# GENERALIZING RANDOM REAL FORCING FOR INACCESSIBLE CARDINALS

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Abstract. The two parallel concepts of "small" sets of the real line are meagre sets and null sets. Those are equivalent to Cohen forcing and Random real forcing for $\aleph_0^{\aleph_0}$; in spite of this similarity, the Cohen forcing and Random Real Forcing have very different shapes. One of these differences is in the fact that the Cohen forcing has an easy natural generalization for ${}^{\lambda}2$ while $\lambda > \aleph_0$, corresponding to an extension for the meagre sets, while the Random real forcing didn't seem to have a natural generalization, as Lebesgue measure doesn't have a generalization for space $2^{\lambda}$ while $\lambda > \aleph_0$. In work [1], Shelah found a forcing resembling the properties of Random Real Forcing for $2^{\lambda}$ while $\lambda$ is a weakly compact cardinal. Here we describe, with additional assumptions, such a forcing for $2^{\lambda}$ while $\lambda$ is an Inaccessible Cardinal; this forcing is $< \lambda$-complete and satisfies the $\lambda^{+}$- c.c hence preserves cardinals and cofinalities, however unlike Cohen forcing, does not add an undominated real.

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## Introduction

There are two classical ways of defining what is a small set of the real line $2^\omega$; the topological definition of a small set is a meagre set, which is a countable union of nowhere stationary sets. The second definition uses measure and defines a set to be small if it is a null set, which means that it has Lebesgue measure zero.

Both the collection of meagre sets and the collection of null sets are ideals in the set $2^\omega$ (or $\omega^\omega$); the forcing modulo the ideal of meagre sets is the Cohen Forcing while the forcing modulo the ideal of null sets is Random Real Forcing [4].

Looking at $\lambda$- reals for $\lambda > \aleph_0$, so elements of the set:

${}^\lambda 2 = \{\eta : \eta$ is a sequence of 0's and 1's of length $\lambda\}$, there is a natural extension to a Cohen Forcing; that would be a forcing modulo sets that are $\lambda$- meagre [2]. Unlike this case, Lebesgue measure has no natural extension in $2^\lambda$ for cardinals $\lambda > \aleph_0$, thus there is no generalization of Random Real Forcing for those cardinals.

An important and useful property of a Random Real Forcing is not adding undominated real; recall that Cohen Forcing adds $f : \lambda \to \lambda$ not smaller (meaning, modulo finite set) than all the original reals (where $\lambda$- reals here are functions $\lambda \to \lambda$). However Random Real Forcing has the property that every "new" real ($\in^\lambda \lambda$) is bounded by a real in the ground model. One of the uses of this property is for cardinal invariants; the bounding number $\mathfrak{d}$ [3] does not change after forcing with random real forcing.

In paper [1], Shelah described a generalization of the null ideal (meaning, the ideal of L sebesgue measure zero sets) for a weakly compact cardinal $\lambda$; that was done by constructing a forcing that has the properties of random real forcing in $2^\lambda$ for a weakly compact $\lambda$; this result is surprising since there is no clear similarity in the definition of the forcing in [1] and Random Real Forcing.

By "having properties of random real forcing" we mean a forcing for which: (1) the $\lambda^+$- chain condition holds and (2) the forcing is strategically $< \lambda$- complete, we can get even $\lambda$- complete; by those conditions it follows that the forcing preserves cardinals and cofinalities when $\lambda = \lambda^{<\lambda}$. Moreover, any new real added in the forcing shall be bounded by a real in the ground model, that will be condition (3): the forcing is $\lambda$- bounding. An additional important property is symmetry, but we delay treating it.

The purpose of this work is to find a forcing as in [1] for Mahlo, and even any inaccessible cardinal (therefore may be smaller than the first weakly compact cardinal). In section 2 we shall describe a construction for which the properties of Random Real Forcing hold for any inaccessible and in particular Mahlo cardinal; those are cardinals whose existence is a weaker condition than the existence of a weakly compact cardinal [5]. However compare to [1] we need some parameter $X \subseteq \lambda$ so the definition is not "pure" as in [1].

We shall assume the existence of a stationary set that reflects only in inaccessibles and has a diamond sequence. Note that this demand can be gotten by an easy forcing [6] and if $V = L$ this is equivalent to not being weakly compact. For a Mahlo cardinal there is a stationary set of inaccessible cardinals below it so in particular this set reflects only in inaccessibles and then we still need to assume the existence of diamond sequence for it. In [1], the main use of the weak compactness was by reflecting an antichain of conditions to an antichain in a corresponding forcing for a smaller cardinal; the purpose of the diamond sequence here will be to overcome this inablity.

Furthermore, for convinience we shall assume that the conditions of the desired forcing are trees that are pruned only in levels of the stationary set (we demand the stationary set to only contain limit ordinals). However it is possible to allow pruning in successor levels, as long as the prune is only of a bounded set.

We may like to make our forcing $< \lambda$- complete (rather than strategically $< \lambda$-complete), this can be done with a minor change: $p \leq q$ iff $p = q$ or $q \subseteq p \wedge tr(p) \triangleleft tr(q)$.

This work is a part of what was promised in [1], the ideas of the construction where stated in Rutgers in 2011.

We intend to deal later with accessible $\lambda = \lambda^{<\lambda} > \aleph_0$, (under reasonable condition) with symmetry; also we can use $|\epsilon|^+$- complete filter on $\theta_\epsilon$ (or on $\{suc_{\boldsymbol{T}}(\eta), \lg(\eta) = \epsilon\}$, as in [1], or Remark 21 below).

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## 1. Preliminaries

In paper [1] Shelah showed a method of finding, for a weakly compact cardinal $\lambda$, a forcing that generalizes the properties of a Random Real forcing for $\aleph_0$. In section 2 we add the assumption of a diamond principle and then see a similar forcing that generalizes the same properties for inaccessible cardinals, with the assumption that there exists a stationary set that reflects only to inaccessibles; so in particular for Mahlo cardinals it follows. Here we show some general definitions that will be used throughout this paper.

**Definition 1.** The cardinal $\lambda$ is said to be inaccessible when it is an uncountable strong limit regular cardinal.

**Definition 2.** The cardinal $\lambda$ is said to be Mahlo cardinal when it is inaccessible and moreover there is a stationary set of inaccessibles in $\lambda$.

*Remark* 3. An inaccessible cardinal and a Mahlo cardinal are large cardinals; so their existence is independent of the axioms of set theory.

**Definition 4.** A forcing resembling random real forcing for a regular cardinal $\lambda = \lambda^{<\lambda}$ will be a forcing for which the following conditions hold:

(1) The forcing is not trivial and the $\lambda^+$- chain condition holds.
(2) The forcing is $< \lambda$- strategically complete.
(3) The forcing is $\lambda$- bounding.
(4) The forcing does not add $\lambda$- Cohen reals (follows from 3).

This definition reflects the properties of Random Real forcing in the case of $\lambda = \aleph_0$.

*Remark* 5. At this point we shall ignore another desired property, symmetry. This property states that for all $\eta_1, \eta_2$ and a model $M$: $\eta_1$ is generic over $M$ and $\eta_2$ is generic over $M[\eta_1]$ if and only if $\eta_2$ is generic over $M$ and $\eta_1$ is generic over $M[\eta_2]$.

Now we can define the terms used for Definition 4:

**Definition 6.** For a cardinal $\kappa$, we will say that the $\kappa$- chain condition holds for a forcing $\mathbb{P}$, when for every antichain $\mathcal{A}$ in $\mathbb{P}$, it holds that $|\mathcal{A}| < \kappa$.

**Definition 7.** Let $\alpha$ be an ordinal.

(1) For a forcing notion $\mathbb{P}$ and condition $p \in \mathbb{P}$, we define a game $\partial_\alpha(p, \mathbb{P})$ as follows. A play of the game has $\alpha$ moves and for $\beta < \alpha$ first the player COM chooses a condition $p_\beta \in \mathbb{P}$ such that:
   (a) $p \leq p_\beta$.
   (b) For all $\gamma < \beta$ it holds that $q_\gamma \leq p_\beta$.
   
   Next, the player INC plays and chooses $q_\beta \in \mathbb{P}$ such that $p_\beta \leq q_\beta$.
   
   The player COM wins the game if she survived; i.e. had a legal move for all $\beta < \alpha$.
(2) A forcing $\mathbb{P}$ is said to be strategically complete in $\alpha$ (or $\alpha$- strategically complete) if for all $p \in \mathbb{P}$ it holds that in the game $\partial_\alpha(p, \mathbb{P})$ between players COM and INC, player COM has a winning strategy.

**Definition 8.** A forcing $\mathbb{P}$ is $< \lambda$- strategically complete if it is $\alpha$- strategically complete for all $\alpha < \lambda$.

**Definition 9.** For a cardinal $\lambda$, a forcing $\mathbb{P}$ will be called $\underline{\lambda\text{-bounding}}$ when the following holds: $\Vdash_{\mathbb{P}} (\forall f : \lambda \to \lambda)(\exists g \in ({}^{\lambda}\lambda)^V : (\forall \alpha < \lambda)(f(\alpha) \leq g(\alpha)))$.

**Definition 10.** A set of ordinals $S$ will be called tenuous (or "nowhere stationary" as in [1]) if for each ordinal $\delta$ of uncountable cofinality, the set $S \restriction \delta$ is not a stationary set in $\delta$.

**Definition 11.** Let $\lambda$ be a cardinal and $S \subseteq \lambda$ a stationary set of $\lambda$. Then $S$ is said to be non-reflecting when for each ordinal $\delta < \lambda$ of cofinality $> \aleph_0$ the set $S \restriction \delta$ is not stationary in $\delta$.

*Remark* 12. Let $\lambda$ be a cardinal and let $S_*$ be a non-reflecting stationary subset of $\lambda$. Then the set $S \subseteq S_*$ is tenuous if and only if $S$ is not stationary.

*Claim* 13. Let $\lambda$ be a cardinal and $S_*$ be a non-reflecting stationary subset of $\lambda$.

(1) If $\bar{S} = \langle S_i : i < i(*) \rangle$ is such that for all $i < i(*)$, $S_i \subseteq \lambda$ is a non-stationary with $i(*) < cf(\lambda)$; then $S = \bigcup_{i<i(*)} S_i$ is not stationary.
(2) If $\bar{S} = \langle S_i : i < i(*) \rangle$ is such that for all $i < i(*)$, $S_i \subseteq S_*$ is a tenuous set with $i(*) < cf(\lambda)$; then $S = \bigcup_{i<i(*)} S_i$ is tenuous.

*Proof.* We see:

(1) For each $i < i(*)$, there is a club $E_i$ such that $S_i \cap E_i = \emptyset$ (as $S_i$ is not stationary); so let $E = \bigcap_{i<i(*)} E_i$. $E$ is a club in $\lambda$, as the intersection of $i(*) < cf(\lambda)$ clubs. In addition, $S \cap E = \emptyset$, thus $S$ is not stationary.
(2) From clause 1, $S$ is not stationary. In addition for each $\alpha < \lambda$, $S_* \cap \alpha$ is non-stationary and so does $S \cap \alpha$, as a subset of it.

$\square$

## 2. New $\lambda$ Real for Inaccessible Cardinal $\lambda$

To find a forcing resembling Random Real Forcing for Mahlo Cardinal, we need to add an additional assumption to those of the Weak Compact Cardinal case in [1]; the new assumption will be a diamond sequence indexed on a stationary set of inaccessible cardinals (such a set exists for a Mahlo Cardinal). For the more general case of any Inaccessible Cardinal, there is still a need to assume the existence of a diamond sequence; however here it will be indexed on a stationary set that only reflects in inaccessible cardinals. Those two cases are unified here, dealing with an Inaccessible Cardinal with a stationary set that only reflects to inaccessible cardinals- a Mahlo Cardinal will be a special case of this.

### 2.1. **Useful Definitions.**

**Definition 14.** A good structure $\mathfrak{r}$ contains:

1. An inaccessible cardinal $\lambda = \lambda_{\mathfrak{r}} > \aleph_0$.
2. A stationary set $S_* = S_*^{\mathfrak{r}} \subseteq \lambda$ of strong limit cardinals, such that if $S_* \cap \delta$ is stationary in $\delta$ then $\delta$ is inaccessible.
3. An increasing sequence of cardinals $\bar{\theta} = \bar{\theta}_{\mathfrak{r}} = \langle \theta_\epsilon : \epsilon < \lambda \rangle$ such that for all $\epsilon < \lambda$: $2 \leq \theta_\epsilon < \lambda$ and if $\epsilon \in S_*$ then for all $\zeta < \epsilon$, $\theta_\zeta < \epsilon$.
4. We assume the diamond principle for $S_*$ and let $\bar{X} = \bar{X}_{\mathfrak{r}}$ be a sequence witnessing it, i.e. $\bar{X} = \langle X_\delta : \delta \in S_* \rangle$; $X_\delta \subseteq \mathscr{H}(\lambda)$.

*Remark* 15. Observe:

1. For $\lambda$ Mahlo there is a stationary set $S_* \subseteq \lambda$ that only contains inaccessible cardinals, thus in particular its reflection will only be to inaccessible cardinals.
2. For $\lambda$ inaccessible that isn't Mahlo, we can assume the existence of a non-reflecting stationary set; we can force it using initial segments as in [6].
3. It is possible to assume that $S_*$ is a set of just limit ordinals (maybe not strong limit) and the only difference will be that for all $\delta \in S_*$ the forcing $\mathbb{Q}_\delta$ (will be defined later) will have the $|\boldsymbol{T}_{<\delta}|^+$- chain condition rather than the $\delta^+$- chain condition as we have here; however the forcing $\mathbb{Q}_\lambda$ will still have the $\lambda^+$- chain condition.

*Remark* 16. When $S_*$ is non-reflecting, the proofs are simpler.

Next, the forcing will be defined in several steps; those will be tree forcings for each $\delta \in S_* \cup \{\lambda\}$. First, we shall define the "biggest" forcing $\mathbb{Q}^0_\delta$; later we will define two additional forcing $\mathbb{Q}_\delta \subseteq \mathbb{Q}'_\delta \subseteq \mathbb{Q}^0_\delta$. For each of those forcing the forcing relation will be of inverse inclusion.

**Definition 17.** Given a good structure $\mathfrak{r}$, we shall define for each $\alpha \leq \lambda$ the collection of vertices of level $\alpha$: $\boldsymbol{T}_\alpha = \prod\limits_{\epsilon<\alpha} \theta_\epsilon$; for $\alpha \leq \lambda$ we will define the complete tree up to $\alpha$ to be the union of those sets: $\boldsymbol{T}_{<\alpha} = \bigcup\{\boldsymbol{T}_\beta : \beta < \alpha\}$.

*Remark* 18. We assume we have a good structure $\mathfrak{r}$ until the end of section 2.

**Convention.** *We let:*

1. *For all* $\delta_1 < \delta \leq \lambda$ *and* $\nu \in \boldsymbol{T}_\delta$ *let* $\nu \restriction \delta_1$ *the restriction of* $\nu$ *to* $\delta_1$.
2. *For each* $\delta \in S_* \cup \{\lambda\}$ *and a set* $u \subseteq \boldsymbol{T}_{<\delta}$ *we write* $\lim_\delta(u) = \{\nu \in \boldsymbol{T}_\delta : \forall \alpha < \delta, \nu \restriction \alpha \in u\}$.

(3) *For all $\delta \le \lambda$ and a set $u \subseteq \boldsymbol{T}_{<\delta}$, for $\delta_1 < \delta$ we shall write $u \restriction \delta_1 = u \cap \boldsymbol{T}_{<\delta_1}$.*
(4) *Assume $\alpha < \delta$ and $u \subseteq \boldsymbol{T}_{<\alpha}$ is a tree: non-empty set closed under taking initial segments. Let $\eta \in u$ be some node; we write $u^{[\eta]} = \{\nu \in u : \eta \trianglelefteq \nu \vee \nu \triangleleft \eta\}$.*

**Definition 19.** We can now define the forcing $\mathbb{Q}^0_\delta$ for each $\delta \in S_* \cup \{\lambda\}$.

(1) A condition in the forcing will be a tree $p \subseteq \boldsymbol{T}_{<\delta}$, that is, a non-empty subset closed under taking initial segments, where there is a unique witness set $S_p$ such that:
   (a) There is a trunk $tr(p)$; this is the unique element $\eta \in p$ with the following properties:
      (i) For all $\nu \in p$ it holds that $\nu \trianglelefteq \eta$ or $\eta \trianglelefteq \nu$.
      (ii) For every $\eta'$ with the property 1(a)i, we have that $\eta' \trianglelefteq \eta$.
   (b) In addition:
      (i) For each $\eta \in p$ and an ordinal $\beta < \delta$ such that $\lg(\eta) < \beta$, there exists an $\nu \in \boldsymbol{T}_\beta \cap p$ such that $\eta \trianglelefteq \nu$.
      (ii) Not only that, it even holds that for each $\eta \in p$ there is a $\nu \in \lim_\delta(p)$ with $\eta \triangleleft \nu$.
   (c) For each node extending the trunk (or equal to it), the set of its immediate successors contains all possible extensions: for $tr(p) \trianglelefteq \eta \in p$; $\{j \in \theta_{\lg(\eta)} : \eta\hat{}\langle j \rangle \in p\} = \theta_{\lg(\eta)}$.
   (d) The set $S_p \subseteq S_*$ is tenuous, this is the set of limit ordinals $\delta_1 \in (\lg(tr(p)), \delta)$ such that there exists $\eta \in \boldsymbol{T}_{\delta_1}$ with $\eta \notin p$, however all of its initial segments are in the tree: $(\forall \epsilon < \delta_1)(\eta \restriction \epsilon \in p)$.
(2) For all $p, q \in \mathbb{Q}^0_\delta$ we say that $p \le q$ if and only if $p \supseteq q$.

*Remark* 20. We can think of a tree $p \in \mathbb{Q}^0_\delta$ for $\delta \in S_* \cup \{\lambda\}$ as a complete tree from the level $\lg(tr(p))$ that we are pruning: in successor levels we are not allowed to prune. On limit levels we are allowed to prune the tree only if the level is an ordinal in $S_p$, so in most limit levels we take all the limits while in stages in $S_p$ we are allowed to cut as much as we want as long as 1b holds; so there will be a continuation to each node in each level higher than its length.

*Remark* 21. An alternative definition can be such that in successor levels there might be prunings, as long as those are not too big, that is, $\{j \in \theta_{\lg(\eta)} : \eta\hat{}\langle j \rangle \notin p\}$ is bounded in $\theta_{\lg(\eta)}$, or even belong to $D_\eta$, a $|\lg(\eta)|^+$- complete filter on $\theta_{\lg(\eta)}$. Not a serious difference.

*Claim* 22. For all $\delta \in S_* \cup \{\lambda\}$, the forcing $\mathbb{Q}^0_\delta$ has the following properties:

(1) The whole tree $\boldsymbol{T}_{<\delta} \in \mathbb{Q}^0_\delta$ and is weaker than any other condition in the forcing $\mathbb{Q}^0_\delta$.
(2) If $p \in \mathbb{Q}^0_\delta$ and $\eta \in p$, then $p^{[\eta]} \in \mathbb{Q}^0_\delta$ and $p \le_{\mathbb{Q}^0_\delta} p^{[\eta]}$.
(3) Let $\epsilon < \delta$; then the set $\{(\boldsymbol{T}_{<\delta})^{[\eta]} : \eta \in \boldsymbol{T}_\epsilon\}$ is a maximal antichain of the forcing $\mathbb{Q}^0_\delta$.
(4) Let $p \in \mathbb{Q}^0_\delta$ and $\epsilon < \delta$, then $\{p^{[\eta]} : \eta \in p \cap \boldsymbol{T}_\epsilon\}$ is a maximal antichain above $p$ (if $\epsilon \le \lg(tr(p))$, $p = p^{[\eta]}$ and this set is a singleton).

*Proof.* Let $\delta \in S_* \cup \{\lambda\}$:

(1) Clearly $p = \boldsymbol{T}_{<\delta} \subseteq \boldsymbol{T}_{<\delta}$ is a tree, let $S_p = \emptyset$ and show that this is a witness: for clause 1a of Definition 19, let the empty sequence $<>$ be the

trunk. Indeed for all $\nu \in \boldsymbol{T}_{<\delta}$, $<>\trianglelefteq \nu$ and there is no other node with this property as the successors of the empty set are $<\alpha>$ for all $\alpha \in \theta_0$. For clause 1b of Definition 19, let $\eta \in \boldsymbol{T}_{<\delta}$ and let $\beta > \lg(\eta)$, any $\nu \in \boldsymbol{T}_\beta$ with $\eta \trianglelefteq \nu$ is in $\boldsymbol{T}_{<\delta}$ so in the condition; same for the limit case. Observe that all of the extensions of each level are in the condition, so clause 1c of Definition 19 holds and so does clause 1d as $S_p = \emptyset$.

(2) Assume $p \in \mathbb{Q}^0_\delta$ and $\eta \in p$; if $\eta \trianglelefteq tr(p)$ then $p^{[\eta]} = p$ so wlog $tr(p) \trianglelefteq \eta$; let $S = S_p \setminus (\lg(\eta)+1)$. We will now show that the requirements in Definition 19 hold for $p^{[\eta]}$ with $S$ withnessing it, thus $p^{[\eta]} \in \mathbb{Q}^0_\delta$.
   (a) For clause 1a of the definition, the trunk will be $\eta$: clause 1(a)i clearly holds from the definition of $p^{[\eta]}$; assume there is some other node $\eta' \in p^{[\eta]}$ with the same property. Then in particular $\eta \triangleleft \eta'$ or $\eta' \trianglelefteq \eta$, if $\eta \triangleleft \eta'$, it follows that $\lg(\eta') > \lg(\eta)$ therefore the set $\{\nu \in p^{[\eta]} : \lg(\nu) = lg(\eta)+1\} = \{\nu \in p : (\lg(\nu) = \lg(\eta)+1) \wedge \eta \triangleleft \nu\}$ is a singleton, a contradiction to the assumption $p \in \mathbb{Q}^0_\delta$ recalling clause 1c of Definition 19 so $\eta' \trianglelefteq \eta$.
   (b) For clause 1b, for all $\nu \in p^{[\eta]}$ and $\beta > \lg(\nu)$, $\nu \in p$ so there exists $\nu \trianglelefteq \nu' \in p \cap \boldsymbol{T}_\beta$. If $\beta < \lg(\eta)$ then $\nu' \trianglelefteq \eta$ so $\nu' \in p^{[\eta]}$; otherwise $\eta \trianglelefteq \nu \trianglelefteq \nu'$ and then from the definition of $p^{[\eta]}$ it follows that $\nu' \in p^{[\eta]}$; similarly for the limit case.
   (c) For clause 1c, let $\eta \trianglelefteq \nu \in p^{[\eta]}$ and since in particular $tr(p) \trianglelefteq \nu \in p$, it holds that $\{j \in \theta_{\lg(\nu)} : \hat{\nu\langle j\rangle} \in p\} = \theta_{\lg(\nu)}$. Observing that for each $j \in \theta_{\lg(\nu)}$, $\eta \trianglelefteq \hat{\nu\langle j\rangle} \in p^{[\eta]}$ it holds that $\{j \in \theta_{\lg(\nu)} : \hat{\nu\langle j\rangle} \in p^{[\eta]}\} = \theta_{\lg(\nu)}$.
   (d) For clause 1d of the forcing definition, we shall show that $S$ is as required; there are no prunings at levels lower than the level of $\lg(\eta)+1$ so indeed $S_{p^{[\eta]}} \subseteq \delta \setminus (\lg(\eta)+1)$; moreover assume $\delta_1 \in (\lg(\eta), \delta)$ and $\nu \in \boldsymbol{T}_{\delta_1}$is such that $\nu \notin p^{[\eta]}$ and $(\forall \epsilon < \delta_1)(\nu \restriction \epsilon \in p^{[\eta]})$, so in particular $(\forall \epsilon < \delta_1)(\nu \restriction \epsilon \in p)$ as $\eta \trianglelefteq \nu$ and $\nu \notin p$ as $p^{[\eta]} \subseteq p$, thus $S = S_{p^{[\eta]}}$.

   Necessarily $p^{[\eta]} \subseteq p$ so $p \leq_{\mathbb{Q}^0_\delta} p^{[\eta]}$.

(3) Let $\epsilon < \delta$; the set is an antichain since for any $\eta \neq \nu \in \boldsymbol{T}_\epsilon$, clearly $(\boldsymbol{T}_{<\delta})^{[\eta]}$ and $(\boldsymbol{T}_{<\delta})^{[\nu]}$ are not compatible. Let $p \in \mathbb{Q}^0_\delta$ and let $\eta \in p \cap \boldsymbol{T}_\epsilon$ be a node. There exists such a node recalling clauses 1a and 1b of Definition 19, then $p^{[\eta]} \in \mathbb{Q}^0_\delta$ by previous clause; $p \leq_{\mathbb{Q}^0_\delta} p^{[\eta]}$ and clearly $(\boldsymbol{T}_{<\delta})^{[\eta]} \leq_{\mathbb{Q}^0_\delta} p^{[\eta]}$; thus $\{(\boldsymbol{T}_{<\delta})^{[\eta]} : \eta \in \boldsymbol{T}_\epsilon\}$ is a maximal antichain in $\mathbb{Q}^0_\delta$.

(4) Similar to the previous clause; $\{p^{[\eta]} : \eta \in p \cap \boldsymbol{T}_\epsilon\}$ is an antichain above $p$ as for $\eta \neq \nu \in \boldsymbol{T}_\epsilon \cap p$, easily $p^{[\eta]}$ and $p^{[\nu]}$ are not compatible. Then for all $p \leq_{\mathbb{Q}^0_\delta} q \in \mathbb{Q}^0_\delta$, $q \subseteq p$ so let $\eta \in q \cap \boldsymbol{T}_\epsilon$ (exists as $q$ is a condition in the forcing $\mathbb{Q}^0_\delta$) and observe that $q, p^{[\eta]} \leq_{\mathbb{Q}^0_\delta} q^{[\eta]}$ so this antichain is maximal.

□

Next, define a structure that will fulfill the roll of using the diamond principle; the structure will be a collection of objects that contain elements that are antichains with additional properties. In the weak compact case [1] there was an important roll for the maximal antichains; in the proof of the $\lambda$- bounding of the forcing there was a maximal antichain that reflected to an antichain in the forcing corresponding to a smaller cardinal.

In the inaccessible case which we are dealing with here, we will have to use diamond to gain a similar property. Each element is an antichain in the forcing $\mathbb{Q}^0_\delta$.

**Definition 23.** For any ordinal $\delta \in S_* \cup \{\lambda\}$, $\Xi_\delta$ will be the collection of objects $\bar{q}$, for which the following conditions hold:

(1) $\bar{q} = \langle q_\eta : \eta \in \Lambda \rangle$,
(2) $\Lambda \subseteq \boldsymbol{T}_{<\delta}$,
(3) for each $\eta \in \Lambda$ it holds that $q_\eta \in \mathbb{Q}^0_\delta$ and $\eta = tr(q_\eta)$,
(4) for each $\eta, \nu \in \Lambda$, $\eta \neq \nu$: $\eta = tr(q_\eta) \notin q_\nu \vee \nu = tr(q_\nu) \notin q_\eta$,
(5) the union of all the conditions from the set will be an element in the forcing: $r^*_{\bar{q}} = \{\rho \in \boldsymbol{T}_{<\delta} : (\exists \eta \in \Lambda)(\rho \in q_\eta)\} \in \mathbb{Q}^0_\delta$.

**Definition 24.** For all $\delta \in S_* \cup \{\lambda\}$ and $\bar{q} \in \Xi_\delta$, the coder $X_{\bar{q}}$ holds the information on the antichain $\bar{q}$: $X_{\bar{q}} = \{(\eta, \nu) : (\eta \in \Lambda) \wedge (\nu \in q_\eta)\}$.

**Definition 25.** Let $\delta \in S_* \cup \{\lambda\}$; we call $\delta$ weakly successful when $\delta = \lambda$ or there is $\bar{q} \in \Xi_\delta$ with $X_{\bar{q}} = X_\delta$, recalling $X_\delta$ is from the good structure $\mathfrak{r}$ defined in clause 4 of 14.

*Claim* 26. For a weakly succesfull $\delta \in S_* \cup \{\lambda\}$, the $\bar{q}$ of Definition 25 is unique.

*Proof.* Observe that the coder $X_{\bar{q}}$ has all the information on $\bar{q}$, therefore such a $\bar{q}$ must be unique. □

**Definition 27.** (1) For a weakly successful $\delta \in S_* \cup \{\lambda\}$:
- look at the unique sequence $\bar{q} = \langle q_\eta : \eta \in \Lambda \rangle$ for which $X_{\bar{q}} = X_\delta$ and write $\Lambda^*_\delta = \Lambda$; for all $\eta \in \Lambda^*_\delta$ let $q^*_{\delta,\eta} = q_\eta$ and lastly $\bar{q}^*_\delta = \langle q^*_{\delta,\eta} : \eta \in \Lambda^*_\delta \rangle$,
- let $r^*_\delta = r^*_{\bar{q}^*_\delta} = \{\rho \in \boldsymbol{T}_{<\delta} : (\exists \nu \in \Lambda^*_\delta)(\rho \in q^*_{\delta,\nu})\}$,
- finally, for each $\eta \in \Lambda^*_\delta$ and $\nu \in \boldsymbol{T}_{<\delta}$ with: $\eta \trianglelefteq \nu \in q^*_{\delta,\eta}$, let $q^*_{\delta,\nu} = (q^*_{\delta,\eta})^{[\nu]}$ .

(2) For $\delta \in S_* \cup \{\lambda\}$ which is not weakly successful, let $r^*_\delta = \boldsymbol{T}_{<\delta}$ and for all $\eta \in r^*_\delta$: $q^*_\eta = (r^*_\delta)^{[\eta]}$.

Now we can use the $\bar{X}$, being a diamond sequence:

*Claim* 28. For all $\bar{q} = \langle q_\eta : \eta \in \Lambda \rangle \in \Xi_\lambda$ there is a stationary set of $\delta \in S_*$ for which $\bar{q}^*_\delta = \langle q_\eta \cap \boldsymbol{T}_{<\delta} : \eta \in \Lambda \cap \boldsymbol{T}_{<\delta} \rangle$.

*Proof.* Recall that $\bar{X}$ is a diamond sequence, therefore for the set $X_{\bar{q}}$, there is a stationary set of $\delta \in S_*$ for which $X_\delta = X_{\bar{q}} \cap (\boldsymbol{T}_{<\delta} \times \boldsymbol{T}_{<\delta})$, from the definition of the coder, and as $X_\delta$ is the coder of $\bar{q}^*_\delta$ the conclusion follows: $\bar{q}^*_\delta = \langle q_\eta \cap \boldsymbol{T}_{<\delta} : \eta \in \Lambda \cap \boldsymbol{T}_{<\delta} \rangle$. □

### 2.2. Defining the Main Forcing.

*Remark* 29. Below the main forcing will be defined, however prior to the definition we would like to state the properties that this forcing is expected to have; this remark is meant to describe the general structure of the forcings $\mathbb{Q}'_\delta$ amd $\mathbb{Q}_\delta$ for each $\delta \in S_* \cup \{\lambda\}$.

(1) We would like those forcing to be subforcings of $\mathbb{Q}^0_\delta$ (but not nesseccarily complete subforcings), where $\mathbb{Q}_\delta \subseteq \mathbb{Q}'_\delta \subseteq \mathbb{Q}^0_\delta$.
(2) For a condition $p \in \mathbb{Q}_\delta$ and a node $\eta \in p$, we have $p^{[\eta]} \in \mathbb{Q}_\delta$; the same holds for $\mathbb{Q}'_\delta$.

(3) The complete tree $\boldsymbol{T}_{<\delta}$ belongs to $\mathbb{Q}_\delta$; so in particular it belongs to $\mathbb{Q}'_\delta$.

We are now ready to finally define the desired forcings.

**Definition 30.** This is the main definition of the forcing; the definition will be inductive on $\delta$ and we will define the subforcings of $\mathbb{Q}^0_\delta$; $\mathbb{Q}_\delta$ and $\mathbb{Q}'_\delta$ for all $\delta \in S_* \cup \{\lambda\}$, in addition we will define the term successful for ordinals and for each $\eta \in \boldsymbol{T}_{<\delta}$ and a tenuous $S \subseteq S_* \cap \delta$ we will define $p^*_{\eta,\delta,S} \in \mathbb{Q}^0_\delta$.

(1) The forcing $\mathbb{Q}'_\delta$:
   (a) For each condition $p \in \mathbb{Q}^0_\delta$, it holds that $p \in \mathbb{Q}'_\delta$ if and only if for each $\delta_1 \in \delta \cap S_*$: if $\lg(tr(p)) < \delta_1$ then $p \restriction \delta_1 \in \mathbb{Q}_{\delta_1}$,
   (b) For $p, q \in \mathbb{Q}'_\delta$, we say that $p \leq_{\mathbb{Q}'_\delta} q$ if and only if $p \supseteq q$.

   In fact the forcing $\mathbb{Q}'_\delta$ is derived from the forcings $\mathbb{Q}_{\delta_1}$ for $\delta_1 \in \delta \cap S_*$.
(2) The forcing $\mathbb{Q}_\delta$:
   (a) For each condition $p \in \mathbb{Q}^0_\delta$, it holds that $p \in \mathbb{Q}_\delta$ if and only if $p = p^*_{\eta,\delta,S}$ where $\eta \in \boldsymbol{T}_{<\delta}$ is some node, the set $S \subseteq S_* \cap \delta$ is tenuous; such $p^*_{\eta,\delta,S}$ is defined below in clause 4,
   (b) For $p, q \in \mathbb{Q}_\delta$ we say that $p \leq_{\mathbb{Q}_\delta} q$ if and only if $p \supseteq q$.
(3) We call $\delta$ <u>successful</u> when it is weakly successful and in addition: $r^*_\delta, q^*_{\delta,\eta} \in \mathbb{Q}'_\delta$ for all $\eta \in \Lambda^*_\delta$.

   <u>Explanation:</u> The successful ordinals represent the levels in which there will be a special pruning, determined by the diamond condition, so there is a "control" on the conditions defined uniquely, and in relation to $r^*_{\delta_1}$ of the corresponding levels $\delta_1$.
(4) We assume that the forcing $\mathbb{Q}_{\delta'}$ is defined for all $\delta' \in S_* \cap \delta$; the condition $p^*_{\eta',\delta',S'}$ is defined for all $\eta' \in \boldsymbol{T}_{<\delta'}$ and $S' \subseteq S_* \cap \delta'$ tenuous; we shall define $p^*_{\eta,\delta,S} \in \mathbb{Q}^0_\delta$ in the following way:
   (a) If $\sup(S) \leq \lg(\eta)$ then $p^*_{\eta,\delta,S} = \boldsymbol{T}^{[\eta]}_{<\delta}$.
   (b) If $\sup(S) > \lg(\eta)$ and $S$ has no last element, then for each $\nu \in \boldsymbol{T}_{<\delta}$ it holds that $\nu \in p^*_{\eta,\delta,S}$ if and only if one of the following conditions holds:
      (i) $\nu \trianglelefteq \eta$,
      (ii) $\eta \triangleleft \nu$ and there exists $\lg(\nu) < \delta_1 \in S$ such that $\nu \in p^*_{\eta,\delta,S\cap\delta_1}$,
      (iii) $\eta \triangleleft \nu$, $\lg(\nu) \geq \sup(S)$ and for all $\delta_1 \in S \setminus (\lg(\eta)+1)$ and $\zeta < \delta_1$ it holds that $\nu \restriction \zeta \in p^*_{\eta,\delta_1,S\cap\delta_1}$.
   (c) If $\sup(S) > \lg(\eta)$ and $S$ has a last element $\delta_1 < \delta$, such that $\lg(\eta) < \delta_1 \in S$ and $\delta_1$ is <u>not successful</u>, then for each $\nu \in \boldsymbol{T}_{<\delta}$ it holds that $\nu \in p^*_{\eta,\delta,S}$ if and only if one of the followings holds:
      (i) On levels lower than the level corresponding to the last element, we use induction to take the previous condition from the forcing; that is: $\lg(\nu) < \delta_1$ and $\nu \in p^*_{\eta,\delta_1,S\cap\delta_1}$,
      (ii) On the levels higher or equal to the level of the last element, the choice will be according to the condition on level $\delta_1$: $\lg(\nu) \geq \delta_1$ and $\nu \restriction \delta_1 \in \lim_{\delta_1}(p^*_{\eta,\delta_1,S\cap\delta_1})$.
   (d) If $\sup(S) > \lg(\eta)$ and $S$ has a last element $\delta_1 < \delta$, such that $\lg(\eta) < \delta_1 \in S$ and $\delta_1$ is <u>successful</u>, then for each $\nu \in \boldsymbol{T}_{<\delta}$ it holds that $\nu \in p^*_{\eta,\delta,S}$ if and only if one of the following hold:

(i) On levels lower than the level corresponding to the last element, we use induction to take the nodes in the corresponding condition of the previous forcing; that is: $\lg(\nu) < \delta_1$ and $\nu \in p^*_{\eta,\delta_1,S\cap\delta_1}$,

(ii) On the levels higher than the level of the last element, every possible node will be chosen: $\lg(\nu) > \delta_1$ and $\nu \restriction \delta_1 \in p^*_{\eta,\delta,S} \cap \boldsymbol{T}_{\delta_1}$ according to the definition of $p^*_{\eta,\delta,S} \cap \boldsymbol{T}_{\delta_1}$ in the following clause,

(iii) On the level of the last element of $S$, the process is more interesting; for $\lg(\nu) = \delta_1$:
   (A) If $\nu \notin \lim_{\delta_1}(r^*_{\delta_1})$ then $\nu \in \lim_{\delta_1}(p^*_{\eta,\delta_1,S\cap\delta_1})$,
   (B) If $\nu \in \lim_{\delta_1}(r^*_{\delta_1})$ then $\nu \in (\bigcup\{\lim_{\delta_1}(q^*_{\delta_1,\eta'}) : \eta' \in \Lambda^*_{\delta_1}\}) \cap \lim_{\delta_1}(p^*_{\eta,\delta_1,S\cap\delta_1})$.

**Definition 31.** For $\delta \in S_* \cup \{\lambda\}$, define $\underset{\sim}{\eta}_\delta$ to be a $\mathbb{Q}_\delta$-name: $\underset{\sim}{\eta} = \bigcup\{tr(p) : p \in \underset{\sim}{G}_{\mathbb{Q}_\delta}\}$ where $\underset{\sim}{G}_{\mathbb{Q}_\delta}$ is a $\mathbb{Q}_\delta$- name of a generic set of the forcing. If $\delta$ is clear from the context, we may write $\underset{\sim}{\eta}$ instead $\underset{\sim}{\eta}_\delta$.

*Claim* 32. For all $\delta \in S_* \cup \{\lambda\}$, $\eta \in \boldsymbol{T}_{<\delta}$ and tenuous $S \subseteq S_* \cap \delta$; if $p = p^*_{\eta,\delta,S}$ (so $p \in \mathbb{Q}_\delta$) <u>then</u>:

(1) If $\delta_0 \in S_* \cap \delta$ is such that $\eta \in \boldsymbol{T}_{<\delta_0}$ then $p^*_{\eta,\delta,S} \restriction \delta_0 = p^*_{\eta,\delta_0,S\cap\delta_0}$.
(2) $\eta$ is the trunk of $p^*_{\eta,\delta,S}$.
(3) $p^*_{\eta,\delta,S} \in \mathbb{Q}^0_\delta$ , moreover $p^*_{\eta,\delta,S} \in \mathbb{Q}'_\delta$.
(4) The tenuous set $S$ contains the set of pruning levels corresponding to the condition, $S_p \subseteq S$.

*Proof.* We prove by induction on the ordinals $\delta \in S_* \cup \{\lambda\}$, assume that the claim it true for $\mathbb{Q}_{\delta'}$, so for all conditions $p^*_{\eta',\delta',S'}$ where $\delta' \in \delta \cap S_*$, $\eta' \in \boldsymbol{T}_{<\delta'}$ and $S' \subseteq S_* \cap \delta'$; we will now prove it for $p = p^*_{\eta,\delta,S}$ where $\eta \in \boldsymbol{T}_{<\delta}$ and $S \subseteq S_* \cap \delta$:

(1) Assume that $\delta_0 \in S_* \cap \delta$ is such that $\eta \in \boldsymbol{T}_{<\delta_0}$ and look at the different cases in the definition of $p^*_{\eta,\delta,S}$:
   (a) For case 4a, $p^*_{\eta,\delta,S} \restriction \delta_0 = (\boldsymbol{T}_{<\delta_0})^{[\eta]} = p^*_{\eta,\delta_0,S\cap\delta_0}$.
   (b) For case 4b, the initial segments of $\eta$ are clearly both in $p^*_{\eta,\delta,S} \restriction \delta_0$ and in $p^*_{\eta,\delta_0,S\cap\delta_0}$; for all $\eta \trianglelefteq \nu \in \boldsymbol{T}_{<\delta_0}$ it holds by clause 4(b)ii of the definition that $\nu \in p^*_{\eta,\delta,S} \iff \nu \in p^*_{\eta,\delta_0,S\cap,\delta_0}$.
   (c) For cases 4c and 4d, for each $\nu \in \boldsymbol{T}_{<\delta_0}$ the relevant clauses are 4(c)i of 4c and 4(d)i of 4d. Those clauses trivially imply $\nu \in p^*_{\eta,\delta,S} \iff \nu \in p^*_{\eta,\delta_0,S\cap,\delta_0}$.

(2) For all $\nu \in p^*_{\eta,\delta.S}$, we will show that $\nu \trianglelefteq \eta$ or $\eta \triangleleft \nu$, split to cases according to the cases in the definition of the forcing 4:
   (a) For case 4a it is clear.
   (b) For case 4b it is also clear from the definition.
   (c) For case 4c:
      (i) For 4(c)i by the induction hypothesis.
      (ii) For 4(c)ii also, by the induction hypothesis- each such $\nu$ has $\eta \triangleleft \nu$.
   (d) For case 4d:
      (i) For $\nu$ chosen in clause 4(d)i we have $\eta \triangleleft \nu$ or $\nu \trianglelefteq \eta$ by the induction hypothesis.
      (ii) For $\nu$ chosen in clause 4(d)ii it holds that $\eta \triangleleft \nu$, again using the induction hypothesis.

(iii) For a node $\nu$ chosen in clause 4(d)iii, since $\nu \in \lim_{\delta_1}(p^*_{\eta,\delta_1,S\cap\delta_1})$ and by the induction hypothesis, $\eta \trianglelefteq \nu$.

Now it remains to prove that $\eta$ is the maximal node for which each other branch is an extension or an initial segment of it.

In case 4a it is clear; in cases 4b and 4c it follows from the induction hypothesis, the node $\eta$ is the trunk of the condition $p^*_{\eta,\delta_1,S\cap\delta_1}$ for each $\delta_1 \in \delta \cap S_*$ and so it has extensions to levels higher than the level of the trunk; those extensions will be in the new condition $p^*_{\eta,\delta,S}$ thus $\eta$ will be a trunk there as well. For case 4d recall that $\delta_1$ is a limit cardinal $> \lg(\eta)$, we can use the induction hypothesis again observing that before the $\delta_1$-th level there are no new prunings that didn't exist in $p^*_{\eta,\delta_1,S\cap\delta_1}$, therefore if there were a different trunk containing $\eta$, it would have been a trunk of $p^*_{\eta,\delta_1,S\cap\delta_1}$ as well- a contrudiction.

(3) Using induction, first we will show that $p^*_{\eta,\delta,S} \in \mathbb{Q}^0_\delta$, checking the clauses in Definition 19:
   (a) For clause 1a; $p^*_{\eta,\delta,S}$ is obviously a tree and it has a trunk $\eta$ by part 2 of this claim.
   (b) To show clause 1b, let $\nu \in p^*_{\eta,\delta,S}$ and $\lg(\nu) < \beta < \delta$. Assume $\eta \trianglelefteq \nu$ (the case of $\nu \triangleleft \eta$ follows from it trivially) then there is an extension of $\nu$ to the level $\beta$ and also to the limit level:
      (i) In case 4a of Definition 30, let $\nu \trianglelefteq \nu' \in \boldsymbol{T}_\beta$, then it holds that also $\eta \trianglelefteq \nu'$ thus $\nu' \in p^*_{\eta,\delta,S}$. Moreover, let $\nu' \in \boldsymbol{T}_\delta$ be such that $\nu \trianglelefteq \nu'$, then $\nu' \in \lim_\delta(p^*_{\eta,\delta,S})$ as $\lim_\delta(p^*_{\eta,\delta,S}) = \{\nu' \in \boldsymbol{T}_\delta : \eta \triangleleft \nu'\}$.
      (ii) In case 4b of Definition 30 for $\beta < \sup(S)$, there is $\beta < \delta' \in S$, thus using the induction hypothesis there is a $\nu'$, $\nu \trianglelefteq \nu' \in p^*_{\eta,\delta',S\cap\delta'}$ such that $\lg(\nu') = \beta$; by clause 4(b)ii $\nu' \in p^*_{\eta,\delta,S}$. For $\beta \geq \sup(S)$ we can use the induction hypothesis again and by clause 4(b)iii the conclusion follows. Similarly for the limit case.
      (iii) In case 4c of Definition 30, for $\beta < \delta_1$, by the induction hypothesis there is a $\nu' \in p^*_{\eta,\delta_1,S\cap\delta_1} \cap \boldsymbol{T}_\beta$, and from the definition in clause 4(c)i it follows that $\nu' \in p^*_{\eta,\delta,S}$. For $\beta \geq \delta_1$, we can use the induction hypothesis again and by clause 4(c)ii the conclusion follows. Similarly for the limit case.
      (iv) In case 4d of Definition 30:
         - For $\beta < \delta_1$ by the induction hypothesis there is a $\nu' \in p^*_{\eta,\delta_1,S\cap\delta_1} \cap \boldsymbol{T}_\beta$ and by 4(d)i it follows that $\nu' \in p^*_{\eta,\delta,S}$.
         - For $\beta = \delta_1$, if $\nu \in r^*_{\delta_1}$ then for some $\eta' \in \Lambda^*_{\delta_1}$, $\nu \in q^*_{\delta_1,\eta'}$, therefore by the induction hypothesis for $\delta_1$, there is a node $\nu' \in \lim_{\delta_1}(q^*_{\delta_1,\eta'})$ extending $\nu$ and this node will be in $p^*_{\eta,\delta,S}$ by clause 4(d)iiiB. Otherwise, $\nu \notin r^*_{\delta_1}$ and for each $\nu' \in \boldsymbol{T}_{\delta_1}$ such that $\nu \triangleleft \nu'$, $\nu' \notin \lim_{\delta_1}(r^*_{\delta_1})$. Since each such $\nu'$ is in $\lim_{\delta_1}(p^*_{\eta,\delta_1,S\cap\delta_1})$ and by clause 4(d)iiiA, $\nu' \in p^*_{\eta,\delta,S}$.
         - For $\beta > \delta_1$ every possible extension is being chosen after the level of height $\delta_1$, so by the preivous clause certainly there is an element in the $\beta$ level and also in the limit by clause 4(d)ii.
   (c) In successor levels all the extensions are taken, as defined in $\mathbb{Q}^0_\delta$.

(d) The set $S$ is tenuous and it holds that $S_p \subseteq S$ by the next clause, so $S_p$ (the set of the levels with the prunes) is also tenuous.

Now we can see that $p^*_{\eta,\delta,S} \in \mathbb{Q}'_\delta$:

- Let $\delta'$ be $\lg(tr(p)) < \delta' \in S_*$; observe that in all the cases of the definition it holds that $p^*_{\eta,\delta,S} \cap \boldsymbol{T}_{\delta'} = p^*_{\eta,\delta',S\cap\delta'} \in \mathbb{Q}_{\delta'}$ and so we are done.

(4) Looking at the definition, in case 4a trivial; for case 4b we will have that $S_{p^*_{\eta,\delta,S}} = \bigcap_{\delta' \in S} S_{p^*_{\eta,\delta',S\cap\delta'}}$ so by induction the required in the claim holds; In case 4c, $S_{p^*_{\eta,\delta.S}} = S_{p^*_{\eta,\delta_1,S\cap\delta_1}}$ and in case 4d, $S_{p^*_{\eta,\delta.S}} = S_{p^*_{\eta,\delta_1,S\cap\delta_1}}$ or $S_{p^*_{\eta,\delta.S}} = S_{p^*_{\eta,\delta_1,S\cap\delta_1}} \cup \{\delta_1\}$. Using the induction hypothesis and as $\delta_1 \in S$, we are done.

$\square$

Recall the required properties of the forcings discussed in Remark 29 and show that those indeed hold:

*Claim* 33. Let $\delta \in S_* \cup \{\lambda\}$;

(1) $\mathbb{Q}_\delta \subseteq \mathbb{Q}'_\delta \subseteq \mathbb{Q}^0_\delta$.
(2) For a condition $p \in \mathbb{Q}_\delta$ and a node $\nu \in p$ , we have $p \leq_{\mathbb{Q}_\delta} p^{[\nu]} \in \mathbb{Q}_\delta$ and $tr(p^{[\nu]}) = \max\{tr(p), \nu\}$, the same holds for $\mathbb{Q}'_\delta \subseteq \mathbb{Q}^0_\delta$ .
(3) Let $p, q \in \mathbb{Q}'_\delta$, then $p, q$ are compatible if and only if $tr(p) \in q \wedge tr(q) \in p$.
(4) Let $p, q \in \mathbb{Q}_\delta$, then $p, q$ are compatible if and only if $tr(p) \in q \wedge tr(q) \in p$.
(5) It holds that $\boldsymbol{T}_{<\delta} \in \mathbb{Q}_\delta$ and $p \in \mathbb{Q}_\delta \Rightarrow \boldsymbol{T}_{<\delta} \leq_{\mathbb{Q}_\delta} p$; in addition $\boldsymbol{T}_{<\delta} \in \mathbb{Q}'_\delta$ and $p \in \mathbb{Q}'_\delta \Rightarrow \boldsymbol{T}_{<\delta} \leq_{\mathbb{Q}'_\delta} p$.
(6) If $p \in \mathbb{Q}_\delta$ and $\lg(tr(p)) < \alpha < \delta$ then $\{p^{[\eta]} : \eta \in p \cap \boldsymbol{T}_\alpha\}$ is a maximal antichain of $\mathbb{Q}_\delta$ above $p$, the same holds for $\mathbb{Q}'_\delta$.

*Proof.* In fact we saw the existence of most of the statements in this claim already. Observe:

(1) From the previous claim, it follows that $\mathbb{Q}_\delta \subseteq \mathbb{Q}'_\delta$, and from the definition of $\mathbb{Q}'_\delta$ clearly $\mathbb{Q}'_\delta \subseteq \mathbb{Q}^0_\delta$ so we are done.
(2) Assume $p = p^*_{\eta,\delta,S} \in \mathbb{Q}_\delta$ and let $\nu \in p$.
   - If $\nu \trianglelefteq \eta$ then $p^{[\nu]} = p \in \mathbb{Q}_\delta$; in particular $tr(p^{[\nu]}) = \eta$.
   - Else, $\eta \triangleleft \nu$. In that case $p^{[\nu]} = p^*_{\nu,\delta,S}$, we will show that using induction, looking at the clauses of definition 30(4):
     (a) If, as in case 4a, $p = \boldsymbol{T}^{[\eta]}_{<\delta}$ then $p^{[\nu]} = \boldsymbol{T}^{[\nu]}_{\delta}$ which is in fact $p^*_{\nu,\delta,\emptyset}$ and thus belongs to $\mathbb{Q}_\delta$ and $tr(p^{[\nu]}) = \nu$.
     (b) If, as in case 4b, there is a $\lg(\eta) < \delta' \in S$ with $\nu \in p^*_{\eta,\delta',S\cap\delta'}$, then by the induction hypothesis $p^{[\nu]} \cap \boldsymbol{T}_{<\delta'} \in \mathbb{Q}_{\delta'}$. In addition, $p^{[\nu]} = p^*_{\nu,\delta,S}$ and therefore belongs to $\mathbb{Q}_\delta$. If $\lg(\nu) \geq \sup(S)$ then for all $\delta_1 \in S$ and $\zeta < \delta_1$ it holds that $\nu \restriction \zeta \in p^*_{\eta,\delta_1,S\cap\delta_1}$. Then $p^{[\nu]} = \boldsymbol{T}^{[\nu]}_{<\delta} = p^*_{\nu,\delta,S}$ and $tr(p^{[\nu]}) = \nu$.
     (c) If, as in cases 4c and 4d, $S$ has a last element $\delta_1 < \delta$, such that $\lg(\eta) < \delta_1 \in S$, then if $\delta_1 \leq \lg(\nu)$, $p^{[\nu]} = \boldsymbol{T}^{[\nu]}_{<\delta} = p^*_{\nu,\delta,S}$. For $\lg(\nu) < \delta_1$, by definition, the condition $p^{[\nu]}$ has prunings in level $\delta_1$ iff there is a node $\nu' \in \lim_{\delta_1}(p^*_{\eta,\delta_1,S\cap\delta_1})$ such that

$\nu' \in \lim_{\delta_1}(r^*_{\delta_1}) \setminus (\bigcup_{\eta' \in \Lambda^*_{\delta_1}} \lim_{\delta_1}(q^*_{\delta_1,\eta'}))$. By the induction assumption, $(p^*_{\eta,\delta_1,S\cap\delta_1})^{[\nu]} = p^*_{\nu,\delta_1,S\cap\delta_1}$ and if there is no pruning in level $\delta_1$ then $(p^*_{\eta,\delta,S})^{[\nu]} = p^*_{\nu,\delta,S}$ follows. Otherwise, those nodes in $A_\nu = \lim_{\delta_1}(p^*_{\nu,\delta_1,S\cap\delta_1}) \cap \lim_{\delta_1}(r^*_{\delta_1}) \setminus (\bigcup_{\eta' \in \Lambda^*_{\delta_1}} \lim_{\delta_1}(q^*_{\delta_1,\eta'}))$ are pruned in $p^*_{\nu,\delta,S}$ and letting $A_\eta = \lim_{\delta_1}(p^*_{\eta,\delta_1,S\cap\delta_1}) \cap \lim_{\delta_1}(r^*_{\delta_1}) \setminus (\bigcup_{\eta' \in \Lambda^*_{\delta_1}} \lim_{\delta_1}(q^*_{\delta_1,\eta'}))$ while $A_\eta$ is the set of nodes pruned in $p^*_{\eta,\delta,S}$ in level $\delta_1$, we can see that $A_\nu = A_\eta \cap \{\nu' : \nu \trianglelefteq \nu'\}$ and so it follows that $(p^*_{\eta,\delta,S})^{[\nu]} = p^*_{\nu,\delta,S}$, thus $p^{[\nu]} \in \mathbb{Q}_\delta$ and $tr(p^{[\nu]}) = \nu$, and we are done.

In particular it holds that $tr(p^{[\nu]}) = \max\{\eta, \nu\}$.

We have finished showing that $\nu \in p \in \mathbb{Q}_\delta \Rightarrow p^{[\nu]} \in \mathbb{Q}_\delta$. What about $\mathbb{Q}'_\delta$?

Let $p \in \mathbb{Q}'_\delta$, then for each $\delta' \in \delta \cap S_*$ it holds that $p \restriction \delta' \in \mathbb{Q}_{\delta'}$. Next, observe that $q = p^{[\nu]}$ for $\eta \trianglelefteq \nu \in p$; then for all $\lg(\nu) \leq \delta' \in \delta \cap S_*$, $q \restriction \delta' = (p \restriction \delta')^{[\nu]}$. Observe that $p \restriction \delta' \in \mathbb{Q}_{\delta'}$ and by the first part of this clause also $(p \restriction \delta')^{[\nu]} \in \mathbb{Q}_{\delta'}$. For $\nu \trianglelefteq \eta$, it holds that $p^{[\nu]} = p \in \mathbb{Q}'_\delta$ and in particular $tr(p^{[\nu]}) = \eta$; so indeed $tr(p^{[\nu]}) = \max\{\eta, \nu\}$.

By the definition of $p^{[\nu]}$, $p^{[\nu]} \subseteq p$ and since the order of both forcing $\mathbb{Q}_\delta$ and $\mathbb{Q}'_\delta$ is inverse inclusion and by what we just showed if $p \in \mathbb{Q}_\delta$, then $p \leq_{\mathbb{Q}_\delta} p^{[\nu]}$ and if $p \in \mathbb{Q}'_\delta$ then $p \leq_{\mathbb{Q}'_\delta} p^{[\nu]}$.

(3) This clause and the next one are shown by simultaneous induction on $\delta$. Considering the forcing $\mathbb{Q}'_\delta$:
- For the first direction, assume $p$ and $q$ are compatible; thus there exists a condition $r \in \mathbb{Q}'_\delta$: $r \subseteq p, q$. In particular, $tr(p), tr(q) \trianglelefteq tr(r)$ thus $tr(p), tr(q) \in r \subseteq p \cap q$.
- For the other direction, assume $tr(p) \in q \wedge tr(q) \in p$, let $r = p \cap q$ and show that $r \in \mathbb{Q}'_\delta$: let $\lg(tr(r)) < \delta_1 \in \delta \cap S_*$, wlog $tr(q) \trianglelefteq tr(p)$, thus $tr(r) = tr(p)$. In particular $\lg(tr(p)), \lg(tr(q)) < \delta_1$ and since $p, q \in \mathbb{Q}'_\delta$ it implies that $p \restriction \delta_1, q \restriction \delta_1 \in \mathbb{Q}_{\delta_1}$. We can use the induction hypothesis to conclude that $(p \restriction \delta_1) \cap (q \restriction \delta_1) = r \restriction \delta_1 \in \mathbb{Q}_{\delta_1}$; therefore indeed $r \in \mathbb{Q}'_\delta$.

(4) Considering the forcing $\mathbb{Q}_\delta$:
- For the first direction, assume that $p$ and $q$ are compatible; thus there exists $r \in \mathbb{Q}_\delta$: $r \subseteq p, q$. In particular, $tr(p), tr(q) \trianglelefteq tr(r)$ thus $tr(p), tr(q) \in r \subseteq p \cap q$.
- For the other direction, assume $tr(p) \in q \wedge tr(q) \in p$ and remember that for some nodes $\eta_1, \eta_2 \in \boldsymbol{T}_{<\delta}$ and tenuous sets $S_1, S_2 \subseteq S_* \cap \delta$, $p = p^*_{\eta_1,\delta,S_1}$, $q = p^*_{\eta_2,\delta,S_2}$. Recall the assumption and assume by symmetry that $\eta_1 \trianglelefteq \eta_2$. Let $r = p \cap q$ and show that $r = p^*_{\eta_2,\delta,S_1 \cup S_2}$, we will show that this is indeed a condition in the forcing $\mathbb{Q}_\delta$ looking at Definition 4:
  - The tree $r$ has a trunk $\eta_2$ since for all $\nu \trianglelefteq \eta_2$, $\nu \in r$ and in addition: $\{j \in \theta_{\lg(\eta_2)} : \eta_2\hat{\ }\langle j \rangle \in p\} = \{j \in \theta_{\lg(\eta_2)} : \eta_2\hat{\ }\langle j \rangle \in q\} = \theta_{\lg(\eta_2)}$; so $\eta_2$ is indeed the trunk of $r$.

– The tree $r$ is pruned precisely in the successful levels $\delta'$ of $S_1 \cup S_2 \setminus (\lg(\eta_2)+1)$ such that $\lim_{\delta'}(p^*_{\eta_2,\delta',(S_1\cup S_2)\cap\delta'}) \cap \lim_{\delta'}(r^*_{\delta'}) \setminus (\bigcup_{\eta'\in\Lambda^*_{\delta'}} \lim_{\delta'}(q^*_{\delta',\eta'})) \neq \emptyset$; in levels higher than the trunk, $r$ is pruned preciesly where $p$ or $q$ are as their intersection. Let $\delta' \in S_1 \cup S_2 \setminus (\lg(\eta_2)+1)$ be successful and observe that as $\eta_2 \in p$, $r = p \cap q = p^{[\eta_2]} \cap q$, let $\nu \in r \cap \boldsymbol{T}_{\delta'}$:
  * If $\nu \in \lim_{\delta'}(r^*_{\delta'})$:
    · If $\delta' \in S_1 \cap S_2$: $\nu \in \bigcup\{\lim_{\delta'}(q^*_{\delta',\eta'}) : \eta' \in \Lambda^*_{\delta'}\} \cap \lim_{\delta'}(p^*_{\eta_1,\delta',S_1\cap\delta'}) \cap \lim_{\delta'}(p^*_{\eta_2,\delta',S_2\cap\delta'})$ so $\nu \in \bigcup\{\lim_{\delta'}(q^*_{\delta',\eta'}) : \eta' \in \Lambda^*_{\delta'}\} \cap \lim_{\delta'}(p^*_{\eta_2,\delta',(S_1\cup S_2)\cap\delta'})$.
    · If $\delta' \in S_1 \setminus S_2$: $\nu \in \bigcup\{\lim_{\delta'}(q^*_{\delta',\eta'}) : \eta' \in \Lambda^*_{\delta'}\} \cap \lim_{\delta'}(p^*_{\eta_1,\delta',S_1\cap\delta'})$ and $\nu \in \lim_{\delta'}(p^*_{\eta_2,\delta',S_2\cap\delta'})$ so again $\nu \in \bigcup\{\lim_{\delta'}(q^*_{\delta',\eta'}) : \eta' \in \Lambda^*_{\delta'}\} \cap \lim_{\delta'}(p^*_{\eta_2,\delta',(S_1\cup S_2)\cap\delta'})$.
    · Similarly if $\delta' \in S_2 \setminus S_1$.
  * If $\nu \notin \lim_{\delta'}(r^*_{\delta'})$ then $\nu \in \lim_{\delta'}(p^*_{\eta_1,\delta',S_1\cap\delta'}) \cap \lim_{\delta'}(p^*_{\eta_2,\delta',S_2\cap\delta'})$ so $\nu \in \lim_{\delta'}(p^*_{\eta_2,\delta',(S_1\cup S_2)\cap\delta'})$.

Easily $S_1 \cup S_2$ is tenuous, so it holds that $r = p^*_{\eta_2,\delta,S_1\cup S_2}$ and $p, q$ are compatible.

(5) It holds that $\boldsymbol{T}_{<\delta} = p^*_{\langle\rangle,\delta,\emptyset}$ so trivially it belongs to $\mathbb{Q}_\delta$; for all $p \in \mathbb{Q}_\delta$ we have $p \subseteq \boldsymbol{T}_{<\delta}$ therefore by the order of $\mathbb{Q}_\delta$ it holds that $\boldsymbol{T}_{<\delta} \leq_{\mathbb{Q}_\delta} p$. For the first clause of this claim it follows that $\boldsymbol{T}_{<\delta} \in \mathbb{Q}'_\delta$ so for all $p \in \mathbb{Q}'_\delta$, in particular $p \subseteq \boldsymbol{T}_{<\delta}$ and by the order of $\mathbb{Q}'_\delta$, $\boldsymbol{T}_{<\delta} \leq_{\mathbb{Q}'_\delta} p$.

(6) Let $\eta, \nu \in p \cap \boldsymbol{T}_\alpha$ be different, then $\eta \notin p^{[\nu]}$ and $\nu \notin p^{[\eta]}$; recalling clause 4 it follows that $p^{[\eta]}, p^{[\nu]}$ are incompatible and so the set $\{p^{[\eta]} : \eta \in p \cap \boldsymbol{T}_\alpha\}$ is an antichain in $\mathbb{Q}_\delta$ above $p$. In addition, let $p \leq_{\mathbb{Q}_\delta} q \in \mathbb{Q}_\delta$ and let $\eta_0 \in q \cap \boldsymbol{T}_\alpha \subseteq p \cap \boldsymbol{T}_\alpha$; then $p^{[\eta_0]}$ is compatible with $q$: their common upper bound is $q^{[\eta_0]}$ and this is in $\mathbb{Q}_\delta$ by what we just showed. Clearly $p^{[\eta_0]} \in \{p^{[\eta]} : \eta \in p \cap \boldsymbol{T}_\alpha\}$ so this set is indeed a maximal antichain. The proof for $\mathbb{Q}'_\delta$ is identical.

$\square$

**Corollary 34.** *Let $\delta \in S_* \cup \{\lambda\}$, if $\delta$ is successful <u>then</u> the set $\bar{q}^*_\delta = \langle q^*_{\delta,\eta} : \eta \in \Lambda^*_\delta\rangle$ is an antichain of $\mathbb{Q}'_\delta$ above $r^*_\delta$.*

### 2.3. **Properties of the Forcing.**

*Claim* 35. Let $\delta \in S_*$ be such that $S_* \cap \delta$ is non-stationary (in $\delta$) and let $\alpha \leq cf(\delta)$, <u>then</u> the forcing $\mathbb{Q}_\delta$ is strategically complete in $\alpha$.

*Remark* 36. Remember that if $\delta \in S_*$ is not inaccessible then $S_* \cap \delta$ is not stationary in $\delta$.

*Proof.* First, there is a club $E$ of $\delta$ such that $E \cap S_* = \emptyset$. Let $p \in \mathbb{Q}_\delta$, we shall play the game $\Game_\alpha(p, \mathbb{Q}_\delta)$, determining a strategy for COM;

(1) At the first step player COM will choose a condition $p_0 = p$.
(2) In successor step $i+1 < \alpha$: look at the condition $q_i$ that player INC chose in the $i$-th step; let $\beta_i = tr(q_i)$ . In addition let $\gamma_i = \min(E \setminus (\beta_i + 1))$. Now choose some $\eta_{i+1} \in q_i \cap \boldsymbol{T}_{\gamma_i}$; <u>player COM will choose</u> $p_{i+1} = (q_i)^{[\eta_{i+1}]}$, this is a condition of the forcing $\mathbb{Q}_\delta$ by claim 33. Observe that $tr(q_i) \trianglelefteq \eta_{i+1}$,

$q_i \leq_{\mathbb{Q}_\delta} p_{i+1}$ and by the choice player COM made, she forced player INC to have $\eta_{i+1} \trianglelefteq tr(q_{i+1})$.

(3) In limit step $i(*) < \alpha$: player COM will choose $p_{i(*)} = \bigcap_{i<i(*)} q_i$, let $S_{i(*)} = \bigcup_{i<i(*)} S_{q_i} \setminus \lg(\nu_{i(*)})$ and $\nu_{i(*)} = \bigcup_{i<i(*)} tr(q_i)$.
   (a) The node $\nu_{i(*)}$ belongs to all the conditions that player INC had chosen in the steps $i < i(*)$: observe that $\delta' = \sup\{\beta_i : i < i(*)\} = \sup\{\gamma_i : i < i(*)\}$ then $\delta' \in E$ since $E$ is a club disjoint to $S_i$ there is no pruning in the level $\delta'$, in particular $\nu_{i(*)}$ is not being pruned. Thus $\nu_{i(*)} \in q_i$ for all $i < i(*)$.
   (b) It remains to show that $p_{i(*)}$ is indeed a condition in the forcing, first observe that $cf(\delta') = cf(i(*))$:
      (i) For each node $\nu' \in p_{i(*)}$ such that $\lg(\nu') < \delta'$ there is $i < i(*)$ such that $\lg(\nu') < \lg(tr(q_i))$ and as $p_{i(*)}$ is the intersection, we get $\nu' \triangleleft tr(q_i)$ and so $\nu' \trianglelefteq \bigcup_{i<i(*)} tr(q_i)$. Thus $\bigcup_{i<i(*)} tr(q_i)$ is a node such that there is no splitting before it in $p_{i(*)}$. However in each level above this node there are splittings as those splittings exist for each $q_i$. In addition, for each $i < j < i(*)$ any splitting in the tree $q_j$ exists in the tree $q_i$ as well: this is an increasing sequence of conditions and $q_j \subseteq q_i$. It follows that $p_{i(*)}$ is a tree with trunk $\nu_{i(*)}$.
      (ii) The set $S_{i(*)}$ is tenuous: the set $S_* \cap \delta$ is non-stationary by the claim assumption, $S_{i(*)} \subseteq S_* \cap \delta$, thus $S_{i(*)}$ is non-stationary in $\delta$. For all $\delta' < \epsilon < \delta$, if $S_*$ doesn't reflect to $\epsilon$ then $S_{i(*)} \restriction \epsilon \subseteq S_*$ is non-stationary in $\epsilon$ by 13(1); if $S_*$ reflects to $\epsilon$ then $\epsilon$ is inaccessible and thus $S_{i(*)} \restriction \epsilon$ is a union of $i(*)$ sets, non-stationary in $\epsilon$ so by 13(1) $S_{i(*)} \restriction \epsilon$ is non-stationary and $S_{i(*)}$ is tenuous.

   Observe that $p^*_{\nu_{i(*)},\delta,S_{i(*)}} = p_{i(*)}$ since $\nu_{i(*)}$ is the trunk and prunings are only possible in levels of $S_{i(*)} = \bigcup_{i<i(*)} S_{q_i}$, so the equality follows. In addition for all $i < i(*)$, $q_i \leq_{\mathbb{Q}_\lambda} p_{i(*)}$ so easily $p_{i(*)}$ is the smallest supremum of those conditions.

□

**Theorem 37.** *If $\delta \in S_* \cup \{\lambda\}$ is inaccessible and $\alpha \leq \delta$, then the forcing $\mathbb{Q}_\delta$ is strategically complete in $\alpha$.*

*Proof.* For $\delta \in S_* \cup \{\lambda\}$ and $p \in \mathbb{Q}_\delta$, we shall play the game $\eth_\alpha(p, \mathbb{Q}_\delta)$; wlog we can assume that $\alpha = \delta$. We shall construct inductively the sequence $\langle p_i, q_i, E_i : i < \delta\rangle$ where $p_i$ is the $i$-th move of player COM, $q_i$ is the $i$-th move of player INC, $E_i$ is a club in $\delta$ chosen by COM after INC plays his $i$-th move; it shall be disjoint to $S_{q_i}$; assume that for all $i' < j' < i$: $E_{j'} \supseteq E_{i'}$.

(1) At the first step player COM will choose a condition $p_0 = p$.
(2) In successor step $i+1 < \alpha$: look at the condition $q_i$ that player INC chose in the $i$-th step; $E_i$ is a club disjoint to $S_{q_i}$ s.t. $E_i \subseteq \bigcap_{j<i} E_j$ (such a club exists since $\delta$ is inaccessible by the Theorem assumption); let $\beta_i = \lg(tr(q_i))$ and let $\gamma_i = \min(E_i \setminus (\beta_i + 1))$. Next, for some node $\eta_{i+1} \in q_i \cap \boldsymbol{T}_{\gamma_i}$; player COM

will choose $p_{i+1} = (q_i)^{[\eta_{i+1}]}$, this is a condition of the forcing $\mathbb{Q}_\delta$ by Claim 33. Observe that $tr(q_i) \trianglelefteq \eta_{i+1}$, $q_i \leq_{\mathbb{Q}_\delta} p_{i+1}$ and by the choice player COM made, she forced player INC to have $\eta_{i+1} \trianglelefteq tr(q_{i+1})$. Finally, after INC will play his $i+1$-th turn, COM will let $E_{i+1}$ be a club: $E_{i+1} \subseteq E_i \setminus S_{q_{i+1}}$; this is possible as $E_i$ is a club of $\delta$, $S_{q_{i+1}}$ is tenuous.

(3) In limit step $i(*) < \alpha$: player COM will choose $p_{i(*)} = \bigcap\limits_{i<i(*)} q_i$, let $S_{i(*)} = \bigcup\limits_{i<i(*)} S_{q_i}$, $\nu_{i(*)} = \bigcup\limits_{i<i(*)} tr(q_i)$ and $E_{i(*)} = \bigcap\limits_{i<i(*)} E_i$. Observe that $\lg(\nu_{i(*)}) < \delta$ since $\delta$ is inaccessible, in addition $E_{i(*)}$ is a club in $\delta$ as an intersection of $i(*) < cf(\delta)$ clubs.
   (a) The node $\nu_{i(*)}$ belongs to all the conditions that player INC had chosen in the steps $i < i(*)$: observe that $\delta' = \sup\{\beta_i : i < i(*)\} = \sup\{\gamma_i : i < i(*)\}$ then $\delta' \in E_{i(*)}$. Since $E_{i(*)}$ is a club that is a decreasing intersection of clubs, note $i < i(*) \Rightarrow E_i \cap S_{q_i} = \emptyset \Rightarrow E_{i(*)} \cap S_{q_i} = \emptyset$, there are no prunings in the level $\delta'$, in particular $\nu_{i(*)}$ is not being pruned. Thus $\nu_{i(*)} \in q_i$ for all $i < i(*)$.
   (b) It remains to show that $p_{i(*)}$ is indeed a condition in the forcing, first observe that $cf(\delta') = cf(i(*))$:
      (i) For each node $\nu' \in p_{i(*)}$ such that $\lg(\nu') < \delta'$ there is $i < i(*)$ such that $\lg(\nu') < \lg(tr(q_i))$ and as $p_{i(*)}$ is the intersection, we get $\nu' \triangleleft tr(q_i)$ and so $\nu' \trianglelefteq \bigcup\limits_{i<i(*)} tr(q_i)$. We get that $\bigcup\limits_{i<i(*)} tr(q_i)$ is a node such that there is no splitting before it in $p_{i(*)}$. However in each level above it there are splittings as there are such splittings for each $q_i$. In addition, for each $i < j < i(*)$ any splitting in the tree $q_j$ exists in the tree $q_i$ as well: this is an increasing sequence of conditions and $q_j \subseteq q_i$. It follows that $p_{i(*)}$ is a tree with trunk $\nu_{i(*)}$.
      (ii) The set $S_{i(*)}$ is tenuous: as a union of $i(*) < \delta = cf(\delta)$ non-stationary sets, $S_{i(*)}$ is non-stationary in $\delta$ by 13(1). For all $\delta' < \epsilon < \delta$, if $S_*$ doesn't reflect to $\epsilon$ then $S_{i(*)} \restriction \epsilon \subseteq S_*$ is non-stationary in $\epsilon$ by 13(1); if $S_*$ reflects to $\epsilon$ then $\epsilon$ is inaccessible and thus $S_{i(*)} \restriction \epsilon$ is a union of $i(*)$ sets, non-stationary in $\epsilon$ so by 13(1) $S_{i(*)} \restriction \epsilon$ is non-stationary and $S_{i(*)}$ is tenuous.

   Observe that $p^*_{\nu_{i(*)},\delta,S_{i(*)}} = p_{i(*)}$ since $\nu_{i(*)}$ is the trunk and prunings are only possible for levels of $S_{i(*)} = \bigcup\limits_{i<i(*)} S_{q_i}$, the equality follows. In addition for all $i < i(*)$, $q_i \leq_{\mathbb{Q}_\lambda} p_{i(*)}$ so easily $p_{i(*)}$ is the smallest supremum of those conditions.

Finally we can see that player COM has a legal move for each $i < \alpha$ thus the forcing $\mathbb{Q}_\delta$ is strategically complete in $\alpha$.

□

**Corollary 38.** *By 35 and 37, for all $\delta \in S_* \cup \{\lambda\}$, $\alpha \leq cf(\delta)$ the forcing $\mathbb{Q}_\delta$ is strategically complete in $\alpha$.*

*Claim* 39. Let $\delta \in S_* \cup \{\lambda\}$, and let $p, q \in \mathbb{Q}_\delta$ be conditions with the same trunk, then their intersection $p \cap q$ is in $\mathbb{Q}_\delta$ and will be their minimal upper bound.

*Proof.* As the conditions have the same trunk, so does the intersection. In addition if $p = p^*_{\eta,\delta,S_1}$ and $q = p^*_{\eta,\delta,S_2}$ then it is easy to see, by the forcing definition, that $p \cap q = p^*_{\eta,\delta,S_1 \cup S_2}$ and $S_1 \cup S_2$ is tenuous. Then $p, q \leq_{\mathbb{Q}_\delta} p \cap q$ and it clearly is the minimal condition with this property. $\square$

**Theorem 40.** *If $\delta \in S_* \cup \{\lambda\}$, then the $\delta^+$- chain condition holds for the forcing $\mathbb{Q}_\delta$ .*

*Proof.* Let $\mathcal{A} \subseteq \mathbb{Q}_\delta$ be an antichain, then for all $p, q \in \mathcal{A}$ by Claim 39, $tr(p) \neq tr(q) \in \boldsymbol{T}_{<\delta}$; recalling the definition of the good structure $\mathfrak{r}$ it holds that for each $\zeta < \delta$: $\theta_\zeta < \delta$ and as $\delta$ is a strong limit, $|\bigcup\limits_{\epsilon<\delta} \prod\limits_{i<\epsilon} \theta_i| = |\boldsymbol{T}_{<\delta}| \leq \delta$; in particular for any antichain $\mathcal{A} \subseteq \mathbb{Q}_\delta$, $|\mathcal{A}| \leq \delta$. $\square$

**Corollary 41.** *By 38 and 40, the forcing $\mathbb{Q}_\lambda$ is $\leq \lambda$-strategically complete and the $\lambda^+$-chain condition holds for it.*

**Definition 42.** For functions $f, g \in {}^\lambda\lambda$, say that $f \leq_* g$ when: $\sup\{\alpha < \lambda : f(\alpha) > g(\alpha)\} < \lambda$.

**Theorem 43.** *If $\lambda$ is an inaccessible cardinal, then the forcing $\mathbb{Q}_\lambda$ is $\lambda$- bounding.*

*Proof.* Let $p_* \in \mathbb{Q}_\lambda$ and $\underset{\sim}{\tau}$ a $\mathbb{Q}_\lambda$- name for a function from $\lambda$ to $\lambda$. We would like to have a condition $q \geq_{\mathbb{Q}_\lambda} p_*$, $q \in \mathbb{Q}_\lambda$ and a function $g : \lambda \to \lambda$ such that $q \Vdash_{\mathbb{Q}_\lambda}$ "$\underset{\sim}{\tau} \leq g$". We will write in this proof $\leq$ instead of $\leq_{\mathbb{Q}_\lambda}$ when comparing forcing conditions.

- We will find a sequence $\langle p_\epsilon, S_\epsilon, E_\epsilon, \alpha_\epsilon \rangle$ for each $\epsilon < \lambda$ such that:
  (1) it holds that $p_0 = p_*$,
  (2) $p_\epsilon = p^*_{\varrho,\lambda,S_\epsilon}$ for $\varrho = tr(p_*)$,
  (3) the sequence $\langle p_\zeta : \zeta \leq \epsilon \rangle$ is increasing and continuous,
  (4) $E_\epsilon$ is a club disjoint to $S_\epsilon$,
  (5) the sequence $\langle E_\epsilon : \epsilon < \lambda \rangle$ is decreasing,
  (6) for $\epsilon = \zeta + 1 < \lambda$ it holds that $\alpha_\epsilon \in E_\zeta$ and $\alpha_\epsilon \in S_* \setminus S_\zeta \setminus (\alpha_\zeta + 1)$,
  (7) for a limit $\epsilon < \lambda$, $\alpha_\epsilon \in E_\epsilon$,
  (8) the interesting levels will be $\langle \alpha_\zeta : \zeta \leq \epsilon \rangle$, an increasing continuous sequence of ordinals with $\alpha_0 > \lg(\varrho)$,
  (9) for $\zeta < \epsilon < \lambda$ it holds that $S_\zeta \cap (\alpha_\zeta + 1) = S_\epsilon \cap (\alpha_\zeta + 1)$,
  (10) the set $S_\epsilon \subseteq S_*$ is tenuous,
  (11) for $\epsilon = \zeta + 1$, the ordinal $\alpha_\epsilon$ represents the 'level, in which in the corresponding tree the value of the function will be determined', that is:
    (a) for all $\nu \in p_\epsilon \cap \boldsymbol{T}_{\alpha_\epsilon}$ it holds that $p^{[\nu]}$ forces a value for $\underset{\sim}{\tau}(\epsilon)$,
    (b) it holds that $p_\epsilon \Vdash_{\mathbb{Q}_\lambda}$ "$\underset{\sim}{\tau}(\zeta) \in u_\zeta$" where $u_\zeta$ a set of ordinals of cardinality $< \lambda$.
- Next we will show that this construction is possible, by induction:
  - For the basis $\epsilon = 0$:
    We have that $p_0 = p$, $\alpha_0 = \lg(\varrho)$ so 1 holds; $S_\epsilon$ is the tenuous set corresponding to $p$ and let $E_\epsilon$ be a club in $\lambda$ disjoint to $S_\epsilon$ (as $S_\epsilon$ is tenuous).
  - For $\epsilon < \lambda$ limit:

Start with the set $S_\epsilon$: let $S_\epsilon = \bigcup\limits_{\zeta<\epsilon} S_\zeta \subseteq S_*$. Then it is easy to see that 9 holds (by the induction hypothesis); let also $\alpha_\epsilon = \bigcup\limits_{\zeta<\epsilon} \alpha_\zeta$ and $E_\epsilon = \bigcap\limits_{\zeta<\epsilon} E_\zeta$ , observe that $E_\epsilon$ is a club disjoint to $S_\epsilon$, so clauses 4 and 5 hold.

Now we will show that the set $S_\epsilon$ is indeed tenuous: first, the set $S_\epsilon$ is not stationary in $\lambda$ as a union of $\epsilon < \lambda = cf(\lambda)$ sets that are not stationary in $\lambda$ and by Remark 12 when $S_*$ is non-reflecting, $S_\epsilon$ is also tenuous, but we have to prove it in general.

Next, let $\gamma < \lambda$ be an ordinal of uncountable cofinality and look at $S_\epsilon \restriction \gamma$:

If there exists $\zeta < \epsilon$ for which $\gamma < \alpha_\zeta$ then as $S_\epsilon \cap (\alpha_\zeta+1) = S_\zeta \cap (\alpha_\zeta+1)$ it follows that $S_\epsilon \cap \gamma = S_\zeta \cap \gamma$ and since $S_\zeta$ is tenuous this set is non-stationary.

For $\gamma = \alpha_\epsilon$, first observe that by the definition of $E_\epsilon$ as the limit of the clubs $\langle E_\zeta : \zeta < \epsilon\rangle$ and since the sequence of clubs is decreasing, and by 6 of the induction hypothesis it holds that $\alpha_\epsilon \in \bigcap\limits_{\zeta<\epsilon} E_\zeta = E_\epsilon$, this was clause 7, and so $\alpha_\epsilon \notin S_\epsilon$.

* When $\alpha_\epsilon$ is regular (and thus inaccessible): by 8 in the induction hypothesis, the set $\{\alpha_\zeta : \zeta \text{ is a limit ordinal } < \epsilon\}$ is a club of $\alpha_\epsilon$, in addition by clause 7 in the induction hypothesis, for all $\zeta < \epsilon$ limit: $\alpha_\zeta \notin S_\zeta$ and by clause 9 in the induction hypothesis, for every $\zeta < \xi < \epsilon$ it holds that $\alpha_\zeta \notin S_\xi$ and therefore $\alpha_\zeta \notin S_\epsilon$ and this club is disjoint to $S_\epsilon \restriction \alpha_\epsilon$, so this is not a stationary set.
* When $\alpha_\epsilon$ is singular, the set $S_*$ doesn't reflect to $\alpha_\epsilon$ by definition, so $S_* \restriction \alpha_\epsilon$ is a non-stationary set, and in particular $S_\epsilon \restriction \alpha_\epsilon \subseteq S_* \restriction \alpha_\epsilon$ is not a stationary set by 12.

Lastly for $\gamma > \alpha_\epsilon$:

* If $cf(\gamma) > \epsilon$ then for all $\zeta < \epsilon$ it holds that $S_\zeta \restriction \gamma$ is a non-stationary set from clause 10 of the induction hypothesis, so there is a club of $\gamma$ disjoint to it, call it $C_\zeta$. Letting $C_\epsilon = \bigcap\limits_{\zeta<\epsilon} C_\zeta$, this is a club as the intersection of $\epsilon$ clubs, disjoint to $S_\epsilon$ by its definition, so $S_\epsilon \restriction \gamma$ is non-stationary.
* Otherwise, if $\gamma > \epsilon \geq cf(\gamma)$ in particular it follows that $\gamma$ is singular, thus $S_*$ doesn't reflect to $\gamma$ and so also $S_\epsilon \subseteq S_*$ using Claim 12.

Let $p_\epsilon = p^*_{\varrho,\lambda,S_\epsilon}$ so clauses 2 and 3 hold.

– For $\epsilon = \zeta + 1$:
This is the main case, as here we deal with clause 11 that is responsible for determining the values of the function.
Define the following set:

$$\mathscr{I}_\epsilon = \{r \in \mathbb{Q}_\lambda : r \text{ forces a value on } \underset{\sim}{\tau}(\zeta) \wedge p_\zeta \leq_{\mathbb{Q}_\lambda} r \wedge \lg(tr(r)) > \alpha_\zeta\}$$

and observe:

(a) This set is dense above $p_\zeta$: for all $p \in \mathbb{Q}_\lambda$ with $p_\zeta \leq p$, we will find a condition $r$ stronger than $p$ that forces a value on $\underset{\sim}{\tau}(\zeta)$ and if $\lg(tr(r)) > \alpha_\zeta$ doesn't hold, we can extend $r$ to a stronger condition with long enough trunk.
(b) The set is open: for all $q \in \mathscr{I}_\epsilon$ and $r \geq q$, $q$ forces a value on $\underset{\sim}{\tau}(\zeta)$ and therefore, so does $r$, $\lg(tr(r)) \geq \lg(tr(q)) > \alpha_\zeta$ and of course that $p_\zeta \leq q \leq r$ .
Now define a set $\Lambda_\epsilon = \{tr(r) : r \in \mathscr{I}_\epsilon\}$ and for every $\eta \in \Lambda_\epsilon$ choose some $q_{\epsilon,\eta} \in \{r \in \mathscr{I}_\epsilon : tr(r) = \eta\}$.
Choose a set $\Lambda^1_\epsilon \subseteq \Lambda_\epsilon$ that is maximal under the restriction that for any different $\eta, \nu \in \Lambda^1_\epsilon$, $\nu \notin q_{\epsilon,\eta} \vee \eta \notin q_{\epsilon,\nu}$ ; let $\bar{q}_\epsilon = \langle q_{\epsilon,\eta} : \eta \in \Lambda^1_\epsilon\rangle$.

* Observe that the sequence $\bar{q}_\epsilon = \langle q_{\epsilon,\eta} : \eta \in \Lambda^1_\epsilon\rangle \in \Xi_\lambda$ because:
  (1) $\Lambda^1_\epsilon \subseteq \boldsymbol{T}_{<\lambda}$,
  (2) for all $\eta \in \Lambda^1_\epsilon$ it holds that $q_{\epsilon,\eta} \in \mathbb{Q}_\lambda \subseteq \mathbb{Q}^0_\lambda$ and $tr(q_{\epsilon,\eta}) = \eta$,
  (3) if $\eta, \nu \in \Lambda^1_\epsilon$ are different, then by the definition of $\Lambda^1_\epsilon$ it holds that $tr(q_{\epsilon,\nu}) = \nu \notin q_{\epsilon,\eta} \vee tr(q_{\epsilon,\eta}) = \eta \notin q_{\epsilon,\nu}$,
  (4) the condition $r^*_{\bar{q}_\epsilon} = \{\rho \in \boldsymbol{T}_{<\lambda} : (\exists \eta \in \Lambda^1_\epsilon)(\rho \in q_{\epsilon,\eta})\}$ belongs to the forcing $\mathbb{Q}_\lambda \subseteq \mathbb{Q}^0_\lambda$; even $r^*_{\bar{q}_\epsilon} = p_\zeta$: observe that for all $\eta \in \Lambda^1_\epsilon$, it holds that $q_{\epsilon,\eta} \subseteq p_\zeta$ and so $r^*_{\bar{q}_\epsilon} \subseteq p_\zeta$. Assume via contradiction that $\nu \in p_\zeta \setminus r^*_{\bar{q}_\epsilon}$ then there is $p^{[\nu]}_\zeta \leq_{\mathbb{Q}_\lambda} q$ that forces a value for $\underset{\sim}{\tau}(\zeta)$ and its trunk is longer than $\alpha_\zeta$, so $q \in \mathscr{I}_\epsilon$ and $tr(q) \in \Lambda_\epsilon$. If $tr(q) \in \Lambda^1_\epsilon$ we get $tr(q) \in r^*_{\bar{q}_\epsilon}$, a contradiction to the assumption; there is $\nu' \in \Lambda^1_\epsilon$ such that $tr(q) \in q_{\epsilon,\nu'}$ so again we get $tr(q) \in r^*_{\bar{q}_\epsilon}$ and $\nu \in r^*_{\bar{q}_\epsilon}$, a contradiction.

For all $\eta \in \Lambda^1_\epsilon$ it holds that $q_{\epsilon,\eta}$ forces a value on $\underset{\sim}{\tau}(\zeta)$; call this value $\gamma_{\epsilon,\eta}$. In addition let $C_\eta$ be a club disjoint to $S_{q_{\epsilon,\eta}}$.
**First, define an approximation for the club $E_\epsilon$.**
$E'_\epsilon = \{\delta \in E_\zeta : \delta > \alpha_\zeta$ is a limit ordinal such that $(\forall \nu' \in \Lambda^1_\epsilon)(\lg(\nu') < \delta \Rightarrow \delta \in C_{\nu'})$ and $(\forall \nu \in p_\zeta \cap \boldsymbol{T}_{<\delta})(\exists \eta \in \boldsymbol{T}_{<\delta} \cap \Lambda'_\epsilon)(\nu \in q_{\epsilon,\eta})\}$
The set $E'_\epsilon$ is a club in $\lambda$:

* Closed- for every increasing sequence of ordinals $\langle \delta_i : i < \zeta^*\rangle$ such that for all $i < \zeta^*$: $\delta_i \in E'_\epsilon$ and $\zeta^* < \lambda$, their limit $\delta = \lim_{i<\zeta^*} \delta_i$ is of course a limit ordinal. In addition, for all $\nu' \in \Lambda^1_\epsilon$ with $\lg(\nu') < \delta$ there is $j_0 < \zeta^*$ such that for all $j_0 < j < \zeta^*$ it holds that $\lg(\nu') < \delta_j$ (as $\delta$ is defined to be the limit of those), then $\delta_j \in C_{\nu'}$ and since $C_{\nu'}$ is a club it follows that $\delta \in C_{\nu'}$, as the limit of $\langle \delta_j : j_0 < j < \zeta^*\rangle$.
* Unbounded- otherwise, the set $E'_\epsilon$ was bounded by some $\xi < \lambda$; then for every limit $\xi < \delta \in E_\zeta$, $\delta \notin \bigcap_{\nu' \in \Lambda^1_\epsilon \cap \boldsymbol{T}_{<\delta}} C_{\nu'} \subseteq \bigcap_{\nu' \in \Lambda^1_\epsilon \cap \boldsymbol{T}_{<\xi}} C_{\nu'} = C$; so it holds that $\{\delta \in (\xi, \lambda) \cap E_\zeta : \delta$ is a limit ordinal$\} \cap C = \emptyset$, however $C$ is a club as the intersection of less than $\lambda$ clubs- a contradiction.

Finally let $E_\epsilon = E'_\epsilon \setminus (\alpha_\epsilon + 1)$ for $\alpha_\epsilon < \lambda$ defined below, notice that also $E_\epsilon$ is a club in $\lambda$.
**Define the level.**

We would like to have an ordinal $\delta$ for which the following properties hold:
a. $\delta \in E'_\epsilon \cap S_*$,
b. $\alpha_\zeta < \delta$ (follows from a.),
c. $r^*_\delta = p_\zeta \cap \boldsymbol{T}_{<\delta}$,
d. It holds that $\bar{q}^*_\delta = \langle q_{\epsilon,\eta} \cap \boldsymbol{T}_{<\delta} : \eta \in \Lambda^1_\epsilon \cap \boldsymbol{T}_{<\delta} \rangle$.
An ordinal with those properties exists:
First, by Claim 28 there is a stationary set of $\delta \in S_*$ such that clause d. holds for and call it $S^+$; as $E'_\epsilon$ is a club, we get that $S^+ \cap E'_\epsilon$ is stationary. Observe that for all $\delta \in S^+ \cap E'_\epsilon$ from clause d. it follows that $r^*_\delta = \bigcup\limits_{\eta \in \Lambda^*_\delta} q^*_\eta = \bigcup\limits_{\nu \in \Lambda^1_\epsilon \cap \boldsymbol{T}_{<\delta}} q_{\epsilon,\nu} \cap \boldsymbol{T}_{<\delta}$, in addition by the definition of $E'_\epsilon$ it holds that $p_\zeta \cap \boldsymbol{T}_{<\delta} = \bigcup\limits_{\nu \in \Lambda^1_\epsilon \cap \boldsymbol{T}_{<\delta}} q_{\epsilon,\nu} \cap \boldsymbol{T}_{<\delta}$, so for all $\delta \in S^+ \cap E'_\epsilon$ clause c. holds, as this set is not empty (as a stationary set) there is such $\delta$, and we are done.
<u>Let $\alpha_\epsilon = \delta$.</u> Observe that in particular it follow that $\Lambda^1_\epsilon \cap \boldsymbol{T}_{<\alpha_\epsilon} = \Lambda^*_{\alpha_\epsilon}$.
**Define the tenuous set of $p_\epsilon$.**
First in the $\alpha_\epsilon$-th level we will define the set of all the limits formed from the conditions of $\bar{q}^*_{\alpha_\epsilon}$:

$$\Lambda^2_\epsilon = \{\eta \in p_\zeta : (\lg(\eta) = \alpha_\epsilon) \wedge ((\exists \nu \in \Lambda^1_\epsilon \cap \boldsymbol{T}_{<\alpha_\epsilon} = \Lambda^*_{\alpha_\epsilon})(\eta \in \lim(q^*_{\alpha_\epsilon,\nu})))\}$$

For $\eta \in \Lambda^2_\epsilon$, by the definition above and the definition of the level there is unique $\nu \in \Lambda^*_{\alpha_\epsilon}$ with $\eta \in \lim(q^*_{\alpha_\epsilon,\nu})$ so $\eta \in q_{\epsilon,\nu}$ and let $r_\eta := (q_{\epsilon,\nu})^{[\eta]}$. Now, define $S^1_\epsilon = \cup\{S_{r_\eta} \setminus (\alpha_\epsilon + 1) : \eta \in \Lambda^2_\epsilon\}$. Observe that for every $\eta \in \Lambda^2_\epsilon$, $S_{r_\eta} \subseteq S_{q_{\epsilon,\nu}}$ for some $\nu \in \Lambda^*_{\alpha_\epsilon} \subseteq \boldsymbol{T}_{<\alpha_\epsilon}$. Thus $S^1_\epsilon \subseteq \cup\{S_{q_{\epsilon,\nu}} : \nu \in \Lambda^*_{\alpha_\epsilon}\}$ and this is a union of $\leq |\boldsymbol{T}_{<\alpha_\epsilon}| \leq \alpha_\epsilon$ sets, each one is a tenuous subset of $S_* \setminus (\alpha_\epsilon + 1)$ and in particular non-stationary in $\lambda$. So their union will be the union of $\leq \alpha_\epsilon < \lambda$ (as $\lambda$ is inaccessible) non-stationary sets, and as $\lambda = cf(\lambda)$ and by Claim 13 it follows that $S^1_\epsilon$ is a non-stationary set.
Next, let $\alpha_\epsilon < \delta < \lambda$:

- $*$ If $\delta$ is an inaccessible cardinal in $S_*$, we want to show that $S^1_\epsilon \restriction \delta$ is non-stationary in $\delta$: as $2^{\alpha_\epsilon} < \delta$ (by inaccessibility of $\delta$) and since for all $\eta \in \Lambda^2_\epsilon$ the set $S_{r_\eta}$ is tenuous, in particular $S_{r_\eta} \restriction \delta$ is non-stationary so $S^1_\epsilon$ is the union of $< \delta = cf(\delta)$ non-stationary set and by Claim 13 it is not stationary.
- $*$ Else, in particular $S_*$ does not reflect to $\delta$, then the set $S_* \restriction \delta$ is non-stationary in $\delta$ and so also in $S^1_\epsilon \restriction \delta$ by 12.

Finally $S^1_\epsilon$ is tenuous and let $S_\epsilon = S_\zeta \cup \{\alpha_\epsilon\} \cup S^1_\epsilon$ that is also tenuous. Moreover we can see that $E_\epsilon$ is disjoint to $S_\zeta \cup \{\alpha_\epsilon\}$ as a subset of $E_\zeta \setminus (\alpha_\epsilon + 1)$ and by the induction hypothesis; in addition for all $\delta \in E_\epsilon$, $\delta \in \bigcap\limits_{\nu' \in \Lambda^1_\epsilon \cap \boldsymbol{T}_{<\delta}} C_{\epsilon,\nu'}$. For all $\eta \in \Lambda^2_\epsilon$ it holds that $S_{r_\eta} \subseteq S_{q_{\epsilon,\nu}}$ for some $\nu \in \Lambda^1_\epsilon \cap \boldsymbol{T}_{<\alpha_\epsilon} \subseteq \Lambda^1_\epsilon \cap \boldsymbol{T}_{<\delta}$, so the set $C_{\epsilon,\nu}$ is disjoint to $S_{r_\eta}$ and in particular $\delta \notin S_{r_\eta}$. Finally we have that $S_\epsilon \cap E_\epsilon = \emptyset$.
**Define the condition.**

The condition will be $p_\epsilon = p^*_{\varrho,\lambda,S_\epsilon}$ and so $p_\epsilon \in \mathbb{Q}_\lambda$; we want of course to have that $p_\epsilon \subseteq p_\zeta$ for the condition to be stronger than the previous one, so the difference between them is in the prunes different than the prunings of $p_\zeta$, those are the prunings in the levels $\{\alpha_\epsilon\} \cup S^1_\epsilon$.
From the forcing definition it holds that for all $\rho \in p_\zeta$, $\rho \in p_\epsilon$ if and only if (1) $\rho \trianglelefteq \varrho$ or (2) $\varrho \trianglelefteq \rho$ also if $\alpha_\epsilon \leq \lg(\rho)$ then $p \restriction \alpha_\epsilon \in \Lambda^2_\epsilon \wedge \rho \in r_\eta$.

* We indeed have that $p_\zeta \leq_{\mathbb{Q}_\lambda} p_\epsilon$.
* The set $\{r_\eta : \eta \in \Lambda^2_\epsilon\}$ is predense above $p_\epsilon$ in $\mathbb{Q}_\lambda$, as in fact the pruning had been to get $p_\epsilon$ exactly by this set.
* Thus, as for all $\eta \in \Lambda^2_\epsilon$ it holds that $r_\eta \Vdash \underset{\sim}{\tau}(\zeta) = \gamma_{\epsilon,\nu_\eta}$ for some $\nu_\eta \trianglelefteq \eta$, we can write $u_\zeta = \{\gamma_{\epsilon,\nu_\eta} : \eta \in \Lambda^2_\epsilon\}$ and have $p_\epsilon \Vdash$ "$\underset{\sim}{\tau}(\zeta) \in u_\zeta$".

Clause 11 holds and so the construction is possible.

- Let $S' = \bigcup_{\epsilon<\lambda} S_\epsilon$, this is not stationary because $\Delta_{\epsilon<\lambda} E_\epsilon \cap S' = \emptyset$ and a tenuous set as for all $\delta < \lambda$ there is $\epsilon < \lambda$ with $S' \cap \delta = S_\epsilon \cap \delta$ (by clause 9).
- Lastly, let $q = p^*_{\varrho,\lambda,S'}$ then indeed $p \leq q$ and we can define $g : \lambda \to \lambda$ by: if $\epsilon = \zeta + 1$ successor, then $g(\epsilon) = \sup\{u_\zeta\}$. For limit $\epsilon$ let $g(\epsilon) = \sup\{g(\delta) : \delta < \epsilon\}$. Observe: $q \Vdash$ "$\underset{\sim}{\tau} \leq g$" so we are done.

$\square$

**Corollary 44.** *The forcing $\mathbb{Q}_\lambda$ resembles random real forcing.*

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