Coding with canonical functions

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Abstract

A function f from ω_1 to the ordinals is called a canonical function for an ordinal α if f represents α in any generic ultrapower induced by forcing with $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$. We introduce here a method for coding sets of ordinals using canonical functions from ω_1 to ω_1. Combining this approach with arguments from [3], we show, assuming the Continuum Hypothesis, that for each cardinal κ there is a forcing construction preserving cardinalities and cofinalities forcing that every subset of κ is an element of the inner model $L(\mathcal{P}(\omega_1))$.

1 Introduction

Results in set theory over the last forty years show that the existence of certain large cardinals implies that there are subsets of ω_1 which are not elements of the inner model $L(\mathcal{P}(\omega))$ (for instance, those coding an ω_1-sequence of distinct subsets of ω; see Chapter 6 of [1]). A natural question, asked of us by several researchers, is whether a similar phenomenon happens at higher cardinals. Here we show that this is not the case. In particular, we show, assuming the Continuum Hypothesis, that for any infinite cardinal κ there is an (ω, ∞)-distributive, $\aleph_2$-c.c. partial order forcing that $\mathcal{P}(\kappa) \subseteq L(\mathcal{P}(\omega_1))$. We employ a coding mechanism using canonical functions from ω_1 to ω_1 to code subsets of the given cardinal κ. We introduce this mechanism in Section 2 and give the proof in the case $\kappa = \omega_2$, which is simpler, at the end of the section. The result for arbitrarity κ is proved in Section 3. This proof involves combining the coding technique introduced here with previous work of the second author from [3], although in our context the argument is considerably simpler.

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2 Coding sets of ordinals by subsets of ω_1

In this section we introduce the forcing construction which is used in this paper to code sets of ordinals. Given functions f and g on ω_1, we let $\mathrm{eq}(f, g)$ denote the set of $\alpha < \omega_1$ for which $f(\alpha) = g(\alpha)$. Suppose that η is an ordinal and that

$$\bar{f} = \langle f_\alpha : \alpha < \eta \rangle$$

and

$$\bar{C} = \langle C_{\{\alpha,\beta\}} : \{\alpha, \beta\} \in [\eta]^2 \rangle$$

are such that

- each f_α is a function from ω_1 to ω_1;
- $\bar{C}$ *witnesses that* $\bar{f}$ is a *mod*-NS_{ω_1}-*distinct sequence*, that is, each $C_{\{\alpha,\beta\}}$ is a club subset of ω_1 disjoint from $\mathrm{eq}(f_\alpha, f_\beta)$;

Let Y be a subset of η. We define a partial order $\mathbb{P}_{\bar{f},\bar{C},Y}$ which adds a function from $\omega_1 \times \omega_1 \to 2$ coding Y via $\langle [f_\alpha]_{\mathrm{NS}_{\omega_1}} : \alpha < \eta \rangle$, where $[f]_{\mathrm{NS}_{\omega_1}}$ (for f a function from ω_1 to ω_1) is the set of functions g from ω_1 to ω_1 for which $\mathrm{eq}(f, g)$ contains a club.

2.1 Definition. A condition in $\mathbb{P}_{\bar{f},\bar{C},Y}$ is a tuple $p = \langle u_p, i_p, h_p, \bar{E}_p, s_p \rangle$ such that

- $u_p \in [\eta]^{\aleph_0}$;
- $i_p \in \bigcap \{C_{\{\alpha,\beta\}} : \{\alpha, \beta\} \in [u_p]^2\}$;
- letting
$$j_p = \sup\{f_\alpha(\xi) + 1 : (\alpha, \xi) \in u_p \times i_p\},$$
h_p is a function from $i_p \times j_p$ to 2;
- $\bar{E}_p$ is a sequence
$$\langle E_{p,\beta} : \beta \in u_p \rangle$$
such that each $E_{p,\beta}$ is a closed subset of i_p;
- s_p is a subset of i_p;
- for all $(\alpha, \xi) \in u_p \times i_p$, if $\xi \notin s_p$ then $h(\xi, f_\alpha(\xi)) = 0$.

Given $p, q \in \mathbb{P}_{\bar{f},\bar{C},Y}$, $p \leq q$ (p is stronger than q) if

- $u_q \subseteq u_p$;
- $i_p \geq i_q$;
- $s_p \cap i_q = s_q$;
- for all $\beta \in u_q$, $E_{p,\beta} \cap i_q = E_{q,\beta}$;

- $h_q \subseteq h_p$;
- for all $\alpha \in u_q$ and all $\xi \in (s_p \cap E_{p,\alpha}) \setminus i_q$,

$$h_p(\xi, f_\alpha(\xi)) = 1 \Leftrightarrow \alpha \in Y.$$

Lemmas 2.2, 2.3 and 2.4 show that various sets are dense in $\mathbb{P}_{\bar{f},\bar{C},Y}$.

Lemma 2.2. *For each $\gamma \in \eta$, the set of $p \in \mathbb{P}_{\bar{f},\bar{C},Y}$ with $\gamma \in u_p$ is dense in $\mathbb{P}_{\bar{f},\bar{C},Y}$.*

Proof. Fix a condition $\langle u, i, h, \bar{E}, s\rangle$ in $\mathbb{P}_{\bar{f},\bar{C},Y}$. If $\gamma \in u$ we are done, so suppose otherwise. Let

- $u' = u \cup \{\gamma\}$;
- $i' = \min \bigcap \{C_{\{\alpha,\beta\}} : \{\alpha, \beta\} \in [u']^2\} \setminus i$;
- $\bar{F} = \langle F_\beta : \beta \in u'\rangle$ be such that $F_\beta = E_\beta$ for all $\beta \in u$, and $F_\gamma = \emptyset$;
- $h' = h \cup (((i' \times \sup\{f_\alpha(\xi) + 1 : (\alpha, \xi) \in u' \times i'\}) \setminus \mathrm{dom}(h)) \times \{0\})$.

Then $\langle u', i', h', \bar{F}, s\rangle$ is a condition in $\mathbb{P}_{\bar{f},\bar{C},Y}$ below $\langle u, i, h, \bar{E}, s\rangle$. □

Lemma 2.3. *For every $p \in \mathbb{P}_{\bar{f},\bar{C},Y}$ and every $\gamma < \omega_1$, there exists a $q \leq p$ in $\mathbb{P}_{\bar{f},\bar{C},Y}$ with $i_q \geq \gamma$ and*

$$i_p \in s_q \cap \bigcap\{E_{q,\beta} : \beta \in u_q\}.$$

Proof. Given p, define q by setting $u_q = u_p$,

$$i_q = \min(\bigcap\{C_{\{\alpha,\beta\}} : \{\alpha, \beta\} \in [u_p]^2\}) \setminus (\gamma \cup (i_p + 1)),$$

and $E_{q,\beta} = E_{p,\beta} \cup \{i_p\}$ for all $\beta \in u_p$. Let $s_q = s_p \cup \{i_p\}$.

It remains to extend h_p (whose domain is $i_p \times j_p$) to

$$h_q \colon i_q \times \sup\{f_\alpha(\xi) + 1 : (\alpha, \xi) \in u_p \times i_q\} \to 2$$

so that, for all $\alpha \in u_p$, $h_q(i_p, f_\alpha(i_p)) = 1$ if and only if $\alpha \in Y$. Since $i_p \in C_{\{\alpha,\beta\}}$ for all $\{\alpha, \beta\} \in [u_p]^2$, there cannot be distinct $\alpha, \beta \in u_p$ such that $f_\alpha(i_p) = f_\beta(i_p)$. It follows that such an h_q exists. □

Lemmas 2.2 and 2.3 give the following.

Lemma 2.4. *The following sets are dense in $\mathbb{P}_{\bar{f},\bar{C},Y}$.*

1. *For each $\gamma < \omega_1$, the set of $p \in \mathbb{P}_{\bar{f},\bar{C},Y}$ such that $i_p \geq \gamma$.*
2. *For each $\gamma < \omega_1$ and each $\alpha < \eta$, the set of $p \in \mathbb{P}_{\bar{f},\bar{C},Y}$ such that $\alpha \in u_p$ and $\sup(E_{p,\alpha}) \geq \gamma$.*

2.5 Remark. The partial order $\mathbb{P}_{\bar{f},\bar{C},Y}$ is σ-closed; moreover, every descending ω-sequence of conditions in $\mathbb{P}_{\bar{f},\bar{C},Y}$ has a greatest lower bound. To see this, suppose that $\langle p_n : n \in \omega\rangle$ is a sequence of conditions in $\mathbb{P}_{\bar{f},\bar{C},Y}$ such that $p_n \geq p_{n+1}$ for all $n \in \omega$. Let

- $u_q = \bigcup_{n\in\omega} u_{p_n}$;
- $i_q = \sup_{n\in\omega} i_{p_n}$;
- $s_q = \bigcup_{n\in\omega} s_{p_n}$;
- $E_{q,\beta} = \bigcup_{n\in\omega} E_{p_n,\beta}$, for each $\beta \in u_q$;
- $h_q = \bigcup_{n\in\omega} h_{p_n}$.

Then

$$q = \langle u_q, i_q, h_q, \langle E_{q,\beta} : \beta \in u_q\rangle, s_q\rangle$$

is the greatest lower bound for $\{p_n : n \in \omega\}$ in $\mathbb{P}_{\bar{f},\bar{C},Y}$.

Now we can apply the proof of Lemma 2.3 to q. That is, let

$$E_{q',\beta} = E_{q,\beta} \cup \{i_q\}$$

for each $\beta \in u_q$ and let $s_{q'} = s_q \cup \{i_q\}$. Let

$$i_{q'} = \min(\bigcap\{C_{\{\alpha,\beta\}} : \{\alpha,\beta\} \in [u_q]^2\}) \setminus i_q.$$

Since i_q is in $C_{\{\alpha,\beta\}}$ for each $\{\alpha,\beta\} \in [u_q]^2$, there is a function $h_{q'}$ on

$$i_{q'} \times \sup\{f_\alpha(\xi) + 1 : (\alpha,\xi) \in u_q \times i_{q'}\}$$

extending h_q such that, for each $\alpha \in u_q$, $h_{q'}(i_q, f_\alpha(i_q)) = 1$ if and only if $\alpha \in Y$. Then

$$q' = \langle u_q, i_q + 1, h_{q'}, \langle E_{q',\beta} : \beta \in u_q\rangle, s_{q'}\rangle$$

is in $\mathbb{P}_{\bar{f},\bar{C},Y}$, and $q' \leq q$.

Given a V-generic filter $G \subseteq \mathbb{P}_{\bar{f},\bar{C},Y}$, we let

$$S_G = \bigcup\{s_p : p \in G\},$$

$$h_G = \bigcup\{h_p : p \in G\}$$

and, for each $\beta \in \eta$, we let

$$E_{G,\beta} = \bigcup\{E_{p,\beta} : p \in G, \beta \in u_p\}.$$

Remark 2.5 shows that $\omega_1^{V[G]} = \omega_1^V$, and that S_G is a stationary subset of ω_1^V in $V[G]$. Lemma 2.4 shows that each $E_{G,\beta}$ ($\beta \in \eta$) is a club subset of ω_1^V, and that

$$\mathrm{dom}(h_G) = \omega_1^V \times \sup\{f_\alpha(\xi) + 1 : \alpha < \eta, \xi < \omega_1\}$$

(which will be $\omega_1^V \times \omega_1^V$ in our applications). By Remark 2.5 and Lemma 2.11 below, forcing with $\mathbb{P}_{\bar{f},\bar{C},Y}$ preserves all cardinalities and cofinalities.

2.6 Remark. If $G \subseteq \mathbb{P}_{\bar{f},\bar{C},Y}$ is a V-generic filter, then, in $V[G]$, Y is the set of $\alpha < \eta$ such that $h_G(\xi, f_\alpha(\xi)) = 1$ for stationarily many $\xi < \omega_1$ (all of which are in S_G). This implies that Y is in any inner model of ZF containing h_G which is correct about NS_{ω_1} and contains a sequence $\langle F_\alpha : \alpha < \eta\rangle$ such that each F_α is nonempty subset of the corresponding $[f_\alpha]_{\mathrm{NS}_{\omega_1}}$. The inner model $L(\mathcal{P}(\omega_1))$ is one such model, if $\eta \leq \omega_2$ and for each $\alpha < \eta$ the function f_α is a canonical function for α (see Theorem 2.12 and the paragraph before it). Note that it is not necessary for $\bar{f}$ and $\bar{C}$ to be elements of the inner model.

2.7 Remark. In the definition of the partial order $\mathbb{P}_{\bar{f},\bar{C},Y}$, the functions f_α are required only to differ pairwise on clubs, not necessarily to dominate one another. We use canonical functions (from ω_1 to ω_1) only because their NS_{ω_1}-classes are definable in $L(\mathcal{P}(\omega_1))$.

We refer the reader to [3] for the definition of countable support iterations. It is a standard fact, and easy to see, that a countable support iteration of σ-closed partial orders is σ-closed. We formulate an abstract consequence of Remark 2.5.

2.8 Remark. Suppose that P is a partial order and that $\underset{\sim}{Q}$ is a P-name for a partial order on a subset of the ground model. We say that $\underset{\sim}{Q}$ has *generic lower bounds* if whenever

- θ is a cardinal greater than $2^{|P*\underset{\sim}{Q}|}$;
- X is a countable elementary submodel of $H(\theta)$ with $P * \underset{\sim}{Q} \in X$;
- $G \subseteq (P * \underset{\sim}{Q}) \cap X$ is X-generic for $P * \underset{\sim}{Q}$;
- $p \in P$ is a lower bound for the restriction of G to P,

there exists an x such that $(p, \check{x})$ is a lower bound for G. The construction of the condition q in Remark 2.5 shows that whenever $\underset{\sim}{Q}$ is a P-name for a partial order of the form $\mathbb{P}_{\bar{f},\bar{C},Y}$, $\underset{\sim}{Q}$ has generic lower bounds.

Given a condition p in a forcing iteration, we let $\mathrm{supp}(p)$ denote the support of p (we also use $\sup(u)$ to mean the supremum of a set of ordinals u).

2.9 Remark. We say that a condition p in an forcing iteration $\langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \theta\rangle$ is *fully realized* if, for each $\alpha \in \mathrm{supp}(p)$, $p(\alpha)$ is $\check{x}$ (relative to P_α), for some set x. If $\langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \theta\rangle$ is a countable support iteration such that each $\underset{\sim}{Q}_\alpha$ has generic lower bounds, then (letting P_θ be the countable support limit of this iteration) whenever

- χ is a cardinal greater than $2^{|P_\theta|}$;
- X is a countable elementary submodel of $H(\chi)$ with $P_\theta \in X$;

- $G \subseteq P_\theta \cap X$ is X-generic for $P * \underset{\sim}{Q}$;

there exists fully realized q which is a lower bound for G. This follows easily by induction on the elements of $X \cap \theta$.

We turn now to showing that a countable support iteration of partial orders of the form $\mathbb{P}_{\bar{f},\bar{C},Y}$ is $\aleph_2$-c.c.. We start by noting a sufficient condition for compatibility.

2.10 Remark. Let p and q be conditions in $\mathbb{P}_{\bar{f},\bar{C},Y}$ such that $h_p = h_q$ and $s_p = s_q$. Then $i_p = i_q$ and $j_p = j_q$. Suppose also that for all $\beta \in u_p \cap u_q$, $E_{p,\beta} = E_{q,\beta}$. Let

- u' be $u_p \cup u_q$;
- i' be $\min(\bigcap\{C_{\{\alpha,\beta\}} : \{\alpha,\beta\} \in [u']^2\} \setminus (i_p + 1))$;
- j' be $\sup\{f_\alpha(\xi) + 1 : (\alpha,\xi) \in u' \times i'\}$;
- $h' = h_p \cup (((i' \times j') \setminus \mathrm{dom}(h_p)) \times \{0\})$.

Then the condition

$$\langle u', i', h', \langle E_{p,\beta} \cup \{i_p\} : \beta \in u_p\rangle \cup \langle E_{q,\beta} \cup \{i_p\} : \beta \in u_q\rangle, s_p\rangle$$

is a lower bound for $\{p, q\}$.

Let us say that partial order $\mathbb{P}$ satisfies the *regressive* $\aleph_2$-*chain condition (c.c.) on cofinality* $\aleph_1$ if for any sequence

$$\langle p_\alpha : \alpha < \omega_2\rangle$$

of conditions from $\mathbb{P}$, for some club $D \subseteq \omega_2$, there exists a regressive function r on the members of D of cofinality $\aleph_1$ such that for all $\gamma, \eta \in D$, if $r(\gamma) = r(\eta)$ then p_γ and p_η are compatible (this is a weakening of condition (c) from the first page of [2]).

Lemma 2.11. *Suppose that the Continuum Hypothesis holds. Let θ be an ordinal, and let $\langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \theta\rangle$ be a countable support forcing iteration such that each $\underset{\sim}{Q}_\alpha$ is a P_α-name for a partial order of the form $\mathbb{P}_{\bar{f},\bar{C},Y}$, where, in the P_α-extension,*

- *$\bar{f}$ is an η-sequence $\langle f_\alpha : \alpha < \eta\rangle$, for some ordinal η, of functions from ω_1 to ω_1;*
- *$\bar{C}$ witnesses that $\bar{f}$ is a mod-NS_{ω_1}-distinct sequence;*
- *Y is a subset of η.*

Let P_θ be the countable support limit of this iteration. Then P_θ satisfies the regressive $\aleph_2$-c.c. on cofinality $\aleph_1$.

Proof. Fix a sequence $\langle p_\alpha : \alpha < \omega_2\rangle$ consisting of distinct conditions in P_θ. By Remark 2.9, it suffices to consider the case where each p_α is fully realized (so for each $\alpha < \omega_2$ and $\gamma < \theta$, we can let $u_{p_\alpha(\gamma)}$, etc., refer to the parts of $p_\alpha(\gamma)$). Let U be the union of all sets of the form $u_{p_\alpha(\gamma)}$ for $\alpha < \omega_2$ and $\gamma < \theta$. By strengthening the conditions p_α if necessary (while keeping them fully realized) we may assume that $|U| = \aleph_2$. Let $\langle \xi_\delta : \delta < \omega_2\rangle$ enumerate U (not necessarily in increasing order). Let κ be a regular cardinal with P_θ and $\langle p_\alpha : \alpha < \omega_2\rangle$ in $H(\kappa)$. Let $\langle X_\nu : \nu < \omega_2\rangle$ be a continuous, $\subseteq$-increasing sequence of elementary submodels of $H(\theta)$ such that

- $\langle p_\alpha : \alpha < \omega_2\rangle$ and $\langle \xi_\delta : \delta < \omega_2\rangle$ are elements of X_0;
- for each $\nu < \omega_2$, $|X_\nu| = \aleph_1$ and $X_\nu \cap \omega_2 \in \omega_2$;
- for each $\nu < \omega_2$ not of countable cofinality, X_ν is closed under ω-sequences (here is where we use the Continuum Hypothesis).

Let D be the set of $\nu < \omega_2$ for which $X_\nu \cap \omega_2 = \nu$. Then D is a club subset of ω_2. For each $\nu \in D$ of cofinality $\aleph_1$ there exists an $\alpha < \nu$ such that, for all $\gamma \in \mathrm{supp}(p_\nu) \cap X_\nu$,

- $\gamma \in \mathrm{supp}(p_\alpha)$;
- $h_{p_\alpha(\gamma)} = h_{p_\nu(\gamma)}$;
- $s_{p_\alpha(\gamma)} = s_{p_\nu(\gamma)}$;
- $u_{p_\nu(\gamma)} \cap \{\xi_\delta : \delta < \nu\} \subseteq u_{p_\alpha(\gamma)}$;
- $\bar{E}_{p_\alpha(\gamma),\beta} = \bar{E}_{p_\nu(\gamma),\beta}$ for all $\beta \in u_{p_\nu(\gamma)} \cap \{\xi_\delta : \delta < \nu\}$.

Let $r(\nu)$ be any such α, for each such ν. Now suppose that $\nu < \mu$ in D have cofinality $\aleph_1$, and that $r(\nu) = r(\mu)$. To see that p_ν and p_μ are compatible, it suffices to show that for each $\gamma \in \mathrm{supp}(p_\nu) \cap \mathrm{supp}(p_\mu)$, $p_\nu(\gamma)$ and $p_\mu(\gamma)$ satisfy the conditions in Remark 2.10. Fix such a γ. Since $p_\nu \in X_\mu$, $\mathrm{supp}(p_\nu) \subseteq X_\mu$, so γ is in $\mathrm{supp}(p_\mu) \cap X_\mu$, and therefore in $\mathrm{supp}(p_{r(\nu)})$. Similarly, if β is in $u_{p_\nu(\gamma)} \cap u_{p_\mu(\gamma)}$, then since $u_{p_\nu(\gamma)} \subseteq X_\mu$, $\beta = \xi_\delta$ for some $\delta < \mu$, so β is in $u_{p_{r(\nu)}(\gamma)}$. This gives the desired conditions from Remark 2.10. □

We give an application of the material in this section, in the case $\eta = \omega_2$. Fix a sequence π_α ($\alpha < \omega_2$) such that each π_α is a surjection from ω_1 to α. For each $\alpha < \omega_2$, let $f_\alpha \colon \omega_1 \to \omega_1$ be such that $f_\alpha(\beta)$ is the ordertype of $\pi_\alpha[\beta]$, for each $\beta < \omega_1$ (such a function f_α is called a *canonical function* for α; it is not hard to see that it represents α in all generic ultrapowers formed by forcing with $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$). Then $\langle [f_\alpha]_{\mathrm{NS}_{\omega_1}} : \alpha < \omega_2\rangle$ is in $L(\mathcal{P}(\omega_1))$. Let $\bar{f} = \langle f_\alpha : \alpha < \omega_2\rangle$ (we do not need $\bar{f}$ to be in $L(\mathcal{P}(\omega_1))$ for Theorem 2.12 below). For each $\{\alpha, \beta\} \in [\omega_2]^2$, let $C_{\{\alpha,\beta\}}$ be a club of countable limit ordinals such that $f_{\min\{\alpha,\beta\}}(\gamma) < f_{\max\{\alpha,\beta\}}(\gamma)$ for all $\gamma \in C_{\{\alpha,\beta\}}$. Assuming that the Continuum Hypothesis holds, for each regular $\theta > 2^{\aleph_2}$ there is a countable support forcing iteration $\langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \theta\rangle$ such that each $\underset{\sim}{Q}_\alpha$ is a P_α-name for a partial order of the form $\mathbb{P}_{\bar{f},\bar{C},Y_\alpha}$, for $\bar{f}$ and $\bar{C}$ the fixed sets introduced in this paragraph, where by suitable bookkeeping the sets Y_α range through all subsets of ω_2 appearing in the final extension. Putting together the material in this section, then, we get the following.

Theorem 2.12. *If the Continuum Hypothesis holds, then there is a σ-closed, $\aleph_2$-c.c. partial order forcing that $\mathcal{P}(\omega_2) \subseteq L(\mathcal{P}(\omega_1))$.*

3 Coding $\mathcal{P}(\kappa)$ for larger κ

In this section we combine the argument of the previous section with the arguments of Section XVII §4 of [3] to produce a coding of the subsets of a cardinal κ larger than ω_2. The object is to obtain the coding in the previous section along with the existence of a canonical function from ω_1 to ω_1 for each $\alpha < \kappa$.

3.1 Remark. The forcing construction from Section XVII §4 of [3] makes the constant function from ω_1 to $\{\omega_1\}$ into a canonical function. Here we need only that there are canonical functions for each $\alpha < \kappa$, so our job is considerably easier.

Let us say that a *sequence of canonical functions*[1] is a sequence of functions $f_\alpha\colon \omega_1 \to \omega_1$ $(\alpha < \gamma)$ such that

- for all $\alpha < \beta < \gamma$, $\{\delta < \omega_1 : f_\alpha(\delta) < f_\beta(\delta)\}$ contains a club;
- for all $\alpha < \gamma$, all stationary $A \subseteq \omega_1$ and all $g\colon \omega_1 \to \omega_1$, if f_α *dominates g on A* (i.e. $g(\delta) < f_\alpha(\delta)$ for all $\delta \in A$), then there exists a $\beta < \alpha$ such that $A \cap \mathrm{eq}(g, f_\beta)$ is stationary.

The paragraph before Theorem 2.12 shows that a sequence of canonical functions of length ω_2 exists.

3.2 Remark. Arguing by induction on γ, one can show that if $\langle f_\alpha : \alpha < \gamma\rangle$ is a sequence of canonical functions, then the sequence $\langle [f_\alpha]_{\mathrm{NS}_{\omega_1}} : \alpha < \gamma\rangle$ is in $L(\mathcal{P}(\omega_1))$, and is the same for every sequence of canonical functions of length γ.

A partial order P is said to be (ω, ∞)-*distributive* if forcing with P adds no new ω-sequences of ordinals. The rest of this section proves the following theorem.

[1] Standard arguments, working by induction on α, show then that each f_α represents α in any generic ultrapower formed by forcing with $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$. The usual definition of canonical function allows functions which map into the ordinals, as opposed to only the countable ordinals. Since we will not need such functions, we modify the definition for this paper. It would interesting if the current argument could be modified to work with canonical functions in this generalized sense, while having $2^{\aleph_1} < \kappa$.

Theorem 3.3. *Suppose that the Continuum Hypothesis holds, and that κ is an infinite cardinal. There exists an (ω,∞)-distributive, $\aleph_2$-c.c. countable support iterated forcing construction forcing the following statements.*

- *There exists a sequence of canonical functions of length κ.*
- $\mathcal{P}(\kappa) \subseteq L(\mathcal{P}(\omega_1))$

The previous section establishes the theorem in the case that $\kappa \leq \omega_2$, so we assume that $\kappa > \omega_2$ here. The second conclusion of the theorem will be obtained by forcing that for each $Y \subseteq \kappa$ there exists an $h\colon \omega_1 \times \omega_1 \to \omega_1$ such that for each $\alpha < \kappa$, $\alpha \in Y$ if and only if $h(\xi, f_\alpha(\xi)) = 1$ for stationarily many $\xi < \omega_1$, for some fixed sequence of functions f_α ($\alpha < \kappa$) witnessing the first conclusion (and applying Remark 3.2).

We consider countable support forcing iterations $\bar{P} = \langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \theta\rangle$, for some ordinal $\theta \geq \kappa$, satisfying the following conditions, where for notational convenience we let G_α denote a generic filter for P_α.

1. For $\alpha < \kappa$, $\underset{\sim}{Q}_\alpha$ is a P_α-name for the partial order which adds a function from ω_1 to ω_1, by countable initial segments. We let $\dot{f}_\alpha$ be the natural $P_{\alpha+1}$-name for this function, and write f_α for $\dot{f}_{\alpha,G_{\alpha+1}}$ when this causes no confusion.

2. For $\gamma \in [\kappa, \kappa\cdot\kappa)$, letting $\alpha, \beta \in \kappa$ be such that $\gamma = \kappa\cdot(1+\alpha)+\beta$, if $\alpha \geq \beta$ then $\underset{\sim}{Q}_\gamma$ is a P_γ-name for the trivial partial order (which we take to be the partial order on $\{\emptyset\}$), otherwise it is a P_γ-name for the partial order which adds by countable initial segments a club subset of the set $\{\xi < \omega_1 : f_\alpha(\xi) < f_\beta(\xi)\}$. Again, we let $\dot{C}_{\{\alpha,\beta\}}$ be the natural $P_{\gamma+1}$-name for this club, and we write $C_{\{\alpha,\beta\}}$ for $\dot{C}_{\{\alpha,\beta\},G_{\gamma+1}}$ when convenient.

3. For $\alpha \in [\kappa\cdot\kappa, \theta)$ of the form $\gamma + 2n$, for γ a limit ordinal and $n \in \omega$, $\underset{\sim}{Q}_\alpha$ is a P_α-name such that, for some $\delta < \kappa$ and some P_α-names $\dot{g}_\alpha$ and $\dot{A}_\alpha$,
   - $\dot{A}_\alpha$ is a P_α-name for a subset of ω_1;
   - $\dot{g}_\alpha$ is a P_α-name for a function from ω_1 to ω_1 which is dominated by f_δ on $\dot{A}_{\alpha,G_\alpha}$;
   - if, in $V[G_\alpha]$, $\dot{A}_{\alpha,G_\alpha}$ is stationary and there does not exist $\beta < \delta$ such that $\dot{A}_{\alpha,G_\alpha} \cap \mathrm{eq}(f_\beta, \dot{g}_{\alpha,G_\alpha})$ is stationary, then $\underset{\sim}{Q}_{\alpha,G_\alpha}$ is the partial order which adds a club subset of ω_1 disjoint from $\dot{A}_{\alpha,G_\delta}$, by countable initial segments;
   - if, in $V[G_\alpha]$, $\dot{A}_{\alpha,G_\alpha}$ is not stationary or there does exist such a β, then $\underset{\sim}{Q}_{\alpha,G_\alpha}$ is the trivial partial order.

4. For $\alpha \in [\kappa\cdot\kappa, \theta)$ of the form $\gamma + 2n + 1$, for γ a limit ordinal and $n \in \omega$, $\underset{\sim}{Q}_\alpha$ is a P_α-name for the partial order $\mathbb{P}_{\bar{f},\bar{C},\dot{Y}_\alpha}$, where

- $\bar{f} = \langle f_\alpha : \alpha < \kappa \rangle$;
- $\bar{C} = \{C_{\{\alpha,\beta\}} : \{\alpha, \beta\} \in [\kappa]^2\}$;
- $\dot{Y}_\alpha$ is a P_α-name for a subset of κ.

We let P_θ denote the countable support limit of $\bar{P}$, and we let P'_θ denote the set of fully realized conditions in P_θ (see Remark 2.9). We will show by induction on $\theta \geq \kappa$ that for any iteration satisfying conditions (1)-(4) above, the following properties hold.

(a) P_θ is (ω, ∞)-distributive;

(b) P'_θ is dense in P_θ;

(c) For every α as in case (4) of the iteration, if S_α is the generic set S added at stage α, then the remainder of the iteration P_θ after stage α preserves the stationarity of S_α.

Given that (b) holds, one can show that the corresponding P_θ satisfies the regressive $\aleph_2$-c.c. on cofinality $\aleph_1$ by a routine modification of the proof of Lemma 2.11. Given this, we may fix a cardinal θ such that $\theta^\kappa = \theta$, and an iteration satisfying conditions (1)-(4) along with the following two conditions.

5. For all $\beta \in [\kappa \cdot \kappa, \theta)$ and $\delta < \kappa$, if $\dot{g}$ and $\dot{A}$ are such that $\dot{A}$ is a P_β-name for a subset of ω_1 and $\dot{g}$ is a P_β-name for a function from ω_1 to ω_1 which is dominated by f_δ on $\dot{A}_{\alpha,G_\alpha}$, there are cofinally many $\alpha \in [\beta, \theta)$ in case (3) above such that $\dot{g}_\alpha$ and $\dot{A}_\alpha$ are the natural reinterpretations of $\dot{g}$ and $\dot{A}$ as P_α-names.

6. For every $\beta \in [\kappa \cdot \kappa, \theta)$, and every P_β-name $\dot{Y}$ for a subset of κ, there is an $\alpha \in [\beta, \theta)$ in case (4) such that $\dot{Y}_\alpha$ is $\dot{Y}$ (again, reinterpreted).

The proof of Theorem 3.3 will then be complete. We briefly review the argument that each f_δ is a canonical function for δ. The first part of the definition is satisfied by the definition of the first $\kappa \cdot \kappa$ stages of the iteration. For the second, suppose that $G \subseteq P_\theta$ is a V-generic filter, and that, in $V[G]$, A is a subset of ω_1 and g is a function from ω_1 to ω_1 dominated by f_δ on A. We may fix a $\gamma \in [\kappa \cdot \kappa, \theta)$ and P_γ-names $\dot{g}$ and $\dot{A}$ such that $\dot{g}_G = g$ and $\dot{A}_G = A$. Then the sets $A \cap \mathrm{eq}(g, f_\beta)$ exist in the P_γ-extension, for each $\beta < \delta$. If $\rho \in [\gamma, \theta)$ is a stage in case (3) with respect to $\dot{g}$ and $\dot{A}$, and either A is nonstationary or one of the sets $A \cap \mathrm{eq}(g, f_\beta)$ is stationary in $V[G_\rho]$, then $\underset{\sim}{Q}_{\rho,G_\rho}$ is the trivial partial order. Otherwise, the partial order $\underset{\sim}{Q}_{\rho,G_\rho}$ destroys the stationarity of A. The cofinality condition in case (5) is then essential, but our argument does not require us to preserve the stationarity of a witness to a challenge given by g and A, since all possible witnesses exist as soon as g and A do, and if they all fail then the stationarity of A is destroyed.

A standard elementary submodel argument shows that the statements (a) and (b) hold for $\theta = \kappa$ (note that (c) holds trivially for $\theta \leq \kappa \cdot \kappa$). By Remark

2.8, the statements (a) and (b) hold for $\theta + 1$ if they hold for a given $\theta \geq \kappa$. It suffices then to show that (a)-(c) hold in the case where $\theta > \kappa$ is a limit ordinal, assuming that (a) and (b) hold for all $\alpha \in [\kappa \cdot \kappa, \theta)$. Let us fix such a θ for the rest of the paper.

Fix a regular cardinal χ greater than $2^{|P_\theta|}$. Let p be a condition in P_θ, let τ be a P_θ-name for an ω-sequence of ordinals and let σ be a P_θ-name for a club subset of ω_1. Let N be a countable elementary submodel of $H(\chi)$ with p, τ and σ in N. Let G be an N-generic filter for P_θ with $p \in G$. For each $\alpha \in N \cap (\theta+1)$, let G_α be $\{p \restriction \alpha : p \in G\}$. Let q be the function on θ which takes the value $1_{\underset{\sim}{Q}_\alpha}$ for all $\alpha \in \theta \setminus N$ (which is $\check{\emptyset}$ is all cases except for (4), and $\check{x}$ for $x = \langle \emptyset, \emptyset, \emptyset, \emptyset, \emptyset \rangle$ in case (4)), and which is defined as follows on $N \cap \theta$.

- For each $\alpha \in N \cap \kappa$, $q(\alpha)$ is $\check{x}$, where x is the function determined by the α-th coordinates of the elements of G, extended by taking the value o.t.$(N \cap \alpha)$ at $N \cap \omega_1$.
- For each $\alpha \in N \cap [\kappa, \kappa \cdot \kappa)$ in the trivial case, $q(\alpha)$ is $\check{\emptyset}$.
- For each $\alpha \in N \cap [\kappa, \kappa \cdot \kappa)$ in the nontrivial case, $q(\alpha)$ is $\check{x}$, where x is the club subset of $N \cap \omega_1$ determined by the α-th coordinates of the elements of G, extended by adding the ordinal $N \cap \omega_1$.
- For each $\alpha \in N$ in case (3) for which $\underset{\sim}{Q}_\alpha$ is forced by some condition in G_α to be trivial, $q(\alpha) = \check{\emptyset}$.
- For each $\alpha \in N$ in case (3) for which $\underset{\sim}{Q}_\alpha$ is forced by some condition in G_α to be nontrivial, $q(\alpha)$ is $\check{x}$, where x is the club subset of $N \cap \omega_1$ determined by the α-th coordinates of the elements of G, extended by adding the ordinal $N \cap \omega_1$.
- For each $\alpha \in N$ in case (4), $q(\alpha)$ is $\check{x}$, for x the condition q described in Remark 2.5, with respect to any cofinal ω-sequence among the collection of sets y such that $\check{y}$ is the α-th coordinate of a condition in $G_{\alpha+1} \cap P'_{\alpha+1}$.

To check that (a) and (b) hold, it suffices to see that q is in P_θ, as then it is easy to see that it is fully realized and below each element of G, (so it decides all of τ). We show by induction on $\alpha \in N \cap (\theta + 1)$ that $q \restriction \alpha$ is in P_α and that $q \restriction \alpha$ is below each element of G_α. This is straightforward in the cases where $\alpha = 0$ or α is a limit ordinal. The successor cases of the form $\alpha + 1$, where α is in cases (1) or (4), or $\underset{\sim}{Q}_\alpha$ is forced by some condition in G_α to be the trivial partial order, are also straightforward. For α in the nontrivial subcase of case (2), the fact that the sequence of values $\langle \text{o.t.}(N \cap \beta) : \beta \in N \cap \kappa \rangle$ is increasing implies that $N \cap \omega_1$ is forced by $q \restriction \alpha$ to be in the desired set.

Now we consider $\alpha \in N$ in case (3) for which $\underset{\sim}{Q}_\alpha$ is forced by some condition p_0 in $G \cap P_\alpha$ to be nontrivial. We want to see that $q \restriction \alpha$ forces that $N \cap \omega_1$ is not in $\dot{A}_\alpha$. To see this, suppose to the contrary that $r \leq q \restriction \alpha$ is in P_α and forces the opposite. We may fix $\delta \in N \cap \kappa$ such that $q \restriction \alpha$ forces that $\dot{g}_\alpha$ is dominated on $\dot{A}_\alpha$ by $\dot{f}_\delta$. Strengthening r, we may assume that it forces a value ξ to $\dot{g}_\alpha(N \cap \omega_1)$. Since $q \restriction \alpha$ forces that $f_\delta(N \cap \omega_1) = \text{o.t.}(N \cap \delta)$, ξ is less than o.t.$(N \cap \delta)$, so ξ is o.t.$(N \cap \gamma)$ for some $\gamma \in N \cap \delta$. Then there is a P_α-name $\dot{C}$ in N for a subset of ω_1 which p_0 forces to be a club disjoint from the set of $\zeta \in \dot{A}_\alpha$ for which $f_\gamma(\zeta) = \dot{g}_\alpha(\zeta)$. Then r forces that $N \cap \omega_1$ is in $\dot{C}$, but also that $\dot{f}_\gamma(N \cap \omega_1) = \dot{g}_\alpha(N \cap \omega_1)$ and that $N \cap \omega_1$ is in $\dot{A}_\alpha$, giving a contradiction.

Finally, we let q' be the condition q defined above, except that for each $\alpha \in N$ in case (4), $q'(\alpha)$ is a P_α-name for the condition q' described in Remark 2.5, with respect to $q(\alpha)$. Then q' is not fully realized (note that the i and h parts of the α-th coordinates of q' may not be decided), but it does force the ordinal $N \cap \omega_1$ into S_α, for each such α. Since q' also forces $N \cap \omega_1$ into the realization of σ, σ is not forced by p to make any such S_α nonstationary.

References

[1] A. Kanamori, **The Higher Infinite**, Springer-Verlag, 2003

[2] S. Shelah, *A weak generalization of MA to higher cardinals*, Israel J. Math. 30 (1978) 4, 297-306. Publication number 80.

[3] S. Shelah, **Proper and Improper Forcing**, Springer-Verlag, 1998