

A NEW LOOK AT INTERPRETABILITY AND SATURATION

M. MALLIARIS AND S. SHELAH

ABSTRACT. We investigate the interpretability ordering \leq^* using generalized Ehrenfeucht-Mostowski models. This gives a new approach to proving inequalities and investigating the structure of types.

$T_0 \leq^*_\kappa T_1$ in the interpretability order if, for sufficiently large regular λ , there is some T_* which interprets both theories and which has the property that for any κ -saturated model $M_* \models T_*$, if the reduct of M_* to $\tau(T_1)$ is λ -saturated, then so is the reduct to $\tau(T_0)$. It was introduced in the mid-90s as a potential help to the study of Keisler's order \leq , which is defined via saturation of regular ultrapowers.

Encouraged by our recent characterization of the maximal class in \leq^* under instances of GCH [13] building on [2] and [21], here we look further at \leq^* . We prove a series of results about its structure, focusing on results which may give us insight into the structure of unstable theories, especially simple unstable theories. We prove the theory of the random graph is minimum among the unstable theories, and prove T_{feq} is minimum among the non-simple theories. We prove directly, i.e. without appealing to Keisler's order, that the theory of the random graph is not maximal. Finally, we prove directly that for any simple theory T_0 and any non-simple theory T_1 , T_1 is not below T_0 . To quote Keisler's order for this result would require assuming existence of a supercompact cardinal, so here both the proof and the theorem are new. (As indicated, \leq^* is often given with cardinal subscripts: as we'll explain in §1, here our main focus is $\kappa = 1$, i.e., $\leq^*_{\lambda, \aleph_0, 1}$.)

The proofs of the two separation results depend on sharpening the tool of Ehrenfeucht-Mostowski models so as to allow for a certain relative measurement of types. We plan to study this further in a companion manuscript in preparation.

CONTENTS

1. What is the interpretability order \leq^* ?	1
2. What was known about \leq^* ?	3
3. GEM models and indiscernible sequences	7
4. T_{rg} is minimal among the unstable theories	10
5. T_{rg} is not maximal	11
6. T_{feq} is minimal among the non-simple theories	16
7. Non-simple theories are not below simple theories	18
8. Incomparability in \leq and \leq^*	24
9. Discussion: weak definability of types	27
10. Some open problems	28
References	29

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1. WHAT IS THE INTERPRETABILITY ORDER \leq^* ?

The interpretability order \leq^* was introduced in Shelah 1996 [18] as a weakening of Keisler's order \leq [6]. It was then studied in several subsequent papers, notably Džamonja and Shelah 2004 [2] and Shelah and Usvyatsov 2008 [21]. In this section we will define \leq^* , following [2], and in the next section we will record what was known. All theories are complete.

Definition 1.1. (Interpretations, c.f. [2] 1.1) *Let T_0 and T_* be complete first-order theories. Suppose that*

$$\bar{\varphi} = \langle \varphi_R(\bar{x}_R) : R \text{ a predicate or function symbol of } \tau(T_0), \text{ or } = \rangle$$

is such that each $\varphi_R(\bar{x}_r) \in \tau(T_*)$.

(a) For any model $M_* \models T_*$, we define the model $N = M_*^{\bar{\varphi}}$ as follows:

- N is a $\tau(T_0)$ -structure
- $\text{Dom}(N) = \{a : M_* \models \varphi_=(a, a)\} \subseteq M_*$
- for each predicate symbol R of $\tau(T_0)$, $R^N = \{\bar{a} : M_* \models \varphi_R[\bar{a}]\}$
- for each function symbol f of $\tau(T_0)$ and each $b \in N$, $N \models "f(\bar{a}) = b"$ iff $M_* \models \varphi_f(\bar{a}, b)$, and $M_* \models "\varphi_f(\bar{a}, b) \wedge \varphi_f(\bar{a}, c) \implies b = c"$.¹

(b) We call $\bar{\varphi}$ an interpretation of T_0 in T_* if:

- each $\varphi_R(\bar{x}_r) \in \tau(T_*)$
- for any model $M_* \models T_*$, we have that $M_*^{\bar{\varphi}} \models T_0$

(c) " T_* interprets T_0 " means: there exists $\bar{\varphi}$ which is an interpretation of T_0 in T_* .

In the definition of \leq^* , note there are naturally three parameters: the amount of saturation to be transferred, the size of the interpreting theory T_* , and a base level of saturation required for models of T_* before we ask about transfer of saturation. These are denoted by λ, χ, κ respectively.

Definition 1.2. (The interpretability order \leq^* , c.f [2] 1.2 and [18] 2.10)

Let T_0 and T_1 be complete first-order theories (in this paper they will be countable but this isn't necessary).

- (1) λ be an infinite regular cardinal, κ an infinite cardinal or 1, and $\lambda \geq \kappa$.
 $T_0 \leq_{\lambda, \chi, \kappa}^* T_1$ means there is T_* of cardinality $\leq |T_0| + |T_1| + \chi$ such that:²
 - (a) T_* interprets T_0 via $\bar{\varphi}_0$, and T_* interprets T_1 via $\bar{\varphi}_1$, and without loss of generality the signatures of the two interpretations are disjoint.
 - (b) For every κ -saturated model $M_* \models T_*$, if $M_*^{\bar{\varphi}_1}$ is λ -saturated, then $M_*^{\bar{\varphi}_0}$ is λ -saturated.
- (2) if $\kappa = 1$, "for every κ -saturated model" should be read as "for every model."
- (3) Omitting χ means $\chi = |T_0| + |T_1|$.
- (4) Omitting λ means "for all large enough regular λ ."
- (5) Omitting κ means $\kappa = (|T_0| + |T_1|)^+$ (but we rarely omit κ).
- (6) In this paper, **if there is one cardinal in the subscript**, it is κ : we will consider two main cases,

$$\leq_1^* \text{ and } \leq_{\aleph_1}^* .$$

¹This clause allows us to restrict to vocabularies with only relation symbols.

²i.e. when all three cardinals are displayed, λ is the amount of saturation transferred, χ bounds the size of the interpreting theory T_* , and κ is the base level of saturation we require in any relevant model M_* of T_* .

Regarding the subscripts in Definition 1.2, we will focus here on countable theories, and our investigations here show that the two cases $\kappa = 1$ and $\kappa = \aleph_1$ are interesting for different reasons; these might be called the superstable and stable case, respectively. The case $\kappa = \aleph_1$ retains a stronger analogy to regular ultrapowers, whereas the case $\kappa = 1$ allows for the introduction of powerful techniques from EM models, and will be our focus here. However, future investigations may illuminate other aspects, and so to allow for easy quotation, we have written κ throughout the paper. We will use freely:

Observation 1.3. *If $T_0 \trianglelefteq_{\lambda, \chi, \kappa}^* T_1$ and $\chi' \geq \chi$, $\kappa' \geq \kappa$ then $T_0 \trianglelefteq_{\lambda, \chi', \kappa'}^* T_1$.*

Corollary 1.4. *In particular, if $T_0 \trianglelefteq_1^* T_1$, then $T_0 \trianglelefteq_{\aleph_1}^* T_1$, and if $\neg(T_0 \trianglelefteq_{\aleph_1}^* T_1)$, then $\neg(T_0 \trianglelefteq_1^* T_1)$.*

For easy reference, we include the following immediate translation of Definition 1.2.

Summary 1.5.

- (1) To show $T_0 \trianglelefteq_{\kappa}^* T_1$ means to show that for all large enough regular λ , there exists T_* of size $\leq |T_0| + |T_1|$ interpreting T_0 via some $\bar{\varphi}_0$ and T_1 via some $\bar{\varphi}_1$, such that for every $M_* \models T_*$ which is κ -saturated, if $M_*^{\bar{\varphi}_1}$ is λ -saturated then $M_*^{\bar{\varphi}_0}$ is λ -saturated.
- (2) To show $\neg(T_0 \trianglelefteq_{\kappa}^* T_1)$ means to show that for arbitrarily large regular λ , for every T_* (of size no more than $|T_0| + |T_1|$) interpreting T_0 via some $\bar{\varphi}_0$ and T_1 via some $\bar{\varphi}_1$, there exists some κ -saturated $M_* \models T_*$ such that $M_*^{\bar{\varphi}_1}$ is λ -saturated but $M_*^{\bar{\varphi}_0}$ is not λ -saturated. (And clearly it suffices to show that for arbitrarily large regular λ , for every T_* interpreting our two theories, there exists some extension $T_{**} \supseteq T_*$ of the same cardinality, e.g. with Skolem functions, and some κ -saturated $M_* \models T_{**}$ such that $M_*^{\bar{\varphi}_1}$ is λ -saturated but $M_*^{\bar{\varphi}_0}$ not λ -saturated.)
- (3) T_0 and T_1 are $\trianglelefteq_{\kappa}^*$ -equivalent when $T_0 \trianglelefteq_{\kappa}^* T_1$ and $T_1 \trianglelefteq_{\kappa}^* T_0$, and they are $\trianglelefteq_{\kappa}^*$ -incomparable when $\neg(T_0 \trianglelefteq_{\kappa}^* T_1)$ and $\neg(T_1 \trianglelefteq_{\kappa}^* T_0)$.
- (4) $T_0 \triangleleft^* T_1$ (i.e. strictly less than) when $T_0 \trianglelefteq^* T_1$ and $\neg(T_0 \trianglelefteq^* T_1)$.

Though in many cases this hypothesis won't be necessary, our focus here will be complete countable theories, because of the connection to Keisler's order, which is stated for such theories, and makes most sense for them, see [16].

Convention 1.6. *Unless otherwise stated, in this paper all theories are complete and countable, and $\chi = \aleph_0$, meaning any relevant T_* is countable.*

2. WHAT WAS KNOWN ABOUT \trianglelefteq^* ?

In this section we describe the state of knowledge on \trianglelefteq_1^* and $\trianglelefteq_{\aleph_1}^*$ as work on this paper began. Not all these results were previously known, e.g. as recently as [13] we didn't record the structure on the stable theories, Theorem 2.17, or that \trianglelefteq_1^* strictly refines \trianglelefteq , Conclusion 2.16. It may be most correct to say they could have been known: the results proved in this section may be deduced with a little thought from results in the literature.

The interpretability order \trianglelefteq^* refines Keisler's order \trianglelefteq in a natural sense as we now explain, but because the quantification over λ in the two orders is different (all sufficiently large vs. all), we will keep track of λ in the next few claims. Recall that

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1124 revision:2017-10-29

Keisler's order \leq on complete countable theories sets $T_0 \leq T_1$ if $T_0 \leq_\lambda T_1$ for every infinite cardinal λ , where $T_0 \leq_\lambda T_1$ means that for every regular ultrafilter \mathcal{D} on λ , every model $M_1 \models T_0$, and every model $M_2 \models T_1$, if $(M_2)^\lambda/\mathcal{D}$ is λ^+ -saturated, then $(M_1)^\lambda/\mathcal{D}$ is λ^+ -saturated.³ In the next claim, we use subscripts i, j for easier quotation later.

Claim 2.7. *If $\neg(T_j \leq_\lambda T_i)$ in Keisler's order, $\neg(T_j \leq_{\lambda^+, \aleph_1}^* T_i)$. Thus by monotonicity, $\neg(T_j \leq_{\lambda^+, 1}^* T_i)$.*

Proof. By monotonicity, it will suffice to prove this for $\kappa = \aleph_1$ (though the same proof will work replacing \aleph_1 everywhere by 1). Suppose that there is a regular ultrafilter \mathcal{D} on λ so that for any $M_\ell \models T_\ell$ for $\ell = i, j$, $(M_i)^\lambda/\mathcal{D}$ is λ^+ -saturated but $(M_j)^\lambda/\mathcal{D}$ is not λ^+ -saturated. Suppose for a contradiction that there were T_* interpreting both T_j and T_i such that in any model of T_* which is \aleph_1 -saturated, if the reduct to $\tau(T_i)$ is λ^+ -saturated then so is the reduct to $\tau(T_j)$. Let $M_* \models T_*$ and let $N_* = M_*^\lambda/\mathcal{D}$. Because N_* is a regular ultrapower, it will be \aleph_1 -saturated. As ultrapowers commute with reducts, $N_* \upharpoonright_{\tau(T_i)}$ will be λ^+ -saturated but $N_* \upharpoonright_{\tau(T_j)}$ will not be λ^+ -saturated. This contradicts the assumption on T_* and shows no such T_* exists, i.e. $\neg(T_j \leq_{\lambda^+, \aleph_1}^* T_i)$. \square

Corollary 2.8. *Let $\kappa \in \{1, \aleph_1\}$. If $T_0 \leq_\kappa^* T_1$, that is, if $T_0 \leq_{\mu, \kappa}^* T_1$ for all sufficiently large regular μ , then $T_0 \leq_\lambda T_1$ for all sufficiently large λ .*

Proof. For all sufficiently large λ , $T_0 \leq_{\lambda^+, \kappa}^* T_1$, so apply the contrapositive of Claim 2.7 with $j = 0, i = 1$ to conclude $T_0 \leq_\lambda T_1$. \square

Corollary 2.9. *Let $\kappa \in \{1, \aleph_1\}$. If for arbitrarily large λ we have $T_0 \triangleleft_\lambda T_1$ in Keisler's order, then $\neg(T_1 \leq_\kappa^* T_0)$.*

Proof. If $T_0 \triangleleft_\lambda T_1$, the strict inequality means that $\neg(T_1 \leq_\lambda T_0)$, so apply Claim 2.7 with $j = 1, i = 0$. \square

So \leq^* refines \leq in a natural sense:

Corollary 2.10. *If T_0 and T_1 are equivalent in $\leq_{\mu, 1}^*$ or in \leq_{μ, \aleph_1}^* for all sufficiently large μ , then they are equivalent in \leq_λ for all sufficiently large λ .*

To show the equivalence relation induced by \leq^* is strictly finer than \leq , we will need a few facts.

Discussion 2.11. *At least a priori, Corollary 2.10 does not imply that the ordering on the Keisler classes must be inherited by the \leq^* -classes. A priori, all or many of the \leq^* -classes could be pairwise incomparable.*

First recall that if $T_* \supseteq T$, the class $PC(T_*, T)$ is the class of reducts to $\tau(T)$ of models of T_* .

Fact 2.12 ([17], Theorem VI.5.3 p. 383). *If T is countable, superstable, and does not have the f.c.p., then there is T_* , $T \subseteq T_*$, $|T| = 2^{\aleph_0}$ such that $PC(T_*, T)$ is categorical in every cardinality $\geq 2^{\aleph_0}$. [The proof goes by showing one can choose T_* so that the reduct to T of any model of T_* of cardinality at least continuum is saturated in its own cardinality].*

³A discussion of this order may be found in e.g. [11] §2.

As an immediate corollary (proved just as in the proof of 2.16) we have:

Corollary 2.13. *If T is countable, superstable, and does not have the f.c.p., T is $\leq_{\lambda, 2^{\aleph_0}, 1}^*$ -minimal among complete countable theories for every $\lambda \geq 2^{\aleph_0}$.*

Second, following [1], and note we are now considering $\chi = \aleph_0$,

Definition 2.14. *We say that saturation is (μ, κ) transferrable in T (the interesting case is $\mu < \kappa$) if there is an expansion $T_* \supseteq T$ with $|T_*| = |T|$ such that if M is a μ -saturated model of T_* and $|M| \geq \kappa$ then the reduct of M to $\tau(T)$ is κ -saturated.*

Using this notion of transfer of saturation, Baldwin, Grossberg, and Shelah characterized four classes of countable theories, one of which we'll use here.

Fact 2.15 ([1], p. 11). *Let λ be an uncountable cardinal and T a countable theory. Then (T is superstable and does not have the finite cover property) \iff saturation is (\aleph_0, λ) -transferrable for T .*

Conclusion 2.16. *Let T_0, T_1 be complete countable theories. If T_0 is superstable and does not have the finite cover property, and T_1 is strictly stable and does not have the finite cover property, then T_0 and T_1 are equivalent in Keisler's order \leq but not equivalent in $\leq_{\lambda, \aleph_0, \aleph_0}^*$, thus not equivalent in \leq_1^* .*

Thus, the equivalence relation on countable theories induced by $\leq_{\lambda, \aleph_0, 1}^*$ is strictly finer than that induced by Keisler's order, already on the stable theories.⁴

Proof. The class of complete countable theories without the f.c.p. form an equivalence class in Keisler's order, see [17] VI.5.1, so it will suffice to show that for T_0, T_1 as in the statement, $\neg(T_1 \leq_1^* T_0)$. Suppose for a contradiction there were some theory T_* interpreting both T_0 and T_1 witnessing that $T_1 \leq_1^* T_0$ for all regular λ above some λ_* . Choose a regular $\lambda > \lambda_*$. There is no harm in expanding T_* to T_{**} witnessing that saturation is (\aleph_0, λ) -transferrable for T_0 , i.e. if M is an \aleph_0 -saturated model of T_{**} of cardinality $\geq \lambda$ then the reduct of M to $\tau(T_0)$ is λ -saturated. Now, by the characterization of Fact 2.15, there can be no theory witnessing such a transfer of saturation for T_1 ; in particular, T_{**} cannot be such a theory, and so there must be a counterexample, namely a model $N \models T_{**}$ of cardinality $\geq \lambda$ such that N is \aleph_0 -saturated but the reduct of N to $\tau(T_1)$ is not λ -saturated. However, since $N \models T_{**}$, its reduct to $\tau(T_0)$ is λ -saturated. Since N is a model of T_* , this contradiction completes the proof. \square

In the companion paper [15] being written we sort out the analogous cases, and will conclude there that \leq_1^* has six classes on the countable stable theories, including incomparable classes.

Returning to the case of $\leq_{\aleph_1}^*$, on the stable theories the picture is the same as that given by Keisler's order. The ordering $\leq_{\aleph_1}^*$ does have incomparable classes on the unstable theories, however; see §8 below for a proof under a set theoretic hypothesis, and a forthcoming ZFC proof in [15].

Theorem 2.17. *On the complete, countable, stable theories, $\leq_{\aleph_1}^*$ has precisely two classes, those theories without the finite cover property and those theories which are stable but have the finite cover property.*

⁴We will prove this for $\kappa = \aleph_0$, which by monotonicity implies it for $\kappa = 1$.

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Proof. We adapt the proof that Keisler’s order on the stable theories has precisely two classes, [17] VI.5.1 (note that regular ultrapowers of countable theories are always \aleph_1 -saturated). As there, the key background theorem says that a model of a countable stable theory is λ -saturated iff it is \aleph_1 -saturated and every maximal indiscernible set has size at least λ . For one direction, by [1] Theorem 2.2, a complete countable stable T has nfcf iff saturation is (\aleph_1, λ) -transferrable for all uncountable λ . By a proof similar to that of 2.16, this shows that the theories without the fcp are precisely the $\leq_{\lambda, \aleph_0, \aleph_1}^*$ -minimum class.

For the other side it suffices to show that if T_1, T_2 are both stable with the fcp then $T_1 \leq_{\aleph_1}^* T_2$. Here we may follow [17] VI.5.1(2) p. 379, which proves that for M a model of a countable stable theory T with fcp, any regular ultrapower M^I/\mathcal{D} is λ -saturated, where λ is the minimum product of an unbounded sequence of finite cardinals mod \mathcal{D} . (In fact, the proof there also shows that the ultrapower will not be λ^+ -saturated; this is true here too, but it’s simpler to just get two classes by quoting 2.9.) The argument will go through essentially verbatim except for one point: since we are not in a regular ultrapower, we’ll need to justify the following: there is T_* interpreting both T_1 and T_2 such that if N is any model of T_* whose reduct to $\tau(T_2)$ is λ -saturated, then in $N \upharpoonright \tau(T_1)$, every pseudofinite set has size at least λ . (What does “pseudofinite” mean? We require that T_* expands the theory of $(\mathbb{N}, <)$, and so we ask that any infinite, bounded subset of \mathbb{N}^N definable, with parameters, in T_* , has size at least λ .)

Let’s now justify this point. Recall that in a stable theory, having the fcp is equivalent to having (perhaps in an imaginary sort) a definable equivalence relation with a class of size n for each n . Call this equivalence relation E . Let M be a countable model expanding $(\mathbb{N}, <)$. Expand M so that its theory interprets T_1 and T_2 , without loss of generality in disjoint signatures. Suppose finally that $Th(M)$ codes enough set theory, or number theory, so that there is a parametrized family of functions $F : E \times E \rightarrow \mathbb{N}$ witnessing that for each finite n and each definable subset X of M of size $\geq n$ there is a definable injection of the E -class of size n into X . Let $T_* = Th(M)$. Suppose now that $N \models T_*$, that $N \upharpoonright \tau(T_2)$ is λ -saturated, and let $\varphi(x, \bar{c})$ be a bounded, definable subset of \mathbb{N}^N . Then by overspill, for some a in some infinite E^N -equivalence class, $F^N(a, -)$ will map $E^N(a, -)$ injectively into $\varphi[N, \bar{c}]$. By our saturation hypothesis, $E^N(a, -)$ has size at least λ , so $\varphi[N, \bar{c}]$ does too, as desired. \square

In the present paper we have focused on the unstable theories, with an eye towards simple theories and Keisler’s order.

The papers [2] and [21] investigated maximality under \leq^* . Building on the first, the second established that under relevant instances⁵ of GCH, any theory with $NSOP_2$ is non-maximal in \leq^* .

Fact 2.18 ([21] 3.15(2)). *Assuming relevant instances of GCH, if T is $NSOP_2$ then T is not maximal in $\leq_{\aleph_1}^*$.*

Next, we quote a characterization of the maximal class under \leq^* . The proof of the complementary direction to 2.18 given in [13] is in ZFC and uses some ideas from the proof that SOP_2 is \leq -maximum, from [9]. Our proof there shows that SOP_2 suffices for \leq_1^* -maximality, so a fortiori for $\leq_{\aleph_1}^*$ -maximality.

⁵The proof depends on a theorem from [2], which assumes relevant instances of GCH.

Fact 2.19 ([13] Theorem 7.13). *Any theory with SOP_2 is \leq_1^* -maximal. Thus, assuming relevant instances of GCH , T is $\leq_{\aleph_1}^*$ -maximal if and only if it has SOP_2 .*

It would be nice to remove GCH from 2.18 and therefore from the characterization of the maximal class, see §10 Problem 10 below, but this seems to require new ideas: currently, GCH contributes to simplifying the structure of trees and thus to extracting a suitable amalgamation condition from the $NSOP_2$ hypothesis.

The remainder of what (we believe) is known on \leq^* comes from quoting the known results on Keisler’s order, in each case invoking 2.7 or 2.9 as appropriate, and noting that the known ZFC dividing lines in Keisler’s order (including the infinitely many classes of [12]) all hold for arbitrarily large λ .

3. GEM MODELS AND INDISCERNIBLE SEQUENCES

In this section we review the basic setup of generalized Ehrenfeucht-Mostowski models and establish that the classes of index models we’ll use later in the paper have the desired properties. We roughly follow [19] §1, but it seemed best to make the paper self contained. We prefer to call the models generalized EM models rather than just EM models to stress that we use different index models.

Definition 3.1. *We say that $\mathbf{a} = \{\bar{a}_t : t \in I\} \subseteq {}^{\omega}N$ generates the model N if every element of N is in the definable closure of $\{\bar{a}_{t,\ell} : t \in I, \ell < \text{lgn}(\bar{a}_t)\}$.*

In the next definition, “quantifier-free types” means “...of finite tuples.”

Definition 3.2. *A template Φ is a map whose domain is the set of quantifier-free types of one structure and whose range is contained in the set of quantifier-free types of a possibly different structure.*

Example 3.3. *The classical example of a template comes when $(I, <)$ is an infinite linear order, N is a model, and $\langle \bar{a}_t : t \in I \rangle$ is an indiscernible sequence in N : the map Φ taking $\text{tp}_{\text{qf}}(\langle t_0, \dots, t_{k-1} \rangle, \emptyset, I)$ to $\text{tp}_{\text{qf}}(\bar{a}_{t_0} \wedge \dots \wedge \bar{a}_{t_{k-1}}, \emptyset, N)$ is a template.*

A related but much richer source of examples arise as follows.

Definition 3.4 (GEM models and proper templates, [19] Definition 1.8). *We say $N = \text{GEM}(I, \Phi) = \text{GEM}(I, \Phi, \mathbf{a})$ is a generalized Ehrenfeucht-Mostowski model with skeleton \mathbf{a} when for some vocabulary $\tau = \tau_\Phi$ we have:*

- (1) I is a model, called the index model.
- (2) N is a τ_Φ -structure and $\mathbf{a} = \{\bar{a}_t : t \in I\}$ generates N .
- (3) $\langle \bar{a}_t : t \in I \rangle$ is quantifier free indiscernible in N .
- (4) Φ is a template, taking (for each $n < \omega$) the quantifier free type of $\bar{t} = \langle t_0, \dots, t_{n-1} \rangle$ in I to the quantifier free type of $\bar{a}_{\bar{t}}$ in N . (So Φ determines τ_Φ uniquely, and also a theory T_Φ , the maximal τ_Φ -theory which holds in every such N .)

The skeleton \mathbf{a} generating a given GEM model may not be unique, so we often display it. Note also that templates are simply possible instructions, which may not be ‘coherent’ or give rise to a model. Templates which do have a special name.

Definition 3.5. *The template Φ is called proper for I if there is M such that $M = \text{GEM}(I, \Phi)$. We say Φ is proper for a class \mathcal{K} if Φ is proper for all $I \in \mathcal{K}$.*

Here are some helpful properties we will assume our templates have.

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1124 revision:2017-10-29

Remark 3.6. Let Φ be a template proper for \mathcal{K} . When T_Φ has Skolem functions,

- (1) Φ is also nice (by transitivity of \prec), which implies:
 - (a) T_Φ is complete.
 - (b) for every $I \in \mathcal{K}$, $\langle \bar{a}_t : t \in I \rangle$ is indiscernible, not just quantifier-free indiscernible, in $\text{GEM}(I, \Phi)$.
 - (c) for every $J \subseteq I$ from \mathcal{K} we have $\text{GEM}(J, \Phi) \preceq \text{EM}(I, \Phi)$.
- (2) $\text{GEM}(I, \Phi)$ is unique in the sense that it depends, up to isomorphism, on Φ and the isomorphism type of I .

To summarize in the usual terminology of EM models:

Convention 3.7. All templates we consider are assumed to be very nice, meaning: they are non-trivial (i.e. we may add in 3.4(1) the condition that $\text{lgn}(\bar{a}_t) \geq 1$ and $\langle \bar{a}_t : t \in I \rangle$ is without repetition), and T_Φ is well defined and has Skolem functions, thus 3.6 applies (so T_Φ is complete, etc., as there).

Next we explain our conventions and requirements on the class \mathcal{K} of index models.

Convention 3.8. In what follows \mathcal{K} will always denote a nonempty class of infinite models, called index models, which are expansions of linear orders, to a vocabulary $\tau = \tau(\mathcal{K})$, so \prec belongs to τ . In particular \mathcal{K} need not be elementary. We will use I, J, \dots for elements of \mathcal{K} .

Convention 3.9. As \mathcal{K} may not be elementary, the phrase “ J in \mathcal{K} is \aleph_0 -saturated” will always abbreviate “ J is \aleph_0 -homogeneous and \aleph_0 -universal for elements of \mathcal{K} .”

In this setup a crucial property of a class \mathcal{K} is being *Ramsey*. To motivate this property, consider again the example 3.3 of an indiscernible sequence $\bar{a} = \langle \bar{a}_t : t \in I \rangle$ in a model N , and its associated template Φ . Suppose we were to expand N to N^+ in some larger language. The sequence \bar{a} might no longer be indiscernible in N^+ , but we could find in some elementary extension N_* of N^+ an indiscernible sequence $\bar{b} = \langle \bar{b}_t : t \in I \rangle$ such that the template Ψ associated to \bar{b} is an extension of Φ in a natural sense:

$$\text{if } \Phi(\text{tp}_{\text{qf}}(\bar{t}, \emptyset, I)) = p \text{ and } \Psi(\text{tp}_{\text{qf}}(\bar{t}, \emptyset, I)) = q \text{ then } p \subseteq q \upharpoonright \tau(\Phi).$$

The right analogue for GEM models is given by the Ramsey property, which both tells us certain templates exist, and ensures that these templates reflect the given base structure. This definition is somewhat less general than [19] Definition 1.15, but it suffices for our purposes here.

Definition 3.10. We say the class \mathcal{K} is *Ramsey*, or *simplified Ramsey*, when: given any

- a) $J \in \mathcal{K}$ which is \aleph_0 -saturated,
- b) model M , and
- c) sequence $\mathbf{b} = \langle \bar{b}_t : t \in J \rangle$ of finite sequences from M with the length of \bar{b}_t determined by $\text{tp}_{\text{qf}}(t, \emptyset, J)$,

there exists a template Ψ which is proper for \mathcal{K} such that:

- i) $\tau(M) \subseteq \tau(\Psi)$
- ii) Ψ reflects \mathbf{b} in the following sense:
 - for any s_0, \dots, s_{n-1} from J ,
 - any $\varphi = \varphi(x_0, \dots, x_{m-1}) \in \mathcal{L}(\tau(M))$,
 - and any $\tau(M)$ -terms $\sigma_\ell(\bar{y}_0, \dots, \bar{y}_{n-1})$ for $\ell = 0, \dots, m-1$,

if $M \models \varphi[\sigma_0(\bar{b}_{t_0}, \dots, \bar{b}_{t_{n-1}}), \dots, \sigma_{m-1}(\bar{b}_{t_0}, \dots, \bar{b}_{t_{n-1}})]$
 for every t_0, \dots, t_{n-1} realizing $\text{tp}_{\text{qf}}(s_0 \hat{\ } \dots \hat{\ } s_{n-1}, \emptyset, J)$ in J ,
then $\text{GEM}(J, \Psi) \models \varphi[\sigma_0(\bar{a}_{s_0}, \dots, \bar{a}_{s_{n-1}}), \dots, \sigma_{m-1}(\bar{a}_{s_0}, \dots, \bar{a}_{s_{n-1}})]$.

We will generally use this definition in the form of Corollary 3.11. For a definition of $\Psi \geq \Phi$, see 3.13 below.

Corollary 3.11. *If \mathcal{K} is Ramsey, whenever we are given:*

- a) $J \in \mathcal{K}$ is \aleph_0 -saturated
- b) Φ a template proper for \mathcal{K}
- c) $M = \text{GEM}(J, \Phi)$ with skeleton \mathbf{a}
- d) N^+ , an elementary extension or expansion of M , or both

then there is a template Ψ proper for \mathcal{K} with $\tau(\Psi) \supseteq \tau(N^+)$ and $\Psi \geq \Phi$. Moreover, Ψ reflects \mathbf{a} in the sense described in 3.10 ii), with \mathbf{a} here replacing \mathbf{b} there.

Convention 3.12. *Given a class \mathcal{K} , we may write $\Upsilon_{\mathcal{K}}$ for the class of templates proper for \mathcal{K} .*

Definition 3.13. *Given a class of templates Υ , let \leq_{Υ} be the natural partial order on Υ , that is, $\Phi \leq_{\Upsilon} \Psi$ means that $\tau(\Phi) \subseteq \tau(\Psi)$ and $\text{GEM}(I, \Phi) \subseteq \text{GEM}(I, \Psi)$, $\text{GEM}_{\tau(\Phi)}(I, \Phi) \preceq \text{GEM}_{\tau(\Phi)}(I, \Psi)$. We may just use \leq when Υ is clear from context.*

The reader will notice that in the proofs below, before applying 3.11, we expand the models in question to have Skolem functions, and this is to ensure 3.7 above.

As we are assuming all templates are nice (3.7), what we have called ‘‘Ramsey’’ here is sometimes called ‘‘nicely Ramsey’’ in the literature.

In this language, Ehrenfeucht and Mostowski [3] proved that:

Fact 3.14. *Let \mathcal{K} be the class of linear orders. Then \mathcal{K} is Ramsey.*

Definition 3.15. *Define $\mathcal{K}_{\mu}(\mathbf{n})$ to be the class of*

$$I = (I, <_I, Q_{\alpha, n}^I)_{\alpha < \mu, n < 1 + \mathbf{n}}$$

where:

- (1) $\mathbf{n} \in [1, \omega]$.
- (2) $(I, <_I)$ is a linear order.
- (3) For each $n < \mathbf{n}$, $\langle Q_{\alpha, n}^I : \alpha < \mu \rangle$ is a partition of $\text{inc}_n(I)$.

As before, the class of index models is not an elementary class, since we require that the predicates $Q_{n, \alpha}$ partition the the n -tuples.

Fact 3.16. *For each finite \mathbf{n} , $\mathcal{K}_{\mu}(\mathbf{n})$ is Ramsey. (Note: in this paper, we will only use the case $\mathbf{n} = 1$.)*

Proof. [19] Theorem 1.18(5), i.e. by Nešetřil-Rödl. □

Definition 3.17.

- (1) $\mathcal{K}_{\aleph_7} = \mathcal{K}_{\aleph_7}^{\text{tr}}$ is the class of trees with \aleph_7 levels and lexicographic order which are normal, meaning⁶ that a member η at a limit level is determined by $\{\nu : \nu \trianglelefteq \eta\}$. (So the tree has the function $\cap(\eta, \nu) = \min\{\rho : \rho \trianglelefteq \nu, \rho \trianglelefteq \eta\}$.)

⁶We can waive normality assuming only successor levels.

- (2) We call $I \in \mathcal{K}$ standard when the i th level, P_i^I , of I consists of sequences of length i and $n \in P_i$, $j < i$, $\eta \upharpoonright j \in P_j$ and $\eta \upharpoonright j \trianglelefteq_I \eta$, so every $I \in \mathcal{K}_{\aleph_7}$ is isomorphic to a standard one (this is justified by the assumption of normality).

Fact 3.18. \mathcal{K}_κ^{tr} is Ramsey.

Proof. This is [17] VII §3, see 3.7 p. 424, and see 2.4 of the Appendix. \square

Discussion 3.19. Whereas one source of reflection onto model theory traditionally comes from set theory, our methods here emphasize the direct usefulness of combinatorics for model theory: proving new Ramsey theorems to allow for new classes \mathcal{K} on the scene may increase the possible comparison of theories via GEM-models.

Notation 3.20.

- 1) When $\bar{t} = \langle t_{i_0}, \dots, t_{i_{k-1}} \rangle \in {}^k I$, write $\bar{a}_{\bar{t}}$ for $\bar{a}_{t_{i_0}} \hat{\ } \dots \hat{\ } \bar{a}_{t_{i_{k-1}}}$.
- 2) We may write $\text{GEM}_\tau(I, \Phi)$ to denote the reduct to τ .
- 3) If φ is a formula, φ^0 abbreviates $\neg\varphi$ and φ^1 abbreviates φ .

4. T_{rg} IS MINIMAL AMONG THE UNSTABLE THEORIES

Lemma 4.1 (Minimality of the random graph).

- (1) T_{rg} , the theory of the random graph, is \trianglelefteq_1^* -minimal among the complete, countable unstable theories.
- (2) Thus, by monotonicity, T_{rg} is $\trianglelefteq_{\aleph_1}^*$ -minimal among such theories.
- (3) Moreover, for $T_0 = T_{\text{rg}}$ and any unstable T_1 , $T_0 \trianglelefteq_1^* T_1$ is witnessed by a theory T_* expanding $\text{Th}(\mathbb{N}, <)$ with the property that if $N \models T_*$ and $N \upharpoonright \tau(T_1)$ is λ -saturated, then \mathbb{N}^N has cofinality at least λ .

For the proof of Lemma 4.1, we'll need a claim.

Claim 4.2. Let T be a complete countable theory and suppose $\varphi(\bar{x}; \bar{y})$ has the independence property for T . Then there is a countable model M of T and a sequence $\langle \bar{b}_n : n < \omega \rangle$ with $\ell(\bar{b}_n) = \ell(\bar{y})$ contained in M , over which φ has the independence property, and such that for any $\bar{a} \in {}^{\ell(\bar{x})}M$, there is a truth value $\mathbf{t} \in \{0, 1\}$ such that for all but finitely many n , $M \models \varphi[\bar{a}, \bar{b}_n]^{\mathbf{t}}$.

Proof. For simplicity, we will write this proof as if \bar{x}, \bar{y} were singletons, but the proof is identical for finite tuples. Let N_1 be an \aleph_1 -saturated model of T . So we can choose $\langle c_\alpha : \alpha < \omega \rangle$ such that in N_1 the set of formulas $\{\varphi(x, c_\alpha) : \alpha < \omega\}$ is independent. Let $N_0 \preceq N_1$ be countable and contain $\{c_\alpha : \alpha < \omega\}$. Let $\langle d_i : i < \omega \rangle$ list the elements of N_0 . Let \mathcal{D} be a uniform ultrafilter on ω . For each i let $\mathbf{t}(d_i)$ be such that $A_i := \{k : N_0 \models \varphi[d_i, c_k]^{\mathbf{t}(d_i)}\} \in \mathcal{D}$.

By induction on $n < \omega$ choose b_n , the choices increasing (in the enumeration of N_0) with n , such that $b_n \in \bigcap_{j < n} A_j$. The sequence of b_n 's is chosen as a subsequence of the d_i 's, so φ will a fortiori have the independence property on this sequence. Moreover, if $a \in N_0$ then for some i , $a = d_i$. So once $n \in (i, \omega)$, $b_n \in \bigcap_{i < n} A_i \subseteq A_i$, thus $N_0 \models \varphi[a, b_n]^{\mathbf{t}(a)}$ as desired. \square

Proof of Lemma 4.1. We will prove that $T_{\mathbf{rg}} \leq_1^* T_1$ for T_1 any countable, unstable, complete first order theory. If T_1 has *SOP*, then it is already maximal, so it will suffice to prove this in the case that T_1 has the independence property.

Let $\varphi(\bar{x}, \bar{y})$ be a formula which has the independence property for T_1 . (In what follows, we'll write as if $\ell(x) = \ell(y) = 1$, but this is only for simplicity of notation.) We'll also assume the three theories T_1 , $T_{\mathbf{rg}}$, and $Th(\mathbb{N}, <)$ have disjoint signatures.

Let M be a countable model whose theory T satisfies:

- (a) M expands $(\mathbb{N}, <)$.
- (b) $M \upharpoonright \tau(T_{\mathbf{rg}})$ is a countable random graph.
That is, for some unary predicate P such that P^M is countably infinite, and some binary relation R , $T \vdash \text{"}(P, R) \text{ is } n\text{-random for every } n\text{"}$ i.e. for every finite n , for any two disjoint subsets $U, V \subseteq P^M$ of size n , there exists $a \in P^M$ such that $R^M(a, u)$ for all $u \in U$ and $\neg R^M(a, v)$ for all $v \in V$.
- (c) $M \upharpoonright \tau(T_1)$ is a model of T_1 satisfying the conclusion of Claim 4.2. The domain of this model is named by Q^M , and S^M is a binary relation with $T \vdash S \subset Q \times Q$ and $T \vdash \text{"}S \text{ has the } n\text{-independence property for each } n\text{"}$ that is, for each n , there exist a_1, \dots, a_n in Q^M such that the formulas $\{S(x, a_i) : 1 \leq i \leq n\}$ are independent. (*We just let S name φ ; in other words, for simplicity, we forget everything about the model of T_1 except for its domain and the formula with the independence property.*)
- (d) F^M is an injective function from \mathbb{N} into Q^M such that for every $a \in Q^M$, for some truth value \mathbf{t} , for every n large enough, $M \models \varphi[a, F^M(n)]^{\mathbf{t}}$.
(*It suffices to let $F^M(n) = b_n$ where $\langle b_n : n < \omega \rangle$ is from Claim 4.2.*)
- (e) G^M is a one to one and onto function from \mathbb{N} onto P^M from part (b).

Now let N be any model of T . It will suffice to prove that if $N \upharpoonright \tau(T_1)$, or really just $N \upharpoonright \{Q, S\}$, is μ -saturated, then the following three facts hold.

- (1) \mathbb{N}^N has cofinality $\geq \mu$.

If not let $\langle a_\alpha : \alpha < \kappa \rangle$ be $<^N$ -increasing and cofinal in \mathbb{N}^N , with $\kappa < \mu$. Let $p(x) = \{Q(x)\} \cup \{S(x, F^N(a_{2\alpha})) \wedge \neg S(x, F^N(a_{2\alpha+1})) : \alpha < \kappa\}$. Then p is a finitely satisfiable type of cardinality $\kappa < \mu$. But it can't be realized because, by item (d), every element of Q has an eventual S -truth value with respect to the image of F . (This was expressible in T , so remains true in N .) So $N \upharpoonright \tau(T_1)$ is not saturated, contradiction.

- (2) (P^N, R^N) is μ -saturated.

If (P^N, R^N) is not μ -saturated, let $p(x)$ be a 1-type of cardinality $< \mu$ there which is omitted. Without loss of generality $p(x) = \{(R(x, a_\alpha))^{\text{if } \eta(\alpha)} : \alpha < \alpha_*\}$ for some $\eta \in \alpha_*2$ and $\langle a_\alpha : \alpha < \alpha_* \rangle$ with no repetition. Let $b_\alpha = (G^{-1})^N(a_\alpha)$ for $\alpha < \alpha_*$. So $\{b_\alpha : \alpha < \alpha_*\}$ is a subset of \mathbb{N}^N of cardinality $< \mu$ with no repetition, so it has an upper bound b_* by (1). Let $c_\alpha = F^N(b_\alpha)$ for $\alpha < \alpha_*$, so $\langle c_\alpha : \alpha < \alpha_* \rangle$ is a sequence of $< \mu$ elements with no repetition, and $\{S(x, c_\alpha)^{\text{if } \eta(\alpha)} : \alpha < \alpha_*\}$ is a type over Q^N . By assumption that $N \upharpoonright \tau(T_1)$ is saturated, this type is realized, say by d . So the set $U = \{b < b_* : S(d, F^N(b))\}$ is a first-order definable subset of N (with parameters) hence by our choice of T [i.e. noting that \mathbb{N}^N is pseudofinite, and the following would be true in M if b_* were finite] there is a_{α_*} in range G such that $(\forall y < b_*)(R(a_{\alpha_*}, G(y)) \iff y \in U)$ i.e. $(\forall y < b_*)(R(a_{\alpha_*}, G(y)) \iff S(d, G(y)))$ so a_{α_*} realizes p .

(3) N satisfies $< \mu$ -regularity, meaning that every set of $< \mu$ elements is contained in some pseudofinite set.

This simply restates the fact that the model expands \mathbb{N} , and \mathbb{N}^N has cofinality $\geq \mu$, so every small subset of the model is contained in some bounded set of (nonstandard) integers. \square

5. $T_{\mathbf{rg}}$ IS NOT MAXIMAL

Next we turn to a direct proof of the following theorem.

Theorem 5.3. *$T_{\mathbf{rg}}$, the theory of the random graph, is not \leq_1^* -maximal.*

The theorem, though to our knowledge not previously recorded, already follows from known results:

Indirect proof of Theorem 5.3. First, we know that $T_{\mathbf{rg}}$ is minimum among the unstable theories in Keisler's order [8]. [10, Theorem 12.1] proves that for arbitrarily large λ , the random graph is strictly \leq_λ -smaller than any non-low or non-simple theory in Keisler's order. By Observation 2.9 above, we have that $T_{\mathbf{rg}}$ is not in the \leq_1^* -maximum class. \square

The direct proof below is new, and illustrates a method which will inspire the rest of the paper. Since the theory of linear order is \leq^* -maximum, we have the following sufficient condition:

Observation 5.4. *To show $T_{\mathbf{rg}}$, the theory of the random graph, is not \leq_1^* -maximum, it will suffice to show that for $T_{\mathbf{dlo}} = \text{Th}(\mathbb{Q}, <)$,*

$$\neg(T_{\mathbf{dlo}} \leq_1^* T_{\mathbf{rg}})$$

i.e. for arbitrarily large regular μ , for every countable theory T_ interpreting both $T_{\mathbf{rg}}$ and $T_{\mathbf{dlo}}$ in the sense of Definition 1.2, there is some model $M \models T_*$ such that $M \upharpoonright \tau(T_{\mathbf{rg}})$ is μ -saturated but $M \upharpoonright \tau(T_{\mathbf{dlo}})$ is not μ -saturated. Recalling 1.5(2), we may assume that T_* has Skolem functions.*

Hypothesis 5.5. *In the rest of this section we will assume:*

- (a) $\lambda = \lambda^{< \mu}$ and $\lambda \geq 2^\mu$.
- (c) T_* is some theory containing Skolem functions, $|T_*| \leq \lambda$, such that: ⁷
 - (i) T_* interprets $T_{\mathbf{rg}}$, with edge relation $R = R_{\mathbf{rg}}$, and
 - (ii) T_* interprets $T_{\mathbf{dlo}}$, with ordering $< = <_{\mathbf{dlo}}$.
- (e) The class of index models $\mathcal{K} = \mathcal{K}_{\aleph_4}$ has elements of the form

$$I = (|I|, <_I, P_\alpha^I)_{\alpha < \lambda}$$

where $<_I$ is a linear order, the P_α are all unary predicates, and $\langle P_\alpha^I : \alpha < \lambda \rangle$ is a partition of I . Note that as we require that the predicates partition I , \mathcal{K} is not an elementary class.

- (f) We will say “ $I \in \mathcal{K}$ is separated” when for all $\alpha < \lambda$, $|P_\alpha^I| \leq 1$.

⁷The proof works for a larger class of T_* than just the countable ones.

Our present choice of index models \mathcal{K} , which are Ramsey by 3.16, may be thought of as linear orders in which every element has one of λ possible colors. Separated I have exactly no more than one element of each color. Note that separation puts no restrictions on the linear order other than size, so e.g. there are separated $I \in \mathcal{K}$ which contain (κ_1, κ_2) -cuts for any $\kappa_1 + \kappa_2 \leq \lambda$.

We will use the following class of templates.

Definition 5.6. Let $\Upsilon = \Upsilon_{\S 4}$ be the class of templates Φ such that:

- (a) Φ is proper for $\mathcal{K}_{\S 4}$,
- (b) $I \in \mathcal{K}_{\S 4}$ implies $\text{GEM}(I, \Phi) \models T_*$,
- (c) $s < t$ implies $a_s <_{\text{dlo}} a_t$, informally, “the template represents the order.”

We use $\leq = \leq_{\Upsilon}$ for the natural partial order on elements of $\Upsilon_{\S 4}$, as in 3.13.

The next claim will be useful for us here and later. Note it does not use anything about the particular \mathcal{K} or Υ we’ve chosen besides the global convention 3.7.

Definition 5.7. We say $\Phi \in \Upsilon$ is smooth when for every $N = \text{GEM}(I, \Phi)$, if $I_1, I_2 \subseteq I$ and $I_0 = I_1 \cap I_2$, and $N_\ell = \text{GEM}(I_\ell, \Phi) \subseteq \text{GEM}(I, \Phi)$ for $\ell = 0, 1, 2$ (by our convention 3.7), then $N_1 \cap N_2 = N_0$.

Claim 5.8. At the cost of possibly expanding $\tau(\Phi)$ by function symbols, without loss of generality, we may assume Φ is smooth. More precisely, for every $\Phi \in \Upsilon$ there is $\Psi \in \Upsilon$ such that

- (1) $\Phi \leq \Psi$
- (2) $\tau(\Psi) \supseteq \tau(\Phi)$, $|\tau(\Psi) \setminus \tau(\Phi)| \leq \aleph_0 + \tau(\Phi)$
- (3) Ψ is smooth.

Proof. It will suffice to find Ψ such that for any $I \in \mathcal{K}$, in $N = \text{GEM}(I, \Psi)$, whenever there are finite $\bar{u}, \bar{v} \subseteq I$ and $\tau(\Psi)$ -terms σ, τ such that $\sigma^N(\bar{a}_{\bar{u}}) = \tau^N(\bar{a}_{\bar{v}})$ then there is also a $\tau(\Psi)$ -term ρ such that $\sigma^N(\bar{a}_{\bar{u}}) = \tau^N(\bar{a}_{\bar{v}}) = \rho^N(\bar{a}_{\bar{u} \cap \bar{v}})$.

Towards this, let $J \in \mathcal{K}$ be \aleph_0 -saturated and let $M = \text{GEM}(J, \Phi)$ with skeleton $\langle a_t : t \in J \rangle$, where $\ell(a_t)$ is not necessarily equal to 1.

First note that whenever $\sigma(x_0, \dots, x_{n-1})$ and $\tau(x_0, \dots, x_{m-1})$ are terms of $\tau(\Phi)$, and whenever $\bar{u} \in \text{inc}_n(J)$, $\bar{v} \in \text{inc}_m(J)$, if

$$\sigma^M(\bar{a}_{\bar{u}}) = \tau^M(\bar{a}_{\bar{v}}) =: b$$

then as we are in a GEM model, for any other finite sequences \bar{u}', \bar{v}' of elements of J such that $\text{tp}_{\text{qf}}(\bar{u}' \wedge \bar{v}', \bar{u} \cap \bar{v}, J) = \text{tp}_{\text{qf}}(\bar{u} \wedge \bar{v}, \bar{u} \cap \bar{v}, J)$ we have

$$\sigma^M(\bar{a}_{\bar{u}'}) = \tau^M(\bar{a}_{\bar{v}'}).$$

Second, if $\bar{u}' \subseteq J$ is increasing and \bar{u}', \bar{u} agree on $\bar{u} \cap \bar{v}$, then $\sigma^M(\bar{a}_{\bar{u}}) = \sigma^M(\bar{a}_{\bar{u}'})$. (By transitivity of equality, we may assume there exists a unique i such that \bar{u} and \bar{u}' differ in the i th-place; and since J is saturated, we can find \bar{v}' such that $\text{tp}_{\text{qf}}(\bar{u} \wedge \bar{v}', \emptyset, J) = \text{tp}_{\text{qf}}(\bar{u}' \wedge \bar{v}', \emptyset, J) = \text{tp}_{\text{qf}}(\bar{u} \wedge \bar{v}, \emptyset, J)$. Then $\sigma^M(\bar{a}_{\bar{u}}) = \tau^M(\bar{a}_{\bar{v}}) = \sigma^M(\bar{a}_{\bar{u}'})$.) A similar argument shows that if $\bar{v}' \subseteq J$ is increasing and \bar{v}', \bar{v} agree on $\bar{u} \cap \bar{v}$, then $\tau^M(\bar{a}_{\bar{v}}) = \tau^M(\bar{a}_{\bar{v}'})$.

Combining the above, we conclude that for any finite sequences \bar{u}', \bar{v}' of elements of J such that $\text{tp}_{\text{qf}}(\bar{u}' \wedge \bar{v}', \bar{u} \cap \bar{v}, J) = \text{tp}_{\text{qf}}(\bar{u} \wedge \bar{v}, \bar{u} \cap \bar{v}, J)$,

$$\sigma^M(\bar{a}_{\bar{u}'}) = \sigma^M(\bar{a}_{\bar{u}}) = \tau^M(\bar{a}_{\bar{v}}) = \tau^M(\bar{a}_{\bar{v}'}) = b.$$

So, if it exists, b is unique, once we have specified σ, τ , the three quantifier-free types $\mathfrak{r} = \text{tp}_{\text{qf}}(\bar{u}, \emptyset, J)$, $\mathfrak{p} = \text{tp}_{\text{qf}}(\bar{v}, \emptyset, J)$, $\mathfrak{q} = \text{tp}_{\text{qf}}(\bar{u} \hat{\ } \bar{v}, \emptyset, J)$, and $\bar{u} \cap \bar{v}$. We may write $b = b_{\sigma, \tau, \mathfrak{r}, \mathfrak{p}, \mathfrak{q}, \bar{u} \cap \bar{v}}$.

We are ready to define the expansion of M to M^* : for each $\sigma, \tau, \mathfrak{r}, \mathfrak{p}, \mathfrak{q}$ as above, add a new function symbol $F_{\sigma, \tau, \mathfrak{r}, \mathfrak{p}, \mathfrak{q}}$ and interpret it so that for any two finite sequences \bar{u}, \bar{v} from J , if $\text{tp}_{\text{qf}}(\bar{u} \hat{\ } \bar{v}, \emptyset, J) = \mathfrak{q}$, $\text{tp}_{\text{qf}}(\bar{u}, \emptyset, J) = \mathfrak{r}$, and $\text{tp}_{\text{qf}}(\bar{v}, \emptyset, J) = \mathfrak{p}$ (or instead of the last two, specifying $\text{lgn}(\bar{u})$ would suffice), then

$$F_{\sigma, \tau, \mathfrak{r}, \mathfrak{p}, \mathfrak{q}}^{M^*}(\bar{a}_{\bar{u} \cap \bar{v}}) = b_{\sigma, \tau, \mathfrak{r}, \mathfrak{p}, \mathfrak{q}, \bar{u} \cap \bar{v}}.$$

If desired also expand by Skolem functions. Let Ψ be the template returned by applying the Ramsey property to J, Φ, M, M^* . \square

Here is the key step in the saturation half of the argument. A comment on strategy: we might wish to simply realize the given p in some elementary extension, expand by a constant naming this realization⁸, and apply 3.11. We are prevented from doing this by 3.11(a), which requires the index model to be sufficiently saturated, which I is not. However, this can work if we first replace I by a sufficiently saturated J , and replace p by a corresponding larger type q . As the proof will show, the point is to choose q in such a way that first, it is a type, and second, that the rule in 3.10(ii) ensures that not only does $c \in \tau(\Psi)$ but also that p remains realized by c “after averaging out” in the models $\text{GEM}(J, \Psi)$ or $\text{GEM}(I, \Psi)$.

Claim 5.9. *Assume $\Phi \in \Upsilon$, $I \in \mathcal{K}$ is separated, $M = \text{GEM}(I, \Phi)$ and $p \in \mathbf{S}_R(M)$. Then there is $\Psi \in \Upsilon$ and a constant $c \in \tau(\Psi) \setminus \tau(\Phi)$ such that $\Phi \leq_{\Upsilon} \Psi$ and if $J \in \mathcal{K}$ and h embeds I into J then c realizes $h(p)$, defined naturally.*

Proof. By Claim 5.8, without loss of generality Φ is smooth.

We begin with $M = \text{GEM}(I, \Phi)$ and $p \in \mathbf{S}_R(M)$. Fix J such that $I \subseteq J \in \mathcal{K}$ and J is \aleph_0 -saturated. Let $N = \text{GEM}(J, \Phi)$. By 3.7 $M \preceq N$, so we will identify the sequence $\langle \bar{a}_t : t \in I \rangle$ which generates M with a subsequence of $\langle \bar{a}_t : t \in J \rangle$.

By quantifier elimination, p is of the form $\{R(x, b_\alpha)^{\mathbf{i}_\alpha} : \alpha < \kappa\}$ for some κ , where each $\mathbf{i}_\alpha \in \{0, 1\}$. As M is generated by $\{\bar{a}_t : t \in I\}$, each b_α may be written as $\sigma_\alpha^M(\bar{a}_{\bar{t}_\alpha})$ for some $\tau(\Phi)$ -term σ_α and some $\bar{t}_\alpha \in \text{inc}(I)$. This representation may not be unique; choose one, subject to $\text{lgn}(\bar{t}_\alpha)$ being minimal but ≥ 1 . Then we may write p as

$$(1) \quad p(x) = \{R(x, \sigma_\alpha^M(\bar{a}_{\bar{t}_\alpha}))^{\mathbf{i}_\alpha} : \alpha < \kappa\}.$$

Now working in N , consider the set of formulas

$$(2) \quad q(x) = \{R(x, \sigma_\alpha^N(\bar{a}_{\bar{s}}))^{\mathbf{i}_\alpha} : \alpha < \kappa, \text{tp}_{\text{qf}}(\bar{s}, \emptyset, J) = \text{tp}_{\text{qf}}(\bar{t}_\alpha, \emptyset, I)\}.$$

Let's show that $q(x)$ is consistent. It suffices to prove that whenever

$$(3) \quad R(x, \sigma_\alpha^N(\bar{a}_{\bar{v}}))^{\mathbf{i}_\alpha} \in q \text{ and } R(x, \sigma_\beta^N(\bar{a}_{\bar{w}}))^{\mathbf{i}_\beta} \in q$$

if $\sigma_\alpha^N(\bar{a}_{\bar{v}}) = \sigma_\beta^N(\bar{a}_{\bar{w}})$ then $\mathbf{i}_\alpha = \mathbf{i}_\beta$. Fix for awhile α and β and suppose for a contradiction that

$$(4) \quad \sigma_\alpha^N(\bar{a}_{\bar{v}}) = \sigma_\beta^N(\bar{a}_{\bar{w}}) = b \text{ but } \mathbf{i}_\alpha \neq \mathbf{i}_\beta.$$

⁸Note: just before invoking 3.11, we expand again by Skolem functions; this is sufficient to ensure the template returned satisfies 3.7.

Since we know p is consistent and N is a GEM model,⁹ we may assume that neither $\sigma_\alpha^N(\bar{a}_{\bar{v}})$ nor $\sigma_\beta^N(\bar{a}_{\bar{w}})$ are in $\text{acl}(\emptyset)$. So by our assumption $\text{lgn}(\bar{v}), \text{lgn}(\bar{w})$ are minimal, but at least 1, and \bar{v}, \bar{w} are increasing. Since we assumed Φ is smooth, necessarily

$$(5) \quad \bar{v} = \bar{w}$$

(if not, note that the element $b = \sigma_\alpha^N(\bar{a}_{\bar{v}}) = \sigma_\beta^N(\bar{a}_{\bar{w}})$ must belong to $\text{GEM}(\bar{v} \cap \bar{w}, \Phi)$, contradicting minimal length for at least one of \bar{v}, \bar{w}). Let $\bar{t} \in \text{inc}(I)$ be such that $\text{tp}_{\text{qf}}(\bar{t}, \emptyset, I) = \text{tp}_{\text{qf}}(\bar{v}, \emptyset, J)$. As I is singular, \bar{t} is uniquely determined, and equal to \bar{t}_α . By the parallel argument, $\text{tp}_{\text{qf}}(\bar{t}_\beta, \emptyset, I) = \text{tp}_{\text{qf}}(\bar{w}, \emptyset, J)$. Combining with (5),

$$\text{tp}_{\text{qf}}(\bar{t}_\beta, \emptyset, I) = \text{tp}_{\text{qf}}(\bar{w}, \emptyset, J) = \text{tp}_{\text{qf}}(\bar{v}, \emptyset, J) = \text{tp}_{\text{qf}}(\bar{t}_\alpha, \emptyset, I).$$

Since I is singular, $\text{tp}_{\text{qf}}(\bar{t}_\alpha, \emptyset, I) = \text{tp}_{\text{qf}}(\bar{t}_\beta, \emptyset, I)$ entails $\alpha = \beta$, contradicting $\mathbf{i}_\alpha \neq \mathbf{i}_\beta$. This contradiction proves that q is consistent.

Since q is consistent, in some elementary extension N' of N there is an element which realizes it. Let N'_c be N' expanded by this constant c , and by Skolem functions. (Note that once we add this constant, \bar{a}_J may no longer be indiscernible.) Apply 3.11 in the case where $N^+ = N'_c$. Let Ψ be the template returned, which will be proper for \mathcal{K} . By 3.11 and equation (2) above, Ψ has the property that for every $\alpha < \kappa$, if $\tau_\alpha = \text{tp}_{\text{qf}}(\bar{t}_\alpha, \emptyset, I)$ then the formula $R(c, \sigma_\alpha(-))^{i_\alpha}$ belongs to $\Psi(\tau_\alpha)$. Thus, $\Phi \leq_\Upsilon \Psi$ and if $J \in \mathcal{K}$ and h embeds I into J then c realizes $h(p)$ in $\text{GEM}(J, \Psi)$. \square

Corollary 5.10. *For every $\Phi \in \Upsilon$ there is $\Psi \in \Upsilon$ such that $\Phi \leq \Psi$ and for every separated $I \in \mathcal{K}$, $\text{GEM}(I, \Phi) \upharpoonright \{R_{\text{rg}}\}$ is μ -saturated.*

Proof. This is a counting argument. As we'll appeal to similar arguments again, here let us give the details. Let Φ be given. Fix for a moment some $I \in \mathcal{K}$ which is separated, therefore of size $\leq \mu$. Recall from our hypotheses for the section that λ, μ are fixed, $\lambda = \lambda^{<\mu}$ and $|\tau(\Phi)| \leq \lambda$. So $M = \text{GEM}(I, \Phi)$ satisfies $|M| \leq \lambda$. This means M has $\lambda^{<\mu} = \lambda$ subsets of size $< \mu$, and over each such subset A , it has at most $2^{|A|} \leq \lambda$ types. Let $\langle p_\alpha : \alpha < \lambda \rangle$ be an enumeration of all such types. By induction on α , we may build a \leq -continuous increasing chain of templates Φ_α , where $\Phi_{\alpha+1}$ is the result of applying Claim 5.9 in the case $\Phi = \Phi_\alpha$, I , and $p = p_\alpha$. Let Ψ be the union of this chain of templates, recalling 7.4. Let $N = \text{GEM}(I, \Psi)$. Then $|N| \leq \lambda$. Moreover, $M \upharpoonright \{R_{\text{rg}}\} \preceq N \upharpoonright \{R_{\text{rg}}\}$ and all random graph types over subsets of M of size $< \mu$ are realized in N . By repeating this construction λ -many times, we obtain a template

$$(6) \quad \Psi_{I, \Phi}$$

which is $\geq \Phi$ and has the property that for our given I , $\text{GEM}(I, \Psi)$ has size $\leq \lambda$ and is saturated for all random graph types over subsets of size $< \mu$. (This is already enough for the proof of Theorem 5.3 below.)

To find a single Ψ which works for all separated I , first note that there are a bounded number of separated $I \in \mathcal{K}$, up to isomorphism. (In fact, there are no more than λ : any separated I has size $\leq \mu$, and any separated J occurs as a subset of some separated I of size μ . So there are no more than 2^μ separated I of size exactly μ , up to isomorphism; each has $\leq 2^\mu$ subsets, up to isomorphism, and recall

⁹If $\sigma_\alpha^N(\bar{a}_{\bar{v}}) = \sigma_\beta^N(\bar{a}_{\bar{w}})$ were in the algebraic closure of the empty set, say equal to c^N for some $c \in \tau(\Phi)$, then since N is a GEM model, it would also have to be the case that $\sigma_\alpha^N(\bar{a}_{\bar{t}_\alpha}) = \sigma_\beta^N(\bar{a}_{\bar{t}_\beta}) = c^N$, so as $M \preceq N$ by 3.7, $\sigma_\alpha^M(\bar{a}_{\bar{t}_\alpha}) = \sigma_\beta^M(\bar{a}_{\bar{t}_\beta})$, and if $i_\alpha \neq i_\beta$ then p is inconsistent.

$\lambda \geq 2^\mu$.) So we can enumerate all such I as $\langle I_\alpha : \alpha < \lambda \rangle$, each occurring cofinally often, and build a \leq_Υ -increasing continuous chain of templates Φ_α , where each $\Phi_{\alpha+1}$ is built as Ψ_{I, Φ_α} from (6) above. The union of this chain will be our desired template. \square

Now for the non-saturation half of the argument. In the next claim, recall that “cut” means “unfilled cut”. A comment on strategy: in any $J \in \mathcal{K}$, the quantifier free type of a tuple is determined by its order type and the colors of its elements. I is separated, so each element has its own color. We consider I as a subset of some saturated J where, say, the order is dense and some fixed color occurs densely often. Note that the f we find isn’t an embedding of $Y \cup Z$ to J , because all elements in the range of f will have the same color. It would suffice to let f choose a sequence suitably cofinal in each side, all of the same color.

Claim 5.11. *Suppose $I \in \mathcal{K}$ is separated, κ is an infinite regular cardinal, and $(\langle s_\alpha : \alpha < \kappa \rangle, \langle t_\alpha : \alpha < \kappa \rangle)$ is a cut of I . Then $\text{GEM}(I, \Phi) \upharpoonright \{<_{dlo}\}$ is not κ^+ -saturated, and in fact omits the type*

$$q(x) = \{ (s_\alpha <_{dlo} x <_{dlo} t_\alpha) : \alpha < \kappa \}.$$

Proof. We’ll prove the a priori stronger claim that for some sufficiently saturated J with $I \subseteq J$, p is not realized in $EM(J, \Phi)$. This suffices under our global assumption (3.7) that $EM(I, \Phi) \preceq EM(J, \Phi)$.

Recall assumption 5.6(d) which says that our templates “represent” $<_{dlo}$: when $s < t$ are from the index model then $a_s <_{dlo} a_t$. Let $Y = \{s_\alpha : \alpha < \kappa\}$ and let $Z = \{t_\alpha : \alpha < \kappa\}$. Observe that we can find some sufficiently saturated $J \in \mathcal{K}$, containing I , and a function f such that:

- (a) f is a function from $Y \cup Z$ to J
- (b) if $s \in Y$ then $s <_J f(s) <_J (Y)_{>s} := \{d \in Y : s <_{I_1} d\}$.
- (c) if $t \in Z$ then $(Z)_{<t} <_J f(t) <_J t$
- (d) if $s \in Y, t \in Z$ then $f(s), f(t)$ realize the same quantifier free type in J over $(J)_{<s} \cup (J)_{>t}$.

Note that $p(x)$ implies $\{(a_{f(s)} <_{dlo} x) \wedge (x <_{dlo} a_{f(t)}) : (s, t) \in Y \times Z\}$. In $\text{GEM}(J, \Phi)$ if there were $a = \sigma(\bar{a}_{\bar{u}})$ realizing p , choose $s \in Y$ and $t \in Z$ such that $[s, t] \cap \text{range}(\bar{u}) = \emptyset$, which possible simply because \bar{u} is finite. Then because we have assumed a realizes p , a must satisfy the formula $(a_{f(s)} <_{dlo} x)$ and also the formula $\neg(a_{f(t)} <_{dlo} x)$. This contradicts (d), so completes the proof. \square

We now prove Theorem 5.3 from the beginning of the section.

Proof of Theorem 5.3. Recall Observation 5.4, and recall from 7.3 the assumption $\lambda = \lambda^{<\mu}$. It suffices to show that for every $\kappa^+ \leq \mu$, there are $\Phi \in \Upsilon$ and $I \in \mathcal{K}$ such that for $M = \text{GEM}_{\tau(T_*)}(I, \Phi)$, we have that $M \upharpoonright_{\{R_{\text{rg}}\}}$ is μ -saturated but $M \upharpoonright_{\{<_{dlo}\}}$ is not κ^+ -saturated. Choose $I \in \mathcal{K}$ which is separated and has a (κ, κ) -cut (note that there is a cardinality restriction on separated I , but in our case $\kappa < \mu$). Let Φ be from 5.10 and apply it to the selected I . By 5.10 and 5.11, $M_{\tau(T_*)}(I, \Phi)$ is as desired. \square

Conclusion 5.12. *T_{rg} is minimum among the (complete, countable) unstable theories in \leq_1^* and does not belong to the maximum class.*

Proof. By Lemma 4.1 and Theorem 5.3. \square

6. T_{feq} IS MINIMAL AMONG THE NON-SIMPLE THEORIES

Theorem 6.13. T_{feq} is \leq_1^* -minimum among the complete countable non-simple theories.

Proof. Let T_1 be a complete, countable, non-simple first order theory. Without loss of generality, T_1 has TP_2 and not SOP_2 , as SOP_2 is already sufficient for maximality, Fact 2.19 above. We'll build on the proof of Lemma 4.1.

Let $\varphi(\bar{x}, \bar{y})$ be a formula which has TP_2 for T_1 . (In what follows, we'll write as if $\ell(x) = \ell(y) = 1$, but this is again only for simplicity of notation.) That is, in some model of T_1 , there is an array $\{a_{i,j} : i < \omega, j < \omega\}$ of elements of M (i.e. of ${}^{\ell(y)}M$) such that for any $X \subseteq \omega \times \omega$, $\{\varphi(x, a_{i,j}) : (i,j) \in X\}$ is consistent if and only if $(i,j) \in X \wedge (i',j) \in X \implies i = i'$, i.e. X does not contain more than one element from each column.

We'll assume that $T_1, T_{\text{rg}}, T_{\text{feq}}$, and $Th(\mathbb{N}, <)$ have disjoint signatures.

Let M be a countable model whose theory T satisfies:

- (a) M expands $(\mathbb{N}, <)$.
- (b) $M \upharpoonright \tau(T_{\text{feq}})$ is a countable model of T_{feq} .

That is, there is a unary predicate P^M which is countably infinite. P^M is partitioned into two infinite sets, P_0^M and P_1^M . On P_1^M , there is an equivalence relation E_0^M which has infinitely many classes, all of which are infinite. Finally, there is a function $F_0^M : P_0^M \times P_1^M \rightarrow P_1^M$ which essentially chooses, for each a in the set P_0 , a path through the equivalence classes. More formally, for each $(a,b) \in P_0^M \times P_1^M$, $E_0^M(F_0^M(a,b), b)$ and for any finitely many $b_1, \dots, b_n, b'_1, \dots, b'_m$ from P_1^M which are pairwise E_0^M -inequivalent, there is $a \in P_0^M$ such that

$$M \models \bigwedge_{1 \leq i \leq n} F_0(a, b_i) = b_i \wedge \bigwedge_{1 \leq j \leq m} \neg F_0(a, b'_j) = b'_j.$$

- (c) $M \upharpoonright \tau(T_1)$ is a countable model of T_1 , containing a sequence $\langle b_i : i < \omega \rangle$ satisfying the conclusion of Claim 4.2.

The domain of this model is Q^M . S^M is a binary relation with $T \vdash S \subseteq Q \times Q$ and $T \vdash$ “ S has TP_2 ”, that is, there is an array $\{a_{i,j} : i < \omega, j < \omega\}$ in Q^M as described above. (As before, we just let S name φ . Note that the sequence \bar{b} will not necessarily be a sequence on which φ , or S , has TP_2 ; we simply need it to be a sequence on which φ , so S , has the independence property, which is fine as TP_2 implies IP.)

- (d) F^M is an injective function from \mathbb{N} into Q^M such that for every $a \in Q^M$, for some truth value \mathbf{t} , for every n large enough, $M \models \varphi[a, F^M(n)]^{\mathbf{t}}$. (It suffices to let $F^M(n) = b_n$ where $\langle b_n : n < \omega \rangle$ is from Claim 4.2.)
- (e) G^M is a one to one and onto function from P_1^M to $\{a_{i,j} : i < \omega, j < \omega\} \subseteq Q^M$ which respects consistency and inconsistency in the natural way, i.e. such that for $b, b' \in P_1^M$, $\{S^M(x, G^M(b)), S^M(x, G^M(b'))\}$ is inconsistent. (Note that T will record that for all finite k , if $b_1, \dots, b_k \in P_1^M$ then

$$\{S(x, G(b_1)), \dots, S(x, G(b_k))\}$$

is consistent in M iff the b_i are pairwise E_0^M -inequivalent.)

- (f) G_*^M is a one to one and onto function from P_1^M to \mathbb{N}^M . (Note that T will record that for every $n \in \mathbb{N}^M$, for every definable subset $U \subseteq n$, if $\{(G_*^{-1})^M(a) : a \in U\}$ are pairwise E_0^M -inequivalent, then the set of

$\{F_0(x, G^{-1}(a)) = a : a \in U\} \cup \{F_0(x, G^{-1}(b)) : b < n, b \notin U\}$ is consistent and moreover realized by an element of P^M . Note that definable means with parameters in $\mathcal{L}(\tau(T))$.

- (g) Finally, though we won't need to refer to the rest by name, for every instance of the word "infinite" in the above catalogue, add a new function symbol interpreted as a bijection between \mathbb{N} and the given infinite set. In the case of the equivalence relation, it will be a parametrized family of functions.

Now let N be any model of T . It will suffice to prove that if $N \upharpoonright \tau(T_1)$, or really just $N \upharpoonright \{Q, S\}$, is μ -saturated, then the following are true. Since our theory simply expands that described in the proof of Lemma 4.1 (in the case, say, where $T_1 = T_{\text{rg}}$) we have by the same proof that (1) and (2) where:

- (1) \mathbb{N}^N has cofinality $\geq \mu$.

Note: this is the only place we use the sequence from Claim 4.2.

- (2) N satisfies $< \mu$ -regularity, meaning that every set of $< \mu$ elements is contained in some pseudofinite set.

- (3) $N \upharpoonright \tau(T_{\text{feq}})$ is μ -saturated.

Suppose P^M is not μ -saturated. In the most interesting case, there is an omitted 1-type p of cardinality $< \mu$ of the form:

$$\{(F_0(x, b_\alpha) = b_\alpha)^{\text{if } \eta(\alpha)} : \alpha < \alpha_* < \mu\}$$

for some $\eta \in {}^{\alpha_*}2$ and $\langle b_\alpha : \alpha < \alpha_* \rangle$ pairwise E_0^N -inequivalent. Invoking the bijection G_*^N from item (f) from Q^N onto \mathbb{N}^N , we know that by item (1), the image of $\{a_\alpha : \alpha < \alpha_*\}$ is bounded in \mathbb{N}^N , say by b_* .

As before we translate to a type in T_1 . Let $a_\alpha = G^N(b_\alpha)$ for $\alpha < \alpha_*$. Then $\{a_\alpha : \alpha < \alpha_*\}$ is a subset of Q^N , and $\{S(x, a_\alpha)^{\text{if } \eta(\alpha)} : \alpha < \alpha_*\}$ is consistent by our definition of T . By our assumption that $N \upharpoonright \{Q, S\}$ is saturated, this type is realized, say by d .

And just as before, we have that $U = \{b \in P_1^N : G_*^N(b) < b_*, S^N(d, G^N(b))\}$ is a first-order definable subset of N (with parameters). By our choice of T , as explained in the comment to item (f), this is enough to show p is realized in P^M , which proves (3). This completes the proof of the theorem. \square

Corollary 6.14. T_{feq} is $\leq_{\aleph_1}^*$ -minimum among the complete countable non-simple theories.

7. NON-SIMPLE THEORIES ARE NOT BELOW SIMPLE THEORIES

In this section we extend methods from the previous sections to prove Theorem 7.1, which says that non-simple theories are not below simple theories in the interpretability ordering \leq^* . The relevant precedent in the literature is the main theorem of [11], Theorem 8.2, which shows that assuming existence of an uncountable supercompact cardinal σ , there exist regular ultrafilters which saturate all simple theories of size $< \sigma$ and do not saturate any non-simple theories. This implies that under a large cardinal hypothesis, $\neg(T_1 \leq_{\aleph_1}^* T_0)$. The proof we give here is in ZFC, so is a strict improvement on this quotation.

Theorem 7.1. *Let T_0 be any simple theory and T_1 any non-simple theory. Then*

$$\neg(T_1 \trianglelefteq_1^* T_0).$$

We will prove the theorem at the end of the section, after several intermediate lemmas. As in §4, we may, without loss of generality, assume the T_* we are considering has Skolem functions.

Observation 7.2. *Since any theory which is not simple has either TP_2 or SOP_2 , and any theory with SOP_2 is \trianglelefteq^* maximal by 2.19, it will suffice to prove the theorem in the case where T_1 has TP_2 . In fact, by Theorem 6.13 above, it would suffice to prove the theorem in the case where $T_1 = T_{\text{feq}}$.*

Here are our hypotheses for the section.

Hypothesis 7.3.

- (1) T_0 is a simple theory. (We can use many such theories simultaneously.)
- (2) T_1 is a non-simple theory with TP_2 .
- (3) $\kappa = \text{cof}(\kappa) \geq \kappa(T_0)$.
- (4) $\kappa \leq \mu, \lambda = \lambda^{<\mu}$.
- (5) T_* is a theory which interprets both T_0 and T_1 , i.e. a potential candidate for showing $T_0 \trianglelefteq^* T_1$. We assume (1.5 above) T_* has Skolem functions.
- (6) F_* is a binary function symbol of $\tau(T_*) \setminus \tau(T_0) \setminus \tau(T_1)$ and there is an identification between some formula of T_1 with TP_2 and the graph of F in the sense that:
 - (a) for any $M \models T_0, M \models$ “ F_* is a 2-place function such that any finite function is represented by some $F(-, a)$.”
 - (b) if $M \upharpoonright \{F_*\}$ omits a type, then $M \upharpoonright \tau(T_1)$ omits one of the same size.
- (7) $\mathcal{K} = \mathcal{K}_\kappa^{\text{tr}}$, the class of normal trees with κ levels, lexicographic order, tree order and predicates for levels, see 3.17 above.
- (8) Υ is the class of templates proper for \mathcal{K} which satisfy our global hypotheses 3.7 and also satisfy: for every $I \in \mathcal{K}$ and $\Phi \in \Upsilon, \text{GEM}(I, \Phi) \models T_*$.
- (9) $\leq = \leq_\Upsilon$ is the natural order on this class, as in 3.13.

Observation 7.4. Υ is closed under unions of increasing sequences of length $\leq \lambda$ (and more but this is all we need here).

A comment on strategy: To prove the saturation direction in the case of the random graph, we began with a type in $\text{GEM}(I, \Phi)$, moved to a larger $\text{GEM}(J, \Phi)$ where we took many copies of the original type in an appropriate sense, and showed that the union of these many copies was itself a type, which we could realize in an elementary extension and so apply the Ramsey property. Our idea here will be similar, but the proof will have several stages. First, in 7.5, we’ll show that we may increase the given template Φ to Ψ to ensure types have a finite satisfiability property. Note that Claim 7.5(2) tells us types in $\text{GEM}(\dots, \Phi)$ are finitely satisfiable in $\text{GEM}(\dots, \Psi)$; later, a simple induction in Claim 7.8 allows us to “catch our tail” and use the same template in both halves of the statement. The saturation argument is carried out in Lemma 7.9, and essentially comes down to showing that if we take a type over (the Skolem hull of those parts of the skeleton whose indices lie in) a single branch and look at many copies of such a type, their union is consistent. In that proof the independence theorem plays a key role, and its use will be justified by the finite satisfiability ensured by 7.5 and 7.8.

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Claim 7.5. *For every $\Phi \in \Upsilon$, there is $\Psi \in \Upsilon$ such that:*

- (1) $\Phi \leq \Psi$
- (2) *for every standard $I \in K$ and $\eta \in I$ of level $i < \kappa$, every type of $\tau(T)$ which $\text{GEM}(I^{\geq \eta} \cup I^{\leq \eta}, \Phi)$ realizes over $\text{GEM}(I^{\perp \eta} \cup I^{\leq \eta}, \Phi)$ inside $\text{GEM}(I, \Phi)$ is finitely satisfiable in $\text{GEM}(I^{\leq \eta}, \Psi)$ where:*

$$I^{\perp \eta} = \{\eta \in I : \neg(\eta \leq \nu)\},$$

$$I^{\geq \eta} = \{\nu \in I : \eta \leq \nu\},$$

$$I^{\leq \eta} = \{\nu \in I : \nu \leq \eta\}.$$

Proof. To begin, let's carefully choose $I_0, I_1 \in \mathcal{K}$. Towards this, fix J_0 to be any infinite \aleph_0 -saturated linear order. Let J_1 be the linear order given by $J_0 \times \mathbb{Q}$, with the usual (lexicographic) order.

Let $I_1 \in \mathcal{K}$ be the index model whose domain is ${}^{\kappa} J_1$. Then I_1 is a tree of sequences [of pairs, though we can't refer to the pairing in $\tau(\mathcal{K})$] with predicates P_i naming level i for $i < \kappa$, the tree order \leq , and the lexicographic order $<_{\text{lex}}$, i.e. lexicographic order on the tree. Let $I_0 = {}^{\kappa} (J_0 \times \{0\}) \subseteq I_1$, the sequences of pairs with second coordinate constantly 0. Let $M_1 = \text{GEM}(I_1, \Phi)$ with skeleton $\mathbf{a} = \{\bar{a}_\eta : \eta \in I_1\}$ and let $M_0 = \text{GEM}(I_0, \Phi)$ with skeleton $\mathbf{a} \upharpoonright I_0$.

This construction accomplishes:

- (a) I_0 is \aleph_0 -saturated for \mathcal{K} (so later we may apply 3.11).
- (b) $M_0 = \text{GEM}(I_0, \Phi) \preceq M_1 = \text{GEM}(I_1, \Phi)$, immediate by $I_0 \subseteq I_1$, see 3.7.
- (c) M_1 acts like a larger saturated model around M_0 in a sense we now explain.

Working in M_1 , let's "pad" M_0 by building in witnesses to finite satisfiability, as follows. Define a new set of function symbols

$$(1) \quad \mathcal{F} = \{F_{i+1,j,\nu} : i+1 < j < \kappa, \nu \in {}^{[i+1,j]}(\mathbb{Q} \setminus \{0\})\}.$$

Let M_1^+ be M_1 expanded to a model of $\tau(\Phi) \cup \mathcal{F}$ in the following way.¹⁰ For every $i+1 < j < \kappa$, and for every $\nu \in {}^{[i+1,j]}(\mathbb{Q} \setminus \{0\})$, expand M_1 by defining $F_{i+1,j,\nu}$ to be the function with domain $\{\bar{a}_\eta : P_{i+1}(\eta)\} = \{\bar{a}_\eta : \eta \in I_0, \ell(\eta) = i+1\}$ such that $F_{i+1,j,\nu}(\bar{a}_\eta) = \bar{a}_\rho$ when $\eta \triangleleft \rho \in {}^j(J_1)$ and $(\forall j)(i+1 \leq i' < j \implies \rho(i') = (t, \nu(i')))$, where t is such that $\eta(i) = (t, 0)$.

Let M^* be the submodel of M_1^+ generated by $\{\bar{a}_\eta : \eta \in I_0\}$. We now argue that M^* has the following key property.

Subclaim 7.6. *For every quantifier free formula $\varphi(\bar{x}_0, \dots, \bar{x}_{k-1}, \bar{y}_0, \dots, \bar{y}_{m-1})$ of $\tau(\Phi)$, every $\eta \in I_0$, every $\eta_0, \dots, \eta_{m-1} \in I_0^{\perp \eta} \cup I_0^{\leq \eta}$ and every $\eta_0^*, \dots, \eta_{k-1}^* \in I_0^{\geq \eta} \cup I_0^{\leq \eta}$ there exist function symbols $F_0, \dots, F_{k-1} \in \mathcal{F}$ such that*

$$\text{if } M^* \models \varphi[\bar{a}_{\eta_0^*}, \dots, \bar{a}_{\eta_{k-1}^*}, \bar{a}_{\eta_0}, \dots, \bar{a}_{\eta_{m-1}}]$$

$$\text{then } M^* \models \varphi[F_0(\bar{a}_\eta), \dots, F_{k-1}(\bar{a}_\eta), \bar{a}_{\eta_0}, \dots, \bar{a}_{\eta_{m-1}}].$$

Moreover, the choice of functions is an invariant of the set of types

$$\{\text{tp}_{\text{qf}}(\langle \eta, \eta_0, \dots, \eta_{m-1} \rangle, \emptyset, I_0), \text{tp}_{\text{qf}}(\langle \eta, \eta_0, \dots, \eta_{m-1}, \eta_0^*, \dots, \eta_{k-1}^* \rangle, \emptyset, I_1)\}.$$

¹⁰Informally, for every $\eta \in I_0$ of successor length $i+1$, and every given sequence ν of j additional non-zero rationals, the function $F_{i+1,j,\nu}$ sends \bar{a}_η to \bar{a}_ρ where ρ is obtained by concatenating onto η a sequence of j additional elements whose first coordinate just repeats the last first coordinate of η and whose second coordinates are those given by ν . The reason to use $i+1$ is to have a last first coordinate to repeat.

Proof of Subclaim 7.6. We unwind the definitions. As M^* is a submodel of M_1^+ and φ is quantifier free in $\tau(\Phi)$,

$$\begin{aligned} M^* &\models \varphi[\bar{a}_{\eta_0^*}, \dots, \bar{a}_{\eta_{k-1}^*}, \bar{a}_{\eta_0}, \dots, \bar{a}_{\eta_{m-1}}] \\ &\iff M_1^+ \models \varphi[\bar{a}_{\eta_0^*}, \dots, \bar{a}_{\eta_{k-1}^*}, \bar{a}_{\eta_0}, \dots, \bar{a}_{\eta_{m-1}}]. \end{aligned}$$

As for the elements in the index model, quantifier free type depends only on level, tree-order, and lexicographic order, for each $\ell < k$ we may find $i_\ell, j_\ell, \nu_\ell, \rho_\ell$ such that first, $i_\ell + 1 < j_\ell < \kappa$ and $\nu_\ell \in {}^{[i_\ell+1, j_\ell]}(\mathbb{Q} \setminus \{0\})$, second, $F_{i_\ell+1, j_\ell, \nu_\ell}^{M_1^+}(\bar{a}_\eta) = \bar{a}_\rho$ for $\ell < k$, and third,¹¹

$$(2) \quad \begin{aligned} \text{tp}_{\text{qf}}(\eta, \eta_0, \dots, \eta_{m-1}, \eta_0^*, \dots, \eta_{k-1}^*, \emptyset, I_1) = \\ \text{tp}_{\text{qf}}(\eta, \eta_0, \dots, \eta_{m-1}, \rho_0, \dots, \rho_{k-1}, \emptyset, I_1). \end{aligned}$$

Now by definition of GEM model, since the skeleton is quantifier-free indiscernible,

$$\begin{aligned} M_1^+ &\models \varphi[\bar{a}_{\eta_0^*}, \dots, \bar{a}_{\eta_{k-1}^*}, \bar{a}_{\eta_0}, \dots, \bar{a}_{\eta_{m-1}}] \\ &\iff M_1^+ \models \varphi[\bar{a}_{\rho_0}, \dots, \bar{a}_{\rho_{k-1}}, \bar{a}_{\eta_0}, \dots, \bar{a}_{\eta_{m-1}}]. \end{aligned}$$

By our choice of ρ_ℓ , the last equation above holds if and only if

$$(3) \quad M_1^+ \models \varphi[F_{i_0+1, j_0, \nu_0}(\bar{a}_\eta), \dots, F_{i_{k-1}+1, j_{k-1}, \nu_{k-1}}(\bar{a}_\eta), \bar{a}_{\eta_0}, \dots, \bar{a}_{\eta_{m-1}}].$$

so recalling the definition of M^* and the fact that φ is quantifier free, (3) holds if and only if

$$(4) \quad M^* \models \varphi[F_{i_0+1, j_0, \nu_0}(\bar{a}_\eta), \dots, F_{i_{k-1}+1, j_{k-1}, \nu_{k-1}}(\bar{a}_\eta), \bar{a}_{\eta_0}, \dots, \bar{a}_{\eta_{m-1}}]$$

which proves the subclaim. *Proof of Subclaim 7.6.* \square

Before continuing, we record the following immediate corollary to the proof of Subclaim 7.6. We'll use x 's and y 's for arbitrary elements of $\tau(T)$ -models and s 's and t 's and v 's for arbitrary elements of index models.

Subclaim 7.7. *Let $\varphi(\bar{x}_0, \dots, \bar{x}_{k-1}, \bar{y}_0, \dots, \bar{y}_{m-1})$ be a quantifier free formula of $\tau(\Phi)$. Suppose $\mathfrak{r}(t, t_0, \dots, t_{k-1}, s_0, \dots, s_{m-1}) \in D_{\text{qf}}(I_0)$ is a type which satisfies $\mathfrak{r} \vdash 't \triangleleft t_\ell \vee t_\ell \trianglelefteq t'$ for each $\ell < k$ and $\mathfrak{r} \vdash 's_i \perp t \vee s_i \trianglelefteq t'$ for each $i < m$. Then there exist functions $F_0, \dots, F_{k-1} \in \mathcal{F}$ such that the formula $\psi = \psi_{\mathfrak{r}}(x, x_0, \dots, x_{k-1}, y_0, \dots, y_{m-1})$ given by*

$$\varphi(x_0, \dots, x_{k-1}, y_0, \dots, y_{m-1}) \implies \varphi(F_0(x), \dots, F_{k-1}(x), y_0, \dots, y_{m-1})$$

belongs to $\text{tp}_{\text{qf}}(\bar{a}_{\bar{v}}, \emptyset, M_1^+)$ for any \bar{v} from I_0 realizing \mathfrak{r} .

We are ready to find Ψ . Expand M^* to a model M^{**} whose theory has Skolem functions. By the Ramsey property 3.11 applied with I_0, M_0 and $\mathbf{a} \upharpoonright I_0$, and M^{**} here for J, M and \mathbf{a} , and N^+ there, there exists a template $\Psi \geq \Phi$ which is proper for I_0 and which has the property that for each \mathfrak{r} satisfying the hypothesis of Subclaim 7.7, the formula $\psi_{\mathfrak{r}}$ from that Subclaim belongs to $\Psi(\mathfrak{r})$.

Let us verify that Ψ satisfies the property of the claim. Let $I \in \mathcal{K}$ be any standard index model. Let $N = \text{GEM}(I, \Psi)$. Let a quantifier-free formula $\theta(\bar{x}, \bar{y})$

¹¹Note that in (2) the η_i 's are elements of $I_0 \subseteq I_1$ while the ρ_ℓ 's are just elements of I_1 , the index model for M_1 . Elements of the form \bar{a}_{ρ_ℓ} belong to the skeleton of M_1 , and a fortiori to the expanded model M_1^+ . These elements *also* belong to the smaller model M^+ by virtue of being equal to $F_{i_\ell+1, j_\ell, \nu_\ell}^{M_1^+}(\bar{a}_\eta)$. However, it would be misleading to say " $\bar{a}_{\rho_\ell} \in M^+$ " because the notation would suggest it is an element of the skeleton, which it is not since $\rho_\ell \notin I_0$.

of $\mathcal{L}(\tau(T))$ be given; this will suffice for the claim as T_Φ has Skolem functions. Note that by definition of \leq_Υ , $\text{GEM}_{\tau(T)}(I, \Phi) \preceq \text{GEM}_{\tau(T)}(I, \Psi)$. Suppose $N \models \theta[\bar{b}, \bar{c}]$ where for some $\eta \in I$, \bar{b} is a finite sequence of elements of $\text{GEM}(I^{\geq \eta} \cup I^{\leq \eta}, \Phi)$ and \bar{c} is a finite sequence of elements of $\text{GEM}(I^{\perp \eta} \cup I^{\leq \eta}, \Phi)$. We would like to find \bar{b}' from $\text{GEM}(I^{\leq \eta}, \Psi)$ such that $N \models \theta[\bar{b}', \bar{c}]$. By definition of GEM-model, there are elements $\eta_0^*, \dots, \eta_{k-1}^* \in I^{\geq \eta} \cup I^{\leq \eta}$ and $\tau(\Phi)$ -terms $\sigma_0^*, \dots, \sigma_{\ell-1}^*$ such that

$$\langle \sigma_0^*(\bar{a}_{\eta_0^*}, \dots, \bar{a}_{\eta_{m-1}^*}), \dots, \sigma_{j-1}^*(\bar{a}_{\eta_0^*}, \dots, \bar{a}_{\eta_{m-1}^*}) \rangle = \bar{b}$$

and also elements $\eta_0, \dots, \eta_{m-1} \in I^{\perp \eta} \cup I^{\leq \eta}$ and $\tau(\Phi)$ -terms $\sigma_0, \dots, \sigma_{j-1}$ such that

$$\langle \sigma_0(\bar{a}_{\eta_0}, \dots, \bar{a}_{\eta_{k-1}}), \dots, \sigma_{\ell-1}(\bar{a}_{\eta_0}, \dots, \bar{a}_{\eta_{k-1}}) \rangle = \bar{c}.$$

Let $\varphi(x_0, \dots, x_{k-1}, y_0, \dots, y_{m-1})$ be the quantifier-free formula equivalent to $\theta(\sigma_0^*(x_0, \dots, x_{k-1}), \dots, \sigma_{\ell-1}^*(x_0, \dots, x_{k-1}), \sigma_0(y_0, \dots, y_{m-1}), \dots, \sigma_{j-1}(y_0, \dots, y_{m-1}))$. By construction it is still a $\tau(\Phi)$ -formula. Let

$$\mathfrak{r} = \text{tp}_{\text{qf}}(\eta \hat{\ } \eta_0 \hat{\ } \dots \hat{\ } \eta_m \hat{\ } \eta_0^* \hat{\ } \dots \hat{\ } \eta_{k-1}^*, \emptyset, I).$$

Recall I_0 and M^* from earlier in the proof. Because I_0 was \aleph_0 -saturated, there is some sequence $\bar{\eta}$ of elements of I_0 realizing \mathfrak{r} . Because $\Psi \geq \Phi$, $M_1 = \text{GEM}(I_1, \Phi) \preceq \text{GEM}(I_1, \Psi)$ and recall that M^* is a submodel of M_1^+ , so a fortiori $M^* \upharpoonright_{\tau(\Phi)} \subseteq M_1$. As φ is a quantifier-free $\tau(\Phi)$ -formula, it must be that $M^* \models \varphi[\bar{a}_{\bar{\eta}}]$. Apply Subclaim 7.7 to finish the proof. (Note: we've written finitely satisfiable in " $I^{\leq \eta}$," but we've used " I^η .")

Proof of Claim 7.5. □

Corollary 7.8. *Let I be standard. For every $\Phi \in \Upsilon$, there is $\Psi \in \Upsilon$ with $\Phi \leq \Psi$ such that every type of $\tau(T)$ which $\text{GEM}(I^{\geq \eta} \cup I^{\leq \eta}, \Psi)$ realizes over $\text{GEM}(I^{\perp \eta} \cup I^{\leq \eta}, \Psi)$ inside $\text{GEM}(I, \Psi)$ is finitely satisfiable in $\text{GEM}(I^{\leq \eta}, \Psi)$.*

Proof. Let $\Phi_0 = \Phi$. Choose Φ_n by induction on $1 \leq n < \omega$ to be the result of applying Claim 7.5 with $\Phi = \Phi_{n-1}$. Then $\Psi = \bigcup_n \Phi_n$ is the desired template, and $\Phi \leq \Psi \in \Upsilon$ recalling Observation 7.4. □

Lemma 7.9. *Let I be standard with universe ${}^{\kappa >} \{0\}$. For every $\Phi \in \Upsilon$, there is $\Psi \in \Upsilon$ with $\Phi \leq \Psi$ such that $M = \text{GEM}_{\tau(T)}(I, \Psi)$ is μ -saturated.*

Proof. Let I and Φ be given. Without loss of generality Φ satisfies the conclusion of Corollary 7.8. Let $M = \text{GEM}_{\tau(T)}(I, \Phi)$. It will suffice to show that if $p \in \mathbf{S}(M)$ is a type over a set of size $< \mu$ then¹² we can find $\Psi \geq \Phi$ such that p is realized in $\text{GEM}(I, \Psi)$. We can then iterate to obtain the template producing a μ -saturated model just as in Claim 5.10.

The first use of simplicity will be not forking over a small set. For $i < \kappa$, let $M_i = \text{GEM}_{\tau(T)}(\{\eta \in I : \text{lgn}(\eta) < i\}, \Phi)$, so the sequence $\langle M_i : i \leq \kappa \rangle$ is \preceq -increasing continuous and its union $M_\kappa = M$. As T is simple and complete and $\kappa \geq \kappa(T)$, there is $i_* < \kappa$ such that p dnf over M_{i_*} . For simplicity, we may assume i_* is a successor.

Towards finding Ψ , we move to work in a saturated index model. Let χ be infinite so $J = {}^{\kappa >} \chi$ is \aleph_0 -saturated. Let $N = \text{GEM}_{\tau(T)}(J, \Phi)$, so $M \preceq N$. Let \mathbf{a} denote the skeleton of N , extending that of M . For every $\eta \in {}^{\kappa} \chi$ let h_η be the canonical isomorphism from I [recalling it is a single branch] onto $J_\eta = J \upharpoonright \{\eta \upharpoonright i : i < \kappa\}$.

¹²We won't really use the size of p when realizing a single type, but just as in 5.10, it's important to keep track of size when iterating to produce saturation.

Let \hat{h}_η be the induced isomorphism from M to $N_\eta = \text{GEM}_{\tau(T)}(J_\eta, \Phi)$ and let $p_\eta = \hat{h}_\eta(p)$.

We may likewise write these models as unions of chains: let $N_{\eta,i} = \text{GEM}_{\tau(T)}(\{ \eta \in J_\eta : \text{lgn}(\eta) < i \}, \Phi)$, for each $i < \kappa$. It remains true for each η that $p_\eta \in \mathbf{S}(N_\eta)$ dnf over N_{η,i^*} . We arrive to the second use of simplicity, the independence theorem.

Subclaim 7.10. *If $\nu \in {}^{i^*}\chi$ then*

$$q_\nu = \bigcup \{ p_\eta : \eta \in {}^\kappa\chi, \nu \trianglelefteq \chi \}$$

is a partial type which dnf over $N_{\trianglelefteq\nu} = \text{GEM}_{\tau(T)}(J \upharpoonright \{ \rho \trianglelefteq \nu \}, \Phi)$.

Proof. It suffices to consider some finite $\Lambda \subseteq {}^\kappa\lambda$ and prove $q_\Lambda = \bigcup \{ p_\eta : \eta \in \Lambda \}$ dnf over $N_{\trianglelefteq\nu}$. We prove this by induction on $|\Lambda|$. If $|\Lambda| = 1$ this is immediate since each p_η is a type which dnf over $N_{\trianglelefteq\nu}$. So assume $|\Lambda| = n + 1 \geq 2$. Let η_0, \dots, η_n list Λ in lexicographically increasing order. Let $\rho_0 = \eta_{n-1} \cap \eta_n$, and let $\rho = \eta_{n-1} \upharpoonright \text{lgn}(\rho_0) + 1$.

Let $q_* = \bigcup \{ q_\ell : \ell \leq n - 1 \}$, which by inductive hypothesis is a partial type which dnf over $N_{\trianglelefteq\nu}$. Let q_{**} be a complete nonforking extension of q_* to $B = \bigcup \{ N_{\trianglelefteq\eta_\ell} : \ell \leq n - 1 \}$. That is, $q_{**} \in \mathbf{S}(B)$ dnf over $N_{\trianglelefteq\nu}$, so a fortiori dnf over $N_{\trianglelefteq\rho}$.

We have already defined B . For clarity, let $A = N_{\trianglelefteq\rho}$, and let $C = N_{\trianglelefteq\eta_n}$. So $q_n \in \mathbf{S}(C)$ dnf over $N_{\trianglelefteq\nu}$, so a fortiori dnf over A .

Let's first prove that $q_{**} \cup q_n$ is consistent and dnf over A . We have that $A = B \cap C$, and $q_{**} \in \mathbf{S}(B)$ dnf over A , $q_n \in \mathbf{S}(C)$ dnf over A , and $q_{**} \upharpoonright A = q_n \upharpoonright A$ (because they agree on any common initial segment). In order to apply the independence theorem, we need to know B is free from C over A . A, B, C are universes of models of T_0 and by Claim 7.5, $tp(C, B)$ is finitely satisfiable in A , which suffices.

We conclude that $Q = q_{**} \cup q_n$ is a consistent partial type which dnf over $A \supseteq N_{\trianglelefteq\nu}$. Recalling the definition of A , $Q \upharpoonright A$ is a type which dnf over $N_{\trianglelefteq\nu}$ because it is just one of the images of p under one of the automorphisms h . So by transitivity of nonforking for simple theories, Q dnf over $N_{\trianglelefteq\nu}$, and this proves the subclaim. *Proof of Subclaim 7.7.* \square

To complete the proof of Lemma 7.9, let N_* be a sufficiently saturated elementary extension of N (so, also a $\tau(\Phi)$ -model) in which for each $\nu \in {}^{i^*}\chi$ the type q_ν is realized by some b_ν . Add to $\tau(\Phi)$ a new unary function symbol F_{i^*} . Expand N_* to N_*^+ by interpreting F_{i^*} so that $\nu \in {}^{i(*)}\chi$ implies $F_{i^*}^{N_*^+}(a_\nu) = b_\nu$, where a_ν belongs to the skeleton of $N \preceq N_*$. In this language, note that what the subclaim has really shown is that for any finite sequence $\bar{\eta}$ from a single branch of $I^{\geq\nu}$ and any formula $\varphi(x, \bar{a}_{\bar{\eta}})$ in the given type p , whether or not $N_*^+ \models \varphi[F_{i^*}(a_\nu), \bar{a}_{\bar{\eta}}]$ is a property of the quantifier-free type of $\bar{\eta}$. Apply the Ramsey property 3.11, with $J, \text{GEM}(J, \Phi)$ and \mathbf{a}, N_*^+ here for J, M and \mathbf{a}, N^+ there, to obtain a template $\Psi \geq \Phi$ proper for \mathcal{K} . By construction, the template Ψ will have registered from f the correct instructions (definition) to ensure realization. In particular, in the model $\text{GEM}(I, \Psi)$, for $\nu = {}^{i^*}\{0\}$, we have that $F_{i^*}(a_\nu)$ will realize p . *Proof of Lemma 7.9.* \square

Claim 7.11. *Let I be standard with universe ${}^{\kappa>}\{0\}$. For any $\Phi \in \Upsilon$, the model $\text{GEM}_{\{F_{i^*}\}}(I, \Phi)$ is not κ^+ -saturated. More precisely, it omits some partial φ -type of cardinality κ , where $\varphi = \varphi(x, \bar{y}) = (F_*(y_0, x) = y_1)$.*

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Proof. Let $\eta_i \in {}^i\{0\}$, and let

$$p(x) = \{F(a_{\eta_{2i}}, x) = a_{\eta_{2i+1}} : i < \kappa\}$$

be the type of a code for a function which acts as a “successor” operation on even elements in this branch of of the skeleton. Towards contradiction assume $c \in M = \text{GEM}(I, \Phi)$ realizes p . So there is a $\tau(\Phi)$ -term $\sigma(t_0, \dots, t_{n-1})$ and $i_0 < \dots < i_{n-1} < \kappa$ such that

$$M \models “c = \sigma(a_{\eta_{i_0}}, \dots, a_{\eta_{i_{n-1}}}).”$$

Let J be ${}^{\kappa>}\chi$, so J is standard and extends I . Let $N = \text{GEM}(J, \Phi)$. Recalling that the predicates P_k name elements of level k , let $\nu \in P_{2i_{n-1}+4}^J$ be such that $\nu \upharpoonright 2i_{n-1} + 2 = \eta_{2i_{n-1}+2}$, but $\nu \neq \eta_{2i_{n-1}+3}$. By the choice of c ,

$$N \models “F_*(a_{\eta_{2i_{n-1}+2}}) = a_{\eta_{2i_{n-1}+3}}”$$

but then by indiscernibility we must also have

$$N \models “F_*(a_{\eta_{2i_{n-1}+2}}) = a_\nu”$$

contradicting $a_{\eta_{2i_{n-1}+3}} \neq a_\nu$. \square

Corollary 7.12. *Let I be standard with universe ${}^{\kappa>}\{0\}$ and let T_1 be the the non-simple theory fixed at the beginning of the section. If T_1 has TP_2 , then for any $\Phi \in \Upsilon$, $\text{GEM}_{\tau(T_1)}(I, \Phi)$ is not κ^+ -saturated.*

Proof. By our hypothesis 7.3 the theory represents F_* , so apply Claim 7.11. \square

Conclusion 7.13. *Let $I = {}^{\kappa>}\{0\}$. There is $\Phi \in \Upsilon$ such that writing $M = \text{GEM}_{\tau(T)}(I, \Phi)$ and $N = \text{GEM}_{\tau(F_*)}(I, \Phi)$ we have that M is μ -saturated but N is not κ^+ -saturated. Moreover, for this same Φ , if T_1 has TP_2 then $\text{GEM}_{\tau(T_1)}(I, \Phi)$ is not κ^+ -saturated.*

Proof. By Lemma 7.9, Claim 7.11, and Corollary 7.12. \square

Proof of Theorem 7.1. There are two cases. If T_1 has SOP_2 then by 2.19 it is already maximal under \leq_1^* . If T_1 has TP_2 apply Conclusion 7.13. \square

8. INCOMPARABILITY IN \leq AND \leq^*

In this section we prove that, under a set theoretic hypothesis, there are incomparable classes in Keisler’s order \leq and, consequently, in \leq_κ^* for $\kappa = 1$ or \aleph_1 .¹³

A comment on strategy: For any finite k , let $T_{k+1,k}$ be the generic $(k+1)$ -ary hypergraph which forbids a complete hypergraph on $(k+2)$ -vertices. For any tuple of cardinals $(\lambda, \mu, \theta, \sigma)$ called “suitable,” so, written in decreasing order of size and satisfying some basic constraints, there exist a family of regular ultrafilters on λ which have two important features. First, holding σ constant, if the cardinal distance (in the alephs) of λ and μ is ℓ then the ultrafilter will saturate $T_{k+1,k}$ only if $k > \ell$. Second, holding μ constant and changing σ , we have that if $\sigma = \aleph_0$, no such ultrafilter will be flexible (so it will fail to saturate any non-low theory) whereas

¹³This proof dates to 2015 [14], but was not widely circulated. Earlier this year, D. Ulrich [20] closely read our papers [11] and [12] and noticed the same result. He hadn’t been aware of our manuscript, and his discovery was independent.

if $\sigma > \aleph_0$, the ultrafilter will indeed saturate the basic non-low simple theory. By combining these two results, incomparability naturally appears.

Context 8.14. *In this section we assume $(\lambda, \mu, \theta, \sigma)$ are suitable, meaning:*

- (1) $\sigma \leq \theta \leq \mu < \lambda$.
- (2) θ is regular, $\mu = \mu^{<\theta}$ and $\lambda = \lambda^{<\theta}$.
- (3) $(\forall \alpha < \theta)(2^{|\alpha|} < \mu)$.

and in addition we assume $\sigma = \aleph_0$ or σ is uncountable and supercompact.

We will use as a black box the so-called $(\lambda, \mu, \theta, \sigma)$ -optimized ultrafilters of [11]. These are built in the sense of [11] Definition 2.11 from certain quite strong (“optimal”) ultrafilters over free Boolean algebras. To simplify notation, when T is a theory and \mathcal{D} is a regular ultrafilter on λ , say that “ \mathcal{D} is good for T ” to mean that for some, equivalently every model $M \models T$, M^λ/\mathcal{D} is λ^+ -saturated.

Fact 8.15. *Suppose $\mu = \aleph_\alpha$ and $\lambda = \aleph_{\alpha+\ell}$ for α an ordinal and ℓ a nonzero integer. Let \mathcal{D} be a $(\lambda, \mu, \aleph_0, \aleph_0)$ -optimized ultrafilter on λ . Then for any $2 \leq k < \omega$:*

- (i) *If $k < \ell$, then \mathcal{D} is good for $T_{k+1,k}$.*
- (ii) *If T is any non-low or non-simple theory, then \mathcal{D} is not good for T .*

Proof. [11] Conclusion 9.6(b) gives the definition, and together with (a), the existence statement, for a $(\lambda, \mu, \aleph_0, \aleph_0)$ -optimized ultrafilter, and states that any such ultrafilter will satisfy (ii). [11] 9.6(a) tells us the same ultrafilter will be perfected. Quoting [12] Theorem 6.1 then gives (i).¹⁴ \square

Fact 8.16. *Suppose $\mu = \aleph_\alpha$ and $\lambda = \aleph_{\alpha+\ell}$ for α an ordinal and ℓ a nonzero integer. Let \mathcal{D} be a $(\lambda, \mu, \theta, \sigma)$ -optimized ultrafilter. If $\ell < k$, then \mathcal{D} is not good for $T_{k+1,k}$.*

Proof. The proof of the non-saturation condition, [12] Theorem 5.1, depends only on λ and μ , as explained in Remark 5.2 there; it will go through as written for any suitable tuple of cardinals. (Note to the reader: the point is that *any* regular ultrafilter, optimized or not, which is built from $(\mathcal{D}_0, \mathfrak{B}_{2^\lambda, \mu, \sigma}, \mathbf{j})$ in the sense of [11] 2.11, will not be good for $T_{k+1,k}$.) \square

Definition 8.17. *Let \mathcal{T}_* be the following “canonical simple non low theory.” \mathcal{T}_* is the theory of the existential closure of the following model. The domain of M is partitioned into P^M and Q^M , infinite disjoint unary predicates. E^M is an equivalence relation on Q^M with infinitely many classes. $R^M \subseteq P^M \times Q^M$ is a binary relation. Each element of P^M is connected, via R , to precisely n elements of the n -th equivalence class of E^M .*

Claim 8.18. *Let σ be uncountable and supercompact, and let $(\lambda, \mu, \theta, \sigma)$ be suitable. Then there is a $(\lambda, \mu, \theta, \sigma)$ -optimized ultrafilter \mathcal{D} on λ . Moreover \mathcal{D} is flexible and is good for the random graph, and thus, \mathcal{D} is good for \mathcal{T}_* .*

Proof. The existence of the optimal ultrafilter is [11] Theorem 5.9 and Definition 5.14. Flexibility is [11] 5.16. \mathcal{D} is good for the random graph by [11] Theorem 7.3, because the random graph trivially satisfies the condition required in that Theorem of being explicitly simple, as noted in [11] Discussion 3.14.

Let $M \models \mathcal{T}_*$, $p \in \mathbf{S}(N)$ where $N \preceq M^I/\mathcal{D}$ and $\|N\| \leq \lambda$.

¹⁴Optimal and perfect were the relevant properties of the ultrafilters on the free Boolean algebra; regular ultrafilters built from these in the stated way are called optimized and perfected.

Let $\langle \varphi(x, a_\alpha^*) : \alpha < \lambda \rangle$ be an enumeration of p . In the main case we will assume each φ is of the form $R(x, a_\alpha^*)^{\mathbf{t}(\alpha)}$. As \mathcal{D}_* is good for the random graph, it will suffice to consider the case where each $\mathbf{t}(\alpha) = 1$. Let M_* be a countable model over which p does not fork. So M_* contains the prime model, and $p \upharpoonright M_*$ includes the data of which n elements p connects to in the n th class of E , for each finite n . Denote these by $\{a_{\langle i, n \rangle} : i < n, n < \omega\}$, where $\langle \cdot \rangle$ is some fixed coding function from $\omega \times \omega$ to ω , and assume that each $a_{\langle i, n \rangle}$ is a_α^* for $\alpha = \langle i, n \rangle < \omega$. Let $\langle \mathbf{b}_u : u \in [\lambda]^{<\sigma}$ be the continuous sequence given by

$$\mathbf{b}_u = \bigcap_{\alpha \in u, i < n < \omega, \bigwedge_{j < n} \alpha \neq \langle j, n \rangle} \mathbf{a}_{\neg(E(x_\alpha, x_{\langle i, n \rangle}))}.$$

With this sequence we may realize p . □

Definition 8.19. Let \mathcal{T}_n be the theory given by the disjoint union of $T_{k+1, k}$ for $k \geq n$. Let \mathcal{T}_n^* be as above.

Conclusion 8.20. Assuming existence of an uncountable supercompact cardinal, for $2 \leq k < \omega$, \mathcal{T}_k and \mathcal{T}^* are incomparable in Keisler's order.

Proof. First suppose that $\sigma = \theta = \aleph_0$, $\lambda \leq \mu^{+k}$. If \mathcal{D} is $(\lambda, \mu, \aleph_0, \aleph_0)$ -optimal, then \mathcal{D} is good for \mathcal{T}_k by Fact 8.15, but it isn't good for \mathcal{T}^* .

Next suppose that $\sigma = \theta$ is an uncountable supercompact cardinal but $\lambda = \mu^{+k+1}$. If \mathcal{D} is $(\lambda, \mu, \theta, \sigma)$ -optimal, then \mathcal{D} is good for \mathcal{T}^* but it is not good for $\mathcal{T}_{k'}$ for any $k' \leq k$. □

Corollary 8.21. Assuming existence of an uncountable supercompact cardinal, \mathcal{T}_k and \mathcal{T}^* are incomparable in \leq_1^* and $\leq_{\aleph_0}^*$.

9. DISCUSSION: WEAK DEFINABILITY OF TYPES

From our proofs of Theorems 5.3 and 7.1 one may extract the following principle.

Hypothesis 9.22. *Fix for this section:*

- a) a theory T .
- b) a class of index models \mathcal{K} satisfying the Ramsey property 3.11.
- c) a class Υ of templates Φ proper for \mathcal{K} satisfying 3.7, and with $\tau(\Phi) \supseteq T$ for each $\Phi \in \Upsilon$, recalling that 3.7 implies T_Φ is well defined and has Skolem functions.
- d) \leq the natural order on Υ .
- e) thus the set Terms of $\tau(\Phi)$ -terms.

Definition 9.23. *Suppose $I \in \mathcal{K}$, $\Phi \in \Upsilon$, $M = \text{GEM}(I, \Phi)$ with skeleton \mathbf{a} , Δ is a set of $\mathcal{L}(\tau_T)$ -formulas, p a partial type $p \subseteq q \in \mathbf{S}_\Delta(M)$. We may say p has a weak definition if there is a partial function*

$$F : \Delta \times {}^{\omega>}(\text{Terms}) \times D_{\text{qf}}(I) \rightarrow \{0, 1\}$$

such that for some \aleph_0 -saturated $J \in \mathcal{K}$, when evaluated in $N = \text{GEM}_{\tau(T)}(J, \Phi)$, the set of formulas

$$(5) \quad \begin{aligned} \{\varphi(\bar{x}, \bar{\sigma}(\bar{a}_{\bar{t}}))^\mathbf{t} : & \varphi \in \Delta, \bar{\sigma} \in {}^{\omega>}(\text{Terms}), \\ & \bar{t} \subseteq J, \text{tp}_{\text{qf}}(\bar{t}, \emptyset, J) = \mathbf{r} \\ & \text{and } \mathbf{t} = F(\varphi, \bar{\sigma}, \mathbf{r}) \in \{0, 1\}\} \end{aligned}$$

is a partial type which extends p .

Note that if \bar{t} , $\bar{\sigma}$ don't have the appropriate length or size for the given φ , the function F from may be undefined on the tuple $(\varphi, \bar{\sigma}, \mathbf{r})$; but order to meet the condition that (5) extends p , F will need to be defined on all of the tuples $(\varphi, \bar{\sigma}, \mathbf{r})$ which arise from p .¹⁵ So this condition does generalize e.g. (2) from the proof of Claim 5.9.

Remark 9.24. *Definition 9.23 can be extended to include weak definitions over some finite $\bar{t}^* \subseteq I$, but since this was not used in the present proofs, we defer this to the companion paper [15].*

Claim 9.25. *Suppose T_Φ has Skolem functions for T . If p has a definition over the empty set in M , a finite subset of M in the usual sense of stability theory, then p has a weak definition in the sense of Definition 9.23.*

Proof of 9.25. If p is definable over \emptyset , then for each $\varphi(\bar{x}, \bar{y}) \in \Delta$ there is a $\tau(T)$ -formula $d_\varphi(\bar{y})$ giving the definition. Fix $\bar{\sigma} \in {}^{\ell(\bar{y})}(\text{Terms})$ and consider any finite sequence $\bar{t} \in {}^{\omega>}I$ for which $\bar{\sigma}(\bar{a}_{\bar{t}})$ can be evaluated. Let $\mathbf{r} = \text{tp}_{\text{qf}}(\bar{t}, \emptyset, J)$. Since Φ is a template, for all \bar{s} with $\text{tp}_{\text{qf}}(\bar{s}, \emptyset, J) = \mathbf{r}$,

$$\text{tp}_{\text{qf}}(\bar{a}_{\bar{t}}, \emptyset, N) = \text{tp}_{\text{qf}}(\bar{a}_{\bar{s}}, \emptyset, N).$$

The assumption that $\tau(\Phi)$ has Skolem functions for T improves this to

$$\text{tp}(\bar{a}_{\bar{t}}, \emptyset, N) = \text{tp}(\bar{a}_{\bar{s}}, \emptyset, N).$$

¹⁵Moreover, if $\varphi(x_0, \dots, x_{m-1}, y_0, \dots, y_{n-1}) \in \Delta$, and $\bar{\sigma} = \langle \sigma_0, \dots, \sigma_{n-1} \rangle$ is a finite sequence from Terms , then without loss of generality (by adding dummy variables) we may assume these terms all have the same number r of free variables, and so if $\bar{a}_{\bar{t}}$ is from the skeleton and $\ell(\bar{a}_{\bar{t}}) = r$, we may write " $\varphi(\bar{x}, \bar{\sigma}(\bar{a}_{\bar{t}}))$ " for $\varphi(\bar{x}, \sigma_0(\bar{a}_{\bar{t}}), \dots, \sigma_{n-1}(\bar{a}_{\bar{t}}))$.

In particular,

$$N \models d_\varphi(\bar{\sigma}(\bar{a}_{\bar{t}})) \iff N \models d_\varphi(\bar{\sigma}(\bar{a}_{\bar{s}})).$$

Note Skolem functions are not needed for quantifier-free definitions. \square

Observation 9.26. *If p has a weak definition in the sense of Definition 9.23, this does not imply p has a definition in the usual sense of stability theory, even assuming Skolem functions for T .*

Proof. Existence of definitions is characteristic of stability; earlier sections built weak definitions for types in the random graph and in arbitrary simple theories, respectively. \square

We may summarize by noting that in each case, the contribution of weak definability was to prove a lemma of the following kind.

Meta-lemma 9.27. *Suppose $I \in \mathcal{K}$, $M = \text{GEM}_{\tau(T)}(I, \Phi)$ and*

$$p(x) = \{\varphi_\alpha(\bar{x}, \bar{\sigma}_\alpha^M(\bar{a}_{\bar{t}_\alpha})) : \alpha < \kappa\}$$

is a consistent partial type in M . If p has a weak definition, then for any \aleph_0 -saturated J with $I \subseteq J \in \mathcal{K}$, the set of formulas

$$q(x) = \{\varphi_\alpha(\bar{x}, \bar{\sigma}_\alpha^M(\bar{a}_{\bar{s}})) : \alpha < \kappa, \text{tp}_{\text{qf}}(\bar{s}, \emptyset, J) = \text{tp}_{\text{qf}}(\bar{t}_\alpha, \emptyset, I)\}$$

is a consistent partial type in N . So we may realize it in some elementary extension N' of N and name this realization by new constants \bar{c} , and applying 3.11, we may find $\Psi \geq \Phi$ such that $\bar{c} \subseteq \tau(\Psi)$ and p is realized by c in $N = \text{GEM}(I, \Psi)$.

Our proofs have suggested that an interesting means of comparing theories may be to find, in the setup of generalized Ehrenfeucht-Mostowski models, a class \mathcal{K} for which types in one theory T_0 have weak definitions, and those in another T_1 do not.

10. SOME OPEN PROBLEMS

We conclude with some open problems. The careful reader may also have noticed many natural questions which we have not addressed here, for example to extend Lemma 4.1 (and the analogous proof for \trianglelefteq) to show that all stable theories are \trianglelefteq^* -below all unstable theories.

Some problems which seem to us particularly fruitful are the following. Note that no equivalence classes of unstable theories under \trianglelefteq^* have been characterized in ZFC (though maximality uses only instances of GCH) and any result along these lines could potentially be very interesting.

Towards understanding \trianglelefteq^* on the simple unstable theories, for $\kappa = 1$ or $\kappa = \aleph_1$:

- (1) Characterize those theories which are \trianglelefteq_κ^* -equivalent to the theory of the random graph.
- (2) Are there infinitely many incomparable classes of simple unstable theories under \trianglelefteq_κ^* ?
- (3) Is it true that every simple theory is \trianglelefteq_κ^* -below every non-simple theory?

Towards understanding \trianglelefteq^* on the non-simple theories with $NSOP_2$:

- (4) Prove Fact 2.18 in ZFC, which would establish in ZFC that a theory is maximal in $\trianglelefteq_{\aleph_1}^*$ if and only if it is SOP_2 .

- (5) Characterize those theories which are \leq_{κ}^* -equivalent to T_{feq} .
- (6) Is there a property of non-simple theories, which is analogous in a natural sense to f.c.p. in stable theories and to non-lowness in simple unstable theories, and is detected as a division in \leq_{κ}^* ?

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, 5734 S. UNIVERSITY AVENUE, CHICAGO, IL 60637, USA

E-mail address: mem@math.uchicago.edu

EINSTEIN INSTITUTE OF MATHEMATICS, EDMOND J. SAFRA CAMPUS, GIVAT RAM, THE HEBREW UNIVERSITY OF JERUSALEM, JERUSALEM, 91904, ISRAEL, AND DEPARTMENT OF MATHEMATICS, HILL CENTER - BUSCH CAMPUS, RUTGERS, THE STATE UNIVERSITY OF NEW JERSEY, 110 FRELINGHUYSEN ROAD, PISCATAWAY, NJ 08854-8019 USA

E-mail address: shelah@math.huji.ac.il

URL: <http://shelah.logic.at>