THE ABELIANIZATION OF INVERSE LIMITS OF GROUPS

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Abstract. The abelianization is a functor from groups to abelian groups, which is left adjoint to the inclusion functor. Being a left adjoint, the abelianization functor commutes with all small colimits. In this paper we investigate the relation between the abelianization of a limit of groups and the limit of their abelianizations. We show that if $\mathcal{T}$ is a countable directed poset and $G : \mathcal{T} \longrightarrow \mathcal{G}rp$ is a diagram of groups, with surjective connecting homomorphisms, then the kernel and cokernel of the natural map

$$\mathrm{Ab}(\lim_{t\in\mathcal{T}} G_t) \longrightarrow \lim_{t\in\mathcal{T}} \mathrm{Ab}(G_t)$$

are cotorsion abelian groups. In the special case of a product of a countable collection of groups $(H_n)_{n\in\mathbb{N}}$, we show that the natural map

$$\mathrm{Ab}(\prod_{i\in\mathbb{N}} H_i) \longrightarrow \prod_{i\in\mathbb{N}} \mathrm{Ab}(H_i)$$

is surjective, and its kernel is a cotorsion group of Ulm length that does not exceed $\aleph_1$.

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Introduction

The abelianization functor is a very fundamental and widely used construction in group theory and other mathematical fields. This is a functor

$$\mathrm{Ab} : \mathcal{G}\mathrm{rp} \longrightarrow \mathcal{A}\mathrm{b},$$

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from the category of groups to the category of abelian groups, equipped with a natural projection map

$$\pi_G : G \longrightarrow \mathrm{Ab}(G),$$

for every group G. This construction is universal in the sense that for any group G, any abelian group A and any morphism of groups $f : G \to A$, there is a unique morphism of (abelian) groups $\overline{f} : \mathrm{Ab}(G) \longrightarrow A$ such that the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\pi_G"] \arrow[dr, "f"'] & \mathrm{Ab}(G) \arrow[d, "\overline{f}"] \\ & A. \end{tikzcd}$$

Expressed in the language of category theory, the above universal property implies that the functor $\mathrm{Ab} : \mathcal{G}\mathrm{rp} \longrightarrow \mathcal{A}\mathrm{b}$ is left adjoint to the inclusion functor $\mathrm{inc} : \mathcal{A}\mathrm{b} \longrightarrow \mathcal{G}\mathrm{rp}$. Being a left adjoint, the functor Ab commutes with all small colimits. That is, given any small category $\mathcal{D}$, and any functor (diagram) $F : \mathcal{D} \longrightarrow \mathcal{G}\mathrm{rp}$, the natural morphism

$$\operatorname*{colim}_{d\in\mathcal{D}} \mathrm{Ab}(F_d) \longrightarrow \mathrm{Ab}(\operatorname*{colim}_{d\in\mathcal{D}} F_d)$$

is an isomorphism. However, the abelianization functor certainly does not commute with all small limits. That is, given a small category $\mathcal{D}$, and a diagram $G : \mathcal{D} \longrightarrow \mathcal{G}\mathrm{rp}$, the natural morphism

$$\mathrm{Ab}(\lim_{d\in\mathcal{D}} G_d) \longrightarrow \lim_{d\in\mathcal{D}} \mathrm{Ab}(G_d).$$

need not be an isomorphism. Since this is a morphism of abelian groups, a natural way to "measure" how far it is from being an isomorphism is to consider its kernel an cokernel. Thus, a natural question is whether the kernel and cokernel of the natural map above can be any abelian groups, or are there limitations?

In this paper we consider the case where the diagram category $\mathcal{D}$ is a countable directed poset, considered as a category which has a single morphism $t \to s$ whenever $t \geq s$. We show in Corollary 3.2.2 that

Theorem 0.0.1. *Let $\mathcal{T}$ be a countable directed poset and let $G : \mathcal{T} \longrightarrow \mathcal{G}\mathrm{rp}$ be a diagram of groups. Suppose that for every $t \geq s$ in $\mathcal{T}$, the structure map $G_t \to G_s$ is surjective. Then the kernel and cokernel of the natural map*

$$\mathrm{Ab}(\lim_{t\in\mathcal{T}} G_t) \longrightarrow \lim_{t\in\mathcal{T}} \mathrm{Ab}(G_t)$$

are cotorsion groups.

Remark 0.0.2. Actually, in Corollary 3.1.7, we show that the **cokernel** of the natural map is cotorsion for a variety of functors into abelian groups (not just the abelianization). See Example 3.1.8.

Recall that a group G is called *perfect* if $\mathrm{Ab}(G) = 0$. We thus obtain the following corollary:

Corollary 0.0.3. *Let $\mathcal{T}$ be a countable directed poset and let $G : \mathcal{T} \longrightarrow \mathcal{G}\mathrm{rp}$ be a diagram of perfect groups. Suppose that all the structure maps $G_t \to G_s$ are surjective. Then $\mathrm{Ab}(\lim_{t\in\mathcal{T}} G_t)$ is a cotorsion group.*

Cotorsion groups are abelian groups A that satisfy $\mathrm{Ext}(\mathbb{Q}, A) = 0$ (or, equivalently, $\mathrm{Ext}(F, A) = 0$ for any torsion free abelian group). That is, an abelian group A is cotorsion iff for every group B, containing A as a subgroup, and satisfying $B/A \cong \mathbb{Q}$, we have that A is a direct summand in B. This is a very important and extensively studied class of abelian groups. There is a structure theorem, due to Harrison [Har], which classifies cotorsion groups in terms of a countable collection of cardinals together with a reduced torsion group:

Theorem 0.0.4 (Harrison [Har]). *Let G be an abelian group. Then G is cotorsion iff there exist cardinals μ_p, for every $p \in \mathbb{P} \cup \{0\}$ ($\mathbb{P}$ is the set of prime numbers), λ_p, for every $p \in \mathbb{P}$, and a reduced torsion abelian group T, such that*

$$G \cong D_G \oplus R_G,$$

where

(1)
$$D_G \cong (\bigoplus_{\alpha<\mu_0} \mathbb{Q}) \oplus \bigoplus_{p\in\mathbb{P}} \bigoplus_{\alpha<\mu_p} \mathbb{Z}(p^\infty),$$

(2)
$$R_G \cong \operatorname{Ext}(\mathbb{Q}/\mathbb{Z}, T \oplus \bigoplus_{p\in\mathbb{P}} \bigoplus_{\alpha<\lambda_p} \mathbb{Z}_p).$$

Moreover, the cardinals μ_p and λ_p and the group T are uniquely determined by G.

For more details on Harrison's theorem see Theorems 1.3.3, 1.3.7 and 1.3.21. This result reduces the structure problem for cotorsion groups to that of reduced torsion abelian groups.

We prove Theorem 0.0.1 by first proving the following criterion for an abelian group to be cotorsion:

Theorem 0.0.5 (see Theorem 2.0.26). *Let H be an abelian group. Then H is cotorsion iff every system of equations over H, with an infinite matrix of the form*

$$\begin{array}{ccccc}
1 & l_0 & 0 & 0 & \cdots \\
 & 1 & l_1 & 0 & \cdots \\
 & & 1 & l_2 & \cdots \\
 & & & 1 & \cdots \\
 & & & & \cdots
\end{array}$$

has a solution in H. That is, H is cotorsion iff for every vector (l_n) in $\mathbb{Z}^{\mathbb{N}}$ and every vector (f_n) in $H^{\mathbb{N}}$, there exists a vector (g_n) in $H^{\mathbb{N}}$ such that for every $n \in \mathbb{N}$ we have

$$g_n = f_n - l_n g_{n+1}.$$

In the special of a countable product, we are able to say more about the kernel and cokernel of the natural map. Namely, we show

Theorem 0.0.6 (see Corollary 4.0.7). *Let $(H_n)_{n\in\mathbb{N}}$ be a countable collection of groups. Then the natural map*

$$\operatorname{Ab}(\prod_{i\in\mathbb{N}} H_i) \longrightarrow \prod_{i\in\mathbb{N}} \operatorname{Ab}(H_i)$$

is surjective, and its kernel is a cotorsion group of Ulm length that does not exceed $\aleph_1$.

Again, we obtain the immediate corollary:

Corollary 0.0.7. *The abelianization of a countable product of perfect groups is a cotorsion group of Ulm length that does not exceed* $\aleph_1$.

This paper originated from a question posed to the second author by Emmanuel Farjoun, from the field of algebraic topology. For the following discussion, the word space will mean a compactly generated Hausdorff topological space. It is known, that the homology groups of a homotopy **colimit** of spaces are computable from the homology groups of the individual spaces, by means of a spectral sequence, while this is not true for the homology groups of a homotopy **limit**. Farjoun asked what can be said about the natural map from the homology of a homotopy limit to the limit of homologies. Since the **homotopy** groups of a homotopy limit are computable from the homotopy groups of the individual spaces, by means of a spectral sequence, and we have a natural isomorphism

$$H_1(X) \cong \mathrm{Ab}(\pi_1 X),$$

for every pointed connected space X (see [GJ, Corollary 3.6]), a good place to start seems to be the investigation of the behaviour of the abelianization functor under limits. And indeed, using our results, we can say something also about Farjoun's question as the following corollary demonstrates:

Corollary 0.0.8.

(1) *Let* $X : \mathbb{N} \longrightarrow \mathrm{Top}_*$ *be a diagram of pointed connected spaces. Suppose that for every* $n \in \mathbb{N}$ *the structure map* $X_{n+1} \to X_n$ *is a Serre fibrations and both maps*
$$\pi_1 X_{n+1} \to \pi_1 X_n \text{ and } \pi_2 X_{n+1} \to \pi_2 X_n$$
are surjective. Then the kernel and cokernel of the natural map
$$H_1(\lim_{n\in\mathbb{N}} X_n) \longrightarrow \lim_{n\in\mathbb{N}} H_1(X_n)$$
are cotorsion groups.

(2) *Let* $(Y_n)_{n\in\mathbb{N}}$ *be a countable collection of connected spaces. Then the natural map*
$$H_1(\prod_{i\in\mathbb{N}} Y_i) \longrightarrow \prod_{i\in\mathbb{N}} H_1(Y_i)$$
is surjective, and its kernel is a cotorsion group of Ulm length that does not exceed $\aleph_1$.

Proof. We begin with (1). Since X is a tower of pointed fibrations, we have for every $i \geq 0$ an exact sequence (see, for instance, [GJ, VI Proposition 2.15])

$$* \longrightarrow \lim^1_{n\in\mathbb{N}} \pi_{i+1} X_n \longrightarrow \pi_i \lim_{n\in\mathbb{N}} X_n \longrightarrow \lim_{n\in\mathbb{N}} \pi_i X_n \longrightarrow *.$$

Since the maps $\pi_1 X_{n+1} \to \pi_1 X_n$ and $\pi_2 X_{n+1} \to \pi_2 X_n$ are all surjective, we have that

$$\lim^1_{n\in\mathbb{N}} \pi_1 X_n \cong \lim^1_{n\in\mathbb{N}} \pi_2 X_n \cong *.$$

Thus, from the exact sequence in the case $i = 0$ we can deduce that $\lim_{n\in\mathbb{N}} X_n$ is connected, so we have a natural isomorphism

$$H_1(\lim_{n\in\mathbb{N}} X_n) \cong \mathrm{Ab}(\pi_1 \lim_{n\in\mathbb{N}} X_n),$$

while from the exact sequence in the case $i = 1$ we obtain a natural isomorphism

$$\pi_1 \lim_{n\in\mathbb{N}} X_n \cong \lim_{n\in\mathbb{N}} \pi_1 X_n.$$

It is not hard to see that these isomorphisms fit into a commutative diagram

$$\begin{tikzcd}
H_1(\lim_{n\in\mathbb{N}} X_n) \arrow[r, "\cong"] \arrow[d] & \mathrm{Ab}(\lim_{n\in\mathbb{N}} \pi_1 X_n) \arrow[d] \\
\lim_{n\in\mathbb{N}} H_1(X_n) \arrow[r, "\cong"] & \lim_{n\in\mathbb{N}} \mathrm{Ab}(\pi_1 X_n),
\end{tikzcd}$$

where the vertical maps are the natural ones. Now the result follows from Theorem 0.0.1.

The proof of (2) is identical to (1), but we use Theorem 0.0.6 instead of Theorem 0.0.1. $\square$

0.1. **Organization of the paper.** In Section 1 we recall some necessary background, especially from the theory of abelian groups. In Section 2 we prove the criterion mentioned above for an abelian group to be cotorsion (Theorem 0.0.5). In Section 3 we prove Theorem 0.0.1 by showing that the kernel and cokernel of the natural map appearing there satisfy the criterion above. Finally, in Section 4, we prove Theorem 0.0.6.

0.2. **Notations and conventions.** We denote by $\mathbb{N}$ the set of natural numbers (including 0), by $\mathbb{Z}$ the ring of integers and by $\mathbb{Q}$ the field of rational numbers. We denote by $\mathbb{P}$ the set of prime natural numbers and enumerate it as

$$\mathbb{P} = \{p_0, p_1, \cdots\}.$$

If $p \in \mathbb{P}$, we denote by $\mathbb{Q}_p$ the field of p-adic numbers, and by $\mathbb{Z}_p$ its subring of p-adic integers. Note that in [Fu1] the ring $\mathbb{Z}_p$ is denoted by J_p. Whenever we treat a ring as an abelian group we mean its underlying additive group.

If G is a group, we denote its unit element by e_G or just by e if the group is understood.

If $\mathcal{T}$ is a small partially ordered set, we view $\mathcal{T}$ as a small category which has a single morphism $u \to v$ whenever $u \geq v$.

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1. Preliminaries

1.1. **The commutator subgroup and the abelianization functor.** In this subsection we recall the notions of the commutator subgroup and the abelianization of a group.

Definition 1.1.1. Let G be a group. If x and y are elements of G we define their *commutator* to be

$$[x, y] := x^{-1}y^{-1}xy \in G.$$

The *commutator subgroup* of G, denoted $\mathrm{C}(G)$, is defined to be the subgroup of G generated by the commutators, that is,

$$\mathrm{C}(G) := \langle [x, y] \mid x, y \in G \rangle \subseteq G.$$

Let G be a group. Then we have the following obvious properties:

(1) For every x and y in G we have
$$[x, y]^{-1} = [y, x]$$

(2) If $f : G \to H$ is a group homomorphism, then for every x and y in G we have
$$f([x, y]) = [f(x), f(y)].$$

From property (1) we see that

$$\mathrm{C}(G) = \{[x_1, y_1] \cdots [x_n, y_n] \mid n \in \mathbb{N},\ \ x_i, y_i \in G\},$$

and from property (2) we see that if $f : G \to H$ is a group homomorphism, then

$$f|_{\mathrm{C}(G)} : \mathrm{C}(G) \longrightarrow \mathrm{C}(H).$$

It follows that C can be naturally extended to a functor from the category of groups to itself

$$\mathrm{C} : \mathcal{G}\mathrm{rp} \longrightarrow \mathcal{G}\mathrm{rp}\,.$$

If s is an element in G, applying (2) to the s-conjugation homomorphism

$$\phi_s : G \longrightarrow G,$$

given by

$$\phi_s(x) = s^{-1}xs,$$

we easily see that $\mathrm{C}(G)$ is a normal subgroup of G. This observation enables to give the following definition:

Definition 1.1.2. Let G be a group. We define the abelianization of G to be the quotient group

$$\mathrm{Ab}(G) := G/\,\mathrm{C}(G).$$

It is easy to see that $\mathrm{Ab}(G)$ is an abelian group.

If $f : G \to H$ is a group homomorphism, then we have shown that

$$f|_{\mathrm{C}(G)} : \mathrm{C}(G) \longrightarrow \mathrm{C}(H).$$

It follows that we have an induced map

$$\mathrm{Ab}(f) : \mathrm{Ab}(G) \longrightarrow \mathrm{Ab}(H).$$

This turns Ab into a functor from the category of groups to the category of abelian groups

$$\mathrm{Ab} : \mathcal{G}\mathrm{rp} \longrightarrow \mathcal{A}\mathrm{b}\,.$$

Note that we have a projection map

$$\pi_G : G \longrightarrow \mathrm{Ab}(G),$$

and this map is natural in G in the sense that it defines a natural transformation of functors from groups to groups

$$\pi : \mathrm{Id}_{\mathcal{G}\mathrm{rp}} \longrightarrow \mathrm{Ab}\,.$$

The functor Ab and the natural transformation $\pi : \mathrm{Id}_{\mathcal{G}\mathrm{rp}} \longrightarrow \mathrm{Ab}$ are universal in the following sense:

Proposition 1.1.3. *Let G be a group and let A be an abelian group. Then for every morphism of groups $f : G \to A$, there exists a unique morphism of (abelian) groups $\overline{f} : \mathrm{Ab}(G) \longrightarrow A$ such that the following diagram commutes:*

$$\begin{tikzcd} G \arrow[r, "\pi_G"] \arrow[dr, "f"'] & \mathrm{Ab}(G) \arrow[d, "\overline{f}"] \\ & A. \end{tikzcd}$$

Expressed in the language of category theory, the above proposition implies that the functor $\mathrm{Ab} : \mathcal{G}\mathrm{rp} \longrightarrow \mathcal{A}\mathrm{b}$ is left adjoint to the inclusion functor $\mathrm{inc} : \mathcal{A}\mathrm{b} \longrightarrow \mathcal{G}\mathrm{rp}$. That is, we have an adjoint pair

$$\mathrm{Ab} : \mathcal{G}\mathrm{rp} \rightleftarrows \mathcal{A}\mathrm{b} : \mathrm{inc}\,.$$

This just means that for every group G and every abelian group A we have an isomorphism of sets

$$\mathcal{A}\mathrm{b}(\mathrm{Ab}(G), A) \cong \mathcal{G}\mathrm{rp}(G, \mathrm{inc}(A)),$$

that is natural in both G and A. The map $\pi : \mathrm{Id}_{\mathcal{G}\mathrm{rp}} \longrightarrow \mathrm{inc} \circ \mathrm{Ab}$ is the *unit* of the above adjunction.

1.2. **Limits in categories.** In this subsection we recall the notion of limit in general categories. We then specialize to limits in the categories of groups and abelian groups. For more detail the reader may consult [ML].

Definition 1.2.1. Let $\mathcal{D}$ be a category. The category $\mathcal{D}^{\triangleleft}$ has as objects $\mathrm{Ob}(\mathcal{D})\coprod\{\infty\}$, and the morphisms are the morphisms in $\mathcal{D}$, together with a unique morphism $\infty \to d$, for every object d in $\mathcal{D}$.

A category is called *small* if it has a set of objects and a set of morphisms. Let $\mathcal{C}$ be a category, $\mathcal{D}$ a small category and $F : \mathcal{D} \longrightarrow \mathcal{C}$ a functor. The functor F is called a diagram in $\mathcal{C}$ of shape $\mathcal{D}$, and $\mathcal{D}$ is called the diagram category of F.

A cone over F is a functor $\mathcal{D}^{\triangleleft} \longrightarrow \mathcal{C}$ extending the functor F. That is, a cone over F consists of an object c in $\mathcal{C}$, together with morphisms $c \to F(d)$ for every d in $\mathcal{D}$, that are compatible in the sense that for every morphism $d \to d'$ in $\mathcal{D}$ the following diagram commutes:

$$\begin{tikzcd} c \arrow[r] \arrow[d] & F(d'). \\ F(d) \arrow[ur] & \end{tikzcd}$$

A *limit* of the functor F is a cone F_∞ over F, which is universal in the sense that if F' is any other cone over F, there exists a unique morphism $F'(\infty) \to F_\infty(\infty)$ such that for every object d in $\mathcal{D}$ the following diagram commutes:

$$\begin{tikzcd} F'(\infty) \arrow[r] \arrow[d] & F(d). \\ F_\infty(\infty) \arrow[ur] & \end{tikzcd}$$

It easily follows that every two limits of F are isomorphic via a canonical isomorphism.

Definition 1.2.2. Let $p : \mathcal{J} \to \mathcal{I}$ be a functor between categories. The functor p is said to be (left) cofinal if for every object i in $\mathcal{I}$ the over category $p_{/i}$ is nonempty and connected.

Cofinal functors play an important role when studying limits in categories because of the following well-known lemma:

Lemma 1.2.3. *Let $\mathcal{C}$ be a category which admits small limits and let $p : \mathcal{J} \to \mathcal{I}$ be a cofinal functor between small categories. Then for every diagram $X : \mathcal{I} \to \mathcal{C}$, the natural map*

$$\lim_{\mathcal{I}} X \longrightarrow \lim_{\mathcal{J}} X \circ p$$

is an isomorphism.

Definition 1.2.4. A poset $\mathcal{T}$ is called directed if for every $t, s \in \mathcal{T}$ there exists $r \in \mathcal{T}$ such that $r \geq t$ and $r \geq s$.

Proposition 1.2.5. *Let $\mathcal{T}$ be a countable directed poset. Then there exists a cofinal functor $\mathbb{N} \longrightarrow \mathcal{T}$ (see Section 0.2).*

In the category $\mathcal{G}\mathrm{rp}$ of groups all (small) limits exist, and they can be given an explicit construction. Let us recall here the construction of limits in $\mathcal{G}\mathrm{rp}$.

Let $\mathcal{D}$ a small category and $F : \mathcal{D} \longrightarrow \mathcal{G}\mathrm{rp}$ a $\mathcal{D}$-shaped diagram in $\mathcal{G}\mathrm{rp}$. Let $\lim F$ denote the following subgroup of the product group:

$$\lim F := \{(x_d)_{d\in\mathrm{Ob}(\mathcal{D})} \in \prod_{d\in\mathrm{Ob}(\mathcal{D})} F(d) \mid \forall \text{ morphism } f : d \to d' \text{ in } \mathcal{D} \text{ we have } F(f)(x_d) = x_{d'}\}.$$

Restricting the natural projections

$$\pi_d : \prod_{e\in \mathrm{Ob}(\mathcal{D})} F(e) \longrightarrow F(d),$$

for $d \in \mathrm{Ob}(\mathcal{D})$, we obtain maps which we denote

$$\psi_d : \lim F \longrightarrow F(d).$$

It is not hard to see that $\lim F$, together with the maps $\psi_d : \lim F \longrightarrow F(d)$, is a limit of F in the category of groups $\mathcal{G}\mathrm{rp}$.

If for every object d in $\mathcal{D}$ the group $F(d)$ is abelian, that is, if $F : \mathcal{D} \longrightarrow \mathcal{A}\mathrm{b}$, then it is not hard to see that $\lim F$ is also abelian. In fact, $\lim F$, together with the maps $\psi_d : \lim F \longrightarrow F(d)$, is a limit of F in the category of **abelian** groups $\mathcal{A}\mathrm{b}$.

Remark 1.2.6. We only considered the notion of *limits* in categories, but there is also a dual notion of *colimits*, which can be defined analogously by reversing all the arrows.

1.3. **Abelian groups.** In this subsection we review the necessary background from the theory of abelian groups. For more detail the reader may consult [Fu1].

1.3.1. *Divisibility and the Ulm length.*

Definition 1.3.1. Let A be an abelian group.

(1) If $n \in \mathbb{Z}$, we denote $nA := \{na \mid a \in A\}$.
(2) If $a \in A$ and $n \in \mathbb{Z}$, we denote by $n|a$ the statement that $a \in nA$.
(3) We say that A is *divisible* if

$$\bigcap_{n=1}^{\infty} nA = A,$$

that is, if for every $a \in A$ and every $n \geq 1$ we have $n|a$.

Example 1.3.2.

(1) *The additive group of any field of characteristic 0 is divisible, so in particular* $\mathbb{Q}$ *is divisible, and* $\mathbb{Q}_p$ *is divisible for every prime* p.
(2) *Any quotient group of a divisible group is divisible, so in particular*
   (a) *The group* $\mathbb{Q}/\mathbb{Z}$ *is divisible.*
   (b) *For any prime* p, *the group*

$$\mathbb{Z}(p^\infty) := \mathbb{Q}_p/\mathbb{Z}_p$$

is divisible. Note that $\mathbb{Z}(p^\infty)$ *is just the group of roots of unity of order* p^k *for some* $k \geq 0$.

The divisible groups $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$ (for $p \in \mathbb{P}$) will be called *basic divisible groups*, because, as the following theorem shows, any divisible group can be decomposed uniquely into a direct sum of such groups.

Theorem 1.3.3 ([Fu1, Theorem 23.1])**.** *Let* A *be an abelian group. Then* A *is divisible iff there exist cardinals* λ_p, *for every* $p \in \mathbb{P} \cup \{0\}$, *such that*

$$A \cong (\bigoplus_{\alpha<\lambda_0} \mathbb{Q}) \oplus \bigoplus_{p\in\mathbb{P}} \bigoplus_{\alpha<\lambda_p} \mathbb{Z}(p^\infty).$$

Moreover, the cardinals λ_p *are uniquely determined by* A.

Definition 1.3.4. Let A be an abelian group. We define, recursively, for every ordinal λ, a subgroup $A^\lambda \subseteq A$, called the λ*th Ulm subgroup* of A, by:

(1) $A^0 := A$.
(2) For every ordinal λ we define $A^{\lambda+1} := \bigcap_{n=1}^{\infty} nA^\lambda$.
(3) For every limit ordinal δ we define $A^\delta := \bigcap_{\lambda<\delta} A^\lambda$.

Clearly we have defined a function $A^{(-)}$ from ordinals to sets, that is monotone decreasing and continuous. In particular, $A^{(-)}$ stabilizes, that is, there exists an ordinal λ such that $A^{\lambda+1} = A^\lambda$. The smallest such ordinal is called the *Ulm length* of A, and is denoted by $u(A)$. We always have $u(A) \leq |A|$.

Clearly, for every ordinal λ, we have that A^λ is divisible iff $u(A) \leq \lambda$. We also have that $p^{u(A)}A$ is the biggest divisible subgroup of A, and we denote $D_A := p^{u(A)}A$.

Theorem 1.3.5 ([Fu1, Theorem 24.5])**.** *Let A be an abelian group. Then the following conditions are equivalent:*

(1) *A is divisible.*
(2) *A is an injective $\mathbb{Z}$-module.*
(3) *A is a direct summand of every group containing A.*

Definition 1.3.6. Let A be an abelian group. Then A is called *reduced* if A has no divisible subgroups other then 0.

Theorem 1.3.7. *Let A be an abelian group. Then there exists a reduced subgroup R_A of A, unique up to isomorphism, such that*

$$A = D_A \oplus R_A.$$

Proof. Follows easily from Theorem 1.3.5. □

1.3.2. *p-divisibility and p-length.*

Definition 1.3.8. Let A be an abelian group and p a prime number. We say that A is *p-divisible* if $pA = A$, that is, if for every $a \in A$ we have $p|a$.

Definition 1.3.9. Let A be an abelian group and p a prime number. We define, recursively, for every ordinal λ, a subgroup $p^\lambda A \subseteq A$, by:

(1) $p^0A := A$.
(2) For every ordinal λ we define $p^{\lambda+1}A := p(p^\lambda A)$.
(3) For every limit ordinal δ we define $p^\delta A := \bigcap_{\lambda<\delta} p^\lambda A$.

Clearly we have defined a function $p^{(-)}A$ from ordinals to sets, that is monotone decreasing and continuous. In particular, $p^{(-)}A$ stabilizes, that is, there exists an ordinal λ such that $p^{\lambda+1}A = p^\lambda A$. The smallest such ordinal is called the *p-length* of A, and is denoted $l_p(A)$.

Clearly, for every ordinal λ, we have that $p^\lambda A$ is p-divisible iff $l_p(A) \leq \lambda$. We also have that $p^{l_p(A)}A$ is the biggest p-divisible subgroup of A.

Lemma 1.3.10. *Let A be an abelian group. Then for every ordinal λ we have $A^\lambda = \bigcap_{p\in\mathbb{P}} p^{\omega\lambda}A$.*

Proposition 1.3.11. *Let A be an abelian group and p a prime number. Suppose that $(y_m)_{m\in\mathbb{N}}$ is an object in $A^\mathbb{N}$ such that for every $m \geq 0$ we have $y_m = py_{m+1}$. Then $y_0 \in p^{l_p(A)}A$.*

Proof. Clearly it is enough to show that for every ordinal λ we have $y_0 \in p^\lambda A$. We show this by induction on λ.

Clearly $y_0 \in p^0A = A$. Let λ be an ordinal and suppose we have shown that $y_0 \in p^\beta A$, for every $\beta \leq \lambda$. Applying what we have shown to $(y_m)_{m\geq1}$ we see that $y_1 \in p^\lambda A$. Thus we obtain

$$y_0 = py_1 \in p(p^\lambda A) = p^{\lambda+1}A.$$

Now suppose that λ is a limit ordinal and we have shown that $y_0 \in p^\beta A$ for every $\beta < \lambda$. Then

$$y_0 \in \bigcap_{\beta<\lambda} p^\beta A = p^\lambda A,$$

which finishes the proof by induction. $\square$

Lemma 1.3.12. *Let $q \neq p$ be prime numbers and let G be a module over the p-adic integers. Then the underlying abelian group of G is q-divisible.*

Proof. We have

$$\mathbb{Z}_p = \{c \in \mathbb{Q}_p \mid |c|_p \leq 1\}.$$

Consider $1/q \in \mathbb{Q} \subseteq \mathbb{Q}_p$. Then $\nu_p(1/q) = 0$ so $|1/q|_p = 1$ and thus $1/q \in \mathbb{Z}_p$. Now let $g \in G$. Since G is a module over $\mathbb{Z}_p$ we have that $(1/q)g \in G$ and $q((1/q)g) = g$. $\square$

Proposition 1.3.13. *Let p be a prime number and let G be an abelian group that is a module over the p-adic integers. Then for every ordinal λ we have $u(G) \leq \lambda$ iff $l_p(G) \leq \omega\lambda$.*

Proof. By Lemmas 1.3.12 and 1.3.10, we see that

$$G^\lambda = \bigcap_{q\in\mathbb{P}} q^{\omega\lambda} G = p^{\omega\lambda} G.$$

Using transfinite induction it is easily seen that G^λ is a sub $\mathbb{Z}_p$-module of G, so we obtain from Lemma 1.3.12 that G^λ is q-divisible for every prime $q \neq p$. Thus, G^λ is divisible, iff it is p-divisible.

Now suppose that $u(G) \leq \lambda$. Then G^λ is divisible. Thus $p^{\omega\lambda}G = G^\lambda$ is p-divisble, and $l_p(G) \leq \omega\lambda$.

Conversely, suppose that $l_p(G) \leq \omega\lambda$. Then $p^{\omega\lambda}G = G^\lambda$ is p-divisble, and as we have shown, also divisible, so $u(G) \leq \lambda$. $\square$

1.3.3. *The* Ext *functor.*

Definition 1.3.14. Let A and C be abelian groups. We denote by $\mathrm{Ext}(C, A)$ the set of equivalence classes of short exact sequences of the form

$$0 \to A \to B \to C \to 0,$$

where an exact sequence as above is called equivalent to an exact sequence

$$0 \to A \to B' \to C \to 0,$$

if there exists an isomorphism $B \to B'$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0.
\end{tikzcd}$$

The set $\mathrm{Ext}(C, A)$ can be given the structure of an abelian group by defining

$$[0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0] + [0 \to A \xrightarrow{f'} B' \xrightarrow{g'} C \to 0],$$

to be

$$[0 \to A \xrightarrow{f \prod f'} B \oplus B' \xrightarrow{g \coprod g'} C \to 0].$$

The zero element in $\mathrm{Ext}(C, A)$ is the splitting short exact sequence

$$[0 \to A \to A \oplus C \to C \to 0].$$

It is also possible to lift the above construction to a functor

$$\mathrm{Ext} : \mathcal{A}\mathrm{b}^{\mathrm{op}} \times \mathcal{A}\mathrm{b} \to \mathcal{A}\mathrm{b},$$

using pullbacks and pushouts in the category of abelian groups.

Note that if A and C are abelian groups, then we have $\mathrm{Ext}(C, A) = 0$ iff every short exact sequence of the form

$$0 \to A \to B \to C \to 0$$

splits, or in other words, iff for every group B containing A and satisfying $B/A \cong C$ we have that A is a direct summand in B. Let us recall two important properties of the Ext functor that we will use later on.

Theorem 1.3.15 ([Fu1, Theorem 51.3])**.** *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian groups and let G be an abelian group. Then we have an exact sequence*

$$0 \to \mathrm{Hom}(G, A) \to \mathrm{Hom}(G, B) \to \mathrm{Hom}(G, C) \to \mathrm{Ext}(G, A) \to \mathrm{Ext}(G, B) \to \mathrm{Ext}(G, C) \to 0,$$

where the morphisms (except the middle one) are the ones induced by the functors Hom *and* Ext.

Theorem 1.3.16 ([Fu1, Theorem 52.2])**.** *Let A be an abelian group and $(A_i)_{i\in I}$ a collection of abelian groups. Then there are natural isomorphisms*

(1)
$$\mathrm{Ext}(\bigoplus_{i\in I} A_i, A) \cong \prod_{i\in I} \mathrm{Ext}(A_i, A),$$

(2)
$$\mathrm{Ext}(A, \prod_{i\in I} A_i) \cong \prod_{i\in I} \mathrm{Ext}(A, A_i).$$

1.3.4. *Cotorsion groups.*

Definition 1.3.17. An abelian group G is called *cotorsion* if for every torsion free abelian group A, we have $\mathrm{Ext}(A, G) = 0$.

In other words, an abelian group G is cotorsion iff for every abelian group H, that contains G as a subgroup, such that H/G is torsion free, we have that G is a direct summand of H.

Proposition 1.3.18. *An abelian group G is cotorsion iff* $\mathrm{Ext}(\mathbb{Q}, G) = 0$

Proof. This follows easily from Theorems 1.3.15 and 1.3.16, using the fact that every torsion free abelian group can be embedded in a group of the form $\bigoplus_{i\in I} \mathbb{Q}$. □

In other words, Proposition 1.3.18 says that an abelian group G is cotorsion iff for every abelian group H, that contains G as a subgroup, such that $H/G \cong \mathbb{Q}$, we have that G is a direct summand of H.

Proposition 1.3.19. *A product $\prod_{i\in I} G_i$ of abelian groups is cotorsion iff G_i is cotorsion for every $i \in I$.*

Proof. Follows easily from Theorem 1.3.16. □

As we have mentioned, divisible abelian groups are classified by a collection of cardinal invariants indexed by $\mathbb{P} \cup \{0\}$ (see Theorem 1.3.3). By decomposing a cotorsion group into its divisible part and reduced part, as in Theorem 1.3.7, and using Proposition 1.3.19, the classification problem for cotorsion groups can be reduced to that of reduced cotorsion groups. In [Har], Harrison proves the following structure theorem for reduced cotorsion groups.

Theorem 1.3.20 (Harrison [Har])**.** *Let G be an abelian group. Then G is reduced cotorsion iff there exist a divisible torsion abelian group D and a reduced torsion abelian group T, such that*

$$G \cong \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, D) \oplus \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T).$$

Moreover, we have:

(1) *The group D satisfies*
$$D \cong (\mathbb{Q}/\mathbb{Z}) \otimes \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, D) \cong (\mathbb{Q}/\mathbb{Z}) \otimes G,$$
and $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, D)$ *is torsion free.*

(2) *The group T satisfies*
$$T \cong \mathrm{Tor}(\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T)) \cong Tor(G),$$
and $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T)/T$ *is divisible.*

We can restate Harrison's theorem in an alternative way:

Theorem 1.3.21. *Let G be an abelian group. Then G is reduced cotorsion iff there exist cardinals λ_p, for every $p \in \mathbb{P}$, and a reduced torsion abelian group T, such that*
$$G \cong \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T \oplus \bigoplus_{p\in\mathbb{P}} \bigoplus_{\alpha<\lambda_p} \mathbb{Z}_p).$$
Moreover, we have:

(1) *The cardinals λ_p are uniquely determined by G*
(2) *The group T satisfies $T \cong Tor(G)$.*

Proof. The divisible torsion abelian group D, appearing in Theorem 1.3.20, is classified by a collection of cardinal invariants $(\lambda_p)_{p\in\mathbb{P}}$ as
$$D \cong \bigoplus_{p\in\mathbb{P}} \bigoplus_{\alpha<\lambda_p} \mathbb{Z}(p^\infty).$$

By considering the exact sequence
$$0 \longrightarrow \bigoplus_{p\in\mathbb{P}} \bigoplus_{\alpha<\lambda_p} \mathbb{Z}_p \longrightarrow \bigoplus_{p\in\mathbb{P}} \bigoplus_{\alpha<\lambda_p} \mathbb{Q}_p \longrightarrow \bigoplus_{p\in\mathbb{P}} \bigoplus_{\alpha<\lambda_p} \mathbb{Z}(p^\infty) \longrightarrow 0$$
and using Theorems 1.3.5 and 1.3.15 we see that we have a natural isomorphism
$$\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \bigoplus_{p\in\mathbb{P}} \bigoplus_{\alpha<\lambda_p} \mathbb{Z}(p^\infty)) \cong \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, \bigoplus_{p\in\mathbb{P}} \bigoplus_{\alpha<\lambda_p} \mathbb{Z}_p).$$
Now the result follows from Theorem 1.3.16 □

Remark 1.3.22. Theorem 1.3.21 reduces the structure problem for reduced cotorsion (and hence for general cotorsion) groups to that of reduced torsion groups.

Corollary 1.3.23. *Every reduced cotorsion group G can be written as a product of the form*
$$G \cong \prod_{p\in\mathbb{P}} G_p,$$
where for each $p \in \mathbb{P}$ we have that G_p is a module over the p-adic integers.

Proof. By Theorem 1.3.21 there exists an abelian group B such that
$$G \cong \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, B).$$
From Theorem 1.3.3 it is easily seen that
$$\mathbb{Q}/\mathbb{Z} \cong \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty).$$
Thus from Theorem 1.3.16 we obtain that
$$G \cong \prod_{p\in\mathbb{P}} \mathrm{Ext}(\mathbb{Z}(p^\infty), B).$$

Now the result follows from the fact that for each prime p, $\mathrm{Ext}(\mathbb{Z}(p^\infty), B)$ is a module over the p-adic integers through the action of $\mathbb{Z}_p$ on $\mathbb{Z}(p^\infty) \cong \mathbb{Q}_p/\mathbb{Z}_p$. □

Theorem 1.3.24. *Let G be a cotorsion group and let λ be an ordinal. Then $u(G) \leq \lambda$ iff $l_p(G) \leq \omega\lambda$ for every $p \in \mathbb{P}$.*

Proof. By decomposing G into its divisible part and reduced part, as in Theorem 1.3.7, we may assume that G is reduced.

By Corollary 1.3.23, G can be written as a product of the form $G \cong \prod_{p\in\mathbb{P}} G_p$, where every G_p is a module over the p-adic integers. Thus, using Proposition 1.3.13, we see that the following statements are equivalent:

(1) $u(G) \leq \lambda$.
(2) G^λ is divisible.
(3) G_p^λ is divisible for every prime p.
(4) $u(G_p) \leq \lambda$ for every prime p.
(5) $l_p(G) \leq \omega\lambda$ for every prime p.

□

2. The equation criterion

The purpose of this section is to prove that an abelian group is cotorsion iff certain types of systems of equations have a solution in it (see Theorem 2.0.26). This result will be our main tool in proving Theorem 0.0.1.

In [Fu1, Section 22], Fuchs defines the notion of a *system of equations* over an abelian group. There is an equation for every $i \in I$, with unknowns $(x_j)_{j\in J}$, where I and J can be arbitrary sets. We will only be using this notion with $I = J = \mathbb{N}$.

Definition 2.0.25. Let H be an abelian group. A *system of equations* over H is the following data:

(1) An $\mathbb{N} \times \mathbb{N}$ matrix with entries in $\mathbb{Z}$, denoted $(l_{n,m})$, such that for every $n \in \mathbb{N}$ the set

$$\{m \in \mathbb{N} \mid l_{n,m} \neq 0\}$$

is finite.
(2) An object (vector) (a_n) in $H^{\mathbb{N}}$.

Note that for every $n \in \mathbb{N}$ we have an equation

$$\sum_{m=0}^{\infty} l_{n,m} x_m = a_n,$$

with unknowns (x_m).

A *solution*, in H, to the system of equations is a vector (g_n) in $H^{\mathbb{N}}$ such that substituting $(x_m) = (g_m)$ in the system above yields correct statements.

Theorem 2.0.26. *Let H be an abelian group. Then the following conditions are equivalent:*

(1) *The group H is cotorsion.*

(2) *Every system of equations over H, with a matrix of the form*

$$\begin{matrix} 1 & l_0 & 0 & 0 & \cdots \\ & 1 & l_1 & 0 & \cdots \\ & & 1 & l_2 & \cdots \\ & & & 1 & \cdots \\ & & & & \cdots \end{matrix}$$

has a solution in H.

(3) *Every system of equations over H, with a matrix that is upper triangular and has identities in its diagonal, has a solution in H.*

Proof. (2) $\Rightarrow$ (1). Let G be an abelian group, that contains H as a subgroup, and suppose $G/H \cong \mathbb{Q}$. Using Proposition 1.3.18, we are left to show that H is a direct summand of G.

Let us choose a fixed isomorphism from G/H to $\mathbb{Q}$, and use it to identify

$$G/H = \mathbb{Q}.$$

For every $n \in \mathbb{N}$ let us choose an element $x_n \in G$ such that

$$[x_n] := x_n + H = (2^n 3^n \cdots p_n^n)^{-1} \in \mathbb{Q} = G/H$$

(See Section 0.2).

Clearly, for every $n \in \mathbb{N}$ there exists $k_n \in \mathbb{Z}$ such that $[x_n] = k_n[x_{n+1}]$. We define

$$f_n := x_n - k_n x_{n+1} \in H.$$

Using condition (2) with $l_n := -k_n$ and (f_n), we see that there exists a vector (g_n) in $H^{\mathbb{N}}$ such that for every $n \in \mathbb{N}$

$$g_n = f_n + k_n g_{n+1}.$$

For every $n \in \mathbb{N}$ we define

$$y_n := x_n - g_n \in G.$$

Then for every $n \in \mathbb{N}$ we have

$$y_n - k_n y_{n+1} = (x_n - g_n) - k_n(x_{n+1} - g_{n+1})$$
$$= (x_n - k_n x_{n+1}) - (g_n - k_n g_{n+1}) = f_n - f_n = 0.$$

Thus $y_n = k_n y_{n+1}$ and $[x_n] = [y_n]$. Note that we have turned equations modulo H (in $[x_n]$) to equations in G (in y_n) using condition (2).

For every $n \in \mathbb{N}$, we thus obtain that

$$\mathbb{Z}y_n \subseteq \mathbb{Z}y_{n+1}.$$

It follows that

$$L := \bigcup_{n=0}^{\infty} \mathbb{Z}y_n$$

is a subgroup of G. Clearly we have

$$G/H = \bigcup_{n=0}^{\infty} \mathbb{Z}[y_n].$$

It follows that

$$G = H + \bigcup_{n=0}^{\infty} \mathbb{Z}y_n = H + L.$$

We are thus left to show that $H \cap L = \{0\}$. Let $y \in H \cap L$. Then there exists $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that $y = ky_n$. It follows that $k|y$ in G. Since G/H is torsion free, it is not hard to see that $k|y$ also in H. Thus, there exists $y'_n \in H$ such that

$$\begin{aligned} y = ky'_n = ky_n \ &\Rightarrow \\ k(y_n - y'_n) = 0 \ &\Rightarrow \\ k[y_n] = 0. \end{aligned}$$

Since $\mathbb{Q}$ is torsion free and $[y_n] \neq 0$ we deduce that $k = 0$, so $y = ky_n = 0$.

(1) $\Rightarrow$ (3). Let $(l_{n,m})$, (b_n) be a system of equations over H, and suppose that $l_{n,n} = 1$ and $l_{n,m} = 0$ if $n > m$. Let us recall from [Fu1] the notion of an *algebraically compact group*. By [Fu1, Theorem 38.1], an algebraically compact group can be defined as an abelian group G such that every system of equations over G, for which every finite subsystem has a solution in G, also has a global solution in G. Since H is cotorsion, we know, by [Fu1, Proposition 54.1], that there exists an algebraically compact group G and a surjective homomorphism $p : G \to H$. For every $n \in \mathbb{N}$ let us choose $c_n \in G$, such that $p(c_n) = b_n$. Then $(l_{n,m})$, (c_n) is a system of equations over G, and it is easy to see that every finite subsystem of it has a solution in G. (If $N \in \mathbb{N}$ we can find a solution to the first N equations as follows: First define $x_{N+1}, x_{N+2}, \cdots$ to be any elements in G. Now define $x_N \in G$ according to the N'th equation. Then define $x_{N-1} \in G$ according to the $(N-1)$'th equation, and so on.) Thus, we have a solution (g_n) to this system of equations in G. Now it is easily seen that $(p(g_n))$ is a solution to our original system of equations in H.

(3) $\Rightarrow$ (2) is obvious so we are done. □

3. Countable directed limits

3.1. **Sequential limits and cotorsion groups.** Recall that $\mathbb{N} = \{0, 1, 2, \cdots\}$ denotes the poset of natural numbers and that, according to our convention (see Section 0.2), we view $\mathbb{N}$ as a small category which has a single morphism $m \to n$ whenever $m \geq n$.

Let $H : \mathbb{N} \longrightarrow \mathcal{G}rp$ be a diagram of shape $\mathbb{N}$ in the category of groups. If $m \leq n$, then we have a unique morphism $n \to m$ in $\mathbb{N}$. We define

$$\phi_{m,n} := H(n \to m) : H_n \to H_m.$$

We denote by H_ω the limit

$$H_\omega = \lim_{n \in \mathbb{N}} H_n$$

and by

$$\phi_n : H_\omega \to H_n$$

the natural map, for every $n \in \mathbb{N}$ (see Section 1.2).

Lemma 3.1.1. *Suppose that $F < H_\omega$ is a subgroup and for every $n \in \mathbb{N}$ the restricted map*

$$\phi_n|_F : F \to H_n$$

is surjective. Then for every $f \in H_\omega$ and every $n \in \mathbb{N}$, there exists $\bar{f} \in H_\omega$, such that:

(1) *$F\bar{f} = Ff$.*
(2) *For every $i < n$ we have $\phi_i(\bar{f}) = e_{H_i}$.*

Proof. Let $f \in H_\omega$ and let $n \in \mathbb{N}$. If $n = 0$ let us choose $\bar{f} := f$. Suppose $n > 0$. By the hypothesis of the lemma we know that

$$\phi_{n-1}|_F : F \to H_{n-1}$$

is surjective. Since $\phi_{n-1}(f) \in H_{n-1}$, we obtain that there exists $f'' \in F$ such that

$$\phi_{n-1}(f'') = \phi_{n-1}(f).$$

We now define

$$\bar{f} := (f'')^{-1} f \in H_\omega,$$

so clearly $F\bar{f} = Ff$.

Now let $i < n$. The following diagram commutes

$$\begin{tikzcd} H_\omega \arrow[r, "\phi_{n-1}"] \arrow[dr, "\phi_i"'] & H_{n-1} \arrow[d, "\phi_{i,n-1}"] \\ & H_i. \end{tikzcd}$$

It follows that

$$\phi_i(f'') = \phi_{i,n-1}(\phi_{n-1}(f'')) = \phi_{i,n-1}(\phi_{n-1}(f)) = \phi_i(f),$$

so we have

$$\phi_i(\bar{f}) = (\phi_i(f''))^{-1}\phi_i(f) = e_{H_i},$$

which finishes the proof of our lemma. $\square$

We now come to the main result of this section, which will be our main tool in showing that certain abelian groups are cotorsion.

Theorem 3.1.2. *Suppose that $F \lhd H_\omega$ is a normal subgroup such that $T := H_\omega/F$ is abelian and for every $n \in \mathbb{N}$ the restricted map*

$$\phi_n|_F : F \to H_n$$

is surjective. Then the abelian group T is cotorsion.

Proof. Let (k_n) be an object in $\mathbb{Z}^{\mathbb{N}}$ and let (f_n) be an object in $T^{\mathbb{N}}$. By Theorem 2.0.26 (setting $l_n = -k_n$), we need to show that there exists an object (g_n) in $T^{\mathbb{N}}$ such that for every $n \in \mathbb{N}$ we have

$$g_n = f_n + k_n g_{n+1}.$$

Let $n \in \mathbb{N}$. We have

$$f_n \in T = H_\omega/F,$$

so we can choose $f'_n \in H_\omega$ such that

$$[f'_n] := Ff'_n = f_n.$$

By Lemma 3.1.1, there exists $\bar{f}_n \in H_\omega$, such that:

(1) $[\bar{f}_n] = [f'_n] = f_n$.
(2) For every $l < n$ we have $\phi_l(\bar{f}_n) = e_{H_l}$.

For every $n > l$ we define $\bar{g}_{n,l} := e_{H_l}$.

Now, let $l \geq 0$ be fixed. We have defined $\bar{g}_{n,l} \in H_l$ for every $n > l$. Let us now define $\bar{g}_{n,l} \in H_l$ for every $n \leq l$ recursively, using the formula

$$\bar{g}_{n,l} = \phi_l(\bar{f}_n)\bar{g}_{n+1,l}^{k_n}.$$

That is, we define

$$\bar{g}_{l,l} = \phi_l(\bar{f}_l)\bar{g}_{l+1,l}^{k_l} = \phi_l(\bar{f}_l),$$

$$\bar{g}_{l-1,l} = \phi_l(\bar{f}_{l-1})\bar{g}_{l,l}^{k_{l-1}} = \phi_l(\bar{f}_{l-1})\phi_l(\bar{f}_l)^{k_{l-1}},$$

and so on.

We have now defined $\bar{g}_{n,l} \in H_l$ for every $n, l \in \mathbb{N}$, and clearly the formula

$$\bar{g}_{n,l} = \phi_l(\bar{f}_n)\bar{g}_{n+1,l}^{k_n}$$

is now satisfied for every $n, l \in \mathbb{N}$. (Note that whenever $n > l$ we have $\phi_l(\bar{f}_n) = e_{H_l}$ by (2) above.) For every $n \in \mathbb{N}$ we now define

$$\bar{g}_n := (\bar{g}_{n,l})_{l\in\mathbb{N}} \in \prod_{l\in\mathbb{N}} H_l.$$

Recall from Section 1.2, that

$$H_\omega \cong \varprojlim_{l\in\mathbb{N}} H_l \cong \{(x_l)_{l\in\mathbb{N}} \in \prod_{l\in\mathbb{N}} H_l \mid \forall\ l \geq m\ .\ \phi_{m,l}(x_l) = x_m\}$$

$$= \{(x_l)_{l\in\mathbb{N}} \in \prod_{l\in\mathbb{N}} H_l \mid \forall l \in \mathbb{N}\ .\ \phi_{l,l+1}(x_{l+1}) = x_l\}.$$

We want to show that for every $n \in \mathbb{N}$ we actually have $\bar{g}_n \in H_\omega$, that is, that for every $n, l \in \mathbb{N}$ we have

$$\phi_{l,l+1}(\bar{g}_{n,l+1}) = \bar{g}_{n,l}.$$

Clearly, this follows from the following lemma, taking $i = l + 2$:

Lemma 3.1.3. *Let $l \in \mathbb{N}$ be fixed. Then for every $i \leq l + 2$ and every $n > l - i + 1$ we have*

$$\phi_{l,l+1}(\bar{g}_{n,l+1}) = \bar{g}_{n,l}.$$

Proof. We prove the lemma by induction on i. When $i = 0$ then $n > l + 1$ so we have

$$\bar{g}_{n,l+1} = e_{H_{l+1}},\ \ \bar{g}_{n,l} = e_{H_l}$$

and the lemma is clear. Now suppose we have proven the lemma for some $i < l + 2$, and let us prove it for $i + 1$. Let $n > l - i$. We know that

$$\bar{g}_{n,l+1} = \phi_{l+1}(\bar{f}_n)\bar{g}_{n+1,l+1}^{k_n}.$$

It follows that

$$\phi_{l,l+1}(\bar{g}_{n,l+1}) = \phi_{l,l+1}(\phi_{l+1}(\bar{f}_n))\phi_{l,l+1}(\bar{g}_{n+1,l+1})^{k_n}.$$

Since $n + 1 > l - i + 1$, we can use the induction hypothesis to obtain

$$\phi_{l,l+1}(\bar{g}_{n,l+1}) = \phi_l(\bar{f}_n)\bar{g}_{n+1,l}^{k_n} = \bar{g}_{n,l},$$

which proves our lemma. □

Let $n \in \mathbb{N}$. We have shown that $\bar{g}_n \in H_\omega$. We define

$$g_n := [\bar{g}_n] \in H_\omega/F = T.$$

For every $l \in \mathbb{N}$ we have an equality in H_l

$$\bar{g}_{n,l} = \phi_l(\bar{f}_n)\bar{g}_{n+1,l}^{k_n}.$$

Thus in H_ω we have

$$\bar{g}_n = \bar{f}_n\bar{g}_{n+1}^{k_n}.$$

Passing to equivalence classes we obtain the following equality in $T = H_\omega/F$:

$$[\bar{g}_n] = [\bar{f}_n][\bar{g}_{n+1}]^{k_n}.$$

But T is abelian, so in additive notation we obtain

$$g_n = f_n + k_n g_{n+1}.$$

which finishes the proof of our theorem. □

The following definition is non standard, but will be useful for us:

Definition 3.1.4. A *quasi concrete category* is a category $\mathcal{C}$, which admits countable directed limits, together with a functor $U : \mathcal{C} \longrightarrow \mathcal{S}\mathrm{et}$ that commutes with countable directed limits.

Examples of quasi concrete categories are easy to find. Many known categories, whose objects consist of sets with some extra structure and whose morphisms are set-functions that preserve this structure, are quasi concrete, if we take U to be the functor that assigns to every object its underlying set. These examples include sets, pointed sets, monoids, groups, rings, Lie algebras etc. In fact, if T is any *Lawvere algebraic theory* [Law], then the category of T-algebras is quasi concrete.

Definition 3.1.5. Let $(\mathcal{C}, U)$ be a quasi concrete category. A morphism f in $\mathcal{C}$ is called surjective if the morphism $U(f)$ is surjective in $\mathcal{S}\mathrm{et}$.

Theorem 3.1.6. *Let $(\mathcal{C}, U)$ be a quasi concrete category and let $A : \mathcal{C} \to \mathcal{A}\mathrm{b}$ be a functor that sends surjective morphisms to surjective morphisms. Let $F : \mathbb{N} \longrightarrow \mathcal{C}$ be a diagram such that for every $n \in \mathbb{N}$ the structure map $F_{n+1} \to F_n$ is surjective. Then the cokernel of the natural map*

$$\rho : A(\lim_{n\in\mathbb{N}} F_n) \longrightarrow \lim_{n\in\mathbb{N}} A(F_n)$$

is cotorsion.

Proof. By Theorem 3.1.2, we only need to show that for every $m \in \mathbb{N}$ the restriction of the natural map

$$\lim_{n\in\mathbb{N}} A(F_n) \to A(F_m)$$

to $\operatorname{Im}\rho$ is surjective. It thus suffices to show that for every $m \in \mathbb{N}$ the map

$$A(\lim_{n\in\mathbb{N}} F_n) \to A(F_m)$$

is surjective.

So let $m \in \mathbb{N}$. Since, by our hypothesis, the map $U(F_{n+1}) \to U(F_n)$ is surjective, for every $n \in \mathbb{N}$, we have that the map

$$U(\lim_{n\in\mathbb{N}} F_n) \cong \lim_{n\in\mathbb{N}} U(F_n) \longrightarrow U(F_m)$$

is surjective. By definition, this implies that the map $\lim_{n\in\mathbb{N}} F_n \longrightarrow F_m$ is surjective. But A sends surjective maps to surjective maps so the map

$$A(\lim_{n\in\mathbb{N}} F_n) \to A(F_m)$$

is also surjective. □

Corollary 3.1.7. *Let $(\mathcal{C}, U)$ be a quasi concrete category and let $A : \mathcal{C} \to \mathcal{A}\mathrm{b}$ be a functor that sends surjective morphisms to surjective morphisms. Let $\mathcal{T}$ be a countable directed poset and let $F : \mathcal{T} \longrightarrow \mathcal{C}$ be a diagram such that for every $t \geq s$ in $\mathcal{T}$, the structure map $F_t \to F_s$ is surjective. Then the cokernel of the natural map*

$$A(\lim_{t\in\mathcal{T}} F_t) \longrightarrow \lim_{t\in\mathcal{T}} A(F_t)$$

is cotorsion.

Proof. By Proposition 1.2.5, there exists a cofinal functor $\mathbb{N} \longrightarrow \mathcal{T}$. Now the result follows from Lemma 1.2.3 and Theorem 3.1.6. □

Example 3.1.8. *Let us note a few cases where Corollary 3.1.7 applies:*

(1) $\mathcal{C} = \mathcal{S}\mathrm{et}$ *and A is the free abelian group functor.*

(2) $\mathcal{C} = \mathcal{S}\mathrm{et}_*$ *is the category of pointed sets and* A *is the functor that assigns to every pointed set* X *the free abelian group on* $X \setminus \{*\}$*, where* $* \in X$ *is the special point.*
(3) $\mathcal{C} = \mathcal{AM}\mathrm{on}$ *is the category of abelian monoids and* A *is the group completion functor.*
(4) $\mathcal{C} = \mathcal{G}\mathrm{rp}$ *and* A *is the abelianization functor (see Section 1.1).*
(5) $\mathcal{C} = \mathcal{A}\mathrm{b}$ *and* $A = G \otimes (-)$ *is the functor which takes the tensor product with a fixed abelian group* G*.*
(6) *Let* $\mathcal{C}$ *be the category of unbounded chain complexes of abelian groups. For a chain complex*

$$G = \cdots \longrightarrow G_{n+1} \xrightarrow{d_{n+1}} G_n \xrightarrow{d_n} \cdots$$

let us define
 (a) $C_n(G) := G_n$,
 (b) $Z_n(G) := \ker(d_n)$,
 (c) $B_n(G) := Im\,(d_{n+1})$,
 (d) $H_n(G) := Z_n(G)/B_n(G)$.
Now let $m \in \mathbb{Z}$ *be fixed. Then we can either take* $U = C_{m+1}$ *and* $A = B_m$ *or* $U = Z_m$ *and* $A = H_m$.

Corollary 3.1.9. *Let* $\mathcal{C}$ *be the category of unbounded chain complexes of abelian groups and let* C_n*,* Z_n*,* B_n *and* H_n *be as in Example 3.1.8 (6). Let* $\mathcal{T}$ *be a countable directed poset and let* $m \in \mathbb{Z}$*. Let* $F : \mathcal{T} \longrightarrow \mathcal{C}$ *be a diagram such that for every* $t \geq s$ *in* $\mathcal{T}$*, the maps* $C_{m+1}(F_t) \to C_{m+1}(F_s)$ *and* $Z_m(F_t) \to Z_m(F_s)$ *are surjective. Then the kernel and cokernel of the natural map*

$$H_m(\lim_{t\in\mathcal{T}} F_t) \longrightarrow \lim_{t\in\mathcal{T}} H_m(F_t)$$

is cotorsion.

Proof. This follows from Corollary 3.1.7 and Example 3.1.8 (6), using the fact that the kernel of the natural map above is isomorphic to the cokernel of the natural map

$$B_m(\lim_{t\in\mathcal{T}} F_t) \longrightarrow \lim_{t\in\mathcal{T}} B_m(F_t).$$

□

3.2. **The abelianization of countable directed limits.** We now turn to the main result of this section

Theorem 3.2.1. *Let* $G : \mathbb{N} \longrightarrow \mathcal{G}\mathrm{rp}$ *be a diagram of shape* $\mathbb{N}$ *in the category of groups and suppose that for every* $n \in \mathbb{N}$*, the structure map* $G_{n+1} \to G_n$ *is surjective. Then the kernel and cokernel of the natural map*

$$\rho : \mathrm{Ab}(\lim_{n\in\mathbb{N}} G_n) \longrightarrow \lim_{n\in\mathbb{N}} \mathrm{Ab}(G_n)$$

are cotorsion.

Proof. The fact that $\mathrm{coker}\,(\rho)$ is cotorsion follows easily from Theorem 3.1.6.

We turn to $\ker(\rho)$. Let us begin by giving a more concrete description of the natural map ρ. Recall from Section 1.1, the commutator subgroup functor

$$\mathrm{C} : \mathcal{G}\mathrm{rp} \longrightarrow \mathcal{G}\mathrm{rp}\,.$$

We thus have an induced diagram $\mathrm{C} \circ G : \mathbb{N} \longrightarrow \mathcal{G}\mathrm{rp}$ and a natural map

$$\chi : \mathrm{C}(\lim_{n\in\mathbb{N}} G_n) \longrightarrow \lim_{n\in\mathbb{N}} \mathrm{C}(G_n).$$

Let us denote $G_\omega := \lim_{n\in\mathbb{N}} G_n$ and $\phi_n : G_\omega \to G_n$ the natural map, for every $n \in \mathbb{N}$.

We denote
$$C_\omega := \bigcap_{n=0}^{\infty} \phi_n^{-1}(\mathrm{C}(G_n)) \subseteq G_\omega.$$
Clearly $\mathrm{C}(G_\omega) \subseteq C_\omega$. It is also easily verified that
$$C_\omega \cong \lim_{n\in\mathbb{N}} \mathrm{C}(G_n),$$
and the inclusion $\mathrm{C}(G_\omega) \subseteq C_\omega$ is just the natural map χ above.

For every $n \in \mathbb{N}$, the map $\phi_n : G_\omega \to G_n$ induces a map
$$\psi_n : G_\omega/C_\omega \longrightarrow G_n/\,\mathrm{C}(G_n) = \mathrm{Ab}(G_n).$$
The maps (ψ_n) are compatible and thus induce a map
$$\psi : G_\omega/C_\omega \longrightarrow \lim_{n\in\mathbb{N}} \mathrm{Ab}(G_n).$$
It is not hard to see that ψ is injective. We have a natural surjective map
$$p : G_\omega/\,\mathrm{C}(G_\omega) \longrightarrow G_\omega/C_\omega,$$
and the composition $\psi \circ p$ is just the natural map
$$\rho : \mathrm{Ab}(\lim_{n\in\mathbb{N}} G_n) \longrightarrow \lim_{n\in\mathbb{N}} \mathrm{Ab}(G_n).$$

We know that $\mathrm{C}(G_\omega)$ is a normal subgroup of G_ω and thus it is also a normal subgroup of C_ω. Since ψ is injective we get that
$$\ker(\rho) = \ker(\psi \circ p) = \ker(p) = C_\omega/\,\mathrm{C}(G_\omega) \cong \lim_{n\in\mathbb{N}} \mathrm{C}(G_n)/\,\mathrm{C}(\lim_{n\in\mathbb{N}} G_n).$$

Let $n \in \mathbb{N}$. Since for every $m \in \mathbb{N}$, the map $G_{m+1} \to G_m$ is surjective, it follows that the map $\phi_n : G_\omega \to G_n$ is surjective. Thus
$$\phi_n|_{\mathrm{C}(G_\omega)} : \mathrm{C}(G_\omega) \to \mathrm{C}(G_n)$$
is also surjective. Using Theorem 3.1.2 with $H := \mathrm{C} \circ G : \mathbb{N} \longrightarrow \mathcal{G}\mathrm{rp}$, we see that $\ker(\rho) = C_\omega/\,\mathrm{C}(G_\omega)$ is cotorsion. $\square$

Corollary 3.2.2. *Let $\mathcal{T}$ be a countable directed poset and let $G : \mathcal{T} \longrightarrow \mathcal{G}\mathrm{rp}$ be a diagram of groups. Suppose that for every $t \geq s$ in $\mathcal{T}$, the structure map $G_t \to G_s$ is surjective. Then the kernel and cokernel of the natural map*
$$\mathrm{Ab}(\lim_{t\in\mathcal{T}} G_t) \longrightarrow \lim_{t\in\mathcal{T}} \mathrm{Ab}(G_t)$$
are cotorsion.

Proof. By Proposition 1.2.5, there exists a cofinal functor $\mathbb{N} \longrightarrow \mathcal{T}$. Now the result follows from Lemma 1.2.3 and Theorem 3.2.1. $\square$

4. Countable products

In the previous section we considered a diagram $G : \mathbb{N} \longrightarrow \mathcal{G}\mathrm{rp}$, and showed that if all the structure maps $G_{n+1} \to G_n$ are surjective, then the kernel and cokernel of the natural map
$$\rho : \mathrm{Ab}(\lim_{n\in\mathbb{N}} G_n) \longrightarrow \lim_{n\in\mathbb{N}} \mathrm{Ab}(G_n),$$
are cotorsion.

In this section we will concentrate on a special case of the last situation. Namely, let $(H_n)_{n\in\mathbb{N}}$ be a countable collection of groups. For every $n \in \mathbb{N}$ we define
$$G_n := \prod_{i\leq n} H_i = H_1 \times \cdots \times H_n.$$

We can turn G into a functor $G : \mathbb{N} \longrightarrow \mathcal{G}\mathrm{rp}$ by letting $G_m \to G_n$ be the natural projection, for every $m \geq n$. Since for every $n \in \mathbb{N}$ we have

$$\mathrm{Ab}(\prod_{i\leq n} H_i) \cong \prod_{i\leq n} \mathrm{Ab}(H_i),$$

we see that the natural map above is

$$\rho : \mathrm{Ab}(\prod_{i\in\mathbb{N}} H_i) \longrightarrow \prod_{i\in\mathbb{N}} \mathrm{Ab}(H_i).$$

It is not hard to see that ρ in this case is surjective. Clearly in this case all the structure maps are surjective, so $\ker(\rho)$ is cotorsion. However, we show below that $\ker(\rho)$ cannot be any cotorsion group. Namely, since a reduced torsion group T can have arbitrary large p-length, for every prime p, we see by Theorem 1.3.21 that a cotorsion group can also have arbitrary large p-length, for every prime p. However, we show in Theorem 4.0.6 that for every prime p the we have $l_p(\ker(\rho)) \leq \aleph_1$.

Definition 4.0.3. Let α be an ordinal. We define

$$\mathrm{des}(\alpha) := \{(\mu_1, ..., \mu_n) \mid n \geq 0 \text{ and } \alpha > \mu_1 > \cdots > \mu_n\}.$$

Note that $\mathrm{des}(\alpha)$ contains also the empty string ϕ, corresponding to $n = 0$.

Let $\mu = (\mu_1, ..., \mu_n) \in \mathrm{des}(\alpha)$. We define

$$l(\alpha) := n \geq 0,$$

$$\min(\mu) := \begin{cases} \alpha & \text{if } \mu = \phi, \\ \mu_n & \text{if } \mu \neq \phi, \end{cases}$$

and if $n \geq m \geq 0$ we define $\mu|_m := (\mu_1, ..., \mu_m)$.

Proposition 4.0.4. *Let N be an infinite set and κ a cardinal such that $\kappa > |N|$. Suppose we are given $k_\mu \in N$, for every $\mu \in \mathrm{des}(\kappa)$. Then for every $n \in \mathbb{N}$ there exists a triple $(k_n, S_n, \mu(n))$, such that:*

(1) *$k_n \in N$.*
(2) *$S_n \subseteq \kappa$ and $|S_n| = \kappa$.*
(3) *$\mu(n) = (\mu(n)_\alpha)_{\alpha\in S_n}$ and for every $\alpha \in S_n$ we have*
 (a) *$\mu(n)_\alpha \in \mathrm{des}(\kappa)$.*
 (b) *$l(\mu(n)_\alpha) = n + 1$.*
 (c) *$\min(\mu(n)_\alpha) = \alpha$.*
 (d) *For every $l \leq n$ we have $k_{\mu(n)_\alpha|_{l+1}} = k_l$.*

Proof. We define the triple $(k_n, S_n, \mu(n))$ recursively with n.

We begin with $n = 0$. For every $\alpha < \kappa$ we have $(\alpha) \in \mathrm{des}(\kappa)$, so $k_{(\alpha)} \in N$. Since κ is a cardinal and $\kappa > |N|$, there exists $k_0 \in N$ such that

$$|\{\alpha < \kappa \mid k_{(\alpha)} = k_0\}| = \kappa.$$

We define

$$S_0 := \{\alpha < \kappa \mid k_{(\alpha)} = k_0\},$$

and for every $\alpha \in S_0$ we define

$$\mu(0)_\alpha := (\alpha) \in \mathrm{des}(\kappa).$$

Clearly, for every $\alpha \in S_0$ we have

(1) $l(\mu(0)_\alpha) = 1$.
(2) $\min(\mu(0)_\alpha) = \alpha$.
(3) $k_{\mu(0)_\alpha|_1} = k_{(\alpha)} = k_0$.

Now let $m \in \mathbb{N}$ and suppose we have defined a triple $(k_n, S_n, \mu(n))$ for every $n \leq m$ such that the conditions in the proposition are satisfied where they are defined. Let us define the triple $(k_{m+1}, S_{m+1}, \mu(m+1))$.

First we define recursively a strictly increasing function $f_m : \kappa \to S_m$ such that for every $\alpha < \kappa$ we have $\alpha < f_m(\alpha)$.

We define

$$f_m(0) := \min(S_m \setminus \{\min(S_m)\}).$$

Clearly $0 < f_m(0)$.

Let $\beta < \kappa$ and suppose we have defined $f_m(\alpha) \in S_m$ for every $\alpha \leq \beta$. Let us now define $f_m(\beta + 1) \in S_m$. We have $f_m(\beta) < \kappa$. Since κ is a cardinal, it follows that $|f_m(\beta)| < \kappa$ and thus

$$|S_m \setminus f_m(\beta)| = |\{\lambda \in S_m \mid \lambda \geq f_m(\beta)\}| = \kappa.$$

In particular

$$\{\lambda \in S_m \mid \lambda > f_m(\beta)\} \neq \phi$$

and we can define

$$f_m(\beta + 1) := \min\{\lambda \in S_m \mid \lambda > f_m(\beta)\} \in S_m.$$

Clearly we have

$$f_m(\beta + 1) > f_m(\beta) \geq \beta + 1.$$

Suppose that $\beta < \kappa$ is a limit ordinal and we have defined $f_m(\alpha) \in S_m$ for every $\alpha < \beta$. Since κ is a cardinal, we have that $|f_m(\alpha)| < \kappa$ for every $\alpha < \beta$, and also $|\beta| < \kappa$. Thus

$$|\bigcup_{\alpha<\beta} f_m(\alpha)| < \kappa,$$

so

$$|S_m \setminus \bigcup_{\alpha<\beta} f_m(\alpha)| = |\{\lambda \in S_m \mid \forall \alpha < \beta . \lambda \geq f_m(\alpha)\}| = \kappa.$$

Since $|\beta| < \kappa$ we obtain in particular that

$$\{\lambda \in S_m \mid \forall \alpha < \beta . \lambda > f_m(\alpha)\} \setminus \{\beta\} \neq \phi$$

and we can define

$$f_m(\beta) := \min(\{\lambda \in S_m \mid \forall \alpha < \beta . \lambda > f_m(\alpha)\} \setminus \{\beta\}) \in S_m.$$

For every $\alpha < \beta$ we thus have

$$f_m(\beta) > f_m(\alpha) > \alpha,$$

so $f_m(\beta) \geq \beta$. But $f_m(\beta) \neq \beta$ so we obtain $f_m(\beta) > \beta$. This finishes our recursive definition of $f_m : \kappa \to S_m$.

For every $\alpha < \kappa$ we have $(\mu(m)_{f_m(\alpha)}, \alpha) \in \mathrm{des}(\kappa)$, so $k_{(\mu(m)_{f_m(\alpha)}, \alpha)} \in N$. Since κ is a cardinal and $\kappa > |N|$, there exists $k_{m+1} \in N$ such that

$$|\{\alpha < \kappa \mid k_{(\mu(m)_{f_m(\alpha)}, \alpha)} = k_{m+1}\}| = \kappa.$$

We define

$$S_{m+1} := \{\alpha < \kappa \mid k_{(\mu(m)_{f_m(\alpha)}, \alpha)} = k_{m+1}\},$$

and for every $\alpha \in S_{m+1}$ we define

$$\mu(m+1)_\alpha := (\mu(m)_{f_m(\alpha)}, \alpha) \in \mathrm{des}(\kappa).$$

Let $\alpha \in S_{m+1}$. Using the induction hypothesis, we have:

(1) $l(\mu(m+1)_\alpha) = l(\mu(m)_{f_m(\alpha)}) + 1 = m + 2$.
(2) $\min(\mu(m+1)_\alpha) = \alpha$.

Let $l \leq m+1$. If $l = m+1$ we have

$$k_{\mu(m+1)_\alpha|_{l+1}} = k_{\mu(m+1)_\alpha} = k_{m+1} = k_l,$$

while if $l \leq m$ we have, using the induction hypothesis,

$$k_{\mu(m+1)_\alpha|_{l+1}} = k_{\mu(m)_{f'_m(\alpha)}|_{l+1}} = k_l,$$

which finishes the proof of our proposition. □

Proposition 4.0.5. *Let H be an abelian group and let α be an ordinal. Then for every $x \in p^\alpha H$ there exist*

$$\bar{x} = (x_\mu)_{\mu \in \mathrm{des}(\alpha)} \in \prod_{\mu \in \mathrm{des}(\alpha)} p^{\min(\mu)} H,$$

such that the following hold:

(1) $x_\phi = x$.
(2) *If $\mu \in \mathrm{des}(\alpha)$ and $l(\mu) = n > 0$, then $px_\mu = x_{\mu|_{n-1}}$.*

Proof. We define $x_\mu \in p^{\min(\mu)} H$ for every $\mu \in \mathrm{des}(\alpha)$ recursively, relative to $l(\mu)$.

Suppose first that $l(\mu) = 0$. Then $\mu = \phi$ and we define

$$x_\phi := x \in p^{\min(\phi)} H = p^\alpha H.$$

Let $n \geq 0$ and suppose that we have defined $x_\mu \in p^{\min(\mu)} H$ for every $\mu \in \mathrm{des}(\alpha)$ with $l(\mu) \leq n$, in such a way that condition (2) above holds where it is defined.

Now let $\mu \in \mathrm{des}(\alpha)$ such that $l(\mu) = n+1$. Clearly $\min(\mu) < \min(\mu|n)$ so $\min(\mu)+1 \leq \min(\mu|n)$ and we have

$$x_{\mu|n} \in p^{\min(\mu|n)} H \subseteq p^{\min(\mu)+1} H = p(p^{\min(\mu)} H).$$

Thus there exist $x_\mu \in p^{\min(\mu)} H$ such that $px_\mu = x_{\mu|n}$. □

Theorem 4.0.6. *The natural map*

$$\rho : \mathrm{Ab}(\prod_{i \in \mathbb{N}} H_i) \longrightarrow \prod_{i \in \mathbb{N}} \mathrm{Ab}(H_i)$$

is surjective and $\ker(\rho)$ *is cotorsion and satisfies $l_p(\ker(\rho)) \leq \aleph_1$ or every prime p.*

Proof. It is not hard to see that the map ψ defined in Section 3.2 is surjective, so ρ is also surjective. Clearly in this case all the structure maps are surjective, so $\ker(\rho)$ is cotorsion by Theorem 3.2.1.

Now let $p \in \mathbb{P}$. For convenience of notation let us denote $S := \ker(\rho)$. Recall that

$$S \cong \lim_{n \in \mathbb{N}} \mathrm{C}(G_n)/\,\mathrm{C}(\lim_{n \in \mathbb{N}} G_n).$$

Since for every $n \in \mathbb{N}$ we have

$$\mathrm{C}(G_n) \cong \mathrm{C}(H_1 \times \cdots \times H_n) \cong \mathrm{C}(H_1) \times \cdots \times \mathrm{C}(H_n),$$

we see that

$$S := \prod_{n \in \mathbb{N}} \mathrm{C}(H_n)/\,\mathrm{C}(\prod_{n \in \mathbb{N}} H_n).$$

We need to show that $p^{\aleph_1} S$ is p-divisible. It is clearly enough to show $p^{\aleph_1} S \subseteq p^{l_p(S)} S$.

So let $x \in p^{\aleph_1} S$. We define

$$\bar{x} = (x_\mu)_{\mu \in \mathrm{des}(\aleph_1)} \in \prod_{\mu \in \mathrm{des}(\aleph_1)} p^{\min(\mu)} S,$$

as in Proposition 4.0.5.

For every $\mu \in \text{des}(\aleph_1)$ we have

$$x_\mu \in p^{\min(\mu)} S \subseteq S = \prod_{n\in\mathbb{N}} \mathrm{C}(H_n)/\,\mathrm{C}(\prod_{n\in\mathbb{N}} H_n),$$

so let us choose a representative

$$f_\mu = (f_\mu(n))_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathrm{C}(H_n)$$

such that $[f_\mu] = x_\mu$.

Let $\mu \in \text{des}(\aleph_1)$ with $l(\mu) = n > 0$. Then by Proposition 4.0.5 we have (in multiplicative notation)

$$x_\mu^p = x_{\mu|_{n-1}}.$$

Thus

$$x_{\mu|_{n-1}}^{-1} x_\mu^p = \mathrm{e} \in S,$$

so

$$f_{\mu|_{n-1}}^{-1} f_\mu^p \in \mathrm{C}(\prod_{n\in\mathbb{N}} H_n).$$

It follows that there exist $k_\mu \in \mathbb{N}$ and

$$g_{\mu,t} = (g_{\mu,t}(n))_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} H_n$$

for every $t < 2k_\mu$ such that

$$f_{\mu|_{n-1}}^{-1} f_\mu^p = \prod_{l<k_\mu} [g_{\mu,2l}, g_{\mu,2l+1}].$$

Let us define $k_\phi := 0$. By Proposition 4.0.4, applied for the set $N = \mathbb{N}$, the cardinal $\kappa = \aleph_1$, and $(k_\mu)_{\mu\in\text{des}(\aleph_1)}$ defined above, we see that for every $n \in \mathbb{N}$ there exists a triple $(k_n, S_n, \mu(n))$, such that for every $n \in \mathbb{N}$ we have:

(1) $k_n \in \mathbb{N}$.
(2) $S_n \subseteq \aleph_1$ and $|S_n| = \aleph_1$.
(3) $\mu(n) = (\mu(n)_\alpha)_{\alpha\in S_n}$ and for every $\alpha \in S_n$ we have
  (a) $\mu(n)_\alpha \in \text{des}(\aleph_1)$.
  (b) $l(\mu(n)_\alpha) = n + 1$.
  (c) $\min(\mu(n)_\alpha) = \alpha$.
  (d) For every $l \leq n$ we have $k_{\mu(n)_\alpha|_{l+1}} = k_l$.

For every $m \in \mathbb{N}$ we define $\alpha_m := \min(S_m)$, and we define

$$h_m = (h_m(n))_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathrm{C}(H_n)$$

by

$$h_m(n) := \begin{cases} \mathrm{e}_{H_n} & \text{if } n < m, \\ f_{\mu(n)_{\alpha_n}|_m}(n) & \text{if } n \geq m. \end{cases}$$

For every $m > 0$ and $t < 2k_m$ we define

$$d_{m,t} = (d_{m,t}(n))_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} H_n$$

by

$$d_{m,t}(n) := \begin{cases} \mathrm{e}_{H_n} & \text{if } n < m, \\ g_{\mu(n)_{\alpha_n}|_m,t}(n) & \text{if } n \geq m. \end{cases}$$

Let $n > m \geq 0$. Since $\mu(n)_{\alpha_n}|_{m+1} \in \mathrm{des}(\aleph_1) \setminus \{\phi\}$, we have as above

$$f^{-1}_{\mu(n)_{\alpha_n}|_m} f^p_{\mu(n)_{\alpha_n}|_{m+1}} = \prod_{l < k_{\mu(n)_{\alpha_n}|_{m+1}}} [g_{\mu(n)_{\alpha_n}|_{m+1},2l}, g_{\mu(n)_{\alpha_n}|_{m+1},2l+1}].$$

In particular, we get an equality in H_n:

$$f_{\mu(n)_{\alpha_n}|_m}(n)^{-1} f_{\mu(n)_{\alpha_n}|_{m+1}}(n)^p = \prod_{l < k_{\mu(n)_{\alpha_n}|_{m+1}}} [g_{\mu(n)_{\alpha_n}|_{m+1},2l}(n), g_{\mu(n)_{\alpha_n}|_{m+1},2l+1}(n)].$$

But $n \geq m+1$ so we obtain

$$h_m(n)^{-1} h_{m+1}(n)^p = \prod_{l < k_m} [d_{m+1,2l}(n), d_{m+1,2l+1}(n)].$$

For every fixed $m \in \mathbb{N}$, the equality above holds for almost all $n \in \mathbb{N}$. Passing to equivalence classes in

$$S = \prod_{n \in \mathbb{N}} \mathrm{C}(H_n) / \, \mathrm{C}(\prod_{n \in \mathbb{N}} H_n)$$

we thus obtain

$$[h_m^{-1} h_{m+1}^p] = [\prod_{l < k_m} [d_{m+1,2l}, d_{m+1,2l+1}]] = \mathrm{e},$$

or

$$[h_m] = [h_{m+1}]^p.$$

By Proposition 1.3.11 we obtain

$$[h_0] \in p^{l_p(S)} S.$$

But for every $n \in \mathbb{N}$ we have $h_0(n) = f_\phi(n)$ so $h_0 = f_\phi$ and

$$x = x_\phi = [f_\phi] = [h_0] \in p^{l_p(S)} S,$$

as required. □

Corollary 4.0.7. *The natural map*

$$\rho : \mathrm{Ab}(\prod_{i \in \mathbb{N}} H_i) \longrightarrow \prod_{i \in \mathbb{N}} \mathrm{Ab}(H_i)$$

is surjective and $\ker(\rho)$ *is cotorsion and satisfies* $u(\ker(\rho)) \leq \aleph_1$.

Proof. By Theorem 4.0.6 ρ is surjective and $\ker(\rho)$ is cotorsion. By Theorem 1.3.24, applied for the ordinal $\lambda := \aleph_1$, we see that $u(\ker(\rho)) \leq \aleph_1$ iff $l_p(\ker(\rho)) \leq \omega\aleph_1 = \aleph_1$ for every $p \in \mathbb{P}$. □

Remark 4.0.8. A cotorsion group A is called called *algebraically compact* if $u(A) \leq 1$ (see [Fu1, Proposition 54.2]), or equivalently, by Theorem 1.3.24, if $l_p(A) \leq \omega$ for every $p \in \mathbb{P}$. The question of whether our bound on $u(S)$ given in Corollary 4.0.7 is strict or can be improved will be addressed in a future paper.

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