SMALL–LARGE SUBGROUPS OF THE REALS

ANDRZEJ ROSŁANOWSKI AND SAHARON SHELAH

Abstract. We are interested in subgroups of the reals that are small in one and large in another sense. We prove that, in ZFC, there exists a non–meager Lebesgue null subgroup of $\mathbb{R}$, while it is consistent that there there is no non–null meager subgroup of $\mathbb{R}$. This answers a question from Filipczak, Rosłanowski and Shelah [5].

1. Introduction

Subgroups of the reals which are small in one and large in another sense were crucial in Filipczak, Rosłanowski and Shelah [5]. If there is a non–meager Lebesgue null subgroup of $(\mathbb{R},+)$, then there is no translation invariant Borel hull operation on the σ–ideal $\mathcal{N}$ of Lebesgue null sets. That is, there is no mapping ψ from $\mathcal{N}$ to Borel sets such that for each null set $A \subseteq \mathbb{R}$:

- $A \subseteq \psi(A)$ and $\psi(A)$ is null, and
- $\psi(A+t) = \psi(A) + t$ for every $t \in \mathbb{R}$.

Parallel claims hold true if "Lebesgue null" is interchanged with "meager" and/or $(\mathbb{R},+)$ is replaced with $({}^{\omega}2,+_2)$.

If $\mathcal{M}$ is the σ–ideal of meager subsets of $\mathbb{R}$ (and $\mathcal{N}$ is the null ideal on $\mathbb{R}$) and $\{\mathcal{I},\mathcal{J}\} = \{\mathcal{N},\mathcal{M}\}$, then various set theoretic assumptions imply the existence of a subgroup of $\mathbb{R}$ which belongs to $\mathcal{I}$ but not to $\mathcal{J}$. But in [5, Problem 4.1] we asked if the existence of such subgroups can be shown in ZFC. This question is interesting *per se*, regardless of its connections to translation invariant Borel hulls.

The present paper presents two theorems. First, in Theorem 2.3 we give ZFC examples of null non-meager subgroups of $({}^{\omega}2,+_2)$ and $(\mathbb{R},+)$, respectively. Next in Theorem 4.1 we show that it is consistent with ZFC that every meager subgroup of $({}^{\omega}2,+_2)$ and/or $(\mathbb{R},+)$ has Lebesgue measure zero. This answers

Date: May 23, 2016.

2010 *Mathematics Subject Classification.* Primary 03E35; Secondary: 28A05, 54A05.

Key words and phrases. Null ideal, meager ideal, additive subgroups of the reals, Borel hulls, forcing.

Both authors acknowledge support from the United States-Israel Binational Science Foundation (Grant no. 2010405). Publication 1081 of the second author.

[5, Problem 4.1]. Also, our results give another example of a strange asymmetry between measure and category.

Notation Our notation is rather standard and compatible with that of classical textbooks (like Jech [6] or Bartoszyński and Judah [1]). However, in forcing we keep the older convention that *a stronger condition is the larger one.*

(1) The Cantor space ${}^{\omega}2$ of all infinite sequences with values 0 and 1 is equipped with the natural product topology, the product measure λ and the group operation of coordinate-wise addition $+_2$ modulo 2.
(2) Ordinal numbers will be denoted be the lower case initial letters of the Greek alphabet $\alpha, \beta, \gamma, \delta$. Finite ordinals (non-negative integers) will be denoted by letters i, j, k, ℓ, m, n while integers will be called L, M.
(3) Most of our intervals will be intervals of non-negative integers, so $[m, n) = \{k \in \omega : m \leq k < n\}$ etc. They will be denoted by letter J (with possible indices). However, we will also use the notation $[0, 1)$ to denote the unit interval of reals.
(4) The Greek letter κ will stand for an uncountable cardinal such that $\kappa^{\aleph_0} = \kappa \geq \aleph_2$.
(5) For a forcing notion $\mathbb{P}$, all $\mathbb{P}$–names for objects in the extension via $\mathbb{P}$ will be denoted with a tilde below (e.g., $\underset{\sim}{\tau}$, $\underset{\sim}{X}$), and $\underset{\sim}{G}_{\mathbb{P}}$ will stand for the canonical $\mathbb{P}$–name for the generic filter in $\mathbb{P}$.
(6) We fix a well ordering $\prec^*$ of all hereditarily finite sets.
(7) The set of all partial finite functions with domains included in ω and with values in 2 is denoted ${}^{\underline{\omega}}2$.

2. Null non–meager

Here we will give a ZFC construction of a non–meager Lebesgue null subgroup of the reals. The main construction is done in ${}^{\omega}2$ and then we transfer it to $\mathbb{R}$ using the standard binary expansion $\mathbf{E}$.

Definition 2.1. Let $D_0^\infty = \{x \in {}^{\omega}2 : (\exists^\infty i < \omega)(x(i) = 0)\}$ and for $x \in D_0^\infty$ let $\mathbf{E}(x) = \sum\limits_{i=0}^{\infty} x(i) 2^{-(i+1)}$.

Proposition 2.2. (1) *The function $\mathbf{E} : D_0^\infty \longrightarrow [0, 1)$ is a continuous bijection, it preserves both the measure and the category.*

(2) *Assume that*
  (a) *$x, y, z \in D_0^\infty$, $\mathbf{E}(z) = \mathbf{E}(x) + \mathbf{E}(y)$ modulo 1, and*
  (b) *$n < m < \omega$ and both $x{\restriction}[n, m]$ and $y{\restriction}[n, m]$ are constant.*
  Then $z{\restriction}[n, m-1]$ is constant.
(3) *Assume that*
  (a) *$x, y \in D_0^\infty$, $0 < \mathbf{E}(x)$ and $\mathbf{E}(y) = 1 - \mathbf{E}(x)$,*
  (b) *$n < m < \omega$ and $x{\restriction}[n, m]$ is constant.*

Then $y\restriction[n, m-1]$ *is constant.*

Proof. (1) Well known, cf. Bukovský [4, §2.4].
(2,3) Straightforward (just consider the possible constant values and analyze how the addition is performed). □

Theorem 2.3. (1) *There exists a null non-meager subgroup of* $({}^{\omega}2, +_2)$.
(2) *There exists a null non-meager subgroup of* $(\mathbb{R}, +)$.

Proof. (1) For $k \in \omega$ let $n_k = \frac{1}{2}k(k+1)$ and let D be a non-principal ultrafilter on ω. Define

$$H_D = \Big\{x \in {}^{\omega}2 : \big(\exists m < \omega\big)\big(\exists j < 2\big)\big(\big\{k > m : x\restriction[n_k, n_{k+1} - m) \equiv j\big\} \in D\big)\Big\}.$$

(i) H_D is a subgroup of $({}^{\omega}2, +_2)$.

Why? Suppose that $x_0, x_1 \in H_D$ and let $m_\ell < \omega$ and $j_\ell < 2$ be such that

$$A_\ell \stackrel{\rm def}{=} \big\{k > m_\ell : x_\ell\restriction[n_k, n_{k+1} - m_\ell) \equiv j_\ell\big\} \in D.$$

Let $m = \max(m_0, m_1)$ and $j = j_0 -_2 j_1$. Then $A_0 \cap A_1 \in D$ and for each $k \in A_0 \cap A_1$ we have $(x_0 -_2 x_2)\restriction[n_k, n_{k+1} - m) \equiv j$. Hence $x_0 -_2 x_1 \in H_D$.

(ii) $H_D \in \mathcal{N}$.

Why? For each $m < k < \omega$ and $j < 2$ we have

$$\lambda(\{x \in {}^{\omega}2 : x\restriction[n_k, n_{k+1} - m) \equiv j\}) = 2^{m-(k+1)}$$

and therefore for each $m < \omega$ and $j < 2$

$$\lambda(\{x \in {}^{\omega}2 : (\exists^\infty k)(x\restriction[n_k, n_{k+1} - m) \equiv j)\}) = 0.$$

Now note that $H_D \subseteq \bigcup\limits_{m<\omega}\bigcup\limits_{j<2}\big\{x \in {}^{\omega}2 : (\exists^\infty k)(x\restriction[n_k, n_{k+1} - m) \equiv j)\big\}$.

(iii) $H_D \notin \mathcal{M}$.

Why? Suppose that W is a dense Π^0_2 subset of ${}^{\omega}2$. Then we may choose an increasing sequence $\langle k_i : i \in \omega\rangle$ and a function $f \in {}^{\omega}2$ such that

$$\Big\{x \in {}^{\omega}2 : \big(\exists^\infty i\big)\big(x\restriction[n_{k_i}, n_{k_{i+1}}) = f\restriction[n_{k_i}, n_{k_{i+1}})\big)\Big\} \subseteq W.$$

Let $A = \bigcup\{[k_{2i}, k_{2i+1}) : i \in \omega\}$ and $B = \bigcup\{[k_{2i+1}, k_{2i+2}) : i \in \omega\}$. Then either $A \in D$ or $B \in D$. Let $x_A, x_B \in {}^{\omega}2$ be such that, for each $i \in \omega$,

$$\begin{array}{ll} x_A\restriction[n_{k_{2i}}, n_{k_{2i+1}}) \equiv 0, & x_A\restriction[n_{k_{2i+1}}, n_{k_{2i+2}}) = f\restriction n_{k_{2i+1}}, n_{k_{2i+2}}) \quad \text{and} \\ x_B\restriction[n_{k_{2i+1}}, n_{k_{2i+2}}) \equiv 0, & x_B\restriction[n_{k_{2i}}, n_{k_{2i+1}}) = f\restriction n_{k_{2i}}, n_{k_{2i+1}}). \end{array}$$

Then $x_A, x_B \in W$ and either $x_A \in H_D$ or $x_B \in H_D$. Consequently, $W \cap H_D \neq \emptyset$.

(2) Consider $H^*_D = \mathbf{E}[H_D \cap D^\infty_0] + \mathbb{Z}$. It follows from 2.2(1) that H^*_D is a Lebesgue null meager subset of $\mathbb{R}$. We will show that it is a subgroup of $(\mathbb{R}, +)$.

Suppose that $x_0, x_1 \in H_D \cap D_0^\infty$ and $L_0, L_1 \in \mathbb{Z}$ and we will argue that $(\mathbf{E}(x_0) + L_0) + (\mathbf{E}(x_1) + L_1) \in H_D^*$. Let $m_\ell < \omega$ be such that

$$A_\ell \stackrel{\rm def}{=} \big\{k > m_\ell : x_\ell{\restriction}[n_k, n_{k+1} - m_\ell) \text{ is constant } \big\} \in D$$

and let $m = \max(m_0, m_1) + 1$. Choose $y \in D_0^\infty$ and $M \in \{0,1\}$ such that $\mathbf{E}(x_0)+\mathbf{E}(x_1) = \mathbf{E}(y)+M$. It follows from 2.2(2) that for every $k \in A_0 \cap A_1$, $k > m$, we have that $y{\restriction}[n_k, n_{k+1} - m)$ is constant and since $A_0 \cap A_1 \in D$ we conclude $y \in H_D$. Consequently, $(\mathbf{E}(x_0)+L_0)+(\mathbf{E}(x_1)+L_1) = \mathbf{E}(y)+(M+L_0+L_1) \in H_D^*$.

Now assume that $x \in H_D \cap D_0^\infty$, $L \in \mathbb{Z}$ and we will argue that $-(\mathbf{E}(x)+L) \in H_D^*$. If $\mathbf{E}(x) = 0$ then the assertion is clear, so assume also $\mathbf{E}(x) > 0$. Let $m < \omega$ be such that

$$A \stackrel{\rm def}{=} \big\{k > m : x{\restriction}[n_k, n_{k+1} - m) \text{ is constant } \big\} \in D.$$

Choose $y \in D_0^\infty$ such that $1 - \mathbf{E}(x) = \mathbf{E}(y)$. It follows from 2.2(3) that for every $k \in A$, $k > m+1$, we have that $y{\restriction}[n_k, n_{k+1} - (m+1))$ is constant. Consequently, $y \in H_D$ and $-(\mathbf{E}(x) + L) = \mathbf{E}(y) - 1 - L \in H_D^*$. □

Remark 2.4. A somewhat simpler non–meager null subgroup of $({}^\omega 2, +_2)$ is

$$H_D^- = \Big\{x \in {}^\omega 2 : \big\{k \in \omega : x{\restriction}[n_k, n_{k+1}) \equiv 0\big\} \in D\Big\}.$$

The group H_D, however, was necessary for our construction of $H_D^* < \mathbb{R}$.

Corollary 2.5. *There exists no translation invariant Borel hull for the null ideal on ${}^\omega 2$ and/or on $\mathbb{R}$.*

3. Some technicalities

Here we prepare the ground for our consistency results.

3.1. **Moving from $\mathbb{R}$ to ${}^\omega 2$.** First, let us remind connections between the addition in $\mathbb{R}$ and that of ${}^\omega 2$ (via the binary expansion $\mathbf{E}$, see 2.1).

Definition 3.1. Let $J = [m, n)$ be a non-empty interval of integers and $c \in \{0,1\}$. For sequences $\rho, \sigma \in {}^J 2$ we define $\rho \circledast_c \sigma$ as the unique $\eta \in {}^J 2$ such that

$$\Big(\sum_{i=m}^{n-1} \rho(i) 2^{-(i+1)} + \sum_{i=m}^{n-1} \sigma(i) 2^{-(i+1)} + c \cdot 2^{-n}\Big) - \sum_{i=m}^{n-1} \eta(i) 2^{-(i+1)} \in \{0, 2^{-m}\}.$$

For notational convenience we also set $\rho \circledast_2 \sigma = \rho +_2 \sigma$ (coordinate-wise addition modulo 2).

The operation $\circledast_c$ is defined on the set ${}^J 2$, so it does depend on J. We may, however, abuse notation and use that same symbol $\circledast_c$ for various J.

Observation 3.2. *Let m, ℓ, n be integers such that $m < \ell < n$ and let $J = [m, n)$.*

(1) *For each* $c\in\{0,2\}$, $({}^J2,\circledast_c)$ *is an Abelian group.*
(2) *If* $\rho,\sigma\in {}^J2$ *and* $\rho(\ell)=\sigma(\ell)$, *then* $(\rho\circledast_0\sigma)\restriction[m,\ell)=(\rho\circledast_1\sigma)\restriction[m,\ell)$.
(3) *If* $\rho,\sigma\in {}^J2$ *and* $(\rho\circledast_0\sigma)(\ell)=0$, *then* $(\rho\circledast_0\sigma)\restriction[m,\ell)=(\rho\circledast_1\sigma)\restriction[m,\ell)$.
(4) *Suppose that* $r,s\in[0,1)$, $\rho,\sigma,\eta\in D_0^\infty$, $\mathbf{E}(\rho)=r$, $\mathbf{E}(\sigma)=s$ *and* $\mathbf{E}(\eta)=r+s$ *modulo 1. Then*
   - *if* $\sum\limits_{i\geq n}\big((\rho(i)+\sigma(i))/2^{i+1}\big)\geq 2^{-n}$, *then* $\eta\restriction J=(\rho\restriction J)\circledast_1(\sigma\restriction J)$;
   - *if* $\sum\limits_{i\geq n}\big((\rho(i)+\sigma(i))/2^{i+1}\big)< 2^{-n}$, *then* $\eta\restriction J=(\rho\restriction J)\circledast_0(\sigma\restriction J)$.

3.2. **The combinatorial heart of our forcing arguments.** For this subsection we fix a strictly increasing sequence $\bar{n}=\langle n_j:j<\omega\rangle\subseteq\omega$.

Definition 3.3. We define $\bar{m}[\bar{n}]=\langle m_i:i<\omega\rangle$, $\bar{N}[\bar{n}]=\langle N(i):i<\omega\rangle$, $\bar{J}[\bar{n}]=\langle J_i:i<\omega\rangle$, $\bar{H}[\bar{n}]=\langle H_i:i<\omega\rangle$, $\pi[\bar{n}]=\langle\pi_i:i<\omega\rangle$ and $\mathbf{F}[\bar{n}]$ as follows.

We set $m_0=0$ and then inductively for $i<\omega$ we let

$(*)_1$ $m_{i+1}=2^{n_{m_i}+1081}$.

Next, for $i<\omega$,

$(*)_2$ $N(i)=n_{m_i}$, $J_i=\big[N(2^i),N(2^{i+1})\big)$, and

$(*)_3$ $H_i=\big\{a\subseteq {}^{J_i}2:(1-2^{-N(2^i)})\cdot 2^{|J_i|}\leq|a|\big\}$.

We also set $\pi_i:|H_i|\longrightarrow H_i$ to be the $\prec^*$–first bijection from $|H_i|$ onto H_i. Finally, for $\eta\in\prod\limits_{m<\omega}(m+1)$ we let

$(*)_4$ $\mathbf{F}_0[\bar{n}](\eta)=\big\{x\in{}^\omega2:\big(\forall i<\omega\big)\big(x\restriction J_i\in\pi_i(\eta(|H_i|-1))\big)\big\}$ and
$\mathbf{F}[\bar{n}](\eta)=\big\{x\in{}^\omega2:\big(\forall^\infty i<\omega\big)\big(x\restriction J_i\in\pi_i(\eta(|H_i|-1))\big)\big\}$.

Lemma 3.4. *For every* $\eta\in\prod\limits_{m<\omega}(m+1)$, $\mathbf{F}_0[\bar{n}](\eta)\subseteq{}^\omega2$ *is a closed set of positive Lebesgue measure, and* $\mathbf{F}[\bar{n}](\eta)$ *is a* Σ^0_2 *set of Lebesgue measure 1.*

Proof. Note that $J_i\cap J_j=\emptyset$ and $|H_i|<|H_j|$ for $i<j$, and $\sum\limits_{i=0}^{\infty}2^{-N(2^i)}<1$. □

Lemma 3.5. *Let* $i<\omega$, $c\in\{0,2\}$ *and let* $\eta\in{}^{J_i}2$. *Suppose that for each* $\ell<2^i$ *and* $x<2$ *we are given a function* $\mathcal{Z}^x_\ell:H_i\longrightarrow{}^{J_i}2$ *such that* $\mathcal{Z}^x_\ell(a)\in a$ *for each* $a\in H_i$. *Then there are* $a^0,a^1\in H_i$ *such that for every* $\ell<2^i$ *there is* $k\in[m_{2^i+\ell},m_{2^i+\ell+1})$ *satisfying*

$$\big(\mathcal{Z}^0_\ell(a^0)\restriction[n_k,n_{k+1})\big)\circledast^k_c\big(\mathcal{Z}^1_\ell(a^1)\restriction[n_k,n_{k+1})\big)=\eta\restriction[n_k,n_{k+1}),$$

where $\circledast^k_c$ *denotes the operation* $\circledast_c$ *on* ${}^{[n_k,n_{k+1})}2$.

Proof. We start the proof with the following Claim.

Claim 3.5.1. *If* $\mathcal{A}\subseteq H_i$, $|\mathcal{A}|\leq 2^{|J_i|-N(2^i)-i}$ *and* $x<2$, *then there is* $b\in H_i$ *such that* $\mathcal{Z}^x_\ell(b)\notin\{\mathcal{Z}^x_\ell(a):a\in\mathcal{A}\}$ *for each* $\ell<2^i$.

Proof of the Claim. Note that $|\{\mathcal{Z}^x_\ell(a): \ell<2^i \ \&\ a\in\mathcal{A}\}| \leq 2^i\cdot 2^{|J_i|-N(2^i)-i} = 2^{|J_i|-N(2^i)}$, so letting $b={}^{J_i}2\setminus\{\mathcal{Z}^x_\ell(a):\ell<2^i \ \&\ a\in\mathcal{A}\}$ we have $b\in H_i$. Since $\mathcal{Z}^x_\ell(b)\in b$ we see that b is as required in the claim. $\square$

It follows from Claim 3.5.1 that we may pick sequences $\langle a^0_j : j<j^*\rangle\subseteq H_i$ and $\langle a^1_j : j<j^*\rangle\subseteq H_i$ with $\mathcal{Z}^x_\ell(a^x_{j_1})\neq\mathcal{Z}^x_\ell(a^x_{j_2})$ for $j_1<j_2<j^*$, $\ell<2^i$, $x<2$ and such that $j^*>2^{|J_i|-N(2^i)-i}$. Now, by induction on $\ell<2^i$, we choose sets $X_\ell, Y_\ell\subseteq j^*$ and integers $k_\ell\in[m_{2^i+\ell}, m_{2^i+\ell+1})$ such that the following demands are satisfied.

(i) $X_{\ell+1}\subseteq X_\ell\subseteq j^*$, $Y_{\ell+1}\subseteq Y_\ell\subseteq j^*$,

(ii) if $j_0\in X_\ell$ and $j_1\in Y_\ell$ then

$$\big(\mathcal{Z}^0_\ell(a^0_{j_0})\restriction[n_{k_\ell},n_{k_\ell+1})\big)\circledast^{k_\ell}_c\big(\mathcal{Z}^1_\ell(a^1_{j_1})\restriction[n_{k_\ell},n_{k_\ell+1})\big)=\eta\restriction[n_{k_\ell},n_{k_\ell+1}),$$

(iii) $\min\big(|X_\ell|,|Y_\ell|\big)\geq j^*\cdot 2^{N(2^i)-N(2^i+\ell+1)-\ell-1}$.

We stipulate $X_{-1}=Y_{-1}=j^*$ and we assume that $X_{\ell-1},Y_{\ell-1}$ have been already determined (and $\min\big(|X_{\ell-1}|,|Y_{\ell-1}|\big)\geq j^*\cdot 2^{N(2^i)-N(2^i+\ell)-\ell}$ if $\ell>0$). Let

$$\begin{aligned}X^*=\big\{&j\in X_{\ell-1}: |X_{\ell-1}|\cdot 2^{N(2^i+\ell)-N(2^i+\ell+1)-1}\leq\\ &\big|\{j'\in X_{\ell-1}:\mathcal{Z}^0_\ell(a^0_{j'})\restriction[N(2^i+\ell),N(2^i+\ell+1))=\mathcal{Z}^0_\ell(a^0_j)\restriction[N(2^i+\ell),N(2^i+\ell+1))\}\big|\big\},\\ Y^*=\big\{&j\in Y_{\ell-1}: |Y_{\ell-1}|\cdot 2^{N(2^i+\ell)-N(2^i+\ell+1)-1}\leq\\ &\big|\{j'\in Y_{\ell-1}:\mathcal{Z}^1_\ell(a^1_{j'})\restriction[N(2^i+\ell),N(2^i+\ell+1))=\mathcal{Z}^1_\ell(a^1_j)\restriction[N(2^i+\ell),N(2^i+\ell+1))\}\big|\big\}.\end{aligned}$$

Claim 3.5.2. $|X^*|\geq\frac12|X_{\ell-1}|$ *and* $|Y^*|\geq\frac12|Y_{\ell-1}|$.

Proof of the Claim. Assume towards contradiction that $|X^*|<\frac12|X_{\ell-1}|$. Then for some $\nu_0\in{}^{[N(2^i+\ell),N(2^i+\ell+1))}2$ we have

$$\big|\big\{j\in X_{\ell-1}\setminus X^*:\nu_0\subseteq\mathcal{Z}^0_\ell(a^0_j)\big\}\big|\geq|X_{\ell-1}\setminus X^*|\cdot 2^{N(2^i+\ell)-N(2^i+\ell+1)}>\frac12|X_{\ell-1}|\cdot 2^{N(2^i+\ell)-N(2^i+\ell+1)}.$$

Let $j\in X_{\ell-1}\setminus X^*$ be such that $\nu_0\subseteq\mathcal{Z}^0_\ell(a^0_j)$. Then $j\in X^*$, a contradiction.

Similarly for Y^*. $\square$

Claim 3.5.3. *For some* $k\in[m_{2^i+\ell},m_{2^i+\ell+1})$ *we have that both* $\big|\big\{\mathcal{Z}^0_\ell(a^0_j)\restriction[n_k,n_{k+1}): j\in X^*\big\}\big|>2^{n_{k+1}-n_k-1}$ *and* $\big|\big\{\mathcal{Z}^1_\ell(a^1_j)\restriction[n_k,n_{k+1}): j\in Y^*\big\}\big|>2^{n_{k+1}-n_k-1}$.

Proof of the Claim. Let

$$K^X=\big\{k\in[m_{2^i+\ell},m_{2^i+\ell+1}):|\{\mathcal{Z}^0_\ell(a^0_j)\restriction[n_k,n_{k+1}):j\in X^*\}|\leq 2^{n_{k+1}-n_k-1}\big\}$$

and

$$K^Y=\big\{k\in[m_{2^i+\ell},m_{2^i+\ell+1}):|\{\mathcal{Z}^1_\ell(a^1_j)\restriction[n_k,n_{k+1}):j\in Y^*\}|\leq 2^{n_{k+1}-n_k-1}\big\}.$$

Assume towards contradiction that $|K^X|\geq\frac12(m_{2^i+\ell+1}-m_{2^i+\ell})$. Then

$$|X^*|=|\{\mathcal{Z}^0_\ell(a^0_j):j\in X^*\}|\leq 2^{-1/2(m_{2^i+\ell+1}-m_{2^i+\ell})}\cdot 2^{|J_i|}<2^{|J_i|}\cdot 2^{-4N(2^i+\ell)}.$$

(Remember 3.3$(*)_1$.) Hence $|X_{\ell-1}| \leq 2^{|J_i|-4N(2^i+\ell)+1}$. If $\ell = 0$ then we get $2^{|J_i|-2N(2^i)} < j^* \leq 2^{|J_i|-4N(2^i)+1}$, which is impossible. If $\ell > 0$, then by the inductive hypothesis (iii) we know that $|X_{\ell-1}| \geq j^* \cdot 2^{N(2^i)-N(2^i+\ell)-\ell} > 2^{|J_i|-i-N(2^i+\ell)-\ell}$, so $3N(2^i+\ell)-1 < i+\ell$, a clear contradiction. Consequently $|K^X| < \frac{1}{2}(m_{2^i+\ell+1} - m_{2^i+\ell})$, and similarly $|K^Y| < \frac{1}{2}(m_{2^i+\ell+1} - m_{2^i+\ell})$. Pick $k \in [m_{2^i+\ell}, m_{2^i+\ell+1})$ such that $k \notin K^X \cup K^Y$. □

Now, let $k_\ell \in [m_{2^i+\ell}, m_{2^i+\ell+1})$ be as given by Claim 3.5.3. Necessarily the sets $\big\{\rho \in {}^{[n_{k_\ell}, n_{k_\ell+1})}2 : (\exists j \in X^*)((\mathcal{Z}^0_\ell(a^0_j){\restriction}[n_{k_\ell}, n_{k_\ell+1})) \circledast^{k_\ell}_c \rho = \eta{\restriction}[n_{k_\ell}, n_{k_\ell+1}))\big\}$ and $\big\{\mathcal{Z}^1_\ell(a^1_j){\restriction}[n_{k_\ell}, n_{k_\ell+1}) : j \in Y^*\big\}$ have non-empty intersection. Therefore, we may find $j_X \in X^*$ and $j_Y \in Y^*$ such that

$$\big(\mathcal{Z}^0_\ell(a^0_{j_X}){\restriction}[n_{k_\ell}, n_{k_\ell+1})\big) \circledast^{k_\ell}_c \big(\mathcal{Z}^1_\ell(a^1_{j_Y}){\restriction}[n_{k_\ell}, n_{k_\ell+1})\big) = \eta{\restriction}[n_{k_\ell}, n_{k_\ell+1}).$$

Set

$$X_\ell = \big\{j \in X_{\ell-1} : \mathcal{Z}^0_\ell(a^0_j){\restriction}[N(2^i+\ell), N(2^i+\ell+1)) = \mathcal{Z}^0_\ell(a^0_{j_X}){\restriction}[N(2^i+\ell), N(2^i+\ell+1))\big\},$$

and

$$Y_\ell = \big\{j \in Y_{\ell-1} : \mathcal{Z}^1_\ell(a^1_j){\restriction}[N(2^i+\ell), N(2^i+\ell+1)) = \mathcal{Z}^1_\ell(a^1_{j_Y}){\restriction}[N(2^i+\ell), N(2^i+\ell+1))\big\}.$$

By the definition of X^*, Y^* and by the inductive hypothesis (iii) we have

$$|X_\ell| \geq |X_{\ell-1}| \cdot 2^{N(2^i+\ell)-N(2^i+\ell+1)-1} \geq j^* \cdot 2^{N(2^i)-\ell-N(2^i+\ell+1)-1}$$

and similarly for Y_ℓ. Consequently, X_ℓ, Y_ℓ and k_ℓ satisfy the inductive demands (i)–(iii).

After the above construction is completed fix any $j_0 \in X_{2^i-1}$, $j_1 \in Y_{2^i-1}$ and consider $a^0 = a_{j_0}$ and $a^1 = a_{j_1}$. For each $\ell < 2^i$ we have $j_0 \in X_\ell$, $j_1 \in Y_\ell$ so

$$\big(\mathcal{Z}^0_\ell(a^0){\restriction}[n_{k_\ell}, n_{k_\ell+1})\big) \circledast^{k_\ell}_c \big(\mathcal{Z}^1_\ell(a^1){\restriction}[n_{k_\ell}, n_{k_\ell+1})\big) = \eta{\restriction}[n_{k_\ell}, n_{k_\ell+1}).$$

Hence $a^1, a^2 \in H_i$ are as required. □

3.3. **The $*$–Silver forcing notion.** The consistency result of the next section will be obtained using CS product of the following forcing notion $\mathbb{S}_*$.

Definition 3.6. (1) We define the $*$–Silver forcing notion $\mathbb{S}_*$ as follows.
A condition in $\mathbb{S}_*$ is a partial function $p : \mathrm{dom}(p) \longrightarrow \omega$ such that $\mathrm{dom}(p) \subseteq \omega$ is coinfinite and $p(m) \leq m$ for each $m \in \mathrm{dom}(p)$.
The order $\leq = \leq_{\mathbb{S}_*}$ of $\mathbb{S}_*$ is the inclusion, i.e., $p \leq q$ if and only if $p \subseteq q$.

(2) For $p \in \mathbb{S}_*$ and $1 \leq n < \omega$ we let $u(n, p)$ be the set of the first n elements of $\omega \setminus \mathrm{dom}(p)$ (in the natural increasing order). Then for $p, q \in \mathbb{S}_*$ we let $p \leq_n q$ if and only if $p \leq q$ and $u(n, q) = u(n, p)$.
We also define $p \leq_0 q$ as equivalent to $p \leq q$.

(3) Let $p \in \mathbb{S}_*$. We let $S(n, p)$ be the set of all functions $s : u(n, p) \longrightarrow \omega$ with the property that $s(m) \leq m$ for all $m \in u(n, p)$.

(4) We let $\mathop{\eta}\limits_{\sim}$ to be the canonical $\mathbb{S}_*$–name such that

$$\Vdash \mathop{\eta}\limits_{\sim} = \bigcup\{p : p \in \mathop{G}\limits_{\sim}{}_{\mathbb{S}_*}\}.$$

Remark 3.7. The forcing notion $\mathbb{S}_*$ may be represented as a forcing of the type $\mathbb{Q}^*_{\mathrm{w}\infty}(K,\Sigma)$ for some finitary creating pair (K,Σ) which captures singletons, see Rosłanowski and Shelah [8, Definition 2.1.10]. It is a close relative of the Silver forcing notion and, in a sense, it lies right above all $\mathbb{S}_n$'s studied for instance in Rosłanowski [7] and Rosłanowski and Steprāns [9].

Lemma 3.8. (1) $(\mathbb{S}_*, \leq_{\mathbb{S}_*})$ *is a partial order of size* $\mathfrak{c}$. *If* $p \in \mathbb{S}_*$ *and* $s \in S(n,p)$ *then* $p \cup s \in \mathbb{S}_*$ *is a condition stronger than* p.

(2) $\Vdash_{\mathbb{S}_*} \mathop{\eta}\limits_{\sim} \in \prod\limits_{m<\omega}(m+1)$ *and* $p \Vdash_{\mathbb{S}_*} p \subseteq \mathop{\eta}\limits_{\sim}$ *(for* $p \in \mathbb{S}_*$*).*

(3) *If* $p \in \mathbb{S}_*$ *and* $1 \leq n < \omega$, *then the family* $\{p \cup s : s \in S(n,p)\}$ *is an antichain pre-dense above* p.

(4) *The relations* $\leq_n$ *are partial orders on* $\mathbb{S}_*$, $p \leq_{n+1} q$ *implies* $p \leq_n q$.

(5) *Assume that* $\mathop{\tau}\limits_{\sim}$ *is an* $\mathbb{S}_*$*–name for an ordinal,* $p \in \mathbb{S}_*$, $1 \leq n, m < \omega$. *Then there is a condition* $q \in \mathbb{S}_*$ *such that* $p \leq_n q$, $\max\big(u(n+1,q)\big) > m$ *and for all* $s \in S(n,q)$ *the condition* $q \cup s$ *decides the value of* $\mathop{\tau}\limits_{\sim}$.

(6) *The forcing notion* $\mathbb{S}_*$ *satisfies Axiom A of Baumgartner* [2, §7] *as witnessed by the orders* $\leq_n$, *it is* ${}^\omega\omega$*–bounding and, moreover, every meager subset of* ${}^\omega 2$ *in an extension by* $\mathbb{S}_*$ *is included in a* Σ^0_2 *meager set coded in the ground model.*

Proof. Straightforward - the same as for the Silver forcing notion. □

Definition 3.9. Assume $\kappa^{\aleph_0} = \kappa \geq \aleph_2$.

(1) $\mathbb{S}_*(\kappa)$ is the CS product of κ many copies of $\mathbb{S}_*$. Thus
a condition p in $\mathbb{S}_*(\kappa)$ is a function with a countable domain $\mathrm{dom}(p) \subseteq \kappa$ and with values in $\mathbb{S}_*$, and
the order $\leq$ of $\mathbb{S}_*(\kappa)$ is such that
$p \leq q$ if and only if $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and $(\forall \alpha \in \mathrm{dom}(p))(p(\alpha) \leq_{\mathbb{S}_*} q(\alpha))$.

(2) Suppose that $p \in \mathbb{S}_*(\kappa)$ and $F \subseteq \mathrm{dom}(p)$ is a finite non-empty set and $\mu : F \longrightarrow \omega \setminus \{0\}$. Let $v(F,\mu,p) = \prod\limits_{\alpha \in F} u(\mu(\alpha), p(\alpha))$ and $T(F,\mu,p) = \prod\limits_{\alpha \in F} S(\mu(\alpha), p(\alpha))$.

If $\sigma \in T(F,\mu,p)$ then let $p|\sigma$ be the condition $q \in \mathbb{S}_*(\kappa)$ such that $\mathrm{dom}(q) = \mathrm{dom}(p)$ and $q(\alpha) = p(\alpha) \cup \sigma(\alpha)$ for $\alpha \in F$ and $q(\alpha) = p(\alpha)$ for $\alpha \in \mathrm{dom}(q) \setminus F$.

We let $p \leq_{F,\mu} q$ if and only if $p \leq q$ and $v(F,\mu,p) = v(F,\mu,q)$.

If μ is constantly n then we may write n instead of μ.

(3) Suppose that $p \in \mathbb{S}_*(\kappa)$ and $\bar{\underset{\sim}{\tau}} = \langle \underset{\sim}{\tau}_n : n < \omega\rangle$ is a sequence of names for ordinals. We say that p *determines* $\bar{\underset{\sim}{\tau}}$ *relative to* $\bar{F}$ if
- $\bar{F} = \langle F_n : n < \omega\rangle$ is a sequence of finite subsets of $\mathrm{dom}(p)$, and
- p forces a value to $\underset{\sim}{\tau}_0$ and for $1 \leq n < \omega$ and $\sigma \in T(F_n, n, p)$ the condition $p|\sigma$ decides the value of $\underset{\sim}{\tau}_n$.

Lemma 3.10. (1) *The forcing notion $\mathbb{S}_*(\kappa)$ satisfies $\mathfrak{c}^+$–chain condition.*

(2) *Suppose that $p \in \mathbb{S}_*(\kappa)$, $F \subseteq \mathrm{dom}(p)$ is finite non-empty, $\mu : F \longrightarrow \omega \setminus \{0\}$ and $\underset{\sim}{\tau}$ is a name for an ordinal. Then there is a condition $q \in \mathbb{S}_*(\kappa)$ such that $p \leq_{F,\mu} q$ and for every $\sigma \in T(F, \mu, q)$ the condition $q|\sigma$ decides the value of $\underset{\sim}{\tau}$.*

(3) *Suppose that $p \in \mathbb{S}_*(\kappa)$ and $\bar{\underset{\sim}{\tau}} = \langle \underset{\sim}{\tau}_n : n < \omega\rangle$ is a sequence of $\mathbb{S}_*(\kappa)$–names for objects from the ground model $\mathbf{V}$. Then there is a condition $q \geq p$ and a $\subseteq$–increasing sequence $\bar{F} = \langle F_n : n < \omega\rangle$ of finite subsets of* $\mathrm{dom}(q)$ *such that q determines $\bar{\underset{\sim}{\tau}}$ relative to $\bar{F}$.*

(4) *Assume $p, \bar{\underset{\sim}{\tau}}$ are as in (3) above and $p \Vdash$ "$\bar{\underset{\sim}{\tau}}$ is a sequence of elements of ${}^{\underline{\omega}}2$ with disjoint domains". Then there are a condition $q \geq p$ and an increasing sequence $\bar{F}$ of finite subsets of* $\mathrm{dom}(q)$ *and a function $f = (f_0, f_1) : \bigcup\limits_{1\leq n<\omega} T(F_n, n, q) \longrightarrow \omega \times {}^{\underline{\omega}}2$ such that $q|\sigma \Vdash \underset{\sim}{\tau}_{f_0(\sigma)} = f_1(\sigma)$ (for all $\sigma \in \mathrm{dom}(f)$) and the elements of $\langle \mathrm{dom}(f_1(\sigma)) : \sigma \in \bigcup_{n<\omega} T(F_n, n, q)\rangle$ are pairwise disjoint.*

Proof. The same as for the CS product of Silver or Sacks forcing notions, see e.g. Baumgartner [3, §1]. □

Corollary 3.11. *Assume $\kappa = \kappa^{\aleph_0} \geq \aleph_2$. The forcing notion $\mathbb{S}_*(\kappa)$ is proper and every meager subset of ${}^{\omega}2$ in an extension by $\mathbb{S}_*(\kappa)$ is included in a Σ^0_2 meager set coded in the ground model.*
If CH holds, then $\mathbb{S}_*(\kappa)$ preserves all cardinals and cofinalities and $\Vdash_{\mathbb{S}_*(\kappa)} 2^{\aleph_0} = \kappa$.

4. Meager non–null

The goal of this section is to present a model of ZFC in which every meager subgroup of $\mathbb{R}$ or ${}^{\omega}2$ is also Lebesgue null.

Theorem 4.1. *Assume CH. Let $\kappa = \kappa^{\aleph_0} \geq \aleph_2$. Then*

(1) $\Vdash_{\mathbb{S}_*(\kappa)}$ *" $2^{\aleph_0} = \kappa$ and each meager subgroup of $({}^{\omega}2, +_2)$ is Lebesgue null."*

(2) $\Vdash_{\mathbb{S}_*(\kappa)}$ *" every meager subgroup of $(\mathbb{R}, +)$ is Lebesgue null."*

Proof. For $\alpha < \kappa$ let $\underset{\sim}{\eta}_\alpha$ be the canonical name for the $\mathbb{S}_*$–generic function in $\prod\limits_{m<\omega} (m+1)$ added on the $\alpha^{\rm th}$ coordinate of $\mathbb{S}_*(\kappa)$.

(1) Suppose towards contradiction that for some $p_0 \in \mathbb{S}_*(\kappa)$ and a $\mathbb{S}_*(\kappa)$–name $\underset{\sim}{H}$ we have

$$p_0 \Vdash_{\mathbb{S}_*(\kappa)} \text{“ } \underset{\sim}{H} \text{ is a meager non–null subgroup of } ({}^\omega 2, +_2). \text{ ”}$$

By Corollary 3.11 (or, actually, Lemma 3.10(4)) we may pick a condition $p_1 \geq p_0$, a strictly increasing sequence $\bar{n} = \langle n_j : j < \omega \rangle \subseteq \omega$ and a function $f \in {}^\omega 2$ such that

$(*)_0$ $p_1 \Vdash_{\mathbb{S}_*(\kappa)}$ “ $\underset{\sim}{H} \subseteq \big\{x \in {}^\omega 2 : (\forall^\infty j < \omega)\big(x{\restriction}[n_j, n_{j+1}) \neq f{\restriction}[n_j, n_{j+1})\big)\big\}$. ”

Let $\bar{m} = \bar{m}[\bar{n}]$, $\bar{N} = \bar{N}[\bar{n}]$, $\bar{J} = \bar{J}[\bar{n}]$, $\bar{H} = \bar{H}[\bar{n}]$, $\pi = \pi[\bar{n}]$ and $\mathbf{F} = \mathbf{F}[\bar{n}]$ be as defined in Definition 3.3 for the sequence $\bar{n}$. Also let $A = \{|H_i| - 1 : i < \omega\}$ and $r^+ \in \mathbb{S}_*$ be such that $\mathrm{dom}(r^+) = \omega \setminus A$ and $r^+(k) = 0$ for $k \in \mathrm{dom}(r^+)$.

Since, by Lemma 3.4, we have $\Vdash$“ $\mathbf{F}(\underset{\sim}{\eta}_\alpha) \subseteq {}^\omega 2$ is a measure one set ”, we know that $p_1 \Vdash_{\mathbb{S}_*(\kappa)}$ “ $(\forall \alpha < \kappa)(\mathbf{F}(\underset{\sim}{\eta}_\alpha) \cap \underset{\sim}{H} \neq \emptyset)$ ”. Consequently, for each $\alpha < \kappa$, we may choose a $\mathbb{S}_*(\kappa)$–name $\underset{\sim}{\rho}_\alpha$ for an element of ${}^\omega 2$ such that

$$p_1 \Vdash_{\mathbb{S}_*(\kappa)} \text{“ } \underset{\sim}{\rho}_\alpha \in \underset{\sim}{H} \ \&\ \underset{\sim}{\rho}_\alpha \in \mathbf{F}(\underset{\sim}{\eta}_\alpha) \text{ ”}.$$

Let us fix $\alpha \in \kappa \setminus \mathrm{dom}(p_1)$ for a moment. Let $p_1^\alpha \in \mathbb{S}_*(\kappa)$ be a condition such that $\mathrm{dom}(p_1^\alpha) = \mathrm{dom}(p_1) \cup \{\alpha\}$, $p_1^\alpha(\alpha) = r^+$ and $p_1 \subseteq p_1^\alpha$. Using the standard fusion based argument (like the one applied in the classical proof of Lemma 3.10(3) with 3.10(2) used repeatedly), we may find a condition $q^\alpha \in \mathbb{S}_*(\kappa)$, a sequence $\bar{F} = \langle F_n^\alpha : n < \omega \rangle$ of finite sets, a sequence $\langle \mu_n^\alpha : n < \omega \rangle$ and an integer $i^\alpha < \omega$ such that the following demands $(*)_1$–$(*)_6$ are satisfied.

$(*)_1$ $q^\alpha \geq p_1^\alpha$, $\mathrm{dom}(q^\alpha) = \bigcup\limits_{n<\omega} F_n^\alpha$, $F_n^\alpha \subseteq F_{n+1}^\alpha$ and $F_0^\alpha = \{\alpha\}$.

$(*)_2$ $\mu_n^\alpha : F_n^\alpha \longrightarrow \omega$, $\mu_n^\alpha(\alpha) = n + 1$, $\mu_n^\alpha(\beta) = n$ for $\beta \in F_n^\alpha \setminus \{\alpha\}$.

$(*)_3$ $\min\big(\omega \setminus \mathrm{dom}(q^\alpha(\alpha))\big) > |H_{i^\alpha}|$ and
if $\max\big(u(n+1, q^\alpha(\alpha))\big) = |H_i| - 1$ and $n \geq 1$, then $|T(F_n, n, q^\alpha)|^2 < 2^i$,

$(*)_4$ $q^\alpha \Vdash \big(\forall i \geq i^\alpha\big)\big(\underset{\sim}{\rho}_\alpha{\restriction}J_i \in \pi_i(\underset{\sim}{\eta}_\alpha(|H_i| - 1))\big)$, and

$(*)_5$ q^α determines $\underset{\sim}{\rho}_\alpha$ relative to $\bar{F}$, moreover

$(*)_6$ if $\sigma \in T(F_n^\alpha, \mu_n^\alpha, q^\alpha)$ and $\max\big(u(n+1, q^\alpha(\alpha))\big) = |H_i| - 1$, then $q^\alpha|\sigma$ decides the value of $\underset{\sim}{\rho}_\alpha{\restriction}J_i$.

Unfixing α and using a standard Δ–system argument with CH we may find distinct $\gamma, \delta \in \kappa \setminus \mathrm{dom}(p_1)$ such that $\mathrm{otp}(\mathrm{dom}(q^\gamma)) = \mathrm{otp}(\mathrm{dom}(q^\delta))$ and if $g : \mathrm{dom}(q^\gamma) \longrightarrow \mathrm{dom}(q^\delta)$ is the order preserving bijection, then the following demands $(*)_7$–$(*)_9$ hold true.

$(*)_7$ $i^\gamma = i^\delta$, $g{\restriction}\big(\mathrm{dom}(q^\gamma) \cap \mathrm{dom}(q^\delta)\big)$ is the identity, $g(\gamma) = \delta$,

$(*)_8$ $q^\gamma(\beta) = q^\delta(g(\beta))$ for each $\beta \in \mathrm{dom}(q^\gamma)$, and $g[F_n^\gamma] = F_n^\delta$,

$(*)_9$ if $F \subseteq \mathrm{dom}(q^\delta)$ is finite, $\mu : F \longrightarrow \omega \setminus \{0\}$, $i < \omega$, $\sigma \in T(F, \mu, q^\delta)$, then

$$q^\delta|\sigma \Vdash \underset{\sim}{\rho}_\delta{\restriction}J_i = z \quad \text{if and only if} \quad q^\gamma|(\sigma \circ g) \Vdash \underset{\sim}{\rho}_\gamma{\restriction}J_i = z.$$

Clearly $q^* \stackrel{\rm def}{=} q^\gamma \cup q^\delta$ is a condition stronger than both q^γ and q^δ. Let $F^*_n = F^\gamma_n \cup F^\delta_n$ for $n<\omega$.

Let $\langle k_\ell : \ell<\omega\rangle$ be the increasing enumeration of $\omega\setminus {\rm dom}(q^\gamma(\gamma)) = \omega\setminus {\rm dom}(q^\delta(\delta))$. Note that by the choice of r^+ and p^γ_1, we have $\omega\setminus{\rm dom}(q^\gamma(\gamma))\subseteq A$, so each k_ℓ is of the form $|H_i|-1$ for some i. Now we will choose conditions $r_\delta, r_\gamma\in\mathbb{S}_*$ so that

$$
{\rm dom}(r_\delta)={\rm dom}(r_\gamma)={\rm dom}(q^\delta(\delta))\cup\{k_{2\ell}:\ell<\omega\},
$$

$q^\delta(\delta)\le r_\delta$, $q^\gamma(\gamma)\le r_\gamma$ and the values of $r_\delta(k_{2\ell}), r_\gamma(k_{2\ell})$ are picked as follows.

Let i be such that $k_{2\ell}=|H_i|-1$. If $x\in\{\gamma,\delta\}$ and $\sigma\in T(F^x_{2\ell},\mu^x_{2\ell},q^x)$ then $q^x|\sigma$ decides the value of $\underset{\sim}{\rho}_x{\restriction}J_i$ (by $(*)_6$) and this value belongs to $\pi_i\big(\sigma(x)(k_{2\ell})\big)$ (by $(*)_4+(*)_3$). Consequently, for $x\in\{\gamma,\delta\}$ and $\tau\in T(F^*_{2\ell},2\ell,q^*)$ we may define a function $\mathcal{Z}^x_\tau: H_i\longrightarrow {}^{J_i}2$ so that

$(*)_{10}$ **if** $a\in H(i)$, $\mu:F^*_{2\ell}\longrightarrow\omega$ is such that $\mu(x)=2\ell+1$ and $\mu(\alpha)=2\ell$ for $\alpha\neq x$, and $\tau_a\in T(F^*_{2\ell},\mu,q^*)$ is such that $\tau_a(\alpha)=\tau(\alpha)$ for $\alpha\in F^*_{2\ell}\setminus\{x\}$ and $\tau_a(x)=\tau(x)\cup\{(k_{2\ell},a)\}$,
then $q^*|\tau_a\Vdash_{\mathbb{S}_*(\kappa)}\underset{\sim}{\rho}_x{\restriction}J_i=\mathcal{Z}^x_\tau(a)$ and $\mathcal{Z}^x_\tau(a)\in a$.

Since $|T(F^*_{2\ell},2\ell,q^*)|\le|T(F^\gamma_{2\ell},2\ell,q^\gamma)|^2<2^i$ (remember $(*)_3$), we may use Lemma 3.5 to find $r_\delta(k_{2\ell}), r_\gamma(k_{2\ell})\le k_{2\ell}$ such that

$(*)_{11}$ for every $\tau\in T(F^*_{2\ell},2\ell,q^*)$ there is $k\in[m_{2^i},m_{2^{i+1}})$ satisfying

$$
\big(\mathcal{Z}^\gamma_\tau(\pi_i(r_\gamma(k_{2\ell}))){\restriction}[n_k,n_{k+1})\big)+_2\big(\mathcal{Z}^\delta_\tau(\pi_i(r_\delta(k_{2\ell}))){\restriction}[n_k,n_{k+1})\big)=f{\restriction}[n_k,n_{k+1}).
$$

(Remember, f was chosen in $(*)_0$.)

This completes the definition of r_γ and r_δ. Let $q^+\in\mathbb{S}_*(\kappa)$ be such that ${\rm dom}(q^+)={\rm dom}(q^*)={\rm dom}(q^\gamma)\cup{\rm dom}(q^\delta)$ and $q^+(\alpha)=q^*(\alpha)$ for $\alpha\in{\rm dom}(q^+)\setminus\{\gamma,\delta\}$ and $q^+(\gamma)=r_\gamma$ and $q^+(\delta)=r_\delta$. Then q^+ is a (well defined) condition stronger than both q^γ and q^δ and such that

$(\clubsuit)$ $q^+\Vdash\Big(\exists^\infty k<\omega\Big)\Big(\big(\underset{\sim}{\rho}_\gamma{\restriction}[n_k,n_{k+1})\big)+_2\big(\underset{\sim}{\rho}_\delta{\restriction}[n_k,n_{k+1})\big)=f{\restriction}[n_k,n_{k+1})\Big)$

(by $(*)_{10}+(*)_{11}$). Consequently, by $(*)_0$,

$(\heartsuit)$ $q^+\Vdash$" $\underset{\sim}{\rho}_\gamma,\underset{\sim}{\rho}_\delta\in\underset{\sim}{H}$ and $\underset{\sim}{\rho}_\gamma+_2\underset{\sim}{\rho}_\delta\notin\underset{\sim}{H}$ and $(\underset{\sim}{H},+_2)$ is a group",

a contradiction.

(2) The proof is a small modification of that for the first part, so we describe the new points only. Assume towards contradiction that for some $p_0\in\mathbb{S}_*(\kappa)$ and a $\mathbb{S}_*(\kappa)$–name $\underset{\sim}{H}^*$ we have

$$
p_0\Vdash_{\mathbb{S}_*(\kappa)}\text{“ }\underset{\sim}{H}^*\text{ is a meager non–null subgroup of }(\mathbb{R},+)\text{ ”}.
$$

Let $\underset{\sim}{H}_0,\underset{\sim}{H}_1$ be $\mathbb{S}_*$–names for subsets of D^∞_0 such that

$$
p_0\Vdash_{\mathbb{S}_*(\kappa)}\text{“ }\underset{\sim}{H}_0=\mathbf{E}^{-1}[\underset{\sim}{H}^*\cap[0,1/2)]\text{ and }\underset{\sim}{H}_1=\mathbf{E}^{-1}[\underset{\sim}{H}^*\cap[0,1)]\text{ ”}.
$$

Necessarily $p_0 \Vdash$ " $\underset{\sim}{H}^* \cap [0,1/2)$ is not null ", so it follows from 2.2(1) that

$$p_0 \Vdash_{\mathbb{S}_*(\kappa)} \text{`` } \underset{\sim}{H}_0 \notin \mathcal{N} \text{ and } \underset{\sim}{H}_1 \in \mathcal{M} \text{ and } \underset{\sim}{H}_0 \subseteq \underset{\sim}{H}_1 \text{ ''}.$$

Clearly we may pick a condition $p_1 \geq p_0$, a sequence $\bar{n} = \langle n_j : j < \omega\rangle \subseteq \omega$ and a function $f \in {}^\omega 2$ such that

$(\oplus)_0$ $n_{j+1} > n_j + j + 1$ for each j,
$(\oplus)_1$ $f(n_{j+1} - 1) = 0$ for each j, and
$(\oplus)_2$ $p_1 \Vdash_{\mathbb{S}_*(\kappa)}$"$\underset{\sim}{H}_1 \subseteq \big\{x \in {}^\omega 2 : (\forall^\infty j < \omega)\big(x{\restriction}[n_j, n_{j+1}-1) \neq f{\restriction}[n_j, n_{j+1}-1)\big)\big\}$."
(Note: "$[n_j, n_{j+1} - 1)$" not "$[n_j, n_{j+1})$".)

Like in part (1), let $\bar{m} = \bar{m}[\bar{n}]$, $\bar{N} = \bar{N}[\bar{n}]$, $\bar{J} = \bar{J}[\bar{n}]$, $\bar{H} = \bar{H}[\bar{n}]$, $\pi = \pi[\bar{n}]$ and $\mathbf{F} = \mathbf{F}[\bar{n}]$. Let $A = \{|H_i| - 1 : i < \omega\}$ and $r^+ \in \mathbb{S}_*$ be such that $\mathrm{dom}(r^+) = \omega \setminus A$ and $r^+(k) = 0$ for $k \in \mathrm{dom}(r^+)$. Then each $\alpha < \kappa$ fix a $\mathbb{S}_*(\kappa)$–name $\underset{\sim}{\rho}_\alpha$ such that $p_1 \Vdash_{\mathbb{S}_*(\kappa)}$" $\underset{\sim}{\rho}_\alpha \in \underset{\sim}{H}_0 \cap \mathbf{F}(\underset{\sim}{\eta}_\alpha)$ ".

Now repeat the arguments of the first part (with $(*)_1$–$(*)_{11}$ there applied to our $\bar{n}, f, \underset{\sim}{\rho}_\alpha$ and the operation $\circledast_0$ here) to find $q^+ \geq p_1$ and $\gamma, \delta \in \mathrm{dom}(q^+)$ such that

$$(\lozenge)\quad q^+ \Vdash \text{`` } \big(\exists^\infty k < \omega\big)\big((\underset{\sim}{\rho}_\gamma{\restriction}[n_k, n_{k+1})) \circledast_0 (\underset{\sim}{\rho}_\delta{\restriction}[n_k, n_{k+1})) = f{\restriction}[n_k, n_{k+1})\big) \text{ ''}.$$

Let $G \subseteq \mathbb{S}_*(\kappa)$ be a generic over $\mathbf{V}$ such that $q^+ \in G$ and let us work in $\mathbf{V}[G]$. Let $\eta \in D_0^\infty$ be such that $\mathbf{E}(\underset{\sim}{\rho}_\gamma^G) + \mathbf{E}(\underset{\sim}{\rho}_\delta^G) = \mathbf{E}(\eta)$ (remember $\mathbf{E}(\underset{\sim}{\rho}_\gamma^G), \mathbf{E}(\underset{\sim}{\rho}_\delta^G) < 1/2$). We know from $(\lozenge)$ that there are infinitely many $k < \omega$ satisfying

$$(\blacklozenge)\quad (\underset{\sim}{\rho}_\gamma^G{\restriction}[n_k, n_{k+1})) \circledast_0 (\underset{\sim}{\rho}_\delta^G{\restriction}[n_k, n_{k+1})) = f{\restriction}[n_k, n_{k+1}).$$

Since $f(n_{k+1} - 1) = 0$ (see $(\oplus)_1$), we get from 3.2(3) that for each k as in $(\blacklozenge)$ we also have

$$\big((\underset{\sim}{\rho}_\gamma^G{\restriction}[n_k, n_{k+1})) \circledast_0 (\underset{\sim}{\rho}_\delta^G{\restriction}[n_k, n_{k+1}))\big){\restriction}[n_k, n_{k+1} - 1) =$$
$$\big((\underset{\sim}{\rho}_\gamma^G{\restriction}[n_k, n_{k+1})) \circledast_1 (\underset{\sim}{\rho}_\delta^G{\restriction}[n_k, n_{k+1}))\big){\restriction}[n_k, n_{k+1} - 1) = f{\restriction}[n_k, n_{k+1} - 1).$$

Therefore (by 3.2(4)) for each k satisfying $(\blacklozenge)$ we have $\eta{\restriction}[n_k, n_{k+1} - 1) = f{\restriction}[n_k, n_{k+1} - 1)$, so

$$\big(\exists^\infty k < \omega\big)\big(\eta{\restriction}[n_k, n_{k+1} - 1) = f{\restriction}[n_k, n_{k+1} - 1)\big).$$

Consequently, by $(\oplus)_2$, we have that $\eta \notin \underset{\sim}{H}_1^G$, i.e., $\mathbf{E}(\eta) \notin (\underset{\sim}{H}^*)^G \cap [0,1)$. This contradicts the fact that $\mathbf{E}(\underset{\sim}{\rho}_\gamma^G), \mathbf{E}(\underset{\sim}{\rho}_\delta^G) \in (\underset{\sim}{H}^*)^G$, $\mathbf{E}(\eta) = \mathbf{E}(\underset{\sim}{\rho}_\gamma^G) + \mathbf{E}(\underset{\sim}{\rho}_\delta^G)$ and $(\underset{\sim}{H}^*)^G$ is a subgroup of $(\mathbb{R}, +)$. □

Remark 4.2. Instead of the CS product of forcing notions $\mathbb{S}_*$ we could have used their CS iteration of length ω_2. Of course, that would restrict the value of the continuum in the resulting model.

5. Problems

Both theorems 2.3(1) and 4.1(1) can be repeated for other product groups. We may consider a sequence $\langle H_n : n < \omega \rangle$ of finite groups and their coordinate-wise product $H = \prod\limits_{n<\omega} H_n$. Naturally, H is equipped with product topology of discrete H_n's and the product probability measure. Then there exists a null non–meager subgroup of H but it is consistent that there is no meager non–null such subgroup. It is natural to ask now:

Problem 5.1. (1) Does every locally compact group (with complete Haar measure) admit a null non–meager subgroup?
(2) Is it consistent that no locally compact group has a meager non–null subgroup?

In relation to Theorem 4.1, we still should ask:

Problem 5.2. Is it consistent that there exists a translation invariant Borel hull for the meager ideal on ${}^{\omega}2$? On $\mathbb{R}$?

References

[1] Tomek Bartoszyński and Haim Judah. *Set Theory: On the Structure of the Real Line.* A K Peters, Wellesley, Massachusetts, 1995.
[2] James E. Baumgartner. Iterated forcing. In A. Mathias, editor, *Surveys in Set Theory*, volume 87 of *London Mathematical Society Lecture Notes*, pages 1–59, Cambridge, Britain, 1978.
[3] James E. Baumgartner. Sacks forcing and the total failure of Martin's axiom. *Topology and its Applications*, 19:211–225, 1985.
[4] Lev Bukovský. *The Structure of the Real Line.* Birkhäuser, 2011.
[5] Tomasz Filipczak, Andrzej Roslanowski, and Saharon Shelah. On Borel hull operations. *Real Analysis Exchange*, 40:129–140, 2015. arxiv:1308.3749.
[6] Thomas Jech. *Set theory.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.
[7] Andrzej Rosłanowski. n–localization property. *Journal of Symbolic Logic*, 71:881–902, 2006. arxiv:math.LO/0507519.
[8] Andrzej Roslanowski and Saharon Shelah. Norms on possibilities I: forcing with trees and creatures. *Memoirs of the American Mathematical Society*, 141(671):xii + 167, 1999. arxiv:math.LO/9807172.
[9] Andrzej Rosłanowski and Juris Steprāns. Chasing Silver. *Canadian Mathematical Bulletin*, 51:593–603, 2008. arxiv:math.LO/0509392.

Department of Mathematics, University of Nebraska at Omaha, Omaha, NE 68182-0243, USA
E-mail address: roslanow@member.ams.org
URL: http://www.unomaha.edu/logic

Institute of Mathematics, The Hebrew University of Jerusalem, 91904 Jerusalem, Israel, and Department of Mathematics, Rutgers University, New Brunswick, NJ 08854, USA
E-mail address: shelah@math.huji.ac.il
URL: http://www.math.rutgers.edu/∼shelah