CORRECTED ITERATION
SH1126

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Abstract. For \( \lambda \) inaccessible, we may consider \((< \lambda)\)-support iteration of some specific \((< \lambda)\)-complete \( \lambda^+\)-c.c. forcing notions. But this fails “preservation by restricting to a sub-sequence of the iterated forcing; to regain it we “correct” the iteration to regain it. We prove this for a characteristic case.
§ 0. Introduction

This work is dedicated to proving a theorem on \((< \lambda\)-support iterations of \((< \lambda\)-complete “nicely” definable \(\lambda^+\)-c.c. forcing notions for \(\lambda\) inaccessible. Assume \(Q\) is such a definition, \(\langle P_\alpha, \tilde{Q}_\beta : \alpha \leq \alpha_\ast, \beta < \alpha_\ast \rangle\) is such an interation, \(\tilde{Q}_\beta = \tilde{Q}_{V[P_\beta]}\) has generic \(\eta_\beta\). So \(\langle \eta_\beta : \beta < \alpha_\ast \rangle\) is generic for \(P_\alpha\), but letting \(\beta_\ast\) be maximal such that \(2\beta_\ast \leq \alpha_\ast\) is generic for the iteration \(\langle P_\alpha, \tilde{Q}_\beta : \alpha \leq \beta_\ast, \beta < \beta_\ast \rangle\)?

The point is that in the parallel case for \(\lambda = \aleph_0\) so far FS-iterated forcing such a claim is true. In fact, by Judah-Shelah [JdSh:292], if \(\langle P_\alpha, \tilde{Q}_\beta : \alpha \leq \alpha(\ast), \beta < \alpha(\ast) \rangle\) is FS-iteration of Suslin-c.c.c. forcing notion, \(\tilde{Q}_\beta\) with the generic \(\eta_\beta \in \omega\) and for notational transparency, its definition is with no parameter and the function \(\zeta : \beta(\ast) \rightarrow \alpha(\ast)\) is increasing and \(P = \langle P_\alpha, \tilde{Q}_\beta : \alpha \leq \beta(\ast), \beta < \beta(\ast) \rangle\) is FS iteration, \(\tilde{Q}_\beta\) defined exactly as \(\tilde{Q}_{\zeta(\beta)}\) but now in \(V[P_\beta]\) rather than \(V[P_{\zeta(\beta)}]\), then \(\vDash_{P_\alpha(\ast)} \langle \eta_{\zeta(\beta)} : \beta < \beta(\ast) \rangle\) is generic for \(P_\beta(\ast)\) over \(V^\beta\).

Now this is not clear to us for \((< \lambda\)-support iteration of \((< \lambda\)-strategically complete forcing notions. The solution is essentially to change the iteration to what we call “corrected iteration” we use a “quite generic” \((< \lambda\)-support iteration which “includes” the one we like and use the complete subforcing it generates. Here we deal with a characteristic case (used in [Sh:945]). The proof applies also to partial memory iteration. The proof applies also to partial memory iteration. On wide generalization and application (for \(\lambda = \aleph_0\)) this is continued in a work with H. Horowitz in preparation.

The problem arises as follows. In [Sh:945] it is proved that for \(\lambda\) inaccessible, consistently \(\text{cov}_\lambda(\text{meagre})\), the covering number of the meagre ideal on \(\lambda\) is strictly smaller than \(\text{\textbf{d}}_\lambda\), the dominating number. The result here is used there but the editor prefers to separate it. In §4 we explain how this work is used there.

We try to use standard notation. We use \(\theta, \kappa, \lambda, \mu, \chi\) for cardinals and \(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta\) for ordinals. We use also \(i\) and \(j\) as ordinals. We adopt the Cohen convention that \(p \leq q\) means that \(q\) gives more information, in forcing notions. The symbol \(<\) is preserved for “being an initial segment”. Also recall \(BA = \{ f : f \text{ a function from } B \text{ to } A \}\) and let \(\text{\textbf{a}} > A = \cup \{ P_A : \beta < \alpha \}\), some prefer \(\text{\textbf{a}} < A\), but \(\text{\textbf{a}} > A\) is used systematically in the author’s papers. Lastly, \(J^\text{bd}_\lambda\) denotes the ideal of the bounded subsets of \(\lambda\).

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§ 1. Iteration Parameters

Explanation 1.1. For \( m \in M \) below (Definition 1.7):

(a) we use \( L_m \) as the index set for the iteration; always a well founded partial order

(b) \( M_m \) is the part of the index set we are really interested in, it may be \((\kappa, <)\) in §1

(c) the other part in the interesting case is "generic enough \( m \)", more accurately enough existentially closed so that the iteration restricted to \( M \) will be "stabilize" under further extensions; inspite of \( L_m \) being required to be well founded this will be well defined.

Hypothesis 1.2. 1) \( \lambda = \lambda^{<\lambda} \) strongly inaccessible.

2) \( \theta = (\theta_\varepsilon : \varepsilon < \lambda) \).

3) \( \theta_\varepsilon \) is an infinite regular cardinal > \( \varepsilon \) and < \( \lambda \).

4) Assume \( \lambda_2 > \lambda_1 \geq \lambda \) are such that \(^1\) \( \lambda_2 > (\lambda_2)^{<\lambda} \geq \beth_1(\lambda_1) \) so pendentically all notations should have the parameter \( \lambda = (\lambda_2, \lambda_1) \) and even \( \lambda = (\lambda_2, \lambda_1, \lambda, \theta) \).

Notation 1.3. \( L, M \) denote partial orders, usually well founded.

Remark 1.4. Here no harm in adding

(a) \( \theta_\varepsilon > \prod_{\zeta < \varepsilon} 2^{\theta_\zeta} + 2^{\aleph_0} \) for \( \varepsilon < \lambda \)

or just

(b) \( \bar{\theta} \) is increasing fast enough

(c) \( M \) a linear order, well founded (it suffices to assume even \( M \cong (\kappa, <), \kappa \) regular > \( \lambda \)).

Definition 1.5. 1) For a partial order \( L \) let

(a) \( dp(L) = \cup \{ dp_L(t) + 1 : t \in L \} \), see below,

(b) \( dp_L(t) = dp(t, L) \in \text{Ord} \cup \{ \infty \} \) be defined by \( dp_L(t) = \cup \{ dp_L(s) + 1 : s < L t \} \).

(\gamma) \( L_{<L} t = L \{ s \in L : s < L t \} \),

(\zeta) \( L_{\leq L} t = L \{ s \in L : s \leq L t \} \).

2) Let \( L^+ = L(+) \) be \( L \cup \{ \infty \} \) with the natural order (but we may write \( t < L \infty \) instead of \( t < L(+) \infty \)).

3) We say the set \( L \) is an initial segment of the partial order \( L_* \) when

- \( L \subseteq L_* \), i.e. \( s \in L \Rightarrow s \in L_* \)
- \( s <_{L_*} t \land t \in L \Rightarrow s \in L \).

Discussion 1.6. Concerning the aim of the following choice, note the following.

1) By the partial order we already can get partial memory, so why the \( u_s \)’s (in 1.7)? Because \( \bar{u} \) is not necessarily transitive, that is, \( s \in u_t \Rightarrow u_s \subseteq u_t \). By partial order we cannot get it.

2) In [Sh:700] we use \( \mathcal{P}_\theta \)’s which are ideals, but here not necessarily: this makes a difference but it uses "\( Q_\theta \) is close to being \( \lambda \)-centered", i.e. any subset of \( \{ p \in Q_\theta : \text{tr}(p) = \eta \} \) of cardinality < \( \theta_{\text{tr}(\eta)} \) has a lub in this subset.

\(^1\)usually \( \lambda_2 \geq \lambda_1 \) suffice but see 3.11, 3.21
**Definition 1.7.** Let \(M\) be the class of objects \(m\), called iteration parameters, of the following form (so really \(M = M[\lambda]\) and if we omit clauses \((\theta), (\iota), (\lambda)\) we may write \(M[\ast]\)):

1. \(L\), a partial order,
2. \(M \subseteq L\), as partial orders,
3. \((\alpha)\) \(\bar{u} = (u_t : t \in L)\) and \(\bar{P} = \langle \bar{P}_t : t \in L \rangle\) and each \(\bar{P}_t\) is closed under subsets,
4. \((\beta)\) \(u_t \subseteq \{s \in L : s < L, t\}\) and \(u \in \bar{P} \Rightarrow u \subseteq u_t,
5. \((\gamma)\) \(dp(L) < \infty\), that is \(L\) is well founded,
6. \((\alpha)\) \(E'\) is a two-place relation (on \(L\)),
7. \((\beta)\) \(E'' := E'[(L \setminus M)\) is an equivalence relation on \(L \setminus M\)
8. \((\gamma)\) if \(s, t \in L \setminus M\) are not \(E''\)-equivalent then \((s < L, t) \Leftrightarrow (\exists r \in M)(s < r < t)\)
9. \((\delta)\) if \(sE't\) then \(s \notin M \vee t \notin M\)
10. \((\varepsilon)\) if \(t \in L \setminus M\) then \(\{s \in L : sE't\} = \{s \in L : tE's\}\); we call it \(t/E'\); so \(E'\) is a symmetric relation
11. \((\zeta)\) if \(s, t \in L \setminus M\) are \(E''\)-equivalent then \(s/E' = t/E'\)
12. \((\eta)\) if \(t \in L \setminus M\) then \(u_t \subseteq t/E'\)
13. \((\theta)\) if \(t \in L \setminus M\) then \(t/E'\) has cardinality \(\leq \lambda_2\)
14. \((\iota)\) \(\|M\| \leq \lambda_1\)
15. \((\kappa)\) if \(t \in L\) and \(u \in \bar{P}_t\) then \(u \notin M \Rightarrow (\exists s)(s \in L \setminus M \land u \subseteq s/E')\)
16. \((\lambda)\) \(\bar{P}_t\) has cardinality \(\leq \lambda_2\) for \(t \in L \setminus M\) and for simplicity \(\bar{P}_t \subseteq [u_t]^{\leq \lambda}\)
as well as those sets matter.

**Notation 1.8.** For \(m \in M\).

\(0\) In 1.7 we let \(m = (L_m, M_m, u_m, \bar{P}_m, E_m)\) and \(\bar{m} = (u_m : t \in L_m)\), \(\bar{P}_m = (\bar{P}_t : t \in L_m)\) and for \(t \in L \setminus M\) let \(E_m = \{t/E'_m\} \cup M_m\) and for \(t \in M_m\) let \(t/E'_m = M_m\); so there is no relation \(E_m\) but \(t/E'_m\) for \(t \in L_m\) is well defined.

1) In 1.7, let \(dp_m(t) = dp_{L_m}(t), dp_m = dp_{L_m}\) and \(\leq_m = \leq_{L_m}\).
2) For \(L \subseteq L_m\):
   (a) let \(n = m\)\(|L\) means \(n \in M, L_n = L, \leq_n = \leq_m, L_{n,u} = u_{m,t} \cap L, \bar{P}_{n,t} = \bar{P}_{m,t} \cap [L]^{\leq \lambda}\) for \(t \in L\) and \(M_n = M_m \cap L;\)
   (b) let \(dp_m(L) = dp_{L_m}(L)\) and we may write \(dp(L)\) for \(L \subseteq L_m\).
3) For \(t \in L_m\), let \(m_{<t} = m(<t) = m|L_{<t}\) where \(L_{<t} = L_{m(<t)} = L_{m,t} = \{s : s < m t\}\) so \(u_m(<t), s = u_m, s\) for \(s \in L_{<t}\), etc.
4) Also \(m_{\leq t} = m(\leq t) = m|L_{\leq t}\) where \(L_{\leq t} = L_{m(\leq t)} = L_{\leq t} \cup \{t\}\); let \(L_{<\infty} = L, L_{\leq \infty} = L^+,\) etc.
5) \(M_{\leq \mu}, M_{= \mu}, M_{> \mu}, M_{\geq \mu}\); let \(M_{\mu} = M_{=\mu}\).
6) For \(m, n \in M\) let \(m \approx n\), and we may say \(m, n\) are equivalent mean that \(L_m = L_n\) and \(t \in L_n \Rightarrow u_{m,t} = u_{n,t} \land \bar{P}_{m,t} = \bar{P}_{n,t}\); note that there are no demands on \(M\) and \(E'\).
7) We say \(f\) is an isomorphism from \(m_1 \in M\) onto \(m_2 \in M\) when:
(a) $f$ is an isomorphism from the partial order $L_m$ onto the partial order $L_{m_2}$

(b) for $s, t \in L_m$, we have $s \in u_{m_1, t} \iff f(s) \in u_{m_2, f(t)}$ and $\mathcal{P}_{m_2, f(t)} = \{ \{ f(s) : s \in u \} : u \in \mathcal{P}_{m_1, t} \}$

(c) for $s, t \in L_m$, we have $sE_{m_1}^* t \iff f(s)E_{m_2}^* f(t)$

(d) $M_{m_2} = \{ f(s) : s \in M_{m_1} \}$.

7) We define weak isomorphisms as in part (6) omitting clauses (c),(d).

**Definition 1.9.** For $m \in M$ let $L = L_m$ and we define the iteration $q_m$ to consist of:

(a) a forcing notion $P_t = P_{m, t}$ for $t \in L^+$; we let $\mathbb{P}_m = \mathbb{P}_\infty$

(b) $Q_t$ is the $P_t$-name of a subforcing of $Q_{\mathbb{P}}$ in the universe $V^{P_t}$, even $\leq_{ic}$ (i.e. incomaptibility and compatibility are preserved)

(c) $p \in P_t$ if

\( \begin{align*}
\text{(a)} & \ p \text{ is a function} \\
\text{(b)} & \ \text{Dom}(p) \subseteq L_{<t} \text{ has cardinality } \lambda \\
\text{(c)} & \ \text{if } s \in \text{Dom}(p) \text{ then } p(s) \text{ consists of } \text{tr}(p(s)) \in \prod_{\varepsilon < \zeta} \theta_\varepsilon \text{ for some } \zeta = \\
& \ \ (\forall \varepsilon < \zeta) \ \zeta(<s) < \lambda \text{ and } \varepsilon = \zeta(p(s)) = \xi(p(s)) \leq \lambda \text{ and } B_{p(s)}(\zeta) \text{ and } p_\zeta = p_\zeta(p(s)) = \langle \zeta : \zeta < \xi(p(s)) \rangle \in \xi(u_s) \text{ list the coordinates used in computing } p(s) \text{ and } (B_{p(s), t}, p_\zeta, t : t < \xi(p(s))) \text{ are such that:} \\
\text{• } & \ B_{p(s)} \text{ is a } \lambda \text{-Borel function, } B = B_{p(s)} : \xi(\prod_{\varepsilon < \zeta} \theta_\varepsilon) \to \prod \theta_\zeta \text{ moreover over into } (\prod \theta_\zeta)^{\text{tr}(p(s))} \\
\text{• } & \ \text{and considering } (d)(\alpha) \text{ below less pedantically } p(s) = (\text{tr}(p(s)), f_{p(s)}), \text{ where } f_{p(s)} = B_{p(s)}(\zeta_1, \ldots, \zeta_\lambda : \zeta_\lambda < \xi(p(s))) \text{ which means: absolutely, i.e. in every } V^{Q_t}, Q \text{ a } (\lambda, t) \text{-strategically complete (which is } \lambda^+\text{-c.c.) forcing notion, } B_{p(s)} \text{ is such a } (\lambda, t) \text{-Borel function; we may write } \xi_{p,s} \text{ instead of } \xi(p(s), s), \text{ etc.} \\
\text{• } & \ \nu(p(s)) < \lambda \text{ moreover } \nu(p(s)) < \theta_{\nu(\text{tr}(p(s)))} \\
\text{• } & \ r_{p(s), t} = r_{p(s), t} \text{ so } w_{p(s), t} = w(p(s), t) = \text{dom}(r_{p(s), t}) = \xi(p(s)) \text{ and } r_{p(s), t} \text{ is a subsequence of } r_{p(s)} \\
\text{• } & \ B_{p(s), t} \text{ is a Borel function from } \nu(p(s), t) \text{ into } (\prod_{\varepsilon < \zeta} \theta_\varepsilon)^{\text{tr}(p(s))} \\
\text{• } & \ B_{p(s)}(\langle \eta_{r_{p(s), t}}(\zeta) : \zeta < \xi(p(s)) \rangle) = \sup(B_{p(s), t}(\langle \eta_{r_{p(s), t}}(\zeta) : \zeta < w_{p(s), t} \rangle)) = \\
\text{• } & \ \text{so } f_{p(s)} = \sup(f_{p(s), t} : t < \nu(p(s))) = B_{p(s)}(\langle \eta_{r_{p(s), t}}(\zeta) : \zeta < w_{p(s), t} \rangle) \\
\text{• } & \ \text{for each } t < \nu(p(s)) \text{ for some } u \in \mathcal{P}_{m, s} \text{ we have } \{ r_{p(s)}(\zeta) : \zeta < \xi(p(s)), \zeta < w_{p(s), t} \} \subseteq u \text{ so is a subset of } u_s \\
\text{• } & \ \text{if } t < \nu(p(s)) \text{ and } \varepsilon \in w_{p(s), t}, r_{p(s), t}(\varepsilon) \in L_m \setminus M_m \implies \{ r_{p(s)}(\zeta) : \zeta < w_{p(s), t} \} \subseteq r_{p(s), t}(\varepsilon) / E_{m_2}, \text{ (follows)} \\
\text{• } & \ \text{the sup will not given in } \varepsilon \text{ the value } \theta_\zeta, \text{ etc.} \\
\end{align*} \)
Proof. Straightforward. For the relevant parts of (a), (b), (c), (d), (e).

Claim 1.11. For $m \in M$ (so $P_t = P_{m,t}$, etc.)

(a) $f_{\text{supp}}(p)$, the full support of $p$ be $\cup\{\{r_{\beta_0}(s) : \zeta < \zeta_{s_0}\} \cup \{s : s \in \text{Dom}(p)\}$

(b) $w_{\text{supp}}(p)$, the wide support of $p$ be $\cup\{t/E_m : t \in f_{\text{supp}}(p)\}$.

(d) (α) $\eta_s$ is the $P_t$-name, when $t \in L^+_{m,s} \in L < t$ defined by $\cup\{\text{tr}(p(s)) : p \in G_r\}$.

(β) For $p \in P_t$ and $s \in \text{Dom}(p)$ we interpret $p(s)$ as a $P_{s,r}$-name $(\text{tr}(p(s)), B_{p,s}(\ldots, \eta_{p,s}(\zeta), \ldots), \zeta < L_{p,s})$.

(c) $P_t \models "p \leq q"$ iff

(α) $p, q \in P_t$

(β) $\text{Dom}(p) \subseteq \text{Dom}(q)$

(γ) if $t \in \text{Dom}(p)$ then $(q\upharpoonright L_{< t}) \Vdash_{p_m(\leq t)} "p(t) \leq q(t)"$.

{c7}

Definition 1.10. 1) For $p \in P_t$ let

(a) $f_{\text{supp}}(p)$, the full support of $p$ be $\cup\{\{r_{\beta_0}(s) : \zeta < \zeta_{s_0}\} \cup \{s : s \in \text{Dom}(p)\}$

(b) $w_{\text{supp}}(p)$, the wide support of $p$ be $\cup\{t/E_m : t \in f_{\text{supp}}(p)\}$.

{c6}

2) For $m \in M$ let $P^m_t = P_{m,t}$, etc., in Definition 1.9.

3) For $L \subseteq L_m$ let $P_m(L) = P_m\{p \in P_m : f_{\text{supp}}(p) \subseteq L\}.$

4) For $m \in M$ and $t \in L_m$ let $Q_t = Q_{m,t}$ be the $P_{r,t}$-name of $Q_{\bar{Q}_t}\{t, f \in \bar{Q}_t : f = \sup\{f_t : t < t(*)\}$ where $t(*) < \theta_{\bar{Q}_t}(v)$ and $f_t \in \bigcap \{\theta_t\}^{\mathbb{V}_{[\bar{Q}_t] \cap \left[\eta_2:s \in u\right]}$ for some $u \in P_{m,t}\}$.

{c8}

Claim 1.11. For $m \in M$ (so $P_t = P_{m,t}$, etc.)

(a) the iteration $Q_m$ is well defined, i.e. exist and is unique

(b) (α) if $t \in L^+_{m,s}$ then $P_t$ is indeed a forcing notion and is equal to $P_{m,L}.$

(β) the $P_{r,t}$-name $\eta_s$ does not depend on $t$ as long as $s < t \in L^+_{m,s}$.

(γ) $\eta_t$ is a $P_{m,L}$-name

(c) if $s < L_t$ then $P_t \models "p \leq q"$ iff $P_s \models "p \leq q"$.

(α) $p, q \in P_t$

(β) $\text{Dom}(p) \subseteq \text{Dom}(q)$

(γ) if $t \in \text{Dom}(p)$ then $(q\upharpoonright L_{<t}) \Vdash_{P_{m,L}} "p(t) \leq q(t)"$.

(d) if $L$ is an initial segment of $L_m$ then $P_m\mid L = P_m\{p \in P_m : \text{dom}(p) \subseteq L\}$, equivalently $\text{supp}(p) \subseteq L$; this holds in particular for $L_{<t}, m_{<t}$.

(e) if $L_1 \subseteq L_2$ are initial segments of $L_m$ then the parallel of clause (c) holds replacing $P_{m,s}, P_{m,t}$ by $P_{m,L_{<1}}, P_{m,L_{<t}}$, respectively.

Proof. Straightforward. For $t \in L^+_{m,s}$ by induction on $d_{P_{m,t}}(t)$, define $P_t$ and prove the relevant parts of (α), (β), (c), (d), (e).

Note

{c13} not used, could have used it in 1.15.
Observation 1.12. If $B$ is a $\lambda$-Borel function from $\xi(\Pi\theta)$ to $\mathcal{P}(\lambda)$ or even $\mathcal{H}(\lambda^+)$ where $\xi \leq \lambda$ then there is a $\lambda$-Borel function $B'$ from $\xi(\Pi\theta)$ to $\mathcal{Q}_\theta$ (so absolutely to $\mathcal{Q}_\theta$) such that for any $\eta \in \xi(\Pi\theta)$ we have, absolutely:

- if $B(\eta) \in \mathcal{Q}_\theta$ then $B'(\eta) = B(\eta)$
- if $B(\eta) \notin \mathcal{Q}_\theta$ then $B'(\eta) = (0,0,\lambda)$, the minimal member of $\mathcal{Q}_\theta$.

Proof. Easy. \(\square\)
(β) if \( \ell g(\text{tr}(q(s))) < \ell g(\text{tr}(p(s))) \) then for some ordinal \( \varepsilon \), \( \ell g(\text{tr}(q(s))) \leq \varepsilon < \ell g(\text{tr}(p(s))) \) and \( r \models L_{m(<s)} \Rightarrow \text{tr}(p(s))(\varepsilon) < f_{q(s)}(\varepsilon) \).

(γ) if \( \ell g(\text{tr}(q(s))) > \ell g(\text{tr}(p(s))) \) then for some ordinal \( \varepsilon \), \( \ell g(\text{tr}(q(s))) > \varepsilon \geq \ell g(\text{tr}(p(s))) \) and \( r \models L_{m(<s)} \Rightarrow \text{tr}(q(s))(\varepsilon) < f_{p(s)}(\varepsilon) \).

9) \( \models P_m \langle [g_s : s \in L_m] = V[G] \rangle \).

\[ \{ \text{c53} \} \]

Remark 1.14. What is the use of e.g. (6),(6A)? See 2.11(A)(b) and 1.15.

Proof. We prove all parts by induction on \( dp_m \).

{c8}

1) For clause (a) for each \( m \), using the induction hypothesis and 1.11(e), the problem is only when \( dp_m(t) = dp_m - 1 \) and use part (5A) proved below. For clause (β) use also part (6A) for \( P_{m(<t)} \) proved below. In both cases the proof of the parts quoted does not rely on part (1).

2) If \( p \in P_m \) for \( \varepsilon < \lambda^+ \) then we by the \( \Delta \)-system lemma can find \( u \) and unbounded \( S \subseteq \lambda^+ \) such that \( \varepsilon \neq \zeta \in S \Rightarrow \text{Dom}(p_\zeta) \cap \text{Dom}(p_\varepsilon) = u \) and \( \langle \text{tr}(p_\beta) : \beta \in \varepsilon \rangle \) is the same for all \( \varepsilon \in S \). Now \( p_\varepsilon, p_\zeta \) has a common upper bound for every \( \varepsilon, \zeta \in u \), i.e. we define \( r \) by

- \( \text{Dom}(r) = \text{Dom}(p_\varepsilon) \cup \text{Dom}(p_\zeta) \)
- \( r(s) = p_\varepsilon(s) \) if \( s \in \text{Dom}(p_\varepsilon) \) \( \setminus \text{Dom}(p_\zeta) \)
- \( r(s) = p_\zeta(s) \) if \( s \in \text{Dom}(p_\zeta) \) \( \setminus \text{Dom}(p_\varepsilon) \)
- \( r(s) = \text{tr}(p_\varepsilon(s)), \max \{ f_{p_\varepsilon(s)}, f_{p_\zeta(s)} \} \) if \( s \in \text{Dom}(p_\varepsilon) \cap \text{Dom}(p_\zeta) \)

3) By (4), the second sentence + (5B) below which use only the induction hypothesis.

4) We define \( p \) by:

- \( \text{Dom}(p) = \cup \{ \text{Dom}(p_i) : i < \delta \} \)
- \( \text{tr}(p(s)) = \cup \{ \text{tr}(p_i(s)) : i < \delta \text{ satisfies } s \in \text{Dom}(p_i) \} \)
- \( f_{p(s)} = \sup \{ f_{p_i(s)} : i < \delta \text{ satisfies } s \in \text{Dom}(p_i) \} \).

Note that here having to really start with \( \langle f_{p_i(s)}, \epsilon < i(p_i(s)) \rangle \) and get \( \langle f_{p(s)}, \epsilon < i(p(s)) \rangle \), see 1.9(c)(γ) causes no problem, similarly in the proof of part (2) - just take the union.

{c6}

5A) Obvious by the definition of \( P_m \) and 1.11(e).

5B) The proof is split to cases.

Case 1: \( dp_m = 0 \)

So \( L_m \) is empty.

Case 2: \( dp_m = \alpha + 1 \)

Hence \( L_2 = \{ s \in L : dp_m(s) < \alpha \} \) is non-empty and letting \( L_1 = L_m \setminus L_2 \); clearly \( s \in L_1 \Rightarrow dp_m(s) < \alpha \), so \( dp_m|L_1 \leq \alpha \). Let \( \zeta_* = \sup(\{ \ell g(\text{tr}(p(s))) + 1 : s \in \text{dom}(p) \} \cup \{ \zeta + 1 \}) \). Hence applying (4) and (5B) to \( m|L_1 \), i.e. the induction hypothesis we can find \( q_1 \) such that \( P_{m|L_1} \models \langle p|L_1 \rangle < q_1 \) and \( [s \in \text{Dom}(q)] \Rightarrow \ell g(\text{tr}(q_1(s))) > \zeta_* \) and \( q_1 \) forces a value to \( f_{p(s)} \mid \zeta_* \), call it \( p_s \) for \( s \in \text{Dom}(p) \cap L_2 \) and \( i < i(p(s)) \).

Define \( q \in P_m \) by \( \text{Dom}(q) = \text{Dom}(q_1) \cup (L_2 \cap \text{dom}(p)), q|L_1 = q_1 \) and if \( s \in L_2 \cap \text{Dom}(p) \) then \( q(s) = (p_s, p_* \mid f_{p(s)} \mid \zeta, \lambda) \), recalling 1.11.

Easily \( q \) is as required.
Case 3: $\delta = d_p \gamma$ is a limit ordinal of cofinality $\geq \lambda$

So $\alpha = \sup \{d_p \gamma(s) + 1 : s \in \text{Dom}(p)\}$ is an ordinal $< \delta$ and let $L = \{s \in L_\gamma : d_p \gamma(s) < \alpha\}$, so $L$ is an initial segment of $L_\gamma$ and applying the induction hypothesis to $m \mid L, p$ we get $q$ as required in $\mathbb{P}_{m/L}$ hence in $\mathbb{P}_m$.

Case 4: $\delta = d_p \gamma$ is a limit ordinal of cofinality $< \lambda$.

Let $\langle \alpha_i : i < \text{cf}(\delta) \rangle$ be increasing continuous with limit $\delta$, let $\alpha_{i, \text{cf}(\delta)} = \delta$ and for $i \leq \text{cf}(\delta)$ let $L_i := \{s \in L_\gamma : d_p \gamma(s) < 1 + \alpha_i\}$.

Now we choose $(p_i, \zeta_i)$ by induction on $i < \text{cf}(\delta)$ such that:

(a) $p_i \in \mathbb{P}_{m/L_i}$
(b) $\mathbb{P}_{m/L_i} \models \langle \gamma(L_i) \leq p_i \rangle$ when $j < i$
(c) if $i$ is a limit ordinal then $p_i$ is gotten from $\langle p_j : j < i \rangle$ as in part (4)
(d) if $i \in \text{Dom}(p_i)$ then $\ell_g(\text{tr}(p_i(s))) \geq \zeta_i$
(e) $\langle \zeta_j : j < i \rangle$ is an increasing continuous sequence of ordinals $< \lambda$ and if $i$ is non-limit then $\zeta_i$ is $> \zeta$ and $\geq \sum_{j < i} |\text{Dom}(p_j)| + |\text{Dom}(p)|$ and $\sup(\{\ell_g(\text{tr}(p_j(s))) : j < i \text{ and } s \in p_j\}) \cup \{\ell_g(\text{tr}(p(s))) : s \in \text{Dom}(p)\})$.

Using 1.11 and the induction hypothesis this is easy.

6) For transparency assume $\models \langle y \in \prod_{\varepsilon < \lambda} \theta_{\varepsilon} \rangle$ or just $\models \lambda^\varepsilon V$, By parts (4) + (5B), i.e. part (3), for each $\zeta < \lambda$ the following subset of $\mathbb{P}_{m,t}$ is open and dense: $\mathscr{F}_\zeta = \{p \in \mathbb{P}_{m,t} : \text{for some } \nu \in \prod_{\varepsilon < \zeta} \theta_{\varepsilon} \text{ or } \varepsilon \in \nu^\varepsilon (\text{from } V!) \text{ we have } p \models \gamma^\varepsilon \simeq \langle y | \nu | = v \rangle\}.$

Clearly there is a maximal antichain $\langle p_{\xi, e} : e < \xi \rangle$ of $\mathbb{P}_{m,t}$ included in $\mathscr{F}_\zeta$ and by part (2) without loss of generality $\xi_{\zeta} \leq \lambda$, the rest should be clear. In the general case we can code $y$ as a subset of $\lambda$, etc.

6A) This too should be clear as $\mathbb{P}_t$ satisfies the $\lambda^+=\text{c.c.}$.

7) Look at the definitions.

8) Using parts (4) and (5B) and the definition this is easy.

9) Suppose toward contradiction that $G_1 \neq G_2$ are generic subsets of $\mathbb{P}_m$ but $s \in L_\gamma \Rightarrow \eta s[G_1] = \eta s[G_2]$.

Let $p_1 \in G_1 \setminus G_2$ hence there is $p_2 \in G_2$ such that $p_2 \models \gamma^{\text{P}_m} \langle p_1 \notin G_2 \rangle$ hence $p_1, p_2$ are incompatible. Let $L_a = \{s \in L_\gamma : G_1 \cap \mathbb{P}_{<s} = G_2 \cap \mathbb{P}_{<s}\}$ so $L_a$ is an initial segment of $L_\gamma$. If $L_a = L_\gamma$ we can easily get a contradiction, so $L_a \neq L_\gamma$ and let $r \in L_\gamma \setminus L_a$ be such that $L_a < r \subseteq L_\gamma$. Now as in part (8) we can get a contradiction having found a common to upper bound to $p_1, p_2$.

Alternatively use part (6).

Conclusion 1.15. Let $m \in M$ and for notational transparency for some ordinal $\beta(\ast), t \in L_\gamma \Leftrightarrow t \in \beta(\ast)$ and $s <_m t \Rightarrow s < t$. Then $q$ is essentially a $(< \lambda)$-support iteration of length $\beta(\ast)$ with $Q_\alpha = \{(\nu, f) \in Q_\beta^{\mathbb{V}[\eta_{\alpha} : \beta < \alpha]} : \nu \in f, f = \sup \{f_\iota : \iota < \iota(\alpha), \iota(\alpha) < \lambda, \nu \in f_\iota \text{ and } \{f_\iota : \iota < \iota(\alpha)\} \subseteq \cup \{Q_\beta^{\mathbb{V}[\eta_{\alpha} \in u]} : \iota < \iota(\alpha)\} \subseteq \cup \{Q_\beta^{\mathbb{V}[u]} : u \in \mathbb{P}_{m, \alpha}\}\} with the natural order, i.e. the order of $Q_\beta^{\mathbb{V}[u]}$ restricted to this set.

Proof. Should be clear by 1.13.

Till now $(E_m, M_m)$ have played no role and we could have omitted them.
Definition 1.16. 1) We define the two-place relation $\leq_{M}$ on $M$ as follows:

(a) $L_{m} \subseteq L_{n}$ as partial orders of course,
(b) $M_{m} = M_{n}$, yes! equal,
(c) $u_{m,t} = u_{n,t} \cap L_{m}$ and $P_{m,t} = \{u \cap L_{m} : u \in P_{n,t}\}$ for $t \in M_{m}$,
(d) $u_{m,t} = u_{n,t}$ and $P_{m,t} = P_{n,t}$ for $t \in L_{m} \setminus M_{m}$
(e) if $t \in L_{m} \setminus M_{m}$ then $t/E_{m} = t/E_{n}$ hence $E_{m}' = E_{n}'|L_{m}$
(f) • if $t \in L_{m} \setminus M_{m}$ then $\mathcal{P}_{m,t} = \mathcal{P}_{n,t}$
• if $t \in M_{m}$ and $s \in L_{m} \setminus M_{m}$ then $\{u \cap L_{m} : u \subseteq s/E_{m}\}$ =
$\{u \in \mathcal{P}_{m,t} : u \subseteq s/E_{n}\}$

2) We define the two-place relation $\leq_{M}$ as in part (1) omitting clauses (b),(d),(e) and (f); natural but not used.

Claim 1.17. 1) $\leq_{M}$ is a partial order.
2) If $(m_{\alpha} : \alpha < \delta)$ is $\leq_{M}$-increasing, then its union $m_{\delta}$ (naturally defined) is a
$\leq_{M}$-lub and $|L_{m_{\alpha}}| \leq \Sigma(|L_{m_{\alpha}}| : \alpha < \delta)$.
3) If $m \leq_{M} n$ and $L \subseteq L_{m}$ then $p \in P_{m}(L) \Leftrightarrow p \in P_{n}(L)$ for every $p$.
4) If $m \leq_{M} n$ and $P_{m} \subseteq P_{n}$ and $L \subseteq L_{m}$ then $P_{m}(L) = P_{n}(L)$ as quasi orders.

Proof. Easy, e.g.
1) Why is $L_{m_{\alpha}} := \cup \{L_{m_{\alpha}} : \alpha < \delta\}$ well founded? Toward contradiction assume
$t = \{t_{n} : n < \omega\}$ is $<_{L_{m_{\alpha}}}$-decreasing. We can replace $t$ by an infinite subsequence.
So without loss of generality

(*) either (a) or (b) where
(a) for every $n < m$ there is $s_{n,m} \in M_{m_{\alpha}}$ such that $t_{m} <_{L_{\alpha}} s_{n,m} <_{L_{\alpha}} t_{n}$
(b) for no $n < m$ this holds.

If clause (a) holds, then $(s_{n,m+1} : n < \omega)$ is a $<_{M_{\alpha}}$-decreasing sequence contradiction.
If (b) holds for any $n < m$, let $\alpha(n) = \min\{s : t_{n} \in L_{m_{\alpha}}\}$; without
loss of generality it is monotonically increasing or constant so as $M_{m_{\alpha}}(\alpha) = M_{m_{\alpha}}$;
by 1.16(1)(e) we get $t_{n}/E_{m_{\alpha}(n+1)} = t_{n+1}/E_{m_{\alpha}(n+1)}$ hence $t_{n+1} \in L_{m_{\alpha}(n)}$

3) See the proof of $\aleph_{\alpha}$ in the proof of 1.22.

Claim 1.18. $(M, \leq_{M})$ has amalgamation.
That is, if $m_{0} \leq_{M} m_{1}, m_{0} \leq_{M} m_{2}$ and $L_{m_{1}} \cap L_{m_{2}} = L_{m_{0}}$ then there is $m \in M$
such that $m_{1} \leq_{M} m_{2} \leq_{M} m$ and $L_{m} = L_{m_{1}} \cup L_{m_{2}}$.

Proof. Note that by clause (e)(γ) of Definition 1.7 and clause (e) of Definition
1.16(1):

(*) assume $(s_{1} \in L_{m_{1}} \setminus L_{m_{0}}) \cap (s_{3} \in L_{m_{2}} \setminus L_{m_{0}})$ and $s_{2} \in L_{m_{0}}$;

• if $s_{1} <_{m_{0}} s_{2} \land s_{2} <_{m_{2}} s_{3}$ then for some $s_{1}', s_{2}', s_{3} \in M_{m_{0}}$ we have $s_{2}' <_{m_{0}}$
$s_{2} \leq_{m_{0}} s_{2}', s_{1} <_{m_{1}} s_{2}', s_{2} \land s_{2}' <_{m_{2}} s_{3}$

6This is covered by clause (i) but see part (2).
Remark Obvious recalling the properties of Claim 1.21.

We now define \( m \) by:

\[(s, a) \text{ (a)} t \in L_m \text{ iff } t \in L_{m_1} \vee t \in L_{m_2}
\]
\[(b) \quad s \leq m \text{ t}
\]
\[(c) \quad u_{m,t} \text{ is}
\]
\[(d) \quad E_{m_1} = E_{m_1} \cup E_{m_2}
\]
\[(e) \quad \mathcal{P}_{m_1,t} \text{ is}
\]

Clearly

\[\circ \quad m \in M \text{ and } m_1 \leq_M m \text{ and } m_2 \leq_M m.\]

So we are done.

Observation 1.19. For \( p, q \in \mathbb{P}_m \) the following conditions are equivalent:

\[(a) \quad q \vDash "p \in G_{\mathbb{P}_m}"\]
\[(b) \quad \text{if } s \in \text{Dom}(p) \text{ then either } s \in \text{Dom}(q) \text{ and } (q \upharpoonright L_{m < s}) \vDash \mathbb{P}_{m < s} "p(s) \leq q(s)" \]
\[\text{ or } s \notin \text{Dom}(q), \text{tr}(p(s)) = \emptyset \text{ and } q \upharpoonright L_{m < s} \vDash \mathbb{P}_{m < s} "p(s) \text{ is trivial, i.e. } f_p(s) \text{ is constantly zero}"
\]
\[(c) \quad \mathbb{P}_m \vDash "p \leq q^+" \text{ where } \text{Dom}(q^+) = \text{Dom}(q) \cup \text{Dom}(p) \text{ and } q^+(s) \text{ is}
\]
\[(\alpha) \quad q(s) \text{ if } s \in \text{Dom}(q)
\]
\[(\beta) \quad \text{the trivial condition if } s \in \text{dom}(p) \setminus \text{dom}(q); \text{ note that } \text{supp}(q^+) = \text{supp}(q) \cup \text{Dom}(p).
\]

Remark 1.20. We shall use this freely. Maybe better to change the order.

Proof. Obvious recalling the properties of \( Q_\theta \).

Claim 1.21. For \( m \in M \), recalling 1.10(3), we have \( \mathbb{P}_m(L_1) < \mathbb{P}_m(L_3) \) when:

\[(s) \quad (a) \quad L_2 \subseteq L_3 \text{ are initial segments of } L_m
\]

\[\quad \text{but recall that } t \in L_m \setminus M_m \Rightarrow u_{m_1,t} = u_{m_0,t} \wedge \mathcal{P}_{m_1,t} = \mathcal{P}_{m_0,t}
\]
(b) $L_1 \subseteq L_3$ and $L_0 = L_1 \cap L_2$

(c) $L_0$ is an initial segment of $L_1$.

(d) $\mathbb{P}_m(L_0) \prec \mathbb{P}_m(L_2)$

(c) $L_1 \setminus L_0$ is disjoint from $M_m$

(f) if $t \in L_1 \setminus L_0$ then $(t/E_m) \cap L_{m,c,t} \subseteq L_1$.

Proof. As $d_p_m(L_1) < \infty$ it suffices to prove by induction on the ordinal $\gamma$ that:

1. If $\langle \ell : \ell \leq 3 \rangle$ satisfies $(*)$ of the claim and $d_p_m(L_1) \leq \gamma$ then:
   
   (a) $\mathbb{P}_m(L_1) \prec \mathbb{P}_m(L_3)$
   
   (b) we have $p_1 \in \mathbb{P}_m(L_1)$ and $p_1 \leq q_1 \in \mathbb{P}_m(L_1) \Rightarrow p_3, q_1$ are compatible in $\mathbb{P}_m(L_3)$ when:
      
      (α) $p_1 \in \mathbb{P}_m(L_1)$
      
      (β) $p_0 \in \mathbb{P}_m(L_0)$
      
      (γ) if $p_0 \leq q_0 \in \mathbb{P}_m(L_0)$ then $p_2 := p_3 \setminus L_2$ and $q_0$ are compatible in $\mathbb{P}_m(L_2)$
   
   (δ) $p_1 = p_0 \cup (p_3 \setminus L_1 \setminus L_0)$.

Why this holds? Assume we have arrived to $\gamma$.

Clause (b): Recalling clause (f) of the assumption, indeed, $p_1 = p_0 \cup (p_3 \setminus (L_1 \setminus L_0)) \in \mathbb{P}_m(L_1)$ by the definitions (clauses (b), (α), (β), (δ) of $\mathbb{P}_m$), e.g. why fsupp$(p_1) \subseteq L_1$?

Note that if $s \in \text{dom}(p_3 \setminus (L_1 \setminus L_0))$ then $s \in L_1 \setminus L_0 \subseteq L_1$ and $\{r_{p_3}(\zeta) : \zeta < \xi_p(s)\}$ is included in $L_3$ because $p \in \mathbb{P}_m(L_3)$ and in $L_{<s}$ by Definition 1.9. As $s \in L_1 \setminus L_0$ by $(*)$ we have $s \notin M_m$ hence by Definition 1.9 we have $\{r_{p_3}(\zeta) : \zeta < \xi_p(s)\} \subseteq u_s \subseteq s/E_m$. By $(*)$ we have $(s/E_m) \cap L_{<s} \subseteq L_1$ hence together $\{r_{p_3}(\zeta) : \zeta < \xi_p(s)\} \subseteq L_1$, and we are done proving fsupp$(p_1) \subseteq L_1$.

So the first statement in $\mathbb{P}_m(\mathbb{P}_m)$ holds; what about the second? Toward contradiction assume $q_1$ contradicts the desired conclusion then by 1.13(8) there are $s$ and $p_3^+$ such that:

1. $s \in \text{dom}(q_1) \cap \text{dom}(p_3)$
2. $p_3^+ \in \mathbb{P}_m(L_{m,<s})$
3. $p_3^+$ is above $p_3 \setminus L_{m,<s}$ and above $q_1 \setminus L_{m,<s}$
4. $p_3^+ \models \varphi_{m,<s}$ “$p_3(s), q_1(s) \in Q_\theta$ are incompatible (in $Q_\theta$)”

So $s \in \text{dom}(q_1) \subseteq L_1$ and as $L_2$ is an initial segment of $L_m$ and clause (γ) of (b) (of $\mathbb{P}_m$), clearly $s \in L_0$ is impossible, so $s \in \text{dom}(q_1) \setminus L_0 \subseteq L_1 \setminus L_0$. As $\mathbb{P}_m \models \langle t_1 \leq q_1 \rangle$, necessarily $q_1 \setminus L_{m,<s} \models \varphi_{m,<s}$ “$p_3(s), q_1(s)$”, so as $q_1 \setminus L_{m,<s} \leq p_3^+ \setminus L_{m,<s}$ (by $\oplus(c)$), also $p_3^+ \setminus L_{m,<s} \models \varphi_{m,<s}$ “$p_3(s), q_1(s)$”. As $s \notin L_0$ clearly $p_1(s) = p_3(s)$ by clauses $\mathbb{P}_m(\mathbb{P}_m)$, (b), (δ), so $p_3^+ \setminus L_{m,<s} \models \varphi_{m,<s}$ “$p_3(s), q_1(s)$” and again easy contradiction to $\oplus(d)$.

Clause (α):

Clearly $\mathbb{P}_m(L_1) \subseteq \mathbb{P}_m(L_3)$ as quasi orders. Next we shall prove $\mathbb{P}_m(L_1) \leq_{\text{ec}} \mathbb{P}_m(L_3)$, so assume $q_1, q_2 \in \mathbb{P}_m(L_1)$ has a common upper bound $p_3$ in $\mathbb{P}_m(L_3)$, and we should find one in $\mathbb{P}_m(L_1)$. Hence (see 1.9(e)(β) we have Dom$(q_1) \cup \text{Dom}(q_2) \subseteq \text{Dom}(p_3)$.

As $p_3 \setminus L_2 \in \mathbb{P}_m(L_2)$ by $(*)$ and we are assuming $\mathbb{P}_m(L_0) \prec \mathbb{P}_m(L_2)$, see $(*)$ then there is $p_0 \in \mathbb{P}_m(L_0)$ such that $p_0 \leq q \in \mathbb{P}_m(L_0) \Rightarrow q, p_3 \setminus L_2$ are compatible
in $\mathbb{P}_m(L_2)$ and let $p_1 = p_0 \cup (p_3 \setminus (L_1 \setminus L_0))$. By $\exists_\beta(b)$, which we have proved noting that clauses $(\alpha) - (\delta)$ of $\exists_\gamma(b)$ holds, we know that $p_1 \in \mathbb{P}_m(L_1)$ and $p_1 \leq p'_1 \in \mathbb{P}_m(L_1) \Rightarrow p_3, p_1'$ are compatible in $\mathbb{P}_m(L_3)$. It suffices to prove that $p_1$ is a common upper bound of $q_1, q_2$.

We could have replaced $p_0$ by $p_0'$ whenever $p_0 \leq p'_0 \in \mathbb{P}_m(L_0)$. So without loss of generality for $\ell = 1, 2$ we have $\text{dom}(q_\ell \cap L_0) \subseteq \text{dom}(p_0) \subseteq \text{dom}(p_1)$, also recall $\text{dom}(q_\ell) \setminus L_0 \subseteq \text{dom}(p_2) \cap L_1 \setminus L_0$ and by the choice of $p_1$ we have $\text{dom}(p_3) \cap L_1 \setminus L_0 \subseteq \text{dom}(p_1) \setminus L_0$.

So recalling $\text{dom}(q_\ell) \subseteq L_1$ together $\text{dom}(q_\ell) \subseteq \text{dom}(p_1)$.

As we are assuming $\mathbb{P}_m(L_0) < \mathbb{P}_m(L_2)$ without loss of generality $p_0$ is above $q_\ell \setminus L_0$. If toward contradiction we assume that $\ell \in \{1, 2\}$ and $q_\ell \not\preceq p_1$ then for some $s \in \text{Dom}(q_\ell)$ we have $(q_\ell \setminus L_{m,s}) \leq (p_1 \setminus L_{m,s})$ but $p_1 \setminus L_{m,s} \not\models \mathbb{P}_m(L_{m,s})$ “$q_\ell(s) \leq p_1(s)$”. Clearly, $s \in L_0$, hence $s \not\in M_\mathbb{P}$ by clause $(*)$.

Let $L'_0 = L_0, L'_1 = L_0 \cup (L_1 \cap L_{m,s}), L'_2 = L_2, L'_3 = L_3$ so $(L'_0, L'_1, L'_2, L'_3)$ satisfies the assumptions of the present claim and $\text{dp}_m(L'_1) < \gamma$, hence by the induction hypothesis, $\mathbb{P}_m(L'_1) < \mathbb{P}_m(L'_3)$.

Recall $s \in L_1 \setminus L_0$ hence $(s/E_m) \cap L_{m,s} \subseteq L_1$ by clause $(f)$ of the assumption of the claim, so $\text{fsupp}(p_1 \setminus \{s\}) \setminus \{s\}$ are $\subseteq L_1$ hence $p_1(s), q_\ell(s)$ are $\mathbb{P}_m(L'_1)$-names. So recalling $p_1 \setminus L_{m,s} \not\models \mathbb{P}_m(L_{m,s})$ “$q_\ell(s) \leq p_1(s)$” and $\mathbb{P}_m(L'_1) < \mathbb{P}_m(L'_3)$ and $L_{m,s} \subseteq L_3 = L_3$ we have $p_1[L'_1] \not\models \mathbb{P}_m(L'_1)$ “$q_\ell(s) \leq p_1(s)$”. Hence there is $p'_1$ such that $p_1[L'_1] \leq p'_1 \in \mathbb{P}_m(L'_1)$ such that $p'_1 \models \mathbb{P}_m(L'_1)$ “$q_\ell(s) \not< p_1(s)$” so recalling $\mathbb{P}_m(L'_1) < \mathbb{P}_m(L'_3)$ we have $p'_1 \models \mathbb{P}_m(L'_3)$ “$q_\ell(s) \not< p_1(s)$”.

But by $\exists_\gamma(b)$ for $\gamma_1 = \text{dp}_m(L'_1)$, we know that $p'_1$ and $p_3 \setminus L_{m,s}$ are compatible (in $\mathbb{P}_m$, equivalently $\mathbb{P}_m(L_{m,s})$) so let $p'_3 \in \mathbb{P}_m(L_{m,s})$ be a common upper bound of $p'_1, p_3 \setminus L_{m,s}$. Now $p'_3 \models \mathbb{P}_m(L'_1)$ “$q_\ell(s) \not< p_1(s)$” because: $q_\ell \not\preceq p_3$ by the choice of $p_3$; $p_1(s) = p_3(s)$ by the choice of $p_1$ and $p_3 \leq p'_3$, see above. Moreover, $p'_3 \models \mathbb{P}_m(L'_1)$ “$q_\ell(s) \not< p_1(s)$” as $p'_1 \preceq p'_3$, see above.

So we have proved $\mathbb{P}_m(L'_1) \leq \text{ie} \mathbb{P}_m(L'_3)$.

To finish proving clause $\exists_\beta(a)$ that is $\mathbb{P}_m(L'_1) \leq \mathbb{P}_m(L'_3)$ note that clause $\exists_\gamma(b)$ does this as for every $p_3 \in \mathbb{P}_m(L'_3)$ there is $p_0$ as in $\exists_\beta(\beta), (\gamma)$ by clause (d) of the claim’s assumption and let $p_1$ be as defined in $\exists_\gamma(b)(\delta)$. \hfill \[c33s\]

Claim 1.22. We have $\mathbb{P}_m(L_1) = \mathbb{P}_m(L_2)$ (i.e. as quasi orders) and $\mathbb{P}_m(L_1) < \mathbb{P}_m$ for $\ell = 1, 2$ when:

\(\square\) (a) $m_1 \leq M m_2$

(b) $L_0 \subseteq L_1 \subseteq L_{m_1}$

(c) $L_0$ is an initial segment of $L_1$

(d) $\mathbb{P}_m(L_0) = \mathbb{P}_m(L_0)$

(e) $\mathbb{P}_m(L_0) < \mathbb{P}_m$ for $\ell = 1, 2$

\[c33s\]
(f) if $t \in L_1 \setminus L_0$ then $t \notin M_m^2$ (but see present $\mathbb{B}_n$) and $L_{m_1} \cap (t/E_m) = L_{m_2} \cap (t/E_m) \subseteq L_1$.

\{c3n\}

Remark 1.23. Used only in the proof of $\mathbb{B}_{1,4}$ inside the proof of 3.19, so we could have used $M_\beta, C$ from there.

Proof. For $\ell \in \{1, 2\}$ let $\bar{L}_\ell = \langle L_{\ell,i} : i < 4 \rangle$ be defined by:

\[ \begin{align*}
\oplus_1 & & (a) & L_{\ell,0} = L_0 \\
(b) & & L_{\ell,1} = L_1 \\
(c) & & L_{\ell,2} = \{ s \in L_m : s \leq m, \text{ for some } t \in L_0 \} \\
(d) & & L_{\ell,3} = L_{m_\ell}.
\end{align*} \]

Clearly

\[ \oplus_2 (a) \quad (m_\ell, \bar{L}_\ell) \text{ satisfies the assumptions of 1.21 hence} \]

\[ (b) \quad \mathbb{P}_m(L_{\ell,1}) < \mathbb{P}_m(L_{\ell,3}) \text{ which means } \mathbb{P}_m(L_1) < \mathbb{P}_m \text{ for } \ell = 1, 2. \]

Why $\oplus_2$? Clearly it suffices to prove clause (a), so we just have to check clauses

\[ (+) (a) - (f) \text{ of 1.22.} \]

Clause (+)(a):

By $\oplus_1(d), L_{\ell,3} = L_{m_\ell}$ hence is an initial segment of $L_{m_\ell}$ and by $\oplus(c), L_{\ell,2}$ is an initial segment of $L_{m_\ell}$ which is $L_{\ell,2} \subseteq L_{\ell,3}$.

Clause (+)(b):

For the first statement, $L_{\ell,1} \subseteq L_{\ell,3}$ is trivial by $\oplus_1(d) + \oplus_1(b) + \Box(a), (b)$. The second statement says $L_{\ell,0} = L_{\ell,1} \cap L_{\ell,2}$. Now $L_{\ell,0} \subseteq L_{\ell,1}$ by $\Box(a), (b)$ of the claim and $\oplus_1(a), (b)$. Also $L_{\ell,0} \subseteq L_{\ell,2}$ holds by $\oplus_1(c)$ (and $\oplus_1(a)$). Together $L_{\ell,0} \subseteq L_{\ell,1} \cap L_{\ell,2}$; to prove the inverse inclusion assume $s \in L_{\ell,0} \setminus L_{\ell,1}$, so as $s \in L_{\ell,2}$ by $\oplus_1(c)$ there is $t \in L_0$ such that $s \leq m, t$. But $s \in L_{\ell,1} = L_1$ so by $\Box(e)$ of the claim we have $s \in L_0$ as promised.

Clause (+)(c):

Holds by $\Box(c)$ of the claim.

Clause (+)(d):

By clause $\Box(f)$ of the claim and $\Box_2(c), L_{\ell,3}$ is an initial segment of $L_{m_\ell}$, hence by 1.11(e) we have $\mathbb{P}_m(L_{\ell,2}) < \mathbb{P}_m = \mathbb{P}_m(L_{\ell,3})$. By $\Box(c) \mathbb{P}_m(L_{\ell,0}) < \mathbb{P}_m$; so together as $L_{\ell,0} \subseteq L_{\ell,2}$, we have $\mathbb{P}_m(L_0) < \mathbb{P}_m(L_{\ell,2})$.

Clause (+)(e), (f):

Holds by $\Box(f)$ of the claim.

So $\oplus_2$ holds indeed. So now we deal with the other half.

Proof of: $\mathbb{P}_m(L_1) = \mathbb{P}_m(L_1)$.

Let $\langle s_\alpha : \alpha < \alpha(*) \rangle$ list $L_1 \setminus L_0$ such that $s_\alpha \leq m, s_\beta \Rightarrow \alpha \leq \beta$. This is possible as $L_{m_2}$ is well founded.

Now

$\oplus_3$ for $\ell = 1, 2$ and $\alpha \leq \alpha(*)$ let $\bar{L}_\ell^* = \langle L_{\ell,\alpha,i}^* : i < 4 \rangle$ be (so we can omit $\ell$ if $\ell = 0, 1$)

\[ \begin{align*}
(a) & & L_{\ell,0}^* = L_0 \\
(b) & & L_{\ell,1}^* = L_0 \cup \{ s_\beta : \beta < \alpha \}
\end{align*} \]
Why? Similar to the proof of (Case 3). Easily by the definition of the iteration. That is, first we know

\[ P(m|s) \leq q. \]

This is obvious by the assumption given \( P \), as promised.

Next, assume \( p \in P(m|s) \). Then we prove that \( P(m|s) \iff P(m|s) \leq q. \)

Note that \( p \in P(m|s) \iff P(m|s) \leq q. \) Therefore, by the induction hypothesis.

Hence, we can assume \( s, s \in \text{dom}(q) \), so we are assuming \( P(s, s) = \text{sup}((s, s)) \). Thus, similarly, \( P(s, s) = \text{sup}((s, s)) \).

This, as \( P(s, s) = \text{sup}((s, s)) \), the \( P(s, s) \) theorem.

By the last two sentences, the definition of \( \text{sup}((s, s)) \).

So \( P(m|s) \iff P(m|s) \leq q. \) holds also in this case. Note that if \( s \in \text{dom}(q) \), then \( P(m|s) \iff P(m|s) \leq q. \), so we are done proving (1.2).}

\[ (1.2) \]

\[ \Box \]
Definition 1.24. 0) For $L \subseteq L_m$, $m \in M$ let

(a) $dp_m(L) = \cup \{dp_{M_m(t)} + 1 : t \in L \cap M_m\}$

(b) $L_m^{dp,\gamma} = \{t \in L_m : s = t \wedge s \in M_m \text{ then } dp_{M_m}(s) < \gamma\}$; moreover, $\sup \{dp_{M_m}(s) : s \in M_m \text{ and } s < L_m \} < \gamma$.

1) For an ordinal $\gamma$ let $M^c_\gamma$ be the class of $m \in M$ such that, recalling Definition 1.10(3):

$n \leq \gamma$ implies $m \leq M^c_\gamma$.

2) Let $M^c = M^c_\gamma$ be the class of $m$ which $\in M^c_\gamma$ for every $\gamma$.

3) Let $M^c_{\chi,\gamma} = \{m \in M^c_\gamma : |L_m| \leq \chi\}$, similarly $M^c_{\chi,\infty}$.

Observation 1.25. 1) Of course, $M^c_{\gamma_2} \subseteq M^c_{\gamma_1}$ and $L_m^{dp,\gamma_2} \subseteq L_m^{dp,\gamma_1}$ are initial segments of $L_m$.

2) In 1.24(1), the following are equivalent:

(a) $P_m(L_m^{dp,\gamma_1}) < P_m(L_m^{dp,\gamma_2})$ for every $\gamma$

(b) $P_m < P_m$.

Proof. 1) Easy. 2) First, concerning (a) $\Rightarrow$ (b), note that for $\gamma$ large enough we have $P_m(L_m^{dp,\gamma_1}) = P_m$, so clear. Assume (b), note that $L_m^{dp,\gamma_1}$ is an initial segment of $L_m$, hence $P_m(L_m^{dp,\gamma_1}) < P_m$ for $t = 1, 2$ by 1.11(c), hence we have $P_m(L_m^{dp,\gamma_1}) < P_m$ for $t = 1, 2$ by 1.11(c).

Crucial Claim 1.26. If $\chi \geq 2^{\lambda_2}$ and $m \in M^c_\chi$ then for some $n$ we have $m \leq M$ $n \in M_{\chi}$ and $n \in M^c_\chi$.

Proof. Let $\mathcal{S} = \{n : (m|M_m) \leq M n$ and $L_n \setminus M_m = t/E''_n$ for some $t$ hence $|L_n| \leq \lambda_2\}$. We define a two-place relation $\mathcal{S}$ on $\mathcal{X}$:

- $(n_1, n_2) \in \mathcal{S}$ if there is an isomorphism $h$ from $n_1$ onto $n_2$ over $m|M_m$; that is: an isomorphism from $L_{n_1}$ onto $L_{n_2}$ over $m$ such that $t \in L_{n_1} \Rightarrow u_{n_2,h(t)} = \{h(s) : s \in u_{n_1,t}\}$ and $t \in L_{n_1} \Rightarrow \mathcal{P}_{n_2,h(t)} = \{\{h(s) : s \in u : u \in \mathcal{P}_{n_1,t}\} \text{ and } s, t \in L_{n_1} \Rightarrow (sE''_n t \Leftrightarrow h(s)E''_n t).\}

Clearly $\mathcal{S}$ is an equivalence relation.

By our assumptions $\chi \geq 2^{\lambda_2}$ and $n \in \mathcal{X} \Rightarrow |L_n| \leq \lambda_2 \wedge (\forall t \in L_n)(\mathcal{P}_{n,t} \subseteq [L_n, c]^{\lambda_2})$ hence recalling $\lambda_2 = (\lambda_2)^\chi$ clearly $\mathcal{S}$ has $\leq 2^{\lambda_2}$ equivalence classes and let $(n_\alpha : \alpha < 2^{\lambda_2})$ be a set of representatives (not necessary, but no harm in allowing repetitions).

By 1.17(2) and 1.18 we can find $n$ such that:

- $(*)$ 1) $m \leq M n \in M_{\chi}$

- for every $\alpha < 2^{\lambda_2}$ we can find $\langle t_{\alpha,i} : i < \chi \rangle$ such that
(a) \( t_{\alpha,i} \in L_n \setminus L_m \)

(\beta) (\alpha \neq \beta) \lor (i \neq j) \Rightarrow t_{\alpha,i}/E_n \neq t_{\beta,j}/E_n

(\gamma) n!(t_{\alpha,i}/E_n) is \( \delta \)-equivalent to \( n_2 \), see 1.8(0) on \( t_{\alpha,i}/E_n \). 

Let us prove that \( n \) is as required. Let \( n \leq_{M} n_1 \leq_{M} n_2 \) and define \( \mathcal{F} \) as the set of functions \( f \) such that some \( L_1, L_2 \):

\( (*)_2 \) (a) \( L_\ell \subseteq L_{n_2} \)
(b) \( M_m = M_n \subseteq L_1 \cap L_2 \)
(c) \( L_\ell \setminus M_m \) has cardinality \( \leq \lambda_2 \)
(d) \( L_\ell \) is \( E_{\text{ns}} \)-closed, i.e. \( t \in L_\ell \setminus M_m \Rightarrow t/E_{\text{ns}} \subseteq L_\ell \)
(e) \( f \) is an isomorphism from \( n_2 \rceil L_1 \) onto \( n_2 \rceil L_2 \) over \( M_m \), i.e.

- \( f \) is a one-to-one mapping from \( L_1 \) onto \( L_2 \)
- \( f[M_m] \) is the identity
- \( f \) maps \( \leq_{n_2} \rceil L_1 \) onto \( \leq_{n_2} \rceil L_2 \)
- for \( s, t \in L_1 \) we have \( sE'_{n_2} t \Leftrightarrow f(s)E'_{n_2}f(t) \)
- for \( s, t \in L_1 \) we have \( s \in u_{n_2,t} \Leftrightarrow f(s) \in u_{n_2,f(t)} \)
- for \( t \in L_1 \) we have \( \mathcal{P}_{n_2,f(t)} = \{ \{ f(s) : s \in u \} : u \in \mathcal{P}_{n_2,t} \} \).

Clearly

\( (*)_3 \) if \( f \in \mathcal{F} \) and \( L' \subseteq L_{n_1}, L'' \subseteq L_{n_2} \) and \( |L'| + |L''| \leq \lambda_2 \) then for some \( g \in \mathcal{F} \) extending \( f \) we have \( L' \subseteq \text{Dom}(g), L'' \subseteq \text{Rang}(g) \) and \( \text{Rang}(g) \setminus (L'' \setminus \text{Rang}(f)) \subseteq L_{n_1} \).

We can finish as in the parallel of the Tarski-Vaught criterion for \( L_{\infty, \lambda_2} \). That is, first we can prove by induction on the ordinal \( \gamma < |L_{n_2}| \) and really just \( \gamma < |M_{n_2}| \) that:

\( (*)_4 \) letting \( L_\gamma = L_{n_2}^{d_{\text{pp}}} \), if \( g \in \mathcal{F} \) then

(a) \( g \) maps \( \text{Dom}(g) \cap L_\gamma \) onto \( \text{Rang}(g) \cap L_\gamma \)
(b) \( g \) induces an isomorphism \( \tilde{g} \) from \( \mathbb{P}_{n_2}(\text{Dom}(g) \cap L_\gamma) \) onto \( \mathbb{P}_{n_2}(\text{Rang}(g) \cap L_\gamma) \), that is: \( \tilde{g}(p) = q \) iff

- (\alpha) \( p \in \mathbb{P}_{n_2}(\text{Dom}(g) \cap L_\gamma) \)
- (\beta) \( q \in \mathbb{P}_{n_2}(\text{Rang}(g) \cap L_\gamma) \)
- (\gamma) \( g \) maps \( \text{Dom}(p) \) onto \( \text{dom}(g) \) and \( s \in \text{dom}(p) \Rightarrow \text{tr}(p(s)) = \text{tr}(g(q(s))) \)
- (\delta) if \( s \in \text{Dom}(g) \), \( g(s) = t \in \text{Rang}(g) \) and \( f(p(s)) = B_{p(s)}(\ldots, \eta_{r_{p(s)}(\zeta)}, \ldots) \zeta < \xi_{p(t)} \)

and \( f(q(t)) = B_{q(t)}(\ldots, \eta_{r_{q(t)}(\zeta)}, \ldots) \zeta < \xi_{q(t)} \) then \( \xi_{q(t)} = \xi_{p(s)} \), \( B_{q(t)} = B_{p(s)} \), and \( \zeta < \xi_{p(s)} \Rightarrow r_{q(t)}(\zeta) = g(r_{p(s)}(\zeta)) \)
- (\varepsilon) moreover in (\delta) we have \( \iota(s, p) = \iota(t, q) \) and if \( t < \iota(s, p) \) then \( w_{p,s,t} = w_{q,t,i}, B_{p(s),i} = B_{q(t),i} \).

Second,

\( (*)_5 \) \( \mathbb{P}_{n_2}(L_\gamma \cap L_{n_1}) \leq \mathbb{P}_{n_2}(L_\gamma) \).
[Why? By the definitions $P_n(L \cap L_n) \subseteq P_n(L)$ as quasi orders.
Also if $p_1, p_2 \in P_n(L \cap L_n)$ are compatible in $P_n(L)$ let $q \in P_n(L)$ be a common upper bound there. We can find an $E_n$-closed $L' \subseteq L \cap L_n$ of cardinality $\leq \lambda_2$ (recalling $n \in \mathcal{F}$ \Rightarrow $|L_n| \leq \lambda_2$) such that $p_1, p_2 \in P_n(L')$ and $E_n$-closed $L'' \subseteq L$ of cardinality $\leq \lambda_2$ such that $L' \subseteq L''$ and $q \in P_n(L'')$. Now we can find $f_1 \in F$ such that $p_1, p_2 \in P_n(L'')$ and $E_n$-closed $L' \subseteq L'' \subseteq L_n$ of cardinality $\leq \lambda_2$ such that $q \in P_n(L'')$. Now we can find $f_2 \in F$ extending $f_1$ with $Dom(f_2) = \cup \{ t/E_n : t \in L'' \}$ and $Rang(f_2) \subseteq Rang(f_1) \cup \{ c \}$ and $\hat{f}_2(q) \in P_n(L)$ as required in Definition 1.24. □]

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[Why? We prove this by induction on $\gamma$, as in proving the Tarski-Vaught criterion is sufficient (we shall later in the proof of 3.19, more specifically $\Box_1$ proves a similar statement in detail with weaker assumptions).]

Hence (using $\gamma = |L_n|^+$)

(*) $P_{n_1} \subseteq P_{n_2}$.

Hence for every $L \subseteq L_n$ by 1.17(4) we have $P_{n_1}(L) = P_{n_2}(L)$ as required in 1.26.
\$2. \textbf{The Corrected } \mathbb{P}_m \$

**Definition 2.1.** Let $\mathbb{P}$ be a forcing notion and $Y \subseteq \mathbb{P}$ and $\chi$ a regular cardinal.

1) Let $L_\chi(Y)$ be the set of sentences formed from $\{ p : p \in P_Y \}$ closing under the operations $\neg p$ and $\bigwedge_{i<\alpha} p_i$, for $\alpha < \chi$; so propositional logic.

2) For $G \subseteq \mathbb{P}$ and $\psi \in L_\chi(Y)$ we define the truth value $\psi[G]$ naturally (by induction on $\psi$ starting with $p[G] = \text{true} \iff p \in G$).

3) Let $L_\chi(Y, \mathbb{P})$, the $L_\chi$-closure of $Y$ for $\mathbb{P}$, $(Y \subseteq \mathbb{P};$ if $Y = \mathbb{P}$ we may omit $Y$) be the following partial order:

- set of elements $\{ \psi \in L_\chi(Y, \mathbb{P}) : \models p \text{ false} \}$
- the order $\psi_1 \leq \psi_2$ iff $\models \psi_2 \implies \psi_1$.

4) The completion of $\mathbb{P}$ is the $L_\chi$-closure of $\mathbb{P}$ for $\mathbb{P}, L_\chi(\mathbb{P})$ where $\chi$ is minimal such that $\mathbb{P}$ satisfies the $\chi$-c.c.

**Claim 2.2.** For a $\chi$-c.c. forcing notion $\mathbb{P}$ and $Y \subseteq \mathbb{P}$ we have:

\begin{enumerate}
  \item $L_\chi^+(Y, \mathbb{P})$ is a forcing notion.
  \item $\mathbb{P} \cap L_\chi^+(Y, \mathbb{P})$ under the natural identification\(^9\)
  \item $L_\chi^+(Y, \mathbb{P}) \triangleq L_\chi(Y, \mathbb{P})$
  \item $\mathbb{P} \cap L_\chi^+(Y, \mathbb{P})$ when $\chi_1 \leq \chi_2$ are regular (and $\chi_1 \geq \chi$)
  \item if $\mathbb{P}$ satisfies the $\chi_1$-c.c. and $\chi_1 \leq \chi_2$ are regular then $L_\chi^+(Y, \mathbb{P})$ is essentially equal to $L_\chi^+(Y, \mathbb{P})$, i.e. up to the natural equivalence of elements in a quasi order
  \item if $Y = \mathbb{P}$ then $\mathbb{P}$ is a dense subset of $\mathbb{P}$.
\end{enumerate}

**Definition 2.3.** Let $m \in M$.

1) For $t \in L_m, \varepsilon < \lambda$ and $\eta \in \prod_{i<\varepsilon} \theta_i$, let $p = p^*_\varepsilon, \eta \in \mathbb{P}_m$ be the function with domain $\{t\}$ such that $p(t) = (\eta, \eta_0 \lambda)$, i.e. $f_p(t) \in \prod_{i<\varepsilon} \theta_i$ is defined by $f_p(t) = \eta(\varepsilon)$ if $\varepsilon < \ell(\eta)$ and is zero otherwise.

2) For $L \subseteq L_m$ let $Y_L = Y_{m,L} = \{p^*_\varepsilon, \eta : t \in L \text{ and } \eta \in \prod_{i<\varepsilon} \theta_i \text{ for some } \zeta \leq \lambda\}$.

3) For $L \subseteq L_m$ let $\mathbb{P}_m[L]$ be $L^\lambda \cdot [Y_L, \mathbb{P}_m]$, see Definition 2.1.

4) For $L \subseteq L_m$ let $\mathbb{P}_m(L) = \mathbb{P}_m \cap \{p \in \mathbb{P}_m : \text{supp}(p) \subseteq L\}$, see Definition 1.10(1), recalling 1.10(2),(3).

5) $\mathbb{P}_m$ is the partial order with the same set of elements as $\mathbb{P}_m$ and $\leq_{P_m} = \{ (p, q) : p, q \in \mathbb{P}_m \text{ and } \text{no } r \text{ above } q \text{ is incompatible with } p \}$ and $\mathbb{P}_m(L) = \mathbb{P}_m \cap \{p \in \mathbb{P}_m : \text{supp}(p) \subseteq L\}$, we may “forget” the distinction\(^10\).

6) For quasi orders $Q_1, Q_2$ let $Q_1 \lhd Q_2$ mean that:

\begin{enumerate}
  \item $s \in Q_1 \Rightarrow s \in Q_2$
  \item $s \leq_{Q_1} t \Rightarrow s \leq_{Q_2} t$
\end{enumerate}

7) For quasi orders $Q_1, Q_2$ let $Q_1 \lhd_{ic} Q_2$ means that $Q_1 \lhd Q_2$ and

\begin{enumerate}
  \item if $s, t \in Q_1$ are incompatible in $Q_1$ then they are incompatible in $Q_2$.
\end{enumerate}

\(^9\text{Really } \mathbb{P} \subseteq L_\chi^+[P] \text{ see 2.3, because } L_\chi^+[P] = \{ p \leq q \iff q \models p \in G \}.\)

\(^10\text{Really the only difference is the possibility that } \text{dom}(p) \nsubseteq \text{dom}(q), \text{ see 1.19.}\)
8) We define $\ll$ similarly.

Claim 2.4. Let $m \in \mathcal{M}$ and $L \subseteq L_m$.

1) $\mathbb{P}_m[L_m]$ is equivalent to $\mathbb{P}_m$ as forcing notions, in fact, $\mathbb{P}_m = \mathbb{P}_m(L_m) \ll \mathbb{P}_m[L_m]$ and is a dense subset of it under the natural identification (see 2.1(1)), but we should pedantically use $\mathbb{P}_m(L_m)$ or use $\ll$.

2) $\mathbb{P}_m[L_m]$ is $(< \lambda)$-strategically complete and is $\lambda^+$-c.c.

3) $\mathbb{P}_m(L) \subseteq \mathbb{P}_m[L]$ as sets and $\mathbb{P}_m[L] \ll \mathbb{P}_m[L_m]$ and $\mathbb{P}_m(L) \ll \mathbb{P}_m[L]$.

4) If $G \subseteq \mathbb{P}_m$ is generic over $V$ and $\eta_t = \eta_t(G)$ for $t \in L$ and $G^+_t = \{ \psi \in \mathbb{P}_\lambda^+(Y_L) : \psi[G] = \text{true} \}$, see 2.1(3), then $V[G] = \mathbb{V}[G^+] = \{ \eta_t : t \in L_m \}$.

5) In part (4), moreover $G^+$ is a subset of $\mathbb{P}_m[L]$ generic over $V$.

6) $\mathbb{P}_m(L_1) \subseteq \mathbb{P}_m(L_2)$ and $\mathbb{P}_m[L_1] \ll \mathbb{P}_m[L_2]$ when $L_1 \subseteq L_2 \subseteq L_m$.

7) Assume $I$ be a $\lambda^+$-directed partial order and $\bar{L} = \langle L_r : r \in I \rangle$ be such that $r \in I_s \Rightarrow L_r \subseteq L_{s_0}$ and $r < s \Rightarrow L_r \subseteq L_s$ and $L = \cup \{ L_r : r \in I \}$. Then $\mathbb{P}_m[L] = \cup \{ \mathbb{P}_m[L_r] : r \in I \}$ and $\mathbb{P}_m(L) = \cup \{ \mathbb{P}_m(L_r) : r \in I \}$.

8) If $m \in M_{\text{sc}}$ and $m \leq M_m \leq M_m$ then $\mathbb{P}_m[L_m] \ll \mathbb{P}_m[L_m]$.

Remark 2.5. What about $\mathbb{P}_m(L) \ll \mathbb{P}_m[L]$ and $\mathbb{P}_m(L) \ll \mathbb{P}_m[L]$?

The problem is the mapping $p \mapsto p \upharpoonright L$ defined in 3.1(3) does not have the required properties of preserving order as the forcing appears there.

Proof. 1) Easy.

2) Follows by part (1) and 1.13.

3) The first statement by their definitions, the second statement by part (1).

4), 5), 6) Should be clear recalling 1.13(9).

7) Easy, recalling 1.13(7).

8), 9) Easy.

\[ \square_2 \]

The Uniqueness Claim 2.6. There is an isomorphism from $\mathbb{P}_{m_1}[M_1]$ onto $\mathbb{P}_{m_2}[M_2]$ which (recalling Definition 2.3(1)) maps $p^*_t, \eta$ to $p^*_t, \eta$ for $t \in M_1, \eta \in \{ \prod_{\varepsilon < \zeta} : \zeta < \lambda \}$ when:

\[ \begin{align*}
& (a) \quad m_\ell \in M_{\infty}^\ell \quad \text{for} \quad \ell = 1, 2 \\
& (b) \quad M_\ell = M_m, \quad \text{for} \quad \ell = 1, 2 \\
& (c) \quad h \quad \text{is an isomorphism from} \quad M_1 \quad \text{onto} \quad M_2 \\
\end{align*} \]

Proof. By renaming without loss of generality $M_1 = M_2$ call it $M$ and $h$ is the identity and $L_m \cap L_m = M$. Let $m_0 = m_1 | M = m_2 | M$ so $m_0 \leq_M m_\ell$ for $\ell = 1, 2$ and $L_m = L_m | L_m$.

By 1.18, there is $m$ such that $m_1 \leq_M m$ and $m_2 \leq_M m$. As $m_1, m_2 \in M_{\infty}$ by 2.4(9) we have $\mathbb{P}_m[M] = \mathbb{P}_m[M]$ and $\mathbb{P}_m[M] = \mathbb{P}_m[M]$ so together we get the desired conclusion.

Definition 2.7. 1) We call $m \in M$ reduced when $L_m = M_m$.

2) For $m \in M$ let $\mathbb{P}_m^{\text{cr}}$ be $\mathbb{P}_n[L_m]$ and $\mathbb{P}_m^{\text{cr}}[L]$ be $\mathbb{P}_n[L]$ for $L \subseteq L_m$ when $m \leq M$.

Remark 2.8. 1) Why is $\mathbb{P}_m^{\text{cr}}[L]$ well defined? see below.

2) Here "cr" stands for corrected.

The interest in the definition is because
Claim 2.9. 1) If \( m \in M \) and \( L \subseteq L_m \) then \( \mathcal{P}^e_m[L] \) is well defined.  
2) \( \mathcal{P}^e_m[M_m] \) is well defined and depend only on \( m|M_m \).  
3) If \( m \leq n \) and \( L_1 \subseteq L_2 \subseteq L_m \) then \( \mathcal{P}^e_m[L_1] = \mathcal{P}^e_n[L_1] \leq \mathcal{P}^e_n[L_2] \leq \mathcal{P}^e_n \).

Proof. 1) By 1.26, \( \mathcal{P}^e_m[L] \) has at least one definition so it suffices to prove uniqueness. So assume \( m \leq n \) \( m_\ell \in M_\ell \) for \( \ell = 1, 2 \) and we should prove that \( \mathcal{P}_m(L) = \mathcal{P}_n(L) \). Without loss of generality \( L_m \cap L_{m_2} = L_m \). Now by 1.18 we can find \( n \in M \) such that \( m_1 \leq M n \) and \( m_2 \leq M n \); as \( m_\ell \in M_\ell \) see Definition 1.24 we have \( \mathcal{P}_m \leq \mathcal{P}_n \) for \( \ell = 1, 2 \). As in the end of the proof of 2.6 we are done. 
2) By 2.6. 
3) Follows from Definition 1.24(2) and 2.7(2). \( \square_{2.9} \)

Discussion 2.10. 1) But we like to prove for reduced \( m \in M \) and \( M \subseteq M_m \) that \( \mathcal{P}_{m|M} \leq \mathcal{P}_m \). This is delayed to 3.26. We now prove it suffices. 
2) The reader may understand 2.11 without reading the rest of §2.3 by ignoring clause (A)(d), (e) reading 2.1, 2.2.

Conclusion 2.11. For every ordinal \( \delta \), there is \( q = (\mathcal{P}_n, \eta_n : \alpha \leq \delta) \) such that:

(A) (a) \( \langle \eta_n : \alpha \leq \delta \rangle \) is \( \triangleleft \)-increasing 
(b) \( \eta_n \) is a \( \mathcal{P}_{\alpha+1} \)-name of a member of \( \prod_{\varepsilon < \lambda} \theta_\varepsilon \) which dominates \( (\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{\mathcal{P}_\alpha} \) 
(c) \( \eta_n \) is a generic for \( \mathcal{P}_\alpha+1/\mathcal{P}_\alpha \), moreover \( \langle \eta_\beta : \beta < \alpha \rangle \) is a generic for \( \mathcal{P}_\alpha \) 
(d) \( p \in \mathcal{P}_\alpha \) iff \( p \in L_{\mathcal{P}_\alpha}(Y_\alpha, \mathcal{P}_\alpha) \) where \( Y_\alpha \) is defined as in 2.3(2) with \( \alpha \) here standing for \( L \) there and see 2.1 
(e) \( \mathcal{P}_\alpha \) is \( (\lambda, \alpha) \)-strategically complete and \( \lambda^+ \)-c.c. 
(f) if \( \delta \leq \delta \) has cofinality \( \lambda \) (actually \( \geq \lambda \) suffice) then \( \mathcal{P}_\delta = \cup \{ \mathcal{P}_\alpha : \alpha < \delta \} \) 
(g) \( \mathcal{P}_\delta \), has cardinality \( |\delta|^{\mathcal{P}_\delta} \)

(B) if \( \mathcal{U} \subset \delta \), then the complete subforcing generated by \( \langle \eta_\alpha : \alpha \in \mathcal{U} \rangle \) is isomorphic to \( \mathcal{P}_{\mathcal{U}^{\mathcal{P}_\mathcal{U}}} \)

(C) if \( G \subseteq \mathcal{P}_{\mathcal{U}} \) is generic over \( \mathcal{V} \) and \( \eta_\alpha = \eta_\alpha[G] \) for \( \alpha < \delta \) and \( \eta_\alpha \in \prod_{\varepsilon < \lambda} \theta_\varepsilon \) for \( \alpha < \delta \) and \( \{ \langle \alpha, \varepsilon : \alpha < \delta, \varepsilon < \lambda \land \eta'_\alpha(\varepsilon) \neq \eta_\alpha(\varepsilon) \} \) has cardinality \( < \lambda \) then also \( \langle \eta'_\alpha : \alpha < \delta \rangle \) is a generic for \( \mathcal{P}_{\mathcal{U}} \), determining a different \( G' \) but \( \mathcal{V}[G'] = \mathcal{V}[G] \)

(D) in clause (C), moreover if \( \mathcal{U} \subseteq \delta \) and \( \langle \alpha_i : i < \mathcal{U}^{\mathcal{P}_\mathcal{U}} \rangle \) is an increasing order then for some unique \( \mathcal{G}'' \subseteq \mathcal{P}_{\mathcal{U}^{\mathcal{P}_\mathcal{U}}} \) generic over \( \mathcal{V} \), \( i < \mathcal{U}^{\mathcal{P}_\mathcal{U}} \Rightarrow \eta''_\alpha = \eta_i[G''] \).

Proof. Without loss of generality \( \lambda_1 \geq |\delta| \).

We define \( m \in M \) by:

(*) (a) \( L_m = \delta \), 
(b) \( M_m = \delta \), 
(c) \( \eta_{m_\alpha} = \alpha \) and \( \mathcal{P}_{m_\alpha} = [\alpha]^{\mathcal{P}_m} \) for \( \alpha < \delta \), 
(d) \( E'' = \emptyset \).
It is easy to check that indeed $m \in M$ and let $n \in M_{\infty}$ be such that $m \leq_{M} n$, exists by the Crucial Claim 1.26 and let $P_{\alpha} = P_{n}[\{i : i < \alpha\}]$ for $\alpha \leq \delta_{*}$.

Now clearly clauses (A),(C) hold and $P_{\delta} = P_{\infty}^{m}$ by 2.7(2), 2.9(1) and clause (A)(b) holds by 1.13(6A). As for clause (B), note that for every $L \subseteq \delta_{*}$, for $P_{m}[L]$ the sequence $\eta_{L} = \langle \eta_{\alpha} : \alpha \in L \rangle$ is generic for $P_{m}[L]$ by Definition 2.3.

For $M \subseteq \delta_{*}$ let $\alpha = \text{otp}(M)$ and $h : M \rightarrow \alpha$ be $h(i) = \text{otp}(i \cap M)$ so $h$ is an isomorphism from $m_{\alpha}^{M}$ onto $m_{\alpha}$ for $m_{\alpha}^{M}$ in $\text{cer}[L_{\infty \lambda \alpha}]$ such that $\delta_{*} \leq \delta_{*}$ here standing for $m_{1}, m_{2}, M_{1}, M_{2}$ there we have $h$ induces an isomorphism from $P_{\text{cer}}^{m_{\alpha}}[M]$ onto $P_{\text{cer}}^{m_{\alpha}}[L_{\infty \lambda \alpha}]$. Similarly, $id_{\alpha}$, induces an isomorphism from $P_{\text{cer}}^{m_{\alpha}}$ onto $P_{\infty}^{m_{\alpha}}[\alpha]$. Together we get clause (B). Also Clause (D) follows so we are done. \Box_{2.11}

Similarly we can deal with such iterations with partial memory and spell out how $P_{\text{cer}}^{m_{\alpha}}[L]$ is defined from a $(\langle \alpha \rangle \lambda)$-support iteration with partial memory.

**Conclusion 2.12.** Assume $M$ is a well founded partial order and $\bar{u} = \langle u_{i} : i \in M \rangle$, $u_{i} \subseteq M_{\infty \xi}$ and $P_{h} = \langle P_{h}^{i} : t \in M \}$ with $P_{h}^{i} \subseteq [u_{i}]^{< \lambda}$ is closed under subsets. Then we can find $\beta(\ast), h, P_{\beta} = P_{0, \beta}, P_{1, \beta}, Q_{\alpha}, \eta_{\alpha}, \eta_{\beta}$ and $P_{u}$ (for $\beta \leq \beta(\ast), \alpha < \beta(\ast), s \in M$ and $u \subseteq M$) and $\bar{u}, \bar{P}$ such that:

(A) $\langle P_{\beta}, Q_{\alpha} \mid \beta \leq \beta(\ast), \alpha < \beta(\ast) \rangle$ is $(\langle \alpha \rangle \lambda)$-support iteration

(b) $\bar{u} = \langle u_{\beta} : \beta < \beta(\ast) \rangle$ such that $u_{\beta} \subseteq \beta$

(c) $\eta_{\alpha}$ a $P_{\alpha+1}$-name of a member of $\prod_{\xi < \lambda} \theta_{\xi}$

(d) $\langle \eta_{\alpha} : \alpha < \beta \rangle$ is generic for $P_{\beta}$

(e) $Q_{\alpha}$ is defined as in Definition 1.10(4)

(f) $\bar{u}^{P_{\beta(\ast)}} \quad \eta_{\beta} \in \prod_{\xi < \lambda} \theta_{\xi}$ dominate every $\nu \in \prod_{\xi < \lambda} \theta_{\xi}$ from $V[\{\eta_{\alpha} : \alpha \in u\}]$ when $u \subseteq P_{\beta}$

(B) $h$ is a one-to-one function from $M$ into $M$ by $\beta(\ast)$; stipulate $h(\infty) = \beta(\ast)$

(b) $s \subseteq M \Leftrightarrow h(s) < h(t)$

(c) $\text{Rang}(h) = \{h(s) : s \subseteq u_{i}\}$

(d) $P_{h(t)} \cap [\text{Rang}(h)]^{\subseteq \lambda} = \{h(s) : s \subseteq u_{i}\} : u \subseteq P_{h}$

(C) $P_{1, \beta} = L_{\alpha+1}^{M}(Y_{\beta}, \beta_{\beta})$ where we let $Y_{\beta} = \{p_{\alpha \nu} \mid \alpha < \beta, \nu \in \prod_{\xi \in \lambda} \theta_{\xi}$ for some $\zeta < \lambda \}$, see 2.4, 2.3(1)

(b) $P_{1, u} = L_{\alpha+1}^{u}(Y_{u}, P_{\beta})$, where $Y_{u}$ is defined similarly when $u \subseteq \beta(\ast)$

(c) $P_{u}$ is a forcing notion for $u \subseteq M$ and $y_{u}$ is a $P_{u}$-name for $s \subseteq M$ such

(d) $h$ induces an isomorphism from $P_{u}$ onto $P_{\infty \lambda \alpha}$ for $u \subseteq M$ and $y_{u}$ to $\eta_{h(s)}$ for $s \subseteq M$

(e) $\langle \eta_{h(s)} : s \subseteq u_{i} \rangle$ is generic for $P_{u}$ for $u \subseteq M$

(D) $\langle \beta \rangle \lambda$-strategically complete and $\lambda^{+}$-c.c.

\[11]_{\text{In general not onto!}}
(c) if \( M_1, M_2 \subseteq M \) and \( f \) is an isomorphism from \( M_1 \) onto \( M_2 \) as partial orders such that \( t \in M_1 \Rightarrow u'_{h(t)} \cap M_2 = \{ f(s) : s \in u'_t \cap M_1 \} \) and \\
t \in M_1 \Rightarrow S_{h(t)} \cap [M_2]^{<\lambda} = \{ \{ f(s) : s \in u \cap M_1 \} : u \in S_t \} then \\
the mapping \( h(s) \mapsto h(f(s)) \) induce an isomorphism from the forcing notion \( P'_M \) onto \( P'_M \).

Proof. Similarly. \( \square \)
§ 3. THE MAIN CONCLUSION

We have a debt from §2, i.e. see discussion 2.10. Toward this we explicate what appear in the proof of 1.26.

Definition 3.1. Let \( m \in M \).

1) We say \( m \) is \( \mu \)-wide when for every \( t \in L_m \setminus M_m \) there are \( t_\alpha \in L_m \setminus M_m \) for \( \alpha < \mu \) such that:
   
   (a) \( m\langle t_\alpha / E_m \rangle \) is isomorphic to \( m\langle t/E_m \rangle \) over \( M_m \)
   
   (b) \( \beta < \gamma < \mu \Rightarrow t_\beta / E_m^\prime \neq t_\gamma / E_m^\prime \).

2) Moreover, it is unique.

3) For every \( L \subseteq L_m \) we say \( m \) is wide when it is \( |L_m| \)-wide.

4) Let \( \mathcal{P}_m \) be the set of the functions \( f \) such that for some \( L_1, L_2 \):

   (a) \( f \) is an isomorphism from \( m\langle L_1 \rangle \) onto \( m\langle L_2 \rangle \)
   
   (b) \( L_\ell \) is a subset of \( L_m \) for \( \ell = 1, 2 \)
   
   (c) \( M_m \subseteq L_\ell \) and \( f \) is the identity
   
   (d) \( L_\ell \) is \( E_m \)-closed, i.e. \( M_m \subseteq L_\ell \) and if \( t \in L_m \setminus M_m \) and \( t \in L_\ell \) then \( t/E_m \subseteq L_\ell \) for \( \ell = 1, 2 \)
   
   (e) \( \{ t/E_m : t \in L_\ell \setminus M_m \} \) has cardinality \( \leq \lambda \).

5) If \( L_1, L_2 \subseteq L_m \) and \( f \) is an isomorphism from \( m\langle L_1 \rangle \) onto \( m\langle L_2 \rangle \) then we let \( \bar{f} \) be the one-to-one mapping\(^{12}\) from \( \mathbb{P}_m(L_1) \) onto \( \mathbb{P}_m(L_2) \) as in \((*)_4(b)\) of the proof of 1.26.

6) Let \( \mathbb{P}_m(L) \) be \( \{ p \in \mathbb{P}_m(L) : \text{fsupp}(p) \subseteq M_m \} \) for some \( t \in L_m \setminus M_m \) with the order inherited from \( \mathbb{P}_m \).

Observation 3.2. Let \( m \in M \) and \( L \subseteq L_m \).

1) The projection of \( q \in \mathbb{P}_m \) to \( L \) is well defined and in \( \mathbb{P}_m(L) \).

2) Moreover, it is unique.

3) If \( p \in \mathbb{P}_m(L) \) is the projection of \( q \in \mathbb{P}_m(L_m) \) then \( p \leq q \).

4) For every \( p \in \mathbb{P}_m \), \( p \) is equivalent to \( \mathcal{J}_p := \{ p \upharpoonright L : L = t/E_m \text{ for some } t \in \text{fsupp}(p) \} \cup \{ p \upharpoonright M_m : \text{fsupp}(p) \subseteq M_m \} \), i.e. \( \mathbb{P}_m \) “\( p \in \mathbb{G}_m \) if \( \mathcal{J}_p = \mathbb{G}_m \).”

5) For every \( p \in \mathbb{P}_m \), \( p \) is equivalent to \( \mathcal{J}_p := \{ p[t, \ell] : t \in \text{dom}(p) \} \), i.e. \( \mathbb{P}_m \) “\( p \in \mathbb{G}_m \) if \( \mathcal{J}_p = \mathbb{G}_m \).”

where \( p[t, \ell] \in \mathbb{P}_m \) has domain \( \{ t \} \) and \( p(t) = (\text{tr}(p), \mathcal{B}_p(t), ((\eta_{p(t)}(z)) : z \in \omega_{p(t)})) \); recall Definition 1.9 for the meaning of \( \iota(p(t)) \), \( \mathcal{B}_p(t, \ell) \), etc.

\(^{12}\)We have not said “order preserving”!
Remark 3.3. 1) Note that the choice in Definition 1.9(c)(γ) to require \( (f_{p(t)}: t \in \iota(p)) \) exists, is necessary for 3.2(4), which is crucial in the proof of 3.26.

2) In Definition 3.1(1A) we can choose “\( \lambda \)-wide” as when applying it, if \( X = \text{fsupp}(p) \) then for some \( Y \subseteq L_m \) of cardinality \( < \lambda \), \( X \subseteq \cup \{t/E_m: t \in Y\} \).

Proof. Easy, e.g.

4) If \( \text{fsupp}(p) \subseteq M_m \) the statement says \( p \in G \) iff \( \{p\} \subseteq G \), so trivial hence we assume \( \text{fsupp}(p) \not\subseteq M_m \). Now if \( t \in \text{fsupp}(p) \) then trivially \( p \upharpoonright (t/E_m) \leq p \), hence \( p \in G \) implies \( \mathcal{S}_p \subseteq G \).

For the other direction assume \( q \in \mathbb{P}_m \) forces \( \mathcal{S}_p \subseteq G \subseteq \mathbb{P}_m \) and we shall prove that \( q \) is compatible with \( p \), this suffices, so toward contradiction assume \( q, p \) are incompatible.

Without loss of generality \( \text{Dom}(p) \subseteq \text{Dom}(q) \) and recalling \( t \in \text{fsupp}(p) \Rightarrow q \upharpoonright (t/E_m) \in G \) clearly \( s \in \text{dom}(p) \Rightarrow q \upharpoonright “\text{tr}(p(s)) \subseteq y_s” \) so necessarily \( s \in \text{Dom}(p) \Rightarrow \text{tr}(p(s)) \subseteq \text{tr}(q(s)) \). Recalling 1.13(8), as \( p, q \) are incompatible there are \( s \in \text{Dom}(p) \cap \text{Dom}(q) \) and \( q_1 \) such that \( q|L_m,<s \leq q_1 \in \mathbb{P}_m(L_m,<s) \) and \( q_1 \upharpoonright “q(s),p(s)” \) are incompatible in \( \mathbb{Q}_Q \).

As \( \text{tr}(p(s)) \subseteq \text{tr}(q(s)) \) this implies \( q_1 \upharpoonright “\text{tr}(q(s)),p(s)” \) are incompatible, i.e. \( f_{p(s)}(\text{tr}(q(s))) \not\subseteq \text{tr}(q(s)) \). Recalling Definition 1.9(c)(γ), \( q_1 \upharpoonright “\text{there is } \iota < \iota(s,p) \text{ such that } f_{p(s)},q_1(\text{tr}(q(s))) \text{ are incompatible}” \). Possibly increasing \( q_1 \), we can fix \( \iota \). But letting \( t \in \text{fsupp}(p) \subseteq L_m \) be such that \( r_{p(s)},t \subseteq t/E_m \) this implies that \( q_1 \upharpoonright “p \upharpoonright (t/E_m) \not\subseteq G \text{ or } \text{tr}(q(s)) \not\subseteq y_s” \). However, \( q_1, q \) are compatible and this contradicts the choice of \( q \).

\[ \square_{3.2} \]

Claim 3.4. 1) The \( n \) constructed in 1.26 satisfies: if \( n \leq M \mathbf{n}_1 \text{ then } \mathbf{n}_1 \text{ is wide, } \]

(if \( n_1 \in M \text{ even very wide} \) and full.

2) If \( n \in M_{\mathcal{E}} \text{ and } n \leq M \mathbf{n}_1 \text{ then } \mathbf{n}_1 \in M_{\mathcal{E}} \).

Proof. 1) Holds by the proof of 1.26.

2) Holds by Definition 1.24(1),(2).

\[ \square_{3.4} \]

Claim 3.5. Assume \( m \) is wide.

1) If \( f \in \mathcal{F}_m \) and \( X \subseteq L_m \) has cardinality \( \leq \lambda \text{ then there is } g \text{ such that: } \]

(a) \( g \in \mathcal{F}_m \)

(b) \( f \subseteq g \)

(c) \( \text{Dom}(g) = \text{Rang}(g) \)

(d) \( X \subseteq \text{Dom}(g) \).

2) If \( g \in \mathcal{F}_m \) and \( \text{Dom}(g) = \text{Rang}(g) \text{ then } g^+ = g \cup \text{id}_{\mathbb{P}_m(\text{Dom}(g))} \text{ is an automorphism of } m \).

3) If \( f \) is an automorphism of \( m \) then it naturally induces an automorphism \( \bar{f} \) of \( \mathbb{P}_m(L_m) \) similarly to \( \bar{f} \) from (a)(b) of the proof of 1.26.

4) If \( f \in \mathcal{F}_m \text{ then } \text{it induces an isomorphism } \bar{f} \text{ of } \mathbb{P}_m(\text{Dom}(f)) \text{ onto } \mathbb{P}_m(\text{Rang}(f)). \)

Proof. 1) Easy by the definition of wide in 3.1(1) and of \( \mathcal{F}_m \) in 3.1(4).

2) Just read the definition of \( m \in M \) and of \( f \in \mathcal{F}_m \), in particular:

(a) if \( t_1, t_2 \in L_m \backslash M_m \) are not \( E_m^+ \)-equivalent then \( (t_1/E_m) \cap (t_2/E_m) = M_m \)

and \( \leq M \{(t_1/E_m) \cup (t_2/E_m) \) is determined by \( \leq M |(t_1/E_m), \leq M |(t_2/E_m) \)

(b) \( g|L_m = \text{id}_{L_m} \).
3. Naturally by the definition.
4) Let \( g \in \mathcal{F} \) be as in part (1) and let \( h = g^{+m} \) so an automorphism of \( m \) which extends \( g \) as in part (2). So \( h \) is an automorphism of \( P_m(L_m) \) and clearly \( f = h[\mathcal{P}_m(Dom(f))] \) is as required. \( \square_{3,5} \)

**Claim 3.6.** Let \( m \in M \) and \( L \subseteq L_m \).
If \( f_1, f_2 \in \mathcal{F}_m \) then:

(a) \( f_1 \subseteq f_2 \Rightarrow \hat{f}_1 \subseteq \hat{f}_2 \)
(b) \( f_1 = f_2^{-1} \Rightarrow \hat{f}_1 = (\hat{f}_2)^{-1} \).

**Proof.** Just consider the definition, see 3.1(5) and (5) of the proof of 1.26. \( \square_{3,6} \)

**Observation 3.7.** 1) \( P_m^-(L) \subseteq P_m(L) \), see Definition 3.1(6).
2) For every \( p \in P_m \) there is a sequence \( (p_i : i < i(\ast)) \) of \( \leq \lambda \) members of \( P_m^- \) such that \( \bigwedge P_m^-[L_m^+] = p \in \mathcal{G} \) iff \( \{p_i : i < i(\ast)\} \subseteq \mathcal{G} \).

**Proof.** 1) By their definitions.
2) Should be clear, see Definition 3.1(6) and 3.2(4). \( \square_{3,7} \)

**Definition 3.9.** Assume \( m \in M \).
1) Let \( \mathcal{G}_m \) be the set of pairs \((t, \hat{s})\) such that \( t \in L_m \setminus M_m \) and \( \hat{s} \in \zeta(t/E_m^\prime) \) for some \( \zeta < \lambda^+ \); we may write \( \hat{s} \) instead of \((t, \hat{s})\) as usually \( \hat{s} \) determines \( t \).
2) By induction on the ordinal \( \gamma \) we define when \((t_1, \hat{s}_1), (t_2, \hat{s}_2) \) are \( \gamma \)-equivalent in \( m \) or are \((m, \gamma)\)-equivalent:

(a) if \( \gamma = 0 \), letting \( L_\ell = (M_m \cup Rang(\hat{s}_\ell)) \) for \( \ell = 1, 2 \) there is \( h \) such that

(\( a \)) \( h \) is an isomorphism from \( m|L_1 \) onto \( m|L_2 \)
(\( b \)) \( h \) maps \( \hat{s}_1 \) to \( \hat{s}_2 \)
(\( c \)) \( h|\mathcal{M}_m \) is the identity

(\( d \)) \( h \) induces an isomorphism from \( P_m(L_1) \) onto \( P_m(L_2) \) (as defined in 1.7(4)(b))
(\( e \)) moreover, \( h \) induces an isomorphism from \( P_m[L_1] \) onto \( P_m[L_2] \), as defined in 2.6, \( p^\ast_{\ell, \eta} \mapsto p^{h(\ell), \eta}_{\ell, \eta} \), see 2.3(3)

(\( b \)) if \( \gamma = \beta + 1 \) then for every \( \varepsilon < \lambda^+ \) and \( \ell \in \{1, 2\} \) and \( \hat{s}_\ell' \in \varepsilon(t_\ell/E_m^\prime) \) there is \( \hat{s}_\ell' \in \varepsilon(t_\ell/E_m^\prime) \) such that \((t_1, \hat{s}_1 \hat{s}_\ell), (t_2, \hat{s}_2 \hat{s}_\ell) \) are \( \beta \)-equivalent

(\( c \)) if \( \gamma \) is a limit ordinal then \((t_1, \hat{s}_1), (t_2, \hat{s}_2) \) are \( \beta \)-equivalent for every \( \beta < \gamma \).

**Remark 3.10.** 1) Note above that if \( \hat{s}_\ell \) is the empty sequence then \( t_\ell \) would not be determined by \( \hat{s}_\ell \), still in those cases the equivalence just means \( \hat{s}_1 = \hat{s}_2 \).
2) We can use \( t/E_m^\prime \) or \( t/E_m^\prime \) instead of \( t/E_m^\prime \) as everything is over \( M_m \).

**Claim 3.11.** For \( m \in M \) and ordinal \( \alpha \) the number of equivalence classes of “being \((m, \alpha)\)-equivalent” is \( \leq \beth_1 + \alpha + 1(\lambda_1) \).
Proof. By induction on $\alpha$.

Case 2: $\alpha = 0$

Note that the set of elements of $\mathbb{P}_m(M_m \cup \text{Rang}(\bar{s}))$ has cardinality $\leq 2^{\lambda_1}$ (and even $\leq (\lambda_1)\lambda$) and depends just on $M_m \cup \text{Rang}(\bar{s})$ but there are $\beth_2(\lambda_1)$ possibilities for the quasi order on $\mathbb{P}_m(L_1)$ and even for $\mathbb{P}_m[L_1]$.

Case 2: $\alpha$ is a limit ordinal

By clause (c) of Definition 3.9, the number of $\alpha$-equivalence classes is
\[ \prod_{\beta < \alpha} \beth_{1+\beta+1}(\lambda_1) \leq (\beth_{1+\alpha+1}(\lambda_1))^{\beth_{1+\alpha}} = \beth_{1+\alpha+1}(\lambda_1). \]

Case 3: $\alpha = \beta + 1$

Clearly every $\alpha$-equivalence class can be coded as a set of $\beta$-equivalence classes hence the number of $\alpha$-equivalence classes is $\beth_{2+\alpha+1}(\lambda_1) = \beth_{1+\alpha+1}(\lambda_1)$, as promised.

Definition 3.12. For an ordinal $\beta$, let $\mathcal{F}_{m,\beta}$ be the set of function $f$ such that for some $t^i_1,s^i_1$ for $i < \iota(*)$ and $\ell \in \{1,2\}$ we have:

(a) $\iota(*) < \lambda^+$
(b) $(t_1^i : i < \iota(*))$ is a sequence of pairwise non-$E_m^\alpha$-equivalent members of $L_m \setminus M_m$
(c) $s^i_1 \in \zeta(i)(t_1^i/E_m^\alpha)$ where $\zeta(i) < \lambda^+$
(d) $(t_1^i, s_1^i, t_2^i, s_2^i)$ are $\beta$-equivalent (members of $\mathcal{F}_m$)
(e) $f$ is an isomorphism from $m[L_1]$ onto $m[L_2]$ when $L_\ell = \cup \{\text{Rang}(s_2^i) : i < \iota(*)\} \cup M_m$
(f) $f|_{M_m}$ is the identity
(g) $f$ maps $s^i_1$ to $s_2^i$ for $i < \iota(*)$.

2) For $f \in \mathcal{F}_{m,0}$ we define $\bar{f}$ as the mapping from $\mathbb{P}_m(\text{Dom}(f))$ onto $\mathbb{P}_m(\text{Rang}(f))$ induced by $f$; see clause 3.9(2)(a)(z).

Claim 3.13. Assume $m$ is wide. The conditions $p,q \in \mathbb{P}_m[L_m]$ are compatible when for some $\psi$ the following condition holds:

\[
\text{(suchthat)}_{p,q,\psi} \quad (a) \quad \psi \in \mathbb{P}_m[M_m] \\
\quad (b) \quad \text{w supp}(p) \cap \text{w supp}(q) \subseteq M_m, \text{ see Definition 1.10(1)(b), equivalently} \\
\quad s \in \text{supp}(p) \setminus M_m, t \in \text{supp}(q) \setminus M_m \Rightarrow \neg (s E_m^\alpha t) \\
\quad (c) \quad \psi \leq \varphi \in \mathbb{P}_m[M_m] \text{ then } \varphi,p \text{ are compatible in } \mathbb{P}_m[L_m] \\
\quad (d) \quad \psi,q \text{ are compatible in } \mathbb{P}_m[L_m], \text{ equivalently } q \not\models_{\mathbb{P}_m} \text{"}\psi[G] = \text{false}".
\]

Remark 3.14. 1) We can use (suchthat)$_{p,q,\psi}^{(a)}$ omit clause (d) and add to clause (c):

\[
\text{and } \varphi,q \text{ are compatible in } \mathbb{P}_m[L_m].
\]

2) We use $\lambda > \aleph_0$ in the proof, to eliminate it we can imitate the completeness theorem for $L_{\aleph_0,\aleph_0}$.

Proof. We choose $(p_n,q_n,\psi_n)$ by induction on $n$ such that:

\[
\exists_n (a)(\alpha) \quad \text{(suchthat)}_{p_n,q_n,\psi_n} \text{ holds if } n \text{ is even} \\
\quad (\beta) \quad \text{(suchthat)}_{q_n,\bar{p}_n,\psi_n} \text{ holds if } n \text{ is odd}
\]
(b) \((p_n, q_0, \psi_0) = (p, q, \psi)\n \;
\) if \(n = 2m + 1\) and \(s \in \text{dom}(p_{2m}) \cap M_m\), then \(s \in \text{dom}(q_{2m+1})\)
and \(\text{tr}(p_{2m}(s)) \subseteq \text{tr}(q_{2m+1}(s))\).

(d) \(\text{if } n = 2m + 2 \text{ and } s \in \text{dom}(q_{2m+1}) \cap M_m \text{ then } s \in \text{dom}(p_{2m+2})\)
and \(\text{tr}(q_{2m+1}(s)) \subseteq \text{tr}(p_{2m+2}(s))\).

(e) \(\text{if } n = m + 1 \text{ then } p_m \leq p_n, q_m \leq q_n.\)

Case 1: For \(n = 0\) use clause (b).

Case 2: \(n = 2m + 1.\)

So the triple \((p_{2m}, q_{2m}, \psi_{2m})\) is well defined, let \(u_{2m} = \text{Dom}(p_{2m}) \cap M_m\) and let \(\nu = (\nu_s : s \in u_{2m})\) be defined by \(\nu_s = \text{tr}(p_{2m}(s))\).

Clearly

\[ (*1) \quad \psi_{2m} \vdash p^*_{s, \nu_s} \text{ for } s \in u_{2m}. \]

[Why? Clearly \(p_{2m} \vdash p^*_{s, \nu_s}\), i.e. \(p^*_{s, \nu_s} \leq p_{2m}\) in \(\mathbb{P}_m[L_m]\), hence if \(\psi_{2m} \not\vdash p^*_{s, \nu_s}\), then \(\psi' = \psi_{2m} \land \neg p^*_{s, \nu_s} \in \mathbb{P}_m[M_m] \), \(\supseteq \psi_{2m}\) hence compatible with \(p_{2m}\), contradiction, see clause (c) in (suchthat)\((p, q, \psi)\).]

\[ (*2) \quad \text{there is } q_{2m} \in \mathbb{P}_m[L_m] \text{ which is above } q_{2m}\] and \(\psi_{2m}\) hence \(s \in u_{2m}\)
implies \(\nu_s \subseteq \text{tr}(q_{2m}(s))\) and \(s \in \text{Dom}(q_{2m})\).

[Why? By clause (d) of (suchthat)\((p_{2m}, q_{2m}, \psi_{2m})\) which holds by \(\mathcal{P}_m(a)(\alpha)\) recalling \(\mathbb{P}_m(L_m)\) is dense \(\mathbb{P}_m[L_m]\); the “hence” by \((*)1).\]

\[ (*3) \quad \text{there is } \psi'_{2m} \in \mathbb{P}_m[M_m] \text{ such that:} \]

(a) \(\psi'_{2m} \leq \varphi \in \mathbb{P}_m[M_m] \text{ then } \varphi, q_{2m}' \text{ are compatible in } \mathbb{P}_m[L_m]; \)
(b) \(\text{if } s \in u_{2m} \text{ then } \psi'_{2m} \vdash p^*_{s, \nu_s} \).
(c) \(\psi_{2m} \leq \psi'_{2m}. \)

[Why? Obvious using the \(\lambda^+\)-c.c., i.e. \(\psi'_{2m} = \psi_{2m} \land \neg(\forall \{\varphi : \varphi \in \mathcal{I}\})\) where \(\mathcal{I}\) is a max antichain of members \(\psi \in \mathbb{P}_m[M_m] \) satisfying \(\psi \downarrow q_{2m} \in \mathbb{P}_m[L_m];\)

\[ (*4) \quad \text{without loss of generality } w\text{supp}(q_{2m}) \cap w\text{supp}(p_{2m}) \subseteq M_m. \]

\[ \{ e10 \} \quad \text{[Why? As } \mathbf{m} \text{ is wide using automorphisms of } \mathbf{m}, \text{ i.e. by 3.5.]} \]

\[ (*5) \quad \text{there is } p'_{2m} \in \mathbb{P}_m[L_m] \text{ which is above } p_{2m} \text{ and above } \psi'_{2m}. \]

[Why? By the choice of \(\psi'_{2m}\) and clause (e) of (suchthat)\((p_{2m}, q_{2m}, \psi_{2m})\) which holds by \(\mathcal{P}_m(a)(\alpha).\).]

\[ (*6) \quad \text{without loss of generality } f\text{supp}(p'_{2m}) \cap f\text{supp}(q_{2m}) \subseteq M_m. \]

\[ \{ e10 \} \quad \text{[Why? As } \mathbf{m} \text{ is wide using 3.5.]} \]

Lastly, let \(p_n = p'_{2m}, q_n = q_{2m}', \psi_n = \psi_{2m}'\) and check.

Case 3: \(n = 2m + 2\)

Similar to case 2 the roles of the \(p\)’s and the \(q\)’s interchanged.

Having carried the induction we can find \(p_\ast\) the upper bound of \(\{p_n : n < \omega\}\) as

\[ (*7) \quad \text{Dom}(p_\ast) = \bigcup_n \text{Dom}(p_n); \text{ in fact, also } f\text{supp}(p_\ast) = \bigcup_n f\text{supp}(p_n). \]
(b) if \( s \in \text{Dom}(p_n) \) then \( \text{tr}(p_n(s)) = \bigcup_{k \geq n} \text{tr}(p_k(s)). \)

Similarly let \( q_* \) be the upper bound of \( \{ q_n : n < \omega \} \) as in 1.13(4), so again:

\[
\begin{align*}
(\ast)_8 (a) & \ \quad \text{Dom}(q_*) = \bigcap_{n} \text{Dom}(q_n), \text{ in fact also } \text{fsupp}(q_*) = \bigcup_{n} \text{fsupp}(q_n) \\
(b) & \ \quad \text{if } s \in \text{Dom}(q_n) \text{ then } \text{tr}(q_n(s)) = \bigcup_{k \geq n} \text{tr}(q_k(s)).
\end{align*}
\]

Hence

\[
\begin{align*}
(\ast)_9 (a) & \ \quad p_*, q_* \in \mathbb{P}_m \\
(b) & \ \quad \text{Dom}(p_*) \cap \text{Dom}(q_*) \subseteq M_m, \text{ in fact, fsupp}(p_*) \cap \text{Dom}(q_*) \subseteq M_m \\
(c) & \ \quad \text{Dom}(p_*) \cap M_m = \text{Dom}(q_*) \cap M_m \\
(d) & \ \quad \text{if } s \in \text{Dom}(p_*) \cap M_m, \text{ equivalently, } s \in \text{Dom}(p_*) \cap \text{Dom}(q_*) \text{ then } \text{tr}(p_*(s)) = \text{tr}(q_*(s)).
\end{align*}
\]

[Why? Clause (a) by properties of \( \mathbb{P}_m \) and \( p_n \leq p_{n+1}, q_n \leq q_{n+1} \) see above, clause (b) as \( \text{Dom}(p_{2m}) \cap \text{Dom}(q_{2m}) \subseteq M_m \) as (suchthat) \( p_{2m}, q_{2m}, \psi_{2m} \), clause (c) by \( \mathbb{P}_n(c), (d) \), the first conclusion and clause (d) by \( \mathbb{P}_n(c), (d) \), the second conclusion.]

It follows that \( p_*, q_* \) are compatible in \( \mathbb{P}_m \) but \( p = p_0 \leq p_*, q = q_0 \leq q_*, \) so \( p, q \) are compatible as promised. \[\boxcheck\]

**Claim 3.15.** The set \( \{ \psi_i : i < i(\ast) \} \cup \{ \psi_\ast \} \) has a common upper bound in \( \mathbb{P}_m[L_m] \) when:

\[
\begin{align*}
(\ast) (a) & \ \quad m \in M \text{ is wide} \\
(b) & \ \quad i(\ast) < \lambda \\
(c) & \ \quad t_i \in E_m \setminus M_m \text{ for } i < i(\ast) \\
(d) & \ \quad t_i, t_j \text{ are not } E_m^\ast\text{-equivalence for } i < j < i(\ast) \\
(e) & \ \quad \psi_\ast \in \mathbb{P}_m[M_m] \\
(f) & \ \quad X_i = t_i/E_m \\
(g) & \ \quad \psi_i \in \mathbb{P}_m[X_i] \\
(h) & \ \quad \text{if } \mathbb{P}_m[M_m] = \{ \psi_\ast \leq \varphi \} \text{ and } i < i(\ast) \text{ then } \psi_i, \varphi \text{ are compatible} \\
& \ \quad \text{in } \mathbb{P}_m[L_m] \text{ equivalently in } \mathbb{P}_m[X_i].
\end{align*}
\]

**Remark 3.16.** Note: \( \lambda \)-wide is enough.

**Proof.** As \( \psi_\ast \in \mathbb{P}_m[M_m] \), there is \( p \in \mathbb{P}_m \) such that \( p \models \mathbb{P}_m[\psi_\ast[G_m] = \text{true}. \) As \( m \) is wide by 3.5 there is an automorphism \( f \) of \( m \) such that \( i < i(\ast) \Rightarrow f''(\text{wsupp}(p)) \cap X_i \subseteq M_m \), hence without loss of generality \( i < i(\ast) \Rightarrow \text{wsupp}(p) \cap X_i \subseteq M_m \). Now we choose \( p_i \) by induction on \( i \leq i(\ast) \) such that:

\[
\begin{align*}
(\ast) (a) & \ \quad p_i \in \mathbb{P}_m \\
(b) & \ \quad \langle p_j : j \leq i \rangle \text{ is increasing} \\
(c) & \ \quad \text{if } s \in \text{Dom}(p_i), i < i(\ast) \text{ then } t \geq g(\text{tr}(p_{i+1}(s))) > i(\ast) \\
(d) & \ \quad p_0 = p \\
(e) & \ \quad \text{if } i = j + 1 \text{ then } p_i \models \text{"}_j \psi_j[G_m] = \text{true} \\
(f) & \ \quad \text{wsupp}(p_i) \text{ hence also } \text{wsupp}(p_i) \text{ is disjoint to } \bigcup \{ X_j \setminus M_m : j \in [i, i(\ast)) \}. 
\end{align*}
\]
This is sufficient for the claim as $p_{i(\ast)}$ is as required. So let us carry the induction.

For $i = 0$ use clause (d), for $i$ limit by 1.12(4) we know that $(p_j : j < i)$ has a $\leq_{\text{p}}$-upper bound $p_i$ with domain $= \cup \{ \text{Dom}(p_j) : j < i \}$ and $\text{w supp}(p_i) \subseteq \cup \{ \text{w supp}(p_j) : j < i \}$ by 1.13(4), hence $p_i$ is as required, in particular as in clause (f).

Lastly, assume $i = j + 1$, now there is $\varphi_j \in \mathbb{P}[\mathbb{M}]$ such that $\varphi_j \leq \varphi \in \mathbb{P}[\mathbb{M}]$. By an assumption $p_j \equiv \text{"j"}$ as $p_0$ forces this hence $\psi_\ast \leq \varphi_j$. As $\varphi_j \in \mathbb{P}[\mathbb{M}]$ by clause (h) of the assumption $\psi_j, \varphi_j$ are compatible in $\mathbb{P}[\mathbb{M}]$ hence have a common upper bound $\varphi_j^+ \in \mathbb{P}[X_j]$, so there is $q_j^+ \in \mathbb{P}$ above $\varphi_j$ and $\psi_j$. As $\mathbf{m}$ is wide without loss of generality $\text{w supp}(q_j^+) \cap \text{w supp}(p_j) \subseteq \mathbb{M}_m$. Together (see 3.13) (such that) $p_j, q_j^+$ holds hence by 3.13 $p_j, q_j^+$ has a common upper bound called $p_i$. As $\mathbf{m}$ is wide, without loss of generality $\text{w supp}(p_i) \cap X_i = \mathbb{M}_m$ for $j \in [i + 1, i(\ast))$. 

Clearly $p_i$ is as required so we have finished the induction. So we are done. \(\square\)

**Conclusion 3.17.** If $\mathbf{m}$ is wide and $f \in \mathbb{F}_m, \beta$ and $L_1, L_2$ its domain and range respectively then $f$ induces an isomorphism $f$ from $\mathbb{P}[\mathbb{M}(L_1)]$ onto $\mathbb{P}[\mathbb{M}(L_2)]$.

**Remark 3.18.** 1) See Definition 3.1(5); note that this claim is not covered by Definition 3.1(4).

2) Here we use 3.2(4), so the choice in Definition 1.9(\(\ast\)) is justified (see Remark 3.3(1)) used below in the proof.

3) We could have separated the definition of "analyze" and its properties.

4) Note that in Definition 3.9, we deal only with $L_1 \subseteq t/E_m$ for some $t$.

5) How come even $\beta = 0$ is suitable for 3.17? The point is clause (a)(\(\ast\)) of Definition 3.9(2). But no real harm using larger $\beta$.

**Proof.** By the definitions, clearly $f$ is a one-to-one function from $\mathbb{P}[\mathbb{M}(L_1)]$ onto $\mathbb{P}[\mathbb{M}(L_2)]$. Next assume $p_1, q_1 \in \mathbb{P}[\mathbb{M}(L_1), \text{Dom}(p_1) \subseteq \text{Dom}(q_1)$ and let $p_2 := f(p_1), q_2 := f(q_1)$; clearly they belong to $\mathbb{P}[\mathbb{M}(L_2)]$. We shall prove that $\mathbb{P}_m \models p_1 \leq q_1$ iff $\mathbb{P}_m \models p_2 \leq q_2$.

Let $\langle t_1^i : i < i(\ast) \rangle$ be such that:

\(\oplus_1\) (a) $t_1^i \in \text{fsupp}(q_1) \setminus \mathbb{M}_m \subseteq L_1$ such that $\text{fsupp}(q_1)$ is included in $\cup \{ t_1^i / E_m : i < i(\ast) \}$

(b) $\langle t_1^i : i < i(\ast) \rangle$ are pairwise non $E''_m$-equivalent.

Next let

\(\oplus_2\) (c) $t_1^j = f(t_1^j)$

(d) let $t_2 = \langle t_2^i : i < i(\ast) \rangle$ without loss of generality $\text{fsupp}(p_i) \subseteq \cup \{ t_2^i / E_m : i < j(\ast) \} \cup \mathbb{M}_m$, so $j(\ast) \leq i(\ast)$.

For $i < i(\ast)$ let $\psi_{1,i}^+ \in \mathbb{P}_m[\mathbb{M}_m]$ be such that: $\psi \in \mathbb{P}_m[\mathbb{M}_m]$ is compatible with $q_{1,i} := q_1 \setminus (t_1^i / E_m)$ (the projection!) if and only if $\psi_{1,i}^+ \in \mathbb{P}_m[\mathbb{M}_m]$; clearly exists as $\mathbb{P}_m$ satisfies the $\lambda^\ast$-c.c. Let $\psi_{1,i}^+ = \wedge \{ \psi_{1,i}^+ : i < i(\ast) \}$.

Now $\psi_{1,i}^+ \in \mathbb{P}_m[\mathbb{M}_m]$ as $q_1 \equiv \text{"j"}$ as $p_0$ forces this hence $\psi_\ast \leq \psi_{1,i}^+$. We will say $\psi_{1,i}^+ = \langle \psi_{1,i}^+ : i < i(\ast) \rangle$ which analyze $q_1$ or $\langle q_1, t_1 \rangle$ when the above holds.

Next choose $\psi_{1,i}^+, \langle \psi_{1,i}^+, p_{1,i} : i < j(\ast) \rangle$ which analyze $p_1, \langle t_1^i : i < j(\ast) \rangle$. Why possible? As above.
Lastly, let \( \psi_i^* \) be a sequence \( p_i, q_i, \tilde{\psi}_i^*, \psi_i^* \) where \( \tilde{\psi}_i^* \) and \( \psi_i^* \) are defined by \( \psi \). So by claim 3.15 there is a function \( f \) from \( L^+ \) onto \( L_m^+ \) defined by \( p_i^* \mapsto p_i^{(t)} \).

(*) for \( \ell = 1, 2 \) the sequence \( (p_{\ell}, q_{\ell}, \tilde{\psi}_{\ell}, \psi_{\ell}, \phi_{\ell}) \) where \( \tilde{\psi}_{\ell} = \langle \psi_{\ell}, q_{\ell} : i < i(\ell) \rangle \) satisfy the same demands as listed above for \( \ell = 1, 2 \), that is

(a) \( \phi_{\ell} \) analyze \( (q_{\ell}, t_{\ell}) \) for \( \ell = 1, 2 \)

(b) \( \phi_{\ell} \) satisfies (for \( \ell = 1, 2 \)) for \( \ell = 1, 2 \).

Why? Think, recalling \( f \) is an isomorphism from \( m \) onto \( m \) ((\( t_i^{(l)} / E_m \)) \), so by Definition 3.9, 3.12; so by claim 3.15 to \( (q_{\ell}, j < i(\ell)) \cup \{ \} \) to get \( q_{\ell}^+ \).

Hence by 3.2(4) the condition \( q_{\ell}^+ \) is above \( q_{\ell} \) but \( q_{\ell}^+ \models \phi_{\ell} \wedge \phi_{\ell} \mapsto \psi_{\ell}, \theta \) are compatible. By the two last sentences \( q_{\ell}^+, q_{\ell} \) are incompatible, in \( m \) equivalently in \( m \). So indeed \( \neg (B) \) or \( \neg A \).

For the other direction assume condition \( B \) holds, but condition (A) fails and we shall get a contradiction. So there is \( q_{\ell}^+ \) in \( m \) above \( q_{\ell} \) incompatible with \( q_{\ell} \).

For each \( i < i(\ell) \) the \( q_{\ell}^+ \) is above \( q_{\ell} \) but \( q_{\ell}^+ \models \phi_{\ell} \wedge p_{\ell,i} \models \theta \) hence \( m \models q_{\ell}^+ \models \psi_{\ell}, \theta \) are compatible, and as we are assuming clause (B) we have \( m \models q_{\ell}^+ \models \phi_{\ell} \wedge p_{\ell,i} \leq q_{\ell}^+ \) holds.

Hence by 3.2(4), \( q_{\ell}^+ \) is above \( p_{\ell} \), contradiction. Indeed \( (B) \Rightarrow (A) \).

Together, \( \Box \) holds. Now clearly \( (B) \) or \( (B)_1 \) follows, see Definition 3.9, 3.12; so by \( \Box \) we have \( (A)_1 \Rightarrow (A)_2 \) which is the desired conclusion. □3.17

Claim 3.19. We have \( m_1 \leq m \) when:

(a) \( m_1 \leq m \)

(b) if \( t \in L_m \setminus M_m \) and \( s \in \zeta(t/E_m) \), \( \zeta < \lambda^+ \) then we can find \( t_i, s_i \) for \( i < \lambda^+ \) such that:

(a) \( t_i \in L_m \setminus M_m \)

(b) \( t_i / E_m \neq t_j / E_m \) when \( i \neq j < \lambda^+ \)

(c) \( s_i \in \zeta(t_i / E_m) \)

(\( i, s_i \) is \( \xi \)-equivalent to \( t, s \) in \( m \) where \( 13 \xi = 1 \).

(c) \( m \) is wide.

\(^{13}\) No real harm in using larger \( \xi \).
Remark 3.20. In the proof we use conclusion 3.17 but not clause \((a)(e)\) of Definition 3.9(2).

Proof.

\(\Xi_1\) for \(f \in \mathcal{F}_{m,\beta}\)

(a) \(\hat{f}\) preserves “\(p_2\) is above \(p_1\) in \(P_m\)”, and its negations

(b) if \(\beta > 0\) then \(\hat{f}\) preserves also incompatibility in \(P_m\).

\(\Xi_2\) if \(p_i \in P_{m,i}\) for \(i < \iota(*) < \lambda^+\) and \(p \in P_m\) then there is \(p^*\) such that:

(a) \(p^* \in P_{m,i}\), equivalently \(p^* \in P_m(L_{m,i})\)

(b) \(P_{m,i} \models \text{“}p_i \leq p^*\text{”}\) iff \(P_m \models \text{“}p_i \leq p\text{”}\)

(c) \(P_{m,i} \models \text{“}p_i, p^*\text{”}\) are compatible” iff \(P_m \models \text{“}p_i, p\text{”}\) are compatible”.

[Why? Clause (a) holds by 3.17. For clause (b) use clause (a) and Definitions 3.9 and 3.12 or see the proof of \(\Xi_2\).]

\(\Xi_3\) if \(q_i \in P_m\) be such that: if \(p_i, p\) are compatible in \(P_m\) then \(p_i \leq q_i \land p \leq q_i\).

We can find \(L_1 \subseteq L_2\) such that

\(M_m \subseteq L_1 \subseteq L_{m,i}, |L_1\setminus M_m| \leq \lambda\)

\(|\{p_i : i < \iota(*)\}| \subseteq P_m(L_1)\)

\(L_1 \subseteq L_2 \subseteq L_m, |L_2\setminus M_m| \leq \lambda\) and \(p, q_i \in P_m(L_2)\) for \(i < \iota(*)\).

By the assumption of the claim there is \(f \in \mathcal{F}_{m,1}\) such that:

(a) \(\text{Dom}(f) \subseteq \cup\{(t/E_m^m) \cap L_2 : t \in L_2\} \cup M_m\)

(b) \(t \in L_1 \Rightarrow f|(t/E_m^m) \cap L_2 = \text{id}_{(t/E_m^m)^\cap L_2}\)

(c) \(q \in \{q_i : i < \iota(*)\} \cup \{p\} \cup \{p_i : i < \iota(*)\}\) and \(t \in \text{Dom}(q)\setminus M_m\) then \(\text{fsupp}(q(t)) \subseteq \text{Dom}(f)\)

(d) \(\text{Rang}(f) \subseteq L_{m,1}\).

Let \(p^* = \hat{f}(p)\): by \(\Xi_1(a)\) clearly clauses \((a),(b)\) of \(\Xi_2\) holds; and the choice of the \(q_i\)'s also the implication “if” of clause (c). The “only if” of clause (c) holds by \(\Xi_1(b)\) so we are done.

\(\Xi_3\) if \(p \in P_m\) then \(p \in P_m\) if \(\text{fsupp}(p) \subseteq L_{m,1}\).

[Why? Obvious.]

Recalling Definition 1.24(0)(b)

\(\Xi_4\) for every ordinal \(\gamma\), we have \(P_{m,i}(L_{m,i,\gamma}) \preceq P_m(L_{m,\gamma})\).

[Why? We shall prove this by induction on \(\gamma\) using \(\Xi_2 + \Xi_3\).

Note that

\(\Xi_{4,1}\)

(a) \(L_{m,\gamma}^P \cap L_{m,1} = L_{m,1,\gamma}^P\)

(b) if \(f \in \mathcal{F}_{m,\beta}, s \in \text{Dom}(f)\) and \(\beta\) is an ordinal then

\(s \in L_{m,\gamma}^P \Leftrightarrow f(s) \in L_{m,\gamma}^P\)

(c) the parallel of \(\Xi_2\) holds for \(P_m(L_{m,\gamma})\) so \(p^* \in P_m(L_{m,\gamma})\)

(d) \(L_{m,\gamma}^P\) is an initial segment of \(L_m\)
(e) \( L_{m_1, \gamma}^{dp} \) is an initial segment of \( L_m \),
(f) \( \mathbb{P}_m(L_{m_1, \gamma}^{dp}) \prec \mathbb{P}_m(L_m) \), similarly for \( m \).

We shall use this freely. The inductive proof on \( \gamma \) splits to three cases.

**Case 1**: \( \gamma = 0 \)

So

- \( E = E_{m_1}^{\gamma} \mid L_m^{dp} \) is an equivalence relation on \( L_m^{dp} \)
- \( E \mid L_{m_1, \gamma}^{dp} = E_{m_1}^{\gamma} \mid L_{m_1, \gamma}^{dp} \)
- if \( t \in L_{m_1, \gamma}^{dp} \) then \( t \notin M_{m_1, \gamma} \) \( \Rightarrow \) \( t/E_{m_1}^{\gamma} = (t/E_{m_1}^{\gamma}) \cap L_{m_1, \gamma}^{dp} = (t/E_{m_1}^{\gamma}) \cap L_{m_1, \gamma}^{dp} \)
- \( \mathbb{P}_m(L_{m_1, \gamma}^{dp}) \) is the product with \( \langle \lambda \rangle \)-support of \((\mathbb{P}_m((t/E_{m_1}^{\gamma}) \cap L_{m_1, \gamma}^{dp}) : t \in L_{m_1, \gamma}^{dp}) \)
- similarly for \( m \).

So the result should be clear.

**Case 2**: \( \gamma = \beta + 1 \)

Let \( M_\beta = \{ s \in M_m : dp_m(s) = \beta \} \), clearly

- \( M_\beta \) is a set of pairwise incomparable elements

- \( a \in M_\beta \Rightarrow L_{m_1, < \beta} \subseteq L_{m_1, \beta} \wedge L_{m, < \beta} \subseteq L_{m_1, \beta} \)

- \( M_\beta \) is disjoint to \( L_{m_1, \beta}, L_{m_1, \beta} \)

- \( M_\beta \subseteq L_{m_1, \gamma}^{dp} \)

- \( L_{m_1, \beta} \cup M_\beta \) is an initial segment of \( L_m \)

- \( L_{m_1, \beta} \cup M_\beta \) is an initial segment of \( L_m \)

As first half we prove

- \( \mathbb{P}_m(L_{m_1, \beta}^{dp} \cup M_\beta) \prec \mathbb{P}_m(L_{m_1, \beta}^{dp} \cup M_\beta) \).

Why? Recalling \( \mathbb{P}_1(a) \), note

- \( (a) \) for \( p, q \in \mathbb{P}_m(L_{m_1, \beta}^{dp} \cup M_\beta) \) we have \( \mathbb{P}_m(L_{m_1, \beta}^{dp} \cup M_\beta) \models \ “p \leq q” \)

- \( \mathbb{P}_m(L_{m_1, \beta}^{dp} \cup M_\beta) \models \ “p \leq q” \).

[Why? Immediate by the definition of the order and the induction hypothesis.]

- \( (b) \) for \( p_1, p_2 \in \mathbb{P}_m(L_{m_1, \beta}^{dp} \cup M_\beta) \) then \( p_1, p_2 \) are compatible in \( \mathbb{P}_m(L_{m_1, \beta}^{dp} \cup M_\beta) \)

iff they are compatible in \( \mathbb{P}_m(L_{m_1, \beta}^{dp} \cup M_\beta) \).

[Why? The implication \( \Rightarrow \) holds by clause \( (a) \). So assume \( p_3 \in \mathbb{P}_m(L_{m_1, \beta}^{dp} \cup M_\beta) \) is a common upper bound of \( p_1, p_2 \) in \( \mathbb{P}_m(L_{m_1, \beta}^{dp} \cup M_\beta) \) equivalently in \( \mathbb{P}_m \).

Now there is \( f \in \mathcal{F}_{m, 1} \) such that

- \( f \upharpoonright (fsupp(p_1) \cup fsupp(p_2)) \) is the identity, moreover
- \( s \in wsupp(p_1) \cup wsupp(p_2) \wedge s \in dom(f) \Rightarrow f(s) = s \),
Hence clearly $f | M_3 = \text{id}_{M_3}$ so by $\mathbb{H}_{4.1}(b)$ we have $\text{Rang}(f) \subseteq L_{m_1,3} \cup M_3$ so $\hat{f}(p_3) \in \mathbb{P}_m(L_{m_1,3} \cup M_3)$.

By $\mathbb{H}_1$ the condition $\hat{f}(p_3)$ is a common upper bound of $p_1, p_2$ in $\mathbb{P}_m$ and by the previous sentence also in $\mathbb{P}_m(L_{m_1,3} \cup M_3)$, so by clause (a) the conclusion of (b) holds.

(c) if $\mathcal{I}$ is a maximal antichain in $\mathbb{P}_m(L_{m_1,3} \cup M_3)$ then $\mathcal{I}$ is a maximal antichain of $\mathbb{P}_m(L_{m_2,3} \cup M_3)$.

[Why? As in the proof of (b) and of $\mathbb{H}_2$.]

So we are done proving $\mathbb{H}_{4.3}$.

Now we return to proving $\mathbb{H}_4$, note

$\mathbb{H}_{4.4}$ let $\mathcal{E} = \{(s_1, s_2) : s_1, s_2 \in L_\ast$ and $s_1/E_m = s_2/E_m$ where $L_\ast = L_{m_1,3} \setminus (L_{m_2,3} \cup M_3) \}$ then

(a) $\mathcal{E}$ is an equivalence relation on $L_\ast$

(b) if $s_1, s_2 \in L_\ast$ and $s_1 \not\leq s_2$ then $s_1 \not\sim s_2$

(c) if $s_1, s_2 \in L_\ast$ and $s_1 \sim s_2$ then $s_1 \in L_{m_1,3} \leftrightarrow s_2 \in L_{m_1,3}$ (and both $\not\in M_3$)

(d) if $s \in L_\ast$ then $L_{m_1}, s \subseteq L_{m_1,3} \cup M_3 \cup (s/\mathcal{E})$

(e) if $s \in L_\ast \cap L_{m_1}$ then $L_{m_1}, s \subseteq L_{m_1,3} \cup M_3 \cup (s/\mathcal{E})$.

Hence let $L_0 = L_{m_1,3} \cup M_3$ and $L_1 = L_{m_1,3} \cup L_{m_1,3} \cup M_3$ they satisfy all the assumptions of 1.22 hence its conclusion, so we are done easily proving Case 2 of $\mathbb{H}_4$.

Case 3: $\gamma$ is a limit ordinal

Clearly $p \in \mathbb{P}_m(L_{m_1,3}, \gamma)$ iff $p \in \mathbb{P}_m(L_{m_1,3})$; also each of them implies $p \in \mathbb{P}_m(L_{m_1,3})$ by the induction hypothesis. Also for $p, q \in \mathbb{P}_m(L_{m_1,3})$ we have $\mathbb{P}_m(L_{m_1,3}) = \{p \leq q\}$ iff $\mathbb{P}_m(L_{m_1,3}) = \{p \leq q\}$ by the definition of the order and the induction hypothesis. Together $\mathbb{P}_m(L_{m_1,3}) \subseteq \mathbb{P}_m(L_{m_1,3})$, (as partial orders).

Next assume that $q_1, q_2 \in \mathbb{P}_m(L_{m_1,3})$ and $p_3$ is a common upper bound of $q_1, q_2$ in $\mathbb{P}_m(L_{m_1,3})$.

We shall find $p_1 \in \mathbb{P}_m(L_{m_1,3})$ such that:

$(*)_1$ (a) $p_1$ is above $q_1, q_2$ (in $\mathbb{P}_m(L_{m_1,3})$) or equivalently in $\mathbb{P}_m(L_{m_1,3})$,

(b) if $p_1 \leq p'_1 \in \mathbb{P}_m(L_{m_1,3})$ then $p'_1, p_3$ are compatible in $\mathbb{P}_m(L_{m_1,3})$.

This clearly suffices; why? e.g. if $\{r_i : i < \iota(*)\} \subseteq \mathbb{P}_m(L_{m_1,3})$ is a maximal antichain of $\mathbb{P}_m(L_{m_1,3})$ but not of $\mathbb{P}_m(L_{m_1,3})$, let $q_1 = q_2 = \emptyset$ and $p_3 \in \mathbb{P}_m(L_{m_1,3})$ be incompatible with every $r_i$; let $p_1$ be as in $(*)_1$, it gives a contradiction.

If $\text{cf} (\gamma) \geq \lambda$ then for some $\gamma_1 < \gamma$ we have $q_1, q_2 \in \mathbb{P}_m(L_{m_1,3})$ and $\text{isupp}(p_3) \cap L_{m_1,3} \subseteq L_{m_1,3}$ and use the induction hypothesis on $\gamma_1$ for clause (a) of $(*)_1$; for
(c11) clause (b) of (1.21) we also recall 1.13(8): (alternatively imitate the case $\text{cf}(\gamma) < \lambda$, choosing “changing our minds” $\gamma \varepsilon < \gamma$ with the induction). So assume $\kappa_0 \leq \text{cf}(\gamma) < \lambda$ and let $(\gamma \varepsilon : \varepsilon < \text{cf}(\gamma))$ be increasing continuous with limit $\gamma$.

Now we choose $p_{1,\varepsilon}$ by induction on $\varepsilon \leq \text{cf}(\gamma)$ such that:

\[
\begin{align*}
\text{(a)} & \quad p_{1,\varepsilon} \in \mathbb{P}(L_{m,\gamma_\varepsilon}^{d_p}) \\
\text{(b)} & \quad (\gamma \varepsilon, q_1 | L_{m,\gamma_\varepsilon}^{d_p}, q_2 | L_{m,\gamma_\varepsilon}, p_3 | L_{m,\gamma_\varepsilon}^{d_p}, p_{1,\varepsilon}) \text{ are like } (\gamma, q_1, q_2, p_3, p_1) \text{ in (1.21)} \\
\text{(c)} & \quad p_{1,\varepsilon} \preceq p_{1,\varepsilon} \text{ for } \varepsilon < \varepsilon \\
\text{(d)} & \quad \text{if } \varepsilon = \varepsilon + 1 \text{ and } s \in \text{dom}(p_{1,\varepsilon}) \text{ then } \ell g(\text{tr}(p_\varepsilon(s))) > \text{cf}(\gamma).
\end{align*}
\]

So we are done proving $\Box_4$.]

$\Box_5 \quad \mathbb{P}_{m_1} \preceq \mathbb{P}_m$.

[Why? By $\Box_4$ for $\gamma$ large enough.]

So we are done. $\Box_{3.19}$

\{c73\}

Claim 3.21. If $m \in M$ is reduced or just $L_m$ has cardinality $\lambda_2$ then there is $n \in M_{\text{ec}}$ of cardinality $\leq \lambda_2$ such that $m \preceq M n$.

\{c76\}

Remark 3.22. By this we may restrict ourselves to $M_{\leq \lambda_2}$ (but then similarly in the end of Section 2).

Proof. We choose $\chi$ large enough and $m_\chi \in M_\chi$ which is wide, belongs to $M_{\text{ec}}$ and $m \preceq M m_\chi$; moreover is full and very wide (as constructed in 1.26).

We can choose $n$ such that:

\[
\begin{align*}
\text{(a)} & \quad n \in M \text{ and } n \text{ is wide and } |L_n| = \lambda_2 \\
\text{(b)} & \quad m \preceq M n \preceq M m_\chi \\
\text{(c)} & \quad (n, m_\chi) \text{ satisfies the criterion from 3.19, with } m_1, m \text{ there standing for } n, m_\chi \text{ here.}
\end{align*}
\]

[Why? Let $\xi = 1$ and recalling Definition 3.9(1) choose $\langle (t_{\alpha}, s_{\alpha}) : \alpha < \lambda_2 \rangle$ such that $(t_{\alpha}, s_{\alpha}) \in H_{m_\chi}, t_{\alpha} \in L_m \setminus M_m, (t_{\alpha}/E_m : \alpha < \lambda_2)$ are pairwise distinct and for every $(t, s) \in H_m$ there are $\lambda^+$ ordinals $\alpha < \lambda_2$ such that $(t, s), (t_{\alpha}, s_{\alpha})$ are $\xi$-equivalent, possible by 3.11 recalling $\lambda_2 \geq \aleph_2(\lambda_1)$. Let $L' = \cup \{t_{\alpha}/E_m : \alpha < \lambda_2\} \cup L_m$ and for each $t \in L' \setminus M_m$ let $\langle s_{t, \alpha} : \alpha < \lambda^+ \rangle$ be such that $s_{t, \alpha} \in L_m \setminus M_m$ and $m_\chi[(s_{t, \alpha}/E_m)]$ is isomorphic to $m_\chi[(t/E_m)]$ over $M_m$. Let $L = L' \cup \{s_{t, \alpha} : \alpha < \lambda^+, t \in L' \setminus M_m\}$ and $n = m_\chi[L$. Now it is easy to check that $n$ is as required.]

It suffices to prove that $n$ belongs to $M_{\text{ec}}$, let $n \preceq M n_1 \preceq M n_2$.

Without loss of generality $L_{n_2}$ has cardinality $\leq 2^{\lambda_2}$, by the LST argument (even $\lambda_2$, as we are assuming $\lambda_2 = (\lambda_2)^+ \lambda_2 = \lambda_2$, and as $m_\chi$ is very wide and full without loss of generality $n_2 \preceq M m_\chi$. Now $(n_1, m_\chi)$ satisfies the criterion from 3.19 hence $\mathbb{P}_{n_1} \preceq \mathbb{P}_m$.]

\{e32\}

Also the pair $(n_2, m_\chi)$ satisfies the criterion from 3.19 looking at the criterion. Hence by 3.19 we have $\mathbb{P}_{n_2} \preceq \mathbb{P}_m$.

\{e32\}

As $n_1 \preceq M n_2 \preceq M m_\chi$ from the last two sentences it easily follows that $\mathbb{P}_{n_1} \preceq \mathbb{P}_{n_2}$, so we are done.

$\Box_{3.21}$
Clearly More specifically, 3.19 seems to be problematic; anyhow this does not matter.

**Definition 3.24.** For \( m \in M \) and \( M \subseteq M_m \) of cardinality \( \leq \lambda_1 \) we define \( n := m(M) \in M \) as follows:

(a) \( L_n = L_m \) even as a partial order
(b) \( \bar{u}_n = \bar{u}_m \) and \( \rho_n = \rho_m \)
(c) \( M_n = M \); not \( M_m \)
(d) \( E'_n = \{(s, t) : s, t \in L_m \text{ and } \{s, t\} \not\in M\} \).

**Claim 3.25.** Assume \( m \in M \leq \lambda_2 \) and \( M \subseteq M_m \).
1) If \( n : m(M) \) indeed belongs to \( M \) and is equivalent to \( m \) hence \( P_m(L_m) = P_n(L_m) \).
2) If \( n \leq M n_1 \) then for some \( m_1 \) we have \( m \leq M m_1 \) and \( m_1, n_1 \) are equivalent.
3) If \( m \in M_{ec} \) and \( n = m(M) \) then \( n \in M_{ec} \).

**Proof.** 1) Check, noting that \( t \in L_n, M_n \Rightarrow t \in L_m, M \Rightarrow |t/E'_m| \leq |L_n| = |L_m| \leq \lambda_2 \) and \( |M_m| = |M| \leq |M_m| \leq \lambda_1 \).
2) Given such \( n_1 \) we now define \( m_1 \in M \) by:

\[ (a) \quad L_{m_1} = L_{n_1}, \]
\[ (b) \quad \bar{u}_{m_1} = \bar{u}_n \text{ and } \rho_{m_1} = \rho_{n_1}, \]
\[ (c) \quad M_{m_1} = M_m, \]
\[ (d) \quad E'_{m_1} = \{(s, t) : s \in M \text{ and } s \notin L_m \} \] Clearly

\[ (a) \quad L_{m_1} = L_{n_1}, \]
\[ (b) \quad \bar{u}_{m_1} = \bar{u}_n \text{ and } \rho_{m_1} = \rho_{n_1}, \]
\[ (c) \quad M_{m_1} = M_m \]
\[ (d) \quad E'_{m_1} = \{(s, t) : s \notin L_m \} \]

\[ (a) \quad s \in L_m \backslash M_m \text{ then } \]
\[ (\beta) \quad s/E'_{m_1} = s/E'_m \]
\[ (\gamma) \quad u_{m_1, s} = u_{n, s} = u_{m, s} \]
\[ (\delta) \quad \rho_{m_1, s} = \rho_{n, s} \]

\[ (a) \quad s \in L_m \backslash L_n \]
\[ (\beta) \quad s/E'_{m_1} = s/E'_m \]
\[ (\gamma) \quad u_{m_1, s} = u_{n_1, s} \]
\[ (\delta) \quad \rho_{m_1, s} = \rho_{n_1, s} \]

\[ (a) \quad u_{m_1, s} = u_{n_1, s} \]
and

\((\beta)\) \(\mathcal{P}_{m_1, s} = \mathcal{P}_{n_1, s}\)

and

\((*)_4\)

(a) Indeed \(m_1 \in M\),
(b) \(m \leq_M m_1\),
(c) \(m_1, n_1\) are equivalent.

So we are done.

3) Assume \(n \leq_M n_1 \leq_M n_2\), as in the proof of part (2) there are \(m_1, m_2\) such that \(m_1 \leq_M m_2\) and \(m_1, n_1\) are equivalent for \(\ell = 1, 2\). As \(m \in M_{ec}\) we have \(P_m \preceq P_m\) but this means \(P_n \preceq P_n\), as required. \(\square_{3.25}\)

**Conclusion 3.26.**

1) If \(m \in M, M \subseteq M_m\) and \(n = m\|M\) then \(P^\text{cer}_n \preceq P^\text{cer}_m\).

2) If \(m_\ell \in M\) and \(M_\ell \subseteq M_{m_\ell}\) for \(\ell = 1, 2\) and \(h\) is an isomorphism from \(m_1\|M_1\) onto \(m_2\|M_2\) then \(h\) induces an isomorphism from \(P^\text{cer}_{m_1}\|M_1\) onto \(P^\text{cer}_{m_2}\|M_2\).

**Proof.** 1) As in the proof of 3.21, without loss of generality \(m, n \in M \subseteq M_\lambda\). By 3.21 there is \(m_* \in M_\lambda\) such that \(m \leq_M m_*\) hence \(P^\text{cer}_m = P^\text{cer}_{m_*}\|L_m\).

Let \(n_\ast = m\|\langle M\rangle\), see 3.24, so \(n_\ast\|M = n\) and by 3.25(3) we have \(n_\ast \in M_{ec}\), hence \(P_{n_\ast}\|L_n = P^\text{cer}_{n_\ast}\). But \(n_\ast, m_*\) are equivalent, hence \(P_{n\ast} = P_m\), hence \(P^\text{cer}_{n\ast}\|L = P^\text{cer}_m\|L\) for every \(L \subseteq L_m\), hence by 2.9(3) \(P^\text{cer}_{n\ast} = P_m\|L_n \preceq P_m\|L_m = P^\text{cer}_m\). So the conclusion holds.

2) Easy, too. \(\square_{3.26}\)
§ 4. Comment on [Sh:945]

How we connect to [Sh:945], i.e. justify (⋆)5 in the proof of Lemma [Sh:945, 1.3(2)=La7].

Let $M$ be the linear order $(κ, <)$, $u'_α = α$, $P_α = [α]^{≤ λ}$; we may replace it by an isomorphic copy. Now applying 2.12 there is $m$ as promised in [Sh:945]. As $L_m$ is a well founded partial order, we can let $h$ be a one-to-one order preserving function from $L_m$ onto some ordinal, call it $β(⋆)$.

So without loss of generality $h$ is the identity, so let $L_m = (β(⋆), <_s)$, $U = M_m, P_{1, α} = P_m[β : β < α], P_{0, α} = P_α$ in 2.12.

\{c52\} 2) For every demanded.

Claim 4.2. $q$ a well founded partial order, we can let $α$ consisting of (omitting $α$, means for some $α$, and $ℓθ(q) = α_q = α_*$):

(a) $u = (u_α : α < α_*)$ and $P = ⟨P_α : α < α_*⟩$ where $P_α ⊆ [u_α]^{≤ λ}, u_α ⊆ α$, without loss of generality $P_α$ is closed under subsets (but is not necessarily an ideal)

(b) $⟨P_α, Q_β : α ≤ α_* β < α_*⟩$ is a $(< λ)$-support iteration let $P_q = P_q,q_α(q)$ and $P_{0, α} = P_α, Q_0, α_0 = Q_0$

(c) each of $P_α$ is strategically $(< λ)$-complete and $λ^+$-c.c.

(d) $η_β ∈ [β]^{< λ}$ is the generic of $Q_β$ where $η_β$, the generic of $Q_p$ (defined in clause (c) below) is $∪{η_p : p ∈ G_β}$

(e) if $G ⊆ P_β$ is generic over $V$ then $η_β[G]$ in $[β]^{< λ}$ dominate every $ν ∈ V[η_γ] : γ ∈ u]$ when $u ∈ P_α$; moreover, in $V[G], Q_β[G]$ is the subforcing of $Q_β$ consisting of the $p ∈ Q_β$ such that: for some $s, f, η_p$ (so $η_p = η$, etc.) we have

(a) $p = (η, f) = (η_p, f_p)$ so $η ∈ ∏ ε<ζ θ_ε$ for some $ζ < λ$

(b) $s = (u_i, f_i) : i < i_*$

(c) $i_*, < λ, u_i ∈ P_β, α ∩ η × f_i ∈ Π β$ and $f_i ∈ V[η_γ, G] : γ ∈ u_1$

(d) $δ = sup\{f_i : i < i_∗\}$, i.e. $ε < λ ⇒ f(ε) = ∪{f_i(ε) : i < i_∗}$

(f) for $α ≤ α_*, P_2, α$ is the completion of $P_α$ to a complete Boolean Algebra, or say:

(⋆) the elements of $P_2, α$ are of the form $B(…, γ_1, 1…, i < i_1)$ where:

(a) $i_1(*) ≤ λ$

(β) $γ_i \in W$ for $i < i_∗$

(γ) $B$ is a $λ$-Borel function from $i(*)$ into $\{0, 1\} = \{false, true\}$; $B$ is from $V$, of course, such that $∀P_θ$ “$B(…, γ_i, 1…, i < i_1) = 0”$

(⋆) the order is natural: $P_2, α |= “B_1(…, γ_1(i), 1…, i < i_1) ≤ B_2(…, γ_1(i), 1…, i < i_1) “$

if $P_θ$ “if $B_2(…, γ_1(i), 1…, i < i_1)$ is equal to $1 then so is $B_1(…, γ_1(i), 1…, i < i_1)$”

(g) for $W \subseteq α, let P_θ$ be the subforcing of $P_2, α$ consist of $B(…, γ_1(i), 1…, i < i_1) ∈ P_α(q) : i(∗) ≤ λ$ and $γ_i \in W$ for every $i < i_1(∗)$.\{z35\}

Claim 4.2. 1) For any sequence $⟨u_α, P_α : α < α_*⟩$ as above, i.e. as in clause (a) of Definition 4.1, there is one and only one $q$ as above and the $P, P_q, W$’s are as demanded.

2) For every $α ≤ α_*$ the set $P_α$ of $p ∈ P_α$ satisfying the following is dense:

\{z32\}
(a) \(\eta_p, i_p, \langle u_{p,i} : i < i_p \rangle\) are objects (not just \(P_\alpha\)-names)
(b) each \(f_i\) has the form \(B(\ldots, \eta_i, i_1, \ldots, i_{< j(*)} \leq \lambda\) where \(\{\gamma(i,j) : j < j(i)\} \subseteq u_{p,i}\).

3) Above for every \(v \subseteq \alpha\) and \(j_* < \lambda\) the set of \(p \in P^*\) such that \(v \subseteq \text{dom}(p) \land (\forall \beta \in \text{dom}(p))(\ell(g(\eta_p(\beta))) > j_*\) is dense.
4) \(P_{q,1,\alpha} \preceq P_{q,2,\alpha}\) moreover \(P_{q,1,\alpha}\) is dense in \(P_{q,2,\alpha}\) and \(\mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \alpha \Rightarrow P_{q,\mathcal{V}_1} \preceq P_{q,\mathcal{V}_2}\) so \(P_{q,\beta,\alpha} \preceq P_{q,2,\alpha}\) and \(|P_{q,\mathcal{V}}| \leq |\mathcal{V}|^{\lambda}\).
5) If \(\alpha < \alpha_*\) and \(u \in \mathcal{P}_\alpha\) then \(\eta_\alpha \in \Pi\beta\) dominate every \(\nu \in (\Pi\beta)\mathcal{V}^{[\mathcal{V}]^u}\).
6) Assume \(G \subseteq P_q\) is generic over \(V, \eta_\alpha = \eta_\alpha[G]\) and \(\eta'_\alpha \in (\Pi\beta)\mathcal{V}^{[\mathcal{V}]^u}\) for \(\alpha < \alpha_*\) and \(\{(\alpha, \epsilon) : \alpha < \alpha_*, \epsilon < \alpha\) and \(\eta_\alpha(\epsilon) \neq \eta'_\alpha(\epsilon)\}\) has cardinality \(< \lambda\). Then for some (really unique) \(G'\) we have \(G' \subseteq P_q\) is generic over \(V\) and \(\alpha\).

**Proof.** We prove this claim by induction on \(\alpha_*\).

1) With iteration \(\langle P_\alpha, Q_\beta : \alpha \leq \alpha_*, \beta < \alpha_*\rangle, \langle \eta_\beta : \beta < \alpha_*\rangle\), are defined in clause (b),(d),(e) of Definition 4.1. Now for clause (c) \(P_\beta\) are strategically \((< \lambda)\)-complete and \(\lambda^\ast\)-c.c. follows by the iteration being \((< \kappa)\)-support and the choice of the \(Q_\beta\)'s.

Note that we do not claim \(\models P_\alpha\). “\(Q_\alpha\) is strategically \((< \lambda)\)-complete because the memory is partial; however (recalling the induction hypothesis on \(\alpha_*\)):

- the set \(\{p \in P_\alpha : \beta \in \text{dom}(p)\) then for \(p(\beta)\) there are \(\hat{s}, f, \eta\) as in 4.1(e) such that:
  - (a) \(\eta \in \bigcup \prod \theta_\epsilon\) is an object, not just a \(P_\alpha\)-name
  - (b) \(\hat{s} = \langle (u_i, f_i) : i < i_* \rangle\) is an object
  - (c) \(f_i\) is a \(P_\alpha\)-name of the form \(B_i(\ldots, \eta_{\gamma(i, \epsilon)}, \ldots) \leq \lambda B_i\) an object as well as \(\gamma(i, \epsilon) \in u_i\).

For clause (c) we use “\(P_\alpha\) satisfies the \(\lambda^\ast\)-c.c.”

Also the rest is easy. \(\square_{4.2}\)

**Theorem 4.3.** For any ordinal \(\alpha_*\) there is a quadruple \((q, \delta_*, \mathcal{U}, h)\) such that:

- (A) \(q \in Q_{\lambda, \delta}\) and let \(\delta_* = \ell g(q)\)
- (B) \(U \subseteq \delta_*\) has order type \(\alpha_*\)
- (C) \(h\) is the order preserving function from \(\alpha_*\) onto \(\mathcal{U}\)
- (D) if \(\alpha \in \mathcal{U}\) then \(\mathcal{U} \cap \alpha \in \mathcal{P}_Q, \alpha\)

\(g\) is the order preserving function from \(\mathcal{U}_1\) onto \(\mathcal{U}_2\), then \(g\) induces an isomorphism \(\hat{g}\) from \(P_{q, \mathcal{U}_1}\) onto \(P_{q, \mathcal{U}_2}\) mapping \(\eta_\beta\) to \(\eta_{\hat{g}(\beta)}\) for \(\beta \in \mathcal{U}_2\).

**Proof.** By 2.12. \(\square_{4.3}\)
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