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THE LEFT SIDE OF CICHOŃ'S DIAGRAM

MARTIN GOLDSTERN, DIEGO ALEJANDRO MEJÍA, AND SAHARON SHELAH

ABSTRACT. Using a finite support iteration of ccc forcings, we construct a model of $\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = \mathfrak{c}$.

1. INTRODUCTION

How many (Lebesgue) null sets do you need to cover the real line? How many points do you need to get a non-null set? What is the smallest number of null sets that you need to get a union which is not null any more? The answers to these questions are the cardinals $\mathrm{cov}(\mathcal{N})$, $\mathrm{non}(\mathcal{N})$, $\mathrm{add}(\mathcal{N})$, and similar definitions are possible for other ideals, such as the ideal $\mathcal{M}$ of meager (=first category) sets, the ideal of at most countable sets, or the ideal of σ-compact subsets of the irrationals.

The cardinal $\mathrm{add}(\sigma\text{-compact}) = \mathrm{non}(\sigma\text{-compact})$ is usually called $\mathfrak{b}$; it is the smallest size of a family of functions from ω to ω which is not eventually bounded by a single function. We define $\mathfrak{d} := \mathrm{cov}(\sigma\text{-compact})$, and write $\mathrm{cf}(I)$ for the smallest size of a basis of any ideal I.

Cichoń's diagram (see [CKP85], [Fre84], [BJ95]) is the table of 12 cardinals shown in Figure 1. The arrows show provable inequalities between these cardinals, such as

$$\aleph_1 = \mathrm{non}(\text{countable}) \leq \mathrm{add}(\mathcal{N}) \leq \mathrm{cov}(\mathcal{N}) \leq 2^{\aleph_0} = \mathrm{cov}(\text{countable}).$$

In addition to the inequalities indicated the dotted arrows represent $\mathrm{add}(\mathcal{M}) = \min(\mathfrak{b}, \mathrm{cov}(\mathcal{M}))$ and $\mathrm{cf}(\mathcal{M}) = \max(\mathfrak{d}, \mathrm{non}(\mathcal{M}))$.

For any two of these cardinals, say $\mathfrak{x}$ and $\mathfrak{y}$, the relation $\mathfrak{x} \leq \mathfrak{y}$ is provable in ZFC if and only if this relation can be seen in the diagram. However, the question how many of these cardinals can be different in a single ZFC-universe is still open.

Some models of partial answers to this question are constructed in [Mej13] and [FGKS15]. In this paper, we will construct a model, so far unknown, where the following strict inequalities hold:

$$\omega_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}$$

Moreover, the values of these cardinals can be quite arbitrary.

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$$
\begin{tikzcd}
 & \operatorname{cov}(\mathcal{N}) \arrow[r] & \operatorname{non}(\mathcal{M}) \arrow[r, dashed] & \operatorname{cof}(\mathcal{M}) \arrow[r] & \operatorname{cof}(\mathcal{N}) \arrow[r] & \mathfrak{c} \\
 & & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u, dashed] & & \\
\aleph_1 \arrow[r] & \operatorname{add}(\mathcal{N}) \arrow[uu] \arrow[r] & \operatorname{add}(\mathcal{M}) \arrow[u, dashed] \arrow[r, dashed] & \operatorname{cov}(\mathcal{M}) \arrow[u] \arrow[r] & \operatorname{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}
$$

FIGURE 1. Cichoń's diagram

2. Informal overview

2.1. **Increasing** $\operatorname{add}(\mathcal{N})$**.** Assume for simplicity that GCH holds. For any regular uncountable cardinal κ there is a natural way to force $\operatorname{add}(\mathcal{N}) = \kappa$, namely a finite support iteration $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \kappa)$ of length κ, where in each step α the forcing $\dot{\mathbb{Q}}_\alpha$ will be amoeba forcing $\mathbb{A}$, which will add an amoeba real η_α; this real will code a null set N_α that covers not only all reals from $V^{\mathbb{P}_\alpha}$ but even the union of all Borel null sets whose code is in $V^{\mathbb{P}_\alpha}$. The final model $V^{\mathbb{P}_\kappa}$ will satisfy the following:

- (as κ is regular:) Every small (i.e.: of size $< \kappa$) family of (Borel) null sets will be added before stage κ; hence its union will be covered by one of the sets N_α. So $\operatorname{add}(\mathcal{N}) \geq \kappa$.
- The union of all N_α contains all reals and is in particular not of measure zero; hence also $\operatorname{add}(\mathcal{N}) \leq \kappa$.

This model will of course also satisfy $2^{\aleph_0} = \kappa$. If we are given two regular cardinals κ_{an} and κ_{ct} (we write κ_{an} to indicate that this cardinal is intended to be the **a**dditivity of **n**ull sets, and κ_{ct} for the intended size of the **c**on**t**inuum), then we can construct a ccc poset $\mathbb{P}$ forcing

$$\kappa_{an} = \operatorname{add}(\mathcal{N}) < 2^{\aleph_0} = \kappa_{ct}$$

as the finite support limit of a finite support iteration $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \kappa_{ct})$ as follows:

- For each $\alpha < \kappa_{ct}$ we choose a $\mathbb{P}_\alpha$-name $\dot{X}_\alpha$ of a family of Borel measure zero sets (or really: Borel codes of measure zero sets) of size $< \kappa_{an}$.
- We find a (name for a) transitive model M_α of a sufficient fragment of ZFC of size $< \kappa_{an}$ which is forced to include $\dot{X}_\alpha$.
  We then let $\dot{\mathbb{Q}}_\alpha$ be the $\mathbb{P}_\alpha$-name for $\mathbb{A}^{M_\alpha} = \mathbb{A} \cap M_\alpha$.
  (So $\dot{\mathbb{Q}}_\alpha$ is the $\mathbb{P}_\alpha$-name for amoeba forcing in some small model containing $\dot{X}_\alpha$, where "small" means of size $< \kappa_{an}$ in $V^{\mathbb{P}_\alpha}$.)
- The generic null set N_α added by $\mathbb{Q}_\alpha$ will cover the union of all measure zero sets in $\dot{X}_\alpha$.

If we choose the sets $\dot{X}_\alpha$ appropriately (using a bookkeeping argument), we can ensure that in $V^{\mathbb{P}_{\kappa_{ct}}}$ every union of $< \kappa_{an}$ null sets will be a null set; this shows that $\operatorname{add}(\mathcal{N}) \geq \kappa_{an}$.

The union of all null sets coded in the intermediate model $V^{\mathbb{P}_{\kappa_{an}}}$ (equivalently, the union $\bigcup_{\alpha<\kappa_{an}} N_\alpha$, where we view the N_α as given by Borel codes that are to be interpreted in the final model $V^{\mathbb{P}_{\kappa_{ct}}}$) will be non-null in the final model[1], witnessing $\mathrm{add}(\mathcal{N}) \leq \kappa_{an}$.

This method of using small subforcings (not necessarily complete) of classical forcing notions is well known, see for example [JS90] and [Bre91].

2.2. **Increasing** $\mathrm{cov}(\mathcal{N})$**,** $\mathfrak{b}$**,** $\mathrm{non}(\mathcal{M})$**.** In a similar way we could construct a model where $\mathrm{cov}(\mathcal{N})$ is large. The natural choice for an iterand $\dot{\mathbb{Q}}_\alpha$ would be random forcing.

If we want to get $\mathrm{cov}(\mathcal{N}) = \kappa_{cn} < \kappa_{ct} = 2^{\aleph_0}$, we could use a finite support iteration of length κ_{ct} where each iterand $\dot{\mathbb{Q}}_\alpha$ is the random forcing $\mathbb{B}$ from a small transitive submodel of the intermediate model V^{P_α}. Standard bookkeeping will ensure that the resulting model satisfies $\mathrm{cov}(\mathcal{N}) \geq \kappa_{cn}$.

We can also ensure that the final model $V^{\mathbb{P}_{\kappa_{ct}}}$ will not contain any random reals over the intermediate model $V^{\mathbb{P}_{\kappa_{cn}}}$; thus we also have $\mathrm{cov}(\mathcal{N}) \leq \kappa_{cn}$.

Replacing random forcing with Hechler forcing $\mathbb{D}$, we can get a model where the cardinal $\mathfrak{b}$ has an intermediate value.

Finally, there is a canonical forcing that will increase $\mathrm{non}(\mathcal{M})$, the forcing $\mathbb{E}$ which adds an "eventually different real". Since the properties of this forcing notion will play a crucial role in our arguments, we give an explicit definition.

Definition 2.1. The elements of the forcing notion $\mathbb{E}$ are pairs $p = (s, \varphi) = (s^p, \varphi^p)$ where $s \in \omega^{<\omega}$ and there is some $w \in \omega$ such that φ is a function $\varphi : \omega \to [\omega]^{\leq w}$ satisfying $s(i) \notin \varphi(i)$ for all $i \in \mathrm{dom}(s)$. The minimal such w will be called the *width* of φ, written $w^p = \mathrm{width}(\varphi)$.

A function $f : \omega \to \omega$ is *compatible* with a condition (s, φ) if s is an initial segment of f, and $f(i) \notin \varphi(i)$ holds for all i.

Our intention is that there will be a "generic" function g, such that each condition p forces that g is compatible with p. Motivated by this intention, we define $(s', \varphi') \leq (s, \varphi)$ by

- $s \subseteq s'$.
- $\forall i \in \omega : \varphi(i) \subseteq \varphi'(i)$.

Letting $\dot{g}$ be the name for $\bigcup\{s : (s, \varphi) \in \dot{G}\}$, the following properties are easy to check:

Remark 2.2.
1. (s, φ) indeed forces that $\dot{g}$ is compatible with (s, φ).
2. If we change the definition by requiring φ to be defined on $\omega \setminus \mathrm{dom}(s)$ only (and adding the condition $s'(i) \notin \varphi(i)$ in the definition of $\leq_{\mathbb{E}}$), we get an equivalent forcing notion which is moreover separative.
3. Our forcing $\mathbb{E}$ is an inessential variant of usual "eventually different" forcing notion in [Mil81].

2.3. **Putting things together.** Assume again GCH, and let $\aleph_1 \leq \kappa_{an} \leq \kappa_{cn} \leq \kappa_b \leq \kappa_{nm} \leq \kappa_{ct}$. We want to construct a ccc finite support iteration $\mathbb{P}$ such that $\mathbb{P}$ forces

$$\mathrm{add}(\mathcal{N}) = \kappa_{an}, \quad \mathrm{cov}(\mathcal{N}) = \kappa_{cn}, \quad \mathfrak{b} = \kappa_b, \quad \mathrm{non}(\mathcal{M}) = \kappa_{nm}, \quad \mathrm{cov}(\mathcal{M}) = 2^{\aleph_0} = \kappa_{ct}.$$

[1]Another way to say this is that the reals in $\omega^\omega \cap V^{\mathbb{P}_{\kappa_{an}}}$ are not localized by a single slalom from $S(\omega, \mathcal{H})$, see Example 3.4(4)

A naive approach would use an iteration of length κ_{ct} in which all iterands are "small" versions of Amoeba forcing, random forcing, Hechler forcing and eventually different forcing. Here,

- "small Amoeba" would mean: Amoeba forcing from a transitive model of size $< \kappa_{an}$,
- "small random" would mean: random forcing from a transitive model smaller than κ_{cn},
- "small Hechler" would mean: Hechler forcing from a transitive model smaller than κ_b,
- "small eventually different" would mean: eventually different forcing from a transitive model smaller than κ_{nm}.

If we use suitable bookkeeping, such an iteration will ensure that all the cardinals considered are *at least* their desired value. For example, every small family $\mathcal{F}$ of null sets (i.e., of Borel codes of null sets) will appear in an intermediate model, and the bookkeeping strategy will ensure that $\mathcal{F}$ was considered at some stage α. The amoeba null set added in stage $\alpha + 1$ will cover all null sets coded in $\mathcal{F}$. Similar arguments work for the other cardinal characteristics. Moreover, we could explicitly add Cohen reals cofinally, or use the fact that any finite support iteration adds Cohen reals in every limit step, to conclude that $\mathrm{cov}(\mathcal{M}) \geq \kappa_{ct}$.

That is, the final model will satisfy

$$\mathrm{add}(\mathcal{N}) \geq \kappa_{an}, \quad \mathrm{cov}(\mathcal{N}) \geq \kappa_{cn}, \quad \mathfrak{b} \geq \kappa_b, \quad \mathrm{non}(\mathcal{M}) \geq \kappa_{nm}, \quad 2^{\aleph_0} \geq \mathrm{cov}(\mathcal{M}) \geq \kappa_{ct}.$$

Using well-known iteration theorems (see [JS90], [Bre91], [BJ95, Section 6], or the summary of [Mej13, Section 2] reviewed in Section 3) we can conclude that

- the union of the family of null sets added in the first κ_{an} steps still is not a null set in the final model,
- there is no random real over the model $V^{P_{\kappa_{cn}}}$,
- the reals from the model $V^{P_{\kappa_{nm}}}$ are still nonmeager,
- the iteration does not add more than κ_{ct} reals.

So we also get

$$\mathrm{add}(\mathcal{N}) \leq \kappa_{an}, \quad \mathrm{cov}(\mathcal{N}) \leq \kappa_{cn}, \qquad\qquad \mathrm{non}(\mathcal{M}) \leq \kappa_{nm}, \quad 2^{\aleph_0} \leq \kappa_{ct}.$$

However, it is not immediately obvious that the reals from the model $V^{P_{\kappa_b}}$ stay an unbounded family, or more explicitly: that the eventually different forcing does not add an upper bound to this family. Indeed, it is consistent that a small subforcing of $\mathbb{E}$ (even one of the form $\mathbb{E} \cap M$ for some transitive model M) adds a dominating real, see [Paw92].

The full forcing $\mathbb{E}$, on the other hand, preserves unbounded families, see [Mil81].

A variant of this construction sketched above, where the full forcing $\mathbb{E}$ is used rather than small subsets of $\mathbb{E}$, would preserve the unboundedness of a κ_b-sized family and hence guarantee $\mathfrak{b} = \kappa_b$, at the cost of raising the value of $\mathrm{non}(\mathcal{M})$ to κ_{ct}.

Another variant is described in [Mej13, Theorem 3]: An iteration of length $\kappa_{ct} \cdot \kappa_{nm}$ (ordinal product) in which the full $\mathbb{E}$ forcings are used will yield a model of

$$\mathrm{add}(\mathcal{N}) = \kappa_{an}, \quad \mathrm{cov}(\mathcal{N}) = \kappa_{cn}, \quad \mathfrak{b} = \kappa_b, \quad \mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{N}) = \kappa_{nm}, \quad 2^{\aleph_0} = \kappa_{ct}$$

In this paper we want to additionally get $\mathrm{non}(\mathcal{M}) = \kappa_{nm} < \mathrm{cov}\,\mathcal{M} = \mathfrak{c} = \kappa_{ct}$, so it seems necessary to use small subforcings of $\mathbb{E}$.

The main point in the following section is to ensure that we will preserve an unbounded family of size κ_b in our iteration.

2.4. **Ultrafilters help us decide.** The actual construction that we will use will be given in section 6. It will be an iteration of length $\kappa_{ct} \cdot \kappa_{nm}$ (ordinal product), where in each coordinate a "small" forcing is added, as described above: an amoeba forcing of size $< \kappa_{an}$, etc.

For notational convenience we will start in a ground model where we already have an unbounded family $F = \{f_i : i < \kappa_b\}$. Moreover we will assume that every subfamily of size κ_b is again unbounded.

To simplify the presentation in this section, we will consider an iteration adding small $\mathbb{E}$ reals only. We will sketch how to construct such an iteration that does not destroy the unboundedness of F. Adding other "small" forcings to the iteration will not be a problem, as all these forcings will be smaller than κ_b; only the small $\mathbb{E}$ forcing notions may be of size $\geq \kappa_b$. A detailed proof is given in Main Lemma 4.6.

Now assume that our iteration $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta)$ has finite support limit $\mathbb{P}_\delta$, and that there is a $\mathbb{P}_\delta$-name $\dot{g}$ of a function which bounds all f_α. We can find a family of conditions $(p_i : i < \kappa_b)$ and natural number m_i such that

$$p_i \Vdash \forall n \geq m_i : f_i(n) \leq \dot{g}(n).$$

By thinning out our family we may assume that all m_i are equal, and for notational simplicity we will moreover assume they are all 0.

Moreover, we may assume that the p_i form a Δ-system satisfying a few extra uniformity conditions (i.e., they behave quite uniformly on the root).

We now choose a countable subset $i_0 < i_1 < \cdots$ of κ_b and some ℓ such that $f_{i_k}(\ell) \geq k$ for all k (this is possible, as otherwise our family $(f_i)_{i<\kappa_b}$ would be bounded). Again assume without loss of generality $\ell = 0$, and $i_k = k$ for all k.

We now have a *countable* Δ-system of conditions $\langle p_k \rangle_{k<\omega}$ in P_δ, where $p_k \Vdash \dot{g}(0) \geq f_k(0)$ for all k.

If we can now find a $\mathbb{P}_\delta$-name $\dot{D}_\delta$ of a non-principal ultrafilter and a condition q such that

$$q \Vdash_{\mathbb{P}_\delta} \{k : p_k \in \dot{G}_\delta\} \in \dot{D}_\delta,$$

then we have our desired contradiction, as already the empty condition forces that the set $\{k : p_k \in \dot{G}_\delta\}$ is finite, and in fact $f_k(0)$ is bounded by the number $\dot{g}(0)$ for any k in this set.

To get this ultrafilter $\dot{D}_\delta$ at the end of the proof, we need some preparations when we set up the iteration. The name $\dot{D}_\delta$ will of course depend on the countable sequence $\langle p_k \rangle_{k<\omega}$, but not very much; we will partition the set of all such sequences into a small family $(\Lambda_\epsilon : \epsilon < \kappa_{nm})$ of sets, and for each element Λ_ϵ of this small family we will define a name for an ultrafilter that will work for all countable Δ-systems (coded) in Λ_ϵ.

2.5. **Ultrafilter limits in $\mathbb{E}$.**

Definition 2.3. Let D be an ultrafilter on ω.

For each sequence $\bar{A} = \langle A_k \rangle_{k<\omega}$ of subsets of ω we define $\lim_D \bar{A} \subseteq \omega$ by taking the pointwise limit of the characteristic functions, or in other words:

$$n \in \lim_D \bar{A} \;\Leftrightarrow\; \{k : n \in A_k\} \in D.$$

If $\bar\varphi = \langle \varphi_k \rangle_{k<\omega}$ is a sequence of slaloms (i.e., each φ_k is a function from ω to $[\omega]^{<\omega}$), then we define $\psi := \lim_D \bar\varphi$ as the function with domain ω satisfying

$$\psi(n) = \lim_D \langle \varphi_k(n) \rangle_{k<\omega}$$

In general the ultrafilter limit of a sequence of slaloms may contain infinite sets. However, the following fact gives a sufficient condition for bounding the size of the sets in the ultrafilter limit.

Fact 2.4. *If $\bar A = \langle A_k \rangle_{k<\omega}$ is a sequence of subsets of ω, $b \in \omega$, and all A_k satisfy $|A_k| \le b$, then also $\lim_D \bar A$ will have cardinality at most b.*

Similarly, if $\bar\varphi = \langle \varphi_k \rangle_{k<\omega}$ is a sequence of slaloms with the property that there is a number b with $|\varphi_k(n)| \le b$ for all k, n, then also $\lim_D \bar\varphi$ will be a slalom consisting of sets of size $\le b$.

Definition 2.5. Let $s \in \omega^{<\omega}$, $w < \omega$, D a non-principal ultrafilter on ω and $\bar p = \langle p_k \rangle_{k<\omega}$ a sequence of conditions in $\mathbb{E}$ where $p_k = (s, \varphi_k)$ for some slalom φ_k of width $\le w$. $\lim_D \bar p$, *the D-limit of $\bar p$ in* $\mathbb{E}$, is defined as the condition $(s, \lim_D \langle \varphi_k \rangle_{k<\omega})$.

To explain the connection between a sequence $\bar p = \langle p_k \rangle_{k<\omega}$ and its ultrafilter limit, we point out the following fact. A stronger version will be proved in Claim 5.3.

Fact 2.6. *Let M be a small transitive model of a large enough fragment of ZFC. Let D be an ultrafilter with $D \cap M \in M$, let $\mathbb{Q} = \mathbb{E} \cap M$, $s \in \omega^{<\omega}$ and let $m^* < \omega$.*

Let $\bar\varphi = \langle \varphi_k \rangle_{k<\omega}$ be a sequence of slaloms of width bounded by m^* and assume $\bar\varphi \in M$ and $(s, \varphi_k) \in \mathbb{E}$ for all $k < \omega$.

Then the D-limit q of the sequence $\bar p = \langle p_k \rangle_{k<\omega} = \langle (s, \varphi_k) \rangle_{k<\omega}$ satisfies

- *$q \in \mathbb{E} \cap M$.*
- *q forces in $\mathbb{Q}$ that the set $\{k < \omega : p_k \in \dot G\}$ is infinite.*

Proof. It is clear that $q \in \mathbb{E}$. Since M contains both the sequence $\bar p$ and the set $D \cap M$, we can compute $\lim_D \bar p$ in M, hence $q \in M$.

Now assume that some $q' \le q$ forces that $\{k < \omega : p_k \in \dot G\}$ is bounded by some k_*, so q' is incompatible with all p_k, $k > k_*$.

For each $i \in \mathrm{dom}(s^{q'})$ we have $s^{q'}(i) \notin \varphi^q(i)$, so the set $B_i := \{k < \omega : s^{q'}(i) \notin \varphi_k(i)\}$ is in D. Let $k \in \bigcap_{i \in \mathrm{dom}(s^{q'})} B_i$ be larger than k_*. Then q' and p_k are compatible. $\square$

3. Background on preservation properties

For reader's convenience, we recall the preservation properties summarized in [Mej13, Sect. 2] which will be applied in the proof of the Main Theorem 6.1. These preservation properties were developed for FS (finite support) iterations of ccc posets by Judah and Shelah [JS90], with improvements by Brendle [Bre91]. These are summarized and generalized in [Gol93] and in [BJ95, Sect. 6.4 and 6.5].

Context 3.1. Fix an increasing sequence $\langle \sqsubset_n \rangle_{n<\omega}$ of 2-place closed relations (in the topological sense) in ω^ω such that, for any $n < \omega$ and $g \in \omega^\omega$, $(\sqsubset_n)^g = \{f \in \omega^\omega : f \sqsubset_n g\}$ is (closed) nwd (nowhere dense).

Put $\sqsubset = \bigcup_{n<\omega} \sqsubset_n$. Therefore, for every $g \in \omega^\omega$, $(\sqsubset)^g$ is an F_σ meager set.

For $f, g \in \omega^\omega$, say that g $\sqsubset$-*dominates* f if $f \sqsubset g$. A family $F \subseteq \omega^\omega$ is a $\sqsubset$-*unbounded* if no function in ω^ω $\sqsubset$-dominates all the members of F. Associate with this notion the cardinal $\mathfrak{b}_\sqsubset$, which is the least size of a $\sqsubset$-unbounded family.

Dually, say that $C \subseteq \omega^\omega$ is a $\sqsubset$-*dominating family* if any real in ω^ω is $\sqsubset$-dominated by some member of C. The cardinal $\mathfrak{d}_\sqsubset$ is the least size of a $\sqsubset$-dominating family.

Given a set Y, say that a real $f \in \omega^\omega$ is $\sqsubset$-*unbounded over* Y if $f \not\sqsubset g$ for every $g \in Y \cap \omega^\omega$.

It is clear that $\mathfrak{b}_\sqsubset \leq \mathrm{non}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{d}_\sqsubset$.

Context 3.1 is defined for ω^ω for simplicity, but in general, the same notions apply by changing the space for the domain or the codomain of $\sqsubset$ to another uncountable Polish space of the form of a product of at most countable discrete spaces (or even the real line $\mathbb{R}$ or the unit interval $[0,1]$). Moreover, the codomain can even be restricted to a Borel subset of such a space (see Example 3.4 (4)).

From now on, fix θ an uncountable regular cardinal.

Definition 3.2 (Judah and Shelah [JS90], [BJ95, Def. 6.4.4])**.** A forcing notion $\mathbb{P}$ is θ-$\sqsubset$-*good* if the following property holds[2]: For any $\mathbb{P}$-name $\dot{h}$ for a real in ω^ω, there exists a nonempty $Y \subseteq \omega^\omega$ (in the ground model) of size $< \theta$ such that, for any $f \in \omega^\omega$ $\sqsubset$-unbounded over Y, $\Vdash f \not\sqsubset \dot{h}$.

Say that $\mathbb{P}$ is $\sqsubset$-*good* if it is $\aleph_1$-$\sqsubset$-good.

This is a standard property associated to preserve $\mathfrak{b}_\sqsubset$ small and $\mathfrak{d}_\sqsubset$ large through forcing extensions that have the property. A family $F \subseteq \omega^\omega$ is θ-$\sqsubset$-*unbounded* if, for any $X \subseteq \omega^\omega$ of size $< \theta$, there exists an $f \in F$ which is $\sqsubset$-unbounded over X. It is clear that, if F is such a family, then $\mathfrak{b}_\sqsubset \leq |F|$ and $\theta \leq \mathfrak{d}_\sqsubset$. On the other hand, θ-$\sqsubset$-good posets preserve, in any generic extension, θ-$\sqsubset$-unbounded families of the ground model and, if $\lambda \geq \theta$ is a cardinal and $\mathfrak{d}_\sqsubset \geq \lambda$ in the ground model, then this inequality is also preserved in any generic extension (see, e.g., [BJ95, Lemma 6.4.8]). It is also known (from [JS90], see also [Bre91, Lemma 8]) that the property of Definition 3.2 is preserved under FS iterations of θ-cc posets. Also, for posets $\mathbb{P} \lessdot \mathbb{Q}$, if $\mathbb{Q}$ is θ-$\sqsubset$-good, then so is $\mathbb{P}$.

Lemma 3.3 ([Mej13, Lemma 4])**.** *Any poset of size $< \theta$ is θ-$\sqsubset$-good. In particular, Cohen forcing is $\sqsubset$-good.*

Example 3.4. (1) *Preserving non-meager sets:* For $f, g \in \omega^\omega$ and $n < \omega$, define $f \neq\!\!\!=_n g$ iff $\forall_{k \geq n}(f(k) \neq g(k))$, so $f \neq\!\!\!= g$ iff f and g are eventually different, that is, $\forall^\infty_{k<\omega}(f(k) \neq g(k))$. Recall form [BJ95, Thm. 2.4.1 and 2.4.7] that $\mathfrak{b}_{\neq\!\!\!=} = \mathrm{non}(\mathcal{M})$ and $\mathfrak{d}_{\neq\!\!\!=} = \mathrm{cov}(\mathcal{M})$.

(2) *Preserving unbounded families:* For $f, g \in \omega^\omega$, define $f \leq^*_n g$ iff $\forall_{k \geq n}(f(k) \leq g(k))$, so $f \leq^* g$ iff $\forall^\infty_{k\in\omega}(f(k) \leq g(k))$. Clearly, $\mathfrak{b} = \mathfrak{b}_{\leq^*}$ and $\mathfrak{d} = \mathfrak{d}_{\leq^*}$. Miller [Mil81] proved that $\mathbb{E}$ is $\leq^*$-good. Random forcing $\mathbb{B}$ is also $\leq^*$-good because it is ω^ω-bounding. But, as discussed in Section 2, subforcings of both may add dominating reals.

(3) *Preserving null-covering families:* Let $\langle I_k \rangle_{k<\omega}$ be the interval partition of ω such that $|I_k| = 2^{k+1}$ for all $k < \omega$. For $n < \omega$ and $f, g \in 2^\omega$ define $f \pitchfork_n g \Leftrightarrow \forall_{k \geq n}(f{\restriction}I_k \neq g{\restriction}I_k)$, so $f \pitchfork g \Leftrightarrow \forall^\infty_{k<\omega}(f{\restriction}I_k \neq g{\restriction}I_k)$. Clearly, $(\pitchfork)^g$ is a co-null F_σ meager set. This relation is related to the covering-uniformity of measure because $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{b}_\pitchfork$ and $\mathfrak{d}_\pitchfork \leq \mathrm{non}(\mathcal{N})$ (see [Mej13, Lemma 7]).

It is known from [Bre91, Lemma 1*] that, given an infinite cardinal $\nu < \theta$, every ν-centered forcing notion is θ-$\pitchfork$-good.

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[2] [BJ95, Def. 6.4.4] has a different formulation, which is equivalent to our formulation for θ-cc posets (recall that θ is uncountable regular). See [Mej13, Lemma 2] for details.

(4) *Preserving "union of non-null sets is non-null":* Fix $\mathcal{H} := \{id^{k+1} : k < \omega\}$ (where $id^{k+1}(i) = i^{k+1}$) and let $S(\omega, \mathcal{H}) = \{\psi : \omega \to [\omega]^{<\omega} : \exists_{h\in\mathcal{H}}\forall_{i<\omega}(|\psi(i)| \leq h(i))\}$. For $n < \omega$, $x \in \omega^\omega$ and a slalom $\psi \in S(\omega, \mathcal{H})$, put $x \in^*_n \psi$ iff $\forall_{k\geq n}(x(k) \in \psi(k))$, so $x \in^* \psi$ iff $\forall^\infty_{k<\omega}(x(k) \in \psi(k))$. By Bartoszyński's characterization [BJ95, Thm. 2.3.9] applied to id and to a function g that dominates all the functions in $\mathcal{H}$, $\mathrm{add}(\mathcal{N}) = \mathfrak{b}_{\in^*}$ and $\mathrm{cf}(\mathcal{N}) = \mathfrak{d}_{\in^*}$.

Judah and Shelah [JS90] proved that, given an infinite cardinal $\nu < \theta$, every ν-centered forcing notion is θ-$\in^*$-good. Moreover, as a consequence of results of Kamburelis [Kam89], any subalgebra[3] of $\mathbb{B}$ is $\in^*$-good.

For a relation $\sqsubset$ as in Context 3.1, the following practical results present facts about adding Cohen reals that form $\sqsubset$-unbounded families.

Lemma 3.5. *Let $\langle\mathbb{P}_\alpha\rangle_{\alpha\leq\theta}$ be a $\lessdot$-increasing sequence of ccc posets where $\mathbb{P}_\theta = \mathrm{limdir}_{\alpha<\theta}\mathbb{P}_\alpha$. Assume that $\mathbb{P}_{\alpha+1}$ adds a Cohen real $\dot{c}_\alpha$ over $V^{\mathbb{P}_\alpha}$ for all $\alpha < \theta$. Then, $\mathbb{P}_\theta$ forces that $\{\dot{c}_\alpha : \alpha < \theta\}$ is a θ-$\sqsubset$-unbounded family.*

Corollary 3.6. *Let $\delta \geq \theta$ be an ordinal and $\mathbb{P}_\delta = \langle\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha\rangle_{\alpha<\delta}$ a FS iteration of non-trivial ccc θ-$\sqsubset$-good posets. Then, $\mathbb{P}_\delta$ forces*

(a) $\mathfrak{b}_\sqsubset \leq \theta$ and

(b) $\mathfrak{d}_\sqsubset \geq |\delta|$.

Proof. By Lemma 3.5 (recall that FS iterations of non-trivial ccc posets add Cohen reals at limit stages), for any $\nu \in [\theta, |\delta|]$ regular, $\mathbb{P}_\nu$ adds a ν-$\sqsubset$-unbounded family of size ν (of Cohen reals), which is preserved to be ν-$\sqsubset$-unbounded in $V^{\mathbb{P}_\delta}$ because the iterands are θ-$\sqsubset$-good and ccc. Therefore, $\mathbb{P}_\delta$ forces $\mathfrak{b}_\sqsubset \leq \nu \leq \mathfrak{d}_\sqsubset$ for any regular $\nu \in [\theta, |\delta|]$, so $\mathfrak{b}_\sqsubset \leq \theta$ and $|\delta| \leq \mathfrak{d}_\sqsubset$. □

4. Iteration candidates

Fix, in this section, an uncountable regular cardinal κ_b (which represents the size of an unbounded family we want to preserve).

For our main result (Theorem 6.1), as described in the introduction, we may use a FS iteration alternating between small ccc posets and subposets of $\mathbb{E}$ and find an iteration where we can preserve an unbounded family of a desired size (κ_b). We describe, in general, such iterations as follows.

Definition 4.1. An *iteration candidate* $\mathbf{q}$ consists of

(i) an ordinal $\delta_\mathbf{q}$ (the length of the iteration) partitioned into two sets $S_\mathbf{q}$ and $C_\mathbf{q}$ (the first set represent the coordinates where a subforcing of $\mathbb{E}$ is used, while the second set corresponds to the coordinates where small ccc posets are used),

(ii) ordinals $\langle Q_{\mathbf{q},\alpha}\rangle_{\alpha\in C_\mathbf{q}}$ less that κ_b (the domains of the small ccc posets),

(iii) a FS iteration $\langle\mathbb{P}_{\mathbf{q},\alpha}, \dot{\mathbb{Q}}_{\mathbf{q},\alpha}\rangle_{\alpha<\delta_\mathbf{q}}$ and a sequence $\langle\mathbb{P}'_{\mathbf{q},\alpha}\rangle_{\alpha\in S_\mathbf{q}}$ such that

- for $\alpha \in S_\mathbf{q}$, $\mathbb{P}'_{\mathbf{q},\alpha} \lessdot \mathbb{P}_{\mathbf{q},\alpha}$ and $\dot{\mathbb{Q}}_{\mathbf{q},\alpha}$ is a $\mathbb{P}'_{\mathbf{q},\alpha}$-name for $\mathbb{E}^{V^{\mathbb{P}'_\alpha}}$ and
- for $\alpha \in C_\mathbf{q}$, $\dot{\mathbb{Q}}_{\mathbf{q},\alpha}$ is a $\mathbb{P}_{\mathbf{q},\alpha}$-name of a ccc poset whose domain is $Q_{\mathbf{q},\alpha}$.

The subindex $\mathbf{q}$ may be omitted when it is obvious from the context.

For each $\alpha \leq \delta$, consider the set $\mathbb{P}^*_{\mathbf{q},\alpha} = \mathbb{P}^*_\alpha$ of conditions $p \in \mathbb{P}_\alpha$ that satisfy:

- if $\xi \in \mathrm{supp}(p) \cap C$ then $p(\xi)$ is an ordinal in Q_α (not just a name)

[3]Here, $\mathbb{B}$ is seen as the complete Boolean algebra of Borel sets (in 2^ω) modulo the null ideal.

- if $\xi \in \operatorname{supp}(p) \cap S$, then $p(\xi)$ is of the form $(s, \dot{\varphi})$ where $s \in \omega^{<\omega}$ (not just a name), $\dot{\varphi}$ is a $\mathbb{P}'_\xi$-name of a slalom (not just a $\mathbb{P}_\xi$-name of a slalom in $V^{\mathbb{P}'_\xi}$) and $\operatorname{width}(\dot{\varphi})$ is decided, that is, there is an $n < \omega$ such that $\Vdash_{\mathbb{P}'_\xi} n = \operatorname{width}(\dot{\varphi})$

It is easy to prove (by induction on α) that $\mathbb{P}^*_\alpha$ is dense in $\mathbb{P}_\alpha$.

For $\alpha \leq \delta_{\mathbf{q}}$, $\mathbf{q}{\restriction}\alpha$ denotes the iteration $\mathbf{q}$ restricted up to α, so $\delta_{\mathbf{q}\restriction\alpha} = \alpha$. Clearly, $\mathbf{q}{\restriction}\alpha$ is an iteration candidate.

The beginning of the proof of Main Lemma 4.6 shows a typical argument with a Δ-system to prove that an iteration candidate preserves an unbounded family of size κ_b (as sketched in Subsection 2.4). Therefore, in order to extend Miller's compactness argument [Mil81] to FS iterations, we start by coding the relevant elements of countable Δ-systems of iteration candidates by *stem sequences*, as it is described below.

Definition 4.2. Let α^* be an ordinal. A *stem sequence* $\mathbf{x} \in \mathrm{SS}_{\alpha^*}$ (of a countable Δ-system) consists of

(i) a countable set of ordinals $w_{\mathbf{x}} \subseteq \alpha^* \cup \kappa_b$ (where the relevant information of the coded Δ-system lives),

(ii) a natural number $l^*_{\mathbf{x}}$ (the size of the support of the conditions in the coded Δ-system) partitioned into two sets $v_{\mathbf{x},S}$ and $v_{\mathbf{x},C}$ (the first set indicate the position of coordinates where a subforcing of $\mathbb{E}$ is used, while the second set corresponds to the positions where small ccc posets are used),

(iii) a subset $v_{\mathbf{x}}$ of $l^*_{\mathbf{x}}$ (the set of positions of the coordinates of the root of the Δ-system),

(iv) a subset $\{\alpha_{\mathbf{x},k,l} : k < \omega,\ l < l^*_{\mathbf{x}}\}$ of $w_{\mathbf{x}} \cap \alpha^*$ satisfying: $\langle\{\alpha_{\mathbf{x},k,l} : l < l^*_{\mathbf{x}}\}\rangle_{k<\omega}$ is a Δ-system with root $\Delta_{\mathbf{x}} = \{\alpha^*_{\mathbf{x},l} : l \in v_{\mathbf{x}}\}$ where, for $l \in v_{\mathbf{x}}$ and $k < \omega$, $\alpha_{\mathbf{x},k,l} = \alpha^*_{\mathbf{x},l}$; moreover, $\{\alpha_{\mathbf{x},k,l} : l < l^*\}$ is an increasing enumeration for each $k < \omega$ and $\langle \alpha_{\mathbf{x},k,l} : k < \omega\rangle$ is increasing[4] for each $l \in l^*_{\mathbf{x}} \smallsetminus v_{\mathbf{x}}$,

(v) ordinals $\langle \gamma_{\mathbf{x},k,l} : k < \omega,\ l \in v_{\mathbf{x},C}\rangle$ (the sequence of ordinals used at the l-th position of the k-th condition of the Δ-system) and $\langle \gamma^*_{\mathbf{x},l} : l \in v_{\mathbf{x}} \cap v_{\mathbf{x},C}\rangle$ in $\kappa_b \cap w_{\mathbf{x}}$ such that $\gamma_{\mathbf{x},k,l} = \gamma^*_{\mathbf{x},l}$ for all $l \in v_{\mathbf{x}} \cap v_{\mathbf{x},C}$ and $k < \omega$ (that is, the ordinals used at the positions of the root are the same for all k),

(vi) a sequence $\langle s^*_{\mathbf{x},l} : l \in v_{\mathbf{x},S}\rangle$ of objects from $\omega^{<\omega}$ (the sequence of stems used at the l-th position of a condition, which is the same for all the conditions in the Δ-system) and

(vii) a sequence $\bar{n}^*_{\mathbf{x}} = \langle n^*_{\mathbf{x},l} : l \in v_{\mathbf{x},S}\rangle$ of natural numbers (the sequences of widths of slaloms at the l-th position of a condition in the Δ-system).

When there is no place to confusion, we may omit the subindex $\mathbf{x}$ for the objects of a stem sequence.

The role of $w_{\mathbf{x}}$ is only important in Remark 6.3 where a more difficult argument (that may work for other problems) for a result weaker than Main Theorem 6.1 is summarized.

If $\mathbf{q}$ is an iteration candidate of length δ, then every sufficiently uniform countable Δ-system $\bar{p} = \langle p_k\rangle_{k<\omega}$ from $\mathbb{P}^*_\delta$ will define a stem sequence. But not every stem sequence is realized by some sequence of conditions from $\mathbb{P}^*_\delta$. In the next definition

[4]This is only needed for the proof of Claim 5.5.

we give a sufficient condition for a stem sequence to be realized by an iteration, and we explain how this stem sequence gives partial information about a sequence of conditions.

Definition 4.3. A stem sequence $\mathbf{x} \in \mathrm{SS}_{\alpha^*}$ (as in Definition 4.2) is *legal* for an iteration candidate $\mathbf{q}$ (as in Definition 4.1) if the following hold for each $k < \omega$ and $l < l^*$ such that $\alpha_{k,l} < \delta = \delta_{\mathbf{q}}$:

(i) $\alpha_{k,l} \in C$ iff $l \in v_C$.
(ii) If $l \in v_C$ then $\gamma_{k,l} < Q_{\alpha_{k,l}}$.

In this case, define $P^\infty_{\mathbf{q},\mathbf{x}}$ the set of sequences $\langle p_k \rangle_{k<\omega}$ of conditions in $\mathbb{P}^*_\delta$ such that

- $\mathrm{supp}(p_k) = \{\alpha_{k,l} : l < l^*\} \cap \delta$,
- if $\xi = \alpha_{k,l} \in \mathrm{supp}(p_k) \cap C$ then $p_k(\xi) = \gamma_{k,l}$ and
- if $\xi = \alpha_{k,l} \in \mathrm{supp}(p_k) \cap S$ then $p_k(\xi) = (s^*_l, \dot{\varphi}_{k,l})$ where $\dot{\varphi_{k,l}}$ is a $\mathbb{P}'_\xi$-name of a slalom of width n^*_l.

Note that $\langle \mathrm{supp}(p_k) \rangle_{k<\omega}$ forms a Δ-system.

Here, $P^\infty_{\mathbf{q},\mathbf{x}}$ is the set of countable Δ-systems that *match* the stem sequence $\mathbf{x}$.

If $\mathbf{x}$ is legal for $\mathbf{q}$ and $\alpha \leq \delta$, then $\mathbf{x}$ is legal for $\mathbf{q}{\restriction}\alpha$ and, for any $\langle p_k \rangle_{k<\omega} \in P^\infty_{\mathbf{q},\mathbf{x}}$, we also have $\langle p_k{\restriction}\alpha \rangle_{k<\omega} \in P^\infty_{\mathbf{q}{\restriction}\alpha,\mathbf{x}}$.

Note that a stem sequence has full information about the conditions it represents only on the set C where we use the small forcings. On the set S, the stem sequence only knows the stems of conditions in $\mathbb{E}$ but not the slaloms.

The main idea of preserving unbounded families in $\mathbb{E}$ is that, given a sequence of conditions $\langle p_k \rangle_{k<\omega}$ in $\mathbb{E}$ that agree in the stems and in the width of the slaloms, it is possible to construct a limit q of the sequence such that q forces that infinitely many p_k belong to the generic set (see Fact 2.6). This limit can be found by an ultrafilter limit on the slaloms from an ultrafilter D in the ground model. Moreover, there is an ultrafilter in the extension that contains D as well as all sets of the form $A_{\bar{p}} := \{k < \omega : p_k \in G\}$ (G is the $\mathbb{E}$-generic filter) for such a sequence $\bar{p} = \langle p_k \rangle_{k<\omega}$ with limit q in G. To extend this argument to an iteration candidate, we define a kind of ultrafilter limit of a countable Δ-system that matches a given stem sequence.

Definition 4.4. Let $\mathbf{q}$ be an iteration candidate and $\mathbf{x} \in \mathrm{SS}_{\alpha^*}$ a legal stem sequence for $\mathbf{q}$. Say that $\mathbf{D} = \langle \dot{D}_\alpha \rangle_{\alpha \leq \delta}$ *solves* $\mathbf{x}$ *(with respect to* $\mathbf{q}$*)* if the following holds for each $\alpha \leq \delta$.

(1) $\dot{D}_\alpha$ is a $\mathbb{P}_\alpha$-name for a non-principal ultrafilter on ω.
(2) For $\alpha \in S$, $\Vdash_\alpha \dot{D}_\alpha \cap \mathcal{P}(\omega)^{V^{\mathbb{P}'_\alpha}} \in V^{\mathbb{P}'_\alpha}$.
(3) $\alpha < \beta \leq \delta$ implies $\Vdash_\beta \dot{D}_\alpha \subseteq \dot{D}_\beta$.
(4) If α contains $\Delta_{\mathbf{x}} \cap \delta$ and $\bar{p} = \langle p_k \rangle_{k<\omega} \in P^\infty_{\mathbf{q},\mathbf{x}}$, then $q \Vdash_\alpha \{k < \omega : p_k{\restriction}\alpha \in \dot{G}_\alpha\} \in \dot{D}_\alpha$ where $q = \lim_{\mathbf{D}{\restriction}\alpha} \bar{p}$, the $\mathbf{D}{\restriction}\alpha$*-limit of* $\bar{p}$ (where $\mathbf{D}{\restriction}\alpha = \langle \dot{D}_\xi \rangle_{\xi<\alpha}$), is defined as
   (i) $\mathrm{supp}(q) = \Delta_{\mathbf{x}} \cap \delta$,
   (ii) if $\xi = \alpha^*_l \in \mathrm{supp}(q) \cap C$ then $q(\xi) = \gamma^*_l$ and
   (iii) if $\xi = \alpha^*_l \in \mathrm{supp}(q) \cap S$ then $q(\xi) = (s^*_l, \dot{\psi}_l)$ where $\dot{\psi}_l$ is a $\mathbb{P}_\xi$-name of the $\dot{D}_\xi$-limit of $\langle \dot{\varphi}_{k,l} \rangle_{k<\omega}$ (here, $p_k(\xi) = (s^*_l, \dot{\varphi}_{k,l})$ for each $k < \omega$).

   As each $\dot{\varphi}_{k,l}$ is a $\mathbb{P}'_\xi$-name (because $p_k \in \mathbb{P}^*_\delta$) of width n^*_l, by (2), $\mathbb{P}_\xi$ forces $\dot{\psi}_l \in V^{\mathbb{P}'_\xi}$ of width $\leq n^*_l$ and $q(\xi) \in \mathbb{E} \cap V^{\mathbb{P}'_\xi}$. Therefore, q is a condition in $\mathbb{P}_\alpha$.

Moreover, q does not depend on α (as long as α contains $\Delta_{\mathbf{x}} \cap \delta$), so we can just call q the $\mathbf{D}$-*limit of* $\bar{p}$.

Say that $\mathbf{q}$ is a *nice iteration candidate* if any $\mathbf{x} \in \mathrm{SS}_\delta$ (with $\delta = \delta_{\mathbf{q}}$) legal for $\mathbf{q}$ can be solved by some $\mathbf{D}$.

Remark 4.5. In (4)(iii) of Definition 4.4, if $\dot{\varphi}_{k,l}$ were just a $\mathbb{P}_\xi$-name of a slalom in $V^{\mathbb{P}'_\xi}$ for each $k < \omega$, we would not be able to guarantee that $\langle \dot{\varphi}_{k,l} : k < \omega \rangle$ is a sequence in $V^{\mathbb{P}'_\xi}$, so the ultrafilter limit ψ_l and $q(\xi)$ may not be in $V^{\mathbb{P}'_\xi}$.

On the other hand, in (4), $q \in \mathbb{P}_\alpha$ but it may not be a condition in $\mathbb{P}^*_\alpha$ because, in (iii), $\dot{\psi}_l$ may not be a $\mathbb{P}'_\xi$-name. However, for the nice iteration candidate constructed in Theorem 6.1, there is a $\mathbb{P}'_\alpha$-name of $\dot{D}_\alpha \cap V^{\mathbb{P}'_\alpha}$ for each $\alpha \in S$, which guarantees that, in (4), $q \in \mathbb{P}^*_\alpha$.

Main Lemma 4.6. *Let $B \subseteq \omega^\omega$ in the ground model that satisfies*

$$(*) \quad |B| \geq \kappa_b \text{ and } \forall_{g \in \omega^\omega}(|\{f \in B : f \leq^* g\}| < \kappa_b)$$

Then, any nice iteration candidate forces that B satisfies $(*)$ in the generic extension.

Proof. Let $\mathbf{q}$ be a nice iteration candidate as in Definitions 4.1 and 4.4. Towards a contradiction, let $p \in \mathbb{P}_\delta$ and $\dot{g}$ be a $\mathbb{P}_\delta$-name for a real such that p forces that $\{f \in B : f \leq^* \dot{g}\}$ has size at least κ_b. By recursion on $\eta < \kappa_b$, find $f_\eta \in B$, $m_\eta < \omega$ and $p_\eta \leq p$ in $\mathbb{P}^*_\delta$ such that $f_\eta \neq f_\xi$ for $\xi < \eta$ and $p_\eta \Vdash_\delta \forall_{m \geq m_\eta}(f_\eta(m) \leq \dot{g}(m))$. Let $u_\eta = \mathrm{supp}(p_\eta)$. By the Δ-system lemma, we can find $K \in [\kappa_b]^{\kappa_b}$ such that $\{u_\eta : \eta \in K\}$ forms a Δ-system. Moreover, we may assume:

(a) There is an m^* such that $m_\eta = m^*$ for all $\eta \in K$.
(b) There is an l^* such that $u_\eta = \{\alpha_{\eta,l} : l < l^*\}$ (increasing enumeration) for all $\eta \in K$.
(c) There is a $v \subseteq l^*$ such that the root of the Δ-system is $\{\alpha_{\eta,l} : l \in v\}$ for any $\eta \in K$.
(d) For each $l < l^*$ with $l \notin v$, $\langle \alpha_{\eta,l} \rangle_{\eta \in K}$ is increasing.
(e) There is a $v_S \subseteq l^*$ such that $\mathrm{supp}(p_\eta) \cap S = \{\alpha_{\eta,l} : l \in v_S\}$ for all $\eta \in K$.
(f) For each $l \in v \smallsetminus v_S$ there is an ordinal γ^*_l such that $p_\eta(\alpha_{\eta,l}) = \gamma^*_l$ for all $\eta \in K$. (Why? Recall that the forcing notions $\dot{\mathbb{Q}}_\alpha$ with $\alpha \in C$ live on sets Q_α of cardinality $< \kappa_b$.)
(g) For each $l \in v_S$ there is an $s^*_l \in \omega^{<\omega}$ and an $n^*_l < \omega$ such that $p(\alpha_{\eta,l})$ is of the form $(s^*_l, \dot{\varphi}_{\eta,l})$ for all $\eta \in K$, where $\dot{\varphi}_{\eta,l}$ is a $\mathbb{P}'_{\alpha_{\eta,l}}$-name for a slalom of width n^*_l.

In the ground model, we can find an increasing sequence $\langle \eta_k \rangle_{k<\omega}$ in K and an $m \geq m^*$ such that $\langle f_{\eta_k}(m) : k < \omega \rangle$ is increasing. This is because there is $m \geq m^*$ and infinitely many $a \in \omega$ such that $\{\eta \in K : f_\eta(m) = a\}$ has size κ_b (if this were not the case, then there is a $K' \in [K]^{\kappa_b}$ such that $\{f_\eta : \eta \in K'\}$ is bounded, which contradicts the hypothesis).

Now it is easy to find a legal stem sequence $\mathbf{x} \in \mathrm{SS}_\delta$ for $\mathbf{q}$ such that $\bar{p} := \langle p_{\eta_k} \rangle_{k<\omega} \in P^\infty_{\mathbf{q},\mathbf{x}}$, so there is some $\mathbf{D} = \langle \dot{D}_\alpha \rangle_{\alpha \leq \delta}$ solving $\mathbf{x}$ (as in Definition 4.4). Let $q = \lim_{\mathbf{D} \restriction \delta} \bar{p} \in \mathbb{P}_\delta$, so

$$q \Vdash_\delta \{k < \omega : p_{\eta_k} \in \dot{G}\} \in \dot{D}_\delta,$$

which implies that $q \Vdash \exists^\infty_{k<\omega}(f_{\eta_k}(m) \leq \dot{g}(m))$. This last fact contradicts that $\langle f_{\eta_k}(m) : k < \omega \rangle$ is increasing. $\square$

5. A method to construct nice iteration candidates

In a very general setting, we show how to construct nice iteration candidates. We then apply this method to build the iteration for our main result.

For a nice iteration candidate, any legal stem sequence has to be solved by some sequence of names of ultrafilters. But recall from Definition 4.4(2) that we want all witnesses D_α to be in $V^{\mathbb{P}'_\alpha}$, and in practice this will be a model of size $\leq \kappa_{nm}$ (the value we want to force for non$(\mathcal{M})$). So we want to have as few such sequences of names of ultrafilters as possible, i.e., each sequence should solve many legal stem sequences. For this purpose, we use the following classical result of Engelking and Karłowicz, which essentially says that a product of at most 2^χ discrete spaces of size χ has a dense set of size χ in an appropriate box topology (in our applications, χ is between κ_b and κ_{nm}).

Theorem 5.1 (Engelking and Karłowicz [EK65], see also [AY08])**.** *Assume $\chi^{<\theta} = \chi$, $\delta < (2^\chi)^+$ an ordinal and $\langle A_\alpha\rangle_{\alpha<\delta}$ a sequence of sets of size $\leq \chi$. Then there is a set $\{h_\epsilon : \epsilon < \chi\} \subseteq \prod_{\alpha<\delta} A_\alpha$ such that, for any $x \in \prod^{<\theta}_{\alpha<\delta} A_\alpha := \bigcup_{E\in[\delta]^{<\theta}} \prod_{\alpha\in E} A_\alpha$, there is an $\epsilon < \chi$ such that $x \subseteq h_\epsilon$. Moreover, $\prod^{<\theta}_{\alpha<\delta} A_\alpha$ can be partitioned into sets $\langle L^*_\epsilon\rangle_{\epsilon<\chi}$ such that*

(i) if $x \in L^*_\epsilon$ then $x \subseteq h_\epsilon$ and

(ii) for all $x, y \in L^*_\epsilon$, dom$x$ *and* dom$y$ *have the same order type and the order-preserving isomorphism $g :$* dom$x \to$ dom$y$ *is the identity on* dom$x \cap$ dom$y$.

When $2^\chi = \chi^+$, we additionally have

(ii') for all $x, y \in L^*_\epsilon$, dom$x$ *and* dom$y$ *have the same order type and* dom$x \cap$dom$y$ *is an initial segment of both* dom$x$ *and* dom$y$.

Fix κ_b as in section 4. Assume $\kappa_b \leq \chi = \chi^{\aleph_0}$, $\delta < (2^\chi)^+$ an ordinal and $\delta = S \cup C$ a disjoint union. For each $\alpha < \delta$ let $A_\alpha = \omega^{<\omega} \times \omega$ if $\alpha \in S$, otherwise, $A_\alpha = \kappa_b$. Let $\{h_\epsilon : \epsilon < \chi\}$ and $\langle L^*_\epsilon\rangle_{\epsilon<\chi}$ be as in Theorem 5.1 applied to $\theta = \aleph_1$. Therefore, we can partition SS_δ into the sets $\langle\Lambda_\epsilon\rangle_{\epsilon<\chi}$ such that $\mathbf{x} \in \Lambda_\epsilon$ iff $z_{\mathbf{x}} \in L^*_\epsilon$, where $\mathrm{dom} z_{\mathbf{x}} = \{\alpha_{\mathbf{x},k,l} : k < \omega,\ l < l^*_{\mathbf{x}}\}$ and, for $k < \omega$ and $l < l^*_{\mathbf{x}}$, if $l \in \upsilon_{\mathbf{x},S}$ then $z_{\mathbf{x}}(\alpha_{\mathbf{x},k,l}) = (s^*_{\mathbf{x},l}, n^*_{\mathbf{x},l})$, otherwise, $z_{\mathbf{x}}(\alpha_{\mathbf{x},k,l}) = \gamma_{\mathbf{x},k,l}$ when $l \in \upsilon_{\mathbf{x},C}$.

Here, h_ϵ is seen as a *guardrail* for the countable Δ-systems that matches a stem sequence in Λ_ϵ. All conditions following the same guardrail will be compatible with each other. This is because, for an iteration candidate of length δ where S corresponds to the coordinates where subforcings of $\mathbb{E}$ are used, if $\langle p_k\rangle_{k<\omega}$ is a Δ-system that matches $\mathbf{x} \in \Lambda_\epsilon$, the function h_ϵ describes the behavior of each p_k, that is, if $\zeta \in \operatorname{supp} p_k \smallsetminus S$, $p_k(\zeta) = h_\epsilon(\zeta)$ and, if $\zeta \in S \cap \operatorname{supp} p_k$, then $h_\epsilon(\zeta)$ tells the stem and the width of the slalom corresponding to $p_k(\zeta)$. All this information depends only on ϵ (and the coordinate ζ) and all the Δ-systems matching stem sequences in Λ_ϵ are described by the same information.

We show a way to construct, inductively, a nice iteration candidate $\mathbf{q}$ with $\delta_{\mathbf{q}} = \delta$, $S_{\mathbf{q}} = S$ and $C_{\mathbf{q}} = C$ by using the guardrails $\langle h_\epsilon : \epsilon < \chi\rangle$. Furthermore, we find $\langle \dot{D}^\epsilon_\alpha : \epsilon < \chi,\ \alpha \leq \delta\rangle$ such that, for each $\epsilon < \chi$, $\mathbf{D}^\epsilon_\delta := \langle \dot{D}^\epsilon_\alpha\rangle_{\alpha\leq\delta}$ solves all the legal stem sequences of Λ_ϵ.

Induction basis. When $\delta = 0$, choose an arbitrary non-principal ultrafilter D^ϵ_0 for each $\epsilon < \chi$.

Lemma 5.2 (Successor step)**.** *Assume $\delta=\alpha+1$. Let $\mathbf{q}{\restriction}\alpha$ be a nice iteration candidate of length α with $S_{\mathbf{q}\restriction\alpha}=S\cap\alpha$ and let $\langle\dot{D}^\epsilon_\xi:\epsilon<\chi,\ \xi\leq\alpha\rangle$ be such that, for each $\epsilon<\chi$, $\mathbf{D}^\epsilon_\alpha$ solves all $\mathbf{x}\in\mathrm{SS}_\alpha\cap\Lambda_\epsilon$ that are legal for $\mathbf{q}{\restriction}\alpha$. Let $\mathbf{q}$ be an iteration candidate of length δ that extends $\mathbf{q}{\restriction}\alpha$ such that the following conditions hold.*

(i) $\alpha\in S_\mathbf{q}$ iff $\alpha\in S$.

(ii) In the case $\alpha\in S$, $\mathbb{P}_\alpha$ forces $\dot{D}^\epsilon_\alpha\cap V^{\mathbb{P}'}_\alpha\in V^{\mathbb{P}'_\alpha}$ for all $\epsilon<\chi$.

<u>Then</u>, there are $\mathbb{P}_{\alpha+1}$-names $\langle\dot{D}^\epsilon_{\alpha+1}:\epsilon<\chi\rangle$ such that, for each $\epsilon<\chi$, $\mathbf{D}^\epsilon_{\alpha+1}=\mathbf{D}^\epsilon_\alpha{}^\frown\langle\dot{D}^\epsilon_{\alpha+1}\rangle$ solves all $\mathbf{x}\in\mathrm{SS}_{\alpha+1}\cap\Lambda_\epsilon$ that are legal for $\mathbf{q}$.

Proof. It is enough to prove the following.

Claim 5.3. *$\mathbb{P}_{\alpha+1}$ forces that, for any $\epsilon<\chi$, the family*

$$\dot{D}^\epsilon_\alpha\ \cup\ \left\{\dot{A}_{\bar{p}}:\bar{p}\in P^\infty_{\mathbf{q},\mathbf{x}},\ \mathbf{x}\in\Lambda_\epsilon\cap\mathrm{SS}_{\alpha+1}\text{ legal},\ \lim_{\mathbf{D}^\epsilon_\alpha}\bar{p}\in\dot{G}\right\}$$

(where $\dot{A}_{\bar{p}}:=\{k<\omega:p_k\in\dot{G}\}$ for any $\bar{p}=\langle p_k\rangle_{k<\omega}$) has the finite intersection property.

Proof. In the case $\alpha\in C$, it is enough to prove that, if $\mathbf{x}\in\Lambda_\epsilon\cap\mathrm{SS}_{\alpha+1}$ is legal for $\mathbf{q}$, $\langle p_k\rangle_{k<\omega}\in P^\infty_{\mathbf{q},\mathbf{x}}$ and q is its $\mathbf{D}^\epsilon_\alpha$-limit, then q forces (with respect to $\mathbb{P}_{\alpha+1}$) that $\{k<\omega\ :\ p_k\in\dot{G}\}\in\dot{D}^\epsilon_\alpha$. We may assume that $\alpha\in\operatorname{supp}q$ (if not, $\operatorname{supp}p_k\subseteq\alpha$ for all $k<\omega$ and the claim is straightforward), so $p_k(\alpha)=q(\alpha)=h_\epsilon(\alpha)$ for all $k<\omega$. On the other hand, $q{\restriction}\alpha$ forces that $\{k<\omega:p_k{\restriction}\alpha\in\dot{G}\}\in\dot{D}^\epsilon_\alpha$ (because $q{\restriction}\alpha$ is the $\mathbf{D}^\epsilon_\alpha{\restriction}\alpha$-limit of $\langle p_k{\restriction}\alpha\rangle_{k<\omega}$), so the conclusion is clear.

Now, assume that $\alpha\in S$. Let $i^*<\omega$, $\mathbf{x}^i\in\Lambda_\epsilon\cap\mathrm{SS}_{\alpha+1}$ legal for $\mathbf{q}$ for $i<i^*$, $\langle p_{i,k}\rangle_{k<\omega}\in P^\infty_{\mathbf{q},\mathbf{x}^i}$, $q_i=\lim_{\mathbf{D}^\epsilon_\alpha}p_{i,k}$, a $\mathbb{P}_\alpha$-name $\dot{a}$ of a set in $\dot{D}^\epsilon_\alpha$, a fixed $k^*<\omega$ and a condition $r\in\mathbb{P}_{\alpha+1}$ stronger than q_i for each $i<i^*$. We find an $r'\leq r$ in $\mathbb{P}_{\alpha+1}$ and a $k>k^*$ such that r' forces that $k\in\dot{a}$ and $p_{i,k}\in\dot{G}$ for all $i<i^*$. We may assume that $r{\restriction}\alpha$ forces $\dot{a}\subseteq\bigcap_{i<i^*}\{k<\omega:p_{i,k}{\restriction}\alpha\in\dot{G}\}\in\dot{D}^\epsilon_\alpha$. Without loss of generality, we assume that $\alpha\in\operatorname{supp}(q_i)$ for all $i<i^*$, so, if $h_\epsilon(\alpha)=(s,n)$, then $p_{i,k}(\alpha)=(s,\dot{\varphi}_{i,k})$ for some $\mathbb{P}'_\alpha$-name of a slalom $\dot{\varphi}_{i,k}$ of width n and $q_i(\alpha)=(s,\dot{\psi}_i)$ where $\dot{\psi}_i$ is a $\mathbb{P}_\alpha$-name of the $\dot{D}^\epsilon_\alpha$-limit of $\langle\dot{\varphi}_{i,k}\rangle_{k<\omega}$ (which is forced to be in $V^{\mathbb{P}'_\alpha}$ by (ii)). Let G_α be $\mathbb{P}_\alpha$-generic over V with $r{\restriction}\alpha\in G_\alpha$. In $V_\alpha=V[G_\alpha]$, let $r(\alpha)=(t,\psi')\in\mathbb{E}\cap V^{\mathbb{P}'_\alpha}$, which is stronger than $q_i(\alpha)=(s,\psi_i)$ for all $i<i^*$. As $t(j)\notin\psi_i(j)$ for any $j<|t|$, then the set $a_i=\{k<\omega:\forall_{j<|t|}(t(j)\notin\varphi_{i,k}(j))\}\in D^\epsilon_\alpha$. So choose $k>k^*$ in $a\cap\bigcap_{i<i^*}a_i$ and let $r'(\alpha)=(t,\psi'')$ where $\psi''(j)=\psi'(j)\cup\bigcup_{i<i^*}\varphi_{i,k}(j)$ for all $j<\omega$. Clearly, $r'(\alpha)$ is stronger than $r(\alpha)$ and than $p_{i,k}(\alpha)$ for all $i<i^*$. Back in V, let $r'{\restriction}\alpha\leq r{\restriction}\alpha$ in $\mathbb{P}_\alpha$ forcing the above statement, so $r'=r'{\restriction}\alpha\cup\{(\alpha,r'(\alpha))\}$ is as desired. □

Choose $\dot{D}^\epsilon_{\alpha+1}$ a $\mathbb{P}_{\alpha+1}$-name of an ultrafilter containing the set of the claim. □

Lemma 5.4 (Limit step)**.** *Assume δ is a limit ordinal, $\mathbf{q}$ is an iteration candidate of length δ and $\langle\dot{D}^\epsilon_\alpha:\epsilon<\chi,\ \alpha<\delta\rangle$ a sequence of $\mathbb{P}_\delta$-names such that, for each $\alpha<\delta$ and $\epsilon<\chi$, $\mathbf{D}^\epsilon_\alpha$ solves all $\mathbf{x}\in\mathrm{SS}_\alpha\cap\Lambda_\epsilon$ that are legal for $\mathbf{q}{\restriction}\alpha$. <u>Then</u>, there are $\mathbb{P}_\delta$-names $\langle\dot{D}^\epsilon_\delta:\epsilon<\chi\rangle$ such that, for each $\epsilon<\chi$, $\mathbf{D}^\epsilon_\delta=\mathbf{D}^\epsilon_{<\delta}{}^\frown\langle\dot{D}^\epsilon_\delta\rangle$ solves all $\mathbf{x}\in\mathrm{SS}_\delta\cap\Lambda_\epsilon$ that are legal for $\mathbf{q}$ (here, $\mathbf{D}^\epsilon_{<\delta}=\langle\dot{D}^\epsilon_\alpha\rangle_{\alpha<\delta}$).*

Proof. If δ has uncountable cofinality, let $\dot{D}^\epsilon_\delta$ be a $\mathbb{P}_\delta$-name of $\bigcup_{\xi<\delta}\dot{D}^\epsilon_\xi$. So assume that δ has countable cofinality.

Claim 5.5. $\mathbb{P}_\delta$ *forces that, for any* $\epsilon < \chi$, *the family*

$$\bigcup_{\alpha<\delta} \dot{D}^\epsilon_\alpha \ \cup\ \big\{\dot{A}_{\bar{p}}\ :\ \bar{p} \in P^\infty_{\mathbf{q},\mathbf{x}},\ \mathbf{x} \in \Lambda_\epsilon \cap \mathrm{SS}_\delta \textit{ legal},\ \lim_{\mathbf{D}^\epsilon_{<\delta}} \bar{p} \in \dot{G}\big\}$$

has the finite intersection property.

Proof. Let $\{\mathbf{x}^i : i < i^*\}$ be a finite subset of $\Lambda_\epsilon \cap \mathrm{SS}_\delta$ of legal stem sequences for $\mathbf{q}$, $\langle p_{i,k}\rangle_{k<\omega} \in P^\infty_{\mathbf{q},\mathbf{x}^i}$ for each $i < i^*$, $q_i = \lim_{\mathbf{D}^\epsilon_{<\delta}} p_{i,k}$ and $\dot{a}$ a $\mathbb{P}_\delta$-name of a set in $\bigcup_{\alpha<\delta} \dot{D}^\epsilon_\alpha$. Let $p \in \mathbb{P}_\delta$ be a condition stronger than q_i for all $i < i^*$ and let $k^* < \omega$ be arbitrary. We want to find $p^* \leq p$ and $k > k^*$ such that p^* is stronger than $p_{i,k}$ for all $i < i^*$ and forces $k \in \dot{a}$.

As in the notation of Definition 4.2, for each $i < i^*$ let $w_i = w_{\mathbf{x}^i}$ $l^*_i = l^*_{\mathbf{x}^i}$, $v_{i,S} = v_{\mathbf{x}^i,S}$ and so on. For the nontrivial case, we assume that $\sup_{l<l^*_i,k<\omega}\{\alpha_{i,k,l}\} = \delta$. For $i < i^*$, $l \in v_{i,S}$ and $k < \omega$, $p_{i,k}(\alpha_{i,k,l}) = (s^*_{i,l}, \dot{\varphi}_{i,k,l})$ where $\dot{\varphi}_{i,k,l}$ is a $\mathbb{P}'_{\alpha_{i,k,l}}$-name of a slalom of width $n^*_{i,l}$.

By strengthening p, find $\alpha < \delta$ that contains $\operatorname{supp} p \cup \bigcup_{i<i^*} \operatorname{supp} q_i$ and such that p forces that $\dot{a} \in \dot{D}^\epsilon_\alpha$, so, without loss of generality, $\dot{a}$ can be assumed to be a $\mathbb{P}_\alpha$-name. Even more, $\alpha < \delta$ can be found so that, for any $i < i^*$ and $l < l^*_i$, if there is some $k < \omega$ such that $\alpha_{i,k,l} \geq \alpha$, then $\sup_{k<\omega}\{\alpha_{i,k,l}\} = \delta$. For each $i < i^*$, let $u_i = \{l < l^*_i : \sup_{k<\omega}\{\alpha_{i,k,l}\} = \delta\}$ (note that this is an interval of the form $[l'_i, l^*_i)$ where l'_i is above v_i).

By hypothesis, find $p' \leq p$ in $\mathbb{P}_\alpha$ and $k > k^*$ such that p' is stronger than $p_{i,k}{\restriction}\alpha$ for all $i < i^*$ and forces $k \in \dot{a}$. Moreover, k can be found so that[5] $\alpha_{i,k,l} > \alpha$ for any $i < i^*$ and $l \in u_i$. Thus, because $\mathbb{E}$ is σ-centered and each $z_{\mathbf{x}^i} \subseteq h_\epsilon$, there is a condition $p^* \leq p'$ in $\mathbb{P}_\delta$ stronger than $p_{i,k}$ for all $i < i^*$. Indeed, $\operatorname{supp} p^* = \operatorname{supp} p' \cup \bigcup_{i<i^*} \operatorname{supp} p_{i,k}$, $p^*(\zeta) = p'(\zeta)$ for $\zeta \in \operatorname{supp} p'$, $p^*(\zeta) = p_{i,k}(\zeta) = h_\epsilon(\zeta)$ for $\zeta \in \operatorname{supp} p_{i,k} \cap v_{i,C} \smallsetminus \alpha$ and $p^*(\zeta) = (s_\zeta, \dot{\psi}_\zeta)$ for $\zeta \in \big(\bigcup_{i<i^*} \operatorname{supp} p_{i,k} \cap v_{i,S}\big) \smallsetminus \alpha$ where $h_\epsilon(\zeta) = (s_\zeta, n_\zeta)$ and $\dot{\psi}_\zeta$ is a $\mathbb{P}'_\zeta$-name of the slalom given by $\dot{\psi}(j) = \bigcup\{\dot{\varphi}_{i,k,l}(j) : \alpha_{i,k,l} = \zeta,\ l < l^*_i,\ i < i^*\}$. The condition p^* is as desired because, if $\zeta = \alpha_{i,k,l} = \alpha_{i',k,l'}$, when $\zeta \in C$ then $p_{i,k}(\zeta) = p_{i',k}(\zeta) = h_\epsilon(\zeta)$ and, when $\zeta \in S$, $p_{i,k}(\zeta) = (s^*_{i,l}, \dot{\varphi}_{i,k,l})$, $p_{i',k}(\zeta) = (s^*_{i',l'}, \dot{\varphi}_{i',k,l'})$ and $s^*_{i,l} = s^*_{i',l'} = s_\zeta$. □

Choose $\dot{D}^\epsilon_\delta$ a $\mathbb{P}_\delta$-name of an ultrafilter that contains the set of the previous claim. □

Remark 5.6. In Lemma 5.2, if all the $\dot{D}^\epsilon_\alpha$ $(\epsilon < \chi)$ are (forced to be) equal to some ultrafilter $\dot{D}_\alpha$, then Claim 5.3 can be similarly proven without fixing ϵ, that is, $\mathbb{P}_{\alpha+1}$ forces that $\dot{D}_\alpha \ \cup\ \big\{\dot{A}_{\bar{p}} : \bar{p} \in P^\infty_{\mathbf{q},\mathbf{x}},\ \mathbf{x} \in \mathrm{SS}_{\alpha+1} \text{ legal},\ \lim_{\mathbf{D}^\epsilon_\alpha} \bar{p} \in \dot{G}\big\}$ has the finite intersection property. Nevertheless, in Lemma 5.4 when δ is a limit of countable cofinality, the corresponding statement for Claim 5.5 may not be true when all the $\dot{D}^\epsilon_\alpha$ are the same for each $\alpha < \delta$ so, at that point, it becomes necessary to have different sequences of names of ultrafilters for each $\epsilon < \chi$ and Theorem 5.1 must be used to have as few sequences as possible (each one with respect to a guardrail h_β).

For instance, let $\delta = \omega$ and $\epsilon, \epsilon' < \chi$ such that h_ϵ and $h_{\epsilon'}$ are incompatible everywhere, that is, for each $m < \omega$, if $A_m = \kappa_b$ then $\Vdash_m h_\epsilon(m) \perp_{\dot{\mathbb{Q}}_m} h_{\epsilon'}(m)$ and, when $A_m = \omega^{<\omega} \times \omega$, the first coordinates of both $h_\epsilon(m)$ and $h_{\epsilon'}(m)$ are

[5]This is the only place where we need $\langle \alpha_{i,k,l} : k < \omega\rangle$ increasing.

incompatible. If $\mathbf{x} \in \Delta_\epsilon \cap \mathrm{SS}_\omega$ and $\mathbf{x}' \in \Delta_{\epsilon'} \cap \mathrm{SS}_\omega$ are legal for $\mathbf{q}$ such that $l^*_{\mathbf{x}} = l^*_{\mathbf{x}'} = 1$ and $\alpha_{\mathbf{x},k,0} = \alpha_{\mathbf{x}',k,0} = k$, if $\langle p_k \rangle_{k<\omega} \in P^\infty_{\mathbf{q},\mathbf{x}}$ and $\langle p'_k \rangle_{k<\omega} \in P^\infty_{\mathbf{q},\mathbf{x}'}$, then $\lim_{\mathbf{D}^\epsilon_{<\omega}} p_k = \lim_{\mathbf{D}^{\epsilon'}_{<\omega}} p'_k$ is the trivial condition and it is clear that $\mathbb{P}_\omega$ forces $\{k < \omega : p_k \in \dot{G}\} \cap \{k < \omega : p'_k \in \dot{G}\} = \varnothing$.

6. Proof of the main result

To prove our main result, we construct a nice iteration candidate with the bookkeeping arguments described in Subsection 2.3. Thanks to Main Lemma 4.6, we can guarantee that $\mathfrak{b}$ is the value we want in the extension.

Main Theorem 6.1. *Let $\kappa_{an} \leq \kappa_{cn} \leq \kappa_b \leq \kappa_{nm} = \kappa_{nm}^{\aleph_0}$ be regular uncountable cardinals and $\kappa_{ct} = \kappa_{ct}^{<\kappa_{nm}} \leq 2^\chi$ where $\kappa_b \leq \chi = \chi^{\aleph_0} \leq \kappa_{nm} < \kappa_{ct}$. Assume that there is a family of reals in ω^ω of size κ_b such that all its bounded subsets have size $< \kappa_b$ (so $(*)$ of Lemma 4.6 is satisfied by that family). Then, there is a ccc poset that forces*

$$\mathrm{add}(\mathcal{N}) = \kappa_{an} \leq \mathrm{cov}(\mathcal{N}) = \kappa_{cn} \leq \mathfrak{b} = \kappa_b \leq \mathrm{non}(\mathcal{M}) = \kappa_{nm} < \mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \kappa_{ct}.$$

Note that, assuming GCH, if $\kappa_{an} \leq \kappa_{cn} \leq \kappa_b \leq \kappa_{nm}$ are regular uncountable cardinals and $\kappa_{ct} > \kappa_{nm}$ is a cardinal of cofinality $\geq \kappa_{nm}$, it is not hard to construct a model, by forcing, that satisfies the hypothesis of the theorem with $\chi = \kappa_b$. For instance, adding κ_b-many Cohen reals gives such a family of size κ_b that satisfies $(*)$ of Lemma 4.6.

Proof. We construct a nice iteration candidate $\mathbf{q}$ of length $\delta_{\mathbf{q}} = \kappa_{ct} \cdot \kappa_{nm}$ (ordinal product) that forces our desired statement. Let $\kappa_{ct} = S' \cup C'$ be a partition where both S' and C' have size κ_{ct} and also let $C' = C'_0 \cup C'_1 \cup C'_2 \cup C'_3$ be a partition where each C'_i has size κ_{ct}. Let $S = S_{\mathbf{q}} = \bigcup_{\beta<\kappa_{nm}} (\kappa_{ct} \cdot \beta + S')$, $C_i = \bigcup_{\beta<\kappa_{nm}} (\kappa_{ct} \cdot \beta + C'_i)$ for $i < 4$ and $C = C_{\mathbf{q}} = C_0 \cup C_1 \cup C_2 \cup C_3$.

We construct $\mathbf{q}$ by recursion using the method in Section 5. The induction basis and the limit step are clear by Lemma 5.4, so we only have to describe what we do in the successor step in such a way that Lemma 5.2 can be applied. Assume we have constructed our iteration up to $\alpha < \kappa_{ct} \cdot \kappa_{nm}$ and that $\langle \dot{D}^\epsilon_\xi : \epsilon < \chi,\ \xi \leq \alpha \rangle$ is as in Lemma 5.2. Clearly, α is of the form $\kappa_{ct} \cdot \beta + \zeta$ for some (unique) $\beta < \kappa_{nm}$ and $\zeta < \kappa_{ct}$. Consider:

(i) $\{\dot{\mathbb{A}}_{\beta,\xi} : \xi \in C'_0\}$ lists the $\mathbb{P}_{\kappa_{ct}\cdot\beta}$-names of all ccc posets whose domain is an ordinal $< \kappa_{an}$.

(ii) $\{\dot{\mathbb{B}}_{\beta,\xi} : \xi \in C'_1\}$ lists the $\mathbb{P}_{\kappa_{ct}\cdot\beta}$-names of all subalgebras of random forcing $\mathbb{B}^{V^{\mathbb{P}_{\kappa_{ct}\cdot\beta}}}$ of size $< \kappa_{cn}$.

(iii) $\{\dot{\mathbb{D}}_{\beta,\xi} : \xi \in C'_2\}$ lists the $\mathbb{P}_{\kappa_{ct}\cdot\beta}$-names of all σ-centered subforcings of Hechler forcing $\mathbb{D}^{V^{\mathbb{P}_{\kappa_{ct}\cdot\beta}}}$ of size $< \kappa_b$.

(iv) $\{\dot{F}^\beta_\xi : \xi \in S'\}$ lists the $\mathbb{P}_{\kappa_{ct}\cdot\beta}$-names for all sets of size $< \kappa_{nm}$ of slaloms of finite width in $V^{\mathbb{P}_{\kappa_{ct}\cdot\beta}}$.

If $\alpha \in C$, let

$$\dot{\mathbb{Q}}_\alpha = \begin{cases} \dot{\mathbb{A}}_{\beta,\zeta} & \text{if } \zeta \in C'_0, \\ \dot{\mathbb{B}}_{\beta,\zeta} & \text{if } \zeta \in C'_1, \\ \dot{\mathbb{D}}_{\beta,\zeta} & \text{if } \zeta \in C'_2, \\ \dot{\mathbb{C}} & \text{(Cohen forcing) if } \zeta \in C'_3. \end{cases}$$

If $\alpha \in S$, we need to construct $\mathbb{P}'_\alpha$ and we want[6] it to have size $\leq \kappa_{nm}$. Given $\epsilon < \chi$ and $\dot{a}$ a (nice) $\mathbb{P}_\alpha$-name of a subset of ω, choose a maximal antichain $A^\epsilon_{\dot{a}}$ that decides either $\dot{a} \in \dot{D}^\epsilon_\alpha$ or $\omega \smallsetminus \dot{a} \in \dot{D}^\epsilon_\alpha$. Therefore, by closing under this and other simpler operations, we can find $\mathbb{P}'_\alpha \lessdot \mathbb{P}_\alpha$ of size $\leq \kappa_{nm}$ such that $\dot{F}_\zeta$ is a $\mathbb{P}'_\alpha$-name and, for any $\epsilon < \chi$ and a (nice) $\mathbb{P}'_\alpha$-name $\dot{a}$ of a subset of ω, $A^\epsilon_{\dot{a}} \subseteq \mathbb{P}'_\alpha$ (because $\kappa_{nm}^{\aleph_0} = \kappa_{nm}$), which implies that there is a $\mathbb{P}'_\alpha$-name of $\dot{D}^\epsilon_\alpha \cap V^{\mathbb{P}'_\alpha}$. Let $\dot{\mathbb{Q}}_\alpha = \mathbb{E}^{V^{\mathbb{P}'_\alpha}}$, which adds a real that evades eventually all the slaloms from $\dot{F}_\zeta$. It is clear that Lemma 5.2 applies, which finishes the construction.

From the results already proved or cited, it is easy to check that $\mathbb{P}_\delta$ forces $\kappa_{an} \leq \mathrm{add}(\mathcal{N})$, $\kappa_{cn} \leq \mathrm{cov}(\mathcal{N})$, $\kappa_b \leq \mathfrak{b}$ and $\kappa_{nm} \leq \mathrm{non}(\mathcal{M})$. The relations $\mathrm{add}(\mathcal{N}) \leq \kappa_{an}$ and $\mathrm{cov}(\mathcal{N}) \leq \kappa_{cn}$ in the extension are consequences of Corollary 3.6(a) applied to the pairs $(\theta, \sqsubset) = (\kappa_{an}, \in^*)$ (see Example 3.4(4)) and $(\theta, \sqsubset) = (\kappa_{cn}, \pitchfork)$ (see Example 3.4(3)), respectively. The crucial inequality $\mathfrak{b} \leq \kappa_b$ is a consequence of Lemma 4.6. Besides, $\mathrm{non}(\mathcal{M}) \leq \kappa_{nm}$ holds in the extension because of the κ_{nm}-cofinal many Cohen reals added along the iteration. It is also clear that $\mathfrak{c} \leq \kappa_{ct}$ is forced.

Finally, by Corollary 3.6(b) applied to the pair $(\theta, \sqsubset) = ({\kappa_{nm}}^+, \neq^*)$ (see Example 3.4(1)), $\mathrm{cov}(\mathcal{M}) = \mathfrak{d}_{\neq^*} \geq \kappa_{ct}$. $\square$

Remark 6.2. If we further assume that $\chi < \kappa_{nm}$ and $\mu^{\aleph_0} < \kappa_{nm}$ for all $\mu < \kappa_{nm}$, then we can similarly construct a nice iteration candidate of length κ_{ct} forcing the same as in Theorem 6.1. In the argument of this, for $\alpha \in S$, we can construct $\mathbb{P}'_\alpha$ of size $< \kappa_{nm}$, so we can force $\mathrm{non}(\mathcal{M}) \leq \kappa_{nm}$ by Corollary 3.6(a) applied to $(\theta, \sqsubset) = (\kappa_{nm}, \neq^*)$. However, this is less general because κ_{nm} is not allowed to be a successor of a cardinal with countable cofinality.

Remark 6.3. A similar proof of Theorem 6.1 can be performed using bounded versions of $\mathbb{E}$ to ease the compactness arguments, but it has the disadvantage that we are restricted to $2^\chi = \chi^+ = \kappa_{ct}$ and $\chi = \kappa_{nm}$. The argument is similar but much more difficult, we point out the differences with the presented argument.

(1) Fix $b : \omega \to \omega$ a fast increasing function with $b(0) > 0$. Let $\mathbb{E}^b$ be the poset whose conditions are pairs (s, φ) where s is a finite sequence below b and φ is a slalom of width at most $|s|$. The order is similar to $\mathbb{E}$. Like $\mathbb{E}$, this poset adds an eventually different real (below b) and does not add dominating reals (moreover, it is $\leq^*$-good), however, it is not σ-centered.
(2) In all the arguments, everything related to $\mathbb{E}$ should be respectively modified to the context of $\mathbb{E}^b$.
(3) In Definition 4.2, we additionally have to include Borel functions that code the names of slaloms corresponding to the coordinates of subforcings of $\mathbb{E}^b$ of the conditions of the countable Δ-system that is coded (here, $w_{\mathbf{x}}$ and $w_{\mathbf{x}} \cap \kappa_b$ of Definition 4.2 play a role). In this case, those codes should be called *blueprints*. Moreover, $n^*_l \leq |s^*_l|$ for all $l < l^*$.
(4) $2^\chi = \chi^+$ is assumed since we need (ii') of Theorem 5.1 in this case. Guardrails h_β should also talk about the Borel functions included in the blueprints of Λ_ϵ (by further assuming $\mathfrak{c} = \kappa_b$ in the ground model), so the last part of the proof of Claim 5.5 could be argued.

[6]See Remark 6.2, which explains why we only require $|\mathbb{P}'_\alpha| \leq \kappa_{nm}$ rather than the strict inequality that the reader might have expected.

(5) In the construction of the iteration for the main result, we have to guarantee that the used subforcings of $\mathbb{E}^b$ don't add random reals nor destroy a witness of $\mathrm{add}(\mathcal{N})$ that we want to preserve, that is, that they are both κ_{an}-$\in^*$-good and κ_{cn}-$\pitchfork$-good. For this, a notion of (π,ρ)-linkedness, defined in [KO14], justifies the desired preservation (for κ_{an}-$\in^*$-goodness, see [BM14, Section 5]).

7. Questions

Question 7.1. Is there a model of $\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) < \mathfrak{c}$? or just a model of $\mathfrak{b} < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) < \mathfrak{c}$?

A ZFC model of $\aleph_1 < \mathrm{add}(\mathcal{N}) = \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = \mathrm{cf}(\mathcal{N}) < \mathfrak{c}$ was constructed in [Mej13, Thm. 11] by a matrix iteration (a technique to construct FS iterations in a non-trivial way). The difficulty to answer Question 7.1 lies in the fact that there are no known easy FS iteration constructions that force $\mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) < \mathfrak{c}$.

Question 7.2. Is there a model of $\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathfrak{b} < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) < \mathfrak{c}$? or just a model of $\mathfrak{b} < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M})$?

As pointed out by Judah and Shelah [JS93] (see also [Paw92]), subalgebras of random forcing may add dominating reals, so there are similar difficulties as those described in Section 2 for subforcings of $\mathbb{E}$. It seems that sophisticated techniques as in [She00] may help to deal with this problem.

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Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Wiedner Hauptstrasse 8-10/104, 1040 Vienna, Austria.

E-mail address: goldstern@tuwien.ac.at

URL: http://www.tuwien.ac.at/goldstern/

Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Wiedner Hauptstrasse 8-10/104, 1040 Vienna, Austria.

Current address: Department of Mathematics, Shizuoka University, Ohya 836, Shizuoka, 422-8529 Japan

E-mail address: diego.guzman@tuwien.ac.at

URL: http://www.researchgate.com/profile/Diego_Mejia2

Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The hebrew University of Jerusalem, Jerusalem, 91904, Israel, and Department of Mathematics, Rutgers University, New Brunswick, NJ 08854, USA.

E-mail address: shlhetal@math.huji.ac.il

URL: http://shelah.logic.at