CICHOŃ'S MAXIMUM

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ABSTRACT. Assuming four strongly compact cardinals, it is consistent that all entries in Cichoń's diagram are pairwise different, more specifically that

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0}.$$

INTRODUCTION

Independence. How many Lebesgue null-sets are required to cover the real line? Obviously countably many are not enough, as the countable union of null-sets is null; and obviously continuum many are enough, as $\bigcup_{r\in\mathbb{R}} N_r = \mathbb{R}$ for $N_r := \{r\}$.

The answer to our question is a cardinal number which we call $\mathrm{cov}(\mathcal{N})$. As we have just seen,

$$\aleph_0 = |\mathbb{N}| < \mathrm{cov}(\mathcal{N}) \leq |\mathbb{R}| = 2^{\aleph_0}.$$

In particular, if the Continuum Hypothesis (CH) holds (i.e., if there are no cardinalities strictly between $|\mathbb{N}|$ and $|\mathbb{R}|$, or equivalently: if $\aleph_1 = 2^{\aleph_0}$), then $\mathrm{cov}(\mathcal{N}) = 2^{\aleph_0}$; but without CH, the answer could also be some cardinal less than $2^{\aleph_0}$. According to Cohen's famous result [Coh63], CH is independent of the usual axiomatization of mathematics, the set theoretic axiom system ZFC. I.e., we can prove that the ZFC axioms neither imply CH nor imply ¬CH. For this result, Cohen introduced the method of forcing, which has been continuously expanded and refined ever since. Forcing also proves that the value of $\mathrm{cov}(\mathcal{N})$ is independent. For example, $\mathrm{cov}(\mathcal{N}) = \aleph_1 < 2^{\aleph_0}$ is consistent, as is $\aleph_1 < \mathrm{cov}(\mathcal{N}) = 2^{\aleph_0}$.

Cichoń's diagram. $\mathrm{cov}(\mathcal{N})$ is a so-called cardinal characteristic. Other characteristics include:

- $\mathrm{add}(\mathcal{N})$, the smallest number of Lebesgue null-sets whose union is not null;
- $\mathrm{non}(\mathcal{N})$ is the smallest cardinality of a non-null set, and
- $\mathrm{cof}(\mathcal{N})$ is the smallest size of a cofinal family of null-sets, i.e., a family that contains for each null-set N a superset of N.
- Replacing "null" with "meager", we can analogously define $\mathrm{add}(\mathcal{M})$, $\mathrm{non}(\mathcal{M})$, $\mathrm{cov}(\mathcal{M})$, and $\mathrm{cof}(\mathcal{M})$.
- In addition, we define $\mathfrak{b}$ as the smallest size of an unbounded family, i.e., a family $\mathcal{H}$ of functions from $\mathbb{N}$ to $\mathbb{N}$ such that for every $f : \mathbb{N} \to \mathbb{N}$ there is some $h \in \mathcal{H}$ such that $(\forall n \in \mathbb{N})\,(\exists m > n)\; h(m) > f(m)$.

  Equivalently, $\mathfrak{b} = \mathrm{add}(\mathcal{K}) = \mathrm{non}(\mathcal{K})$, where $\mathcal{K}$ is the σ-ideal generated by the compact subsets of the irrationals.

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- And $\mathfrak{d}$ is the smallest size of a dominating family, i.e., a family $\mathcal{H}$ such that for every $f : \mathbb{N} \to \mathbb{N}$ there is some $h \in \mathcal{H}$ such that $(\exists n \in \mathbb{N})\,(\forall m > n)\; h(m) > f(m)$.
  Equivalently, $\mathfrak{d} = \mathrm{cov}(\mathcal{K}) = \mathrm{cof}(\mathcal{K})$.
- For the ideal ctbl of countable sets, we trivially get $\mathrm{add}(\mathrm{ctbl}) = \aleph_1$ and $\mathrm{cov}(\mathrm{ctbl}) = 2^{\aleph_0}$.

The characteristics we have mentioned so far,[1] and the basic relations between them, can be summarized in Cichoń's diagram:

$$\begin{array}{ccccccccc}
 & & \mathrm{cov}(\mathcal{N}) & \to & \mathrm{non}(\mathcal{M}) & \to & \mathrm{cof}(\mathcal{M}) & \to & \mathrm{cof}(\mathcal{N}) & \to & 2^{\aleph_0} \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & & & \mathfrak{b} & \longrightarrow & \mathfrak{d} & & & & \\
 & & & & \uparrow & & \uparrow & & & & \\
\aleph_1 & \to & \mathrm{add}(\mathcal{N}) & \to & \mathrm{add}(\mathcal{M}) & \to & \mathrm{cov}(\mathcal{M}) & \to & \mathrm{non}(\mathcal{N}) & &
\end{array}$$

An arrow between $\mathfrak{x}$ and $\mathfrak{y}$ indicates that ZFC proves $\mathfrak{x} \le \mathfrak{y}$. Moreover, $\max(\mathfrak{d}, \mathrm{non}(\mathcal{M})) = \mathrm{cof}(\mathcal{M})$ and $\min(\mathfrak{b}, \mathrm{cov}(\mathcal{M})) = \mathrm{add}(\mathcal{M})$. A (by now) classical series of theorems [Bar84; BJS93; CKP85; JS90; Kam89; Mil81; Mil84; RS83] proves these (in)equalities in ZFC and shows that they are the only ones provable. More precisely, all assignments of the values $\aleph_1$ and $\aleph_2$ to the characteristics in Cichoń's Diagram are consistent, provided they do not contradict the above (in)equalities. (A complete proof can be found in [BJ95, ch. 7].)

Note that Cichoń's diagram shows a fundamental asymmetry between the ideals of Lebesgue-null-sets and of meager sets (we will mention another one in the context of large cardinals): Any such asymmetry is hidden if we assume CH, as under CH not only all the characteristics are $\aleph_1$, but even the Erdős-Sierpiński Duality Theorem holds [Oxt80, ch. 19]: There is an involution $f : \mathbb{R} \to \mathbb{R}$ (i.e., a bijection such that $f \circ f = \mathrm{Id}$) such that $A \subseteq \mathbb{R}$ is meager iff $f''A$ is null.

So it is settled which assignments of $\aleph_1$ and $\aleph_2$ to Cichoń's diagram are consistent. It is more challenging to show that the diagram can contain more than two different cardinal values. For recent progress in this direction see, e.g., [Mej13; GMS16; FGKS17; KTT].

The result of this paper is in some respect the strongest possible, as we show that consistently *all* the entries are pairwise different (apart from the two equalities provable in ZFC mentioned above). Of course one can ask more, see the questions in Section 3. In particular, we use large cardinals in the proof.

Large cardinals. As mentioned, ZFC is an axiom system for the whole of mathematics. A much "weaker" axiom system (for the natural numbers) is PA (Peano arithmetic).

Gödel's Incompleteness Theorem shows that a theory such as PA or ZFC can never prove its own consistency. On the other hand, it is trivial to show in ZFC that PA is consistent (as in ZFC we can construct $\mathbb{N}$ and prove that it satisfies PA).

It is possible to arrange things in a way so that ZFC contains exactly the sentences of PA, together with the statement "there is an infinite cardinal". We can say: The existence of an infinite cardinal has a higher consistency strength than just PA.

There are notions of cardinals numbers much "stronger" than just "infinite". Often, such large cardinal assumptions (abbreviated LC in the following) have the following form:

> There is a cardinal $\kappa > \aleph_0$ that behaves towards the smaller cardinals in a similar way as $\aleph_0$ behaves to finite numbers.

A forcing proof shows, e.g.,

[1] There are many other cardinal characteristics, see for example [Bla10], but the ones in Cichoń's diagram seem to be considered to be the most important ones.

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If ZFC is consistent, then ZFC+¬CH is consistent,

and this implication can be proved in a very weak system such as PA. However, we can not prove (not even in ZFC) for any large cardinal

"if ZFC is consistent, then ZFC+LC is consistent";

because in ZCF+LC we can prove the consistency of ZFC. We say: LC has a higher consistency strength than ZFC.

An instance of a large cardinal (in fact the weakest one, a so-called inaccessible cardinal), appears in another striking example of the asymmetry between measure and category: The following statement is equiconsistent to an inaccessible cardinal [Sol70; She84]:

All projective[2] set of reals are Lebesgue measurable.

Whereas according to [She84] no large cardinal assumptions is required to show the consistency of

All projective set of reals have the property of Baire.

So we can assume all (reasonable) sets to have the Baire property "for free", whereas we have to provide additional consistency strength for Lebesgue-measurability.

In the case of our paper, we require (the consistency of) the existence of four compact cardinals to prove our main result. It seems unlikely that any large cardinals are actually required; but a proof without them would most probably be considerably more complicated. It is not unheard of that ZFC results first have (simpler) proofs using large cardinal assumptions, an example can be found in [She04].

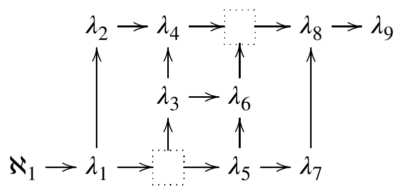
Annotated Contents. From now on, we assume that the reader is familiar with some basic properties of the characteristics defined above, as well as with the associated forcing notions Cohen, amoeba, random, Hechler and eventually different, all of which can be found, e.g., in [BJ95]).

This paper consists of two parts: In Section 1, we present a finite support ccc iteration $\mathbb{P}^5$ forcing that $\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}$. This result is not new: Such a forcing was introduced in [GMS16], and we follow this construction quite closely (however, we need GCH in the ground model, whereas in [GMS16] we require $2^\chi \gg \lambda$ for some $\chi < \lambda$).

In the second part, Section 2, we investigate an (iterated) Boolean ultrapower $\mathbb{P}^9$ of $\mathbb{P}^5$. We show the main result of this paper: Assuming four strongly compact cardinals, this ultrapower (again a finite support ccc iteration) forces

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0},$$

i.e., we get for increasing cardinals λ_i the following form of the diagram:



Boolean ultrapowers as used in this paper were investigated by Mansfield [Man71] and recently applied e.g. by the third author with Malliaris [MS16] and with Raghavan [RS],

[2] This is the smallest family containing the Borel sets and closed under continuous images, complements, and countable unions. In practice, all sets used in mathematics that are defined without using AC. Alternatively we could use the statement: "ZF (without the Axiom of choice) holds and all sets of reals are Lebesgue measurable."

where Boolean ultrapowers of forcing notions are used to force specific values to certain cardinal characteristics. Recently the third author developed a method of using Boolean ultrapowers to control characteristics in Cichoń's diagram. A first (and simpler) application of these methods is given in [KTT].

We mention some open questions in Section 3.

1. The initial forcing

1.1. **Good iterations and the LCU property.** We want to show that some forcing $\mathbb{P}^5$ results in $\mathfrak{x} = \lambda_i$ for certain characteristics $\mathfrak{x}$. So we have to show two "directions", $\mathfrak{x} \leq \lambda_i$ and $\mathfrak{x} \geq \lambda_i$. For most of the characteristics, one direction will use the fact that $\mathbb{P}^5$ is "good"; a notion introduced by Judah and the third author [JS90] and Brendle [Bre91]. We now recall the basic facts of good iterations, and specify the instances of the relations we use.

Assumption 1.1. *We will consider binary relations* R *on* $X = \omega^\omega$ *(or on* $X = 2^\omega$*) that satisfy the following: There are relations* R^n *such that* $\mathrm{R} = \bigcup_{n\in\omega} \mathrm{R}^n$*, each* R^n *is a closed subset (and in fact absolutely defined) of* $X \times X$*, and for* $g \in X$ *and* $n \in \omega$*, the set* $\{f \in X : f\,\mathrm{R}^n\,g\}$ *is nowhere dense (and of course closed). Also, for all* $g \in X$ *there is some* $f \in X$ *with* $f\,\mathrm{R}\,g$*.*

We will actually use another space as well, the space $\mathcal{C}$ of strictly positive rational sequences $(q_n)_{n\in\omega}$ such that $\sum_{n\in\omega} q_n \leq 1$. It is easy to see that $\mathcal{C}$ is homeomorphic to ω^ω, when we equip the rationals with the discrete topology and use the product topology.

We use the following instances of relations R on X; it is easy to see that they all satisfy the assumption (for $X_1 = \mathcal{C}$ we use the homeomorphism mentioned above):

Definition 1.2.
1. $X_1 = \mathcal{C}$: $f\,\mathrm{R}_1\,g$ if $(\forall^* n \in \omega)\, f(n) \leq g(n)$.
   (We use $\forall^* n$ as abbreviation for $(\exists n_0)\,(\forall n > n_0)$.)
2. $X_2 = 2^\omega$: $f\,\mathrm{R}_2\,g$ if $(\forall^* n \in \omega)\, f \restriction I_n \neq g \restriction I_n$,
   where $(I_n)_{n\in\omega}$ is any partition of ω with $|I_n| = 2^{n+1}$.
3. $X_3 = \omega^\omega$: $f\,\mathrm{R}_3\,g$ if $(\forall^* n \in \omega)\, f(n) \leq g(n)$.
4. $X_4 = \omega^\omega$: $f\,\mathrm{R}_4\,g$ if $(\forall^* n \in \omega)\, f(n) \neq g(n)$.

We say "f is bounded by g" if $f\,\mathrm{R}\,g$; and, for $\mathcal{Y} \subseteq \omega^\omega$, "$f$ is bounded by $\mathcal{Y}$" if $(\exists y \in \mathcal{Y})\, f\,\mathrm{R}\,y$. We say "unbounded" for "not bounded". (I.e., f is unbounded by $\mathcal{Y}$ if $(\forall y \in \mathcal{Y})\, \neg f\,\mathrm{R}\,y$.) We call $\mathcal{X}$ an R-unbounded family, if $\neg(\exists g)\,(\forall x \in \mathcal{X}) x\,\mathrm{R}\,g$, and an R-dominating family if $(\forall f)\,(\exists x \in \mathcal{X})\, f\,\mathrm{R}\,x$.

- Let $\mathfrak{b}_i$ be the minimal size of an R_i-unbounded family,
- and let $\mathfrak{d}_i$ be the minimal size of an R_i-dominating family.

We only need the following connection between R_i and the cardinal characteristics:

Lemma 1.3.
1. $\mathrm{add}(\mathcal{N}) = \mathfrak{b}_1$ *and* $\mathrm{cof}(\mathcal{N}) = \mathfrak{d}_1$.
2. $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{b}_2$ *and* $\mathrm{non}(\mathcal{N}) \geq \mathfrak{d}_2$.
3. $\mathfrak{b} = \mathfrak{b}_3$ *and* $\mathfrak{d} = \mathfrak{d}_3$.
4. $\mathrm{non}(\mathcal{M}) = \mathfrak{b}_4$ *and* $\mathrm{cov}(\mathcal{M}) = \mathfrak{d}_4$.

Proof. (3) holds by definition. (1) can be found in [BJ95, 6.5.B]. (4) is a result of [Mil82; Bar87], cf. [BJ95, 2.4.1 and 2.4.7].

To prove (2), note that for fixed $f \in 2^\omega$ the set $\{g \in 2^\omega : \neg f\,\mathrm{R}_2\,g\}$ is a null-set, call it N_f. Let $\mathcal{F}$ be an R_2-unbounded family. Then $\{N_f : f \in \mathcal{F}\}$ covers 2^ω: Fix $g \in 2^\omega$. As g does not bound $\mathcal{F}$, there is some $f \in \mathcal{F}$ unbounded by g, i.e., $g \in N_f$. Let X be a non-null set. Then X is R_2-dominating: For any $f \in 2^\omega$ there is some $x \in X \setminus N_f$, i.e., $f\,\mathrm{R}_2\,x$. □

We will also use:

Lemma 1.4. *[BJ95] Amoeba forcing* $\mathbb{A}$ *adds a dominating element* $\bar{b}$ *of* C*, i.e.,* $\mathbb{A} \Vdash \bar{q}\,\mathrm{R}_1\,\bar{b}$ *for all* $\bar{q} \in C \cap V$*.*

Proof. Let us define a slalom $\mathcal{S}$ to be a function $\mathcal{S} : \omega \to [\omega]^{<\omega}$ such that $|\mathcal{S}(n)| > 0$ and $\sum_{n=1}^{\infty} \frac{|\mathcal{S}(n)|}{n^2} < \infty$.

Amoeba forcing will add a null-set covering all old null-sets, and therefore (according to [BJ95, 2.3.3]) a slalom $\mathcal{S}$ covering all old slaloms. Set $a_n := \frac{|\mathcal{S}(n)|}{n^2}$, $M := \sum_{n=1}^{\infty} a_n$ and $b_n := \frac{a_{n+1}}{M}$. Then it is easy to see that $(b_n)_{n\in\omega} \in C$ dominates every old sequence $(q_n)_{n\in\omega}$ in C. □

Definition 1.5. [JS90] Let P be a ccc forcing, λ an uncountable regular cardinal, and R as above. P is (R, λ)-good, if for each P-name $r \in \omega^\omega$ there is (in V) a nonempty set $\mathcal{Y} \subseteq \omega^\omega$ of size $<\lambda$ such that every f (in V) that is R-unbounded by $\mathcal{Y}$ is forced to be R-unbounded by r as well.

Note that λ-good trivially implies μ-good if $\mu \geq \lambda$ are regular.

How to we get good forcings? Let us just quote the following results:

Lemma 1.6. *A FS iteration of Cohen forcing is good for any* (R, λ)*, and the composition of two* (R, λ)*-good forcings is* (R, λ)*-good.*
Assume that $(P_\alpha, Q_\alpha)_{\alpha<\delta}$ *is a FS ccc iteration. Then* P_δ *is* (R, λ)*-good, if each* Q_α *is forced to satisfy the following:*

1. *For* $\mathrm{R} = \mathrm{R}_1$*:* $|Q_\alpha| < \lambda$*, or* Q_α *is* σ*-centered, or* Q_α *is a sub-Boolean-algebra of the random algebra* $\mathbb{B}$*.*
2. *For* $\mathrm{R} = \mathrm{R}_2$*:* $|Q_\alpha| < \lambda$*, or* Q_α *is* σ*-centered.*
4. *For* $\mathrm{R} = \mathrm{R}_4$*:* $|Q_\alpha| < \lambda$*.*

(Remark: For R_3 the same holds as for R_4, which however is of no use for our construction.)

Proof. (R, λ)-goodness is preserved by FS ccc iterations (in particular compositions), as proved in [JS90], cf. [BJ95, 6.4.11–12]. Also, ccc forcings of size $<\lambda$ are (R, λ)-good [BJ95, 6.4.7]; which takes care of the case of Cohens and of $|Q_\alpha| < \lambda$. So it remains to show that (for $i = 1, 2$) the "large" iterands in the list are (R_i, λ)-good.

For R_1 this follows from [JS90] and [Kam89], cf. [BJ95, 6.5.17–18].

For R_2, this is proven in [Bre91], and as the proof is very short, we give it here: Write Q_α as union $\bigcup_{k\in\omega} Q^k$ of centered sets. Given the Q_α-name r, pick a countable elementary submodel N containing r, and set $\mathcal{Y} = N \cap 2^\omega$. Assume towards a contradiction that f is unbounded in $\mathcal{Y}$, but is forced by p_0 to be bounded by r, i.e., $p_0 \in Q^k$ forces $(\forall n > n_0) f \restriction I_n \neq r \restriction I_n$. In N, we can pick for each $n \in \omega$ some $s_n \in 2^{I_n}$ such that no $q \in Q^k$ forces $r \restriction I_n \neq s_n$. (There are only finitely many $s \in 2^{I_n}$; if each s is forbidden by some q, then the common stronger element would prevent all possibilities for $r \restriction I_n$.) So in N, we get some $g \in 2^\omega$ such that $g \restriction I_n = s_n$. As f is uncovered by N, there is some $n > n_0$ such that $f \restriction I_n = g \restriction I_n = s_n$, which implies that p_0 (as an element of Q^k) does not force $r \restriction I_n \neq f \restriction I_n$, a contradiction. □

Lemma 1.7. *Let* $\lambda \leq \kappa \leq \mu$ *be uncountable regular cardinals. After forcing with* μ *many Cohen reals* $(c_\alpha)_{\alpha\in\mu}$*, followed by an* (R, λ)*-good forcing, we get: For every real* r *in the final extension, the set* $\{\alpha \in \kappa : c_\alpha$ *is unbounded by* $r\}$ *is cobounded in* κ*. I.e.,* $(\exists \alpha \in \kappa)\,(\forall \beta \in \kappa \setminus \alpha)\, \neg c_\alpha\,\mathrm{R}\,r$*.*

(The Cohen real c_β can be interpreted both as Cohen generic element of 2^ω and as Cohen generic element of ω^ω; we use the interpretation suitable for the relation R.)

Proof. Work in the intermediate extension after κ many Cohen reals, let us call it V_κ. The remaining forcing (i.e., $\mu \setminus \kappa$ many Cohens composed with the good forcing) is good; so applying the definition we get (in V_κ) a set $\mathcal{Y}$ of size $<\lambda$.

As the initial Cohen extension is ccc, and $\kappa \geq \lambda$ is regular, we get some $\alpha \in \kappa$ such that each element y of $\mathcal{Y}$ already exists in the extension by the first α many Cohens, call it V_α. The set of reals M_y bounded by y is meager (and absolute). Any c_β for $\beta \in \kappa \setminus \alpha$ is Cohen over V_α, and therefore not in M_y, i.e., not bounded by y, i.e., not by $\mathcal{Y}$. So according to the definition of good, each such c_β is unbounded by r as well. □

In the light of this result, let us revisit Lemma 1.3 with some new notation, the "linearly cofinally unbounded" property LCU:

Definition 1.8. For $i = 1, 2, 3, 4$, γ a limit ordinal, and P a ccc forcing notion, let $\mathsf{LCU}_i(P, \gamma)$ stand for:

> There is a sequence $(x_\alpha)_{\alpha\in\gamma}$ of P-names such that for every P-name y $(\exists\alpha \in \gamma)\,(\forall\beta \in \gamma \setminus \alpha)\; P \Vdash \neg x_\beta\, \mathrm{R}_i\, y)$.

Lemma 1.9.
- $\mathsf{LCU}_i(P, \delta)$ *is equivalent to* $\mathsf{LCU}_i(P, \mathrm{cf}(\delta))$.
- *If λ is regular, then* $\mathsf{LCU}_i(P, \lambda)$ *implies* $\mathfrak{b}_i \leq \lambda$ *and* $\mathfrak{d}_i \geq \lambda$.

In particular:

1. $\mathsf{LCU}_1(P, \lambda)$ *implies* $P \Vdash (\,\mathrm{add}(\mathcal{N}) \leq \lambda\ \&\ \mathrm{cof}(\mathcal{N}) \geq \lambda\,)$.
2. $\mathsf{LCU}_2(P, \lambda)$ *implies* $P \Vdash (\,\mathrm{cov}(\mathcal{N}) \leq \lambda\ \&\ \mathrm{non}(\mathcal{N}) \geq \lambda\,)$.
3. $\mathsf{LCU}_3(P, \lambda)$ *implies* $P \Vdash (\,\mathfrak{b} \leq \lambda\ \&\ \mathfrak{d} \geq \lambda\,)$.
4. $\mathsf{LCU}_4(P, \lambda)$ *implies* $P \Vdash (\,\mathrm{non}(\mathcal{M}) \leq \lambda\ \&\ \mathrm{cov}(\mathcal{M}) \geq \lambda\,)$.

Proof. Assume that $(\alpha_\beta)_{\beta\in\mathrm{cf}(\delta)}$ is increasing continuous and cofinal in δ. If $(x_\alpha)_{\alpha\in\delta}$ witnesses $\mathsf{LCU}_i(P, \delta)$, then $(x_{\alpha_\beta})_{\beta\in\mathrm{cf}(\delta)}$ witnesses $\mathsf{LCU}_i(P, \mathrm{cf}(\delta))$. And if $(x_\beta)_{\beta\in\mathrm{cf}(\delta)}$ witnesses $\mathsf{LCU}_i(P, \mathrm{cf}(\delta))$, then $(y_\alpha)_{\alpha\in\delta}$ witnesses $\mathsf{LCU}_i(P, \mathrm{cf}(\delta))$, where $y_\alpha := x_\beta$ for $\alpha \in [\alpha_\beta, \alpha_{\beta+1})$.

The set $\{x_\alpha : \alpha \in \lambda\}$ is certainly forced to be R_i-unbounded; and given a set $Y = \{y_j : j < \theta\}$ of $\theta < \lambda$ many P-names, each has a bound $\alpha_j \in \lambda$ so that $(\forall\beta \in \lambda \setminus \alpha_j)\; P \Vdash \neg x_\beta\, \mathrm{R}_i\, y_j)$, so for any $\beta \in \lambda$ above all α_j we get $P \Vdash \neg x_\beta\, \mathrm{R}_i\, y_j$ for all j; i.e., Y cannot be dominating. □

1.2. **The inital forcing $\mathbb{P}^5$: Partial forcings and the COB property.** Assume we have a forcing iteration $(P_\beta, Q_\beta)_{\beta<\alpha}$ with limit P_α, where each Q_β is a set of reals such that the generic filter of Q_β is determined (in a Borel way)[3] from some generic real η_β. Fix some $w \subseteq \alpha$. We define the P_α-name Q_α to consist of all random forcing conditions that can be Borel-calculated from generics at w alone.

More explicitly:

Definition 1.10. (1) q is in Q_α if there is in the ground model V a countable subset $u \subseteq w$ and a Borel function $B : \mathbb{R}^u \to \mathbb{R}$ such that $q = B(\,(\eta_\beta)_{\beta\in u}\,)$ is a random condition.

[3]For example, define random forcing $\mathbb{B}$ to be the set of all positive pruned trees T, i.e., trees $T \subseteq 2^{<\omega}$ without leaves such that $[T]$ has positive measure. Then the generic filter $G_{\mathbb{B}}$ for random forcing is determined by the generic real η (the random real), and G consists of those trees T such that $\eta \in [T]$, which is a Borel relation. See [KTT, Sec. 1.2] for a formal definition and more details.

Being a random condition is a Borel property (if we fix some suitable representation of random forcing). Accordingly, we can restrict ourselves to the case that B is a Borel function whose image consists of random conditions only.

(2) We call a pair (B, u) as above "a w-groundmodel-code" or just "code". Note that this code is a ground model object. Q_α consists exactly of the evaluations of such codes.

(3) We call a condition $(p, q) \in P_\alpha * Q_\alpha$ "determined at position α", if there is a code (B, u) such that p forces that (B, u) is a code for q. (Note that generally we only have a P_α-name for a code.) Given some (p, q), we can obviously find $p' \leq p$ such that (p', q) is determined at α.

(4) We will later also consider so-called "groundmodel-code-sequences" for elements of Q, that is (in V) a sequence $(B_n, u_n)_{n\in\omega}$ of codes. Of course not every ω-sequence of Q_α-conditions in the P_α-extension is described by a ground model sequence. (In particular, there will only be few ground model sequences, but many new ω-sequences in the extension.)

Clearly Q_α is a subforcing (not necessarily a complete one) of the full random forcing, and if p, q in Q_α are incompatible in Q_α then they are incompatible in random forcing. (Two compatible conditions p, q have a canonical join $p \wedge q$ (the intersection), and if p and q are both Borel-calculated from w, then so is the the intersection.) In particular Q_α is ccc.

We call this forcing "partial random forcing defined from w". Analogously, we define the "partial Hechler", "partial eventually different"[4] and "partial amoeba" forcings (and the same argument shows that these forcings are also ccc).

Assume that λ is regular uncountable and $\mu < \lambda$ implies $\mu^{\aleph_0} < \lambda$. Then $|w| < \lambda$ implies that the sizes of the partial forcings defined by w are $<\lambda$.

We will assume the following throughout the paper:

Assumption 1.11. *$\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < \lambda_5$ are regular cardinals such that $\mu < \lambda_i$ implies $\mu^{\aleph_0} < \lambda_i$. Furthermore, λ_3 is the successor of a regular cardinal χ.*

We set $\delta_5 = \lambda_5 + \lambda_5$, and partition $\delta_5 \setminus \lambda_5$ into unbounded sets S^1, S^2, S^3 and S^4. Fix for each $\alpha \in \delta_5 \setminus \lambda_5$ some $w_\alpha \subseteq \alpha$ such that each $\{w_\alpha : \alpha \in S^i\}$ is cofinal[5] *in $[\delta]^{<\lambda_i}$.*

The reader can assume that $(\lambda_i)_{i=1,\dots,5}$, $(S^i)_{i=1,\dots,4}$ as well as $(w_\alpha)_{\alpha\in S^i}$ for $i = 1, 2, 3$ have been fixed once and for all (let us call them "fixed parameters"), whereas we will investigate various possibilities for $\bar{w} = (w_\alpha)_{\alpha\in S^4}$ in the following two sections. (We will call such a $\bar{w}$ that satisfies the assumption a "cofinal parameter".)

Definition 1.12. Let $\mathbb{P}^5 = (P_\alpha, Q_\alpha)_{\alpha\in\delta_5}$ be the FS iteration where Q_α is Cohen forcing for $\alpha \in \lambda_5$, and

$$Q_\alpha \text{ is the partial} \begin{Bmatrix} \text{amoeba} \\ \text{random} \\ \text{Hechler} \\ \text{eventually different} \end{Bmatrix} \text{forcing defined from } w_\alpha \text{ if } \alpha \text{ is in} \begin{cases} S^1 \\ S^2 \\ S^3 \\ S^4 \end{cases}$$

According to Lemma 1.6 $\mathbb{P}^5$ is $(\lambda_i, \mathrm{R}_i)$-good for $i = 1, 2, 4$, so Lemmas 1.7 and 1.9 gives us:

[4]See 1.21 for the definition.

[5]i.e., if $\alpha \in S^i$ then $|w_\alpha| < \lambda_i$, and for all $u \subseteq \delta_5$, $|u| < \lambda_i$ there is some $\alpha \in S^i$ with $w_\alpha \supseteq u$.

Lemma 1.13. $\mathsf{LCU}_i(\mathbb{P}^5, \kappa)$ *holds for* $i = 1, 2, 4$ *and each regular cardinal* κ *in* $[\lambda_i, \lambda_5]$.

So in particular, $\mathbb{P}^5$ forces $\text{add}(\mathcal{N}) \leq \lambda_1$, $\text{cov}(\mathcal{N}) \leq \lambda_2$, $\text{non}(\mathcal{M}) \leq \lambda_4$ and $\text{cov}(\mathcal{M}) = \text{non}(\mathcal{N}) = \text{cof}(\mathcal{N}) = \lambda_5 = 2^{\aleph_0}$; i.e., the respective left hand characteristics are small. It is easy to see that they are also large:

For example, the partial amoebas and the fact that $(w_\alpha)_{\alpha \in S^1}$ is cofinal ensure that $\mathbb{P}^5$ forces $\text{add}(\mathcal{N}) \geq \lambda_1$: Let $(N_k)_{k \in \mu}$, $\aleph_1 \leq \mu < \lambda_1$ be a family of $\mathbb{P}^5$-names of null-sets. Each N_k is a Borel-code, i.e., a real, i.e., a sequence of natural numbers, each of which is decided by a maximal antichain (labeled with natural numbers). Each condition in such an antichain has finite support, hence only uses finitely many coordinates in δ_5. So all in all we get a set w^* of size $\leq \mu$ that already decides all N_k. (I.e., for each $k \in \mu$ there is a Borel function B in V and a sequence $(\alpha_j)_{j \in \omega}$ in V of elements of w^* such that $N_i = B(\eta_{\alpha_0}, \eta_{\alpha_1}, \dots)$.) There is some $\beta \in S^1$ such that $w_\beta \supseteq w^*$, and the partial amoeba forcing at β sees all the null-sets N_k and therefore covers their union.

We will reformulate this in a slightly cumbersome manner that can be conveniently used later on, using the "cone of bounds" property COB:

Definition 1.14. For a ccc forcing notion P, regular uncountable cardinals λ, μ and $i = 1, 3, 4$, let $\mathsf{COB}_i(P, \lambda, \mu)$ stand for:

> There is a $<\lambda$-directed partial order $(S, \prec)$ of size μ and a sequence $(g_s)_{s \in S}$ of P-names for reals such that for each P-name f of a real $(\exists s \in S)\,(\forall t \succ s)\, P \Vdash f \mathrel{R_i} g_t$.

So s is the tip of a cone that consists of elements bounding f.

Lemma 1.15. *For* $i = 1, 3, 4$, $\mathsf{COB}_i(P, \lambda, \mu)$ *implies* $P \Vdash (\, \mathfrak{b}_i \geq \lambda \,\&\, \mathfrak{d}_i \leq \mu \,)$.

Proof. $\mathfrak{d}_i \leq \mu$, as the set $(g_s)_{s \in S}$ is a dominating family of size μ. To show $\mathfrak{b}_i \geq \lambda$, assume $(f_\alpha)_{\alpha \in \theta}$ is a sequence of P-names of length $\theta < \lambda$. For each f_α there is a cone of upper bounds with tip $s_\alpha \in S$, i.e., $(\forall t \succ s_\alpha)\, P \Vdash f_\alpha \mathrel{R_i} g_t$. As S is directed, there is some t above all tips s_α. Accordingly, $P \Vdash f_\alpha \mathrel{R_i} g_t$ for all α, i.e., $\{f_\alpha : \alpha \in \theta\}$ is not unbounded. □

So for example, $\mathsf{COB}_1(P, \lambda, \mu)$ implies $\lambda_1 \leq \mathfrak{b}_1 = \text{add}(\mathcal{N})$, etc. The definition and lemma would work for $i = 2$ as well, but would not be useful[6] as we do not have $\mathfrak{b}_2 \leq \text{cov}(\mathcal{N})$. So instead, we define COB_2 separately:

Definition 1.16. For P, λ and μ as above, let $\mathsf{COB}_2(P, \lambda, \mu)$ stand for:

> There is a $<\lambda$-directed partial order $(S, \prec)$ of size μ and a sequence $(g_s)_{s \in S}$ of P-names for reals such that for each P-name f of a null-set $(\exists s \in S)\,(\forall t \succ s)\, P \Vdash g_t \notin f$.

Lemma 1.17.
1. $\mathsf{COB}_1(P, \lambda, \mu)$ *implies* $P \Vdash (\, \text{add}(\mathcal{N}) \geq \lambda \,\&\, \text{cof}(\mathcal{N}) \leq \mu \,)$.
2. $\mathsf{COB}_2(P, \lambda, \mu)$ *implies* $P \Vdash (\, \text{cov}(\mathcal{N}) \geq \lambda \,\&\, \text{non}(\mathcal{N}) \leq \mu \,)$.
3. $\mathsf{COB}_3(P, \lambda, \mu)$ *implies* $P \Vdash (\, \mathfrak{b} \geq \lambda \,\&\, \mathfrak{d} \leq \mu \,)$.
4. $\mathsf{COB}_4(P, \lambda, \mu)$ *implies* $P \Vdash (\, \text{non}(\mathcal{M}) \geq \lambda \,\&\, \text{cov}(\mathcal{M}) \leq \mu \,)$.

Proof. The cases $i \neq 2$ are direct consequences of Lemmas 1.3 and 1.15. The proof for $i = 2$ is analogous to the proof of Lemma 1.15. □

Lemma 1.18. $\mathsf{COB}_i(\mathbb{P}^5, \lambda_i, \lambda_5)$ *holds (for* $i = 1, 2, 3, 4$*)*.

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[6] More specifically: this definition would give us the property $g_t \notin f$ only for the null-sets of the specific form $f = \{h : \neg r \mathrel{R_2} h\} = N_r$ for some $r \in 2^\omega$; whereas we will define COB_2 to deal with all names f of null-sets.

Proof. Set $S = S^i$ and $s \prec t$ if $w_s \subsetneq w_t$, and let g_s be the generic added at s (e.g., the partial random real in case of $i = 2$, etc). A $\mathbb{P}^5$-name f depends (in a Borel way) on a countable index set $w^* \subseteq \delta$. Fix some $s \in S^i$ such that $w_s \supseteq w^*$. Pick any $t \succ s$. Then $w_t \supseteq w_s$, so w_t contains all information to calculate f, so we can show that $P \Vdash f \, \mathrm{R}_i \, g_t$. Let us list the possible cases: $i = 2$: A partial random real g_t will avoid the null-set f. $i = 3$: A partial Hechler real g_t will dominate f. $i = 4$: A partial eventually different real g_t will be eventually different from f. As for $i = 1$, we use[7] Lemma 1.4. □

So to summarize what we know so far about $\mathbb{P}^5$:

- COB_i holds for $i = 1, 2, 3, 4$. So the left hand characteristics are large.
- LCU_i holds for $i = 1, 2, 4$. So the left hand characteristics other than $\mathfrak{b}$ are small.

However, LCU_3 (corresponding to "$\mathfrak{b}$ small") is missing; and we cannot get it by a simple "preservation of $(\mathrm{R}_3, \lambda_3)$-goodness" argument. Instead, we will argue in the following two sections that it is possible to choose the parameter $(w_\alpha)_{\alpha \in S^4}$ in such a way that LCU_3 holds as well.

1.3. **Dealing with $\mathfrak{b}$ without GCH.** In this section, we follow (and slightly modify) the main construction of [GMS16].

In this section (and this section only) we will assume the following (in addition to Assumption 1.11):

Assumption 1.19. *(This section only.)* $2^\chi = |\delta_5| = \lambda_5$.

Set $S^0 = \lambda_5 \cup S^1 \cup S^2 \cup S^3$. So $\delta_5 = S^0 \cup S^4$, and $\mathbb{P}^5$ is a FS ccc iteration along δ_5 such that $\alpha \in S^0$ implies $|Q_\alpha| < \lambda_3$, i.e., $|Q_\alpha| \leq \chi$. Let us fix P_α-names

$$i_\alpha : Q_\alpha \to \chi \text{ injective} \tag{1.20}$$

(for $\alpha \in S^0$). Note that we can strengthen each $p \in \mathbb{P}^5$ to some q such that $\alpha \in \mathrm{supp}(q) \cap S^0$ implies $q \restriction \alpha \Vdash i_\alpha(q(\alpha)) = \check{j}$ for some $j \in \chi$.

For $\alpha \in S^4$, Q_α is a partial eventually different forcing. At this point, we should specify which variant of this forcing we actually use:[8]

Definition 1.21.
- Eventually different forcing $\mathbb{E}$ consists of all pairs (s, k, φ), where $s \in \omega^{<\omega}$, $k \in \omega$, and $\varphi : \omega \to [\omega]^{\leq k}$ satisfies $s(i) \notin \varphi(i)$ for all $i \in \mathrm{dom}(s)$.
- We define $(s', k', \varphi') \leq (s, k, \varphi)$ if $s \subseteq s'$, $k \leq k'$, and $\varphi(i) \subseteq \varphi'(i)$ for all i.
- The generic object $g^* = \bigcup_{(s,k,\varphi) \in G_{\mathbb{E}}} s$ is a function such that each condition (s, k, φ) forces that s is an initial segment of g^*, and $g^*(i) \notin \varphi(i)$ for all i.
- We define the "stem" of (s, k, φ) to be the pair $(s, k) \in \omega^{<\omega} \times \omega$.

A density argument shows that g^* will be eventually different from all functions $f : \omega \to \omega$ from V.

The following is easy to see:

- If $p, q \in \mathbb{E}$ are compatible, then they have a greatest lower bound.
- Any finite set of conditions with the same stem has a lower bound (again with the same stem). So $\mathbb{E}$ is σ-centered.
- If $q = (s', k', \phi')$ and $p = (s, k, \phi)$ and s' extends s, then p and q are compatible iff $s'(i) \notin \phi(i)$ for all $i \in \mathrm{dom}(s')$.

[7] Alternatively, we could use, instead of amoeba, some other Suslin ccc forcing that more directly adds an R_1-dominating element of C.

[8] In the previous section it did not matter which variant we use.

- If a condition $q^* = (s^*, k^*, \varphi^*)$ is compatible with each condition in a finite set $B \subseteq \mathbb{E}$, and s^* extends s for each $(s, k, \varphi) \in B$, then the set $B \cup \{q^*\}$ has a lower bound. (Use the stem of q^*, and take the pointwise union of all φ that occur in $B \cup \{q^*\}$.

We will not force with $\mathbb{E}$, but with a partial version of $\mathbb{E}$. In the P_α-extension (for $\alpha \in S^4$), this partial forcing $Q_\alpha = \mathbb{E}'$ is a (generally not complete) sub-forcing of $\mathbb{E}$ which is easily seen to be closed under conjunctions (i.e., under the partial operation "greatest lower bound" of finite sets of conditions). Note that this implies that compatibility is absolute between $\mathbb{E}$ and $\mathbb{E}'$, and that the previous items also hold for $\mathbb{E}'$. For later reference, let us explicitly state the last item:

Fact 1.22. *Assume $\mathbb{E}' \subseteq \mathbb{E}$ is closed under conjunctions. If a condition $q^* = (s^*, k^*, \varphi^*)$ in $\mathbb{E}'$ is compatible with each condition in a finite set $B \subseteq \mathbb{E}'$, and s^* extends s for each $(s, k, \varphi) \in B$, then the set $B \cup \{q^*\}$ has a lower bound in $\mathbb{E}'$.*

Definition 1.23. Let D be a non-principal ultrafilter on ω, and let $\bar{p} = (p_n)_{n\in\omega} = (s, k, \varphi_n)_{n\in\omega}$ be a sequence of conditions in $\mathbb{E}$ with the same stem. We define $\lim_D \bar{p}$ to be (s, k, φ_∞), where for all i and all j we have $j \in \varphi_\infty(i) \Leftrightarrow \{n : j \in \varphi_n(i)\} \in D$.

The following is easy to see: $\lim_D \bar{p} \in \mathbb{E}$ and if $q \leq \lim_D \bar{p}$, then the set $B \coloneqq \{n \in \omega : p_n \text{ compatible with } q\}$ is in D.

(Proof: $q = (s', k', \phi') \leq \lim_D \bar{p} = (s, k, \phi_\infty)$. So for each $i \in \mathrm{dom}(s')$, $s'(i) \notin \phi_\infty(i)$, and by the definition of the limit, $A^i = \{n : s'(i) \notin \phi_n(i)\} \in D$. If $n \in \bigcap_{i\in\mathrm{dom}(s')} A^i$, then p_n is compatible with q.)

As B is defined using only compatibility, the statement still holds for compatibility preserving subforcings. We state it for later reference in the following form:

Fact 1.24. *Assume that $\mathbb{E}'$ is a subforcing of $\mathbb{E}$ closed under conjunctions, let $\bar{p}$ be a sequence of $\mathbb{E}'$ conditions with the same stem, and assume that $\lim_D(\bar{p}) \in \mathbb{E}'$ and that $q \leq_{\mathbb{E}'} \lim_D(\bar{p})$ Then $B \coloneqq \{n \in \omega : p_n \text{ compatible with } q\}$ is in D.*

Definition 1.25.
- A "partial guardrail" is a function h defined on a subset of δ_5 such that $h(\alpha) \in \chi$ for $\alpha \in S^0 \cap \mathrm{dom}(h)$, and $h(\alpha) \in \omega^{<\omega} \times \omega$ for $\alpha \in S^4 \cap \mathrm{dom}(h)$.
- A "countable guardrail" is a partial guardrail with countable domain. A "full guardrail" is a partial guardrail with domain δ_5.

We will use the following lemma, which is a consequence of the Engelking-Karłowicz theorem [EK65] on the density of box products (cf. [GMS16, 5.1]):

Lemma 1.26. *(As $|\delta_5| \leq 2^\chi$.) There is a family H^* of full guardrails of cardinality χ such that each countable guardrail is extended by some $h \in H^*$. We will fix such an H^* and enumerate it as $(h^*_\varepsilon)_{\varepsilon\in\chi}$.*

Note that the notion of guardrail (and the density property required in Lemma1.26) only depends on χ, δ_5, S^0 and S^4, i.e., on fixed parameters; so we can fix an H^* that will work for all cofinal parameters $\bar{w} = (w_\alpha)_{\alpha\in S^4}$.

Once we have decided on $\bar{w}$, and thus have defined $\mathbb{P}^5$, we can define the following:

Definition 1.27. A condition $p \in \mathbb{P}^5$ follows the full guardrail h, if
- for all $\alpha \in S^0 \cap \mathrm{dom}(p)$, the empty condition of P_α forces that $p(\alpha) \in Q_\alpha$ and $i_\alpha(p(\alpha)) = h(\alpha)$ (where i_α is defined in (1.20)), and
- for all $\alpha \in S^4 \cap \mathrm{dom}(p)$:
  - $p\restriction\alpha$ forces that $p(\alpha)$ has stem $h(\alpha)$, and moreover

– p is determined at α. (This was defined in 1.10(3): We already know in V a code (B, u) that evaluates to $p(\alpha)$.)

As we are dealing with a FS iteration, the set of conditions p determined at each position $\alpha \in \mathrm{dom}(p)$ is easily seen to be dense (by induction). So note that

- the set of conditions p such that there is *some* guardrail h such that p follows h, is dense; while
- for each *fixed* guardrail h, the set of all conditions p such that p follows h, is *centered* (i.e., each finitely many such p are compatible).

Definition 1.28.

- A "Δ-system with root ∇ following the full guardrail h" is a family $\bar{p} = (p_i)_{i\in I}$ of conditions all following h, where $(\mathrm{dom}(p_i) : i \in I)$ is a Δ system with root ∇ in the usual sense (so $\nabla \subseteq \delta_5$ is finite).
- We will be particularly interested in countable Δ-systems. Let $(p_n : n \in \omega)$ be such a Δ-system with root ∇ following h, and assume that $\bar{D} = (D_\alpha : \alpha \in u)$ is a sequence such that $u \supseteq \nabla \cap S^4$ and each D_α is a P_α-name of an ultrafilter on ω. Then we define the $\lim_{\bar{D}} \bar{p}$ to be the following function with domain ∇:
  - If $\beta \in \nabla \cap S^0$, then $\lim_{\bar{D}} \bar{p}(\beta)$ is the common value of all $p_n(\beta)$. (Recall that this value is already determined by the guardrail h.)
  - If $\alpha \in \nabla \cap S^4$, then $\lim_{\bar{D}} \bar{p}(\alpha)$ is (forced by $\mathbb{P}^5_\alpha$ to be) $\lim_{D_\alpha}(p_n(\alpha))_{n\in\omega}$.

Note that in general $\lim_{\bar{D}} \bar{p}$ will not be a condition in $\mathbb{P}^5$: the object $\lim_{\bar{D}} \bar{p}(\alpha)$ will be forced to be in the eventually different forcing $\mathbb{E}$, but not necessarily in the *partial* eventually different forcing $Q_\alpha \subseteq \mathbb{E}$.

Also note the following: If $\bar{p}$ is a countable Δ-system, and $\alpha \in \nabla \cap S^4$, then $(p_n(\alpha))_{n\in\omega}$ is a ground-model-code-sequence (see Definiton 1.10(4)). This follows trivially from the definition of "p_n follows h" and the fact that $\bar{p}$ is in V.

Recall that we assume all of the parameters defining $\mathbb{P}^5 = (P_\alpha, Q_\alpha)_{\alpha\in\delta_5}$ to be fixed, apart from $(w_\alpha)_{\alpha\in S^4}$. Once we fix w_α for $\alpha \in S^4 \cap \beta$, we know P_β.

Lemma/Construction 1.29. *We can construct by induction on $\alpha \in \delta_5$ the sequences $(D^\varepsilon_\alpha)_{\varepsilon\in\chi}$ and, if $\alpha \in S^4$, also w_α, such that:*

(a) Each D^ε_α is a P_α-name of a nonprincipal ultrafilter extending $\bigcup_{\beta<\alpha} D^\varepsilon_\beta$.

(b) For each countable Δ-system $\bar{p}$ in P_α which follows the guardrail $h^*_\varepsilon \in H^*$: $\lim_{(D^\varepsilon_\beta)_{\beta<\alpha}} \bar{p}$ is in $P_\alpha \dots$

(c) $\dots$ and forces that $A_{\bar{p}} \coloneqq \{n \in \omega : p_n \in G_\alpha\}$ is in D^ε_α.

(d) (If $\alpha \in S^4$.) $w_\alpha \subseteq \alpha$, $|w_\alpha| < \lambda_4$, and for all ground-model-code-sequences[9] for elements of Q_α, the D^ε_α-limit is forced to be in Q_α as well (for all $\varepsilon \in \chi$).
(Actually, the set of w_α satisfying this is an ω_1-club set in $[\alpha]^{<\lambda_4}$.[10])

Proof. (b) for α limit: The heart of a Δ-system is finite and therefore below some $\beta < \alpha$, so the limit exists (by induction) already in P_β.

(a+c) for α limit: It is enough to show, for each $\varepsilon \in \chi$, that P_α forces that the following forms a filter basis (i.e., any finite intersection of elements of this set is nonempty):

$$\bigcup_{\beta<\alpha} D^\varepsilon_\beta \,\cup\, \{A_{\bar{p}} : \bar{p} \text{ is a countable } \Delta\text{-system following } h^*_\varepsilon \text{ and } \lim_{(D^\varepsilon_\beta)_{\beta<\alpha}} \bar{p} \in G_\alpha\}.$$

[9] see Definition 1.10(4).

[10] I.e., for each $w^* \in [\alpha]^{<\lambda_4}$ there is a $w_\alpha \supseteq w^*$ satisfying (c), and if $(w^i)_{i\in\omega_1}$ is an increasing sequence of sets satisfying (c), then the limit $w_\alpha \coloneqq \bigcup_{i\in\omega_1} w^i$ satisfies (c) as well.

(Then we let D^ε_α be any ultrafilter extending this set.)

So assume towards a contradiction that $q \in P_\alpha$ forces that $A \cap A_{\bar p^0} \cap \cdots \cap A_{\bar p^{n-1}} = \emptyset$, where $A \in D^\varepsilon_{\beta_0}$ for some $\beta_0 < \alpha$ (we can assume β_0 is already decided in V) and $\bar p^i$ as above with $q \leq \lim_{(D^\varepsilon_\beta)_{\beta<\alpha}} \bar p^i$ for $i < n$. Let $\beta_1 < \alpha$ be the maximum of the union of the hearts of the $\bar p^i$, and set $\beta_2 = \max(\mathrm{supp}(q))$ and $\gamma = \max(\beta_0, \beta_1, \beta_2) + 1$. By the induction hypothesis, q forces $A' \coloneqq A \cap \bigcap_{i<n} A_{\bar p^i \restriction \gamma} \in D^\varepsilon_\gamma$ (as $\lim_{(D^\varepsilon_\beta)_{\beta<\gamma}} \bar p \restriction \gamma = \lim_{(D^\varepsilon_\beta)_{\beta<\alpha}} \bar p$, since the heart lies below γ). As A' is a P_γ-name, we can find $q' \leq q$ in P_γ and $\ell \in \omega$ such that $q' \Vdash \ell \in A'$. We now find $q'' \leq q'$ in P_α by defining $q''(\beta)$ for each element β of the finite set $\bigcup_{i<n} \mathrm{supp}(p^i_\ell) \setminus \gamma$: For such β in S^0, the guardrail gives a specific value $h_\varepsilon(\beta) \in Q_\beta$, which we use for $q''(\beta)$ as well. For $\beta \in S^4$, all conditions $p^k_\ell(\beta)$ (where defined) have the same stem $h_\varepsilon(\beta)$; hence there is a common extension $q''(\beta)$.

Clearly q'' forces that ℓ is in the allegedly empty set, the desired contradiction.

(b) for $\alpha = \gamma + 1$ successor: Assume the nontrivial case, $\gamma \in S^4$: Write the Δ-system as $(p_i, q_i)_{i\in\omega}$ with $(p_i, q_i) \in P_\gamma * Q_\gamma$. As noted above, $(q_n)_{n\in\omega}$ is a ground-model-code-sequence, and by induction (d) holds for w_γ. So it is forced that the D^ε_γ-limit q^* of the q_n is in Q_γ. Again by induction, the limit p^* of the p_n exists as well; and (p^*, q^*) is the required limit.

(a+c) for $\alpha = \gamma + 1$ successor: We again have to show that P_α forces that the following is a filter base, for each $\varepsilon \in \chi$:

$$D^\varepsilon_\gamma \cup \{A_{\bar p} \,:\, \bar p \text{ is a countable } \Delta\text{-system following } h^*_\varepsilon \text{ and } \lim_{(D^\varepsilon_\beta)_{\beta<\alpha}} \bar p \in G_\alpha\}.$$

As above, assume that q forces $A \cap A_{\bar p^0} \cap \cdots \cap A_{\bar p^{n-1}} = \emptyset$.

We can assume that $q \restriction \gamma$ forces that $q(\gamma)$ is stronger than the limit of all $\bar p^i(\gamma)$ (for $i < n$). Thus, by Fact 1.24, each $B_i \coloneqq \{\ell \in \omega \,:\, q(\gamma) \text{ compatible with } p^i_\ell(\gamma)\}$ is forced to be in D^ε_γ.

By induction, $q \restriction \gamma$ forces $A' \coloneqq A \cap \bigcap_{i<n} A_{\bar p^i \restriction \gamma} \in D^\varepsilon_\gamma$, and so that $B' = A' \cap \bigcap_{i<n} B_i$ is in the ultrafilter and in particular nonempty. Work in the P_γ-extension by some generic filter containing $q \restriction \gamma$. Fix some $\ell \in B'$. By the definition of B_i, $q(\gamma)$ is compatible with each $p^i_\ell(\gamma)$ for $i < n$. According to Fact 1.22 there is a common lower bound q''.

$q \restriction \gamma \Vdash_{P_\gamma} q'' \Vdash_{Q_\gamma} \ell \in A_{\bar p^i}$. I.e., $q \restriction \gamma * q'' \leq q$ forces that ℓ is an element of the allegedly empty set.

(d) For any $w \subseteq \alpha$, let Q^w be the (P_α-name for) the partial eventually different forcing defined using w. Start with some $w^0 \subseteq \alpha$ of size $<\lambda_4$. There are $|w^0|^{\aleph_0}$ many ground-model sequences in Q^{w^0}. For any ε and any such sequence, the D^ε_α-limit is a real; so we can extend w^0 by a countable set to some w' such that $Q^{w'}$ contains the limit. We can do that for all $\varepsilon \in \chi$ and all sequences, resulting in some $w^1 \supseteq w^0$ still of size $<\lambda_4$. We iterate this construction and get w^i for $i \leq \omega_1$, taking the unions at limits. Then $w_\alpha \coloneqq w^{\omega_1}$ is as required, as $Q_\alpha \coloneqq Q^{w_\alpha} = \bigcup_{i<\omega_1} Q^{w_i}$.

So this proof actually shows that the set of w_α with the desired property is an ω_1-club. □

After carrying out the construction of this lemma, we get a forcing notion $\mathbb{P}^5$ satisfying the following:

Lemma 1.30. $\mathsf{LCU}_3(\mathbb{P}^5, \kappa)$ *for $\kappa \in [\lambda_3, \lambda_5]$, witnessed by the sequence $(c_\alpha)_{\alpha<\kappa}$ of the first κ many Cohen reals.*

Proof. So we want to show that for every $\mathbb{P}^5$-name y $(\exists\alpha\in\kappa)\,(\forall\beta\in\kappa\setminus\alpha)\,\mathbb{P}^5\Vdash\neg c_\beta\leq^* y)$.

Assume that p^* forces that there are unboundedly many $\alpha\in\kappa$ with $c_\alpha\leq^* y$, and enumerate them as $(\alpha_i)_{i\in\kappa}$. Pick $p^i\leq p^*$ deciding α_i to be some β_i, and also deciding n_i such that $(\forall m\geq n_i)\,c_{\alpha_i}(m)\leq y(m)$. We can assume that $\beta_i\in\mathrm{dom}(p_i)$. Note that β_i is a Cohen position (as $\beta_i<\kappa\leq\lambda_5$), and we can assume that $p_i(\beta_i)$ is a Cohen condition in V (and not just a P_{β_i}-name for such a condition). By thinning out, we may assume:

- All n_i are equal to some n^*.
- $(p_i)_{i\in\kappa}$ forms a Δ system with root ∇.
- $\beta_i\notin\nabla$, hence all β_i are distinct.
  (For any $\beta\in\kappa$, at most $|\beta|$ many p_i can force $\alpha_i=\beta$, as p_i forces that $\alpha_i\geq i$ for all i.)
- $p_i(\beta_i)$ is always the same Cohen condition s, without loss of generality of length $n^{**}\geq n^*$.
  (Otherwise extend s.)

Pick the first ω many elements $(p_i)_{i\in\omega}$ of this Δ-system. Now extend each p_i to p'_i by extending the Cohen condition $p_i(\beta_i)=s$ to $s^\frown i$ (i.e., forcing $c_{\alpha_i}(n^{**})=i$). Note that $(p'_i)_{i\in\omega}$ is still a countable Δ-system, following some new countable guardrail and therefore some full guardrail $h^*_\varepsilon\in H^*$.

Accordingly, the limit $\lim_{(D^\varepsilon_\alpha)_{\alpha\in\delta_5}}\bar{p}'$ forces that infinitely many of the p'_i are in the generic filter. But each such p'_i forces that $c_{\alpha_i}(n^{**})=i\leq y(n^{**})$, a contradiction. □

1.4. **Recovering GCH.** For the rest of the paper we will assume the following for the ground model V (in addition to Assumption 1.11):

Assumption 1.31. *GCH holds.*

(Note that this is incompatible with Assumption 1.19.)

Recall that all parameters used to define $\mathbb{P}^5$ are fixed, apart from $\bar{w}=(w_\alpha)_{\alpha\in S^4}$.

Lemma 1.32. *We can choose $\bar{w}$ such that* $\mathsf{LCU}(\mathbb{P}^5,\kappa)$ *holds for all $\kappa\in[\lambda_3,\lambda_5]$.*

Proof. Let $\mathbb{R}$ be a $<\chi$-closed χ^+-cc p.o. that forces $2^\chi=\lambda_5$.

In the $\mathbb{R}$-extension V^*, Assumption 1.19 holds; and Assumption 1.11 still holds for the fixed parameters.[11]

So in V^*, we can perform the inductive Construction 1.29, where now "ground model" refers to V^*, not V (e.g., when we talk about determined positions, or ground-model-code-sequences, etc). Actually, we can construct in V the following, by induction on $\alpha\in\delta_5$, and starting with some cofinal $\bar{w}^{*,0}=(w^{*,0}_\alpha)_{\alpha\in S^4}$:

- An $\mathbb{R}$-name $(D^\varepsilon_\alpha)_{\varepsilon\in\chi}$ (forced to be constructed) according to 1.29(a,b).
- If $\alpha\in S^4$, some $w_\alpha\supseteq w^*_\alpha$ in V such that R forces w_α satisfies (c).
  (We can do this, as any $\mathbb{R}$-name for an ω_1-clubset of $[\alpha]^{<\lambda_4}$ contains a ground model element.)

So we get in V a cofinal parameter $\bar{w}$ satisfying the following: In the $\mathbb{R}$-extension V^*, the same parameters define a forcing (call it $\mathbb{P}^{*,5}$) satisfying $\mathsf{LCU}_3(\mathbb{P}^{*,5},\kappa)$ in V^*.

$\mathbb{P}^{*,5}$ is basically the same as $\mathbb{P}^5$. More formally:

[11] In particular, $(w_\alpha)_{\alpha\in S^i}$ is still cofinal in $[\delta_5]^{<\lambda_i}$: For $i=1,2$, the forcing $\mathbb{R}$ doesn't add any new elements of $[\delta_5]^{<\lambda_i}$ as $\mathbb{R}$ is λ-closed; for $i=3$ any new subset of δ_5 of size $\theta<\lambda_3$ is contained in a ground model set of size $\leq\theta\times\chi<\lambda_3$, as $\mathbb{R}$ is λ^+-cc.

> In the $\mathbb{R}$-extension V^*, $\mathbb{P}^5 = (P_\alpha, Q_\alpha)_{\alpha<\delta_5}$ (the iteration defined in and element of V) is canonically densely embedded into $\mathbb{P}^{*,5} = (P^*_\alpha, Q^*_\alpha)$ (the iteration defined in V^* with the same parameters).

Proof: By induction, we show (in the $\mathbb{R}$-extension) that P^*_α forces that Q^*_α (evaluated by the P^*_α-generic) is equal to Q_α (evaluated by the induced P_α-generic, as per induction hypothesis): Every element of Q^*_α is a Borel function (which already exists in V) applied to the generics at a countable sequence of indices in w_α (which also already exists in V).

This implies:

> In V, $\mathsf{LCU}_3(\mathbb{P}^5, \kappa)$ holds for all $\kappa \in [\lambda_3, \lambda_5]$, witnessed by the first κ many Cohen reals.

Proof: Let y be a $\mathbb{P}^5$-name of a real. In V^*, we can interpret y as $\mathbb{P}^{*,5}$-name, and as $\mathsf{LCU}_3(\mathbb{P}^{*,5}, \kappa)$ holds, we get $(\exists \alpha \in \kappa)\,(\forall \beta \in \kappa \setminus \alpha)\mathbb{P}^{*,5} \Vdash c_\beta \not\leq^* y$, where c_β is the Cohen added at β. As $\chi < \kappa$, there is in V an upper bound $\alpha^* < \kappa$ for the possible values of α. For any $\beta \in \kappa \setminus \alpha^*$, we have (in V) $\mathbb{P}^5 \Vdash x_\beta \not\leq^* y$ (by absoluteness). □

To summarize:

Theorem 1.33. *Assuming GCH and given λ_i as in Assumption 1.11, we can find parameters*[12] *such that the FS ccc iteration $\mathbb{P}^5$ as defined in 1.12 satisfies, for $i = 1, 2, 3, 4$:*

- $\mathsf{LCU}_i(\mathbb{P}^5, \kappa)$ *holds for any regular cardinal κ in $[\lambda_i, \lambda_5]$.*
- $\mathsf{COB}_i(\mathbb{P}^5, \lambda_i, \lambda_5)$ *holds.*

So in particular $\mathbb{P}^5$ forces $\mathrm{add}(\mathcal{N}) = \lambda_1$, $\mathrm{cov}(\mathcal{N}) = \lambda_2$, $\mathfrak{b} = \lambda_3$, $\mathrm{non}(\mathcal{M}) = \lambda_4$ *and* $\mathrm{cov}(\mathcal{M}) = \mathfrak{d} = \mathrm{non}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \lambda_5 = 2^{\aleph_0}$.

For the rest of the paper we fix these parameters and thus the forcing $\mathbb{P}^5$.

2. The Boolean ultrapower of the forcing

Sections 2.1–2.3 largely follow the presentation of [KTT, Sec. 2], where some additional details and some of the straightforward proofs are given.

2.1. **Boolean ultrapowers.**

Boolean ultrapowers generalize regular ultrapowers by using arbitrary Boolean algebras instead of the power set algebra.

We assume that κ is strongly compact and that $\boldsymbol{B}$ is a κ-distributive, κ^+-cc, atomless complete Boolean algebra. Then every κ-complete filter on $\boldsymbol{B}$ can be extended to a κ-complete ultrafilter $\boldsymbol{U}$.[13] Also, there is a maximal antichain A_0 in $\boldsymbol{B}$ of size κ such that $A_0 \cap \boldsymbol{U} = \emptyset$ (i.e., $\boldsymbol{U}$ is not κ^+-complete).[14]

The Boolean algebra $\boldsymbol{B}$ can be used as forcing notion. As usual, V (or: the ground model) denotes the universe we "start with". In the following, we will not actually force with $\boldsymbol{B}$ (or any other partial order), we always remain in V; but we still use forcing notation. In particular, we call the usual $\boldsymbol{B}$-names "forcing names".

A **BUP-name** (or: labeled antichain) x is a function $A \to V$ whose domain is a maximal antichain of $\boldsymbol{B}$. We may write $A(x)$ to denote A.

---

[12] I.e., we set $\delta_5 = \lambda_5 + \lambda_5$, and find $(S^i)_{i=1,\dots,4}$ and $\bar{w} = (w_\alpha)_{\alpha\in\delta_5}$.

[13] For this, neither κ^+-cc nor atomless is required, and κ-complete is sufficient. The proof is straightforward, the first proof that we are aware of has been published in [KT64].

[14] Proof: Let A_0 be a maximal antichain in the open dense set $\boldsymbol{B} \setminus \boldsymbol{U}$, by κ^+-cc $|A| \leq \kappa$. And A cannot have size $<\kappa$, as otherwise it would meet the κ-complete $\boldsymbol{U}$.

Each BUP-name corresponds to a forcing-name[15] for an element of V. We will identify the BUP-name and the corresponding forcing-name. In turn, every forcing name τ for an element of V has a forcing-equivalent BUP-name.

In particular, we can calculate, for two BUP-names x and y, the Boolean value $[\![x = y]\!]$. We call x and y **equivalent**, if $[\![x = y]\!] \in U$.

In particular there is a unique (up to equivalence) **standard BUP-name** $\check{v}$ for each $v \in V$.

The **Boolean ultrapower** M^- consists of the equivalence classes $[x]$ of BUP-names x; and we define $[x] \in^- [y]$ by $[\![x \in y]\!] \in U$. We are interested in the $\in$-structure $(M^-, \in^-)$. We let $j^- : V \to M^-$ map v to $[\check{v}]$.

Given BUP-names $x_1, \dots, x_n$ and an $\in$-formula φ, the truth value $[\![\varphi^V(x_1, \dots, x_n)]\!]$ is well defined (it is the weakest element of B forcing that in the ground model $\varphi(x_1, \dots, x_n)$ holds, which makes sense as $x_1, \dots, x_n$ are guaranteed to be in the ground model).

A straightforward induction shows:

- Łoś's theorem: $(M^-, \in^-) \Vdash \varphi([x_1], \dots, [x_n])$ iff $[\![\varphi^V(x_1, \dots, x_n)]\!] \in U$.
- $j^- : (V, \in) \to (M^-, \in^-)$ is an elementary embedding.
- In particular, $(M^-, \in^-)$ is a ZFC model.

As U is σ-complete, $(M^-, \in^-)$ is wellfounded. So we let M be the transitive collapse of $(M^-, \in^-)$, and let $j : V \to M$ be the composition of j^- with the collapse. We denote the collapse of $[x]$ by x^U. So in particular $\check{v}^U = j(v)$.

Facts 2.1.
- *$M \vDash \varphi(x_1^U, \dots, x_n^U)$ iff $[\![\varphi^V(x_1, \dots, x_n)]\!] \in U$. In particular, $j : V \to M$ is an elementary embedding.*
- *If $|Y| < \kappa$, then $j(Y) = j''Y$. In particular, j restricted to κ is the identity. M is closed under $<\kappa$-sequences.*
- *$j(\kappa) \neq \kappa$, i.e., $\kappa = \mathrm{cr}(j)$.*

As we have already mentioned, an arbitrary forcing-name for an element of V has a forcing-equivalent BUP-name, i.e., a maximal antichain labeled with elements of V. If τ is a forcing-name for an element of Y ($Y \in V$), then without loss of generality τ corresponds to a maximal antichain labeled with elements of Y. We call such an object y a "BUP-name for an element of $j(Y)$" (and not "for an element of Y", for the obvious reason: unlike in the case of a forcing extension, y^U is generally not in Y, but, by definition of $\in^-$, it is in $j(Y)$).

2.2. The algebra and the filter.

We will now specify the concrete Boolean algebra we are going to use:

Lemma 2.2. *Assume GCH. Let κ be strongly compact and $\theta > \kappa$ regular. Then there is a κ^+-cc, κ-complete Boolean algebra B and a κ-complete ultrafilter U on B such that:*

(a) The Boolean ultrapower gives an elementary embedding $j : V \to M$. M is closed under $<\kappa$-sequences.

(b) The elements x^U of M are exactly (the collapses of equivalence classes of) B-names x for elements of V; more concretely, a function from an antichain (of size κ) to V. We sometimes say "x^U is a mixture of κ many possibilities".

Similarly, for $Y \in V$, the elements x^U of $j(Y)$ correspond to the B-names x of elements of Y, i.e., antichains labeled with elements of Y.

(c) If $|A| < \kappa$, then $j''A = j(A)$. In particular, j restricted to κ is the identity.

(d) j has critical point κ, $\mathrm{cf}(j(\kappa)) = \theta$, and $\theta \leq j(\kappa) \leq \theta^+$.

[15] more specifically, to the forcing-name $\{(\widetilde{x(a)}, a) : a \in A(x)\}$.

(e) If $\lambda > \kappa$ is regular, then $\max(\theta, \lambda) \leq j(\lambda) < \max(\theta, \lambda)^+$.
(f) If S is a $<\lambda$-directed partial order, and $\kappa < \lambda$, then $j''(S)$ is cofinal in $j(S)$.
(g) If $\mathrm{cf}(\alpha) \neq \kappa$ *, then $j''\alpha$ is cofinal in $j(\alpha)$, so in particular* $\mathrm{cf}(j(\alpha)) = \mathrm{cf}(\alpha)$*).*

Proof. Let B be the complete Boolean algebra generated by the forcing notion $P_{\kappa,\theta}$ consisting of partial functions from θ to κ with domain of size $<\kappa$, ordered by extension. Clearly B is κ-distributive (it is even κ-complete) and κ^+-cc.

We have already seen (a)–(c).

(d): The forcings adds a canonical generic function $f^* : \theta \to \kappa$. So for each $\delta \in \theta$, $f^*(\delta)$ is a forcing-name for an element of κ, and thus a BUP-name for an element of $j(\kappa)$.

Let x be some other BUP-name for an element of $j(\kappa)$, i.e., an antichain A of size κ labeled with elements of κ. Let $\delta \in \theta$ be bigger than the supremum of $\mathrm{supp}(a)$ for each $a \in A$. We call such a pair (x, δ) "suitable", and set $b_{x,\delta} \coloneqq [\![f^*(\delta) > x]\!]$. We claim that all these elements form a basis for a κ-complete filter. To see this, fix suitable pairs (x_i, δ_i) for $i < \mu < \kappa$; we have to show that $\bigwedge_{i\in\mu} b_{x_i,\delta_i} \neq \mathbb{0}$. Enumerate $\{\delta_i : i \in \mu\}$ increasing (and without repetitions) as δ^j for $j \in \gamma \leq \mu$. Set $A_j = \{i : \delta_i = \delta^j\}$. Given q_j, define $q_{j+1} \in P_{\kappa,\theta}$ as follows: $q_{j+1} \leq q_j$; $\delta^j \in \mathrm{supp}(q_{j+1}) \subseteq \delta^j \cup \{\delta^j\}$; and $q_{j+1} \restriction \delta^j$ decides for all $i \in A_j$ the values of x_i to be some α_i; and $q_{j+1}(\delta^j) = \sup_{i\in A}(\alpha_i) + 1$. This ensures that q_{j+1} is stronger than b_{x_i,δ_i} for $i \in A_j$. For $j \leq \gamma$ limit, let q_j be the union of $\{q_k : k < j\}$. Then q_γ is stronger than each b_{x_i,δ_i}.

As κ is strongly compact, we can extend the κ-complete filter generated by all b_{x_i,δ_i} to a κ-complete ultrafilter U. Then the sequence $f^*(\delta)^U_{\delta\in\theta}$ is strictly increasing (as $(f^*(\delta), \delta')$ is suitable for all $\delta < \delta'$) and cofinal in $j(\kappa)$ (as we have just seen); so $\mathrm{cf}(j(\kappa)) = \theta$. $|j(\kappa)| < \theta^+$ follows from the next item.

(e): To get an upper bound for $j(\mu)$ for any cardinal μ, we count all possible BUP-names for elements of $j(\mu)$. As we can assume that the antichains are subset of $P_{\kappa,\theta}$, which has size θ, we get the upper bound $|j(\mu)| \leq [\theta]^\kappa \times \mu^\kappa = \max(\theta, \mu^\kappa)$. For $\mu = \kappa$, this gives us $|j(\kappa)| \leq \theta$; for $\mu > \kappa$ regular we get $|j(\mu)| \leq \max(\theta, \mu)$.

(f): An element x^U of $j(S)$ is a mixture of κ many possibilities in S. As $\kappa < \lambda$, there is some $t \in S$ above all the possibilities. Then $j(t) > x^U$.

(g): Set $\mu = \mathrm{cf}(\alpha)$, and pick an increasing cofinal sequence $\bar\beta = (\beta_i)_{i\in\mu}$ in α. $j(\bar\beta)$ is increasing cofinal in $j(\alpha)$ (as this is absolute between M and V). If $\mu < \kappa$, then $j''\bar\beta = j(\beta)$, otherwise use (f). □

2.3. **The ultrapower of a forcing notion.** We now investigate the relation of a forcing notion $P \in V$ and its image $j(P) \in M$, which we use as forcing notion over V. (Think of P as being one of the forcings of Section 1; it has no relation with the boolean algebra B.)

Note that as $j(P) \in M$ and M is transitive, every $j(P)$-generic filter G over V is trivially generic over M as well, and we will use absoluteness between $M[G]$ and $V[G]$ to prove various properties of $j(P)$.

Lemma 2.3. *If $P = (P_\alpha, Q_\alpha)_{\alpha<\delta}$ is a finite support (FS) ccc iteration of length δ, then $j(P)$ is a FS ccc iteration of length $j(\delta)$ (more formally: it is canonically equivalent to one).*

Proof. M certainly thinks that $j(P) = (P^*_\alpha, Q^*_\alpha)_{\alpha<j(\delta)}$ is a FS iteration of length $j(\delta)$.

By induction on α we define the FS ccc iteration $(\tilde P_\alpha, \tilde Q_\alpha)_{\alpha<j(\delta)}$ and show that P^*_α is a dense subforcing of $\tilde P_\alpha$: Assume this is already the case for P^*_α. M thinks that Q^*_α is a P^*_α-name, so we can interpret it as $\tilde P_\alpha$-name and use it as $\tilde Q_\alpha$. Assume that (p, q) is an element (in V) of $\tilde P_\alpha * \tilde Q_\alpha$. So p forces that q is a name in M; we can strengthen p to some p' that decides q to be the name $q' \in M$. By induction we can further strengthen p' to $p'' \in P^*_\alpha$,

then $(p'', q') \in P^*_{\alpha+1}$ is stronger than (p, q). (At limits there is nothing to do, as we use FS iterations.)

$j(P)$ is ccc, as any $A \subseteq j(P)$ of size $\aleph_1$ is in M (and M thinks that $j(P)$ is ccc). □

Similarly, we get:

- If $\tau = x^U$ is in M a $j(P)$-name for an element of $j(Z)$, then τ is a mixture of κ many P-names for an element of Z (i.e., the BUP-name x consists of an antichain $A \subseteq B$ labeled, without loss of generality, with P-names for elements of Z).
  (This is just the instance of "each $x^U \in j(Y)$ is a mixture of elements of Y", where we set Y to be the set[16] of P-names for elements of Z.)
- A $j(P)$-name τ for an element of $M[G]$ has an equivalent $j(P)$-name in M.
  (There is a maximal antichain A of $j(P)$ labeled with $j(P)$-names in M. As M is countably closed, this labeled antichain is in M, and gives a $j(P)$-name in M equivalent to τ.)
- In $V[G]$, $M[G]$ is closed under $<\kappa$ sequences.
  (We can assume the names to be in M and use $<\kappa$-closure.)
- In particular, every $j(P)$-name for a real, a Borel-code, a countable sequence of reals, etc., is in M (more formally: has an equivalent name in M).
- If each iterand is forced to consist of reals, then P forces the continuum to have size at most $|\delta + 2|^{\aleph_0}$ and $j(P)$ forces the continuum to have size at most $|j(\delta) + 2|^{\aleph_0}$.
  (This is satisfied by any FS ccc iteration.)

2.4. **Preservation of values of characteristics.** Recall Definition 1.8 of LCU_i.

Lemma 2.4. $\mathsf{LCU}_i(P, \delta)$ *implies* $\mathsf{LCU}_i(j(P), j(\delta))$. *If $\lambda \neq \kappa$ regular, then* $\mathsf{LCU}_i(P, \lambda)$ *implies* $\mathsf{LCU}_i(j(P), \lambda)$.

Proof. Let $\bar{y} = (y_\alpha)_{\alpha<\delta}$ be the sequence of P-names witnessing $\mathsf{LCU}_i(P, \delta)$. So M thinks: For every $j(P)$-name r of a real $(\exists \alpha \in j(\delta))\,(\forall \beta \in j(\delta) \setminus \alpha)\,\neg(j(\bar{y}))_\beta \,\mathrm{R}_i\, r$. This is absolute, so $j(\bar{y})$ witnesses $\mathsf{LCU}_i(j(P), j(\delta))$.

The second claim follows from the fact that $\mathsf{LCU}_i(j(P), j(\delta))$ is equivalent to $\mathsf{LCU}_i(j(P), \mathrm{cf}(j(\delta)))$ and that $\mathrm{cf}(j(\lambda)) = \lambda$ for regular $\lambda \neq \kappa$. □

Recall Definitions 1.14 and 1.16 of COB_i.

Lemma 2.5. *Assume* $\mathsf{COB}_i(P, \lambda, \mu)$. *If $\kappa > \lambda$, then* $\mathsf{COB}_i(j(P), \lambda, |j(\mu)|)$*; if $\kappa < \lambda$, then* $\mathsf{COB}_i(j(P), \lambda, \mu)$.

Proof. Let $(S, \prec)$ and $\bar{g}$ witness $\mathsf{COB}_i(P, \lambda, \mu)$. M thinks that

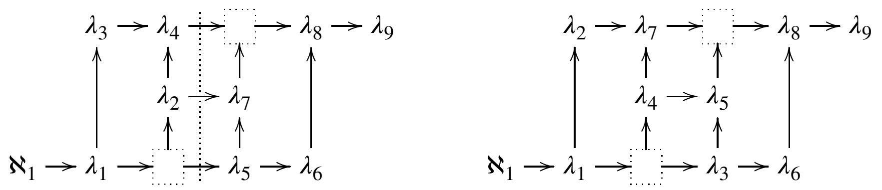
$(*)$ for each $j(P)$-name f $(\exists s \in j(S))\,(\forall t \in j(S))\,(\, t \succ s \rightarrow j(P) \Vdash f \,\mathrm{R}_i\, g_t\,)$

(or, in the case $i = 2$, $j(P) \Vdash g_t \notin f$, where f is the name of a null-set). This is absolute.

If $\kappa > \lambda$, then $j(\lambda) = \lambda$, and $j(S)$ is λ-directed in M and therefore in V as well, so we get $\mathsf{COB}_i(j(P), \lambda, |j(\mu)|)$.

So assume $\kappa < \lambda$. We claim that $j''(S)$ and $j''\bar{g}$ witness $\mathsf{COB}_i(j(P), \lambda, \mu)$. $j''S$ is isomorphic to S, so directedness is trivial. Given a $j(P)$-name f, without loss of generality in M, there is in M a cone with tip $s \in j(S)$ as in $(*)$. As $j''S$ is cofinal in $j(S)$ (according to Lemma 2.2(f)), there is some $s' \in S$ such that $j(s') \succ s$. Then for all $t' \succ s'$, i.e., $j(t) \succ j(s')$, we get $j(P) \Vdash f \,\mathrm{R}_i\, j(g_t)$. (Or, in case $i = 2$, $j(P) \Vdash j(g_t) \notin f$). □

[16]formally: some set containing representatives of each equivalence class of the class Y



(A) An ordering compatible with FS ccc. (B) Another one, incompatible with FS ccc.

FIGURE 1. Alternative orderings of the invariants.

2.5. **The main theorem.** We now have all everything required for the main result:

Theorem 2.6. *Assume GCH and that* $\aleph_1 < \kappa_9 < \lambda_1 < \kappa_8 < \lambda_2 < \kappa_7 < \lambda_3 < \kappa_6 < \lambda_4 < \lambda_5 < \lambda_6 < \lambda_7 < \lambda_8 < \lambda_9$ *are regular,* λ_3 *a successor of a regular, and* κ_i *strongly compact for* $i = 6, 7, 8, 9$. *Then there is a ccc forcing notion* $\mathbb{P}^9$ *resulting in:*

$$\text{add}(\mathcal{N}) = \lambda_1 < \text{cov}(\mathcal{N}) = \lambda_2 < \mathfrak{b} = \lambda_3 < \text{non}(\mathcal{M}) = \lambda_4 <$$
$$< \text{cov}(\mathcal{M}) = \lambda_5 < \mathfrak{d} = \lambda_6 < \text{non}(\mathcal{N}) = \lambda_7 < \text{cof}(\mathcal{N}) = \lambda_8 < 2^{\aleph_0} = \lambda_9.$$

Proof. Let $j_i : V \to M_i$ be the Boolean ultrapower embedding with $\text{cr}(j_i) = \kappa_i$ and $\text{cf}(j_i(\kappa_i)) = \lambda_i$ (for $i = 6, \dots, 9$). We use $\mathbb{P}^5$ of Theorem 1.33, and set $\mathbb{P}^{i+1} \coloneqq j_i(\mathbb{P}^i)$ and $\delta_{i+1} \coloneqq j_i(\delta_i)$ for $i = 5, 6, 7, 8$.

By Lemmas 1.9 and 1.17 it is enough to show the following, where for $i = 1, \dots, 4$ we set $i^* \coloneqq 9 - i$ (which corresponds to the dual characteristic):

- $\mathbb{P}^9$ is a FS ccc iteration of length δ_9 and forces $2^{\aleph_0} = \lambda_9$.
  This was shown in Section 2.3.
- $\mathsf{LCU}_i(\mathbb{P}^9, \lambda_i)$ holds for $i = 1, \dots, 4$; as well as $\mathsf{LCU}_4(\mathbb{P}^9, \lambda_5)$.
  The statements hold for $\mathbb{P}^5$ by Theorem 1.33 and are preserved by Lemma 2.4.
- $\mathsf{LCU}_i(\mathbb{P}^9, \lambda_{i^*})$ holds for $i = 1, 2, 3$.
  Note that $\kappa_{i^*+1} < \lambda_i < \kappa_{i*} < \lambda_5$. So $\mathsf{LCU}_i(\mathbb{P}^5, \kappa_{i^*})$ holds; which implies $\mathsf{LCU}_i(\mathbb{P}^j, \kappa_{i^*})$ for $j = 5, \dots, i^* - 1$, and then $\mathsf{LCU}_i(\mathbb{P}^j, \lambda_{i^*})$ for $j = i^*, \dots, 9$.
- $\mathsf{COB}_i(\mathbb{P}^9, \lambda_i, \lambda_{i^*})$ holds for $i = 1, 2, 3, 4$.
  $\mathsf{COB}_i(\mathbb{P}^5, \lambda_i, \lambda_5)$ implies $\mathsf{COB}_i(\mathbb{P}^j, \lambda_i, \lambda_j)$ for $j = 5, \dots, i^*$ (while $\kappa_j > \lambda_i$), and then $\mathsf{COB}_i(\mathbb{P}^j, \lambda_i, \lambda_{i^*})$ for $j = i^* + 1, \dots, 9$. □

3. QUESTIONS

The result poses some obvious questions:

(a) Can we prove the result without using large cardinals?
It would be quite surprising if compact cardinals are needed, but a proof without them will probably be a lot more complicated.

(b) Does the result still hold for other specific values of λ_i, such as $\lambda_i = \aleph_{i+1}$?
In our construction, the regular cardinals λ_i for $i = 4, \dots, 9$ can be chosen quite arbitrarily (above the compact κ_6, that is). However, λ_1, λ_2 and λ_3 each have to be separated by a compact cardinal (and furthermore λ_3 has to be a successor of a regular cardinal).

(c) Are other linear orders between the characteristics of Cichoń's diagram consistent?
Note that in this paper, we use a FS ccc iteration of (uncountable) length δ, which always results in $\text{non}(\mathcal{M}) \leq \delta \leq \text{cov}(\mathcal{M})$. Under these restrictions, there are only four

possible assignments. After the initial submission of this paper, another one of these was shown to be consistent [KST], the one of Figure 1.A: We construct a different initial forcing that gives us the modified ordering of the left hand side; then the same construction and proof as in this paper gives us the whole diagram. It seems possible that the remaining two variants can be achieved by very similar methods.

Of course there are a lot more[17] possibilities to assign $\lambda_1, \dots, \lambda_8$ to Cichoń's diagram in a way that satisfies the known ZFC-provable (in)equalities. Figure 1.B is an example. Such orders require entirely different methods. (Even to get just the five different values $\aleph_1 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 < \lambda_6 < \lambda_7 < \lambda_8 < \lambda_9$ in this figure turnes out to be rather involved [FGKS17].)

(d) Is it consistent that other cardinal characteristics that have been studied,[18] in addition to the ones in Cichoń's diagram, have pairwise different values as well?

For some characteristics this can be achieved (work in progress) with an extension of the methods of this paper.

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