[Jan 6, 2014 - was F1333 - June 19, 2013]

SUPERSTABLE THEORIES AND REPRESENTATION F1333

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Abstract. In this paper we give characterizations of the super-stable theories, in terms of an external property called representation. In the sense of the representation property, the mentioned class of first-order theories can be regarded as "not very complicated".

Date: February 2, 2014.

2010 *Mathematics Subject Classification.* Primary:03C45; Secondary:03C55.

Key words and phrases. model theory, classification theory, stability, representation, superstable.

The author thanks Alice Leonhardt for the beautiful typing. First typed June 3, 2013.

§ 0. Introduction

We continue Cohen-Shelah [CoSh:919], which deals with the cases of T stable and $\aleph_0$-stable. We give here a complete answer also for the superstable case.

§ 1. Superstable theories

The main theorem in this section is

{d2}

Theorem 1.1. *For a first-order, complete theory T the following are equivalent:*

(1) T *is superstable.*
(2) T *is representable in* $\mathrm{Ex}^2_{2^{|T|},\aleph_0}(\mathfrak{t}^{\mathrm{eq}})$
(3) T *is representable in* $\mathrm{Ex}^1_{2^{|T|},2}(\mathfrak{t}^{\mathrm{eq}})$
(4) T *is representable in* $\mathrm{Ex}^{0,\mathrm{lf}}_{2^{|T|},2}(\mathfrak{t}^{\mathrm{eq}})$
(5) T *is representable in* $\mathrm{Ex}^2_{\mu,\aleph_0}(\mathfrak{k}^{\mathrm{eq}})$ *for some cardinal* μ
(6) T *is representable in* $\mathrm{Ex}^{0,\mathrm{lf}}_{\mu,\kappa}(\mathfrak{k}^{\mathrm{eq}})$ *for some cardinals* μ,κ.

Proof. $2 \Rightarrow 5, 4 \Rightarrow 6$ are immediate.

$2 \Rightarrow 3$ is direct from [CoSh:919, 1.30]

$3 \Rightarrow 4$ direct from [CoSh:919, 1.24]

$5 \Rightarrow 6$

This follows since $\mathrm{Ex}^2_{\mu,\aleph_0}(\mathfrak{k}^{\mathrm{eq}})$ is qf-representable in $\mathrm{Ex}^1_{\mu,2}(\mathfrak{k}^{\mathrm{eq}}) \subseteq \mathrm{Ex}^{0,\mathrm{lf}}_{\mu,2}(\mathfrak{k}^{\mathrm{eq}})$ by [CoSh:919, 1.30] and $\mathrm{Ex}^1_{\mu,2}(\mathfrak{k}^{\mathrm{eq}}) \subseteq \mathrm{Ex}^{0,\mathrm{lf}}_{\mu,2}(\mathfrak{k}^{\mathrm{eq}})$ by [CoSh:919, 1.24] with 2 here standing for κ there.

The rest follows from Theorem 1.2 below giving $1 \Rightarrow 2$ and Theorem 1.3 below giving $6 \Rightarrow 1$. {d8} $\square_{1.1}$

{d5}

Theorem 1.2. *If T is representable in* $\mathrm{Ex}^{0,\mathrm{lf}}_{\mu,\kappa}(\mathfrak{k}^{\mathrm{eq}})$ *for some cardinals* μ,κ *then* T *is superstable.*

Proof. Like the proof of Propositions [CoSh:919, Th.2.4,2.5]. $\square_{1.2}$

{d8}

Theorem 1.3. *Every superstable T is representable in* $\mathrm{Ex}^2_{2^{|T|},\aleph_0}(\mathfrak{k}^{\mathrm{eq}})$.

Proof. Let T be superstable. Let $M \models T$. We choose $B_n, \langle a_s, u_s : s \in S_n\rangle$ by induction on $n < \omega$ such that:

(1) $S_n \cap S_k = \emptyset$ $(k < n)$
(2) $\langle a_s : s \in S_n\rangle \subseteq M$
(3) $B_n = \{a_s : s \in S_{<n}\} \subseteq M$ (where $S_{<n} := \cup\{S_k : k < n\}$, as usual)
(4) $\langle a_s : s \in S_n\rangle$ is without repetitions, disjoint from $\{a_s : s \in S_{<n}\}$ and independent over B_n,
(5) for all $s \in S, u_s \subseteq S_{<n}$ is finite such that $t \in u_s \Rightarrow u_t \subseteq u_s$ and $\mathrm{tp}(a_s, B_n)$ does not fork over $\{a_t : t \in u_s\}$
(6) $\langle a_s : s \in S_n\rangle$ is maximal under conditions 1-5.

Here we make a convention that u, v, w vary on $\mathscr{I}$ defined below:

$\circledast_1$ It is possible to carry the induction

$\circledast_2$ (0) $\mathscr{I} = \{u : u \subseteq S\}, S = \bigcup_n S_n$

(1) we define $\mathscr{I}^{\mathrm{cl}} = \{u \in \mathscr{I} : u = \mathrm{cl}(u)\}$;

(2) for $v \in \mathscr{I}$ let $\mathrm{cl}(v)$ be the minimal $u \supseteq v$ such that $u_t \subseteq u$ holds for all $t \in u$;

$\circledast_3$ (0) if $u \in \mathscr{I}$ then $\mathrm{cl}(u) \in \mathscr{I}$

(1) $v \subseteq u \Rightarrow \mathrm{cl}(v) \subseteq \mathrm{cl}(u)$;

(2) $\mathrm{cl}(u_1 \cup u_2) = \mathrm{cl}(u_1) \cup \mathrm{cl}(u_2)$;

(3) $\mathrm{cl}(\{s\}) = u_s \cup \{s\}$;

(4) $\mathrm{cl}(\mathrm{cl}(u)) = \mathrm{cl}(u)$;

(5) $\mathrm{cl}(u) = \bigcup\{u_s : s \in u\} \cup u$

$\circledast_4$ $|M| = \{a_s : s \in S\}$ (Why? As in §2 - that is, otherwise, there exists $a \in |M| \backslash \{a_s : s \in S\}$ such that (since T is representable) $\mathrm{tp}(a, \{a_s : s \in S\})$ does not fork over $\{a_s : s \in v\}$ for some finite subset $v \subseteq \{a_s : s \in S\}$. Let $u = \mathrm{cl}(v)$, so $u \in \mathscr{I}^{\mathrm{cl}}$ and let n be such that $u \subseteq S_n$ and we get a contradiction to the maximality of $\{a_s : s \in S_n t\}$.)

$\circledast_5$ Let $\langle v_\alpha : \alpha < \alpha(*)\rangle$ enumerate $\mathscr{I}$ (without repetition) such that

(1) $v_\alpha \subseteq v_\beta \Rightarrow \alpha \leq \beta$;

(2) $\alpha < \beta \wedge v_\beta \subseteq S_{<n} \Rightarrow v_\alpha \subseteq S_{<n}$. So, v_α is not necessarily closed.

We choose a model M_{v_α} by induction on α such that:

$\circledast_6$ (1) $M_{v_\alpha} \prec \mathfrak{C}_T$ has cardinality $\leq \aleph_0 + |T|$;

(2) $v_\alpha \subseteq v_\beta \Rightarrow M_{v_\alpha} \prec M_{v_\beta}$;

(3) $\bigcup\{M_{v_\alpha} : \beta < \alpha \wedge v_\beta \subseteq v_\alpha\} \not\supseteq M_{v_\beta}$;

(4) if $\gamma \in v_\alpha$ and $u_s \subseteq v_\alpha$ then $a_s \in M_{v_\alpha}$;

(5) $\mathrm{tp}(M_{v_\alpha}, M \cup \{M_\beta : \beta < \alpha\})$ does not fork over $B_{v_\alpha} := \cup\{M_{v_\beta} : v_\beta \subseteq v_\alpha,\ \beta < \alpha\} \cup \{a_s : \mathrm{cl}(\{s\}) \subseteq v_\alpha\}$

$\circledast_7$ By $\circledast_6(3)$, clearly $\alpha < \beta \Rightarrow M_{v_\alpha} \neq M_{v_\beta}$.

The induction is clearly possible.

A major point is

$\circledast_8$ $\mathrm{tp}(M_{v_\alpha}, \cup\{M_{v_\beta} : u_\beta \subseteq u_\alpha and \beta < \alpha\})$ does not fork over $A_{v_\alpha} := \cup\{M_{v_\beta} : \beta < \alpha\}$.

[Why? If $v_\alpha = \emptyset$ this is trivial so assume $v_\alpha \neq \emptyset$ let n be such that $v_\alpha \subseteq S_{\leq n}, v_\alpha \nsubseteq S_{<n}$ and let $\langle t_\ell : \ell < k\rangle$ list $\{s \in v_\alpha : s \notin S_{<n}$ and $\mathrm{cl}(\{s\}) \subseteq v_\alpha\}$.

First, assume $k = 0$. So if $s \in v_\alpha$ and $\mathrm{cl}(\{s\}) \subseteq v_\alpha$ then $s \in v_\alpha \cap S_{<n}$, this implies that $\mathrm{cl}(\{s\}) \subseteq S_{<n}$ hence $a_s \in M_{v_\alpha \cap S_{<n}} \subseteq A_{v_\alpha}$. This implies that $B_{v_\alpha} \subseteq A_{v_\alpha}$ (in fact equal - see their definitions in $\circledast_6(5), \circledast_8$). Now $\circledast_6$ says that $\mathrm{tp}(M_{v_\alpha}, \cup\{M_{v_\beta} : \beta < \alpha\})$ does not fork over B_{v_α}, so by monotonicity of non-forking and the last sentence, it does not fork over A_{v_α} as desired.

Second, assume $k = 1$ and $\forall(\beta < \alpha)[\mathrm{cl}(\{t_0\}) \nsubseteq v_\beta]$,

Now

$\odot_1$ $B_{v_\alpha} = A_{v_\alpha} \cup \{a_{t_0}\}$; and clearly,

$\odot_2$ $\mathrm{tp}(a_{t_0}, \{a_s : s \in S_{\leq n} \backslash \{t_0\}\})$ does not fork over $\{a_{s_\beta} : s_\beta \in u_s$ and $u_s \subseteq v_\alpha \cap S_{<n}\}$;

$\odot_3$ $\mathrm{tp}(\cup\{M_{v_\beta} : \beta < \alpha\}, \{a_s : s \in S_{\leq n}\}) \subseteq \mathrm{tp}(\cup\{M_{v_\beta} : v_\beta \subseteq S_{\leq n}, \mathrm{cl}(\{t_0\}) \nsubseteq v_\beta\}, \{a_s : s \in S_{\leq n}\})$ do not fork over $\{a_s : s \in S_{\leq n} \backslash \{t_0\}\}$ [by $\circledast_6$].

By the non-forking calculus, $\mathrm{tp}(a_{t_0}, \cup\{M_{v_\beta:\alpha<\beta}\})$ does not fork over $a_s : s \in S_{\leq n}, s \neq t_0$. But we know that $\mathrm{tp}(M_{v_\alpha}, \cup\{M_{v_\beta:\alpha<\beta}\})$ does not fork over $B_{v_\alpha} = A_{v_\alpha} \cup \{t_0\}$. Together, $\mathrm{tp}(M_{v_\alpha}, \cup\{M_{v_\beta:\alpha<\beta}\})$ does not fork over A_{v_α}, as desired in $\circledast_8$.

Third, assume $k = 1, \beta < \alpha$ and $\mathrm{cl}(\{t_0\}) \subseteq v_\beta$. Without loss of generality β is minimal with these properties, so necessarily $v_\beta = \mathrm{cl}(\{t_0\})$ and so again, $B_{v_\alpha} = A_{v_\alpha}$ and we continue as in "First" above.

Fourth, assume $k \geq 2$. In this case, for each $\ell < k, \mathrm{cl}(\{t_\ell\})$ is $v_{\beta(\ell)}$ for some unique $\beta(\ell) < \alpha$, so $a_{t_\ell} \in M_{v_{\beta(\ell)}} \subseteq A_{v_\alpha}$, hence, $B_{v_\alpha} \subseteq A_{v_\alpha}$ (in fact equal) and again $\circledast_6$ gives the desired conclusion.]

Now, $\circledast_8$ is the necessary condition in Fact **??** and so we can conclude that $\langle M_v : v \in \mathscr{I} \rangle$ is a stable system (see Definition **??**). {?} {?}

Now $\langle M_v : v \in \mathscr{I} \rangle$ is a stable system of models. For all $v \in \mathscr{I}$ let $\bar{b}_v$ enumerate M_v. By $\circledast_6(3)$, $\langle \bar{b}_v : v \in \mathscr{I} \rangle$ is without repetitions.

For all $\alpha < \omega \times \omega$ we define $\mathscr{I}_\alpha$ as follows:

$\oplus_1$ (1) $\mathscr{I}_0 = \{\emptyset\}$;
(2) $\mathscr{I}_k = \{v \in \mathscr{I} : v \subseteq S_{<1}, |v| = k\} (k < \omega)$;
(3) $\mathscr{I}_{\omega n+k} = \{v \in \mathscr{I} : v \nsubseteq S_{<n}, v \subseteq S_{<n+1}, |v| = k+1\}, (k < \omega,\ 0 < n < \omega)$.

Now clearly

$\oplus_2$ $w \subseteq v \in \mathscr{I}_\alpha \Rightarrow w \in \mathscr{I}_{<\alpha}$ for all $w, v \in \mathscr{I}$
$\oplus_3$ For all $\alpha < \omega \times \omega$ let $B_\alpha := \cup\{M_v : v \in \mathscr{I}_{<\alpha}\}$.

So,

- B_α is increasing and continuous
- $B_{\alpha+1} = \cup\{\bar{b}_v : v \in \mathscr{I}_\alpha\} \cup B_\alpha$
- $v \in \mathscr{I}_\alpha \wedge w \subseteq v \Rightarrow w \in v$ and most important
  $\boxtimes$ for all $\alpha < \omega \times \omega$ and $v \in \mathscr{I}_\alpha$ the type
  $$p_v := \mathrm{tp}(\bar{b}_v, \cup\{\bar{b}_u : u \in \mathscr{I}_{<\alpha+1} \wedge u \neq v\})$$
  does not fork over $\cup\{\bar{b}_w : w \subseteq v\}$, and p_v is the unique non-forking extension of $p_v \restriction \cup\{\bar{b}_w : w \subseteq v\}$ in $\mathbf{S}^{\ell g(\bar{b}_v)}(\cup\{\bar{b}_u : u \in \mathscr{I}_{<\alpha+1} \wedge u \neq v\})$.

The proof is carried by basic properties of stable systems (see Conclusion 2.12 in [Sh:c, Ch.XII,p.605]).

Now we define an equivalence relation E_α on $\mathscr{I}$, (and $E_\alpha = E \restriction \mathscr{I}_\alpha$) such that $v_1 E_\alpha v_2$ iff for some $g = g_{v_1,v_2}, f = f_{v_1,v_2}$ (really g determines f. we may require g to be order preserving)

$\oplus_2$ (1) $v_1, v_2 \in \mathscr{I}$;
(2) $|v_1| = |v_2|$;
(3) g is one-to-one from v_1 onto v_2 (we may add mapping $v_1 \cap S_{<n}$ onto $v_2 \cap S_{<n}$ for every n such that $g''(u_1) = u_2 \Rightarrow u_B \vee [u_1 \in \mathscr{I}_\beta \equiv u_2 \in \mathscr{I}_\beta]$;
(4) $u_{g(t)} = \{g(a_s) : s \in u_t\}$
(5) f is an elementary mapping of $\mathfrak{C}_T$
(6) if $u_\ell \subseteq v_\ell$ for $\ell = 1,2$ and g maps u_1 onto u_2, then f maps $\bar{b}_{u_1}$ to $\bar{b}_{u_2}$
(7) $\mathrm{Dom}(f) = \cup\{\bar{b}_u : u \subseteq v_1\}$.

(element-by-element, and this implies f_{v_1,v_2} is unique). (So for some bijection $g_{v_1,v_2} : v_1 \to v_2$ which preserves being in I_β for all $\beta < \alpha$, such that f_{v_1,v_2} maps $\bar{b}_{w_1}$ to $\bar{b}_{g_{v_1,v_2}(w_1)}$ for all $w_1 \subset v_1$.) Let $\langle I_{\alpha,i} : i < i(\alpha) \leq 2^{|T|} \rangle$ enumerate the equivalence classes of E_α.

We get that $\boxplus_1 \Rightarrow \boxplus_2$ where

$\boxplus_1$ (α) the sets $\{v_0 \dots v_{n-1}\}, \{u_0 \dots u_{n-1}\} \subset \mathscr{I}$ are closed under subsets

(β) $\bigwedge_{\alpha,i} [v_l \in \mathscr{I}_{\alpha,i} \Leftrightarrow u_l \in \mathscr{I}_{\alpha,i}]$

(γ) $u_{l(1)} \subset u_{l(2)} \Leftrightarrow v_{l(1)} \subset u_{l(2)}$ for all $l(1), l(2) < n$

(δ) for all $v_{l(1)} \subseteq v_{l(2)}$ it holds that $g_{v_{l(2)},u_{l(2)}}$ maps $v_{l(1)}$ onto $u_{l(1)}$.

$\boxplus_2$ The sequences $\bar{b}_{v_0} \frown \dots \frown \bar{b}_{v_{n-1}}$ and $\bar{b}_{u_0} \frown \dots \frown \bar{b}_{u_{n-1}}$ realize the same complete type over $\emptyset$. (This follows from the definitions of the equivalence relations E_α and $\boxtimes$ above).

Without loss of generality $\mathscr{I} \cap S = \emptyset$. We define a structure $\mathscr{A}$ with universe $|\mathscr{A}| = \mathscr{I} \cup S$ as follows:

$$P_\alpha^{\mathscr{A}} := \mathscr{I}_\alpha, P_{\alpha,i}^{\mathscr{A}} := \mathscr{I}_{\alpha,i}, S_n^{\mathscr{A}} d := S_n.$$

Define partial functions $F_\epsilon^{\mathscr{A}}$ for all $\epsilon < |T|, n < \omega$ as $F_\epsilon^{\mathscr{A}}(s) = b^\epsilon_{u_s \cup \{s\}}$. Define partial functions G_n such that $\{G_n(u) : n\} = \{v : v \subseteq u\}$. Now, the function f well defined as $f(a) = s \Leftrightarrow a = a_s$ is a representation of M in $\mathscr{A}$ (proof should be clear from the definitions). $\square_{1.3}$

§ 2. Between stable and superstable

Discussion 2.1. 1) For superstable T, we may wonder about whether "$2^{|T|}$ is optimal". Really, $\lambda(T)$ is sufficient where {e3}

$(*)_{1.1}$ $\lambda(T) = \min\{\lambda : T \text{ is stable in } \lambda\}$.

Note that

$(*)_{1.2}$ If T is countable then $\lambda(T) = \aleph_0$ is equivalent to T is $\aleph_0$-stable and
$(*)_{1.3}$ if T is countable and $\lambda(T) > \aleph_0$ then $\lambda(T) = 2^{\aleph_0}$.

2) Why indeed in Theorem 1.3 does $\mathrm{Ex}^2_{\lambda(T),\aleph_0}(\mathfrak{k}^{\mathrm{eq}})$ suffice? Choosing M_{v_α} by induction on α we now claim {d8}

$\circledast^*_6$ (1) $M_\alpha \prec \mathfrak{C}$ is saturated of cardinality $\lambda(T)$;
(2) $v_\beta \subseteq v_\alpha \Rightarrow M_{v_\beta} \subseteq M_{v_\alpha}$
(3) $(M_\alpha, c)_{c \in \cup\{M_{v_\beta} : v_\beta \subset v_\alpha\}}$ is saturated.

Now before $\oplus_1$ in the proof we can add, without loss of generality

$\circledast_9$ if g is one-to-one from v_α onto v_β then there is a unique $f = fg$ such that
(a) f is an isomorphism from M_{v_α} onto M_{v_β}
(b) if $u_2 = \{g(s) : s \in u_1\}$ then f maps $\bar{b}_{u_1}$ onto $\bar{b}_{u_2}$.

So when we arrive to E, it has at most $\lambda(T)$ equivalence classes.

§ 2(A). **Characterization of $\kappa(T)$.**

Theorem 2.2. *For a complete theory T, and κ the following are equivalent:* {e8}

(1) $\kappa(T) \le \kappa^+$
(2) *T is representable in* $\mathrm{Ex}^2_{2^{|T|},\kappa}(\mathfrak{k}^{\mathrm{eq}})$
(3) *T is representable in* $\mathrm{Ex}^2_{\mu,\kappa}(\mathfrak{k}^{\mathrm{eq}})$ *for some μ*
(4) *T is representable in* $\mathrm{Ex}^1_{2^{|T|},\kappa}(\mathfrak{k}^{\mathrm{eq}})$
(5) *T is representable in* $\mathrm{Ex}^1_{\mu,\kappa}(\mathfrak{k}^{\mathrm{eq}})$ *for some μ.*

Remark 2.3. In Theroem 2.2, demanding κ be regular, we can change clauses (2)-(5) above to (2')-(5') which mean that we use only $\mathbf{I}$ such that the closure of any set of cardinality $< \kappa$ has cardinality $< \kappa$ and change (1) to (1') $\kappa(T) \le \kappa = \delta$. The proof is similar. {e11} {e8}

Proof. By Theorem 1.1, without loss of generality $\kappa > \aleph_0$. The proof is continuing the proof of the stable case and the superstable case. So, cannibalizing the proof of Theorem 1.3 and/or [CoSh:919, 2.15] we have $\langle a_s : s \in s_i\rangle$ for $i < \kappa^+, u_s \in [S_{<i}]^{<\kappa}$ for $s \in S_i, \mathrm{cl}(u) = \cup\{u_s : s \in u\} \cup u$ for $u \subseteq S = S_{<\kappa^+}$ and $\mathscr{I} = \{u : u \subseteq u_s \cup \{s\}$ for some $s \in S\}$, or just $[S]^{<\kappa}$, it does not matter. Letting $\langle v_\alpha : \alpha < \alpha(*)\rangle$ be a listing of $\mathscr{I}$ such that $v_\alpha \subset v_\beta \Rightarrow \alpha < \beta$. Let {d2} {d8}

$$[\mathscr{I}_\alpha = \{u : \quad \text{for some } \beta < \alpha, u \subseteq v_\beta, \text{ and } v \text{ belongs to the closure of} \\ \{u_s, u_\alpha : \cup\{s\} : s \in S\} \text{ under intersection}\}$$

Now by induction on α we choose $\langle M_v : v \in \mathscr{I}_\alpha \rangle$ such that:

$\oplus$ (1) $M_{u_1} \cap M_{u_2} = M_{u_1 \cap u_2}$;
(2) if $\mathrm{tp}(M_u, \cup\{M_{w_\ell} : \ell < n\})$ does not fork over $\cup\{M_{w_\ell \cap u} : \ell < n\}$

(so we refine a stable system and have to prove)

$\boxplus$ We say that $\mathbf{a}$ is an independent system where $\mathbf{a}$ consists of
- $\bar{A} = \langle A_v : v \in \mathscr{I} \rangle$;
- $\mathscr{I}$ is a family of subsets of $S = \cup \mathscr{I}$;
- $\emptyset \in \mathscr{I}$;
- $\mathscr{I}$ is closed under any intersection
- $v \subseteq u \Rightarrow A_v \subseteq A_u$ for all $v, u \in \mathscr{I}$
- $A_{v \cap u} = A_v \cap A_u$ for all $v, u \in \mathscr{I}$
- $\mathrm{tp}(A_v, \cup\{A_{u_\ell} : \ell < n\}$ does not fork over $\cup\{A_{v \cap u_\ell} : \ell < n\}$ for all $v, u_0, \ldots, u_{n-1} \in \mathscr{I}$

$\boxplus_1$ (a) for $v \subseteq S_{\mathbf{a}}$ let $A_v(\mathbf{a}) = \cup\{A_u(\mathbf{a}) : u \subseteq v, u \in \mathscr{I}_{\mathbf{a}}\}$;
(b) $\mathbf{b} = \mathrm{cl}(\mathbf{a})$ where $\mathbf{a}$ is an independent system consists of $\mathscr{I}_{\mathbf{b}} = \{u : u \subseteq S_{\mathbf{a}}\}, A_v = A_v(\mathbf{a})$;
(c) $\mathbf{b} = \mathrm{dcl}(\mathbf{a})$ is as above but $\mathscr{I}_{\mathbf{b}} = \{u : u \in v$ for some $v \in \mathscr{I}_{\mathbf{a}}\}$

$\boxplus_2$ for $\mathbf{a}$ an independent system (as above)
(a) if $v, u_i \in \mathscr{I}$ if $i < i(*)$ then $\mathrm{tp}(A_v, \cup\{A_{u_i} : i < i(*)\}$ does not fork over $\cup\{A_{v \cap u_i} : i < i(*)\}$;
(b) if $v \subseteq S, u_i \in \mathscr{I}$ for $i < i(*$ then $\mathrm{tp}(A_v[\mathbf{a}], \cup\{A_{u_i} : i < i(*)\})$ does not fork over $\cup\{A_{v \cap u_i}[\mathbf{a}] : i < i(*)\}$.

$\square_{2.2}$

Remark 2.4. Why? for clause (a) it suffices to prove that for every finite $\bar{a} \subseteq A_v$ and finite $B \subseteq \cup\{A_{u_i} : i < i(*)\}$, the type $\mathrm{tp}(\bar{a}, \cup\{A_{u_i} : i < i(*)\}$ does not fork over $\cup\{A_{v \cap u_i} : i < i(*)\}$, but for some finite n and $i(0) < \ldots < i_{n-1} < i(*)$ we have $A' \subseteq \cup\{u_{i(\ell)} : \ell < n\}$ and use the definition. For clause (b), it suffices to prove for $n < \omega, v_\ell \subseteq v, v_\ell \in \mathscr{I}$ for $\ell < n$ that $\mathrm{tp}(\cup\{A_{v_\ell} : \ell < n\}, \cup\{A_{u_i} : i < i(*)\}$ does not fork over $\cup\{A_{v \cap u_i}[\mathbf{a}] : i < i(*)\}$. For this it suffices for $k < n$ to prove $\mathrm{tp}(A_{v_k}, \cup\{A_{u_i} : i < i(*)\} \cup \{A_{v_\ell} : \ell < k\})$ does not fork over $\cup\{A_{v \cap u_i} : i < i(*)\} \cup \{A_{v \cap v_\ell} : \ell < k\}$, but this follows by (a)

$\oplus$ if $\mathbf{a}$ is an independent system, then so are $\mathrm{dcl}(\mathbf{a}), \mathrm{cl}(\mathbf{a})$.

For this we need

$\oplus$ assume we are given $\mathbf{a}, \mathscr{I}, P$ satisfying (A), then we can find $\mathbf{b}$ satisfying (B) where
(A) (a) $\mathbf{a}$ is an independent system, $S = S_{\mathbf{a}}$;
(b) $\mathscr{I} = \mathscr{I}_{\mathbf{a}}$ is closed under subsets
(c) $\lambda = \lambda^{<\kappa(T)} + \lambda(T) \geq \Sigma\{M_v : v \in \mathscr{I}\} + 2^{|S_{\mathbf{a}}|}$
(B) (a) $\mathbf{b}$ an independent system;
(b) $\mathbf{a} = \mathbf{b} \restriction \mathscr{I}_{\mathbf{a}}$;
(c) $M_v \prec \mathfrak{C}$ has cardinality λ for $v \in P \backslash \mathscr{I}$;

(d) $(M_v, c)_{c \in A_v}$ is saturated where $A_v = \cup\{M_u \cap M_v : u \subset v\}$.

Remark 2.5. Why? Let AP be the set of $\mathbf{b}, \bar{A} = \langle A_v : v \in P\rangle$ satisfying

- $\mathbf{b}$ an independent system,
- $\mathbf{b} \restriction \mathscr{I} = \mathbf{a}$,
- $|A_v| \leq \lambda$.

So

$\ominus_1$ $AP \neq \emptyset$.

[Why? Let $\mathbf{a} = \mathrm{cl}(\mathbf{a})$.]

Let $\leq_{AP}$ be the following two-place relation on AP:

$\ominus_2$ $\mathbf{b}_1 \leq_{AP} \mathbf{b}_2$ iff both are from AP and if $u \in P, v_i \in P$ for $i < i(*)$ then $\mathrm{tp}(A_u(\mathbf{b}_2), \cup\{A_{v_i}[\mathbf{b}_2] : i < i(*)\} \cup A_u(\mathbf{b}_1)$ does not fork over $\cup\{A_{v_i \cap u}[\mathbf{b}_2] : i < i(*)\} \cup A_u(\mathbf{b}_1)$

$\ominus_3$ $\leq_{AP}$ is a partial order: if $\bar{\mathbf{b}} = \langle \mathbf{b}_\alpha : \alpha < \delta\rangle$ is $\leq_{AP}$-increasing, and $\mathbf{b}_\delta = \cup\{\mathbf{b}_\alpha : \alpha < \delta\}$ is naturally defined, then $\mathbf{b}_\delta$ is an $\leq_{AP}$-upper bound (even least upper bound) of $\bar{\mathbf{b}}$.

[Why? Easy.]

$\ominus_4$ if $\mathbf{b} \in AP, u \in P \backslash \mathscr{I}, p \in \mathbf{S}(A_u(\mathbf{b}))$, then for some $\mathbf{c}$ and $\mathbf{b} \leq_{AP} \mathbf{c}$ and p is realized by some $a \in A_u(\mathbf{c})$.

[Why? Let $q \in \S(A_s(\mathbf{b}))$ extend p and non-forking over $A_u(\mathbf{b})$. Let $a \in \mathfrak{C}$ realize q and define $\mathbf{b}$ by letting

$$A_v(\mathbf{c}) = \begin{cases} A_v(\mathbf{b}) & \text{if} \quad u \nsubseteq v \\ A_v(\mathbf{b}) \cup \{a\} & \text{if} \quad u \subseteq v \end{cases}$$

Now check.]

The rest should be clear.

References

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