On the absoluteness of orbital ω-stability

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Abstract

We show that orbital ω-stability is upwards absolute for $\aleph_0$-presented abstract elementary classes satisfying amalgamation and the joint embedding property (each for countable models). We also show that amalgamation does not imply upwards absoluteness of orbital ω-stability by itself.

Suppose that $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ is an abstract elementary class (or *AEC*; see [1, 8] for a definition), and let (M, a, N) and (P, b, Q) be such that M, N, P and Q are structures in $\boldsymbol{K}_{\aleph_0}$ (where, for a cardinal κ, $\boldsymbol{K}_\kappa$ denotes the members of $\boldsymbol{K}$ of cardinality κ) with $M \preceq_{\boldsymbol{k}} N$, $P \preceq_{\boldsymbol{k}} Q$, $a \in N \setminus M$ and $b \in Q \setminus P$. The triples (M, a, N) and (P, b, Q) are said to be *Galois equivalent* or *orbitally equivalent* if $M = P$ and there exist $R \in \boldsymbol{K}_{\aleph_0}$ and $\preceq_{\boldsymbol{k}}$-embeddings $\pi\colon N \to R$ and $\sigma\colon Q \to R$ such that π and σ are the identity on M, and $\pi(a) = \sigma(b)$. If $\boldsymbol{k}$ satisfies amalgamation (the property that if M, N and P are elements of $\boldsymbol{K}$ such that $M \preceq_{\boldsymbol{k}} N$ and $M \preceq_{\boldsymbol{k}} P$ then there exist $Q \in \boldsymbol{K}$ and $\preceq_{\boldsymbol{k}}$-embeddings $\pi\colon N \to Q$ and $\sigma\colon P \to Q$ such that π and σ are the identity on M) then this relation is an equivalence relation on the class of such triples; each equivalence class is called a *Galois type* or *orbital type* (amalgamation is not necessary for orbital equivalence to be transitive). We say that the AEC $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ is *ω-orbitally stable* if, for each $M \in \boldsymbol{K}_{\aleph_0}$, the set of equivalence classes over M as above for triples (M, a, N) with $N \in \boldsymbol{K}_{\aleph_0}$ is countable.

An abstract elementary class $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ over a countable vocabulary τ is called $\aleph_0$-*presentable* (among other names, including $\mathrm{PC}_{\aleph_0}$ and *analytically presented*) if the class of models $\boldsymbol{K}$ and the class of pairs corresponding to $\preceq_{\boldsymbol{k}}$ are each the set of reducts to τ of the models of an $L_{\aleph_1,\aleph_0}$-sentence in some expanded language (this formulation implies that the Löwenheim-Skolem number of $\boldsymbol{k}$ is $\aleph_0$, which in any case we take to be part of the definition). Equivalently (assuming that the Löwenheim-Skolem number of $\boldsymbol{k}$ is $\aleph_0$), $\boldsymbol{k}$ is $\aleph_0$-presentable if the collections of subsets of ω coding (in some natural fashion) the restrictions of $\boldsymbol{K}$ and $\preceq_{\boldsymbol{k}}$ to countable structures are analytic. If $\boldsymbol{K}$ is $\aleph_0$-presentable, the ω-orbital stability of $\boldsymbol{K}$ is naturally expressed as a Π^1_4 property in a countable parameter for $\boldsymbol{K}$. One might hope that this property has a simpler definition, and moreover that the property is absolute between models of set theory with the same ordinals.

In this note we show that ω-orbital stability is upwards absolute for $\aleph_0$-presentable abstract elementary classes $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ for which $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$ satisfies amalgamation and the joint embedding property (the property that any two elements of $\boldsymbol{K}$ can be $\preceq_{\boldsymbol{k}}$-embedded in a common element of $\boldsymbol{K}$, i.e., that $(\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ is directed). We also present an $\aleph_0$-presented AEC, satisfying amalgamation but not the joint embedding

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property, for which ω-orbital stability is not upwards absolute. In light of Shoenfield's absoluteness theorem for (boldface) Σ^1_2 relations (Theorem 25.20 of [6]), it follows that ω-orbital stability for an $\aleph_0$-presented AEC cannot in general be expressed by a Boolean combination of Σ^1_2 formulas (in a parameter coding the class in question). This contrasts with the fact that almost ω-orbital stability, the property of not having a perfect set of representatives of distinct equivalence classes for orbital equivalence, is Π^1_2 (see [2], for instance). By Burgess's Theorem for analytic equivalence relations (see [4], Theorem 9.1.5), an $\aleph_0$-presented AEC which is almost ω-orbitally stable but not ω-orbitally stable has a countable structure with exactly $\aleph_1$ many orbital types.

In the absence of amalgamation, it is natural to define orbital equivalence using the transitive closure of the relation defined above. We have not been able to resolve the following question.

Question 0.1. *Is ω-orbital stability upwards absolute, for $\aleph_0$-presented AEC's satisfying $\aleph_0$-JEP?*

1 Upward absoluteness with amalgamation and joint embedding

In this section we show that orbital ω-stability is upwards absolute for any $\aleph_0$-presented AEC $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ for which $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$ satisfies amalgamation and the joint embedding property. The proof below uses the notion of model-theoretic forcing from [8] and the following natural generalization of the notion of orbital type to finite sequences.

Suppose that $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ is an abstract elementary class, and let $(M, \langle a_0, \ldots, a_n\rangle, N)$ and $(P, \langle b_0, \ldots, b_q\rangle, Q)$ be such that

- $n, q \in \omega$;
- M, N, P and Q are structures in $\boldsymbol{K}_{\aleph_0}$;
- $M \preceq_{\boldsymbol{k}} N$ and $P \preceq_{\boldsymbol{k}} Q$;
- each a_i is in N and each b_i is in Q.

The triples $(M, \langle a_0, \ldots, a_n\rangle, N)$ and $(P, \langle b_0, \ldots, b_m\rangle, Q)$ are *orbitally equivalent* if $M = P$, $n = q$ and there exist $R \in \boldsymbol{K}_{\aleph_0}$ and $\preceq_{\boldsymbol{k}}$-embeddings $\pi\colon N \to R$ and $\sigma\colon Q \to R$ such that π and σ are the identity on M, and $\pi(a_i) = \sigma(b_i)$ for all $i \leq n$. As above, if $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$ satisfies amalgamation, then this relation is an equivalence relation on the class of such triples, and, for a fixed $M \in \boldsymbol{K}_{\aleph_0}$ the set of equivalence classes *over* M is the set of equivalence classes of triples with M as their first coordinate. By (a special case of) a recent result of Boney [3], if $\boldsymbol{k}$ is orbitally ω-stable (and $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$ satisfies amalgamation), then for each $M \in \boldsymbol{K}_{\aleph_0}$ there are just countably many equivalence classes over M in this generalized sense.[1]

Given an AEC $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$, a subclass $\boldsymbol{K}'$ of $\boldsymbol{K}$ and an $M \in \boldsymbol{K}'$, we say that M is *$\preceq_{\boldsymbol{k}}$-universal* for $\boldsymbol{K}'$ if for each $N \in \boldsymbol{K}'$ there is a $\preceq_{\boldsymbol{k}}$-embedding of N into M; M is *$\preceq_{\boldsymbol{k}}$-maximal* for $\boldsymbol{K}'$ if there does not exist an $N \in \boldsymbol{K}'$ (other than M) such that $M \preceq_{\boldsymbol{k}} N$.

For an $\aleph_0$-presented AEC $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ the following are easily seen to be absolute. The last of these says that there are just countably many orbital types over M.

- The statement that $\boldsymbol{K}_{\aleph_0}$ is nonempty (Σ^1_1 in a code for $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$).
- The statement that $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$ satisfies amalgamation (Π^1_2 in a code for $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$).

[1] Although we won't use this fact here, we note that Boney's arguments go through without change under the assumption that orbital equivalence (for finite tuples) is transitive, in place of amalgamation.

- The statement that $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$ satisfies joint embedding (Π^1_2 in a code for $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$).
- For a fixed $M \in \boldsymbol{K}_{\aleph_0}$, M is a $\preceq_{\boldsymbol{k}}$-universal member of $\boldsymbol{K}_{\aleph_0}$ (Π^1_2 in codes for $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$ and M).
- For a fixed $M \in \boldsymbol{K}_{\aleph_0}$, M is a $\preceq_{\boldsymbol{k}}$-maximal member of $\boldsymbol{K}_{\aleph_0}$ (Π^1_1 in codes for $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$ and M).
- For a fixed $M \in \boldsymbol{K}_{\aleph_0}$, and a fixed countable set of pairs (a, N) with $N \in \boldsymbol{K}_{\aleph_0}$, $M \preceq_{\boldsymbol{k}} N$ and $a \in N \setminus M$, the statement that every orbital type over M contains a member of the set (Π^1_2 in codes for $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$, M and the set) .

In light of these facts, Theorem 1.3 below shows that ω-orbital stability is upwards absolute for an $\aleph_0$-presented AEC $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ for which $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$ satisfies amalgamation and joint embedding, as it is Π^1_2 in codes for $\boldsymbol{K}$ and a $\preceq_{\boldsymbol{k}}$-universal model for $\boldsymbol{K}_{\aleph_0}$. We first show that one direction of the equivalence in Theorem 1.3 follows from weaker hypotheses. Theorems 1.1 and 1.3 do not use the assumption of $\aleph_0$-presentability.

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Theorem 1.1. *Suppose that $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ is an abstract elementary class such that*

- $\boldsymbol{K}_{\aleph_0} \neq \emptyset$;
- $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$ *satisfies the joint embedding property;*
- *for each $M \in \boldsymbol{K}_{\aleph_0}$, the set of orbital types over M (for finite tuples) is countable.*

Then $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$ has a universal element.

Proof. If there exists $\preceq_{\boldsymbol{k}}$-maximal element of $\boldsymbol{K}_{\aleph_0}$, then it is universal, by the joint embedding property, so assume otherwise. We use model-theoretic forcing. We refer the reader to pages 162-163 of [8] for the definition of the relation $N \Vdash \phi(a_0, \ldots, a_n)$, where $N \in \boldsymbol{K}_{\aleph_0}$, $a_0, \ldots, a_n \in N$, $\phi \in L_{\aleph_1,\aleph_0}(\tau)$ and τ is the vocabulary corresponding to $\boldsymbol{k}$. The following facts follow easily from this definition.

1. Since $\boldsymbol{k}$ satisfies the joint embedding property, for each sentence ϕ in $L_{\aleph_1,\aleph_0}(\tau)$ and each $M \in \boldsymbol{K}_{\aleph_0}$, $M \Vdash \phi$ or $M \Vdash \neg\phi$.

{forceablety

2. If $M, N, P \in \boldsymbol{K}_{\aleph_0}$, $n \in \omega$, $a_0, \ldots, a_{n-1} \in M$, $b_0, \ldots, b_{n-1} \in N$, $\phi \in L_{\aleph_1,\aleph_0}(\tau)$ is an n-ary formula and $\pi \colon M \to P$ and $\sigma \colon N \to P$ are $\preceq_{\boldsymbol{k}}$-embeddings such that $\pi(a_i) = \sigma(b_i)$ for all $i < n$, then
$$M \Vdash \phi(a_0, \ldots, a_{n-1}) \Rightarrow \neg(N \Vdash \neg\phi(b_0, \ldots, b_{n-1})).$$

{materialize

3. For every countable subset Ψ of $L_{\aleph_1,\aleph_0}(\tau)$, and every $M \in \boldsymbol{K}_{\aleph_0}$, there is an $N \in \boldsymbol{K}_{\aleph_0}$ such that $M \preceq_{\boldsymbol{k}} N$ and, for all $n \in \omega$, and $a_0, \ldots, a_{n-1} \in N$ and all n-ary formulas $\phi \in \Psi$,
$$N \Vdash \phi(a_0, \ldots, a_{n-1}) \Leftrightarrow N \models \phi(a_0, \ldots, a_{n-1}).$$

By item (3), it suffices to see that there exists a sentence $\phi \in L_{\aleph_1,\aleph_0}(\tau)$ which is the Scott sentence of a countable τ-structure, and which is forced by some (equivalently, every) element of $\boldsymbol{K}_{\aleph_0}$, as then the models of ϕ are $\preceq_{\boldsymbol{k}}$-universal for $\boldsymbol{K}_{\aleph_0}$.

To see that this does hold, we will assume some familiarity with the Scott analysis of a τ-structure (see [5, 7], for instance). This analysis, given a τ-structure M, assigns to each finite tuple $\overline{a}$ from M and each ordinal α a formula $\phi^M_{\overline{a},\alpha}$ in such a way that (among other things)

4. $M \models \phi^M_{\overline{a},\alpha}(\overline{a})$;

5. for any τ-structure N and any finite tuple $\overline{b}$ from N, if $N \models \phi^M_{\overline{a},\alpha}(\overline{b})$, then $\phi^M_{\overline{a},\alpha} = \phi^N_{\overline{b},\alpha}$.

For each ordinal $\alpha < \omega_1$, let Φ_α be the set of all formulas of the form $\phi^M_{\overline{a},\alpha}$, for some τ-structure M and some finite tuple $\overline{a}$ from M. By facts (3) and (5) above, for each ordinal α, each $M \in \boldsymbol{K}_{\aleph_0}$ and each finite tuple $\overline{a} = \langle a_0, \ldots, a_{n-1}\rangle$ from M, there is at most one formula $\psi \in \Phi_\alpha$ such that $M \Vdash \psi(a_0, \ldots, a_{n-1})$. Call this formula $\psi_{M,\overline{a},\alpha}$ if it exists. Let Ψ_α be the set of all formulas of the form $\psi_{M,\overline{a},\alpha}$, for some $M \in \boldsymbol{K}_{\aleph_0}$. Applying the joint embedding property, for each $M \in \boldsymbol{K}_{\aleph_0}$, each member of each Ψ_α is witnessed by a $\preceq_{\boldsymbol{k}}$-extension of M. It follows by item (2) above, and our assumption on the number of orbital types for finite sequences, that each set Ψ_α is countable.

It follows then by induction on α that for each $M \in \boldsymbol{K}_{\aleph_0}$, each $n \in \omega$ and each n-tuple $\overline{a}$ from M, there exists an $N \in \boldsymbol{K}_{\aleph_0}$ such that $M \preceq_{\boldsymbol{k}} N$ and such that the formula $\psi_{N,\overline{a},\alpha}$ exists and is a member of $L_{\aleph_1,\aleph_0}(\tau)$ (for the induction step from α to $\alpha+1$, build N as the union of a countable $\preceq_{\boldsymbol{k}}$-chain starting with M, applying the induction hypothesis for α for each finite tuple $\overline{a}$ from each model in the chain and including, for each $(n+1)$-ary formula $\psi \in \Psi_\alpha$, a structure $P \in \boldsymbol{K}_{\aleph_0}$ such that $P \Vdash \exists b \psi(\overline{a}, b)$, if possible). Again by the assumption of orbital ω-stability for finite tuples, and the joint embedding property, there is a countable ordinal α such that for all $M \in \boldsymbol{K}_{\aleph_0}$ and all finite tuples $\overline{a}$, $\overline{b}$ from M, if $\psi_{M,\overline{a},\alpha+1}$ and $\psi_{M,\overline{b},\alpha+1}$ exist and $\psi_{M,\overline{a},\alpha} = \psi_{M,\overline{b},\alpha}$, then $\psi_{M,\overline{a},\alpha+1} = \psi_{M,\overline{b},\alpha+1}$. To see this, note first that, supposing otherwise, and applying JEP, we may fix an $M \in \boldsymbol{K}_{\aleph_0}$ and, for each $\alpha < \omega_1$, a $\preceq_{\boldsymbol{k}}$-extension N_α of M and finite tuples $\overline{a}_\alpha$, $\overline{b}_\alpha$ from N_α such that $\psi_{N_\alpha,\overline{a}_\alpha,\alpha+1}$ and $\psi_{N_\alpha,\overline{b}_\alpha,\alpha+1}$ exist, $\psi_{N_\alpha,\overline{a}_\alpha,\alpha} = \psi_{N_\alpha,\overline{b}_\alpha,\alpha}$, and $\psi_{N_\alpha,\overline{a}_\alpha,\alpha+1} \neq \psi_{N_\alpha,\overline{b}_\alpha,\alpha+1}$. Applying ω-orbital stability (for the tuples $\overline{a}_\alpha \cup \overline{b}_\alpha$) then gives a contradiction.

Finally, by item (3) above, each $M \in \boldsymbol{K}_{\aleph_0}$ has a $\preceq_{\boldsymbol{k}}$-extension $N \in \boldsymbol{K}_{\aleph_0}$ with the property that for all finite tuples $\overline{a}$ from N, $N \models \psi_{N,\overline{a},\alpha+\omega}(\overline{a})$. This implies that N has Scott rank at most α, and that N forces its own Scott sentence. □

Adding amalgamation, we get an equivalence (Theorem 1.3 below). The forward direction of the theorem follows from Theorem 1.1. The reverse direction follows from the following consequence of amalgamation.

Lemma 1.2. *Suppose that $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ is an AEC satisfying amalgamation, and suppose that $M \preceq_{\boldsymbol{k}} N$ are elements of $\boldsymbol{K}$. Then the set of orbital types over M injects into the union of N with the set of orbital types over N.*

Proof. By amalgamation, we can pick for each orbital type over M a representative of the form (M, a, P), for some $\preceq_{\boldsymbol{k}}$-extension P of N. If $a \in N$, then we can map this type to a. Otherwise, we map the type to the type of (N, a, P). The definition of orbital type shows that this map is an injection. □

{upguy}

Theorem 1.3. *Suppose that $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ is an AEC satisfying amalgamation and the joint embedding property, for which $\boldsymbol{K}_{\aleph_0}$ is nonempty. Then $\boldsymbol{K}$ is orbitally ω-stable if and only if $\boldsymbol{K}_{\aleph_0}$ has a $\preceq_{\boldsymbol{k}}$-universal member over which there are only countably many orbital types.*

2 A counterexample with amalgamation but not joint embedding

{amnotjepsec

In this section we present an $\aleph_0$-presentable AEC which satisfies amalgamation, fails the joint embedding property, and is orbitally ω-stable if $\mathbb{R} \subseteq L$ but not if ω_1^L is uncountable and there exists a nonconstructible real.

Let T_L be the theory of the structure $\langle L_{\omega_1^L}, \in \rangle$. We use the following standard fact, which is a special case of a result of Harvey Friedman. As usual, $\mathbb{Q}$ denotes the set of rational numbers.

{lfact}

Fact 2.1. *If M is a countable illfounded ω-model of T_L then the ordinals of M have ordertype $\alpha + (\mathbb{Q} \times \alpha)$ for some ordinal $\alpha < \omega_1^L$, where $\mathbb{Q} \times \alpha$ is given the lexicographical order.*

Let τ be the vocabulary consisting of $=$, binary symbols E and $<$, and unary symbols W_n ($n \in \omega$). We let $\boldsymbol{K}^\tau$ be the class of τ-structures M of the form

$$\langle |M|, E^M, <^M, W_n^M; n \in \omega \rangle$$

such that

- E^M is an equivalence relation on $|M|$ and $<^M$ is a subset of E^M;
- each W_n^M is either the empty set or all of $|M|$;
- for each $a \in |M|$, there exists an ω-model N of T such that
  - the ordinals of N are $[a]_{E^M}$ and $<^M \restriction [a]_{E^M}$ is the corresponding ordering,
  - $\{n \in \omega : W_n^M \neq \emptyset\}$ is not a member of N (i.e., for no $w \in N$ is it true that $N \models w \subseteq \omega$ and, for all $n \in \omega$, that $N \models$ "the n-th member of ω is in w" if and only if $W_n^M \neq \emptyset$).

Given $M, N \in \boldsymbol{K}^\tau$, let $M \preceq_{\boldsymbol{k}^\tau} N$ if $|M| \subseteq |N|$, $E^M = E^N \cap (|M| \times |M|)$, $<^M = <^N \cap (|M| \times |M|)$, each E^N equivalence class is either contained in or disjoint from $|M|$ and, for each $n \in \omega$, $W_n^N = \emptyset$ if and only if $W_n^M = \emptyset$.

This is an $\aleph_0$-presented AEC, satisfying amalgamation, but not the joint embedding property (by the order condition on the sets W_n^M). By Fact 2.1, if $\mathbb{R} \subseteq L$ there are countably many orbital types over each countable member of $\boldsymbol{K}^\tau$ (since the lengths of longest wellfounded initial segments of the orders $<^M \restriction [a]_{E^M}$ are bounded by the least ordinal α such that $\{n \in \omega : W_n^M \neq \emptyset\}$ is in L_α) .

On the other hand, if $r \subseteq \omega$ is nonconstructible, and $\omega_1^V = \omega_1^L$, consider a countable model $M \in \boldsymbol{K}^\tau$ with $\{n : W_n^M \neq \emptyset\} = r$. The models L_α which are countable elementary submodels of $L_{\omega_1^L}$ then induce uncountably many orbital types over M.

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