TRIVIAL AND NON-TRIVIAL AUTOMORPHISMS OF $\mathcal{P}(\omega_1)/[\omega_1]^{<\aleph_0}$

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ABSTRACT. The following statement is shown to be independent of set theory with the Continuum Hypothesis: There is an automorphism of $\mathcal{P}(\omega_1)/[\omega_1]^{<\aleph_0}$ whose restriction to $\mathcal{P}(\alpha)/[\alpha]^{<\aleph_0}$ is induced by a bijection for every $\alpha \in \omega_1$, but the automorphism itself is not induced by any bijection on ω_1.

1. INTRODUCTION

For any set X let $\mathcal{P}(X)/\mathcal{F}in$ represent the Boolean algebra of all subsets of X modulo the ideal of finite subsets of X. Let $A \equiv^* B$ denote that $A \triangle B$, the symmetric difference of A and B, is finite and, for $A \subseteq X$, let $[A]$ denote the equivalence class $\{B \subseteq X \mid A \equiv^* B\}$. A homomorphism

$$\Psi : \mathcal{P}(X)/\mathcal{F}in \to \mathcal{P}(Y)/\mathcal{F}in$$

is called trivial if there is a function $\psi : Y \to X$ such that $[\Psi(A)] = [\psi^{-1}A]$. Let $\mathbb{AUT}_\kappa$ denote the set of all automorphisms of $\mathcal{P}(\kappa)/\mathcal{F}in$. For $\Psi \in \mathbb{AUT}_\kappa$ let $\mathcal{T}(\Psi)$ denote, as in §2 of [8], the ideal of all subsets $X \subseteq \kappa$ such that $\Psi \restriction \mathcal{P}(X)/\mathcal{F}in$ is trivial.

The study of $\mathbb{AUT}_\omega$ was initiated by W. Rudin in [5, 6] who showed that the Continuum Hypothesis can be used to construct non-trivial autohomeomorphisms of $\beta\mathbb{N} \setminus \mathbb{N}$, in other words, using Stone duality, homeomorphisms $\beta\mathbb{N} \setminus \mathbb{N}$ such that the automorphism of $\mathcal{P}(\mathbb{N})/\mathcal{F}in$ they induce is not trivial. A further advance was provided by S. Shelah in [7] who showed that it is consistent with set theory that $\mathcal{T}(\Psi)$ is not proper — in other words, $\omega \in \mathcal{T}(\Psi)$ — for every $\Psi \in \mathbb{AUT}_\omega$; in more conventional terminology, every $\Psi \in \mathbb{AUT}_\omega$ is trivial. B. Velickovic later showed in [11] that the conjunction of OCA and MA implies that the same is true for every $\Psi \in \mathbb{AUT}_{\omega_1}$ and, assuming PFA, the same is true for every $\Psi \in \mathbb{AUT}_\kappa$. It was later shown in [9] that it is consistent that $\mathcal{T}(\Psi)$ contains an infinite set for every $\Psi \in \mathbb{AUT}_\omega$ yet there are Ψ such that $\mathcal{T}(\Psi)$ is proper.

However, finding extensions of Rudin's result on the existence on non-trivial automorphisms of $\mathcal{P}(\kappa)/\mathcal{F}in$ has proven to be much harder. In [10] it is shown

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that if $\kappa > 2^{\aleph_0}$ and κ is less than the first inaccessible cardinal then for every $\Psi \in \mathbb{AUT}_\kappa$ there is a set $X \in \mathcal{T}(\Psi)$ such that $|\kappa \setminus X| \leq 2^{\aleph_0}$. On the other hand, it has been shown by P. Larson and P. McKenney in [4] that if $\kappa \leq 2^{\aleph_0}$ and $\Psi \in \mathbb{AUT}_\kappa$ and $[\kappa]^{\aleph_1} \subseteq \mathcal{T}(\Psi)$ then Ψ is trivial. It follows that if κ is an uncountable cardinal less than the first inaccessible and $\Psi \in \mathbb{AUT}_\kappa$ is non-trivial then there is $X \in [\kappa]^{\aleph_1}$ such that $\Psi \restriction \mathcal{P}(X)/\mathcal{F}in$ is also non-trivial.

These results leave open the question of whether or not it is consistent that there is some $\Psi \in \mathbb{AUT}_{\omega_1}$ such that $\mathcal{T}(\Psi)$ is proper. Of course, this question must be formulated properly because an easy solution is to use Rudin's result under the Continuum Hypothesis and find a $\Psi \in \mathbb{AUT}_{\omega_1}$ such that $\omega \notin \mathcal{T}(\Psi)$. Hence the proper formulation is Question 7.2 of [10]: Is it consistent that there is some $\Psi \in \mathbb{AUT}_{\omega_1}$ such that $[\omega_1]^{\aleph_0} \subseteq \mathcal{T}(\Psi)$ and $\mathcal{T}(\Psi)$ is proper? A positive answer will be provided by Theorem 1.1. On the other hand, Theorem 4.2 will provide the following companion to Velickovic's result from [11] under the conjunction of OCA and MA: It is even consistent with the Continuum Hypothesis that $\mathcal{T}(\Psi)$ is not proper for any $\Psi \in \mathbb{AUT}_{\omega_1}$ such that $\mathcal{T}(\Psi) \supseteq [\omega_1]^{\aleph_0}$. The following are the main results to be proved:

Theorem 1.1. *Assuming $\diamondsuit^+_{\omega_1}$ (see Definition 2.1) there is $\Psi \in \mathbb{AUT}_{\omega_1}$ such that $\mathcal{T}(\Psi) \supseteq [\omega_1]^{\aleph_0}$ yet Ψ is not trivial.*

Theorem 1.2. *The Continuum Hypothesis, and even $\diamondsuit_{\omega_1}$, does not imply that there is $\Psi \in \mathbb{AUT}_{\omega_1}$ such that $\mathcal{T}(\Psi)$ is a proper ideal containing $[\omega_1]^{\aleph_0}$.*

In §3 the methods of §2 are modified to obtain results giving more information on the possible structure of $\mathcal{T}(\Psi)$.

2. Proof of Theorem 1.1

Definition 2.1. Let $H_{<\aleph_0}(X)$ be the hereditarily finite sets with the elements of X considered as atoms — in other words, $H_{<\aleph_0}(X) = \bigcup_{n\in\omega} A_n(X)$ where $A_0(X) = X$ and $A_{n+1}(X) = [A_n(X)]^{<\aleph_0}$. Following the proof of R. Jensen and K. Kunen in [1] that there is a Kurepa family if $V = L$, a family $\{D_\xi\}_{\xi\in\omega_1}$ will be said to be a $\diamondsuit^+_{\omega_1}$ sequence if:

- each D_ξ is a countable model of set theory without the power set axiom
- $\xi + 1 \subseteq D_\xi$
- for each $X \subseteq H_{<\aleph_0}(\omega_1)$ there is a club $C \subseteq \omega_1$ such that $X \cap H_{<\aleph_0}(\xi) \in D_\xi$ and $C \cap \xi \in D_\xi$ for every $\xi \in C$
- $\varnothing = D_{\xi+1} = D_{\xi+\omega}$ for each $\xi \in \omega_1$.

The last clause is not part of the usual definition, but will avoid technical difficulties that would complicate the proof of Theorem 1.1. The use of $H_{<\aleph_0}(\omega_1)$

instead of ω_1 avoids having to make remarks about coding when trapping more complicated sets, such as functions, instead of just subsets of ω_1.

The following theorem was first proved by R. Jensen and is documented in hand written notes in [2]. A proof can also be found in [3].

Theorem 2.2 (R. Jensen)**.** *There is a $\diamondsuit^+_{\omega_1}$ sequence in the constructible universe.*

Definition 2.3. Suppose that $\sqsubset$ is a tree ordering on $\omega_1 \times \omega$ whose α^{th} level is $\{\alpha\} \times \omega$. If $t \in \{\alpha\} \times \omega$ then α will be denoted by $\mathbf{ht}(t)$. If $\alpha \in \mathbf{ht}(t)$ then $t[\alpha]$ will denote the unique element of $\{\alpha\} \times \omega$ such that $t[\alpha] \sqsubset t$.

Let $\mathfrak{R}$ denote the set of all functions R such that there is some $C(R)$ such that:

$$C(R) \subseteq \omega_1 \text{ is closed} \tag{2.1}$$

$$(\forall \xi)\ \{\xi + 1, \xi + \omega\} \cap C(R) = \varnothing \tag{2.2}$$

$$\mathbf{domain}(R) = C(R) \times \omega \tag{2.3}$$

$$(\forall t \in \mathbf{domain}(R))\ R(t) \subseteq \mathbf{ht}(t) \tag{2.4}$$

$$(\forall t \sqsubset s)\ R(t) = R(s) \cap \mathbf{ht}(t). \tag{2.5}$$

If $R \in \mathfrak{R}$ and $\eta \in C(R)$ then define $R \perp \eta = R \restriction (C(R) \cap (\eta + 1)) \times \omega$ and note that $R \perp \eta \in \mathfrak{R}$. Let

$$\mathfrak{R}_\xi = \left\{ R \in \mathfrak{R} \;\middle|\; \sup(C(R)) \leq \xi \text{ and } (\forall \zeta \in C(R) \cap \xi + 1)\ a \restriction \zeta \in D_\zeta \right\}$$

noting that the dependence on $\sqsubset$ has been suppressed in the notation. Note also that it may happen that $\mathfrak{R}_\xi \neq \varnothing$ even when $D_\xi = \varnothing$.

Notation 2.4. For any function F and A a subset of the domain of F let $F\langle A \rangle$ denote the image of A under F.

The main part of the proof will be to construct the tree order $\sqsubset$ as well as mappings π_t for $t \in \omega_1 \times \omega$ and $\psi_\xi : \mathfrak{R}_\xi \to \mathfrak{R}_\xi$ for each $\xi \in \omega_1$. This will be accomplished constructing tree orderings $\sqsubset_\xi$ on $\xi \times \omega$, π_t for $t \in \xi \times \omega$ and $\psi_\xi : \mathfrak{R}_\xi \to \mathfrak{R}_\xi$ by induction on ξ so that the following hold:

(1) if $\eta \in \xi$ then $\sqsubset_\eta = \sqsubset_\xi \cap [\eta \times \omega]^2$
(2) π_t is an involution of $\mathbf{ht}(t)$ such that $\pi_t\langle\zeta\rangle = \zeta$ for every limit ordinal $\zeta \in \mathbf{ht}(t)$
(3) if $\xi + \omega \in \mathbf{ht}(t)$ then $\pi_t(\xi + i) = \xi + i$ for all but finitely many $i \in \omega$
(4) if $t \sqsubset_\xi s$ then $\pi_t \subseteq^* \pi_s$
(5) if $\eta \in \xi$ then $\psi_\eta \subseteq \psi_\xi$
(6) if $R \in \mathfrak{R}_\xi$ (to be precise, it must be specified that $\mathfrak{R}_\xi$ is defined using the tree ordering $\sqsubset_\xi$ in (2.5) of Definition 2.3) then

- $C(R) = C(\psi_\xi(R))$
- $\pi_t \langle R(t) \rangle \equiv^* \psi_\xi(R)(t)$

for all $t \in T_\xi$ such that $\mathbf{ht}(t) \geq \sup(C(R))$

(7) if $R \in \mathfrak{R}_\xi$ and $\eta \in C(R)$ then $\psi_\xi(R) \perp \eta = \psi_\xi(R \perp \eta)$.

It will, furthermore be assumed that if ξ is a limit ordinal then the following conditions will also hold.

(8) if $\mathcal{C} \in D_\xi$ is a maximal antichain in $\sqsubset_\xi$ then for all $t \in \{\xi\} \times \omega$ there is some $\zeta \in \xi$ such that $t[\zeta] \in \mathcal{C}$

(9) if $g \in D_\xi$ is a function with domain $\xi \times \omega$ such that $g(t) : \mathbf{ht}(t) \to \xi$ and[1] for each $t \in \xi \times \omega$ there is s such that $\mathbf{ht}(s) = \xi$ and $t \sqsubset_{\xi+1} s$ then for every $\mu \in \xi$ there is some η such that

- $\xi > \eta > \mu$
- $g(s[\eta + \omega])(\eta) \neq \pi_t(\eta)$

(10) if $\mathcal{A} \in [\mathfrak{R}_\xi]^{<\aleph_0}$ and $t \in \xi \times \omega$ then there is t^* such that

- $\mathbf{ht}(t^*) = \xi$
- $t \sqsubset_{\xi+1} t^*$
- $\pi_{t^*} \langle R(t^*) \rangle = \psi(R)(t^*)$

for all $R \in \mathcal{A}$.

If this induction can be completed, then let the tree order $\sqsubset$ be defined to be $\bigcup_{\xi \in \omega_1} \sqsubset_\xi$ and note that condition (8) implies that $\mathbb{S} = (\omega_1 \times \omega, \sqsubset)$ is a Suslin tree. Let $\psi : \mathfrak{R} \to \mathfrak{R}$ be defined by

$$\psi(R) = \bigcup_{\xi \in \omega_1} \psi_\xi(R \perp \xi)$$

using (2) and (7) to conclude that ψ is a well defined function from $\mathfrak{R}$ to itself .

Observe that if $\dot{A}$ is an $\mathbb{S}$-name for a subset of ω_1 then, since $\mathbb{S}$ is a Suslin tree, it is possible to find a club $C \subseteq \omega_1$ and R with domain $C \times \omega$ such that if $t \in C \times \omega$ then $R(t) \subseteq \mathbf{ht}(t)$ and for each $\xi \in C$ and each $t \in \{\xi\} \times \omega$

$$t \Vdash_{\mathbb{S}} \text{“}\dot{A} \cap \xi = R(t)\text{”}.$$

Given $R \in \mathfrak{R}$ and letting $\dot{G}$ be a name for the generic set on $\mathbb{S}$ define

$$R(\dot{G}) = \bigcup_{\xi \in \omega_1} R(\dot{G}_\xi)$$

where $\dot{G}_\eta$ is a name for the element of $\{\eta\} \times \omega$ satisfying

$$1 \Vdash_{\mathbb{S}} \text{“}\{\dot{G}_\eta\} = \dot{G} \cap \{\eta\} \times \omega\text{”}.$$

Hence every subset $A \subseteq \omega_1$ in an $\mathbb{S}$ generic extension is equal to $R(\dot{G})$ for some $R \in \mathfrak{R}$. Given a generic set $G \subseteq \mathbb{S}$ let Ψ be the function from $\mathcal{P}(\omega_1)/\mathcal{F}in$ to

[1] In applications it will always be the case that if $t \sqsubset s$ then $g(t) \subseteq g(s)$ but there is no need to assume this at this stage.

$\mathcal{P}(\omega_1)/\mathcal{F}in$ defined by $\Psi([R(\dot{G})]) = [\psi(R)(\dot{G})]$ for $R \in \mathfrak{R}$. Furthermore, in $V[G]$ let π_ξ be defined to be $\pi_{\dot{G}_\xi}$.

Claim 2.5.

(2.6) $1 \Vdash_{\mathbb{S}}$ "*$\dot{\Psi}$ is a well defined automorphism of $\mathcal{P}(\omega_1)/\mathcal{F}in$ such that*

$$(\forall \xi \in \omega_1)\ \dot{\Psi} \restriction \mathcal{P}(\xi)/\mathcal{F}in \text{ is induced by } \dot{\pi}_\xi\text{''}.$$

Moreover, $1 \Vdash_{\mathbb{S}}$ "*$\dot{\Psi}$ is non-trivial*".

Proof. Since it has already been established that if $G \subseteq \mathbb{S}$ is generic over V then in $V[G]$

$$\mathcal{P}(\omega_1) = \left\{ R(\dot{G}) \mid R \in \mathfrak{R} \cap V \right\}$$

the first point to establish is that Ψ is well defined. So suppose that R and R' are in $\mathfrak{R}$ and that

$$t \Vdash_{\mathbb{S}} \text{``}R(\dot{G}) \equiv^* R'(\dot{G})\text{''} \tag{2.7}$$

but that

$$t \Vdash_{\mathbb{S}} \text{``}\psi(R)(\dot{G}) \not\equiv^* \psi(R'(\dot{G}))\text{''}.$$

By extending t if necessary, it may be assumed that there is some $\eta \in \omega_1$ such that $t \Vdash_{\mathbb{S}}$ "$\psi(R)(\dot{G}) \cap \eta \not\equiv^* \psi(R')(\dot{G}) \cap \eta$" and, hence, that there is some $\eta \in \omega_1$ such that $t \Vdash_{\mathbb{S}}$ "$(\psi(R) \perp \eta)(\dot{G}) \not\equiv^* (\psi(R') \perp \eta)(\dot{G})$". By condition (7) it follows that $t \Vdash_{\mathbb{S}}$ "$\psi(R \perp \eta)(\dot{G}) \not\equiv^* \psi(R' \perp \eta)(\dot{G})$". By condition (6) it follows that

$$t \Vdash_{\mathbb{S}} \text{``}\pi_t \left\langle (R \perp \eta)(\dot{G}) \right\rangle \not\equiv^* \pi_t \left\langle (R \perp \eta)(\dot{G}) \right\rangle\text{''}$$

and, hence, that $t \Vdash_{\mathbb{S}}$ "$(R \perp \eta)(\dot{G}) \not\equiv^* (R \perp \eta)(\dot{G})$" contradicting condition (4) and (2.7). The fact that Ψ is one-to-one has a similar proof.

To see that Ψ is an automorphism suppose that $t \Vdash_{\mathbb{S}}$ "$R(\dot{G}) \subseteq^* R'(\dot{G})$" but that $t \Vdash_{\mathbb{S}}$ "$\psi(R(\dot{G})) \not\subseteq^* \psi(R'(\dot{G}))$". As in the argument for well definedness, it can be assumed that there is some $\eta \in \omega_1$ such that $t \Vdash_{\mathbb{S}}$ "$(\psi(R) \perp \eta)(\dot{G}) \not\subseteq^* (\psi(R') \perp \eta)(\dot{G})$". But condition (7) then yields the contradiction that

$$t \Vdash_{\mathbb{S}} \text{``}\psi(R \perp \eta)(\dot{G}) \not\subseteq^* \psi(R' \perp \eta)(\dot{G})\text{''}.$$

Since each π_t is an involution it follows easily that so is Ψ. From this it follows that Ψ is a surjection. To see that Ψ is not trivial, it suffices to show that there is no $g : \omega_1 \to \omega_1$ in $V[G]$ such that $\pi_\xi \subseteq g$ for all $\xi \in \omega_1$. To this end suppose that $s \Vdash_{\mathbb{S}}$ "$\dot{g} : \omega_1 \to \omega_1$" and note that since $\mathbb{S}$ is Suslin, there is a club $B \subseteq \omega_1$ such that for each $\beta \in B$ and $t \in \{\beta\} \times \omega$ there is some $\bar{g}(t) : \beta \to \beta$ such that

$$t \Vdash_{\mathbb{S}} \text{``}\dot{g} \restriction \beta = \bar{g}(t)\text{''}.$$

Let g with domain $\omega_1 \times \omega$ be defined by

$$g(t) = \begin{cases} \bar{g}(t) & \text{if } \mathbf{ht}(t) \in B \\ \bar{g}(t[\sup(B \cap \mathbf{ht}(t))]) & \text{otherwise.} \end{cases}$$

Then use $\diamondsuit^+_{\omega_1}$ to find $\xi \in \omega_1$ and $s^* \in \{\xi\} \times \omega$ such that

- $\xi \in B \setminus \mathbf{ht}(s)$
- $B \cap \xi$ is cofinal in ξ
- $g \restriction (B \times \omega) \in D_\xi$
- $s \sqsubset_\xi s^*$.

Then apply condition (9) to get that there are infinitely many $\gamma \in \xi$ such that

$$\pi_{s^*}(\gamma) \neq g(s^*[\gamma + \omega])(\gamma) = g(s^*)(\gamma).$$

Since $s^* \Vdash_{\mathbb{S}}$ "$\dot{g} \restriction \xi = g(s^*)$" it follows that $s^* \Vdash_{\mathbb{S}}$ "$\dot{g} \not\supseteq^* \pi_{s^*} = \pi_\xi$" as required. □

To begin the induction let $\sqsubset_{\omega+1}$ be an arbitrary tree order on $(\omega+1)\times\omega$ and let $\pi_t(k) = k$ for each $k \in \mathbf{ht}(t)$. Let $\psi_{\omega+1}(R) = R$ for each $R \in \mathfrak{R}_\omega$. It is immediate that conditions (1) to (7) and 10 all hold. Since ω is not a limit of limit ordinals, (8) and (9) are not relevant at this stage.

A very similar argument works if ξ is a limit ordinal and $\sqsubset_{\xi+1}$, $\psi_{\xi+1}$ and $\{\pi_t\}_{\mathbf{ht}(t)\leq\xi}$ have been constructed. In this case let $\sqsubset_{\xi+\omega+1}$ be an arbitrary tree order extending $\sqsubset_{\xi+1}$. If $\xi < \mathbf{ht}(t) < \xi + \omega$ let π_t be defined by

$$\pi_t(\gamma) = \begin{cases} \pi_{t[\xi]}(\gamma) & \text{if } \gamma \leq \xi \\ \gamma & \text{if } \gamma > \xi. \end{cases}$$

Let $\psi_{\xi+\omega+1} = \psi_\xi$ noting that $D_{\xi+\omega} = \varnothing$ and, hence, there are no further requirements on $\psi_{\xi+\omega+1}$ since $(\xi + \omega + 1) \cap C(R) \subseteq \xi + 1$ for all $R \in \mathfrak{R}$. It is again immediate that conditions (1) to (7) all hold. Note that (8) and (9) are again not relevant at this stage since $D_{\xi+\omega} = \varnothing$. In order for (10) to hold it is necessary to define π_t appropriately for $t \in \{\xi + \omega\} \times \omega$.

To do this, let $\{R_j\}_{j\in\omega}$ enumerate $\mathfrak{R}_\xi = \mathfrak{R}_{\xi+\omega}$ and let

$$f : (\xi + \omega) \times \omega \to \{\xi + \omega\} \times \omega$$

be a one-to-one function such that $t \sqsubset_{\xi+\omega+1} f(t, k)$ for each t and k. Let ξ^- be the largest ordinal that is a limit of limit ordinals and $\xi^- \leq \xi$. From Definition 2.3 it follows that

$$(\forall R \in \mathfrak{R}_\xi)\ \sup(C(R)) \leq \xi^-. \tag{2.8}$$

Now fix $t \in (\xi + \omega) \times \omega$ and $k \in \omega$. Let $\rho \in \xi^-$ be a limit ordinal larger than the maximal element of the finite set of all $\gamma \in \xi^-$ such that

$$(2.9) \qquad (\exists j \leq k)\ \pi^{-1}_{t[\xi]}(\gamma) \in R_j(t[\xi]) \text{ if and only if } \gamma \notin \psi_\xi(R_j)(t[\xi]).$$

It follows that the following two equalities hold:

$$(2.10) \qquad R_j(t[\xi]) \cap \rho = R^*_j(t[\rho])$$

$$(2.11) \qquad \psi_\xi(R_j)(t[\xi]) \cap \rho = \psi_\xi(R^*_j)(t[\rho])$$

where $R^*_j = R_j \perp \sup(C(R_j) \cap \rho)$. Then apply (10) and the induction hypotheses to find t^{**} such that $\mathbf{ht}(t^{**}) = \xi$ and $t[\rho] \sqsubset_\xi t^{**}$ such that

$$(2.12) \qquad \pi_{t^{**}} \left\langle R_j(t^{**}) \right\rangle = \psi_\xi(R_j)(t^{**})$$

for each $j \leq k$. Then define $\pi_{f(t,k)}$ by

$$\pi_{f(t,k)}(\gamma) = \begin{cases} \gamma & \text{if } \xi \leq \gamma < \xi + \omega \\ \pi_{t[\xi]}(\gamma) & \text{if } \rho \leq \gamma < \xi \\ \pi_{t^{**}}(\gamma) & \text{if } \gamma \in \rho. \end{cases}$$

It must first be established that $\pi_{f(t,k)}$ is an involution. This follows from the fact both

$$(2.13) \qquad \pi_{t[\xi]} \restriction [\rho, \xi) \text{ and } \pi_{t^{**}} \restriction \rho$$

are involutions of their domains since ρ is a limit ordinal and (2) holds.

Then, by (3) and the fact that $\xi = \xi^- + \omega \cdot m$ for some $m \in \omega$, it follows that $\pi_{f(t,k)}(\gamma) = \pi_t(\gamma)$ for all but finitely many $\gamma \in \mathbf{ht}(t)$; so (4) holds. Next, observe that

$$(2.14) \quad \pi_{t^{**}} \left\langle R_j(t[\xi]) \right\rangle \cap \rho = \pi_{t^{**}} \left\langle R_j(t[\xi]) \cap \rho \right\rangle = \pi_{t^{**}} \left\langle R^*_j(t[\rho]) \right\rangle$$
$$= \pi_{t^{**}} \left\langle R_j(t^{**}) \right\rangle \cap \rho = \psi_\xi(R_j)(t^{**}) \cap \rho = \psi_\xi(R^*_j)(t[\rho]) \cap \rho = \psi_\xi(R^*_j)(t[\xi]) \cap \rho.$$

The first, second, fourth and last equalities follow from (2), (2.10), (2.12) and (2.11) respectively. The others follow from the definition of t^{**} and β. It now follows that $f(t,k)$ witnesses that (10) holds for t and $\mathcal{A} = \{R_j\}_{j \leq k}$. In order to see this keep in mind that (2.8) holds and note that (2.14) implies that

(2.15)
$$\pi_{f(t,k)} \left\langle R_j(f(t,k)) \right\rangle = \left(\pi_{t[\xi]} \left\langle R_j(t[\xi]) \right\rangle \cap [\rho, \xi)\right) \cup \left(\pi_{t^{**}} \left\langle R_j(t[\xi]) \right\rangle \cap \rho\right)$$
$$= \left(\psi_\xi(R_j)(t[\xi]) \cap [\rho, \xi)\right) \cup (\psi_\xi(R^*_j)(t[\xi]) \cap \rho) = \psi_\xi(R_j)(f(t,k))$$

for each $j \leq k$.

So now suppose that $\xi \in \omega_1$ is an arbitrary limit of limit ordinals such that all of the induction hypotheses hold for all $\eta \in \xi$. First, let

$$\mathfrak{R}^* = \left\{ R \in \mathfrak{R}_\xi \mid C(R) \cap \xi \text{ is cofinal in } \xi \text{ or } \sup(C(R)) < \xi \right\}$$

or, in other words, $C(R) \notin \mathfrak{R}^*$ if $\xi \in C(R)$ and ξ has an immediate predecessor in $C(R)$. The first step will be to find $\sqsubset_{\xi+1}$, $\{\pi_t\}_{t\in\{\xi\}\times\omega}$ and $\psi_{\xi+1} \restriction \mathfrak{R}^*$ such that

(11) (1), (2), (3), (4), (8) and (9) all hold
(12) $\psi_\eta \subseteq \psi_{\xi+1} \restriction \mathfrak{R}^*$ for each $\eta \leq \xi$
(13) the versions of (6), (7) and (10) in which $\mathfrak{R}_\xi$ is replaced by $\mathfrak{R}^*$ all hold.

In order to do this begin by letting

- $\xi_n \in \xi$ be such that $\lim_{n\to\infty} \xi_n = \xi$
- $\{t_n\}_{n\in\omega}$ enumerate infinitely often $\xi \times \omega$
- $\{R_n\}_{n\in\omega}$ enumerate $\mathfrak{R}^*$
- $\{C_n\}_{n\in\omega}$ enumerate the antichains of $\sqsubset_\xi$ belonging to D_ξ
- $\{g_n\}_{n\in\omega}$ enumerate infinitely often all the functions g belonging to D_ξ such that $g(t) : \mathbf{ht}(t) \to \xi$ for each $t \in \xi \times \omega$.

Now fix n and construct a sequence $\{b_n(j)\}_{j\in\omega} \subseteq \xi \times \omega$ and involutions $\{\theta_j\}_{j\in\omega}$ such that (denoting $b_n(i)$ by $b(i)$ to simplify notation)

(14) $t_n \sqsubset_\xi b(0)$
(15) $b(i) \sqsubset_\xi b(i+1)$
(16) $\mathbf{ht}(b(j))$ is a limit ordinal at least as large as ξ_j
(17) there is some $s \in C_j$ such that $s \sqsubset^* b(j+1)$
(18) $\theta_0 = \pi_{b(0)}$ and the domain of θ_{i+1} is $[\mathbf{ht}(b(i)), \mathbf{ht}(b(i+1))$ and
   - $\theta_{i+1}(\gamma) = \pi_{b(i+1)}(\gamma)$ for all γ such that $\mathbf{ht}(b(i)) + \omega \leq \gamma < \mathbf{ht}(b(i+1))$
   - $\theta_{i+1}(\gamma) = \pi_{b(i+1)}(\gamma)$ for all but finitely many γ such that $\mathbf{ht}(b(i)) \leq \gamma < \mathbf{ht}(b(i)) + \omega$

(19) for all $j \in \omega$ there is $k \in \omega$ such that

$$\theta_{j+1}(\mathbf{ht}(b(j)) + k) \neq g_j(b(j+1)[\mathbf{ht}(b(j)) + \omega])(\mathbf{ht}(b(j) + k).$$

Furthermore, letting $R_{j,i} = R_j \perp \sup(C(R_j) \cap b(i))$, the following hold:

(20) $\pi_{b(i)} \left\langle R_{j,i}(b(i)) \right\rangle = \bigcup_{k\leq i} \theta_k \left\langle R_{j,i}(b(i)) \right\rangle = \psi_\xi(R_{j,i})(b(i))$ for all i and $j \leq n$
(21) $\pi_{b(i+1)} \left\langle R_{j,i+1}(b(i+1)) \setminus \mathbf{ht}(b(i)) \right\rangle = \theta_{i+1} \left\langle R_{j,i+1}(b(i+1)) \setminus \mathbf{ht}(b(i)) \right\rangle = \psi_\xi(R_{j,i+1})(b(i+1)) \setminus \mathbf{ht}(b(i))$ for all $j \leq i$.

If this can be done, then define $t \sqsubset_{\xi+1} (\xi, n)$ if and only if there is some j such that $t \sqsubset_\xi b(j)$. Then define $\pi_{(\xi,n)} = \bigcup_{j\in\omega} \theta_j$. Conditions (1) to (4) are immediate. Conditions (8) and (9) follow from (17) and (19) respectively and so (11) holds. Then for $R \in \mathfrak{R}^*$ define

$$\psi_{\xi+1}(R) = \begin{cases} \bigcup_{\eta\in\xi} \psi_\xi(R\perp\eta) & \text{if } \sup(C(R) \cap \xi) = \xi \\ \psi_\xi(R) & \text{if } \sup(C(R) \cap \xi) < \xi. \end{cases}$$

It is immediate that $C(R) = \psi_{\xi+1}(C(R))$ and that (12) holds. To see that (13) holds observe that (7) follows directly from the construction, (6) follows from condition (21) and (10) follows from condition (20). Then choose $\{b_m(i)\}$ similarly for all $m \in \omega$.

In order to construct $\{b(i)\}_{i\in\omega}$ use (10) to let $b(0)$ be such that $t_n \sqsubset_\xi b(0)$ and $\pi_{b(0)} \left\langle R_{j,0}(b(0)) \right\rangle = \psi_\xi(R_{j,0})(b(0))$ for $j \leq n$. Let $\theta_0 = \pi_{b(0)}$. It follows that conditions (14) to (16) all hold. Conditions (17), (19) and (21) do not apply in this case. Conditions (18) and (20) are immediate.

Now suppose that $b(i)$ is given. First find $s \in C_i$ such that either $s \sqsubset_\xi b(i)$ or $b(i) \sqsubset_\xi s$. Let $s^* = \max_{\sqsubset_\xi}(s, b(i))$. Then find a limit ordinal $\Xi \geq \xi_i$ such that $\mathbf{ht}(s^*) + \omega < \Xi$. Using (10) of the induction hypothesis let $b(i+1)$ be such that

- $\mathbf{ht}(b(i+1)) = \Xi$
- $s^* \sqsubset_\xi b(i+1)$
- $\pi_{b(i+1)} \left\langle R_{j,i+1}(b(i+1)) \right\rangle = \psi_\xi(R_{j,i+1})(b(i+1))$ for $j \leq \max(i, n)$.

It follows that conditions (15) and (16) both hold and condition (14) is no longer relevant. The choice of s guarantees that condition (17) holds. Let u_m denote $\mathbf{ht}(b(i)) + m$. Using (3) let $K \in \omega$ be such that $\pi_{b(i+1)}(u_m) = u_m$ for $m > K$. Find[2] $\ell_1 > \ell_0 > K$ such that $u_{\ell_0} \in R_j(b(i+1))$ if and only if $u_{\ell_1} \in R_j(b(i+1))$ for all $j \leq \max(i, n)$. Then let

$$\theta_{i+1} = \pi_{b(i+1)} \restriction [\mathbf{ht}(b(i)), \mathbf{ht}(b(i+1))$$

if either $g_i(b(i+1))(u_{\ell_0}) \neq u_{\ell_0}$ or $g_i(b(i+1))(u_{\ell_1}) \neq u_{\ell_1}$. Otherwise define θ_{i+1} with domain $[\mathbf{ht}(b(i)), \mathbf{ht}(b(i+1))$ by

$$\theta_{i+1}(\delta) = \begin{cases} \pi_{b(i+1)}(\delta) & \text{if } \delta \notin \{u_{\ell_0}, u_{\ell_1}\} \\ u_{\ell_1} & \text{if } \delta = u_{\ell_0} \\ u_{\ell_0} & \text{if } \delta = u_{\ell_1}. \end{cases}$$

Observe that

$$\theta_{i+1} \left\langle R_{j,i+1}(b(i+1)) \right\rangle = \psi_\xi(R_{j,i+1})(b(i+1)) \cap [\mathbf{ht}(b(i)), \mathbf{ht}(b(i+1))$$

for each $j \leq \max(i, n)$. Therefore (18), (19), (20) and (21) all hold. This completes the induction.

All that remains to be done is to define $\psi_\xi(R)$ for $R \in \mathfrak{R}_\xi \setminus \mathfrak{R}^*$. In other words, $\psi_\xi(R)$ must be defined when $R \in \mathfrak{R}_\xi$, $\xi \in C(R)$ but $\mu(R) = \sup(C(R) \cap \xi) < \xi$. In this case $\psi_\xi(R)(t)$ must be defined for each $t \in \{\xi\} \times \omega$. Note however, that

[2] The reader wondering why the argument presented here does not apply to ω_2 assuming $\diamondsuit^+_{\omega_2}$, thereby contradicting the results of [10], will note that this the key point that does not extend beyond ω_1.

$\psi(R)(t) \cap \mu(R)$ must be equal to $\psi(R \perp \mu(R))(\mu(R))$ in order for (2.5) to hold. Hence it suffices to define,

$$\psi(R)(t) = \psi(R \perp \mu(R))(\mu(R)) \cup \left([\mu(R), \xi) \cap \pi_t(R)\right).$$

Observe that

$$(\forall t \in \{\xi\} \times \omega)\ \pi_t \langle R(t) \rangle \setminus \mu(R) = \psi_\xi(R)(t) \setminus \mu(R) \tag{2.16}$$

and hence (6) holds. Conditions (5) and (7) are immediate. To see that (10) holds let $\mathcal{A} \in [\mathfrak{R}_\xi]^{<\aleph_0}$ and $t \in T_\xi$ such that $\mathbf{ht}(t) < \xi$. Let

$$\mathcal{A}^* = (\mathcal{A} \cap \mathfrak{R}^*) \cup \{ R \perp \mu(R) \mid R \in \mathcal{A} \setminus \mathfrak{R}^* \}$$

and note that $\mathcal{A}^* \subseteq \mathfrak{R}^*$. It is therefore possible to use the version of (10) for $\mathfrak{R}^*$ to find $t^* \sqsupset_{\xi+1} t$ such that $\mathbf{ht}(t^*) = \xi$ and $\pi_{t^*} \langle R(t^*) \rangle = \psi(R)(t^*)$ for all $R \in \mathcal{A}^*$. Then applying (2.16) yields that $\pi_{t^*} \langle R(t^*) \rangle = \psi(R)(t^*)$ for all $R \in \mathcal{A}$ as required.

3. Other results on $\mathcal{T}(\Psi)$

The methods of §2 can be modified to exert more control over $\mathcal{T}(\Psi)$. This section sketches arguments exhibiting two extreme possibilities for $\mathcal{T}(\Psi)$.

Theorem 3.1. *It is consistent that there is $\Psi \in \mathbb{AUT}_{\omega_1}$ such that $\mathcal{T}(\Psi)$ is a proper ideal, $[\omega_1]^{\leq\aleph_0} \subseteq \mathcal{T}(\Psi)$ but $\mathcal{T}(\Psi)$ is not a σ-ideal — in other words, ω_1 can be covered by countably many elements from $\mathcal{T}(\Psi)$.*

Proof. The only change needed to the proof of §2 is to choose disjoint sets B_n such that $\omega_1 = \bigcup_{n\in\omega} B_n$ such that $B_n \cap [\xi, \xi + \omega)$ is infinite for every $\xi \in \omega_1$ and then to add to (2) the requirement that for every $n \in \omega$ and for all but finitely many $\beta \in B_n \cap \mathbf{ht}(t)$ the equality $\pi_t(\beta) = \beta$ holds. This will guarantee that each B_n belongs to $\mathcal{T}(\Psi)$ but requires modifying (10) of §2 to the following:

(10) if $\mathcal{A} \in [\mathfrak{R}_\xi]^{<\aleph_0}$ and $m \in \omega$ and $t \in \xi \times \omega$ then there is $t^* \sqsupset_{\xi+1} t$ such that $\mathbf{ht}(t^*) = \xi$ and $\pi_{t^*} \langle R(t^*) \rangle = \psi(R)(t^*)$ for all $R \in \mathcal{A}$ and $\pi_{t^*}(\beta) = \beta$ for each $\beta \in \bigcup_{j\leq m} B_j \setminus \mathbf{ht}(t)$.

In choosing the u_{ℓ_i} required to satisfy (19) it will be required that the u_{ℓ_i} come from $\bigcup_{j>m} B_j$ where m is now an additional parameter in the enumeration following (13). □

Theorem 3.2. *It is consistent that there is $\Psi \in \mathbb{AUT}_{\omega_1}$ such that $[\omega_1]^{\leq\aleph_0} = \mathcal{T}(\Psi)$.*

Proof. In order to establish Theorem 3.2 it will be necessary to use $\diamondsuit^+_{\omega_1}$ to trap uncountable partial functions from ω_1 to ω_1 and not just bijections. This will, of course, require weakening (2) because it cannot be expected that any interval of the form $[\xi, \xi + \omega)$ will contain more than one member of the domain of the trapped function, as is necessary in choosing the u_{ℓ_i} to satisfy (19). On the other hand, dispensing with (2) entirely might create problems in finding the limit ρ to satisfy (2.9) because satisfying (2.13) would no longer be automatic. Nevertheless, the following modification of (10) of §2 allows requirement (2) to be removed from the construction:

(10) if $\mathcal{A} \in [\mathfrak{R}_\xi]^{<\aleph_0}$ and $t \in \xi \times \omega$ then there is $t^* \sqsupset_{\xi+1} t$ such that $\mathbf{ht}(t^*) = \xi$ and $\pi_{t^*} \langle R(t^*)\rangle = \psi(R)(t^*)$ for all $R \in \mathcal{A}$ and, furthermore, $\zeta = \pi_{t^*} \langle \zeta \rangle$.

It is easy to check that the construction of §2 actually does yield this stronger induction hypothesis.

Next modify (9) of §2 to the following:

(9) if $g \in D_\xi$ is a function with domain $\Gamma \times \omega$ for some Γ a cofinal subset of ξ and, if $g(t) : \Delta_t \to \gamma$ with Δ_t a cofinal subset of γ for each $\gamma \in \Gamma$ and $t \in \{\gamma\} \times \omega$ then for each $t \in \{\xi\} \times \omega$ the following holds:

$$(\forall \beta \in \xi)(\exists \gamma \in \Gamma)(\exists \delta \in \Delta_{t[\gamma]})\ \beta < \delta \text{ and } g(t[\gamma])(\delta) \neq \pi_t(\delta)$$

In choosing the u_{ℓ_i} required to satisfy (19) it can no longer be expected that they will come from $[\mathbf{ht}(b(j), \mathbf{ht}(b(j) + \omega)$. However, if it is only required that they belong to $\Delta_{b_n(j+1)}$ the construction can proceed as before. □

4. Proof of Corollary 1.2

Notation 4.1. Let $\mathbb{C}(X)$ denote the partial order of countable partial functions from X to 2 ordered by inclusion.

Theorem 4.2. *Given bijections $\pi_\xi : \xi \to \xi$ for each $\xi \in \omega_1$ such that*

(1) if $\xi \in \eta$ then $\pi_\xi \equiv^* \pi_\eta \restriction \xi$
(2) there is no $\pi : \omega_1 \to \omega_1$ such that $\pi_\eta \equiv^* \pi \restriction \eta$ for all $\eta \in \omega_1$
(3) $G \subseteq \mathbb{C}(\omega_1)$ generic

there is no set $B \subseteq \omega_1$ such that

$$\pi_\xi^{-1}(B) \equiv^* \bigcup_{g \in G} g^{-1}\{1\} \cap \xi$$

for each $\xi \in \omega_1$.

Proof. Suppose that $\dot{B}$ is a $\mathbb{C}(\omega_1)$ name such that

$$1 \Vdash_{\mathbb{C}(\omega_1)} \text{“}(\forall \xi \in \omega_1)\ \dot{B} \cap \xi \equiv^* \bigcup_{g \in \dot{G}} \pi_\xi \left\langle g^{-1}\{1\} \right\rangle\text{”}$$

where $\dot{G}$ is a name for the generic set. Let $\mathfrak{M} = (M, \dot{B}, \{\pi_\xi\}_{\xi \in \omega_1}, \in)$ be a countable elementary submodel of $(H(\aleph_2), \dot{B}, \{\pi_\xi\}_{\xi \in \omega_1}, \in)$ and let $\mu = M \cap \omega_1$.

Claim 4.3. *For all $g \in \mathbb{C}(\omega_1) \cap M$ there is $h \in \mathbb{C}(\omega_1) \cap M$ such that $g \subseteq h$ and*

$$h \Vdash_{\mathbb{C}(\omega_1)} \text{“}\dot{B} \cap \mathbf{domain}(h \setminus g) \neq \pi_\mu \left\langle (h \setminus g)^{-1}\{1\} \right\rangle\text{”}. \tag{4.1}$$

Proof. Suppose that $g \in \mathbb{C}(\omega_1) \cap M$ is a counterexample to the claim. Without loss of generality there is $\alpha \in \mu$ such that $\mathbf{domain}(g) = \alpha$. If $\alpha \in \delta \in \mu$ and $X \subseteq [\alpha, \delta)$ then define $F_{X,\delta} \in \mathbb{C}(\omega_1)$ to be the function extending g with domain δ such that if $\alpha \in \eta \in \delta$ then $F_{X,\delta}(\eta) = 1$ if and only if $\eta \in X$. It follows from the failure of (4.1) that if $\alpha \leq \beta < \delta$ then

$$F_{\{\beta\},\delta} \Vdash_{\mathbb{C}(\omega_1)} \text{“}\dot{B} \cap [\alpha, \delta) = \{\pi_\mu(\beta)\}\text{”}$$

and hence it is possible to define in $\mathfrak{M}$ a function θ by letting $\theta(\beta)$ be the unique ordinal such that

$$F_{\{\beta\},\delta} \Vdash_{\mathbb{C}(\omega_1)} \text{“}\dot{B} \cap [\alpha, \delta) = \{\theta(\beta)\}\text{”}$$

for all $\delta > \beta$ and noting that $\theta(\beta)$ is defined for each $\beta \geq \alpha$. Then

$$\mathfrak{M} \models \theta : [\alpha, \omega_1) \to [\alpha, \omega_1) \text{ and } (\forall \beta > \alpha)(\forall \delta > \beta) \quad F_{\{\beta\},\delta} \Vdash_{\mathbb{C}(\omega_1)} \text{“}\dot{B} \cap [\alpha, \delta) = \{\theta(\beta)\}\text{”}. \tag{4.2}$$

By Hypothesis 2 of the theorem, there must be ξ such that

$$\mathfrak{M} \models \pi_\xi \not\equiv^* \theta \restriction \xi \tag{4.3}$$

and since $\theta \subseteq \pi_\mu$ it follows $\pi_\xi \not\equiv^* \pi_\mu \restriction \xi$ contradicting Hypothesis 1. □

Using Claim 4.3 it is easy to find a sequence $\{h_n\}_{n\in\omega}$ of conditions in $\mathbb{C}(\omega_1) \cap M$ such that $h_n \subseteq h_{n+1}$ and

$$h_{n+1} \Vdash_{\mathbb{C}(\omega_1)} \text{“}\dot{B} \cap \mathbf{domain}(h_{n+1} \setminus h_n) \neq \pi_\mu \left\langle (h_{n+1} \setminus h_n)^{-1}\{1\} \right\rangle\text{”}$$

and then to let $h = \bigcup_n h_n$. It follows that $h \Vdash_{\mathbb{C}(\omega_1)}$ “$\dot{B} \cap \mu \not\equiv^* \pi_\mu \left\langle h^{-1}\{1\} \right\rangle$” as required. □

Theorem 1.2 can now be established with the following argument.

Proof. Let V be a model of the Continuum Hypothesis and let G be a subset of $\mathbb{C}(\omega_2)$ that is generic over V. Then $\diamondsuit_{\omega_1}$ holds in $V[G]$. Given $\Psi \in \mathbb{A}UT_{\omega_1}$ such that $\mathcal{T}(\Psi) \supseteq [\omega_1]^{\aleph_0}$ let $X \in [\omega_2]^{\aleph_1}$ be such that for each $\xi \in \omega_1$ there is $\pi_\xi \in V[G \cap \mathbb{C}(X)]$ such that $\Psi \restriction \mathcal{P}(\xi)/\mathcal{F}in$ is induced by π_ξ. If $\mathcal{T}(\Psi)$ is not a proper ideal in $V[G \cap \mathbb{C}(X)]$ then it is also not a proper ideal in $V[G \cap \mathbb{C}(\omega_2)]$ so assume that $\mathcal{T}(\Psi)$ is a proper ideal in $V[G \cap \mathbb{C}(X)]$. Then let $\mu = \sup(X) + 1$ and apply Theorem 4.2 to conclude that if

$$B \in \Psi([\{\beta \in \omega_1 \mid \exists g \in G\ g(\mu + \beta) = 1\}])$$

then there is some $\xi \in \omega_1$ such that $\pi_\xi^{-1}(B) \not\equiv^* g^{-1}\{1\} \cap \xi$ for all $g \in G \cap \mathbb{C}(\mu + \omega_1)$. A standard argument shows that no countably closed forcing can add a set Z such that for every $\xi \in \omega_1$ there is $g \in G \cap \mathbb{C}(\mu + \omega_1)$ such that $\pi_\xi^{-1}(Z) \equiv^* g^{-1}\{1\} \cap \xi$. Hence $[\{\beta \in \omega_1 \mid \exists g \in G\ g(\mu + \beta) = 1\}]$ has no image under Ψ in $V[G]$ contradicting that $\Psi \in \mathbb{AUT}_{\omega_1}$. □

5. Open questions

An examination of the Velickovic's proof of Theorem 3.1 from [11] reveals that it shows that it is consistent that there is some $\Psi \in \mathbb{AUT}_\omega$ such that $\mathcal{T}(\Psi)$ is an ultrafilter. His proof does not generalize to answer the following question though.

Question 5.1. Is it consistent that there is $\Psi \in \mathbb{AUT}_{\omega_1}$ such that $\mathcal{T}(\Psi)$ is an ultrafilter? Can the question be answered when ω_1 is replaced by some other uncountable cardinal?

It was mentioned in the introduction that it is shown in [10] that if $\kappa > 2^{\aleph_0}$ and κ is less than the first inaccessible cardinal then for every $\Psi \in \mathbb{AUT}_\kappa$ there is a set $X \in \mathcal{T}(\Psi)$ such that $|\kappa \setminus X| \leq 2^{\aleph_0}$. The following question remains open though.

Question 5.2. Is it consistent that κ is at least as large as the first inaccessible cardinal and there is $\Psi \in \mathbb{AUT}_\kappa$ such that $T(\Psi)$ is a proper ideal and $[\kappa]^{<\kappa} \subseteq \mathcal{T}(\Psi)$?

However, it will be noted that the remark following Question 7.4 in [10] is strengthened by the following. Recall that if κ is weakly compact then every tree of height κ whose levels all have size less than κ has a branch of length κ.

Proposition 5.3. *If κ is a weakly compact cardinal then every Ψ such that $[\kappa]^{<\kappa} \subseteq \mathcal{T}(\Psi)$ is trivial.*

Proof. If $\Psi \in \mathbb{AUT}_\kappa$ is a counterexample to the proposition then note first that there is an unbounded set $S \subseteq \kappa$ and a finite $F \subseteq \kappa$ such that for each $\xi \in S$ there is a one-to-one function $\pi_\xi : \xi \setminus F \to \xi$ such that π_ξ induces $\Psi \restriction \mathcal{P}(\xi)/\mathcal{F}in$. To see this simply choose a continuous sequence $\{\mathfrak{M}_\xi\}_{\xi \in \kappa}$ of elementary submodels of $(H(\kappa^+), \Psi, \in)$ such that the set of elements of κ in the universe of $\mathfrak{M}_\xi$ is an ordinal $\mu_\xi \in \kappa$ and, if ξ has uncountable cofinality, then the universe of $\mathfrak{M}_\xi$ is closed under countable subsets. Note that since $[\kappa]^{<\kappa} \subseteq \mathcal{T}(\Psi)$, for each $\xi \in \kappa$ there is some $\pi : \mu_\xi \to \kappa$ that induces $\Psi \restriction \mathcal{P}(\mu_\xi)/\mathcal{F}in$. Note also that if ξ has uncountable cofinality and $\pi^{-1}(\kappa \setminus \mu_\xi)$ is infinite then there is some infinite $Z \subseteq \pi^{-1}(\kappa \setminus \mu_\xi)$ such that $Z \in \mathfrak{M}_\xi$. By elementarity there are η and θ in $\mathfrak{M}_\xi$ such that

$$\mathfrak{M}_\xi \models Z \subseteq \eta \text{ and } \theta \text{ induces } \Psi \restriction \mathcal{P}(\eta)/\mathcal{F}in.$$

But then $\theta\langle Z\rangle \subseteq \mu_\xi$ contradicting the fact that $\theta \restriction \eta \equiv^* \pi \restriction \eta$. Therefore $F_\xi = \pi^{-1}(\kappa \setminus \mu_\xi)$ is finite and π_ξ can be defined to be $\pi \restriction \xi \setminus F_\xi$. There is then some fixed F such that

$$S = \left\{ \mu_\xi \;\middle|\; F_\xi = F \text{ and } \xi \in \kappa \text{ and } \mathbf{cof}(\xi) \geq \omega_1 \right\}$$

satisfies the requirement.

Let $\{\sigma_\xi\}_{\xi\in\kappa}$ be an increasing enumeration of S and let

$$L_\xi = \left\{ \pi : \sigma_\xi \setminus F \to \sigma_\xi \;\middle|\; \pi \equiv^* \pi_{\sigma_\xi} \right\}$$

and note[3] that $|L_\xi| \leq 2^{|\sigma_\xi|} < \kappa$. Then let $T = \left(\bigcup_{\xi\in\kappa} L_\xi, \subseteq\right)$.

Note that $L_\eta \neq \varnothing$ since $\pi_{\sigma_\eta} \in L_{\sigma_\eta}$ and that distinct elements of L_η are incomparable under $\subseteq$. Hence it suffices to check that if $\xi \in \eta \in \kappa$ then

$$(\forall \pi \in L_\eta)(\exists \theta \in L_\xi)\, \theta \subseteq \pi \tag{5.1}$$

since this will establish that L_η is precisely the η^{th} level of T. But (5.1) is immediate since $\theta = \pi \restriction \sigma_\xi \setminus F \in L_\xi$. T is therefore a tree of height κ with levels of cardinality less than κ and no branches of length κ, contradicting that κ is weakly compact. □

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[3]Note also that if L_ξ were to be defined as $\left\{ \pi : \sigma_\xi \to \kappa \;\middle|\; \pi \equiv^* \pi_{\sigma_\xi} \right\}$, as would be natural, then it would not be the case that $|L_\xi| < \kappa$.

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