$\leq_{SP}$ Can Have Infinitely Many Classes

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Abstract

Building off of recent results on Keisler's order, we show that consistently, $\leq_{SP}$ has infinitely many classes. In particular, we define the property of $\leq k$-type amalgamation for simple theories, for each $2 \leq k < \omega$. If we let $T_{n,k}$ be the theory of the random k-ary, n-clique free random hyper-graph, then $T_{n,k}$ has $\leq k-1$-type amalgamation but not $\leq k$-type amalgamation. We show that consistently, if T has $\leq k$-type amalgamation then $T_{k+1,k} \not\leq_{SP} T$, thus producing infinitely many $\leq_{SP}$-classes. The same construction gives a simplified proof of the theorem from [8] that consistently, the maximal $\leq_{SP}$-class is exactly the class of unsimple theories. Finally, we show that consistently, if T has $< \aleph_0$-type amalgamation, then $T \leq_{SP} T_{rg}$, the theory of the random graph.

1 Introduction

T is always a complete theory in a countable language. We will fix a monster model $\mathfrak{C} \models T$ and work within it.

The first author introduced the following definition in [8], although he had previously investigated the phenomenon in [7] (without giving it a name):

Definition 1.1. Suppose $\lambda \geq \theta$. Define $SP_T(\lambda, \theta)$ to mean: for every $M \models T$ of size λ, there is a θ-saturated $N \models T$ of size λ extending M.

In this paper, we will restrict to the following special case:

Definition 1.2. Say that (θ, λ) is a nice pair if θ is a regular uncountable cardinal, and $\lambda \geq \theta$ has $\lambda = \lambda^{\aleph_0}$. Given T_0, T_1 complete first order theories, say that $T_0 \leq_{SP} T_1$ if whenever (θ, λ) is a nice pair, if $SP_{T_0}(\lambda, \theta)$ then $SP_{T_1}(\lambda, \theta)$.

Thus, $\leq_{SP}$ is a pre-ordering of theories which measures how difficult it is to build saturated models. The main case of interest is when $\mathrm{cof}(\lambda) < \theta$.

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In [7], the first author proves: the stable theories are the minimal SP-class, and unsimple theories are always maximal. In [8], the first author additionally proves that consistently, unsimple theories are exactly the maximal class.

Recently, there has been substantial progress on Keisler's order $\trianglelefteq$, another pre-ordering of theories which measures how difficult it is to build saturated models; see for instance [5] and [6] by the first author and Malliaris. In particular, in [6] it is shown that Keisler's order has infinitely many classes, these being seperated by certain amalgamation properties. In this paper we use similar ideas to continue investigation of $\leq_{SP}$.

In Section 2 we summarize what is already known on $\leq_{SP}$.

In Section 3, we introduce several amalgamation-related properties of forcing notions (Definition 3.2), and show that it is preserved under iterations in a suitable sense (Theorem 3.5). In light of this, we define a class of forcing axioms (Definition 3.6); these are closely related to the forcing axiom $\mathrm{Ax}\mu_0$, defined by the first author in [9] and used to demonstrate the consistent maximality of unsimple theories under $\leq_{SP}$ in [8]. However, the forcing axioms we develop are designed specifically for what we want and have been simplified somewhat.

In Section 4, we define and prove some helpful facts about non-forking diagrams of models.

In Section 5, we introduce, for each $3 \leq k < \omega$, a property of simple theories called $< k$-type amalgamation (Definition 5.1), and discuss some of its properties. For example, if for $n > k$ we let $T_{n,k}$ be the theory of the k-ary, n-clique free hypergraph, then if $k \geq 3$, $T_{n,k}$ has $< k$-type amalgamation but not $< k+1$-type amalgamation. We also show that if T has $< \aleph_0$-type amalgamation (i.e., $< k$-type amalgamation for all k), then $SP_T(\lambda, \theta)$ holds whenever we have that there is some $\theta \leq \mu \leq \lambda$ with $\mu^{<\theta} \leq \lambda$ and $2^\mu \geq \lambda$ (Theorem 5.6). This implies that if the singular cardinals hypothesis holds, then whenever T has $< \aleph_0$-type amalgamation, then $T \leq_{SP} T_{rg}$, where T_{rg} is the theory of the random graph.

In Section 6, we put everything together to show that consistently, for all $k \geq 3$, if T has the $< k$-type amalgamation property, then $T_{k,k-1} \not\leq_{SP} T$ (Theorem 6.2). In particular, for $k < k'$, $T_{k+1,k} \not\leq_{SP} T_{k'+1,k'}$; this is similar to the situation for Kiesler's order in [6].

By a forcing notion, we mean a pre-ordered set $(P, \leq^P)$ such that P has a least element 0^P (pre-order means that $\leq^p$ is transitive); we are using the convention where $p \leq q$ means q is a stronger condition than p. That is, when we force by P we add a generic ideal, rather than a generic filter. Thus, a finite sequence $(p_i : i < k)$ from P is compatible if it has an upper bound in P.

2 Background

The following theorem is closely related to the classical Hewitt-Marczewski-Pondiczery theorem of topology; it is proved in [1]. It will be central for our investigations.

Theorem 2.1. Suppose $\theta \leq \mu \leq \lambda$ are infinite cardinals such that θ is regular, $\mu = \mu^{<\theta}$, and $\lambda \leq 2^\mu$. Then there is a sequence $(\mathbf{f}_\gamma : \gamma < \mu)$ from ${}^\lambda\mu$ such that for all partial functions f from λ to μ of cardinality less than θ, there is some $\gamma < \mu$ such that $\mathbf{f}_\gamma$ extends f. Additionally, if $\lambda > 2^\mu$ then this fails.

We will also want the following technical device, which will allow us to apply Theorem 2.1 to conclude $SP_T(\lambda, \theta)$ holds. Here is the idea: suppose $M \models T$ with $|M| \leq \lambda$, and we want to find some θ-saturated $N \succeq M$ with $|N| \leq \lambda$. To do this, we will always first find some $N_0 \succeq M$ with

$|N_0| \leq \lambda$ which realizes every type over M of cardinality less than θ, and then we iterate θ-many times. The key step is to find N_0, and the following definitions capture when this is possible.

Definition 2.2. Suppose T is a simple theory, θ is a regular uncountable cardinal, and $M_* \preceq M \models T$. then let Γ^θ_{M,M_*} be forcing notion of all partial types $p(x)$ over M of cardinality less than θ which do not fork over M_*, ordered by inclusion. Also, if $p_*(x)$ is a complete type over M_*, then let $\Gamma^\theta_{M,p_*} \subseteq \Gamma^\theta_{M,M_*}$ be the set of all $p(x)$ which extend $p_*(x)$.

Given (θ, λ) a nice pair and given μ with $\theta \leq \mu \leq \lambda$, define $SP^1_T(\lambda, \mu, \theta)$ to mean: for every $M \models T$ of size $\leq \lambda$ and for every countable $M_* \preceq M$, there are complete types $p_\gamma(x) : \gamma < \mu$ over M which do not fork over M_*, such that whenever $p(x) \in \Gamma^\theta_{M,M_*}$, then $p(x) \subseteq p_\gamma(x)$ for some $\gamma < \mu$. Given a fixed countable $M_* \models T$ and type $p_*(x)$ over M_*, define $SP^1_{T,p_*}(\lambda, \mu, \theta)$ similarly: whenever $M \succeq M_*$ has size at most λ, there are are complete, non-forking extensions $p_\gamma(x) : \gamma < \mu$ of $p_*(x)$ to M, such that whenever $p(x) \in \Gamma^\theta_{M,p_*}$, then $p(x) \subseteq p_\gamma(x)$ for some $\gamma < \mu$.

Note that if $\mu \geq 2^{\aleph_0}$, then $SP^1_T(\lambda, \mu, \theta)$ if and only if $SP^1_{T,p_*}(\lambda, \mu, \theta)$ for every complete type $p_*(x)$ over a countable model M_* (the forward direction is unconditional in μ, but for the reverse direction, we need to concatenate witnesses for each $p_*(x)$, of which there are $2^{\aleph_0}$-many). In particular this holds when $\mu = \lambda$, since $\lambda^{\aleph_0} = \lambda$.

The following is an important example.

Example 2.3. Suppose (θ, λ) is a nice pair and suppose μ is a cardinal with $\mu = \mu^{<\theta}$ and $\theta \leq \mu \leq \lambda$. Then $SP^1_{T_{rg}}(\lambda, \mu, \theta)$ holds if and only if $\lambda \leq 2^\mu$; and this is equivalent to $SP^1_{T_{rg},p_*}(\lambda, \mu, \theta)$ holding for some or any nonalgebraic complete type $p_*(x)$ over a countable model M_*.

Proof. Suppose $M \models T$ has size $\leq \lambda$. Then the nonalgebraic types in $\bigcup\{S^1(A) : A \subseteq M\}$ correspond naturally to partial functions from M to 2, and so this is just a restatement of Theorem 2.1. □

Theorem 2.4. Suppose T is a simple theory (in a countable language, as always). Suppose (θ, λ) is a nice pair.

(A) $SP_T(\lambda, \theta)$ if and only if $SP^1_T(\lambda, \lambda, \theta)$.

(B) Suppose $p_*(x)$ is a complete type over a countable model M_*, and $SP^1_{T,p_*}(\lambda, \lambda, \theta)$ holds, and $\mathrm{cof}(\lambda) < \theta$. Then for some μ with $\theta \leq \mu < \lambda$, $SP^1_{T,p_*}(\lambda, \mu, \theta)$ holds.

Proof. (A), forward direction: Suppose $M \models T$ has size $\leq \lambda$, and $M_* \preceq M$ is countable. Choose $N \succeq M$, a θ-saturated model of size λ. Enumerate $N = (a_\alpha : \alpha < \lambda)$, and for each $\alpha < \lambda$ let $p_\alpha(x) = tp(a_\alpha/M)$. Clearly this works.

(A), reverse direction: suppose $M \models T$ has size $\leq \lambda$. Using $SP^1_T(\lambda, \lambda, \theta)$, we can find $N \succeq M$ of size λ, such that every partial type $p(x)$ over M of cardinality less than θ is realized in N (we are also using $\lambda = \lambda^{\aleph_0}$, so there are only λ-many countable elementary submodels M_* of M). If we iterate this θ-many times then we will get a θ-saturated model of T.

(B): Suppose towards a contradiction that $SP^1_{T,p_*}(\lambda, \mu, \theta)$ failed for all $\theta \leq \mu < \lambda$. Write $\kappa = \mathrm{cof}(\lambda)$, and let $(\mu_\beta : \beta < \kappa)$ be a cofinal sequence of cardinals in λ. For each $\beta < \kappa$, choose $M_\beta \succeq M_*$ with $|M_\beta| \leq \lambda$, witnessing that $SP^1_{T,p_*}(\lambda, \mu_\beta, \theta)$ fails. We can suppose that $(M_\beta : \beta < \kappa)$ is independent over M_*.

Let $N \models T$ have size $\leq \lambda$ such that each $M_\beta \preceq N$. Then by $SP^1_{T,p_*}(\lambda, \lambda, \theta)$, we can find $(q_\alpha(x) : \alpha < \lambda)$ such that whenever $q(x) \in \Gamma^\theta_{N,p_*}$, then $q(x) \subseteq q_\alpha(x)$ for some $\alpha < \lambda$.

For each $\beta < \kappa$, we can by hypothesis choose $p_\beta(x) \in \Gamma^\theta_{M^\beta,p_*}$ such that $p_\beta(x) \not\subseteq q_\alpha(x)$ for any $\alpha < \mu_\beta$. By the independence theorem for simple theories, $p(x) := \bigcup_{\beta<\kappa} p_\beta(x)$ does not fork over M_*. Hence $p(x) \subseteq q_\alpha(x)$ for some $\alpha < \lambda$. Choose $\beta < \kappa$ with $\alpha < \mu_\beta$; then this implies that $p_\beta(x) \subseteq q_\alpha(x)$, a contradiction. □

Finally, the following theorem is a collection of most of what has been previously known on $\leq_{SP}$.

Theorem 2.5. Suppose T is a complete first order theory in a countable language. Suppose (θ, λ) is a nice pair.

(A) If $\lambda = \lambda^{<\theta}$, then $SP_T(\lambda, \theta)$ holds; if T is unsimple then the converse is true as well. Thus unsimple theories are all $\leq_{SP}$-maximal. (This is proved in [7].)

(B) T_{rg} is the $\leq_{SP}$-minimal unstable theory. (This is implicit in [8].)

(C) If T is stable, then $SP_T(\lambda, \theta)$ holds (this is proved in [7]).

(D) If λ is a strong limit with $\mathrm{cof}(\lambda) < \theta$, and if $SP_T(\lambda, \theta)$ holds, then T is stable. (This is implicit in [8].) Thus the stable theories are exactly the minimal $\leq_{SP}$-class. Also, under GCH, all unstable theories are maximal.

(E) If $\theta \leq \mu \leq \lambda$ and $\mu^{<\theta} = \mu$ and $\lambda \leq 2^\mu$, then $SP_{T_{rg}}(\lambda, \theta)$ holds. (This is an exercise in [7].)

(F) It is consistent that there exists a nice pair (θ, λ) such that for all simple T, $SP_T(\theta, \lambda)$ holds. Hence, it is consistent that the unsimple theories are exactly the $\leq_{SP}$-maximal class. (This is proved in [8].)

For the reader's convenience, we prove (A) through (E), making use of the language of SP^1. Theorem (F) will be a special case of our main theorem, namely Theorem 6.2(B).

Proof. (A): By standard arguments, if $\lambda^{<\theta} = \lambda$ then $SP_T(\lambda, \theta)$ holds. Suppose T is unsimple, and $SP_T(\lambda, \theta)$ holds, and suppose towards a contradiction that $\lambda^{<\theta} > \lambda$. Choose a formula $\varphi(x, y)$ with the tree property (possibly y is a tuple).

Let $\kappa < \theta$ be least such that $\lambda^\kappa > \lambda$. Choose $M \models T$ and $(a_\eta : \eta \in {}^{<\kappa}\lambda)$ such that for all $\eta \in {}^\kappa\lambda$, $p_\eta(x) := \{\varphi(x, a_{\eta\restriction_\beta}) : \beta < \kappa\}$ is consistent, and for all $\eta \in {}^{<\kappa}\lambda$ and for all $\alpha < \beta < \lambda$, $\varphi(x, a_{\eta^\frown(\alpha)})$ and $\varphi(x, a_{\eta^\frown(\beta)})$ are inconsistent. Note that each $|p_\eta(x)| < \theta$; but clearly if $N \succeq M$ realizes each $p_\eta(x)$ then $|N| \geq \lambda^\kappa > \lambda$.

(B): Suppose T is unstable; we show $T_{rg} \leq_{SP} T$. By (A), this is true if T is unsimple, so we can suppose that T is simple, hence has the independence property via some formula $\varphi(x, y)$. Now suppose (θ, λ) is a nice pair. By Theorem 2.4(A), it suffices to show that if $SP_T(\lambda, \theta)$ holds, then $SP^1_{T_{rg}}(\lambda, \lambda, \theta)$ holds. (Note we cannot apply Example 2.3 because possibly $\lambda^{<\theta} > \lambda$.) Choose some $(a_\alpha : \alpha < \lambda)$ from $\mathfrak{C}$ such that for all $\mathbf{f} : \lambda \to 2$, $\{\varphi(x, a_\alpha)^{\mathbf{f}(\alpha)} : \alpha < \lambda\}$ is consistent. By $SP_T(\lambda, \theta)$ we can find some θ-saturated $M \models T$ with $|M| \leq \lambda$ and each $a_\alpha \in M$.

Suppose $N \models T_{rg}$, say $N = \{a_\alpha : \alpha < \lambda\}$ without repetitions. For each $b \in M$, $p_b(x)$ to be the complete nonalgebraic type over N, defined by putting $R(x, a_\alpha) \in p_b(x)$ if and only if $M \models \varphi(b, a_\alpha)$. Then this witnesses $SP^1_{T_{rg}}(\lambda, \lambda, \theta)$ holds (since $|M| \leq \lambda$).

(C): Suppose T is stable. It suffices to show that $SP^1_T(\lambda, \theta, \theta)$ holds. But this is clear: given $M \models T$ of size $\leq \lambda$ and $M_* \preceq M$ countable, there are only countable many types over M that do not fork over M_*, seeing as types over M_* are stationary.

(D): Suppose towards a contradiction that $SP_T(\lambda, \theta)$ holds for some unstable T. Then in particular $SP_{T_{rg}}(\lambda, \theta)$ holds. Let $p_*(x)$ be a complete nonalgebraic type over some countable $M_* \models T_{rg}$. By Theorem 2.4 we can find $\theta \leq \mu < \lambda$ such that $SP^1_{T_{rg}, p_*}(\lambda, \mu, \theta)$ holds. By possibly replacing μ with $\mu^{<\theta}$ we can suppose $\mu = \mu^{<\theta}$. Then this contradicts Example 2.3, since $2^\mu < \lambda$.

(E): By Example 2.3 and Theorem 2.4(A). □

If the singular cardinals hypothesis holds, then we can say more. Recall that the singular cardinals hypothesis states that if λ is singular and $2^{\mathrm{cof}(\lambda)} < \lambda$, then $\lambda^{\mathrm{cof}(\lambda)} = \lambda^+$. (Note that $2^{\mathrm{cof}(\lambda)} \neq \lambda$ since $\mathrm{cof}(2^\kappa) > \kappa$ for all cardinals κ, by König's theorem.) The failure of the singular cardinals hypothesis is a large cardinal axiom; see Chapter 5 of [4].

We want the following simple lemma.

Lemma 2.6. Suppose the singular cardinals hypothesis holds. Suppose θ is regular, $\lambda \geq \theta$, $\lambda^{<\theta} > \lambda$, and $2^{<\theta} \leq \lambda$. Then for every $\mu < \lambda$, $\mu^{<\theta} < \lambda$. Further, λ is singular of cofinality $< \theta$.

Proof. First of all, note that $2^{<\theta} < \lambda$, as otherwise $\lambda^{<\theta} = \lambda$.

Now suppose towards a contradiction there were some $\mu < \lambda$ with $\mu^{<\theta} \geq \lambda$; then necessarily $\mu^{<\theta} > \lambda$, as otherwise again $\lambda^{<\theta} = \lambda$. We can choose μ least with $\mu^{<\theta} > \lambda$. Let $\kappa < \theta$ be least such that $\mu^\kappa > \lambda$.

Note that $2^\kappa < \mu$, as otherwise $2^\kappa = (2^\kappa)^\kappa \geq \mu^\kappa > \lambda$, contradicting $2^{<\theta} < \lambda$. Thus, by a consequence of the singular cardinals hypothesis (Theorem 5.22 (ii) (b),(c) of [4]), $\mu^\kappa \leq \mu^+$. But since $\mu < \lambda$, $\mu^+ \leq \lambda$, so this is a contradiction.

To finish, suppose towards a contradiction that $\mathrm{cof}(\lambda) \geq \theta$. Then $\lambda^{<\theta} = \lambda + \sup(\mu^{<\theta} : \mu < \lambda) = \lambda$, a contradiction. □

This allows us to more intimately connect SP and SP^1:

Theorem 2.7. Suppose the singular cardinals hypothesis holds, and suppose (θ, λ) is a nice pair. Then $SP_T(\lambda, \theta)$ holds if and only if either T is stable, or $\lambda = \lambda^{<\theta}$, or else T is simple and for every complete type $p_*(x)$ over a countable model $M_* \models T$, there is some $\theta \leq \mu < \lambda$ with $\mu^{<\theta} = \mu$ and $2^\mu \geq \lambda$, such that $SP^1_{T,p_*}(\lambda, \mu, \theta)$ holds.

Proof. If T is stable or $\lambda = \lambda^{<\theta}$, then $SP_T(\lambda, \theta)$ holds, by Theorem 2.5 (A), (C). Thus we can assume T is unstable and $\lambda > \lambda^{<\theta}$. If T is unsimple, then $SP_T(\lambda, \theta)$ fails by Theorem 2.5(A).

Note that $SP_T(\lambda, \theta)$ iff $SP^1_T(\lambda, \lambda, \theta)$ by Theorem 2.4(A), so it suffices to show that $SP^1_T(\lambda, \lambda, \theta)$ holds if and only if for every complete type $p_*(x)$ over a countable model M_*, there is some $\theta \leq \mu < \lambda$ with $\mu^{<\theta} = \mu$ and $2^\mu \geq \lambda$, such that $SP^1_{T,p_*}(\lambda, \mu, \theta)$ holds.

Suppose first $SP^1_T(\lambda, \lambda, \theta)$ holds, and $p_*(x)$ is given. Since T is unstable, this clearly implies that $2^{<\theta} \leq \lambda$. Hence, by Lemma 2.6, λ is singular with $\mathrm{cof}(\lambda) < \theta$, and there are cofinally many $\mu < \lambda$ with $\mu^{<\theta} = \mu$. By Theorem 2.5 (D), λ is not a strong limit. Thus by Theorem 2.4(B), we can find $\theta \leq \mu < \lambda$ such that $\mu = \mu^{<\theta}$ and $2^\mu \geq \lambda$ and $SP^1_{T,p_*}(\lambda, \mu, \theta)$ holds.

Conversely, we have in particular that each $SP^1_{T,p_*}(\lambda,\lambda,\theta)$ holds; since $\lambda = \lambda^{\aleph_0} \geq 2^{\aleph_0}$ we get that $SP^1_T(\lambda,\lambda,\theta)$ holds. □

3 Forcing Axioms

In this section, we introduce the forcing axioms which will produce the desired behavior in SP. It is well-known that the countable chain condition is preserved under finite support iterations; we aim to find generalizations to the κ-closed, κ^+-c.c. context.

Definition 3.1. For a cardinal θ and sets X, Y, define $P_{XY\theta}$ to the forcing notion of all partial functions from X to Y of cardinality less than θ, ordered by inclusion. Note that $P_{XY\theta}$ has the $|Y^{<\theta}|^+$-c.c. and is θ-closed.

Definition 3.2. Suppose P, Q are forcing notions, and suppose $k \geq 3$ is a cardinal (typically finite). Then say that $P \to_k Q$ if there is a dense subset P_0 of P and a map $F : P_0 \to Q$ such that for all sequences $(p_i : i < i_*)$ from P_0 with $i_* < k$, if $(F(p_i) : i < i_*)$ are compatible in Q, then $(p_i : i < i_*)$ has a least upper bound in P; we write $F : (P, P_0) \to_k Q$. Say that $P \to^w_k Q$ (where w stands for weak) if there is a map $F : P \to Q$ such that whenever $(p_i : i < i_*)$ is a sequence from P with $i_* < k$, if $(F(p_i) : i < i_*)$ is compatible in Q, then $(p_i : i < i_*)$ is compatible in P.

Suppose P is a forcing notion, $\aleph_0 < \theta \leq \mu$ are cardinals with θ regular, and $3 \leq k \leq \theta$ is a cardinal (often finite). Then say that P has the $(< k, \mu, \theta)$-amalgamation property if every ascending chain from P of length less than θ has a least upper bound in P, and for some set X, $P \to_k P_{X\mu\theta}$.

For example, $P_{X\mu\theta}$ has the $(< k, \mu, \theta)$-amalgamation property.

The following lemma sums up several obvious facts.

Lemma 3.3. Suppose $\aleph_0 < \theta \leq \mu$ are cardinals with $\theta > \aleph_0$, and $3 \leq k \leq \theta$ is a cardinal.

1. If $P \to_k Q$ and $Q \to^w_k Q'$ then $P \to_k Q'$.
2. If P, Q have the $(< k, \mu, \theta)$-amalgamation property, then P forces that $\check{Q}$ has the $(< k, |\mu|, \theta)$-amalgamation property. (We write $|\mu|$ because possibly P collapses μ to θ.) (This is where we use $k \leq \theta$.)
3. Suppose P has the $(< k, \mu, \theta)$-amalgamation property for some $k \geq 3$. Then P is $< \theta$-distributive and $(\mu^{<\theta})^+$-c.c.
4. If P is θ-closed and has the least upper bound property, then P has the $(< k, \mu, \theta)$-amalgamation property if and only if $P \to^w_k P_{\lambda\mu\theta}$ for some λ.

We note the following:

Lemma 3.4. Suppose $\aleph_0 < \theta \leq \mu$ are cardinals with θ regular, and $3 \leq k \leq \theta$. Then P has the $(< k, \mu, \theta)$-amalgamation property if and only if P has the $(< k, \mu^{<\theta}, \theta)$-amalgamation property.

Proof. Define $\mu' = \mu^{<\theta}$, and let λ be a cardinal. It suffices to show there is a cardinal λ' such that $P_{\lambda\mu'\theta} \to^w_k P_{\lambda'\mu\theta}$, by Lemma 3.3 (4). Write $Y' = {}^{<\theta}\mu$; it suffices to find a set X' such that $P_{\lambda Y'\theta} \to^w_k P_{X'\mu\theta}$.

Let $X' = \lambda \times (\theta + 1)$. Define $F : P_{\lambda Y'\theta} \to P_{X'\mu\theta}$ as follows. Let $f \in P_{\lambda Y'\theta}$ be given. Let $\mathrm{dom}(F(f)) = \{((\gamma, \delta) : \gamma \in \mathrm{dom}(f)$, and either $\delta < \mathrm{dom}(f(\gamma))$ or $\delta = \theta\}$. Define $F(f)(\gamma, \delta) = f(\gamma)(\delta)$ if $\delta < \theta$, and otherwise $F(f)(\gamma, \theta) = \mathrm{dom}(f(\gamma))$. Clearly this works. □

The following is key; it states that the $(< k, \mu, \theta)$-amalgamation property is preserved under $< \theta$-support iterations. Note that it follows from Lemma 3.3(2) that the $(< k, \mu, \theta)$-amalgamation property is preserved under $< \theta$-support products.

Theorem 3.5. Suppose θ is a regular uncountable cardinal, $\mu \geq \theta$ and $3 \leq k \leq \theta$. Suppose $(P_\alpha : \alpha \leq \alpha_*), (\dot{Q}_\alpha : \alpha < \alpha_*)$ is a $< \theta$-support forcing iteration, such that each P_α forces that $\dot{Q}_\alpha$ has the $(< k, |\mu|, \theta)$-amalgamation property. Then P_{α_*} has the $(< k, \mu, \theta)$-amalgamation property.

Proof. Let λ be large enough.

Inductively, choose $(P^0_\alpha : \alpha \leq \alpha_*, \dot{Q}^0_\alpha : \alpha < \alpha_*)$ a $< \theta$-support forcing iteration, and $(\dot{F}_\alpha : \alpha < \alpha_*)$, such that each P^0_α is dense in P_α, and each P_α forces $\dot{F}_\alpha : (\dot{Q}_\alpha, \dot{Q}^0_\alpha) \to_k \check{P}_{\lambda\mu\sigma}$.

Claim. For each $\gamma_* < \theta$, if $(p_\gamma : \gamma < \gamma_*)$ is an ascending chain from P_{α_*}; then it has a least upper bound p in P_{α_*}, such that $\mathrm{supp}(p) \subseteq \bigcup_{\gamma < \gamma_*} \mathrm{supp}(p_\gamma)$.

Proof. By induction on $\alpha \leq \alpha_*$, we construct $(q_\alpha : \alpha \leq \alpha_*)$ such that each $q_\alpha \in P_\alpha$ with $\mathrm{supp}(q_\alpha) \subseteq \bigcup_{\gamma < \gamma_*} \mathrm{supp}(p_\gamma) \cap \alpha$, and for $\alpha < \beta \leq \alpha_*$, $q_\beta \restriction_\alpha = q_\alpha$, and for each $\alpha \leq \alpha_*$, q_α is a least upper bound to $(p_\gamma \restriction_\alpha : \gamma < \gamma_*)$ in P_α. At limit stages there is nothing to do; so suppose we have defined q_α. If $\alpha \notin \bigcup_{\gamma < \gamma_*} \mathrm{supp}(p_\gamma)$ then let $q_{\alpha+1} = q_\alpha {}^\frown (0^{\dot{Q}_\alpha})$. Otherwise, since q_α forces that $(p_\gamma(\alpha) : \gamma < \gamma_*)$ is an ascending chain from $\dot{Q}_\alpha$, we can find $\dot{q}$, a P_α-name for an element of $\dot{Q}_\alpha$, such that q_α forces $\dot{q}$ is the least upper bound. Let $q_{\alpha+1} = q_\alpha {}^\frown (\dot{q})$. □

Now suppose $p \in P^0_{\alpha_*}$. Note that $\mathrm{supp}(p) \in [\alpha_*]^{<\theta}$.

It is easy to find, for each $n < \omega$, elements $\mathbf{q}_n(p) \in P^0_{\alpha_*}$ with $\mathbf{q}_0(p) = p$, so that for all $n < \omega$:

- $\mathbf{q}_{n+1}(p) \geq \mathbf{q}_n(p)$;
- For all $\alpha < \alpha_*$, $\mathbf{q}_{n+1}(p) \restriction_\alpha$ decides $\dot{F}_\alpha(\mathbf{q}_n(p)(\alpha))$. (This is automatic whenever $\alpha \notin \mathrm{supp}(\mathbf{a}_n)$, since then P forces that $\dot{F}_\alpha(\mathbf{q}_n(p)) = \emptyset$.)

So we can choose $f_{n,\alpha} \in P_{\lambda\mu\sigma}$ such that each $\mathbf{q}_{n+1}(p) \restriction_\alpha$ forces that $\dot{F}_\alpha(\mathbf{q}_n(p)(\alpha)) = \check{f}_{n,\alpha}(p)$.

Let $\mathbf{q}_\omega(p) \in P$ be the least upper bound of $(\mathbf{q}_n(p) : n < \omega)$, which is possible by the claim. Let $P^0 = \{\mathbf{q}_\omega(p) : p \in P^0_{\alpha_*}\}$. For each $q \in P^0$, choose $\mathbf{p}(q) \in P^0_{\alpha_*}$ such that $q = \mathbf{q}_\omega(\mathbf{p}(q))$. For each $n < \omega$, let $\mathbf{p}_n(q) = \mathbf{q}_n(\mathbf{p}(q))$, and for each $\alpha < \alpha_*$, let $f_{n,\alpha}(q) = f_{n,\alpha}(\mathbf{p}(q))$.

Thus we have arranged that for all $q \in P^0$, q is the least upper bound of $(\mathbf{p}_n(q) : n < \omega)$, and for all $n < \omega$ and $\alpha < \alpha_*$, $\mathbf{p}_{n+1}(q) \restriction_\alpha$ forces that $\dot{F}_\alpha(\mathbf{p}_n(q)(\alpha)) = \check{f}_{n,\alpha}(q)$.

Write $X = \omega \times \alpha_* \times \lambda$. Choose $F : P^0 \to P_{X\mu\theta}$ so that for all $q, q' \in P^0$, if $F(q)$ and $F(q')$ are compatible, then for all $n < \omega$ and for all $\alpha < \alpha_*$, $f_{n,\alpha}(q)$ and $f_{n,\alpha}(q')$ are compatible. For instance, let the domain of $F(q)$ be the set of all (n, α, β) such that β is in the domain of $f_{n,\alpha}$, and let $F(q)(n, \alpha, \beta) = f_{n,\alpha}(\beta)$.

Now suppose $(q_i : i < i_*)$ is a sequence from P^0 with $i_* < k$, such that $(F(q_i) : i < i_*)$ are compatible. Write $\Gamma = \bigcup_{i<i_*, n<\omega} \mathrm{supp}(\mathbf{p}_n(q_i))$.

By induction on $\alpha \leq \alpha_*$, we construct a least upper bound s_α to $(\mathbf{p}_n(q_i) \restriction_\alpha : i < i_*, n < \omega)$ in P_α, such that $\mathrm{supp}(s_\alpha) \subseteq \Gamma \cap \alpha$, and for $\alpha < \alpha'$, $s_{\alpha'} \restriction_\alpha = s_\alpha$.

Limit stages of the induction are clear. So suppose we have constructed s_α. If $\alpha \notin \Gamma$ clearly we can let $s_{\alpha+1} = s_\alpha {}^\frown (0^{\dot{Q}_\alpha})$; so suppose instead $\alpha \in \Gamma$. Let $n < \omega$ be given. Then $(f_{n,\alpha}(q_i) : i < i_*)$ are compatible, and s_α forces that $\dot{F}_\alpha(\mathbf{p}_n(q_i)(\alpha)) = \check{f}_{n,\alpha}(\check{q}_i)$ for each $i < i_*$, since $\mathbf{p}_{n+1}(q_i) \restriction_\alpha$ does. Thus s_α forces that $(\mathbf{p}_n(q_i)(\alpha) : i < i_*)$ has a least upper bound $\dot{r}_n$. Now s_α forces that $(\dot{r}_n : n < \omega)$ is an ascending chain in $\dot{Q}_\alpha$, so let $\dot{q}$ be such that s_α forces $\dot{q}$ is a least upper bound to $(\dot{r}_n : n < \omega)$. Let $s_{\alpha+1} = s_\alpha {}^\frown (\dot{q})$.

Thus the induction goes through, and s_{α_*} is a least upper bound $(q_i : i < i_*)$. □

The following class of forcing axioms, for $k = 2$, is related to Shelah's $\mathrm{Ax}\mu_0$ from [9] although the formulation is different. Although it is not relevant for the current paper, we could have allowed $\theta = \aleph_0$ with some minor changes to the proof of Theorem 3.5; this would then give weakenings of Martin's Axiom.

Definition 3.6. Suppose $\aleph_0 < \theta = \theta^{<\theta} \leq \lambda$, and suppose $2 \leq k < \omega$. Then say that $\mathrm{Ax}(k, \theta, \lambda)$ holds if for every forcing notion P such that $|P| \leq \lambda$ and P has the (k, θ, θ)-amalgamation property, if $(D_\alpha : \alpha < \lambda)$ is a sequence of dense subsets of P, then there is an ideal of P meeting each D_α. (By dense, we mean upwards dense: for every $p \in P$, there is $q \in D_\alpha$ with $q \geq p$.) Say that $\mathrm{Ax}(k, \theta)$ holds iff $\mathrm{Ax}(k, \theta, \lambda)$ holds for all $\lambda < 2^\theta$.

By a typical downward Lowenheim-Skolem argument we could drop the condition that $|P| \leq \lambda$ in $\mathrm{Ax}(k, \theta, \lambda)$, but we won't need this. Note that $P_{\theta\mu\theta}$ collapses μ to θ, so this is why there is not a parameter for μ in $\mathrm{Ax}(k, \theta)$. Finally, note that $\mathrm{Ax}(k, \theta, \lambda)$ implies that $2^\theta > \lambda$, easily.

Theorem 3.7. Suppose $\aleph_0 < \theta \leq \mu \leq \lambda$ are cardinals such that θ is regular and $\mu = \mu^{<\theta}$, and suppose $3 \leq k \leq \theta$. Suppose $\kappa \geq 2^\lambda$ has $\kappa^{<\kappa} = \kappa$. Then there is a forcing notion P with the $(< k, \mu, \theta)$-amalgamation property (in particular, θ-closed and μ^+-c.c.), such that P forces that $\mathrm{Ax}(k, \theta)$ holds and that $2^\theta = \kappa$. We can arrange $|P| = \kappa$.

Proof. Let $(P_\alpha : \alpha \leq \kappa), (\dot{Q}_\alpha : \alpha < \kappa)$ be a $< \theta$-support iteration, such that (viewing P_α-names as P_β-names in the natural way, for $\alpha \leq \beta < \kappa$):

- Each P_α forces that $\dot{Q}_\alpha$ has the $(< k, \mu, \theta)$-amalgamation property;
- Whenever $\alpha < \kappa$, and $\dot{Q}$ is a P_α-name such that $|\dot{Q}| < \kappa$ and P_α forces $\dot{Q}$ has the $(< k, \mu, \theta)$-amalgamation property, then there is some $\beta \geq \alpha$ such that P_β forces that $\dot{Q}_\beta$ is isomorphic to $\dot{Q}$;
- Each $|P_\alpha| \leq \kappa$.

This is possible by the μ^+-c.c., as in the proof of the consistency of Martin's axiom, and using Lemma 3.3(2). The point is that at each stage α, if P_α forces that $|\dot{Q}| = \lambda' < \kappa$, then we can choose a P_α-name $\dot{Q}'$ such that P_α-forces $\dot{Q} \cong \dot{Q}'$ and that $\dot{Q}'$ has universe λ'; then there are only $|P_\alpha|^{\lambda' \cdot \mu} \leq \kappa$-many possibilities for $\dot{Q}'$, up to P_α-equivalence. Thus we can eventually deal with all of them.

P_{α_*} then works, easily. □

We now relate this to model theory.

Definition 3.8. Suppose (θ, λ) is a nice pair, and $\theta \leq \mu \leq \lambda$, and T is simple. Then say that T has $(< k, \lambda, \mu, \theta)$-type amalgamation if whenever $M \models T$ has size $\leq \lambda$, and whenever $M_* \preceq M$ is countable, then Γ^θ_{M,M_*} has the $(< k, \mu, \theta)$-amalgamation property, or equivalently, $\Gamma^\theta_{M,M_*} \rightarrow^w_k P_{X\mu\theta}$ for some set X.

We prove some simple facts.

Lemma 3.9. Suppose T fails the $(< k, \lambda, \mu, \theta)$-amalgamation property, and P has the $(< k, \mu, \theta)$-amalgamation property. Then P forces that $\check{T}$ fails the $(< k, \lambda, \mu, \theta)$-amalgamation property.

Proof. It suffices to show that if Q is a forcing notion and P forces that $\check{Q} \rightarrow^w_k \check{P}_{\check{X}\mu\theta}$, then $Q \rightarrow^w_k P_{X'\mu\theta}$ for some X', by Lemma 3.3(4). (We then apply this to $Q = \Gamma^\theta_{M,M_*}$ witnessing the failure of $(< k, \mu, \theta)$-amalgamation.)

Choose some $F_* : (P, P_0) \rightarrow_k P_{X_*\mu\theta}$, and let $\dot{G}$ be a P-name so that P forces $\dot{F} : \check{Q} \rightarrow^w_k P_{\check{Y}\mu\theta}$. For every $q \in Q$, choose $\mathbf{p}(q) \in P_0$ such that $\mathbf{p}(q)$ decides $\dot{F}(\check{q})$, say $\mathbf{p}(q)$ forces that $\dot{F}(\check{q}) = f(q)$. Choose $F : Q \rightarrow P_{X\mu\theta}$ so that if $F(q)$ and $F(q')$ are compatible, then $f(q)$ and $f(q')$ are compatible, and $F_*(\mathbf{p}(q))$ and $F_*(\mathbf{p}(q'))$ are compatible.

Suppose $(q_i : i < i_*)$ is a sequence from Q with $(F(q_i) : i < i_*)$ compatible in $P_{X\mu\theta}$. Then $(F_*(\mathbf{p}(q_i)) : i < i_*)$ are all compatible in $P_{X_*\mu\theta}$, so $(\mathbf{p}(q_i) : i < i_*)$ are compatible in P_0 with the least upper bound p. Then p forces each $\dot{F}(\check{q}_i) = f(q_i)$. But also (by choice of F), $(f(q_i) : i < i_*)$ are compatible in $P_{Y,\mu,\theta}$, so p forces that $(\check{q}_i : i < i_*)$ is compatible in $\check{Q}$, i.e. $(q_i : i < i_*)$ is compatible in Q. □

Theorem 3.10. Suppose T simple, and $\aleph_0 < \theta = \theta^{<\theta} \leq \lambda = \lambda^{\aleph_0}$, and $\mathrm{Ax}(k, \theta)$ holds. Suppose $2^\theta > \lambda^{<\theta}$, and suppose $3 \leq k \leq \aleph_0$. Then the following are equivalent:

(A) T has $(< k, \lambda, \theta, \theta)$-type amalgamation;

(B) $SP^1_T(\lambda, \theta, \theta)$ holds.

Proof. (B) implies (A) is obvious. For (A) implies (B): let $M \models T$ have size at most λ and let $M_* \preceq M$ be countable. Let P be the $< \theta$-support product of Γ^θ_{M,M_*}; then P has the $(< k, \theta, \theta)$-amalgamation property and $|P| \leq \theta^{<\theta}$. For each $p(x) \in \Gamma^\theta_{M,M_*}$ let D_p be the dense subset of P consisting of all $f \in P$ such that for some $\gamma \in \mathrm{dom}(f)$, $f(\gamma)$ extends $p(x)$. By $\mathrm{Ax}(k, \lambda^{<\theta}, \theta)$ we can choose an ideal I of P meeting each D_p. This induces a sequence $(p_\gamma(x) : \gamma < \theta)$ of partial types over M that do not fork over M_*, such that for all $p(x) \in \Gamma^\theta_{M,M_*}$ there is $\gamma < \theta$ with $p(x) \subseteq p_\gamma(x)$. To finish, extend each $p_\gamma(x)$ to a complete type over M not forking over M_*.

The final claim follows from Theorem 2.4(A). □

4 Non-Forking Diagrams

Suppose T is a simple theory in a countable language. We wish to study various type amalgamation properties of T; in particular we will be looking at systems of types $(p_s(x) : s \in P)$ over a system of models $(M_s : s \in P)$, for some $P \subseteq \mathcal{P}(I)$ closed under subsets. For this to be interesting, we need $(M_s : s \in P)$ to be independent in a suitable sense, which we define in this section.

The following definition is similar to the first author's definition of independence in [7] in the context of stable theories, see Section XII.2. In fact we are modeling our definition after Fact 2.5 there (we cannot take the definition exactly from [7] because we allow P to contain infinite subsets of I).

Definition 4.1. Let T be simple.

Suppose I is an index set and $P \subseteq \mathcal{P}(I)$ is downward closed. Say that $(A_s : s \in P)$ is a diagram (of subsets of $\mathfrak{C}$) if each $A_s \subseteq \mathfrak{C}$ and $s \subseteq t$ implies $A_s \subseteq A_t$. Say that $(A_s : s \in P)$ is a non-forking diagram if for all $s_i : i < n, t \in P$, $\bigcup_{i<n} A_{s_i}$ is free from A_t over $\bigcup_{i<n} A_{s_i \cap t}$.. Say that $(A_s : s \in P)$ is a continuous diagram if for every $X \subseteq P$, $\bigcap_{s \in X} A_s = A_{\bigcap X}$. (If X is finite then this is a consequence of non-forking.)

Note that $(A_s : s \in P)$ is continuous if and only if for every $a \in \bigcup_{s \in P} A_s$, there is some least $s \in P$ with $a \in A_s$. Also note that if $(A_s : s \in P)$ is non-forking (continuous) and $Q \subseteq P$ is downward closed then $(A_s : s \in Q)$ is non-forking (continuous).

Lemma 4.2. Suppose $(A_s : s \in P)$ is a diagram of subsets of $\mathfrak{C}$. Then the following are equivalent:

(A) For all downward-closed subsets $S, T \subseteq P$, $\bigcup_{s \in S} A_s$ is free from $\bigcup_{t \in T} A_t$ over $\bigcup_{s \in S \cap T} A_s$.

(B) For all $s_i : i < n, t_j : j < m$ from P, $\bigcup_{i<n} A_{s_i}$ is free from $\bigcup_{j<m} A_{t_j}$ over $\bigcup_{i<n, j<m} A_{s_i \cap t_j}$.

(C) $(A_s : s \in P)$ is non-forking.

Proof. (A) implies (B) implies (C) is trivial. For (B) implies (A), use local character of nonforking and monotonicity.

We show (C) implies (B). So suppose $(A_s : s \in P)$ is non-forking. By induction on m, we show that for all n, if $s_i : i < n, t_j : j < m$ are from P, then $\bigcup_{i<n} A_{s_i}$ is free from $\bigcup_{j<m} A_{s_j}$ over $\bigcup_{i<n, j<m} A_{s_i \cap t_j}$. $m = 1$ is the definition of non-forking diagrams. Suppose true for all $m' \leq m$ and we show it holds at $m + 1$; so we have $s_i : i < n, t_j : j < m + 1$. Let $A_* = \bigcup_{i<n} A_{s_i}$ and let $B_* = \bigcup_{j<m} A_{t_j}$. By inductive hypothesis applies at $(s_i : i < n, t_m), (t_j : j < m)$, we get that $A_* \cup A_{t_m}$ is free from B_* over $(A_* \cup A_{t_m}) \cap B_*$. By monotonicity, A_* is free from $B_* \cup A_{t_m}$ over $(A_* \cap B_*) \cup A_{t_m}$. By the inductive hypothesis applied at $(s_i : i < n), t_m$, we get that A_* is free from A_{t_m} over $A_* \cap A_{t_m}$, so by monotonicity we get that A_* is free from $(A_* \cap B_*) \cup A_{t_m}$ over $A_* \cap (B_* \cup A_{t_m})$. □

The following lemma is similar to Lemma 2.3 from [7] Section XII.2.

Lemma 4.3. Suppose $P \subseteq \mathcal{P}(I)$ is downward closed and $(A_s : s \in P)$ is a continuous diagram of subsets of $\mathfrak{C}$. Suppose there is a well-ordering $<_*$ of $\bigcup_s A_s$ such that for all $a \in \bigcup_s A_s$, a is free from $\{b \in \bigcup_s A_s : b <_* a\}$ over $\{b \in s_a : b <_* a\}$, where s_a is the least element of P with $a \in A_{s_a}$. Then $(A_s : s \in P)$ is non-forking.

Proof. Let $(a_\alpha : \alpha < \alpha_*)$ be the $<_*$-increasing enumeration of $\bigcup_s A_s$, and let s_α be the least element of P with $a_\alpha \in A_{s_\alpha}$. For each $\alpha \leq \alpha_*$ and for each $s \in P$ let $A_{s,\alpha} = A_s \cap \{a_\beta : \beta < \alpha\}$. We show by induction on α that $(A_{s,\alpha} : s \in P)$ is non-forking. In fact we show (B) holds of Lemma 4.2 (due to symmetry it is easier).

Limit stages are clear. So suppose we have shown $(A_{s,\alpha} : s \in P)$ is non-forking. Let $(s_i : i < n), (t_j : j < m) \in P$ be given. We wish to show that $\bigcup_{i<n} A_{s_i,\alpha+1}$ is free from $\bigcup_{j<m} A_{t_j \cap \alpha+1}$ over $\bigcup_{i<n, j<n} A_{s_i \cap t_j, \alpha+1}$. If $a_\alpha \notin s_i$ and $a_\alpha \notin t_j$ for each $i < n$ then we conclude by the inductive hypothesis. If $a_\alpha \in s_{i_*} \cap t_{j_*}$ for some $i_* < n$, $j_* < m$, then we conclude by the inductive hypothesis and the fact that a_α is free from $\bigcup_{i<n} A_{s_i,\alpha} \cup \bigcup_{j<m} A_{t_j,\alpha}$ over $A_{s_{i_*} \cap t_{j_*}, \alpha}$, since $s_{i_*} \cap t_{j_*}$ contains s_α. If $a_\alpha \in s_i$ for some $i < n$ and $a_\alpha \notin t_j$ for any $j < m$, then reindex so that there is $0 < i_* \leq n$ so that $a_\alpha \in s_i$ iff $i < i_*$. Now a_α is free from $\{a_\beta : \beta < \alpha\}$ over s_α, so by monotonicity, $\bigcup_{i<n} A_{s_i,\alpha+1}$ is free from $\bigcup_{j<m} A_{s_j,\alpha+1}$ over $\bigcup_{i<n} A_{s_i,\alpha}$; use transitivity and the inductive hypothesis to finish.

□

For the proof of the following, the reader may find it helpful to bear in mind the special case when T is supersimple, so that every type does not fork over a finite subset of its domain. In that case we can in fact get $(M_s : s \in [\lambda]^{<\aleph_0})$ to cover $\mathbf{A}$.

Theorem 4.4. Suppose T is a simple theory in a countable language, and suppose $\mathbf{A}$ is a set of cardinality λ, where $\lambda = \lambda^{\aleph_0}$. Then we can find a continuous, non-forking diagram of models $(M_s : s \in [\lambda]^{\leq \aleph_0})$ such that $\mathbf{A} \subseteq \bigcup_s M_s$, and such that for all $S \subseteq \lambda$, $\bigcup_{s \in [S]^{\leq \aleph_0}} M_s$ has size at most $|S| \cdot \aleph_0$.

Proof. Enumerate $\mathbf{A} = (a_\alpha : \alpha < \lambda)$.

We define $(\mathrm{cl}(\{\alpha\}) : \alpha < \lambda)$ inductively as follows, where each $\mathrm{cl}(\{\alpha\})$ is a countable subset $\alpha + 1$ with $\alpha \in \mathrm{cl}(\{\alpha\})$. Suppose we have defined $(\mathrm{cl}(\{\beta\}) : \beta < \alpha)$. Choose a countable set $\Gamma \subseteq \alpha$ such that a_α is free from $\{a_\beta : \beta < \alpha\}$ over $\bigcup_{\beta \in \Gamma} a_\beta$; put $\mathrm{cl}(\{\alpha\}) = \{\alpha\} \cup \bigcup_{\beta \in \Gamma} \mathrm{cl}(\{\beta\})$. (So, if T is supersimple, each Γ can be chosen to be finite.)

Now, for each $s \subseteq \lambda$, let $\mathrm{cl}(s) := \bigcup_{\alpha \in s} \mathrm{cl}(\{\alpha\})$. Say that $A \subseteq \lambda$ is closed if $\mathrm{cl}(A) = A$; this satisfies the usual properties of a set-theoretic closure operation, that is $\mathrm{cl}(A) \supseteq A$, and $A \subseteq B$ implies $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$, and $\mathrm{cl}^2(A) = \mathrm{cl}(A)$, and cl is finitary: in fact $\mathrm{cl}(A) = \bigcup_{\alpha \in A} \mathrm{cl}(\{\alpha\})$, which is even stronger. Finally, $|\mathrm{cl}(A)| \leq |A| + \aleph_0$.

For each $s \in [\lambda]^{\leq\omega}$, let $A_s = \{a_\alpha : \alpha < \lambda \text{ and } \mathrm{cl}(\{\alpha\}) \subseteq s\}$. Since each $a_\alpha \in A_{\mathrm{cl}(\{\alpha\})}$, clearly $\bigcup_s A_s = \mathbf{A}$. I claim that $(A_s : s \in [\lambda]^{\leq\omega})$ is a non-forking diagram of sets. But this follows from Lemma 4.3, since each a_α is free from $\{a_\beta : \beta < \alpha\}$ over $A_{\mathrm{cl}(\{\alpha\})} \cap \{a_\beta : \beta < \alpha\}$.

For each $\alpha \leq \lambda$, let $\mathcal{A}_\alpha = \{\mathrm{cl}(s) : s \in [\alpha]^{<\omega}\}$. I show by induction on $\alpha \leq \lambda$ that $(\mathcal{A}_\alpha, \subset)$ is well-founded. Note that since $\mathbf{A} = \bigcup \mathcal{A}_\lambda$, it will follow that $(A_s : s \in [\lambda]^{\leq\aleph_0})$ is continuous. Since $\mathcal{A}_\alpha$ is an end extension of $\mathcal{A}_\beta$ for $\alpha > \beta$, the limit stage is clear. So suppose we have shown $(\mathcal{A}_\alpha, \subset)$ is well-founded.

Write $X = \mathrm{cl}(\{\alpha\}) \cap \alpha$; note that $\mathrm{cl}(X) = X$. Now suppose $s, t \in [\alpha]^{<\omega}$. I claim that $\mathrm{cl}(s \cup \{\alpha\}) \subseteq \mathrm{cl}(t \cup \{\alpha\})$ iff $\mathrm{cl}(s \cup X) \subseteq \mathrm{cl}(t \cup X)$. But this is clear, since $\mathrm{cl}(s \cup \{\alpha\}) = \mathrm{cl}(s) \cup X \cup \{\alpha\}$, and $\mathrm{cl}(t \cup \{\alpha\}) = \mathrm{cl}(t) \cup X \cup \{\alpha\}$, and $\mathrm{cl}(s \cup X) = \mathrm{cl}(s) \cup X$, and $\mathrm{cl}(t \cup X) = \mathrm{cl}(t) \cup X$.

Thus it follows from the inductive hypothesis that $(\{\mathrm{cl}(s \cup \{\alpha\}) : s \in [\alpha]^{<\omega}\}, \subset)$ is well-founded, and hence that $\mathcal{A}_{\alpha+1}$ is well-founded; hence $\mathcal{A}_\lambda$ is well-founded.

Let $<_*$ be a well-order of $\mathcal{A}_\lambda$ refining $\subset$. Now by induction on $<_*$, choose countable models $(M(A) : A \in \mathcal{A}_\lambda)$ so that $M(A) \supseteq A$ and such that $M(A)$ is free from $\mathbf{A} \cup \bigcup\{M(B) : B \in \mathcal{A}_\lambda, B <_* A\}$ over $A \cup \bigcup\{M(B) : B \in \mathcal{A}, B \subset A\}$. Finally, given $s \in [\lambda]^{\leq\omega}$, let $M_s := M(A_s)$. This is a non-forking diagram of models, using Lemma 4.3, and it is clearly continuous.

The final claim follows, since for all $S \subseteq \lambda$, $\{t \in \mathcal{A} : t \subseteq S\}$ has size at most $|S| \cdot \aleph_0$. □

5 Amalgamation properties

Suppose T is a simple theory in a countable language. We now explain what we mean by T having type amalgamation.

Definition 5.1. Given $\Lambda \subseteq {}^n m$, let P_Λ be the set of all partial functions from n to m which can be extended to an element of Λ; so P_Λ is a downward-closed subset of $n \times m$, and Λ is the set of maximal elements of P_λ.

Suppose $(M_u : u \subseteq n)$ is a non-forking diagram of models. Then by a $(\Lambda, \overline{M})$-array, we mean a non-forking diagram of models $(N_s : s \in P_\Lambda)$, together with maps $(\pi_s : s \in P_\Lambda)$ such that each each $\pi_s : M_{\mathrm{dom}(s)} \cong N_s$, and such that $s \subseteq t$ implies $\pi_s \subseteq \pi_t$.

Definition 5.2. Suppose $\Lambda \subseteq {}^n m$. Then T has Λ-type amalgamation if, whenever $(M_u : u \subseteq n)$ is a non-forking diagram of models, and whenever $p(x)$ is a complete type over M_n in finitely many variables which does not fork over M_0, and whenever $(N_s, \pi_s : s \in P_\Lambda)$ is a $(\Lambda, \overline{M})$-array, then $\bigcup_{\eta \in \Lambda} \pi_\eta(p(x))$ does not fork over N_0.

Suppose $3 \leq k \leq \aleph_0$; then say that T has $< k$-type amalgamation if whenever $|\Lambda| < k$, then T has Λ-type amalgamation.

The following lemma is straightforward.

Lemma 5.3. Suppose $\Lambda \subseteq {}^n m$. Then in the definition of Λ-type amalgamation, the following changes would not matter:

(A) We could restrict to just countable models M_u.

(B) We could allow $p(x)$ to be any partial type, or insist it is a single formula. Also, we could replace x by a tuple $\overline{x}$ of arbitrary cardinality.

Example 5.4. Every simple theory has < 3-type amalgamation. T_{rg} has $< \aleph_0$-type amalgamation.

Example 5.5. Suppose $\ell > k \geq 2$. Let $T_{\ell,k}$ be the theory of the random k-ary, ℓ-clique free hypergraph; these examples were introduced by Hrushovski [3], where he proved $T_{\ell,k}$ is simple if and only if $k \geq 3$.

For $k \geq 3$, $T_{\ell,k}$ has $< k$-type amalgamation but not $< k+1$-type amalgamation.

Proof. First suppose $\Lambda \subseteq {}^n m$ with $|\Lambda| < k$, and $(M_u : u \subseteq n)$ are given, and suppose $p(\overline{x})$ is a complete type over M_n. Suppose towards a contradiction there were a $(\Lambda, \overline{M})$-array $(N_s, \pi_s : s \in P_\Lambda)$ with $\bigcup_{\eta \in \Lambda} \pi_\eta[p(\overline{x})]$ inconsistent. Write $q(\overline{x}) = \bigcup_{\eta \in \Lambda} \pi_\eta[p(\overline{x})]$; then $q(\overline{x})$ must create some ℓ-clique $(a_i :< \ell_0), (x_j : j < \ell_1)$, where $\ell_0 + \ell_1 = \ell$, and each $a_i \in N_\eta$ for some $\eta \in \Lambda$, and each $x_j \in \overline{x}$. Clearly we have each $\ell_0, \ell_1 > 0$.

For each $i < \ell_0$, let $h(i)$ be the least $s \in P_\Lambda$ with $a_i \in N_s$. The following must hold:

(I) For every $u \in [\ell_0]^{<k}$, $h[u] \in P_\Lambda$;

(II) $h[\ell_0] \notin P_\Lambda$.

By (II), for each each $\eta \in \Lambda$ we must have $h[\ell_0] \not\subseteq \eta$; thus we can choose $i_\eta < \ell_0$ such that $h(i_\eta) \not\subseteq \eta$. Let $u = \{i_\eta : \eta \in \Lambda\} \in [\ell_0]^{<k}$. Clearly then $h[u] \notin P_\Lambda$, but this contradicts (I).

Now we show that $T_{\ell,k}$ fails $< k+1$-type amalgamation. Indeed, let $\Lambda \subseteq {}^k 2$ be the set of all $f : k \to 2$ for which there is exactly one $i < k$ with $f(i) = 1$; so $|\Lambda| = k$. Also, let $(M_u : u \subseteq k)$ be a non-forking diagram of models so that there are $a_i \in M_{\{i\}}$ for $i < k$ and there are $b_j \in M_0$ for $n < \ell - k - 1$, such that every k-tuple of distinct elements from $(a_i, b_j : i < k, j < \ell - k - 1)$ is in R except for $(a_i : i < k)$. Let $p(x)$ be the partial type over M_k which asserts that $R(x, \overline{a})$ holds for every $k-1$-tuple of distinct elements from $(a_i, b_j : i < k, j < \ell - k - 1)$.

It is not hard to find a $(\Lambda, \overline{M})$-array $(N_s, \pi_s : s \in P_\Lambda)$ such that, if we write $\pi_{\{(i,0)\}}(a_i) = c_i$, then $R(c_i : i < k)$ holds; but now we are done, since $\bigcup_{f \in \Lambda} \pi_f[p(x)]$ is inconsistent. □

The following is the key consequence of $< k$-type amalgamation.

Theorem 5.6. Suppose T is a simple theory with $< k$-type amalgamation. Then for all nice pairs (θ, λ), T has $(< k, \lambda, \theta, \theta)$-type amalgamation.

Proof. By Theorem 4.4, it suffices to show that if $(\mathbf{M}_s : s \in [\lambda]^{<\theta})$ is a continuous non-forking diagram of countable models such that each $|\mathbf{M}_s| < \theta$, then writing $\mathbf{M} = \bigcup_s \mathbf{M}_s$, we have that $\Gamma^\theta_{\mathbf{M},\mathbf{M}_0} \to^w_k P_{X\theta\theta}$ for some X. Let $<_*$ be a well-ordering of $\mathbf{M}$.

Given $A \in [\mathbf{M}]^{<\theta}$ let s_A be the $\subseteq$-minimal $s \in [\lambda]^{<\theta}$ with $A \subseteq M_{s_A}$, possible by continuity.

Let P be the set of all $p(x) \in \Gamma^\theta_{\mathbf{M},\mathbf{M}_0}$ such that for some $s \in [\lambda]^{<\theta}$, $p(x)$ is a complete type over $\mathbf{M}_s$; we write $p(x, \mathbf{M}_s)$ to indicate this. P is dense in $\Gamma^\theta_{\mathbf{M},\mathbf{M}_0}$, so it suffices to show that $P \to^w_k P_{\lambda\theta\theta}$ for some λ.

Choose X large enough, and $F : P \to P_{X\theta\theta}$ so that if $F(p(x), \mathbf{M}_s)$ is compatible with $F(q(x), \mathbf{M}_t)$, then:

- s and t have the same order-type, and if we let $\rho : s \to t$ be the unique order-preserving bijection, then ρ is the identity on $s \cap t$;
- $\mathbf{M}_s$ and $\mathbf{M}_t$ have the same $<_*$-order-type, and the unique $<_*$-preserving bijection from $\mathbf{M}_s$ to $\mathbf{M}_t$ is in fact an isomorphism $\tau : \mathbf{M}_s \cong \mathbf{M}_t$
- For each finite $\overline{a} \in \mathbf{M}_s^{<\omega}$, if we write $s' = s_{\overline{a}}$ and if we write $t' = s_{\tau(\overline{a})}$, then: $\rho[s'] = t'$ and $\tau \restriction_{\mathbf{M}_{s'}} : \mathbf{M}_{s'} \cong \mathbf{M}_{t'}$.
- $\tau[p(x)] = q(x)$.

This is not hard to do. Note that it follows that for every $s' \subseteq s$, $\rho \restriction_{\mathbf{M}_{s'}} : \mathbf{M}_{s'} \cong \mathbf{M}_{\rho[s']}$, since $\mathbf{M}_{s'} = \bigcup\{\mathbf{M}_{s_{\overline{a}}} : \overline{a} \in (\mathbf{M}_{s'})^{<\omega}\}$ and similarly for $\mathbf{M}_{t'}$.

I claim that F works.

So suppose $p_i(x, \mathbf{M}_{s_i}) : i < i_*$ is a sequence from P for $i_* < k$, such that $(F(p_i(x)) : i < i_*)$ is compatible in $P_{\lambda\theta\theta}$.

Let γ_* be the order-type of some or any s_i. Enumerate each $s_i = \{\alpha_{i,\gamma} : \gamma < \gamma_*\}$ in increasing order. Let E be the equivalence relation on γ_* defined by: $\gamma E \gamma'$ iff for all $i, i' < k$, $\alpha_{i,\gamma} = \alpha_{i',\gamma}$ iff $\alpha_{i,\gamma'} = \alpha_{i',\gamma'}$. Let $(E_j : j < n)$ enumerate the equivalence classes of E. For each $i < i_*$, and for each $j < n$, let $X_{i,j} = \{\alpha_{i,\gamma} : \gamma \in E_j\}$. Thus s_i is the disjoint union of $X_{i,j}$ for $j < n$. Moreover, $X_{i,j} \cap X_{i',j'} = \emptyset$ unless $j = j'$; and if $X_{i,j} \cap X_{i',j} \neq \emptyset$ then $X_{i,j} = X_{i',j}$. For each $j < n$, enumerate $\{X_{i,j} : i < i_*\} = (Y_{\ell,j} : \ell < m_i)$ without repetitions. Let $m = \max(m_j : j < n)$; and for each $i < i_*$, define $\eta_i \in {}^n m$ via: $\eta_i(j) =$ the unique $\ell < m_i$ with $X_{i,j} = Y_{\ell,j}$.

Let $\Lambda = \{\eta_i : i < i_*\}$. For each $s \in P_\Lambda$, let $N_s = \mathbf{M}_{t_s}$ where $t_s = \bigcup_{(j,\ell) \in s} Y_{\ell,j}$. Also, define $(M_u : u \subseteq n) := (N_{\eta_0 \restriction u} : u \subseteq n)$. Then the hypotheses on F give commuting isomorphisms $\pi_s : M_{\mathrm{dom}(s)} \cong N_s$ for each $s \in P_\Lambda$, in such a way that $(\overline{N}, \overline{\pi})$ is a $(\lambda, \overline{M})$-array, and each $\pi_{\eta_i}(p_0(x)) = p_i(x)$. It follows by hypothesis on T that $\bigcup_{i<i_*} p_i(x)$ does not fork over N_0, as desired.

$\square$

Corollary 5.7. Suppose T is simple, with $< \aleph_0$-type amalgamation.

(A) Suppose θ is a regular uncountable cardinal. Then for any $M \models T$ and any $M_0 \preceq M$ countable, Γ^θ_{M,M_0} has the $(< \aleph_0, \theta, \theta)$-amalgamation property.

(B) Suppose (θ, λ) is a nice pair, and suppose that $\theta \leq \mu \leq \lambda$ satisfies $\mu = \mu^{<\theta}$ and $2^\mu \geq \lambda$. Then $SP^1_T(\lambda, \mu, \theta)$ holds.

(C) If the singular cardinals hypothesis holds, then $T \leq_{SP} T_{rg}$.

Proof. (A) follows immediately from Theorem 5.6, and (C) follows from (B) by Theorem 2.4(A) and Theorem 2.7. So it suffices to verify (B).

Suppose $M \models T$ has $|M| \leq \lambda$, and suppose $M_0 \preceq M$ is countable. Choose some $F : \Gamma^\theta_{M,M_0} \to^w_k P_{\lambda\theta\theta}$. By Corollary 2.1, we can find $(\mathbf{f}_\gamma : \gamma < \mu)$ such that whenever $f \in P_{\lambda\theta\theta}$ then $f \subseteq \mathbf{f}_\gamma$ for some $\gamma < \mu$; for each $\gamma < \mu$, choose $q_\gamma(x)$, a complete type over M not forking over M_0, and extending $\bigcup\{p(x) : F(p(x)) \subseteq \mathbf{f}_\gamma\}$. Then clearly $(q_\gamma(x) : \gamma < \mu)$ witnesses $SP^1_T(\lambda, \mu, \theta)$. $\square$

6 Conclusion

We begin to put everything together. We aim to produce a forcing extension in which, whenever T has $< k$-type amalgamation, then $T_{k,k-1} \not\leq_{SP} T$. We will choose in advance nice pairs (θ_k, λ_k) to witness this. In order to arrange that $SP_T(\lambda_k, \theta_k)$ holds we will use Theorems 3.10 and 5.6. To arrange that $SP_{T_{k,k-1}}(\lambda_k, \theta_k)$ fails, we will use the following.

Theorem 6.1. Suppose (θ, λ) is a nice pair such that $\theta = \theta^{<\theta}$ and $\lambda > \theta$ is a limit cardinal. Let $3 \leq k < \omega$. Then $P_{\lambda\theta\theta}$ forces that for all $\mu < \lambda$, $\check{T}_{k+1,k}$ fails $(< k+1, \lambda, \mu, \theta)$-type amalgamation.

Proof. Fix $\theta \leq \mu < \lambda$, and write $P = P_{[\lambda]^k\theta\theta}$. We show that P forces $\check{T}_{k+1,k}$ fails $(< k+1, \lambda, \mu, \theta)$-type amalgamation. Since $P \cong P_{\lambda\theta\theta}$, this suffices.

We pass to a P-generic forcing extension $\mathbb{V}[G]$ of $\mathbb{V}$. Let $R \subseteq [\lambda]^k$ be the set of all v with $\{(v, 0)\} \in G$. Choose $M_0 \preceq M \models T_{k+1,k}$, and $(a_{i,\alpha} : i < k, \alpha < \lambda)$ such that, writing $\overline{a}_s = \{a_{i,\alpha} : (i, \alpha) \in s\}$ for $s \subseteq k \times \lambda$:

- M_0 is countable, and $|M| \leq \lambda$ and each $a_{i,\alpha} \in M \backslash M_0$;
- $a_{i,\alpha} = a_{j,\beta}$ iff $\alpha = \beta$ and $i = j$;
- For every $v_* \in [k \times \lambda]^k$, if v_* is not the graph of the increasing enumeration of some $v \in [\lambda]^k$, then $R^M(\overline{a}_{v_*})$ fails. Otherwise, $R^M(\overline{a}_{v_*})$ holds if and only if $v \in R$.

For each $v \in [\lambda]^k$, let $\varphi_u(x, \overline{a}_{k\times v})$ be the formula that asserts that $R(x, \overline{a}_u)$ holds for each $u \in [k \times v]^{k-1}$. Note that $\varphi_v(x, \overline{a}_{k\times v})$ is consistent exactly when $v \notin R$.

It suffices to show that there is no cardinal λ' and function $F_0 : \Gamma^\theta_{M,M_0} \to^w_{k+1} P_{\lambda'\mu\theta}$; so suppose towards a contradiction some such F_0 existed. Then we can find $F : [\lambda]^k \backslash R \to P_{\lambda'\mu\theta}$ such that for all sequences $(w_i : i < k+1)$ from $[\lambda]^k \backslash R$, if $(F(w_i) : i < k)$ is compatible in P then $\bigwedge_{i<k} \varphi_v(x, \overline{a}_{k\times v})$ is consistent. This is all we will need, and so we can replace λ' by λ (since $|[\lambda]^k| = \lambda$).

Pulling back to $\mathbb{V}$, we can find $p_* \in P$, and P-names $\dot{R}, \dot{M}, \dot{M}_0, \dot{a}_{i,\alpha}, \dot{F}$, such that p_* forces these behave as above.

Write $X = \lambda \backslash \bigcup \mathrm{dom}(p_*)$; so $|X| = \lambda$.

Suppose $v \in [X]^k$. Choose $p_v \in P$ such that $p_v \geq p_* \cup \{(v, 1)\}$ (so p_v forces $v \notin \dot{R}$), and so that p_v decides $\dot{F}(v)$, say p_v forces that $\dot{F}(v) = f_v \in P_{\lambda\mu\theta}$.

Choose $F_* : [\lambda]^k \to P_{\lambda\mu\theta}$ so that for all v, v', if $F_*(v)$ and $F_*(v')$ are compatible, then $p_v, p_{v'}$ are compatible, and $f_v, f_{v'}$ are compatible.

Let $\mathcal{B}$ be the Boolean-algebra completion of $P_{\lambda\mu\theta}$. For each $u \in [\lambda]^{k-1}$, let $\mathbf{b}_u$ be the least upper bound in $\mathcal{B}$ of $(F_*(v) : u \subseteq v \in [\lambda]^k)$. Since $\mathcal{B}$ has the μ^+-c.c., we can find $S(u) \in [\lambda]^{\leq\mu}$ such that $\mathbf{b}_u$ is also the least upper bound in $\mathcal{B}$ of $(F_*(v) : u \subseteq v \in [S(u)]^k)$. By expanding $S(u)$, we can suppose that for all $u \subseteq v \in [\lambda]^k$, $\bigcup \mathrm{dom}(p_v) \subseteq S(u)$.

By Theorem 46.1 of [2], we can find some $v \in [\lambda]^k$ such that for all $u \in [v]^{k-1}$, $S(u) \cap v = u$. Now $(\mathbf{b}_u : u \in [v]^{k-1})$ has an upper bound in $\mathcal{B}$, namely $F_*(v)$; thus we can find $(v_u : u \in [v]^{k-1})$ such that each $u \subseteq v_u \in [S(u)]^k$, and $(F_*(v_u) : u \in [w]^{k-1})$ is compatible in $\mathcal{B}$ (i.e. in $P_{\lambda\mu\theta}$). Thus $(p_{v_u} : u \in [v]^{k-1})$ is compatible in P; write $p = \bigcup_{u\in[v]^{k-1}} p_{v_u}$ (recall $P = P_{[\lambda]^k\theta\theta}$). Note that

$v \notin \mathrm{dom}(p)$, since if $v \in \mathrm{dom}(p_{v_u})$ then $v \subseteq \bigcup \mathrm{dom}(p_{v_u}) \subseteq S(u)$, contradicting that $S(u) \cap v = u$. Thus we can choose $p' \geq p$ in P with $p'(v) = 0$.

Now p' forces that each $\dot{F}(v_u) = \check{f}_{v_u}$, and $(f_{v_u} : u \in [v]^{k-1})$ is compatible; thus p' forces that $\varphi(x) := \bigwedge_{u \in [v]^{k-1}} \varphi_{v_u}(\dot{\bar{a}}_{k \times v_u})$ is consistent. But this is impossible, since if we let v_* be the graph of the increasing enumeration of v, then p' forces that $\dot{R}^{\dot{M}}(\dot{\bar{a}}_{v_*})$ holds, and $\varphi(x)$ in particular implies that $\dot{R}^{\dot{M}}(x, \dot{\bar{a}}_{u_*})$ holds for all $u_* \in [v_*]^{k-1}$, thus creating a $k+1$-clique.

□

Theorem 6.2. Suppose *GCH* holds. Then there is a forcing notion P, which forces:

(A) For every $k \geq 3$, if T is a simple theory with $< k$-type amalgamation, then $T_{k,k-1} \not\leq_{SP} T$;

(B) The maximal $\leq_{SP}$-class is the class of simple theories;

(C) If T has $< \aleph_0$-type amalgamation then $T \leq_{SP} T_{rg}$.

Of course, we can also force to make GCH hold (via a proper-class forcing notion). Thus, (A), (B), (C) can consistently hold.

Proof. Write $\theta_2 = \lambda_2 = \aleph_0$. Choose nice pairs $((\theta_k, \lambda_k) : 3 \leq k \leq \omega)$, such that each $\theta_k > \lambda_{k-1}^{++}$, and each λ_k is singular with $\mathrm{cof}(\lambda_k) < \theta_k$ (so each $\lambda_k^{<\theta_k} = \lambda_k^+$).

We will define a full-support forcing iteration $(P_k : 3 \leq k \leq \omega)$, $(\dot{Q}_k : 3 \leq k < \omega)$; for each $3 \leq k < \omega$, we will have that $|P_k| \leq \lambda_{k-1}^{++}$, and P_k will force that $\dot{Q}_k$ is θ_k-closed and has the θ_k^+-c.c.

Having defined P_k, note that P_k forces that (θ_k, λ_k) remains a nice pair and $\mathrm{cof}(\lambda_k) < \theta_k$ and $\theta_k^{<\theta_k} = \theta_k$, since P_k has the θ_{k-1}^+-c.c. Let $\dot{Q}_k^0 = \check{P}_{\lambda_k \theta_k \theta_k}$. By Theorem 3.7, we can choose a $P_k * \dot{Q}_k^0$-name $\dot{Q}_k^1$ for a forcing notion, such that $P_k * \dot{Q}_k^0$ forces $\dot{Q}_k^1$ has the $(< k, \theta_k, \theta_k)$-amalgamation property, and $\mathrm{Ax}(< k, \theta_k)$ holds, and $2^{\theta_k} = \lambda_k^{++}$, and $|\dot{Q}_k^1| = \lambda_k^{++}$. Let $\dot{Q}_k$ be the P_k-name for $\dot{Q}_k^0 * \dot{Q}_k^1$.

Let P_ω be the iteration of $P_k : 3 \leq k < \aleph_0$ with full supports. Also, for each $3 \leq k < \omega$, write $P_\omega = P_k * \dot{P}_{\geq k}$, where $\dot{P}_{\geq k}$ is the P_k-name for the forcing iteration induced by $(\dot{Q}_{k'} : k' \geq k)$. Note that each $\dot{P}_{\geq k}$ is θ_k-closed, and each P_k is θ_{k-1}^+-c.c.

Given $3 \leq k < \omega$, note that since P_k forces that (θ_k, λ_k) is a nice pair, and $\dot{Q}_k$ is θ_k-closed and θ_k^+-c.c., we have that P_{k+1} forces that (θ_k, λ_k) is a nice pair; since $\dot{P}_{\geq k+1}$ is in particular λ_k^+-closed, we have that P_ω forces that (θ_k, λ_k) is a nice pair.

Now P_{k+1} forces that $SP_T(\lambda_k, \theta_k)$ holds whenever T has $< k$-type amalgamation by Theorem 5.6 and Theorem 3.10, and that $SP_{T_{k,k-1}}(\lambda_k, \theta_k)$ fails by Theorem 6.1 and Theorem 3.9. Since $\dot{P}_{\geq k+1}$ is $(\lambda_k^{<\theta_k})^+$-closed, it does not change this, and so we have that P_ω forces that (θ_k, λ_k) is a nice pair, $SP_T(\lambda_k, \theta_k)$ holds and $SP_{T_{k,k-1}}(\lambda_k, \theta_k)$ fails. Thus we have verified that P_ω forces (A) to hold. (B) follows from (A) in the case $k = 3$, since every simple theory has < 3-type amalgamation, and by Theorem 2.5(A), unsimple theories are maximal in $\leq_{SP}$.

To verify (C), it suffices to show that P_ω forces the singular cardinals hypothesis to hold. This is standard, but we give a full argument.

Claim. Suppose the singular cardinals hypothesis holds and P is κ-closed, κ^+-c.c. Then P forces that the singular cardinals hypothesis holds.

Proof. Let $\mathbb{V}[G]$ be a P-generic forcing extension; we work in $\mathbb{V}[G]$. Suppose λ is singular and $2^{\mathrm{cof}(\lambda)} < \lambda$. Note that $|\lambda^{\mathrm{cof}(\lambda)}| = |[\lambda]^{\mathrm{cof}(\lambda)}| \cdot 2^{\mathrm{cof}(\lambda)} = |[\lambda]^{\mathrm{cof}(\lambda)}|$, so it suffices to show that $|[\lambda]^{\mathrm{cof}(\lambda)}| = \lambda^+$. Note that $|[\lambda]^{\mathrm{cof}(\lambda)} \cap \mathbb{V}| = \lambda^+$ since the singular cardinals hypothesis holds in $\mathbb{V}$ (and $\lambda^+ = (\lambda^+)^{\mathbb{V}}$), and so $|[\lambda]^{\mathrm{cof}(\lambda)}| = \lambda^+ \cdot 2^{\mathrm{cof}(\lambda)}$, since every $X \in [\lambda]^{\mathrm{cof}(\lambda)}$ can be covered some $Y \in ([\lambda]^{\mathrm{cof}(\lambda)} \cap \mathbb{V})$, using that P is κ-closed if $|X| < \kappa$, and that P is κ^+-c.c. if $|X| \geq \kappa$. $\square$

Write $\theta = \sup(\theta_k : 3 \leq k < \omega)$. Note that by a trivial induction together with the claim, for all $3 \leq k < \omega$, P_k forces that the singular cardinals hypothesis holds. Thus, given $3 \leq k < \omega$, since $\dot{P}_{\geq k}$ is θ_k-closed, we have that P_ω forces that the singular cardinals hypothesis holds at all singular cardinals $\lambda < \theta_k$. Since this holds for all k, we get that P forces that the singular cardinal hypothesis holds for all singular $\lambda < \theta$. Also, P_ω is θ^{++}-c.c. (since $|P_\omega| = \theta^+$). Thus to finish it suffices to show that $P_\omega \Vdash 2^\theta = \theta^+$, since then P_ω forces that GCH holds above θ.

Let $\mathbb{V}[G]$ be a P_ω-generic forcing extension of $\mathbb{V}$. Easily, $(2^{<\theta})^{\mathbb{V}[G]} = \theta$; also, since P_ω is ω-closed, $(\theta^+)^{\mathbb{V}[G]} = \theta^+$ (as otherwise it would have countable cofinality) and $(|\theta^\omega|)^{\mathbb{V}[G]} = \theta^+$. But then in $\mathbb{V}[G]$, $2^\theta \leq (2^{<\theta})^\omega = \theta^+$, since we can encode $X \subseteq \theta$ by $(X \cap \theta_k : 3 \leq k < \omega)$. $\square$

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