

SUPERSTABLE THEORIES AND REPRESENTATION
F1784

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ABSTRACT. In this paper we give characterizations of the first order complete superstable theories, in terms of an external property called representation. In the sense of the representation property, the mentioned class of first-order theories can be regarded as “not very complicated”. This was done for “stable” and for “ \aleph_0 -stable.” Here we give a complete answer for “superstable”.

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§ 0. INTRODUCTION

Our motivation to investigate the properties under consideration in this paper comes from the following

Thesis: It is very interesting to find dividing lines and it is a fruitful approach in investigating quite general classes of models. A “natural” dividing property “should” have equivalent internal, syntactical, and external properties. (see [Shea] for more)

Of course, we expect the natural dividing lines will have many equivalent definitions by internal and external properties.

The class of stable (complete first order theories) T is well known (see [She90]), it has many equivalent definitions by “internal, syntactical” properties, such as the order property. As for external properties, one may say “for every $\lambda \geq |T|$ for some model M of T we have $\mathbf{S}(M)$ has cardinality $> \lambda$ ” is such a property (characterizing instability). Anyhow, the property “not having many κ -resplendent models (or equivalently, having at most one in each cardinality)” is certainly such an external property (see [Sheb]).

Here we deal with another external property, *representability*. This notion was a try to formalize the intuition that “the class of models of a stable first order theory is not much more complicated than the class of models $M = (A, \dots, E_t, \dots)_{s \in I}$ where E_t^M is an equivalence relation on A refining E_s^M for $s < t$; and I is a linear order of cardinality $\leq |T|$. It was first defined in Cohen-Shelah [SC16], where it was shown that one may characterize stability and \aleph_0 -stability by means of representability. In this paper we give a complete answer also for the superstable case. Moreover if T is uncountable we consider other values of $\kappa(T)$. That is, recall that for a stable (complete first order) theory T , $\kappa(T)$ can be any cardinal in the interval $[\aleph_0, |T|^+]$. So if T is countable there are two possible values- \aleph_0, \aleph_1 , the second is dealt with in [SC16] and the first in Theorem 2.1. But if T is uncountable, the result above gives a representation in a class which depends just on $|T|$, so it is natural to suspect that if $\kappa(T) < |T|^+$ we can restrict this class further. We succeed to do this in Theorem 3.7.

The results are phrased below, and the full definition appears in Definition 1.2, but first consider a simplified version. We say that a model M is \mathfrak{k} -representable for a class \mathfrak{k} when there exists a structure $\mathbf{I} \in \mathfrak{k}$ with the same universe as M such that for any n and two sequences of length n from M , if they realize the same quantifier free type in \mathbf{I} then they realize the same (first order) type in M . Of course, T is \mathfrak{k} -representable if every model of T is \mathfrak{k} -representable. We prove, e.g. that T is superstable iff it is $\mathfrak{k}_\kappa^{\text{unary}}$ -representable for some κ where $\mathfrak{k}_\kappa^{\text{unary}}$ is the class of structures with exactly κ unary functions (and nothing else).

There is also a relative characterizing “ $\kappa(T) \leq \kappa^+$ ”.

This raises various further questions

Problem:

- (1) Can we characterize, by representability “ T is strongly dependent”, similarly for the various relatives (see [Sh:863])
- (2) For a natural number n , what is the class of T representable by \mathfrak{k}_κ^n of structures with just κ n -place functions (or relations).

The main results presented in this paper are:

Characterization of superstable theories (Theorem 2.1):

Characterization of $\kappa(T)$ (Theorem 3.7):

In the attempt to extend the framework of representation it seemed natural, initially, to conjecture that if we consider representation over linear orders rather than over sets, we could find an analogous characterizations for dependent theories. However, such characterizations would imply strong theorems on existence of indiscernible sequences. In [KS], some dependent theories were discovered for which it is provably “quite hard to find indiscernible subsequences”, implying that this conjecture would fail in its original formulation.

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§ 1. STRUCTURE CLASSES AND REPRESENTATIONS

We recall some needed definitions and properties from [SC16].

Convention 1.1. (1) The vocabulary is a set of individual constants, (partial) function symbols and relation symbols (=predicates), each with the number of places (=arity) being finite. Individual constants may be considered as 0-place function symbols; here function symbols are interpreted as partial functions.

(2) A structure $\mathbf{I} = \langle \tau, I, \models \rangle$ is a triple of vocabulary, universe(domain) and the interpretation relation for the vocabulary: let $|\mathbf{I}| = I$, $\|\mathbf{I}\|$ the cardinality of I and $\tau_{\mathbf{I}} = \tau$; \mathbf{I} is called a τ -structure.

(3) \mathfrak{k} denotes a class of structures in a given vocabulary $\tau_{\mathfrak{k}}$, so $\mathbf{I} \in \mathfrak{k} \Rightarrow \mathbf{I}$ is a $\tau_{\mathfrak{k}}$ -structure.

(4) $\mathcal{L}_{\text{qf}}^{\tau}$ denotes all the quantifier-free formulas with terms from $\tau_{\mathfrak{k}}$. That is, finite boolean combinations of atomic formulas, where atomic formulas (for τ) have the form $P(\sigma_0, \dots, \sigma_{n-1})$ or $\sigma_0 = \sigma_1$ for some n -ary predicate $P \in \tau$, $\sigma_0 \dots$ are terms, i.e. they are in the closure of the set of variables by function (and partial function) symbols.

(5) If \mathbf{I} a τ -structure, $\bar{a} = \langle a_i : i < \alpha \rangle \in {}^{\alpha}|\mathbf{I}|$, then

$$\text{tp}_{\text{qf}}(\bar{a}, B, \mathbf{I}) = \left\{ \varphi(\bar{x}, \bar{b}) : \varphi(\bar{x}, \bar{y}) \in \mathcal{L}_{\text{qf}}^{\tau} : \mathbf{I} \models \varphi(\bar{a}, \bar{b}), \bar{b} \in {}^{\ell\varphi(\bar{y})}B \right\}$$

§ 1(A). **Defining representations.** We recall the definition of a representation.

Definition 1.2. Consider a structure \mathbf{I} .

(1) For a structure \mathbf{J} , a function $f : |\mathbf{I}| \rightarrow |\mathbf{J}|$ is called a representation of \mathbf{I} in \mathbf{J} if

$$\text{tp}_{\text{qf}}(f(\bar{a}), \emptyset, \mathbf{J}) = \text{tp}_{\text{qf}}(f(\bar{b}), \emptyset, \mathbf{J}) \Rightarrow \text{tp}(\bar{a}, \emptyset, \mathbf{I}) = \text{tp}(\bar{b}, \emptyset, \mathbf{I})$$

for any two sequences $\bar{a}, \bar{b} \in {}^{<\omega}I$ of equal length.

(2) We say that \mathbf{I} is represented in a class of models \mathfrak{k} if there exists a $\mathbf{J} \in \mathfrak{k}$ such that \mathbf{I} is represented in \mathbf{J} .

(3) For two classes of structures $\mathfrak{k}_0, \mathfrak{k}$ we say that \mathfrak{k}_0 is represented in \mathfrak{k} if every $\mathbf{I} \in \mathfrak{k}_0$ is represented in \mathfrak{k} .

(4) We say that a first-order theory T is represented in \mathfrak{k} if the elementary class $\text{EC}(T)$ of T is represented in \mathfrak{k} .

Definition 1.3. \mathfrak{k}^{eq} denotes the class of structures of the vocabulary $\{=\}$, where eq stands for equality.

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(1043)

{conv:structure-class}

{def:representation}

{def:k^eq}

§ 1(B). **The free algebras** $\mathcal{M}_{\mu,\kappa}$.

Definition 1.4. For a given structure \mathbf{I} , we define the structure $\mathcal{M}_{\mu,\kappa}(\mathbf{I})$ as the structure whose vocabulary is $\tau_{\mathbf{I}} \cup \{F_{\alpha,\beta} : \alpha < \mu, \beta < \kappa\}$, with a β -ary function symbol $F_{\alpha,\beta}$ for all $\alpha < \mu, \beta < \kappa$. (the vocabulary of \mathbf{I} includes a unary relation symbol I for the structure's universe, and we will assume $F_{\alpha,\beta} \notin \tau_{\mathbf{I}}$). The universe for this structure is¹:

$$\mathcal{M}_{\mu,\kappa}(\mathbf{I}) = \bigcup_{\gamma \in \text{Ord}} \mathcal{M}_{\mu,\kappa,\gamma}(\mathbf{I})$$

Where $\mathcal{M}_{\zeta} = \mathcal{M}_{\mu,\kappa,\zeta}(\mathbf{I})$ is defined as follows:

- $\mathcal{M}_0(\mathbf{I}) := |\mathbf{I}|$
- For limit ζ : $\mathcal{M}_{\zeta}(\mathbf{I}) = \bigcup_{\xi < \zeta} \mathcal{M}_{\xi}(\mathbf{I})$
- For $\zeta = \gamma + 1$

$$\mathcal{M}_{\zeta} = \mathcal{M}_{\gamma} \cup \{F_{\alpha,\beta}(\bar{b}) : \bar{b} \in {}^{\beta}\mathcal{M}_{\gamma}, \alpha < \mu, \beta < \kappa\}$$

Where $F_{\alpha,\beta}(\bar{b})$ is treated as a formal object. The symbols in $\tau_{\mathbf{I}}$ have the same interpretation as in \mathbf{I} . In particular, α -ary functions may be interpreted as $(\alpha + 1)$ -ary relations. The β -ary function $F_{\alpha,\beta}(\bar{x})$ is interpreted as the mapping $\bar{a} \mapsto F_{\alpha,\beta}(\bar{a})$ for all $\bar{a} \in {}^{\beta}|\mathcal{M}_{\mu,\kappa}(\mathbf{I})|$, where $F_{\alpha,\beta}(\bar{a})$ on the right side of the mapping is the formal object. If $\mu = \kappa = \aleph_0$ we may omit them.

Remark 1.1. It is shown in [SC16] that $\mathcal{M}_{\mu,\kappa}(S)$ is a set (though defined as a class).

§ 1(C). **Extensions of classes of structures.**

Discussion 1.5. For a class of structures \mathfrak{k} , we define several classes of structures that are based on \mathfrak{k} .

Definition 1.6. $\text{Ex}_{\mu,\kappa}^0(\mathfrak{k})$ is the class of structures \mathbf{I}^+ which, for some $\mathbf{I} \in \mathfrak{k}$ satisfy $|\mathbf{I}^+| = |\mathbf{I}|; \tau_{\mathbf{I}^+} = \tau_{\mathbf{I}} \cup \{P_{\alpha} : \alpha < \mu\} \cup \{F_{\beta} : \beta < \kappa\}$ for new unary relation symbols P_{α} and new unary function symbols F_{β} ; such that if $\mu > 0$ then $\langle P_{\alpha}^{\mathbf{I}^+} : \alpha < \mu \rangle$ is a partition of $|\mathbf{I}|$; and, $\langle F_{\beta}^{\mathbf{I}^+} : \beta < \kappa \rangle$ are **partial** unary functions.

Definition 1.7. $\text{Ex}_{\mu,\kappa}^{0,\text{lf}}(\mathfrak{k})$ is the class of structures in $\text{Ex}_{\mu,\kappa}^0(\mathfrak{k})$ for which the closure of every element under the new functions is finite. (lf stands for “locally finite”).

Definition 1.8. $\text{Ex}_{\mu,\kappa}^1(\mathfrak{k})$ is the class of structures in $\text{Ex}_{\mu,\kappa}^0(\mathfrak{k})$ for which $F_{\beta}(P_{\alpha}) \subseteq P_{<\alpha} := \bigcup_{\gamma < \alpha} P_{\gamma}$ holds for every $\alpha < \mu, \beta < \kappa$.

Definition 1.9. $\text{Ex}_{\mu,\kappa}^2(\mathfrak{k})$ is the class of structures of the form $\mathbf{I}^+ = \mathcal{M}_{\mu,\kappa}(\mathbf{I})$, for some $\mathbf{I} \in \mathfrak{k}$ (cf. Definition 1.4).

Convention 1.10. $\text{Ex}_{\mu,\kappa}$ will be one of the above classes.

§ 2. SUPERSTABLE THEORIES

The main theorem in this section is

Theorem 2.1. *For a first-order, complete theory T the following are equivalent:*

- (1) T is superstable.

¹This defines a set and not a proper class by remark 1.1.

{def:free_algebra_M}

{rem:M_mu_kappa_is_a_set}

{struct_class_enrich_ex0}

{struct_class_enrich_ex0_lf}

{struct_class_enrich_ex1}

{struct_class_enrich_ex2}

{def:free_algebra_M}

{d2}

{rem:M_mu_kappa_is_a_s}

- (2) T is representable in $\text{Ex}_{2|T|, \aleph_0}^2(\mathfrak{t}^{\text{eq}})$
- (3) T is representable in $\text{Ex}_{2|T|, 2}^1(\mathfrak{t}^{\text{eq}})$
- (4) T is representable in $\text{Ex}_{2|T|, 2}^{0, \text{lf}}(\mathfrak{t}^{\text{eq}})$
- (5) T is representable in $\text{Ex}_{\mu, \aleph_0}^2(\mathfrak{t}^{\text{eq}})$ for some cardinal μ
- (6) T is representable in $\text{Ex}_{\mu, \kappa}^{0, \text{lf}}(\mathfrak{t}^{\text{eq}})$ for some cardinals μ, κ .

Proof. $2 \Rightarrow 5, 4 \Rightarrow 6$ are immediate.

$2 \Rightarrow 3$ is direct from [SC16, 1.30]

$3 \Rightarrow 4$ direct from [SC16, 1.24]

$5 \Rightarrow 6$

This follows since $\text{Ex}_{\mu, \aleph_0}^2(\mathfrak{t}^{\text{eq}})$ is qf-representable in $\text{Ex}_{\mu, 2}^1(\mathfrak{t}^{\text{eq}})$ by [SC16, 1.30] and $\text{Ex}_{\mu, 2}^1(\mathfrak{t}^{\text{eq}}) \subseteq \text{Ex}_{\mu, 2}^{0, \text{lf}}(\mathfrak{t}^{\text{eq}})$ by [SC16, 1.24] with 2 here standing for κ there.

The rest follows from Theorem 2.3 below giving $1 \Rightarrow 2$ and Theorem 2.2 below giving $6 \Rightarrow 1$. {d8}

□_{2.1}

{d5}

Theorem 2.2. *If T is representable in $\text{Ex}_{\mu, \kappa}^{0, \text{lf}}(\mathfrak{t}^{\text{eq}})$ for some cardinals μ, κ then T is superstable.*

Proof. Like the proof of Propositions [SC16, Th.2.4,2.5]. □_{2.2}

□_{2.2}

{d8}

Theorem 2.3. *Every superstable T is representable in $\text{Ex}_{2|T|, \aleph_0}^2(\mathfrak{t}^{\text{eq}})$.*

Proof. Let T be superstable. Let $M \prec \mathfrak{C}_T$. We choose $B_n, \langle a_s, u_s : s \in S_n \rangle$ by induction on $n < \omega$ such that:

- ⊗₀
 - (1) $S_n \cap S_k = \emptyset$ ($k < n$)
 - (2) $\langle a_s : s \in S_n \rangle \subseteq M$
 - (3) $B_n = \{a_s : s \in S_{<n}\} \subseteq M$, where $S_{<n} := \cup\{S_k : k < n\}$, as usual
 - (4) $\langle a_s : s \in S_n \rangle$ is without repetitions, disjoint from $\{a_s : s \in S_{<n}\}$ and independent over B_n ,
 - (5) for all $s \in S, u_s \subseteq S_{<n}$ is finite such that $t \in u_s \Rightarrow u_t \subseteq u_s$ and $\text{tp}(a_s, B_n)$ does not fork over $\{a_t : t \in u_s\}$
 - (6) $\langle a_s : s \in S_n \rangle$ is maximal under conditions 1-5.

Here we make a convention that u, v, w vary on \mathcal{I} defined below:

- ⊗₁ Since T is superstable, it is possible to carry the induction.
- ⊗₂ (0) $\mathcal{I} = \{u : u \subseteq S, u \text{ finite}\}$, where $S = \bigcup_n S_n$
 - (1) for $v \in \mathcal{I}$ let $\text{cl}(v)$ be the minimal $u \supseteq v$ such that $u_t \subseteq u$ holds for all $t \in u$;
 - (2) we define $\mathcal{I}^{\text{cl}} = \{u \in \mathcal{I} : u = \text{cl}(u)\}$;
- ⊗₃ (0) if $u \in \mathcal{I}$ then $\text{cl}(u) \in \mathcal{I}$
 - (1) $v \subseteq u \Rightarrow \text{cl}(v) \subseteq \text{cl}(u)$;
 - (2) $\text{cl}(u_1 \cup u_2) = \text{cl}(u_1) \cup \text{cl}(u_2)$;
 - (3) $\text{cl}(\{s\}) = u_s \cup \{s\}$;
 - (4) $\text{cl}(\text{cl}(u)) = \text{cl}(u)$;

- (5) $\text{cl}(u) = \bigcup\{u_s : s \in u\} \cup u$
- ⊗₄ $|M| = \{a_s : s \in S\}$ (Why? Otherwise, there exists $a \in |M| \setminus \{a_s : s \in S\}$ such that (since T is superstable) $\text{tp}(a, \{a_s : s \in S\})$ does not fork over $\{a_s : s \in v\}$ for some finite subset $v \subseteq S$. Let $u = \text{cl}(v)$, so $u \in \mathcal{I}^{\text{cl}}$ and let n be such that $u \subseteq S_n$ and we get a contradiction to the maximality of $\{a_s : s \in S_n\}$.)
- ⊗₅ Let $\langle v_\alpha : \alpha < \alpha(*) \rangle$ enumerate \mathcal{I} (without repetition) such that
- (1) $v_\alpha \subseteq v_\beta \Rightarrow \alpha \leq \beta$;
 - (2) $\alpha < \beta \wedge v_\beta \subseteq S_{<n} \Rightarrow v_\alpha \subseteq S_{<n}$.

We choose a model M_{v_α} by induction on α such that:

- ⊗₆ (1) $M_{v_\alpha} \prec \mathfrak{C}_T$ has cardinality $\leq \aleph_0 + |T|$;
- (2) $v_\alpha \subseteq v_\beta \Rightarrow M_{v_\alpha} \prec M_{v_\beta}$;
 - (3) $\bigcup\{M_{v_\beta} : \beta < \alpha \wedge v_\beta \subseteq v_\alpha\} \subsetneq M_{v_\alpha}$;
 - (4) if $s \in v_\alpha$ and $u_s \subseteq v_\alpha$ then $a_s \in M_{v_\alpha}$;
 - (5) $\text{tp}(M_{v_\alpha}, \bigcup\{M_{v_\beta} : \beta < \alpha\} \cup M)$ does not fork over $B_{v_\alpha} := \bigcup\{M_{v_\beta} : v_\beta \subseteq v_\alpha, \beta < \alpha\} \cup \{a_s : \text{cl}(\{s\}) \subseteq v_\alpha\}$
- ⊗₇ By ⊗₆(3), clearly $\alpha < \beta \Rightarrow M_{v_\alpha} \neq M_{v_\beta}$.

The induction is clearly possible.

A major point is

- ⊗₈ $\text{tp}(M_{v_\alpha}, \bigcup\{M_{v_\beta} : \beta < \alpha\})$ does not fork over $A_{v_\alpha} := \bigcup\{M_{v_\beta} : v_\beta \subsetneq v_\alpha\}$.

[Why? If $v_\alpha = \emptyset$ this is trivial so assume $v_\alpha \neq \emptyset$.

Let n be such that $v_\alpha \subseteq S_{\leq n}, v_\alpha \not\subseteq S_{<n}$ and

- ⊗_{8.1} let $\langle t_\ell : \ell < k \rangle = \langle t_\ell^\alpha : \ell < k_\alpha \rangle$ list $\{s \in v_\alpha : s \notin S_{<n} \text{ and } \text{cl}(\{s\}) \subseteq v_\alpha\}$.

First, assume $k = 0$. So if $s \in v_\alpha$ and $\text{cl}(\{s\}) \subseteq v_\alpha$ then $s \in v_\alpha \cap S_{<n}$, this implies that $u_s \cup \{s\} = \text{cl}(\{s\}) \subseteq S_{<n}$, hence by ⊗₆(4), $a_s \in M_{v_\alpha \cap S_{<n}} \subseteq A_{v_\alpha}$. This implies that $B_{v_\alpha} \subseteq A_{v_\alpha}$ (in fact equal - see their definitions in ⊗₆(5), ⊗₈). Now ⊗₆(5) says that $\text{tp}(M_{v_\alpha}, \bigcup\{M_{v_\beta} : \beta < \alpha\})$ does not fork over B_{v_α} , so by monotonicity of non-forking and the last sentence, it does not fork over A_{v_α} as desired.

Second, assume $k = 1$ and $(\forall \beta < \alpha) (\text{cl}(\{t_0\}) \not\subseteq v_\beta)$,

Since necessarily $v_\alpha = \text{cl}(\{t_0\})$ so $B_{v_\alpha} = A_{v_\alpha} \cup \{a_{t_0}\}$ and clearly $\text{tp}(a_{t_0}, \bigcup\{M_{v_\beta} : \beta < \alpha\})$ does not fork over A_{v_α} , together with ⊗₆(5) we get that $\text{tp}(M_{v_\alpha}, \bigcup\{M_{v_\beta} : \beta < \alpha\})$ does not fork over A_{v_α} , as desired in ⊗₈.

Third, assume $k = 1, \beta < \alpha$ and $\text{cl}(\{t_0\}) \subseteq v_\beta$. Without loss of generality β is minimal with these properties, so necessarily $v_\beta = \text{cl}(\{t_0\})$ and so again, $B_{v_\alpha} = A_{v_\alpha}$ and we continue as in “First” above.

Fourth, assume $k \geq 2$. In this case, for each $\ell < k, \text{cl}(\{t_\ell\})$ is $v_{\beta(\ell)}$ for some unique $\beta(\ell) < \alpha$, so $a_{t_\ell} \in M_{v_{\beta(\ell)}} \subseteq A_{v_\alpha}$, hence, $B_{v_\alpha} \subseteq A_{v_\alpha}$ (in fact equal) and again ⊗₆(5) gives the desired conclusion.]

Now, ⊗₈ is the necessary condition in [She90, XII.2, Lemma 2.3(1)] and so we can conclude that $\langle M_v : v \in \mathcal{I} \rangle$ is a stable system (see [She90, XII.2, Definition 2.1, page 598].

Now $\bar{M} = \langle M_v : v \in \mathcal{I} \rangle$ is a stable system of models. For all $v \in \mathcal{I}$ let \bar{b}_v enumerate M_v . By ⊗₆(3), $\langle \bar{b}_v : v \in \mathcal{I} \rangle$ is without repetitions.

For all $\alpha < \omega \times \omega$ we define \mathcal{I}_α as follows:

- ⊕₁ (1) $\mathcal{I}_0 = \{\emptyset\}$;
- (2) $\mathcal{I}_k = \{v \in \mathcal{I} : v \subseteq S_{<1}, |v| = k\}, (k < \omega)$;
- (3) $\mathcal{I}_{\omega n+k} = \{v \in \mathcal{I} : v \not\subseteq S_{<n}, v \subseteq S_{<n+1}, |v| = k+1\}, (k < \omega, 0 < n < \omega)$.

Now clearly

- ⊕₂ (1) $\langle \mathcal{I}_\alpha : \alpha < \omega \times \omega \rangle$ is a partition of \mathcal{I}
- (2) $w \subsetneq v \in \mathcal{I}_\alpha \Rightarrow w \in \mathcal{I}_{<\alpha}$ for all $w, v \in \mathcal{I}$
- ⊕₃ and most important, for all $\alpha < \omega \times \omega$ and $v \in \mathcal{I}_\alpha$ the type

$$p_v := \text{tp}(\bar{b}_v, \cup\{\bar{b}_u : u \in \mathcal{I}_{<\alpha+1} \wedge u \neq v\})$$

does not fork over $\cup\{\bar{b}_w : w \subsetneq v\}$, and moreover p_v is the unique non-forking extension of $p_v \upharpoonright \cup\{\bar{b}_w : w \subseteq v\}$ in $\mathbf{S}^{\text{lg}(\bar{b}_v)}(\cup\{\bar{b}_u : u \in \mathcal{I}_{<\alpha+1} \wedge u \neq v\})$ (The proof is carried by basic properties of stable systems, see Conclusion 2.12 in [She90, Ch.XII, page 603]).

Now we define an equivalence relation E on \mathcal{I} , (and $E_\alpha = E \upharpoonright \mathcal{I}_\alpha$) such that $v_1 E v_2$ if and only if for some $g = g_{v_1, v_2}, f = f_{v_1, v_2}$ (really g determines f , we may require g to be order preserving)

- ⊕₄ (1) $v_1, v_2 \in \mathcal{I}$;
- (2) $|v_1| = |v_2|$;
- (3) g is one-to-one from v_1 onto v_2 (we may add mapping $v_1 \cap S_{<n}$ onto $v_2 \cap S_{<n}$ for every n such that if $u_i \subseteq v_i$ (for $i = 1, 2$) and $g''(u_1) = u_2 \Rightarrow [u_1 \in \mathcal{I}_\beta \equiv u_2 \in \mathcal{I}_\beta]$;
- (4) $u_{g(t)} = \{g(s) : s \in u_t\}$, when $u_t \cup \{t\} \subseteq v_1$
- (5) f is an elementary mapping of \mathfrak{C}_T
- (6) if $u_\ell \subseteq v_\ell$ for $\ell = 1, 2$ and g maps u_1 onto u_2 , then f maps \bar{b}_{u_1} to \bar{b}_{u_2}
- (7) $\text{Dom}(f) = \cup\{\bar{b}_u : u \subseteq v_1\}$
- (8) if $s_l \in v_l, \text{cl}(s_l) \subseteq v_l$ for $l = 1, 2$ and $g(s_1) = s_2$ then $f(a_{s_1}) = a_{s_2}$.

(element-by-element, and this implies f_{v_1, v_2} is unique). (So for some bijection $g_{v_1, v_2} : v_1 \rightarrow v_2$ which preserves being in \mathcal{I}_β for all $\beta < \alpha$, such that f_{v_1, v_2} maps \bar{b}_{w_1} to $\bar{b}_{g_{v_1, v_2}(w_1)}$ for all $w_1 \subseteq v_1$.) Let $\langle I_{\alpha, i} : i < i(\alpha) \leq 2^{|\mathcal{I}|} \rangle$ enumerate the equivalence classes of E_α .

We get that $\boxplus_1 \Rightarrow \boxplus_2$ where

- ⊕₁ (α) the sets $\{v_0 \dots v_{n-1}\}, \{u_0 \dots u_{n-1}\} \subset \mathcal{I}$ are closed under subsets
- (β) $\bigwedge_{\alpha, i} [v_l \in I_{\alpha, i} \Leftrightarrow u_l \in I_{\alpha, i}]$
- (γ) $u_{l(1)} \subset u_{l(2)} \Leftrightarrow v_{l(1)} \subset v_{l(2)}$ for all $l(1), l(2) < n$
- (δ) if $v_{l(1)} \subseteq v_{l(2)}$ then $g_{v_{l(2)}, u_{l(2)}}$ maps $v_{l(1)}$ onto $u_{l(1)}$.
- ⊕₂ The sequences $\bar{b}_{v_0} \frown \dots \frown \bar{b}_{v_{n-1}}$ and $\bar{b}_{u_0} \frown \dots \frown \bar{b}_{u_{n-1}}$ realize the same complete type over \emptyset . (This follows from the definitions of the equivalence relations E_α and \boxplus_1 above).

Let \mathcal{A} be the free algebra, i.e. a structure of the form $\mathcal{M}_{2^{|T|}, \aleph_0}(\mathfrak{t}^{\text{eq}})$, generated by $\{x_u : u \in \mathcal{I}\}$ and the function symbols $\{F_{e,\zeta} : \zeta < |T|, e = u/E \text{ for some } u \in \mathcal{I}\}$, each one of them $|\{v \in \mathcal{I} : v \subseteq u\}|$ -ary for $u \in \mathcal{I}$ such that $e = u/E$. Since for every $b \in M$ there is a unique (u, ζ) such that $b \in \bar{b}_{u,\zeta}$ (where $\bar{b}_{u,\zeta}$ is the ζ -element of \bar{b}_u), recalling $\oplus_4(8)$, the following gives the desired representation

$$H(b) = \langle \dots x_v \dots \rangle_{v \subseteq u} \frown \langle F_{u/E,\zeta}(\dots, x_v, \dots) \rangle_{v \subseteq u},$$

after choosing some ordering on the elements of \mathcal{I} . □_{2.3}

§ 3. BETWEEN STABLE AND SUPERSTABLE

{e3}

Discussion 3.1. For superstable T , we may wonder about whether “the cardinal $2^{|T|}$ is optimal”. Really, $\lambda(T)$ is sufficient where

$$(*)_{1.1} \quad \lambda(T) = \min\{\lambda : T \text{ is stable in } \lambda\}.$$

Note that

$$(*)_{1.2} \quad \text{If } T \text{ is countable then } \lambda(T) = \aleph_0 \text{ is equivalent to } T \text{ is } \aleph_0\text{-stable and}$$

$$(*)_{1.3} \quad \text{if } T \text{ is countable and } \lambda(T) > \aleph_0 \text{ then } \lambda(T) = 2^{\aleph_0}.$$

{e6}

{d8}

Theorem 3.2. *In Theorem 2.3, $\text{Ex}_{\lambda(T), \aleph_0}^2(\mathfrak{t}^{\text{eq}})$ suffice.*

{d8}

Proof. We repeat the proof of Theorem 2.3 with some changes. Choosing M_{v_α} by induction on α we now demand more than in \otimes_6 in the proof of Theorem 2.3:

{d8}

- $$\begin{aligned} \otimes_6^* \quad (1) \quad & M_{v_\alpha} \prec \mathfrak{C} \text{ is saturated of cardinality } \lambda(T); \\ (2) \quad & v_\beta \subseteq v_\alpha \Rightarrow M_{v_\beta} \subseteq M_{v_\alpha} \\ (3) \quad & (M_\alpha, c)_{c \in \cup\{M_{v_\beta} : v_\beta \subseteq v_\alpha\}} \text{ is saturated.} \end{aligned}$$

{d8}

In $\otimes_{8.1}$ of the proof of Theorem 2.3 we replace v_α with v , i.e.

- $$\otimes_{8.1}^* \quad \text{if } v \in S_n \text{ then } \bar{t}_v = \langle t_{v,\ell} : \ell < k_v \rangle \text{ list } \{s \in v : s \notin S_{<n} \text{ and } \text{cl}(\{s\}) \subseteq v\} \text{ in increasing order.}$$

Now after \oplus_3 in the proof and as in the proof, we choose by induction on $\alpha < \omega \times \omega$, $E_\alpha, f_{v_1, v_2}, g_{v_1, v_2}$ for $(v_1, v_2) \in E_\alpha$ such that

- $$\begin{aligned} \oplus_4^* \quad (a) \quad & E_\alpha \text{ is an equivalence relation on } \mathcal{I}_\alpha \text{ with } \leq \lambda(T) \text{ equivalence classes} \\ (b) \quad & \text{Clauses (1) – (8) of } \oplus_4 \text{ holds with the following modifications: if } \\ & (v_1, v_2) \in E_\alpha \text{ then } f = f_{v_1, v_2}, g = g_{v_1, v_2} \text{ satisfy} \\ (1) \quad & v_1, v_2 \in \mathcal{I}; \\ (2) \quad & |v_1| = |v_2|; \\ (3) \quad & g \text{ is one-to-one from } v_1 \text{ onto } v_2 \text{ (we may add mapping } v_1 \cap S_{<n} \\ & \text{onto } v_2 \cap S_{<n} \text{ for every } n \text{ such that if } u_i \subseteq v_i \text{ (for } i = 1, 2) \text{ and } \\ & g''(u_1) = u_2 \Rightarrow [u_1 \in \mathcal{I}_\beta \equiv u_2 \in \mathcal{I}_\beta] \wedge g \upharpoonright u_1 = g_{u_1, u_2}; \\ (4) \quad & u_{g(t)} = \{g(s) : s \in u_t\}, \text{ when } u_t \cup \{t\} \subseteq v_1 \\ (5) \quad & f_{v_1, v_2} \text{ is an elementary mapping of } \mathfrak{C}_T \\ (6) \quad & \text{if } u_\ell \subseteq v_\ell \text{ for } \ell = 1, 2 \text{ and } g \text{ maps } u_1 \text{ onto } u_2, \text{ then } f_{v_1, v_2} \text{ maps} \\ & M_{u_1} \text{ onto } M_{u_2} \text{ and } f \upharpoonright M_{u_1} = f_{u_1, u_2} \\ (7) \quad & \text{Dom}(f_{v_1, v_2}) = \cup\{M_u : u \subseteq v_1\} \\ (8) \quad & \text{if } s_l \in v_l, \text{cl}(s_l) \subseteq v_l \text{ for } l = 1, 2 \text{ and } g_{v_1, v_2}(s_1) = s_2 \text{ then} \\ & f_{v_1, v_2}(a_{s_1}) = a_{s_2}. \end{aligned}$$

Note that the \bar{b}_v notation does not appear.

The induction is carried in the following way. For $\alpha = 0$ this is obvious, so let $\alpha \in (0, \omega \times \omega)$. First we define a two place relation E'_α on \mathcal{S}_α :

- $\oplus_{5,\alpha}$ $v_1 E'_\alpha v_2$ if and only if
 - (1) $v_1, v_2 \in \mathcal{S}_\alpha$ and $|v_1| = |v_2|$
 - (2) letting $v_\ell = \{t_{\ell,j} : j < |v_\ell|\}$ for $\ell = 1, 2$ list v_ℓ in increasing order we have

$$j < |v_1| \Rightarrow \bigvee_{\beta < \alpha} (v_1 \setminus \{t_{1,j}\}) E_\beta (v_2 \setminus \{t_{2,j}\})$$

- (3) $g = g_{v_1, v_2}$ maps $t_{1,j}$ to $t_{2,j}$ for $j < |v_1|$.

Note that for every α , E'_α is an equivalence relation on \mathcal{S}_α with $\leq \lambda(T)$ equivalence classes. By [She90, XII, 3.2 page 604, 3.5 page 608] and the induction hypothesis, if $(v_1, v_2) \in E'_\alpha$ then $f'_{v_1, v_2} := \bigcup \{f_{v_1 \setminus \{t_{1,j}\}, v_2 \setminus \{t_{2,j}\}} : j < |v_1|\}$ is an elementary map.

- $\oplus_{7,\alpha}$ We define the place relation E_α on \mathcal{S}_α , as $v_1 E_\alpha v_2$ if and only if $v_1 E'_\alpha v_2$ and f_{v_1, v_2} and f'_{v_1, v_2} maps $\text{tp}(\langle a_{t_{v_1,j}} : j < |v_1| \rangle, \bigcup \{M_u : u \subsetneq v_1\})$ onto $\text{tp}(\langle a_{t_{v_2,j}} : j < |v_2| \rangle, \bigcup \{M_u : u \subsetneq v_2\})$.

Since T is stable in $\lambda(T)$, E_α is an equivalence relation on \mathcal{S}_α with $\leq \lambda(T)$ equivalence classes refining E'_α .

Finally, if $(v_1, v_2) \in E_\alpha$ then let f_{v_1, v_2} be an elementary map from M_{v_1} to M_{v_2} extending f'_{v_1, v_2} and mapping $\langle a_{t_{v_1,j}} : j < |v_1| \rangle$ to $\langle a_{t_{v_2,j}} : j < |v_2| \rangle$

Now that E_α, f_{v_1, v_2} and g_{v_1, v_2} are defined inductively, the rest of the proof is as in the proof of Theorem 2.3. □_{3.2} {d8}

§ 3(A). **Characterization of $\kappa(T)$.**

Let T be a stable theory. Recall the definition of $\kappa(T)$ from [She90, III.3]: $\kappa(T) = \sup_m \kappa^m(T)$, where $\kappa^m(T)$ is the first infinite κ for which we do not have an increasing sequence $A_i, i \leq \kappa$ and $p \in S_m(A_\kappa)$ such that for all $i < \kappa, p \upharpoonright A_{i+1}$ forks over A_i .

Remark 3.3. If T is stable then $\kappa(T) \leq |T|^+$ [She90, Corollary 3.3]. T is superstable if and only if $\kappa(T) = \aleph_0$ [She90, Corollary 3.8]. {f5}

Definition 3.4. Assume T is stable.

- (1) We say that \mathbf{a} is an independent system where $\mathbf{a} = \langle \mathcal{S}_\mathbf{a}, \bar{A}_\mathbf{a} \rangle$ consists of
 - ₁ $\bar{A}_\mathbf{a} = \langle A_{\mathbf{a},v} : v \in \mathcal{S}_\mathbf{a} \rangle$;
 - ₂ $\mathcal{S}_\mathbf{a}$ is a family of subsets of $S = \bigcup \mathcal{S}_\mathbf{a}$;
 - ₃ $\emptyset \in \mathcal{S}_\mathbf{a}$;
 - ₄ $\mathcal{S}_\mathbf{a}$ is closed under any intersection
 - ₅ $v \subseteq u \Rightarrow A_{\mathbf{a},v} \subseteq A_{\mathbf{a},u}$ for all $v, u \in \mathcal{S}_\mathbf{a}$
 - ₆ $A_{\mathbf{a},v \cap u} = A_{\mathbf{a},v} \cap A_{\mathbf{a},u}$ for all $v, u \in \mathcal{S}_\mathbf{a}$
 - ₇ $\text{tp}(A_{\mathbf{a},v}, \bigcup \{A_{\mathbf{a},u_\ell} : \ell < n\})$ does not fork over $\bigcup \{A_{\mathbf{a},v \cap u_\ell} : \ell < n\}$ for all $v, u_0, \dots, u_{n-1} \in \mathcal{S}_\mathbf{a}$

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- (2) We may also say that \mathbf{a} is an independent \mathcal{I} -system and let $\mathcal{I}_{\mathbf{a}} = \mathcal{I}$ etc. We say that \mathbf{a} is an independent (θ, \mathcal{I}) -system if in addition we demand that $|A_v| \leq \theta$ for all $v \in \mathcal{I}$.
- (3) Let \mathbf{ab} mean:
- \mathbf{a}, \mathbf{b} are independent systems,
 - $\mathcal{I}_{\mathbf{a}} = \mathcal{I}_{\mathbf{b}}$,
 - $A_{\mathbf{a},v} \subseteq A_{\mathbf{b},v}$ for $v \in \mathcal{I}_{\mathbf{a}}$,
 - $\text{tp}(A_{\mathbf{b},v}, \cup\{A_{\mathbf{a},u} : u \in \mathcal{I}_{\mathbf{a}}\})$ does not fork over $A_{\mathbf{a},v}$.

{3.5}

Proposition 3.5. *Assume T is stable.*

- (1) \leq is a partial order on the the class of independent systems.
- (2) If $\langle \mathbf{a}_\alpha : \alpha < \delta \rangle$ is a \leq -increasing sequence then it has a least upper bound $\mathbf{a}_\delta := \bigcup_{\alpha < \delta} \mathbf{a}_\alpha$, i.e.
- $\mathcal{I}_{\mathbf{a}_\delta} = \mathcal{I}_{\mathbf{a}_\alpha}$ for any $\alpha < \delta$,
 - $A_{\mathbf{a}_\delta, v} = \bigcup\{A_{\mathbf{a}_\alpha, v} : \alpha < \delta\}$ for any $v \in \mathcal{I}_{\mathbf{a}_\delta}$.
- (3) If $\langle \mathbf{a}_\alpha : \alpha < \delta \rangle$ is a \leq -increasing sequence of (θ, \mathcal{I}) -systems and $\delta < \theta^+$ then $\bigcup_{\alpha < \delta} \mathbf{a}_\alpha$ is an independent (θ, \mathcal{I}) -system.

Proof. Should be clear. □_{3.5}

{3.6}

Proposition 3.6. *Assume T is stable and $\theta \geq |T|$. Let \mathbf{a} be an (θ, \mathcal{I}) -system, with $2^{|v|} \leq \theta$ for all $v \in \mathcal{I}$.*

- (1) *There is (θ, \mathcal{I}) -system \mathbf{b} with $\mathbf{a} \leq \mathbf{b}$ and $\bar{M} = \langle M_v : v \in I \rangle$ such that for every $v \in \mathcal{I}$ we have*
- ₁ $A_{\mathbf{a},v} \subseteq M_v \subseteq A_{\mathbf{b},v}$
 - ₂ $M_v \prec \mathfrak{C}_t$
 - ₃ $(M_v, c)_{c \in \bigcup\{A_{\mathbf{a},u} : u \subsetneq v, u \in \mathcal{I}\}}$ is saturated if T is stable in θ .

Proof. Let $\langle v_\alpha : \alpha < \alpha(*) \rangle$ list \mathcal{I} with no repetition. We chose M_α by induction on $\alpha < \alpha(*)$ such that

- (*)₁ ^{α}
- (a) $M_\alpha \prec \mathfrak{C}_T$ has cardinality at most θ
 - (b) $A_{\mathbf{a},v_\alpha} \subseteq M_\alpha$
 - (c) $\text{tp}(M_\alpha, \bigcup\{M_\beta : \beta < \alpha\} \cup \bigcup\{A_{\mathbf{a},u} : u \in \mathcal{I}\})$ does not for over $A_{\mathbf{a},v}$, and
 - (d) $(M_\alpha, c)_{c \in \bigcup\{A_{v_\beta} : \beta < \alpha, v_\beta \subsetneq v_\alpha\}}$ is saturated if T is stable in θ .

Now define \mathbf{b} by setting $\bar{A}_{\mathbf{b}} = \langle A_{\mathbf{b},v_\alpha} : v_\alpha \in \mathcal{I} \rangle$, where $A_{\mathbf{b},v_\alpha} := \bigcup\{M_u : u \subseteq v_\alpha, u \in \mathcal{I}\}$.

Note that by the non-forking calculus

- (*)₂
- (a) \mathbf{b} is an independent (θ, \mathcal{I}) -system and
 - (b) $\mathbf{a} \leq \mathbf{b}$.

□_{3.6}

{e8}

Theorem 3.7. *For a complete stable theory T , and κ the following are equivalent, letting $\theta = 2^{|T|}$:*

- (1) $\kappa(T) \leq \kappa^+$
- (2) T is representable in $\text{Ex}_{\theta, \kappa}^2(\mathfrak{k}^{\text{eq}})$

- (3) T is representable in $\text{Ex}_{\mu,\kappa}^2(\mathfrak{k}^{\text{eq}})$ for some μ
- (4) T is representable in $\text{Ex}_{\theta,\kappa}^1(\mathfrak{k}^{\text{eq}})$
- (5) T is representable in $\text{Ex}_{\mu,\kappa}^1(\mathfrak{k}^{\text{eq}})$ for some μ .

Remark 3.8. In Theroem 3.7, demanding κ be regular, we can change clauses (2)-(5) above to (2')-(5') which mean that we use only \mathbf{I} such that the closure of any set of cardinality $< \kappa$ has cardinality $< \kappa$ and change (1) to (1') $\kappa(T) \leq \kappa = \delta$. The proof is similar, and in the proof we have only S_i for $i < \kappa$. {e11}
{e8}

Proof. By Theorem 2.1, without loss of generality $\kappa > |T| + \aleph_0$. Let {d2}

$$\partial = \min\{\lambda : \lambda \text{ regular, } \lambda \geq \kappa(T)\}.$$

The proof is continuing the proof of the stable case and the superstable case. So, cannibalizing the proof of Theorem 2.3 and/or [SC16, 2.15] we have $\langle S_i : i < \partial \rangle$ is a sequence of pairwise disjoint sets of ordinals, $\langle a_s : s \in S_i \rangle$ for $i < \partial, u_s \in [S_{<i}]^{<\kappa}$ for $s \in S_i, \text{cl}(u) = \cup\{u_s : s \in u\} \cup u$ for $u \subseteq S = S_{<\partial}$ and $\mathcal{S} = \{u : u \subseteq u_s \cup \{s\} \text{ for some } s \in S\}$. {d8}

By the non-forking calculus, we have that $A_{\mathbf{a},v} = \{a_s : s \in S \text{ and } \text{cl}(\{s\}) \subseteq v\}$ for $v \in \mathcal{S}$ defines a (∂, \mathcal{S}) -independent system hence a (θ, \mathcal{S}) -independent system.

Similar to the proof of Theorem 2.3, we build an equivalence relation on \mathcal{S} , but now we have to do it gradually and simultaneously with defining a new independent system. {d8}

- (*)₁ Let E_\bullet be the following equivalence relation on \mathcal{S} : $v_1 E_\bullet v_2$ if and only if
 - (a) The order-type of v_1 is equal to the order-type over v_2 , recalling that v_1 and v_2 are set of ordinals.
 - (b) g_{v_1,v_2} , the unique order preserving map from v_1 onto v_2 , maps $\{u \in \mathcal{S} : u \subseteq v_1\}$ onto $\{u \in \mathcal{S} : u \subseteq v_2\}$.

Note that E_\bullet is an equivalence relation on \mathcal{S} with at most θ equivalence classes. Let $\bar{g} = \langle g_{v_1,v_2} : v_1 E_\bullet v_2 \rangle$.

By using Proposition 3.6 repeatedly, we choose by induction on $n < \omega$ $(\mathbf{b}_n, E_n, \bar{f}_n, \bar{M}_{\{g\}})$ satisfying:

- (*)₂ (a) \mathbf{b}_n is a (θ, \mathcal{S}) -independent system
- (b) (i) E_n is an equivalence relation on \mathcal{S} with $\leq \theta$ equivalence classes refining E_\bullet and E_m for $m < n$.
- (ii) Let $\bar{f}_n = \langle f_{v_1,v_2}^n : v_1 E_n v_2 \rangle$, where
- (iii) f_{v_1,v_2}^n is an elementary mapping from $A_{\mathbf{b}_n,v_1}$ onto $A_{\mathbf{b}_n,v_2}$
- (iv) if $v_2, v_3 \in v_1/E_n$ then $f_{v_2,v_3}^n \circ f_{v_1,v_2}^n = f_{v_1,v_3}^n$
- (v) if $\text{cl}(\{s_1\}) \in v_1$ and $g_{v_1,v_2}^n = s_2$ then $f_{v_1,v_2}^n(a_{s_1}) = a_{s_2}$.
- (c) if $n = m + 1$ then by applying Proposition 3.6, we get a (θ, \mathcal{S}) -independent system $\mathbf{b}_m \leq \mathbf{b}_n$ $\bar{M}_n = \langle M_{n,v} : v \in \mathcal{S} \rangle$ satisfying for every $v \in \mathcal{S}$ $A_{\mathbf{b}_m,v} \subseteq M_{n,v} \subseteq A_{\mathbf{b}_n,v}$. Note that $(M_{n,v}, c)_{c \in \cup\{M_{n,u} : u \subsetneq v, u \in \mathcal{S}\}}$ is saturated. {3.6}
- (d) if $v_1 E_n v_2$ for $n = m + 1$ then $f_{v_1,v_2}^m \subseteq f_{v_1,v_2}^n$

By Proposition 3.5, $\mathbf{b} := \bigcup_m \mathbf{b}_m$ is a (θ, \mathcal{S}) -independent system and $\mathbf{a} \leq \mathbf{b}$. {3.5}
 Clearly, for every $v \in \mathcal{S}$,

$$\begin{aligned} A_{\mathbf{b},v} &= \bigcup\{A_{\mathbf{b}_m,v} : m < \omega\} \subseteq \bigcup\{M_{m,v} : m < \omega\} \\ &\subseteq \bigcup\{A_{\mathbf{b}_{m+1},v} : m < \omega\} = A_{\mathbf{b},v}. \end{aligned}$$

Hence $\mathfrak{C} \upharpoonright A_{\mathbf{b},v} \prec \mathfrak{C}$.

{d8} Let $E = \bigcap_m E_m$ and $f_{v_1, v_2} = \bigcup_m f_{v_1, v_2}^m$. The rest is as in the proof of Theorem 2.3.

□_{3.7}

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