RANDOM REALS AND POLARIZED COLORINGS

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Abstract. We analyze the strong polarized partition relation with respect to several cardinal characteristics and forcing notions of the reals. We prove that random reals (as well as the existence of real-valued measurable cardinals) yield downward negative polarized relations.

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0. Introduction

This paper focuses on two cardinal characteristics of the continuum, the reaping number $\mathfrak{r}$ and the splitting number $\mathfrak{s}$. Let us commence with the basic definitions of these invariants:

Definition 0.1. The reaping number.

(א) Suppose $B \in [\omega]^\omega$ and $S \subseteq \omega$. S splits B if $|S \cap B| = |(\omega \setminus S) \cap B| = \aleph_0$.

(ב) $\{T_\alpha : \alpha < \kappa\}$ is an unreaped family if there is no single $S \in [\omega]^\omega$ so that S splits T_α for every $\alpha < \kappa$.

(ג) The reaping number $\mathfrak{r}$ is the minimal cardinality of an unreaped family.

(ד) $\mathfrak{r}_\sigma$ is the minimal cardinality of a collection $\mathcal{R} \subseteq [\omega]^\omega$ which is not splitted by ω-many sets.

The dual of the reaping number is the splitting number. Recall:

Definition 0.2. The splitting number.

(א) Suppose $B \in [\omega]^\omega$ and $S \subseteq \omega$. S splits B if $|S \cap B| = |(\omega \setminus S) \cap B| = \aleph_0$.

(ב) $\{S_\alpha : \alpha < \kappa\}$ is a splitting family in ω if for every $B \in [\omega]^\omega$ there exists an ordinal $\alpha < \kappa$ so that S_α splits B.

(ג) The splitting number $\mathfrak{s}$ is the minimal cardinality of a splitting family in ω.

In the present paper, combinatorial arguments serve in the area of cardinal invariants of the continuum. The main tool is the following concept. If $\lambda \geq \kappa$ are infinite cardinals then the strong polarized relation $\binom{\lambda}{\kappa} \to \binom{\lambda}{\kappa}^{1,1}_2$ means that for every $c : \lambda \times \kappa \to 2$ there are $A \in [\lambda]^\lambda, B \in [\kappa]^\kappa$ such that $c \restriction (A \times B)$ is constant. We shall make use of the following theorem from [6]:

Claim 0.3. *Strong polarized relations below the splitting number.*
Assume $\kappa < \mathfrak{s}$.
The positive relation $\binom{\kappa}{\omega} \to \binom{\kappa}{\omega}^{1,1}_2$ *holds iff* $\mathrm{cf}(\kappa) > \aleph_0$.

$\square_{0.3}$

In a way, the splitting number $\mathfrak{s}$ is a natural point for proving downward positive relations. The dual notion of the reaping number $\mathfrak{r}$ is a natural point for upward positive relations, as shown in the following theorem from [7]:

Theorem 0.4. *Strong polarized relations above the reaping number.*
Assume $\mathfrak{r} < \kappa \leq \mathfrak{c}$.
If $\mathfrak{r} < \mathrm{cf}(\kappa)$ *then* $\binom{\kappa}{\omega} \to \binom{\kappa}{\omega}^{1,1}_2$.

$\square_{0.4}$

Observe that the requirement about the cofinality of κ is stronger in this theorem, and we do not have full knowledge when $\aleph_0 < \mathrm{cf}(\kappa) \leq \mathfrak{r}$, see below.

However, some kind of duality is reflected in this theorem. The keypoint for the results of this paper is that we can prove also a *negative* downward theorem with respect to $\mathfrak{r}$. It enables us to supply a positive answer to the following open problem from [8] (Problem 3.19 there):

Question 0.5. Suppose $\aleph_1 < \kappa = \mathrm{cf}(\kappa) < \lambda = \mathfrak{c}$ and $\binom{\kappa}{\omega} \rightarrow \binom{\kappa}{\omega}^{1,1}_2$. Let $\mathbb{P}$ be Lévy(κ, λ). Is it possible that $\binom{\kappa}{\omega} \nrightarrow \binom{\kappa}{\omega}^{1,1}_2$ in $\mathbf{V}^{\mathbb{P}}$?

As we shall see, a wide range of positive answers can be given to the above problem, i.e., for every regular cardinal κ we can build a model of ZFC in which the Lévy collapse destroys the strong polarized relation for κ.

Another problem from [8] focuses on the random real forcing. One of the salient properties of random reals is that they are dominated by old reals from the ground model, hence the dominating number $\mathfrak{d}$ is unchanged. Problem 3.9 (in [8]) asks if one can ruin the positive relation for $\mathfrak{d}$ while iterating random real forcing notions:

Question 0.6. Assume $\binom{\mathfrak{d}}{\omega} \rightarrow \binom{\mathfrak{d}}{\omega}^{1,1}_2$, and one adds λ-many random reals (for some $\lambda > \mathfrak{d}$). Does the positive relation $\binom{\mathfrak{d}}{\omega} \rightarrow \binom{\mathfrak{d}}{\omega}^{1,1}_2$ still hold?

As in the former question, we will be able to show that one can force a negative relation for $\mathfrak{d}$ after adding random reals to the universe. This follows, again, from the downward negative relation for $\mathfrak{r}$. Moreover, the implication of negative results upon adding many random reals is wider. This gives rise to the third problem (number 3.12) that we quote:

Question 0.7. Assume κ is a real-valued measurable cardinal.
Is it possible that $\binom{\kappa}{\omega} \rightarrow \binom{\kappa}{\omega}^{1,1}_2$?

Concerning this question, we will be able to supply only a partial answer, by proving many negative relations *below* the real-valued measruable cardinal. We comment that these relations would give the interesting corollary that $\mathfrak{s} = \aleph_1$ whenever a real-valued measurable cardinal exists. Likewise, under the additional assumption that $\kappa = \mathfrak{c}$ is real-valued measurable we shall prove that $\binom{\kappa}{\omega} \nrightarrow \binom{\kappa}{\omega}^{1,1}_2$ as well.

Our notation is standard. We shall follow [1] with respect to cardinal invariants. We employ the Jerusalem forcing notation, so $p \leq q$ means that q is stronger than p. In cases where ambiguity lurks around the corner we shall state the pertinent conventions explicitly. We thank the referee for the careful reading of the manuscript.

1. Continuum reaping

Investigating the interplay between cardinal invariants and strong polarized relations, one may ask whether there exists a natural cardinal invariant $\mathfrak{y}$ so that $\binom{\mathfrak{y}}{\omega} \nrightarrow \binom{\mathfrak{y}}{\omega}^{1,1}_2$ always holds. The answer is negative. It is shown in [7] that for every cardinal invariant, the above polarized relation is independent. Nonetheless, there is a natural characteristic for which the negative relation follows, under the extra assumption of being the continuum. Our basic result says that if the reaping number and the continuum coincide, then a negative polarized parition relation can be proved dwon to the cofinality of the continuum:

Main Claim 1.1. *Negative downward relations.*
If $\mathfrak{r} = \mathfrak{c}$ *Then* $\binom{\mathfrak{r}}{\omega} \nrightarrow \binom{\mathfrak{r}}{\omega}^{1,1}_2$.
Moreover, $\binom{\kappa}{\omega} \nrightarrow \binom{\kappa}{\omega}^{1,1}_2$ *for every* $\kappa \in [\mathrm{cf}(\mathfrak{c}), \mathfrak{c}]$.

Proof.
Enumerate the members of $[\omega]^\omega$ by $\{B_\gamma : \gamma < \mathfrak{r}\}$. For every $\alpha < \mathfrak{r}$ let $\mathcal{B}_\alpha = \{B_\gamma : \gamma < \alpha\}$. The size of $\mathcal{B}_\alpha$ is less than $\mathfrak{r}$, so we can choose a set S_α which splits all the members of $\mathcal{B}_\alpha$. This is rendered for every $\alpha < \mathfrak{r}$, and gives rise to a coloring $d : \mathfrak{r} \times \omega \to \{0, 1\}$ as follows:

$$d(\alpha, n) = 0 \Leftrightarrow n \in S_\alpha.$$

We claim that d exemplifies the negative relation $\binom{\mathfrak{r}}{\omega} \nrightarrow \binom{\mathfrak{r}}{\omega}^{1,1}_2$. Indeed, assume $H \in [\mathfrak{r}]^\mathfrak{r}$ and $B \in [\omega]^\omega$. Assume toward contradiction that $d \restriction (H \times B)$ is constant. If the constant value is 0 then $B \subseteq S_\alpha$ for every $\alpha \in H$, and if the constant value is 1 then $B \subseteq (\omega \setminus S_\alpha)$ for every $\alpha \in H$.

In any case, The set B appears in the above enumeration, so $B \equiv B_\gamma$ for some $\gamma < \mathfrak{r}$. Choose an ordinal $\alpha \in H$ so that $\gamma < \alpha$, and notice that S_α splits B, a contradiction.

Moreover, if $\kappa \geq \mathrm{cf}(\mathfrak{c})$ then we enumerate the members of $[\omega]^\omega$ as $\{B_\gamma : \gamma < \mathfrak{c}\}$. We choose an increasing and unbounded sequence of ordinals $\langle \alpha_\varepsilon : \varepsilon < \kappa \rangle$ in $\mathfrak{c}$ (repetitions are welcome), and define $\mathcal{B}_\varepsilon = \{B_\gamma : \gamma < \alpha_\varepsilon\}$ for every $\varepsilon < \kappa$. Again, $|\mathcal{B}_\varepsilon| < \mathfrak{r}$ for every $\varepsilon < \kappa$ as $\mathfrak{r} = \mathfrak{c}$. For every $\varepsilon < \kappa$ we choose some $S_\varepsilon \in [\omega]^\omega$ which splits all the members of $\mathcal{B}_\varepsilon$.

The coloring $d : \kappa \times \omega \to \{0, 1\}$ is defined in the same way, i.e., $d(\varepsilon, n) = 0 \Leftrightarrow n \in S_\varepsilon$. The same argument shows that d has no monochromatic product of size $\kappa \times \omega$, so we are done.

$\square_{1.1}$

Remark 1.2. For any uncountable cardinal θ define $\mathfrak{r}_\theta$ as the minimal cardinality of a subset of $[\theta]^\theta$ such that no single $B \in [\theta]^\theta$ splits all the members of this family. One can verify that $\mathfrak{r}_\theta > \theta$ for every infinite cardinal θ, and the main claim holds for every θ (under the parallel generalized assumption that $\mathfrak{r}_\theta = 2^\theta$).

$\square_{1.2}$

The main claim is optimal in the sense that the assumption $\mathfrak{r} = \mathfrak{c}$ cannot induce a stronger negative relation in ZFC. The closest attempt would be refuting the positive unbalanced relation $\binom{\mathfrak{r}}{\omega} \to \binom{\mathfrak{r}\ \alpha}{\omega\ \omega}^{1,1}_2$ for every $\alpha < \mathfrak{r}$, but the following claim proves its independence:

Claim 1.3. *Unbalanced negative relations.*
The assumption $\mathfrak{r} = \mathfrak{c}$ is consistent with both $\binom{\mathfrak{r}}{\omega} \to \binom{\mathfrak{r}\ \alpha}{\omega\ \omega}^{1,1}_2$ for every $\alpha < \mathfrak{r}$ and $\binom{\mathfrak{r}}{\omega} \nrightarrow \binom{\mathfrak{r}\ \alpha}{\omega\ \omega}^{1,1}_2$ for some $\alpha < \mathfrak{r}$.

Proof.
For the positive direction force $\mathfrak{p} = \mathfrak{c}$, in which case $\mathfrak{r} = \mathfrak{c}$ as well. However, the relation $\binom{\mathfrak{p}}{\omega} \to \binom{\mathfrak{p}\ \alpha}{\omega\ \omega}^{1,1}_2$ for every $\alpha < \mathfrak{p}$ is established in [10] and holds in ZFC, so $\binom{\mathfrak{r}}{\omega} \to \binom{\mathfrak{r}\ \alpha}{\omega\ \omega}^{1,1}_2$ for every $\alpha < \mathfrak{r}$. Observe that $\mathfrak{r}$ is a regular cardinal in such models.

For the negative direction choose any $\lambda = \lambda^{\aleph_0}$ such that $\lambda > \aleph_1$. Let $\mathbb{Q}$ be a finite support iteration of adding λ-many Cohen reals. It is known that $\mathbf{V}^{\mathbb{Q}} \models \binom{\mu}{\omega} \nrightarrow \binom{\omega_1}{\omega}^{1,1}_2$ for every $\mu \in (\aleph_0, \lambda]$, as shown in [6], Remark 2.4. In particular, it holds for $\mu = \mathfrak{c}$. As $\mathfrak{r} = \mathfrak{c}$ in this generic extension and $\lambda > \aleph_1$ we have the consistency of the negative direction.

$\square_{1.3}$

We turn back to the balanced relation. In the case of a regular continuum, we can characterize now the strong polarized relation for $\mathfrak{c}$ as follows:

Corollary 1.4. *Assume $\mathfrak{c}$ is a regular cardinal.*
Then $\binom{\mathfrak{c}}{\omega} \to \binom{\mathfrak{c}}{\omega}^{1,1}_2$ iff $\mathfrak{r} < \mathfrak{c}$.

Proof.
If $\mathfrak{r} < \mathfrak{c}$ then Theorem 0.4 gives the positive direction of $\binom{\mathfrak{c}}{\omega} \to \binom{\mathfrak{c}}{\omega}^{1,1}_2$, since $\mathfrak{c}$ is a regular cardinal. If $\mathfrak{r} = \mathfrak{c}$ then Claim 1.1 gives the negative relation, so the proof is accomplished.

$\square_{1.4}$

We employ the above corollary in the proof of the following theorem:

Theorem 1.5. *Lévy collapse and random reals.*
Suppose κ is an uncountable regular cardinal.
For every $\lambda = \mathrm{cf}(\lambda) > \kappa$ there is a model of ZFC in which $\mathfrak{c} = \lambda$, $\binom{\kappa}{\omega} \to \binom{\kappa}{\omega}^{1,1}_2$ and if $\mathbb{P}$ =Lévy(κ, λ) then $\mathbf{V}^{\mathbb{P}} \models \binom{\kappa}{\omega} \nrightarrow \binom{\kappa}{\omega}^{1,1}_2$.
Likewise, it is consistent that $\binom{\mathfrak{d}}{\omega} \to \binom{\mathfrak{d}}{\omega}^{1,1}_2$ in the ground model, and after adding λ-many random reals for some $\lambda > \mathfrak{d}$ we have $\binom{\mathfrak{d}}{\omega} \nrightarrow \binom{\mathfrak{d}}{\omega}^{1,1}_2$.

Proof.
If $\kappa = \aleph_1$ then the theorem follows from the fact that $2^{\aleph_0} = \aleph_1$ implies $\binom{\aleph_1}{\aleph_0} \nrightarrow \binom{\aleph_1}{\aleph_0}^{1,1}_2$ (as proved in [3]). So assume that $\kappa > \aleph_1$. We begin with MA + $2^{\aleph_0} = \lambda$. In this case, $\mathfrak{s} = \lambda$ as well, so $\binom{\kappa}{\omega} \to \binom{\kappa}{\omega}^{1,1}_2$ by Claim 0.3. Observe also that $\mathfrak{r} = \lambda$. We claim that after forcing with $\mathbb{P}$ we will get the negative relation $\binom{\kappa}{\omega} \nrightarrow \binom{\kappa}{\omega}^{1,1}_2$.

Indeed, $\mathfrak{r} = \kappa$ in the generic extension. This fact follows from the completeness of $\mathbb{P}$ which ensures that no new sequence of sets of length below κ is introduced. The length λ of $\mathfrak{r}$-sequences in the old universe is collapsed to κ, but no $\mathfrak{r}$-family of size less than κ appears. Hence $\mathfrak{r} = \mathfrak{c} = \kappa$ in $\mathbf{V}^{\mathbb{P}}$. From Claim 1.1 we infer that $\binom{\kappa}{\omega} \nrightarrow \binom{\kappa}{\omega}^{1,1}_2$ as required.

We indicate that if $\mathfrak{r} < \kappa$ in the ground model then the positive relation $\binom{\kappa}{\omega} \rightarrow \binom{\kappa}{\omega}^{1,1}_2$ holds both in the old universe and after the collapse (see Corollary 1.4), so the opposite situation is also consistent for every regular cardinal κ above $\aleph_1$.

For the second assertion, begin with a model in which the positive relation $\binom{\mathfrak{d}}{\omega} \rightarrow \binom{\mathfrak{d}}{\omega}^{1,1}_2$ holds, and $\mathfrak{d} > \aleph_1$. This can be done due to [7], based on the model of [2], upon noticing that $\mathfrak{r} < \mathfrak{d}$ gives the desired result when $\mathfrak{d}$ is a regular cardinal (see Theorem 0.4).

We choose a large enough singular cardinal λ so that $\lambda > \mathfrak{d}$ but $\mathrm{cf}(\lambda) \leq \mathfrak{d}$. By adding λ-many random reals we blow up the continuum to λ but $\mathfrak{d}$ remains in its place. Moreover, $\mathfrak{r} = \lambda$ as well (see, e.g., [1]). Since $\mathrm{cf}(\lambda) \leq \mathfrak{d}$ we conclude that the negative relation $\binom{\mathfrak{d}}{\omega} \nrightarrow \binom{\mathfrak{d}}{\omega}^{1,1}_2$ holds in the generic extension, so the proof is accomplished.

$\square_{1.5}$

Can we incorporate singular cardinals in Corollary 1.4? A good understanding of polarized relations for singular cardinals above $\mathfrak{r}$ is needed. It is consistent that $\kappa > \mathrm{cf}(\kappa) = \mathfrak{r}$ and $\binom{\kappa}{\omega} \rightarrow \binom{\kappa}{\omega}^{1,1}_2$. The opposite direction is not so clear:

Question 1.6. Is it consistent that $\kappa > \mathfrak{r}, \mathrm{cf}(\kappa) > \aleph_0$ and $\binom{\kappa}{\omega} \nrightarrow \binom{\kappa}{\omega}^{1,1}_2$?

The negative downward theorem below $\mathfrak{r}$ (under the assumption that $\mathfrak{r} = \mathfrak{c}$) can be used also for a surprising relationship between $\mathfrak{r}$ and $\mathfrak{s}$. One of the dividing lines in the realm of cardinal invariants is the distinction between small characteristics (which are bounded by $\mathrm{cf}(\mathfrak{c})$) and large characteristics (which are not bounded by $\mathrm{cf}(\mathfrak{c})$). The distributivity number $\mathfrak{h}$ is a typical example of a small invariant, while $\mathfrak{s}$ is a large invariant. Nevertheless, if $\mathfrak{r} = \mathfrak{c}$ then $\mathfrak{s}$ becomes small:

Theorem 1.7. *If* $\mathfrak{r} = \mathfrak{c}$ *then* $\mathfrak{s} \leq \mathrm{cf}(\mathfrak{c})$.
Moreover, if $\mathfrak{r}_\sigma = \mathfrak{c}$ *then* $\mathfrak{s} \leq \mathrm{cf}(\mathfrak{c})$.

Proof.
We prove the first assertion with the aid of the polarized relations, and the second assertion in a direct way. Let κ be $\mathrm{cf}(\mathfrak{c})$. By Claim 1.1 we have $\binom{\kappa}{\omega} \nrightarrow \binom{\kappa}{\omega}^{1,1}_2$, since $\mathfrak{r} = \mathfrak{c}$. This relation excludes the possibility that $\mathfrak{s} > \mathrm{cf}(\mathfrak{c})$, because in this situation we have $\binom{\kappa}{\omega} \rightarrow \binom{\kappa}{\omega}^{1,1}_2$ since κ is an uncountable regular cardinal and due to Claim 0.3.

Assume now that $\mathfrak{r}_\sigma = \mathfrak{c}$. Enumerate the members of $[\omega]^\omega$ by $\{B_\gamma : \gamma < \mathfrak{c}\}$, and choose an increasing unbounded sequence of ordinals of the form

$\langle \alpha_\varepsilon : \varepsilon < \kappa \rangle$ in $\mathfrak{c}$. For every $\varepsilon < \kappa$ let $\mathcal{B}_\varepsilon$ be $\{B_\gamma : \gamma < \alpha_\varepsilon\}$, and we choose a collection of sets $\{S^\varepsilon_n : n \in \omega\}$ which splits the members of $\mathcal{B}_\varepsilon$.

The collection $\mathcal{F} = \{S^\varepsilon_n : \varepsilon < \kappa, n \in \omega\}$ is a splitting family for $[\omega]^\omega$. For this, pick up any $B \in [\omega]^\omega$ and any ordinal $\varepsilon < \kappa$ so that $B \in \mathcal{B}_\varepsilon$. By the choice of $\{S^\varepsilon_n : n \in \omega\}$ there is a set S^ε_n which splits B. But $S^\varepsilon_n \in \mathcal{F}$, hence $\mathcal{F}$ is a splitting family. Consequently, $\mathfrak{s} \leq |\mathcal{F}| \leq \mathrm{cf}(\mathfrak{c})$, so we are done.

$\square_{1.7}$

The above results raise some natural problems. We phrase a couple of them:

Question 1.8. Small cofinality above $\mathfrak{r}$.

(α) Assume $\mathfrak{r} < \kappa \leq \mathfrak{c}$ and $\mathrm{cf}(\kappa) \leq \mathfrak{r}$. Is it possible that $\binom{\kappa}{\omega} \nrightarrow \binom{\kappa}{\omega}^{1,1}_2$? In particular, is it possible for $\kappa = \mathfrak{c}$?

(β) Assume $\mathfrak{r}_\sigma = \mathfrak{c}$. Is it provable that $\binom{\mathfrak{r}_\sigma}{\omega} \nrightarrow \binom{\mathfrak{r}_\sigma}{\omega}^{1,1}_2$?

2. Random reals and real-valued measurability

In this section we try to analyze the polarized relation under the existence of random reals and in the presence of real-valued measurable cardinals. For a general background and notational conventions used below, we refer to [5]. We commence with a negative downward spectrum which issues from adding random reals.

Theorem 2.1. *Random reals and polarized relations.*
Assume $\kappa > \aleph_0$ and $\mathbb{Q}$ is a forcing notion for adding κ-many random reals. Then $\Vdash_{\mathbb{Q}} \binom{\theta}{\omega} \nrightarrow \binom{\theta}{\omega}^{1,1}_2$ for every $\theta \leq \kappa$, and even $\Vdash_{\mathbb{Q}} \binom{\kappa}{\omega} \nrightarrow \binom{\theta}{\omega}^{1,1}_2$.

Proof.
Choose a generic subset $G \subseteq \mathbb{Q}$, and fix a cardinal $\theta \in [\aleph_1, \kappa]$. Let m be the product measure over ${}^\kappa 2$. Recall that $p \in \mathbb{Q}$ iff $p \subseteq {}^\kappa 2$, where p is a Borel set of positive measure, supported by some countable set $u = u_p \in [\kappa]^{\leq \aleph_0}$. We indicate that a support of a given condition p is not unique (every countable subset of κ which contains a support can serve as well), though a minimal support always exists. For $p, q \in \mathbb{Q}$ we define $p \leq q$ iff $q \subseteq p$.

Let $\mathcal{F}$ be the set $\{f : f \text{ is a finite (partial) function from } \kappa \text{ into } 2\}$. For every $f \in \mathcal{F}$ let $\mathcal{O}(f)$ be $\{g \in {}^\kappa 2 : f \subset g\}$, denoted also by $({}^\kappa 2)^{[f]}$. By the definition of the product measure, $m(\mathcal{O}(f)) = \frac{1}{2^{|f|}}$. In particular, each $\mathcal{O}(f)$ belongs to $\mathbb{Q}$.

We define a $\mathbb{Q}$-name $\underset{\sim}{\eta}$ of a function from κ into $\{0,1\}$ by $\underset{\sim}{\eta} \supset f \Leftrightarrow \mathcal{O}(f) \in G$. If $\alpha \in \kappa, f = \{\langle \alpha, 0 \rangle\}$ and $g = \{\langle \alpha, 1 \rangle\}$ then $\mathcal{A} = \{\mathcal{O}(f), \mathcal{O}(g)\}$ forms a maximal antichain, so exactly one member of $\mathcal{A}$ belongs to G. Hence $\alpha \in \operatorname{dom}(\underset{\sim}{\eta})$ for every $\alpha < \kappa$. The fact that $\underset{\sim}{\eta}$ is a function follows from the directness of G. This gives rise to the definition of a name $\underset{\sim}{c}$ for a coloring from $\kappa \times \omega$ into $\{0,1\}$ as follows:

$$\underset{\sim}{c}(\alpha, n) = \underset{\sim}{\eta}(\omega\alpha + n).$$

We claim that $\underset{\sim}{c}$ exemplifies the negative relation $\binom{\kappa}{\omega} \nrightarrow \binom{\theta}{\omega}^{1,1}_2$.

Assume this is not the case. Pick up a condition $p \in \mathbb{Q}$, a color $\ell \in \{0,1\}$ and names of sets $\underset{\sim}{A} \in [\kappa]^\theta, \underset{\sim}{B} \in [\omega]^{\aleph_0}$ so that $p \Vdash \underset{\sim}{c} \restriction (\underset{\sim}{A} \times \underset{\sim}{B}) = \{\ell\}$.

For every $\alpha < \kappa$ we choose a Borel set $B_\alpha \subseteq {}^\kappa 2$ with support $u_\alpha \in [\kappa]^{\leq \aleph_0}$ such that $B_\alpha \subseteq p$ and B_α decides whether $\check{\alpha}$ belongs to $\underset{\sim}{A}$ in the following sense: if $m(B_\alpha) > 0$ then $B_\alpha \Vdash \check{\alpha} \in \underset{\sim}{A}$ and if not then $(p - B_\alpha) \Vdash \check{\alpha} \notin \underset{\sim}{A}$. Similarly, for every $n \in \omega$ we choose a Borel set $B_n \subseteq {}^\kappa 2$ with support $u_n \in [\kappa]^{\leq \aleph_0}$ such that $B_n \subseteq p$ and the following is satisfied: if $m(B_n) > 0$ then $B_n \Vdash \check{n} \in \underset{\sim}{B}$ and if not then $(p - B_n) \Vdash \check{n} \notin \underset{\sim}{B}$.

Let $v \in [\kappa]^{\leq \aleph_0}$ be a support of the condition p so that $u_n \subseteq v$ for every $n \in \omega$. We shall force with the part of $\mathbb{Q}$ above p using conditions with the support v.

For any ordinal $\alpha \in \kappa \setminus v$ let $r_\alpha = \{\eta \in {}^\kappa 2 : \eta \restriction \{\omega\alpha + n : n \in B\} = \ell\}$. Since $\underset{\sim}{B}$ is a name of an unbounded subset of ω, $m(r_\alpha) = 0$. Indeed, for every $n \in \omega$ let $\{b_j : j < n\}$ enumerate the first n members of B, and let

$f_n = \{\langle \omega\alpha + b_j, \ell\rangle : j < n\}$. By definition, $m(\mathcal{O}(f_n)) = \frac{1}{2^n}$, and $m(r_\alpha) \leq m(\mathcal{O}(f_n))$ for every $n \in \omega$, so $m(r_\alpha) = 0$.

Since $p \Vdash \underset{\sim}{A} \in [\kappa]^\theta$, there exists an ordinal α and a condition $q \geq p$ so that $q \Vdash \check{\alpha} \in \underset{\sim}{A}$ and $\alpha \notin v$. Notice that $q \Vdash \underset{\sim}{c}(\alpha, n) = \ell$ for every $n \in B$. It follows that $q \subseteq r_\alpha$, so $m(q) = 0$, which is impossible since $q \in \mathbb{Q}$.

$\square_{2.1}$

The effect of adding random reals is sharpened if we assume the existence of a real-valued measurable cardinal. The classical way to introduce such a cardinal is the random real forcing, as proved by Solovay, but this is not the only way. Gitik and Shelah, [9], introduced a different way to introduce such cardinals, and some of the properties of Solovay's construction are not shared by all real-valued measurable cardinals. We shall see, however, that the mere existence of a real-valued measurable cardinal entails strong negative relations.

We say that κ is real-valued measurable iff there exists an atomless κ-additive measure over κ. The requirement of being atomless implies $\kappa \leq 2^{\aleph_0}$, so κ is not strongly inaccessible. It is known, however, that such κ is weakly inaccessible. Before embarking on the impact of real-valued meaurable cardinals we need some preliminaries. Let (X, Σ, m) be a measure space. The measure algebra associated with it is the Boolean algebra $B = \Sigma/I$ when $I = \{A \subseteq X : m(A) = 0\}$. The following belongs to Maharam:

Theorem 2.2. *Maharam's Theorem.*
Suppose m is a homogeneous σ-additive measure on a σ-complete Boolean algebra $\mathcal{B}$.
The measure algebra $(\mathcal{B}, m)$ is isomorphic to the measure algebra of ${}^\lambda 2$ for some λ, with the product measure.

This fundamental theorem appears in [11]. Let κ be real-valued measurable as witnessed by the measure m, and let $\mathcal{I} = \{a \subseteq \kappa : m(a) = 0\}$. From Maharam's theorem there is some μ for which $\mathcal{P}(\kappa)/\mathcal{I}$ is isomorphic to the Boolean algebra $\mathrm{Borel}({}^\mu 2)/\mathcal{J}$, where $\mathcal{J}$ is the ideal of null sets in ${}^\mu 2$. It has been proved in [9], Section 2, that $\mu > \kappa$.

Theorem 2.3. *Negative relations and real-valued measurable cardinals.*
Let κ be a real-valued measurable cardinal.
Then $\binom{\theta}{\omega} \nrightarrow \binom{\aleph_1}{\omega}^{1,1}_2$ for every $\theta < \kappa$.

Proof.
Let $m : \mathcal{P}(\kappa) \to [0,1]_\mathbb{R}$ be a measure which exemplifies the fact that κ is real-valued measurable. The collection of sets $\mathcal{I} = \{a \subseteq \kappa : m(a) = 0\}$ is a κ-complete ideal over κ. By the facts quoted above, let $\mu > \kappa$ be such that $\mathcal{P}(\kappa)/\mathcal{I}$ is isomorphic to the Boolean algebra $\mathrm{Borel}({}^\mu 2)/\mathcal{J}$, where $\mathcal{J}$ is the ideal of null sets in ${}^\mu 2$. We fix an isomorphism $\jmath$ which exemplifies this fact. Fix any cardinal $\theta < \kappa$.

Viewing the Boolean algebra as a forcing notion, let $\underset{\sim}{\eta} = (\underset{\sim}{\eta}_\alpha : \alpha < \mu)$ be a random sequence of reals (i.e. $\underset{\sim}{\eta}_\alpha \in {}^\omega 2$ for every $\alpha < \mu$, and

$\underset{\sim}{\eta}_\alpha = \langle \underset{\sim}{\eta}(\omega\alpha + n) : n \in \omega \rangle)$. For each $\underset{\sim}{\eta}_\alpha$ we choose a sequence of sets $(B_{\alpha n} : n \in \omega)$ so that $B_{\alpha n} \subseteq \kappa$ and $\jmath(B_{\alpha n}/\mathcal{I}) = ({}^{\mu}2)^{[(\alpha, \underset{\sim}{\eta}_\alpha(n))]}/\mathcal{J}$, i.e. $\jmath(B_{\alpha n}/\mathcal{I}) = \{\nu \in {}^{\mu}2 : \nu(\alpha) = \underset{\sim}{\eta}_\alpha(n)\}/\mathcal{J}$.

For every $\alpha < \mu$ and each $n \in \omega$ we define $e_{\alpha n} \in {}^{\kappa}2$ by $e_{\alpha n}(i) = 1 \Leftrightarrow i \in B_{\alpha n}$. We concentrate on the collection $T = \{e_{\alpha n} : \alpha < \theta, n \in \omega\}$. We define κ colorings c_i for every $i < \kappa$, each c_i is a function from $\theta \times \omega$ into 2, by letting $c_i(\alpha, n) = e_{\alpha n}(i)$ for each member of T. We claim that for some $i < \kappa$ the coloring c_i exemplifies the negative relation $\binom{\theta}{\omega} \nrightarrow \binom{\aleph_1}{\omega}^{1,1}_2$.

For proving this, choose an ordinal $i < \kappa$ for which $\{(e_{\alpha n}(i) : n \in \omega) : \alpha < \theta\}$ is a Sierpiński set, i.e. for every $B \subseteq {}^{\omega}2$ of Lebesgue measure zero we have $(e_{\alpha n}(i) : n \in \omega) \notin B$ apart from a countable set of such sequences. Notice that each sequence of the form $(e_{\alpha n}(i) : n \in \omega)$ is an element in ${}^{\omega}2$. Such i exists by the following paragraph, upon noticing that $\theta < \kappa$.

The existence of a Sierpiński set is a well-known property of real-valued measurable cardinals, see [4] 6F, p. 215. We make the comment that here is the only point along the proof in which we use the assumption $\theta < \kappa$, and we do not know whether the existence of a Sierpiński set can be guaranteed if $\theta = \kappa$.

Assume now that $H_0 \in [\theta]^{\aleph_1}, H_1 \in [\omega]^{\aleph_0}$, and we shall show that $c_i \restriction (H_0 \times H_1)$ is not constant. Let B be $\{e \in {}^{\omega}2 : e \restriction H_1 \text{ is constant}\}$. Since H_1 is unbounded in ω we have $m(B) \leq \frac{1}{2^n}$ for every $n \in \omega$ and hence $m(B) = 0$. Consequently, $(e_{\alpha n}(i) : n \in \omega) \notin B$ apart from a countable set of such sequences. Since $|H_0| > \aleph_0$ we can choose an ordinal $\alpha \in H_0$ so that $(e_{\alpha n}(i) : n \in \omega) \restriction H_1$ is not constant. It means that there are $n_0, n_1 \in H_1$ such that $c_i(\alpha, n_0) \neq c_i(\alpha, n_1)$, so we are done.

$\square_{2.3}$

As a consequence, we deduce the following known fact:

Corollary 2.4. *If there is a real-valued measurable cardinal, then the splitting number is $\aleph_1$.*

Proof.
By Claim 0.3, if $\mathfrak{s} > \aleph_1$ then $\binom{\omega_1}{\omega} \to \binom{\omega_1}{\omega}^{1,1}_2$. However, this is impossible if one assumes the existence of a real-valued measurable cardinal.

$\square_{2.4}$

Finally, we return to Question 0.7 and we ask about the negative relation $\binom{\kappa}{\omega} \nrightarrow \binom{\kappa}{\omega}^{1,1}_2$ for the real-valued measurable κ itself. If we force the existence of such a cardinal by the classical way of Solovay, [12], then $\mathfrak{r} = \kappa$ and the negative relation follows. It turns out that this is always true:

Claim 2.5. *If κ is real-valued measurable then $\mathfrak{r} \geq \kappa$. Consequently, if $\mathfrak{c} = \kappa$ is real-valued measurable then $\binom{\kappa}{\omega} \nrightarrow \binom{\kappa}{\omega}^{1,1}_2$.*

Proof.
Suppose that κ is real-valued measurable and assume toward contradiction

that $\mathfrak{r} < \kappa$. Since κ is a limit cardinal, $\theta = \mathfrak{r}^+ < \kappa$ as well. By Theorem 2.3 we have $\binom{\theta}{\omega} \nrightarrow \binom{\theta}{\omega}^{1,1}_2$. However, $\binom{\theta}{\omega} \rightarrow \binom{\theta}{\omega}^{1,1}_2$ by Theorem 0.4, a contradiction.

In case $\mathfrak{c} = \kappa$ is real-valued measurable we have $\mathfrak{r} = \kappa$, hence $\binom{\kappa}{\omega} \nrightarrow \binom{\kappa}{\omega}^{1,1}_2$ by virtue of Corollary 1.4, so we are done.

$\square_{2.5}$

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