+1)}(a(m(n)), a(m(n+1))) \leq 1/(n+1)$.

Assume $m(i)$ and $k(i)$ for $i \leq n$ were chosen so that (3) and (4) hold and moreover all patterns of $\tilde{\phi}_0(\bar{a}(m(n))_j), \dots, \tilde{\phi}_{n-1}(\bar{a}(m(n))_j)$ occurring in A_t for $t \notin K_{k(n)}$ are relevant. Pick the least $m(n+1) \in Y_{n+1}$ greater than $m(n)$. Since there are infinitely many k such that $d_k(a(m(n)), a(m(n+1))) \leq 1/(n+1)$, we can choose large enough $k(n+1)$ so that both (3) and (4) are satisfied and all patterns of $\tilde{\phi}_0(\bar{a}(m(n+1))_j), \dots, \tilde{\phi}_{n-1}(\bar{a}(m(n+1))_j)$ occurring in A_t for $t \notin K_{k(n+1)}$ are relevant.

Once these objects are chosen one is tempted to define $\bar{a} \in \mathrm{C_b}(X, A)$ via

$$\bar{a}_t = \bar{a}(m(i))_t, \text{ if } t \in K_{k(i+1)} \setminus K_{k(i)}.$$

However, for this function the map $t \mapsto d(\bar{a}_t, 0^{\mathrm{C_b}(X,A)})$ is not necessarily continuous and therefore $\bar{a}$ may not be in $\mathrm{C_b}(X, A)$. Nevertheless, $a_{m(i)}$ and $a_{m(i+1)}$ differ by at most $1/(i+1)$ on $\partial K_{k(i+1)}$ and we proceed as follows.

Fix n. Let ϵ_n be such that $d(\bar{x}, \bar{y}) < \epsilon_n$ implies $\max_{i \leq n} |\phi_i(\bar{x}) - \phi_i(\bar{y})| < 1/n$. By using finiteness of $\partial K_{k(n+1)}$ fix an open neighbourhood U_n of $\partial K_{k(n+1)}$ such that both $\bar{a}_{m(n)}$ and $\bar{a}_{m(n+1)}$ vary by at most ϵ_n on U_n. Also assure that $U_n \subseteq K_{k(n+2)}$ and $U_m \cap U_n = \emptyset$ if $m \neq n$.

By Tietze's extension theorem recursively choose $h_n \colon X \to [0, 1]$ such that for all m and n we have the following (here 1_Z denotes the characteristic function of $Z \subseteq X$)

(5) $0 \leq h_n \leq 1$,
(6) $h_{n-1} + h_n + h_{n+1} \geq 1_{K(k(n)} - 1_{K_{k(n-1)}}$,
(7) $h_n + h_{n+1} \geq 1_{\partial K_{k(n)}}$,
(8) $\sum_n h_n = 1_X$,
(9) $\operatorname{supp} h_m \cap \operatorname{supp} h_n = \emptyset$ if $|m - n| \geq 2$,
(10) $\operatorname{supp} h_n \cap \operatorname{supp} h_{n+1} \subseteq U_n$.

Then

$$\bar{a}_t = \textstyle\sum_n h_n(t) \bar{a}(m(n))_t$$

is an element of $\mathrm{C_b}(X, A)$ which agrees with $a_{m(n)}$ on $K_{k(n+1)} \setminus K_{k(n)}$ up to ϵ_n for all n. Moreover, the set of relevant n-patterns of $\bar{a}$ is equal to $R(m(i), n)$ for infinitely many i. As before, this implies that $\bar{a}$ realizes type $\mathbf{t}$.

This concludes the proof of Theorem 2.5.

2.4. **Continuous fields of models.** In addition to models $\mathrm{C_b}(X, A)$ and $C_0(X, A)$, one can consider the following submodel of the former.

$$C_c(X, A) = \{f \colon X \to A : f \text{ is continuous, its range is included in a single domain of } A \text{, and it has compact closure.}\}.$$

This smaller model has the property that $C_c(X, A) \cong \mathrm{C_b}(X, A)$ if X is compact and $C_c(X, A) \cong \mathrm{C_b}(\beta X, A)$ otherwise.

A possible definition of a *continuous field of models* A_t, for $t \in X$, can be obtained by fixing a sufficiently saturated model (commonly called 'monster model') M, requiring all fibers A_t to be submodels of M and considering the model consisting of all continuous $a \colon Y \to M$ such that $a_t \in A_t$ for all t.

However, one motivating example is given by continuous fields of C*-algebras and in this case one does not expect all fibers to be elementarily equivalent. It is not difficult to modify the definition in [3, IV.6.1] to the general context of continuous fields of metric structures of an arbitrary signature. At the moment we do not have an application for this notion, but as model-theoretic methods are gaining prominence in the theory of operator algebras this situation is likely to change. On a related note, sheaves of metric structures were defined in [28].

2.5. **Reduced products over ideals other than the Fréchet ideal.** Reduced products over a nontrivial proper ideal are countably saturated in both extremal cases: when the ideal is maximal (i.e., when the quotient is an ultrapower) and when we have the Fréchet ideal (Theorem 1.5). We consider some of the intermediate cases.

Assume A_n, for $n \in \omega$, are $\mathcal{L}$-structures as in Theorem 1.5 and that the language has a distinguished symbol $0^{\mathcal{S}}$ for every $\mathcal{L}$-sort $\mathcal{S}$. For an ideal $\mathcal{I}$ on ω define (here $\mathcal{S}$ ranges over all sots in $\mathcal{L}$)

$$\bigoplus_{\mathcal{I}} A_n := \bigcup_{\mathcal{S}} \{\bar{x} \in \mathcal{S}^{\prod_n A_n} : (\forall \epsilon > 0)(\exists X \in \mathcal{I}) \sup_{n \notin X} d(x_n, 0_{\mathcal{S}}) \leq \epsilon\}.$$

Therefore if $\mathcal{I}$ is the Fréchet ideal then $\bigoplus_{\mathcal{I}}$ is the standard direct sum. We shall show that for many ideals $\mathcal{I}$ quotient structure $\prod_n A_n / \bigoplus_{\mathcal{I}} A_n$ is countably saturated, analogously to the situation in the first-order logic. Following [13, Definition 6.5] we say that an ideal $\mathcal{I}$ on ω is *layered* if there is $f \colon \mathcal{P}(\omega) \to [0, \infty]$ such that

(1) $A \subseteq B$ implies $f(A) \leq f(B)$,
(2) $\mathcal{I} = \{A : f(A) < \infty\}$,
(3) $f(A) = \infty$ implies $f(A) = \sup_{B \subseteq A} f(B)$.

The following extends [13, Lemma 6.7].

Theorem 2.7. *Every reduced product $\prod_n A_n / \bigoplus_{\mathcal{I}} A_n$ over a layered ideal is countably saturated.*

Proof. We follow a similar route as in the proof of Theorem 1.5. Assume $\mathbf{t} = \{\phi_i(\bar{x}) = r_i : i \in \omega\}$ is a type and $\bar{a}(m)$ is such that $|\phi_i(q(\bar{a}(m)))^{\mathcal{A}} - r_i| \leq 1/m$ for all $i \leq m$. By Lemma 1.1 we may assume that all ϕ_i are given in prenex normal form.

Lift parameters of ϕ_i, for $i \in \omega$, to $\prod_n A_n$ to obtain formulas $\tilde{\phi}_i$, for $i \in \omega$. Let $P_i(m,n)$ be the n-pattern of $\tilde{\phi}_j(\bar{a}(m))$, for $0 \leq j < n$, in A_i. An n-pattern P of $\tilde{\phi}_j(\bar{a}(m))$, for $0 \leq j < n$, is $\mathcal{I}$-*relevant* (for $\tilde{\phi}_0, \dots, \tilde{\phi}_{m-1}, \bar{a}(m)$) if the set

$$\{i : P = P_i(m,n)\}$$

is $\mathcal{I}$-positive. Let

$$R(m,n) = \{P : P \text{ is an } \mathcal{I}\text{-relevant pattern for } \tilde{\phi}_0, \dots, \tilde{\phi}_{m-1}, \bar{a}(m) \text{ in } \textstyle\prod_n A_n\}.$$

Thus patterns relevant in the sense of §1.3 are Fin-relevant.

The analogue of Lemma 1.8, that for every ϕ and every $\epsilon > 0$ there exists n such that the value of ϕ in $\prod_n A_n / \bigoplus_{\mathcal{I}} A_n$ depends only on relevant n-patterns, is easily checked.

Since $\mathcal{I}$ includes the Fréchet ideal, we can choose sets $\emptyset = Y_0 \subseteq Y_1 \subseteq$ in $\mathcal{I}$, sets $\omega = X_0 \supseteq X_1 \supseteq \dots$ and a sequence $m(i)$, for $i \in \omega$, such that $\bigcup_n Y_n = \omega$ and moreover we have the following for all n.

(1) $R(m,n) = R(m(n),n)$ for all $m \in X_n$.
(2) The set of n-patterns of $\tilde{\phi}_0(\bar{a}(m)_j), \dots, \tilde{\phi}_{n-1}(\bar{a}(m)_j)$ in A_j for $j \in Y_m \setminus Y_{m-1}$ is equal to $R(m,n)$.

Recursive construction of these sequences is analogous to one in the proof of Theorem 1.5. As before, one concatenates $\bar{a}(m(i))$ into $\bar{a}$ and checks that $q(\bar{a})$ realizes $\mathbf{t}$. It is important to note that for every n all n-patterns of $\tilde{\phi}_0, \dots, \tilde{\phi}_{m-1}, \bar{a}$ in A_j for $j \in \omega \setminus Y_n$ coincide. The analogue of Lemma 1.8 stated above then implies that $\bar{a}$ realizes $\mathbf{t}$. $\square$

All F_σ ideals are layered (essentially [27]), but there are Borel layered ideals of an arbitrarily high complexity ([13, Proposition 6.6]). Examples of non-layered ideals are the ideal of asymptotic density zero sets, $\mathcal{Z}_0 = \{X \subseteq \omega : \lim_n |X \cap n|/n = 0\}$ and the ideal of logarithmic density zero sets, $\mathcal{Z}_{\log} = \{X \subseteq \omega : \lim_n (\sum_{k \in X \cap n} 1/k)/\log k = 0\}$. Their quotient Boolean algebras are not countably quantifier-free saturated since any strictly decreasing sequence of positive sets whose upper densities converge to zero has no nonzero lower bound. However, a fairly technical construction of an isomorphism between $\mathcal{P}(\omega)/\mathcal{Z}_0$ and $\mathcal{P}(\omega)/\mathcal{Z}_{\log}$ in [27] (cf. also [13, §5]) using the Continuum Hypothesis resembles a back-and-forth construction of an isomorphism between elementarily equivalent countably saturated structures. Logic of metric structures puts this apparently technical result into the correct context.

A map $\phi\colon \mathcal{P}(I) \to [0,\infty)$ is a *submeasure* if it is subadditive, monotonic and satisfies $\phi(\emptyset) = 0$. If $\omega = \sqcup_n I_n$ is a partition of ω into finite intervals and ϕ_n is a submeasure on I_n for every n, then the ideal

$$\mathcal{Z}_\phi = \{A \subseteq \omega : \limsup_n \phi_n(A \cap I_n) = 0\}$$

is a *generalized density ideal* (see [13, §2.10]). Ideals $\mathcal{Z}_0$, $\mathcal{Z}_{\log}$, and all of the so-called *EU-ideals* ([27]) are (generalized) density ideals ([11, Theorem 1.13.3]).

On the other hand, generalized density ideals are a special case of ideals of the form

$$\mathrm{Exh}(\phi) = \{A \subseteq \omega : \limsup_n \phi(A \setminus n) = 0\}$$

where ϕ is a lower semicontinuous submeasure on ω (take $\phi(A) = \sup_n \phi_n(A \cap I_n)$). All such ideals are $F_{\sigma\delta}$ P-ideals and by Solecki's theorem ([35]), every analytic P-ideal is of this form for some ϕ.

For such ideal quotient Boolean algebra $\mathcal{P}(\omega)/\mathrm{Exh}(\phi)$ is equipped with the complete metric (q denotes the quotient map)

$$d_\phi(q(A), q(B)) = \liminf_n \phi((A \Delta B) \setminus n).$$

See [11, Lemma 1.3.3] for a proof.

Proposition 2.8. *If* $\mathrm{Exh}(\phi)$ *is a generalized density ideal then the quotient* $\mathcal{P}(\omega)/\mathrm{Exh}(\phi)$ *with respect to* d_ϕ *is a countably saturated metric Boolean algebra.*

Proof. Let I_n and ϕ_n be as in the definition of generalized density ideals. Letting A_n be the metric Boolean algebra $(\mathcal{P}(I_n), d_n)$ where $d_n(X, Y) = \phi_n(X \Delta Y)$. We have that $\mathcal{P}(\omega)/\mathrm{Exh}(\phi)$ is isomorphic to $\prod_n A_n / \bigoplus_n A_n$ by [13, Lemma 5.1]. By Theorem 1.5, this is a countably saturated metric structure. □

By Proposition 2.8 the main result of [27] is equivalent to the assertion that the quotients over $\mathcal{Z}_0$ and $\mathcal{Z}_{\log}$ equipped with the canonical metric are elementarily equivalent in logic of metric structures. The latter assertion is more elementary and absolute between transitive models of ZFC. We note that it cannot be proved in ZFC that the quotients over $\mathcal{Z}_0$ and $\mathcal{Z}_{\log}$ are isomorphic ([26], see also [11] and [14]).

Proposition 2.8 and the ensuing discussion beg several questions. Can one describe theories of quotients over analytic P-ideals in the logic of metric structures? How does the theory depend on the choice of the metric? Are those quotients countably saturated whenever the ideal includes the Fréchet ideal? A positive answer would imply that the Continuum Hypothesis implies that a quotient Boolean algebra over a nontrivial analytic P-ideal has $2^{\aleph_1}$ automorphisms, complementing the main result of [11, §3] and partially answering a question of

Juris Steprāns. A related problem is to extend the Feferman–Vaught theorem ([21]) to reduced products of metric structures. The discretization method from §1.3 should be relevant to this problem. Question of the existence of a nontrivial Borel ideal with a rigid quotient Boolean algebra will be treated in an upcoming paper.

3. Automorphisms

Proof of the following theorem is due to Bradd Hart and it is included with his kind permission.

Theorem 3.1. *Assume A is a κ-saturated structure of density character κ. Then A has 2^κ automorphisms.*

Proof. Enumerate a dense subset of A as a_γ for $\gamma < \kappa$. We follow von Neumann's convention and write 2 for $\{0,1\}$, consider $2^{<\kappa} = \bigcup_{\gamma<\kappa} 2^\gamma$ and write $\mathrm{len}(s) = \gamma$ if $s \in 2^\gamma$.

We construct families f_s and A_s for $s \in 2^{<\kappa}$ such that the following holds for all s.

(1) A_s is an elementary submodel of A of density character $< \kappa$ including $\{a_\gamma : \gamma < \mathrm{len}(s)\}$.
(2) $f_s \in \mathrm{Aut}(A_s)$.
(3) If $s \sqsubseteq t$ then $A_s \prec A_t$ and $f_t \restriction A_s = f_s$.
(4) $A_{s^\frown 0} = A_{s^\frown 1}$ but $f_{s^\frown 0} \neq f_{s^\frown 1}$.

The only nontrivial step in the recursive construction is to assure (4). Fix γ such that A_s and f_s for all $s \in 2^\gamma$ have been chosen and satisfy the above requirements. Fix one of these s. Choose a_ξ with the least index ξ which is not in A_s and let $\epsilon = \mathrm{dist}(a_\xi, A_s)$. If $\mathbf{t}(x)$ is the type of a_ξ over A_s, then let $\mathbf{s}(x,y)$ be the 2-type $\mathbf{t}(x) \cup \mathbf{t}(y) \cup \{d(x,y) \geq \epsilon\}$. Since A_s is an elementary submodel of A, every finite subtype of $\mathbf{t}(x)$ is realized in A_γ and therefore $\mathbf{s}(x,y)$ is consistent. By the saturation of A it is realized by a pair of elements b_1, b_2 in A. Again by saturation (and the smallness of A_s) we can extend f_s to an automorphism of A that sends b_1 to b_2. Call this automorphism g_1. Let g_0 be an automorphism of A that extends f_s and sends b_1 to itself. By a Löwenheim–Skolem argument we can now choose an elementary submodel $A_{s^\frown 0}$ of A which includes $A_s \cup \{a_\xi, b_1\}$ and is closed under both g_0 and g_1. Let $A_{s^\frown 1} = A_{s^\frown 1}$. Then $g_0 \restriction A_{s^\frown 0} \neq g_1 \restriction A_{s^\frown 0}$ and $f_{s^\frown i} = g_i \restriction A_{s^\frown i}$ for $i \in \{0,1\}$ are as required.

This describes the recursive construction. For every $s \in 2^\kappa$ we have that $A = \bigcup_{\gamma<\kappa} A_{s\restriction\gamma}$ and $f_s := \bigcup_{\gamma<\kappa} f_{s\restriction\kappa}$ for $s \in 2^\kappa$ are distinct automorphisms of A. $\square$

In the above proof it may be possible to assure that $A_s = A_t$ whenever $\mathrm{len}(s) = \mathrm{len}(t)$. This is not completely obvious since the model $A_{s^\frown 0} = A_{s^\frown 1}$ defined in the course of the proof depends on automorphisms g_0 and g_1, and therefore on s.

Proof of Theorem 1. By Theorem 1 C*-algebra $C([0,1)^*)$ is countably saturated. Therefore by Theorem 3.1 the Continuum Hypothesis implies that it has $2^{\aleph_1}$ automorphisms. By the Gelfand–Naimark duality, each of these automorphisms corresponds to a distinct autohomeomorphism of $[0,1)^*$. □

4. The asymptotic sequence algebra

If A_n, for $n \in \omega$ is a sequence of C*-algebras then the reduced product $\prod_n A_n / \bigoplus_n A_n$ is the *asymptotic sequence algebra.* If $A_n = A$ for all n then we write $\ell_\infty(A)/c_0(A)$ for $\prod_n A / \bigoplus_n A$. Also, algebra A is identified with its diagonal image in $\ell_\infty(A)/c_0(A)$ and one considers the relative commutant

$$A' \cap \ell_\infty(A)/c_0(A) = \{b \in \ell_\infty(A)/c_0(A) : ab = ba \text{ for all } a \in A\}.$$

This is the *central sequence algebra.* The following corollary provides an explanation of why the asymptotic sequence C*- algebras and the central sequence C*-algebras are almost as useful for the analysis of separable C*-algebras as the ultrapowers and the corresponding relative commutants.

Corollary 4.1. *If A is a separable C*-algebra then the asymptotic sequence algebra $\ell_\infty(A)/c_0(A)$ is countably saturated and the corresponding central sequence algebra $A' \cap \ell_\infty(A)/c_0(A)$ is countably quantifier-free saturated.*

The Continuum Hypothesis implies that each of these algebras has $2^{\aleph_1}$ automorphisms and that $\ell_\infty(A)/c_0(A)$ is isomorphic to its ultrapower associated with a nonprincipal ultrafilter on ω.

Proof. Assume A is a nontrivial separable C*-algebra. The asymptotic sequence algebra $\ell_\infty(A)/c_0(A)$ is countably saturated by Theorem 1.5. Countable quantifier-free saturation of the central sequence algebra $A' \cap \ell_\infty(A)/c_0(A)$ follows by [16, Lemma 2.4]. By Theorem 3.1 Continuum Hypothesis implies that the asymptotic sequence algebra has $2^{\aleph_1}$ automorphisms. A diagonal argument shows that for a separable $B \subseteq \ell_\infty(A)/c_0(A)$ the relative commutant $B' \cap \ell_\infty(A)/c_0(A)$ is nonseparable. Then the proof of Theorem 3.1 shows that one can construct $2^{\aleph_1}$ of its automorphisms that pointwise fix A and differ on its relative commutant, and therefore the central sequence algebra has $2^{\aleph_1}$ automorphisms.

Finally, if $\mathcal{U}$ is a nonprincipal ultrafilter then the ultrapower of $\ell_\infty(A)/c_0(A)$ is elementarily equivalent to itself and of cardinality $2^{\aleph_0}$. Since elementarily equivalent saturated structures of the same density character are isomorphic, this concludes the proof. □

We record a (very likely well-known) corollary.

Proposition 4.2. *For unital separable C*-algebras A and B and every nonprincipal ultrafilter $\mathcal{U}$ on ω the following are equivalent.*

(1) *there exists a unital *-homomorphisms of B into the ultrapower $A^{\mathcal{U}}$.*
(2) *There exists a unital *-homomorphism of B into $\ell_\infty(A)/c_0(A)$.*

Proof. Recall that for a countably saturated structure C and a separable structure B of the same language we have that B embeds into C if and only if the *existential theory* of B,

$$\mathrm{Th}_\exists(B) = \{\inf_{\bar{x}} \phi(\bar{x}) : \inf_{\bar{x}} \phi(\bar{x})^B = 0 \text{ and } \phi \text{ is quantifier-free}\}$$

is included in $\mathrm{Th}_\exists(C)$ ([18]). Both $A^{\mathcal{U}}$ and $\ell_\infty(A)/c_0(A)$ are countably saturated. Since A and $A^{\mathcal{U}}$ have the same theory by Łos's theorem and A and $\ell_\infty(A)/c_0(A)$ have the same existential theory by [28] the conclusion follows. □

A tentative definition of a trivial automorphism of a corona of a separable C*-algebra was given in [6]. Every inner automorphism is trivial and there can be at most $2^{\aleph_0}$ trivial automorphisms; this is all information that we need for the following corollary.

Corollary 4.3. *The assertion that all automorphisms of the algebra $\prod_n M_n(\mathbb{C})/\bigoplus_n M_n(\mathbb{C})$ are trivial is independent from ZFC.*

Proof. Relative consistency of this assertion was proved in [22]. Continuum Hypothesis implies that the algebra has $2^{\aleph_1}$ automorphisms by Theorem 1.5 and Theorem 3.1. □

5. The influence of forcing axioms

In this section, if f is a function and X is a subset of its domain we write

$$f[X] = \{f(x) : x \in X\}..$$

Let X be a topological space. Let $\mathcal{F}_X$ and $\mathcal{K}_X$ denote the lattice of closed subsets of X and its ideal of compact sets, respectively. For $F \in \mathcal{F}_X$ let

$$F^* = \overline{F} \setminus X,$$

where closure is taken in βX. Fix a homeomorphism $\Phi\colon X^* \to Y^*$. If there is a function $\Psi\colon \mathcal{F}_X \to \mathcal{F}_Y$ such that $\Phi[F^*] = \Psi(F)^*$ for all $F \in \mathcal{F}_X$ then we say that Ψ is a *representation* of Φ.

This is a very weak assumption since Φ is just an arbitrary function with this property. Nevertheless, Continuum Hypothesis implies that there exists a homeomorphism between Stone–Čech remainders of locally compact Polish spaces with no representation (see §5.2).

If Φ is trivial and $f\colon \beta X \to \beta Y$ is its continuous extension then $\Psi(F) = f(F)$ defines a representation of Ψ. Since points of βX are maximal filters of closed subsets of X, Ψ is uniquely determined by its representation.

We shall show (Theorem 5.3) that PFA implies every homeomorphism Φ between Stone–Čech remainders of locally compact Polish spaces such that both Φ and its inverse have a representation is trivial.

In the remainder of this section we work in ZFC and prove the following.

Lemma 5.1. *Every autohomeomorphism of* $[0,1)^*$ *has a representation.*

In the following lemma we say that a closed subset a of $[0,1)$ is *nontrivial* if neither a nor $[0,1) \setminus a$ is relatively compact.

Lemma 5.2. *Assume* F *is a closed subset of* $[0,1)^*$.

(1) F *includes* a^* *for some nontrivial closed* $a \subseteq [0,1)$ *if and only if* $[0,1)^* \setminus F$ *is disconnected.*
(2) $F = a^*$ *for a nontrivial* $a \subseteq [0,1)$ *if and only if* $[0,1)^* \setminus W$ *is disconnected for every nonempty relatively open* $W \subseteq F$.

Proof. Let us write $X = [0,1)$. (1) If $F \subseteq X^*$ is a nontrivial closed set then its complement is a union of infinitely many nonempty disjoint open sets. Pick a sequence V_n, for $n \in \omega$, of these sets such that $\lim_n \min V_n = 1$. Now let $W_1 = \bigcup_n V_{2n}$ and $W_2 = X \setminus (F \cup W_1)$. Then W_2 is open, being a union of open intervals, and neither W_1 nor W_2 is included in a compact subset of $[0,1)$. Hence $X^* \setminus F$ is partitioned into two disjoint open sets corresponding to W_1 and W_2.

Now we show the converse implication. Let U and V be disjoint open subsets of X^* such that $U \cup V = X^* \setminus F$. If $\tilde{U}$ and $\tilde{V}$ are open subsets of βX such that $U = \tilde{U} \cap X^*$ and $V = \tilde{V} \cap X^*$ then $\tilde{U} \cap \tilde{V}$ is included in a compact set, and we can assume it is empty. Since X is connected $a = X \setminus (\tilde{U} \cup \tilde{V})$ is nonempty and moreover it is not compact. Since both U and V are nonempty the complement of a is not included in a compact set. Therefore a is a nontrivial closed set such that $a^* \subseteq F$.

(2) By (1), only the converse implication requires a proof. Let $a = X \setminus (\tilde{U} \cap \tilde{V})$ be the nontrivial closed subset of X constructed in (1) such that $a^* \subseteq F$. We claim that $a^* = F$. Assume, for the sake of obtaining a contradiction, that $W = F \setminus a^*$ is nonempty. By the assumption, $X^* \setminus W$ is disconnected. By compactness let $G \subseteq W$ be a closed set such that $X^* \setminus G$ is disconnected. By (1) fix a nontrivial closed $b \subseteq X$ such that $b^* \subseteq G$. Then $a^* \cap b^*$ is empty and therefore $a \cap b$ is compact. Therefore b has noncompact intersection with $\tilde{U} \cup \tilde{V}$ used in (1), and therefore b^* has a nonempty intersection with $U \cup V$. But this is absurd since $U \cap V$ is disjoint from F. □

Proof of Lemma 5.1. Let $\Phi\colon [0,1)^* \to [0,1)^*$ be a homeomorphism. Fix a closed $a \subseteq [0,1)$. We need to find closed $b \subseteq [0,1)$ such that $\Phi(a^*) = b^*$. If $a^* = [0,1)^*$ or $a^* = \emptyset$ then this is easy. Otherwise, if both a^* and $[0,1)^* \setminus a^*$ are nonempty, then Lemma 5.2 implies that $[0,1)^* \setminus W$ is disconnected for every nonempty relatively open $W \subseteq a^*$. Therefore $[0,1)^* \setminus V$ is disconnected for every nonempty relatively open $V \subseteq \Phi(a^*)$ and by Lemma 5.2 we have a closed $b \subseteq [0,1)$ such that $b^* = \Phi(a^*)$. □

The following result together with Lemma 5.1 implies Theorem 3.

Theorem 5.3 (PFA)**.** *If X and Y are locally compact, separable, metrizable spaces then every homeomorphism $\Phi\colon X^* \to Y^*$ that has a representation is trivial.*

5.1. **Proof of Theorem 5.3.** Fix X, Y and Φ as in Theorem 5.3. Fix compact sets K_n for $n \in \omega$ such that $X = \bigcup_n K_n$ and K_n is included in the interior of K_{n+1} for all n. For a topological space Z let

$$\mathcal{D}(Z) = \{a \subseteq Z : a \text{ is infinite, closed, and discrete}\}.$$

We start with a straightforward application of Martin's Axiom (write $a \subseteq^* b$ if $a \setminus b$ is finite).

Lemma 5.4 (MA$_\kappa$)**.** *If $X_0 \subseteq X$ is countable. and $A \subseteq \mathcal{D}(X_0)$ has cardinality κ then there exists $a \in \mathcal{D}(X_0)$ such that $b \subseteq^* a$ for all $b \in A$.*

Proof. Let $\mathcal{A}$ be a subset of $\mathcal{D}(X_0)$ of cardinality κ. Define poset $\mathbb{P}$ as follows. It has conditions of the form (s, k, A) where $k \in \omega$, $s \subseteq K_k$ is finite, and $A \subseteq \mathcal{A}$ is finite. We let (s, k, A) extend (t, l, B) if $s = t \cup (\bigcup B \cap (K_k \setminus K_l))$. Conditions with the same working part (s, k) are clearly compatible. Since X_0 is countable $\mathbb{P}$ is σ-centered. For every $a \in \mathcal{A}$ the set of all (s, k, A) such that $a \in A$ is dense and for every

$n \in \omega$ the set of (s, k, A) such that $k \geq n$ is dense. If G is a filter intersecting these dense sets then $a_G = \bigcup_{(s,k,A)\in G} s$ is as required. $\square$

Lemma 5.5 (PFA)**.** *Assume $\Phi\colon X^* \to Y^*$ is as in Theorem 5.3. If $a \in \mathcal{D}(X)$ then there is a map $h_a\colon a \to Y$ such that $\Phi(x) = (\beta h_a)(x)$ for every $x \in a^*$ and $b = h_a[a]$ is in $\mathcal{D}(Y)$. Equivalently, for every $b \subseteq a$ we have*

$$\Phi(b^*) = h_a[b]^*.$$

Proof. Fix $a \in \mathcal{D}(X)$. Since Φ has a representation, there is $b_0 \subseteq Y$ such that $\Phi(a^*) = b_0^*$. By PFA and the main result of [7] (see also [11, §4]), b_0 is homeomorphic to a direct sum of ω and a compact set. By removing this compact set we obtain $b \subseteq b_0$ in $\mathcal{D}(Y)$ such that $\Phi(a^*) = b^*$. By [34] or [36], the restriction of Φ to every $a \in \mathcal{D}(X)$ is trivial and we obtain the required map $h_a\colon a \to Y$. $\square$

By Lemma 5.5 applied with the roles of X and Y reversed for $b \in \mathcal{D}(Y)$ we obtain $g_b\colon b \to X$ such that

$$\Phi^{-1}(y) = \beta g_b(y)$$

for every $y \in b^*$.

For $a \in \mathcal{D}(X)$, a subset of a is compact if and only if it is finite. We note the following immediate consequences.

(A) If a and a' are in $\mathcal{D}(X)$ then $h_a(x) \neq h_{a'}(x)$ for at most finitely many $x \in a \cap a'$.

(B) If $a \in \mathcal{D}(X)$ and $b = h_a[a]$, then $(g_b \circ h_a)(x) \neq x$ for at most finitely many $x \in a$.

Lemma 5.6 (PFA)**.** *Assume X, Y and Φ are as in Theorem 5.3. If X_0 is a countable subset of X with non-compact closure then there exists $h^{X_0}\colon X_0 \to Y$ such that $\Phi(a^*) = h^{X_0}[a]^*$ for every $a \in \mathcal{D}(X_0)$.*

Proof. Let $Y \cup \{\infty\}$ denote the one-point compactification of Y. For $n \in \omega$ define a partition of unordered pairs in $\mathcal{D}(X_0)$ by

$$\{a, b\} \in K_0^n \text{ iff } (\exists x \in (a \cap b) \setminus K_n) h_a(x) \neq h_b(x).$$

Identify $a \in \mathcal{D}(X_0)$ with a function $\tilde{h}_a\colon X_0 \to Y \cup \{\infty\}$ that extends h_a and sends $X_0 \setminus a$ to ∞ and equip $(Y \cup \{\infty\})^{X_0}$ with the product topology. Since $Y \cup \{\infty\}$ is compact and metrizable, with this identification each K_0^n is an open partition. For distinct α and β in 2^ω we denote the least n such that $\alpha(n) \neq \beta(n)$ by $\Delta(\alpha, \beta)$. By PFA and the main result of [10, §3] one of the two following possibilities 5.1.1 or 5.1.2 (corresponding to (b') and (a) of [10, §3], respectively) applies.

5.1.1. *There is $\mathcal{Z} \subseteq 2^\omega$ of cardinality $\aleph_1$ and a continuous injection $\eta\colon \mathcal{Z} \to \mathcal{D}(X_0)$ such that $\{\eta(\alpha), \eta(\beta)\} \in K_0^{\Delta(\alpha,\beta)}$ for all distinct α and β in $\mathcal{Z}$.* We shall prove that this alternative leads to contradiction. Since $\bigcup \eta[Z] \subseteq X_0$ and X_0 is countable, by Lemma 5.4 we can find $a \in \mathcal{D}(X_0)$ such that $\eta(\alpha) \subseteq^* a$ for all $\alpha \in \mathcal{Z}$.

Fix for a moment $\alpha \in \mathcal{Z}$. Then for all but finitely many $y \in \eta(\alpha)$ we have $h_{\eta(\alpha)}(y) = h_a(y)$. By a counting argument we can find $\bar{m} \in \omega$ and an uncountable $\mathcal{Z}_0 \subseteq \mathcal{Z}$ such that for every $\alpha \in \mathcal{Z}_0$ we have $\eta(\alpha) \setminus K_{\bar{m}} \subseteq a$ and for all $y \in \eta(\alpha) \setminus K_{\bar{m}}$ we have $h_{\eta(\alpha)}(y) = h_a(y)$.

Pick α and β in $\mathcal{Z}_0$ such that $\Delta(\alpha, \beta) > \bar{m}$. Then $\{\eta(\alpha), \eta(\beta)\} \in K_0^{\bar{m}}$ and there exists $y \in (\eta(\alpha) \cap \eta(\beta)) \setminus K_{\bar{m}}$ such that $h_{\eta(\alpha)}(y) \neq h_{\eta(\beta)}(y)$, contradicting the fact that both functions agree with h_a past $K_{\bar{m}}$.

5.1.2. *There are $\mathcal{Y}_n$, for $n \in \omega$, such that $\mathcal{D}(X_0) = \bigcup_n \mathcal{Y}_n$ and $[\mathcal{Y}_n]^2 \cap K_0^n = \emptyset$ for all n.* Consider $\mathcal{D}(X_0)$ as a partial ordering with respect to $\subseteq^*$. By Lemma 5.4, every countable subset of $\mathcal{D}(X_0)$ is bounded. Therefore there exists $\bar{n}$ such that $\mathcal{Y}_{\bar{n}}$ is $\subseteq^*$-cofinal in $\mathcal{D}(X_0)$ (see e.g., [11, Lemma 2.2.2 (b)]). For each $a \in \mathcal{Y}_{\bar{n}}$ let

$$\tilde{h}_a = h_a \restriction (X \setminus K_{\bar{n}}).$$

Then $h = \bigcup\{\tilde{h}_a : a \in \mathcal{Y}_{\bar{n}}\}$ is a function since $\{a, b\} \notin K_0^{\bar{n}}$ for all distinct a and b in $\mathcal{Y}_{\bar{n}}$. Then for every $a \in \mathcal{D}(X_0)$ there exists $b \in \mathcal{Y}_{\bar{n}}$ such that $a \subseteq^* b$. We therefore have $a \subseteq^* \mathrm{dom}(h)$ and $h_a(x) = h(x)$ for all but finitely many $x \in a$. This in particular implies that with h^{X_0} as guaranteed by Lemma 5.6 we have $\Phi(a^*) = h^{X_0}[a]^*$ for every $a \in \mathcal{D}(X_0)$. □

Lemma 5.7 (PFA). *Assume X, Y and Φ are as in Theorem 5.3. If X_0 is a countable subset of X with non-compact closure then there exist countable $Y_0 \subseteq Y$, m, and a homeomorphism $h\colon X_0 \setminus K_m \to Y_0$ such that $\Phi(a^*) = h[a]^*$ for every $a \in \mathcal{D}(X_0)$.*

Proof. First apply Lemma 5.6 to X_0 and obtain h^{X_0}. Since the assumptions on X, Y and Φ are symmetric, we can apply Lemma 5.6 with the roles of X and Y reversed and $Y_0 = h^{X_0}[X_0]$ in place of X_0. We obtain a function g from a co-compact subset of Y_0 into X such that for every $a \in \mathcal{D}(Y_0)$ the domain of g includes a modulo finite and $g_a(y) = g(y)$ for all but finitely many $y \in a$. Now define a new function h to be the function whose graph is the intersection of the graphs of h^{X_0} and g^{-1}. That is, $h^{X_0}(x) = y$ if $h(x) = y$ and $g(y) = x$, and undefined otherwise.

We claim that $X_0 \setminus \mathrm{dom}(h)$ is compact. Otherwise there exists $a \in \mathcal{D}(X_0)$ disjoint from $\mathrm{dom}(h)$, but this contradicts the choice of h^{X_0} and g.

We claim that for $a \in \mathcal{D}(X_0)$ we have $h[a] \in \mathcal{D}(Y)$. Since a^* is nonempty, $h[a]$ is not compact. It therefore suffices to show that it has no infinite subset whose closure is included in Y. But if $b \subseteq h[a]$ were infinite and such that $b^* = \emptyset$, then $a_1 = h^{-1}(b)$ would be a non-compact subset of a, contradicting $\Phi(a_1^*) = b^*$.

By removing compact sets from X_0 and Y_0 we may assume that $\mathrm{dom}(h) = X_0$ and $\mathrm{range}(h) = Y_0$.

Claim 5.8. *There is $\bar{k}$ such that the restriction of h to $X_0 \setminus K_{\bar{k}}$ is continuous and the restriction of h^{-1} to $Y_0 \setminus L_{\bar{k}}$ is continuous.*

Proof. Assume that the restriction of h to $X_0 \setminus K_k$ is discontinuous for all k. For every n choose a sequence $\{x_{n,i}\}_i$ in $X_0 \cap (K_{n+2} \setminus K_n)$ converging to x_n such that $\lim_i h(x_{n,i}) \neq h(x_n)$.

Since both h and h^{-1} send relatively compact sets to relatively compact sets and the interior of K_{n+1} includes K_n for all n, for every m there exists n such that $h[\mathrm{dom}(h) \cap K_m] \subseteq L_n$ and $h^{-1}[L_m] \subseteq K_n$. We can therefore go to a subsequence $n(j)$, for $j \in \omega$, such that (with $n(0) = 0$) for $j \geq 1$ and all i we have

$$h(x_{n(j),i}) \in L_{n(j+1)} \setminus L_{n(j-1)} \qquad \text{and } h(x_{n(j)}) \in L_{n(j+1)} \setminus L_{n(j-1)}.$$

The only accumulation points of the set

$$c = \{x_{n(j)}, x_{n(j),i} : j \geq 1, i \in \omega\}$$

are $x_{n(j)}$, for $j \geq 1$. Since $x_{n(j)}$, for $j \geq 1$, form a closed discrete set c is homeomorphic to ω^2 equipped with its ordinal topology. The proof can now be completed by applying the weak Extension Principle (wEP) of [11, §4], but we give an elementary and self-contained proof.

Fix a nonprincipal ultrafilter $\mathcal{U}$ on ω. Then $y_j = \lim_{i \to \mathcal{U}} h(x_{n(j),i})$ exists and is in $L_{n(j+1)}$ by compactness. Since each $L_{n(j+1)}$ is second countable, by a diagonal argument we can choose a sequence $i(k)$, for $k \in \omega$, such that

$$\lim_k h(x_{n(j),i(k)}) = y_j$$

for all j. Since y_j and $h(x_{n(j)})$ are distinct elements of $L_{n(j+1)} \setminus L_{n(j-1)}$ for all j, we can find disjoint open subsets U and V of Y such that $h(x_{n(j)}) \in U$ and $y_j \in V$ for all j. By going to subsequences again and re-enumerating we can assume that $h(x_{n(j),i(k)}) \in U$ for all j, k. If W and S are disjoint open subsets of X such that $\Phi(W^*) = U^*$ and $\Phi(S^*) = V$ then we have that $x_{n(j)} \in W$ for all but finitely many j but $x_{n(j),i(k)} \notin S$ for every j and all but finitely many k—a contradiction. $\square$

By the above argument, for every countable $X_0 \subseteq X$ we can find n and a continuous function $h^{X_0}\colon X_0 \setminus K_n \to Y$ such that βh^{X_0} agrees with Φ on X_0^*. If $X_0 \subseteq X_1$ then h^{X_1} extends $h^{X_0}\restriction(X_0 \setminus K_n)$ for a large enough n. $\square$

Lemma 5.9. *Assume X, Y and Φ are as in the assumption of Theorem 5.3 and $\Phi_1\colon X^* \to Y^*$ is a trivial homeomorphism such that Φ^{-1} and Φ_1^{-1} agree on sets of the form a^* for $a \in \mathcal{D}(Y)$. Then $\Phi = \Phi_1$.*

Proof. Fix a representation Ψ of Φ and a homeomorphism $h\colon X \setminus K \to Y \setminus L$ between co-compact subsets of X and Y such that $(\beta h)[F^*] = \Phi_1(F^*)$ for all $F \in \mathcal{F}_X$. Assume $\Phi \neq \Phi_1$. Then for some $F \in \mathcal{F}_X$ we have that $\Psi(F)\Delta h[F]$ is not compact. We can therefore find $a \in \mathcal{D}(Y)$ such that (i) $a \subseteq \Psi(F) \setminus h[F]$ or (ii) $a \subseteq h[F] \setminus \Psi(F)$.

In either case we have that $\Phi^{-1}(a^*) \cap \Phi_1^{-1}(a^*) = \emptyset$, contradicting our assumption. $\square$

Proof of Theorem 5.3. Fix a countable dense set $X_0 \subseteq X$ and apply Lemma 5.7 to obtain h^{X_0}. Let $\tilde{h}$ be the maximal continuous extension of h^{X_0} to a G_δ subset X_1 of X. We claim that $X_1 \supseteq X_0 \setminus K_n$ for some n. Otherwise, find $a \in \mathcal{D}(X)$ disjoint from X_1 and apply Lemma 5.7 to $X_2 = X_0 \cup a$. The resulting continuous function h^{X_2} agrees with h^{X_0} on $X_0 \setminus K_n$ for a large enough n. Since K_n is included in the interior of K_{n+1}, the restriction of h^{X_2} to $\operatorname{dom}(h^{X_2} \setminus K_{n+1})$ is compatible with $\tilde{h}$ contradicting the assumption that h^{X_0} cannot be continuously extended to the points in a.

Therefore the domain of $\tilde{h}$ contains $X \setminus K_n$ for a large enough n. The analogous argument shows that the range of $\tilde{h}$ includes $Y \setminus K_m$ for a large enough m, and that $\tilde{h}$ is a homeomorphism. The restriction of the map $\beta\tilde{h}$ to X^* is a homeomorphism between X^* and Y^*, and by Lemma 5.9 this trivial homeomorphism coincides with Φ. $\square$

An autohomeomorphism Φ of $(X^*)^\kappa$ is *trivial* if there are a permutation σ of κ and autohomeomorphisms f_ξ, $\xi < \kappa$, of $[0, 1)$ such that $\Phi(x)(\xi) = \beta f_\xi(x(\sigma(\xi))$ for every $x \in (X^*)^\kappa$. Since $[0, 1)^*$ is connected, the following is an immediate consequence of Theorem 3 and the main result of [12].

Corollary 5.10 (PFA). *For an arbitrary cardinal κ, all autohomeomorphisms of $([0, 1)^*)^\kappa$ are trivial.* $\square$

5.2. **A homeomorphism without a representation.** It is now time to give an example promised in the beginning of §5. By Parovičenko's theorem, CH implies that $(\omega^2)^*$ (where ω^2 is taken with respect to the ordinal topology) and ω^* are homeomorphic. However, a homeomorphism $\Phi\colon (\omega^2)^* \to \omega^*$ does not have a representation. If $a \subseteq \omega^2$ is the set of limit ordinals below ω^2, then a^* is closed and nowhere dense. Therefore $\Phi(a^*)$ is a closed nowhere dense subset of ω^*. Since for every $b \subseteq \omega$ the set b^* is clopen, $\Phi(a^*) \neq b^*$ for all b.

On the other hand, Φ^{-1} has a representation. As a matter of fact, whenever $\Psi\colon \omega^* \to X^*$ is a homeomorphism then Ψ has a representation. In order to show this it suffices to prove that if $F \subseteq X^*$ is clopen then $F = b^*$ for closed $b \subseteq X$. But if U and V are open subsets of βX such that $U \cap X^* = F$ and $X^* \setminus V = F$, then clearly $b = X \setminus V$ satisfies $b^* = F$.

6. Concluding remarks

The motivation for this work comes from [6, Conjecture 1.2 and Conjecture 1.3]. We restate the abelian case of these conjectures in its dual form.

Conjecture 6.1 (PFA)**.** *Every homeomorphism between Stone–Čech remainders of locally compact Polish spaces X and Y is trivial.*

Since every trivial homeomorphism has a representation, by Theorem 5.3 this is equivalent to conjecture that under PFA every homeomorphism $\Phi\colon X^* \to Y^*$ between remainders of locally compact, non-compact, Polish spaces X and Y has a representation.

Conjecture 6.2. *Continuum Hypothesis implies that X^* has $2^{\aleph_1}$ non-trivial autohomeomorphisms for every locally compact, non-compact, separable metrizable space X.*

By Proposition 2.1 (5), $C(X^*)$ is not countably quantifier-free saturated for some locally compact Polish spaces X. However, the space constructed there includes a copy of $[0,1)$ as a clopen subset and therefore $C(X^*)$ has at least as many automorphisms as $C([0,1)^*)$. Large families of automorphisms of coronas that are not countably saturated were constructed in [6] using the Continuum Hypothesis. We do not know whether Conjecture 6.2 is true for $X = \mathbb{R}^{n+1}$, for $n \geq 1$. It may be worth mentioning that for $n \geq 1$ we have (with $\mathbb{T}$ denoting the unit circle)

$$C((\mathbb{R}^{n+1})^*) \cong \mathrm{C_b}([0,1), C(\mathbb{T}^n))/C_0([0,1), C(\mathbb{T}^n)).$$

To see this, remove a small open ball containing the origin from $\mathbb{R}^{n+1}$ and note that $\mathrm{C_b}([0,1) \times \mathbb{T}^n) \cong \mathrm{C_b}([0,1), C(\mathbb{T}^n))$ (this follows from [1, 3.4], see also [3, II.7.3.12 (iv)] by noting that in a unital algebra norm and strict topologies coincide).

Thus a relevant question is to what extent the assumption on compactness of domains in A can be removed from Theorem 2.5? Proposition 2.1 (5) gives a warning sign.

By Woodin's Σ^2_1 absoluteness theorem (see [38]), Continuum Hypothesis is the optimal set-theoretic assumption for obtaining autohomeomorphisms as in Conjecture 6.2.

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