CORRECTED ITERATION
SH1126

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ABSTRACT. For λ inaccessible, we may consider $(< \lambda)$-support iteration of some specific $(< \lambda)$-complete λ^+-c.c. forcing notions. But this fails "preservation by restricting to a sub-sequence of the iterated forcing; to regain it we "correct" the iteration to regain it. We prove this for a characteristic case.

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§ 0. Introduction

This work is dedicated to proving a theorem on $(< \lambda)$-support iterations of $(< \lambda)$-complete "nicely" definable λ^+-c.c. forcing notions for λ inaccessible. Assume $\mathbb{Q}$ is such a definition, $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \alpha_*, \beta < \alpha_* \rangle$ is such an interation, $\mathbb{Q}_\beta = \mathbb{Q}^{\mathbf{V}[\mathbb{P}_\beta]}$ has generic $\underset{\sim}{\eta}_\beta$. So $\langle \underset{\sim}{\eta}_\beta : \beta < \alpha_* \rangle$ is generic for $\mathbb{P}_{\alpha_*}$, but letting β_* be maximal such that $2\beta_* \le \alpha_*$ is $\langle \underset{\sim}{\eta}_{2\beta} : \beta$ satisfies $2\beta < \alpha_* \rangle$ generic for the iteration $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \beta_*, \beta < \beta_* \rangle$?

The point is that in the parallel case for $\lambda = \aleph_0$ so far FS-iterated forcing such a claim is true. In fact, by Judah-Shelah [JdSh:292], if $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \alpha(*), \beta < \alpha(*) \rangle$ is FS-iteration of Suslin-c.c.c. forcing notion, $\underset{\sim}{\mathbb{Q}}_\beta$ with the generic $\underset{\sim}{\eta}_\beta \in {}^\omega\omega$ and for notational transparency, its definition is with no parameter and the function $\zeta : \beta(*) \to \alpha(*)$ is increasing and $\mathbb{P} = \langle \mathbb{P}'_\alpha, \underset{\sim}{\mathbb{Q}}'_\beta : \alpha \le \beta(*), \beta < \beta(*) \rangle$ is FS iteration, $\underset{\sim}{\mathbb{Q}}'_\beta$ defined exactly as $\underset{\sim}{\mathbb{Q}}_{\zeta(\beta)}$ but now in $\mathbf{V}^{\mathbb{P}'_\beta}$ rather than $\mathbf{V}^{\mathbb{P}_{\zeta(\beta)}}$ then $\Vdash_{\mathbb{P}_{\alpha(*)}}$ "$\langle \underset{\sim}{\eta}_{\zeta(\beta)} : \beta < \beta(*) \rangle$ is generic for $\mathbb{P}'_{\beta(*)}$ over $\mathbf{V}$".

Now this is not clear to us for $(< \lambda)$-support iteration of $(< \lambda)$-strategically complete forcing notions. The solution is essentially to change the iteration to what we call "corrected iteration" we use a "quite generic" $(< \lambda)$-support iteration which "includes" the one we like and use the complete subforcing it generates. Here we deal with a characteristic case (used in [Sh:945]). The proof applies also to partial memory iteration. The proof applies also to partial memory iteration. On wide generalization and application (for $\lambda = \aleph_0$) this is continued in a work with H. Horowitz in preparation.

The problem arises as follows. In [Sh:945] it is proved that for λ inaccessible, consistently cov_λ(meagre), the covering number of the meagre ideal on λ is strictly smaller than $\mathfrak{d}_\lambda$, the dominating number. The result here is used there but the editor prefers to separate it. In §4 we explain how this work is used there.

We try to use standard notation. We use $\theta, \kappa, \lambda, \mu, \chi$ for cardinals and $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ for ordinals. We use also i and j as ordinals. We adopt the Cohen convention that $p \le q$ means that q gives more information, in forcing notions. The symbol $\triangleleft$ is preserved for "being an initial segment". Also recall ${}^B A = \{f : f$ a function from B to $A\}$ and let ${}^{\alpha >}A = \cup\{{}^\beta A : \beta < \alpha\}$, some prefer ${}^{<\alpha}A$, but ${}^{\alpha >}A$ is used systematically in the author's papers. Lastly, J^{bd}_λ denotes the ideal of the bounded subsets of λ.

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§ 1. Iteration Parameters

{c0x}

Explanation 1.1. For $\mathbf{m} \in \mathbf{M}$ below (Definition 1.7): {c4}

(a) we use $L_{\mathbf{m}}$ as the index set for the iteration; always a well founded partial order

(b) $M_{\mathbf{m}}$ is the part of the index set we are really interested in, it may be $(\kappa, <)$ in §1

(c) the other part in the interesting case is "generic enough $\mathbf{m}$", more accurately enough existentially closed so that the iteration restricted to M will be "stabilize" under further extensions; inspite of $L_{\mathbf{m}}$ being required to be well founded this will be well defined.

{c0}

Hypothesis 1.2. 1) $\lambda = \lambda^{<\lambda}$ strongly inaccessible.
2) $\bar{\theta} = \langle \theta_\varepsilon : \varepsilon < \lambda \rangle$.
3) θ_ε is an infinite regular cardinal $> \varepsilon$ and $< \lambda$.
4) Assume $\lambda_2 \geq \lambda_1 \geq \lambda$ are such that[1] $\lambda_2 = (\lambda_2)^\lambda \geq \beth_3(\lambda_1)$ so pendantically all notations should have the parameter $\bar{\lambda} = (\lambda_2, \lambda_1)$ and even $\bar{\lambda} = (\lambda_2, \lambda_1, \lambda, \bar{\theta})$.

Notation 1.3. L, M denote partial orders, usually well founded.

{c1}

Remark 1.4. Here no harm in adding

(a) $\theta_\varepsilon > \prod\limits_{\zeta < \varepsilon} 2^{\theta_\zeta} + 2^{\aleph_0}$ for $\varepsilon < \lambda$
or just

(b) $\bar{\theta}$ is increasing fast enough

(c) M a linear order, well founded (it suffices to assume even $M \cong (\kappa, <), \kappa$ regular $> \lambda$).

{c2}

Definition 1.5. 1) For a partial order L let

(α) $\mathrm{dp}(L) = \cup\{\mathrm{dp}_L(t) + 1 : t \in L\}$, see below,

(β) $\mathrm{dp}_L(t) = \mathrm{dp}(t, L) \in \mathrm{Ord} \cup \{\infty\}$ be defined by $\mathrm{dp}_L(t) = \cup\{\mathrm{dp}_L(s) + 1 : s <_L t\}$.

(γ) $L_{<t} = L \restriction \{s \in L : s <_L t\}$,

(ζ) $L_{\leq t} = L \restriction \{s \in L : s \leq_L t\}$.

2) Let $L^+ = L(+)$ be $L \cup \{\infty\}$ with the natural order (but we may write $t <_L \infty$ instead of $t <_{L(+)} \infty$).
3) We say the set L is an initial segment of the partial order L_* when

- $L \subseteq L_*$, i.e. $s \in L \Rightarrow s \in L_*$
- $s <_{L_*} t \wedge t \in L \Rightarrow s \in L$.

{c3}

Discussion 1.6. Concerning the aim of the following choice, note the following.
1) By the partial order we already can get partial memory, so why the u_s's (in 1.7)? Because $\bar{u}$ is not necessarily transitive, that is, $s \in u_t \nRightarrow u_s \subseteq u_t$. By partial order we cannot get it. {c4}
2) In [Sh:700] we use $\mathscr{P}_t$'s which are ideals, but here not necessarily: this makes a difference but it uses "$\mathbb{Q}_{\bar{\theta}}$ is close to being λ-centered", i.e. any subset of $\{p \in \mathbb{Q}_{\bar{\theta}} : \mathrm{tr}(p) = \eta\}$ of cardinality $< \theta_{\ell g(\eta)}$ has a lub in this subset.

[1]usually $\lambda_2 \geq \lambda_1$ suffice but see 3.11, 3.21 {e23}

{c4}

Definition 1.7. Let $\mathbf{M}$ be the class of objects $\mathbf{m}$, called iteration parameters, of the following form (so really $\mathbf{M} = \mathbf{M}[\bar\lambda]$ and if we omit clauses $(\theta), (\iota), (\lambda)$ we may write $\mathbf{M}[*]$):

- (a) L, a partial order,
- (b) $M \subseteq L$, as partial orders,
- (c) (α) $\bar u = \langle u_t : t \in L\rangle$ and $\bar{\mathscr{P}} = \langle \bar{\mathscr{P}}_t : t \in L\rangle$ and each $\mathscr{P}_t$ is closed under subsets,
  - (β) $u_t \subseteq \{s \in L : s <_L t\}$ and $u \in \mathscr{P}_t \Rightarrow u \subseteq u_t$,
- (d) $\mathrm{dp}(L) < \infty$, that is L is well founded,
- (e) (α) E' is a two-place relation (on L),
  - (β) $E'' := E' \restriction (L \backslash M)$ is an equivalence relation on $L \backslash M$
  - (γ) if $s, t \in L \backslash M$ are not E''-equivalent then $(s <_L t) \Leftrightarrow (\exists r \in M)(s < r < t)$
  - (δ) if $sE't$ then $s \notin M \vee t \notin M$
  - (ε) if $t \in L \backslash M$ then $\{s \in L : sE't\} = \{s \in L : tE's\}$; we call it t/E'; so E' is a symmetric relation
  - (ζ) if $s, t \in L \backslash M$ are E''-equivalent then $s/E' = t/E'$
  - (η) if $t \in L \backslash M$ then $u_t \subseteq t/E'$
  - (θ) if $t \in L \backslash M$ then t/E' has cardinality $\le \lambda_2$
  - (ι) $\|M\| \le \lambda_1$
  - (κ) if $t \in L$ and $u \in \mathscr{P}_t$ then $u \nsubseteq M \Rightarrow (\exists s)(s \in L \backslash M \wedge u \subseteq s/E')$
  - (λ) $\mathscr{P}_t$ has cardinality $\le \lambda_2$ for $t \in L \backslash M$ and for simplicity $\mathscr{P}_t \subseteq [u_t]^{\le \lambda}$ as only those sets matter.

{c5}

Notation 1.8. For $\mathbf{m} \in \mathbf{M}$.

{c4} 0) In 1.7 we let $\mathbf{m} = (L_{\mathbf{m}}, M_{\mathbf{m}}, \bar u_{\mathbf{m}}, \bar{\mathscr{P}}_{\mathbf{m}}, E'_{\mathbf{m}})$ and $\bar u_{\mathbf{m}} = \langle u_{\mathbf{m},t} : t \in L_{\mathbf{m}}\rangle$, $\bar{\mathscr{P}}_{\mathbf{m}} = \langle \mathscr{P}_{\mathbf{m},t} : t \in L_{\mathbf{m}}\rangle$ and for $t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ let $t/E_{\mathbf{m}} = (t/E'_{\mathbf{m}}) \cup M_{\mathbf{m}}$ and for $t \in M_{\mathbf{m}}$ let $t/E_{\mathbf{m}} = M_{\mathbf{m}}$; so there is no relation $E_{\mathbf{m}}$ but $t/E_{\mathbf{m}}$ for $t \in L_{\mathbf{m}}$ is well defined.

{c4} 1) In 1.7, let $\mathrm{dp}_{\mathbf{m}}(t) = \mathrm{dp}_{L_{\mathbf{m}}}(t), \mathrm{dp}_{\mathbf{m}} = \mathrm{dp}(L_{\mathbf{m}})$ and $\le_{\mathbf{m}} = \le_{L_{\mathbf{m}}}$.

2) For $L \subseteq L_{\mathbf{m}}$:

- (a) let $\mathbf{n} = \mathbf{m} \restriction L$ means $\mathbf{n} \in \mathbf{M}, L_{\mathbf{n}} = L, \le_{\mathbf{n}} = \le_{\mathbf{m}} \restriction L_{\mathbf{n}}, u_{\mathbf{n},t} = u_{\mathbf{m},t} \cap L, \mathscr{P}_{\mathbf{n},t} = \mathscr{P}_{\mathbf{m},t} \cap [L]^{\le \lambda}$ for $t \in L$ and $M_{\mathbf{n}} = M_{\mathbf{m}} \cap L$;
- (b) let $\mathrm{dp}_{\mathbf{m}}(L) = \mathrm{dp}(L_{\mathbf{m}} \restriction L)$ and we may write $\mathrm{dp}(L)$ for $L \subseteq L_{\mathbf{m}}$.

3) For $t \in L_{\mathbf{m}}$, let $\mathbf{m}_{<t} = \mathbf{m}(<t) = \mathbf{m} \restriction L_{<t}$ where $L_{<t} = L_{\mathbf{m}(<t)} = L_{\mathbf{m},t} = \{s : s <_{\mathbf{m}} t\}$ so $u_{\mathbf{m}(<t),s} = u_{\mathbf{m},s}$ for $s \in L_{<t}$, etc.

3A) Also $\mathbf{m}_{\le t} = \mathbf{m}(\le t) = \mathbf{m} \restriction L_{\le t}$ where $L_{\le t} = L_{\mathbf{m}(\le t)} = L_{<t} \cup \{t\}$; let $L_{<\infty} = L, L_{\le \infty} = L^+$, etc.

4) $\mathbf{M}_{<\mu}$ is the class of $\mathbf{m} \in \mathbf{M}$ such that $L_{\mathbf{m}}$ has cardinality $< \mu$. Similarly $\mathbf{M}_{\le\mu}, \mathbf{M}_{=\mu}, \mathbf{M}_{>\mu}, \mathbf{M}_{\ge\mu}$; let $\mathbf{M}_\mu = \mathbf{M}_{=\mu}$.

5) For $\mathbf{m}, \mathbf{n} \in \mathbf{M}$ let $\mathbf{m} \approx \mathbf{n}$, and we may say $\mathbf{m}, \mathbf{n}$ are equivalent mean that $L_{\mathbf{m}} = L_{\mathbf{n}}$ and $t \in L_{\mathbf{n}} \Rightarrow u_{\mathbf{m},t} = u_{\mathbf{n},t} \wedge \mathscr{P}_{\mathbf{m},t} = \mathscr{P}_{\mathbf{n},t}$; note that there are no demands on M and E'.

6) We say f is an isomorphism from $\mathbf{m}_1 \in \mathbf{M}$ onto $\mathbf{m}_2 \in \mathbf{M}$ when:

(a) f is an isomorphism from the partial order $L_{\mathbf{m}_1}$ onto the partial order $L_{\mathbf{m}_2}$

(b) for $s, t \in L_{\mathbf{m}_1}$ we have $s \in u_{\mathbf{m}_1,t} \Leftrightarrow f(s) \in u_{\mathbf{m}_2,f(t)}$ and $\mathscr{P}_{\mathbf{m}_2,f(t)} = \{\{f(s) : s \in u\} : u \in \mathscr{P}_{\mathbf{m}_1,t}\}$

(c) for $s, t \in L_{\mathbf{m}_1}$ we have $sE'_{\mathbf{m}_1}t \Leftrightarrow f(s)E'_{\mathbf{m}_2}f(t)$

(d) $M_{\mathbf{m}_2} = \{f(s) : s \in M_{\mathbf{m}_1}\}$.

7) We define weak isomorphisms as in part (6) omitting clauses (c),(d).

Definition 1.9. For $\mathbf{m} \in \mathbf{M}$ let $L = L_{\mathbf{m}}$ and we define the iteration $\mathbf{q}_{\mathbf{m}}$ to consist of: {c6}

(a) a forcing notion $\mathbb{P}_t = \mathbb{P}_{\mathbf{m},t}$ for $t \in L^+$; we let $\mathbb{P}_{\mathbf{m}} = \mathbb{P}_\infty$

(b) $\underset{\sim}{\mathbb{Q}}_t$ is the $\mathbb{P}_t$-name of a subforcing of $\mathbb{Q}_{\bar\theta}$ in the universe $\mathbf{V}^{\mathbb{P}_t}$, even $\le_{\mathrm{ic}}$ (i.e. incompatibility and compatibility are preserved)

(c) $p \in \mathbb{P}_t$ <u>iff</u>

  (α) p is a function

  (β) $\mathrm{Dom}(p) \subseteq L_{<t}$ has cardinality $< \lambda$

  (γ) if $s \in \mathrm{Dom}(p)$ then $p(s)$ consists of $\mathrm{tr}(p(s)) \in \prod\limits_{\varepsilon<\zeta(s)} \theta_\varepsilon$ for some $\zeta_s = \zeta(s) < \lambda$ and $\xi = \xi_{p(s)} = \xi(p(s)) \le \lambda$ and $\mathbf{B}_{p(s)}$ and $\bar r = \bar r_{p(s)} = \langle r(\zeta) : \zeta < \xi_{p(s)}\rangle = \langle r_{p(s)}(\zeta) : \zeta < \xi_{p(s)}\rangle \in {}^{\xi}(u_s)$ list the coordinates used in computing $p(s)$ and $\langle \mathbf{B}_{p(s),\iota}, \bar r_{p(s),\iota} : \iota < \iota(p(s))\rangle$ are such that:

  - $\mathbf{B}_{p(s)}$ is a λ-Borel function[2], $\mathbf{B} = \mathbf{B}_{p(s)} : {}^{\xi}(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon) \to \Pi\bar\theta$ moreover into $(\Pi\bar\theta)^{[\mathrm{tr}(p(s))]}$; and considering $(d)(\alpha)$ below less pedantically $p(s) = (\mathrm{tr}(p(s)), \underset{\sim}{f}_{p(s)})$, where $\underset{\sim}{f}_{p(s)} = \mathbf{B}_{p(s)}(\ldots, \underset{\sim}{\eta}_{r_{p(s)}(\zeta)}, \ldots)_{\zeta<\xi}$ which means: absolutely, i.e. in every $\mathbf{V}^{\mathbb{Q}}$, $\mathbb{Q}$ a $(<\lambda)$-strategically complete (which is λ^+-c.c.) forcing notion, $\mathbf{B}_{p(s)}$ is such a (λ-Borel) function; we may write $\xi_{p,s}$ instead of $\xi_{p(s)}$, etc.
  - $\iota(p(s)) < \lambda$ moreover[3] $< \theta_{\ell g(\mathrm{tr}(p(s)))}$
  - $\bar r_{p(s),\iota} = \bar r_{p(s)} \restriction w_{p(s),\iota}$ so $w_{p(s),\iota} = w(p(s),\iota) = \mathrm{dom}(\bar r_{p(s),\iota}) \subseteq \xi_{p(s)}$ and $\bar r_{p(s),\iota}$ is a subsequence of $\bar r_{p(s)}$
  - $\mathbf{B}_{p(s),\iota}$ is a Borel function from ${}^{w(p(s),\iota)}(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)$ into $(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)^{[\mathrm{tr}(p(s))]}$
  - $\mathbf{B}_{p(s)}(\langle \underset{\sim}{\eta}_{r_{p(s)}(\zeta)} : \zeta < \xi_{p(s)}\rangle) = \sup\{\mathbf{B}_{p(s),\iota}(\langle \underset{\sim}{\eta}_{r_{p(s)}(\zeta)} : \zeta \in w_{p(s),\iota}\rangle) : \iota < \iota(p(s))\}$ so $f_{p(s)} = \sup\{\underset{\sim}{f}_{p(s),\iota} : \iota < \iota(p(s))\}, \underset{\sim}{f}_{p(s),\iota} = \mathbf{B}_{p(s),\iota}(\langle \underset{\sim}{\eta}_\zeta : \zeta \in w_{p(s),\iota}\rangle)\}$
  - for each $\iota < \iota(p(s))$ for some $u \in \mathscr{P}_{\mathbf{m},s}$ we have $\{r_{p(s)}(\zeta) : \zeta < \xi_{p(s)}, \zeta \in w_{p(s),\iota}\} \subseteq u$ so is a subset of u_s
  - if $\iota < \iota(p(s))$ and $\varepsilon \in w_{p(s),\iota}, r_{p(s)}(\varepsilon) \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ <u>then</u> $\{r_{p(s)}(\zeta) : \zeta \in w_{p(s),\iota}\} \subseteq r_{p(s)}(\varepsilon)/E_{\mathbf{m}}$, (follows)

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[2]that is, a definition of one

[3]This and the rest of $(c)(\gamma)$ are used in the proof of 3.17. The aim is that defining $\mathbf{B}_{p(s)}$ from $\langle \mathbf{B}_{p(s),\iota} : \iota\rangle$, the sup will not given in ε the value θ_ε. {e31}

(d) (α) $\underset{\sim}{\eta}_s$ is the $\mathbb{P}_t$-name, when $t \in L^+_{\mathbf{m}}, s \in L_{<t}$ defined by $\cup\{\text{tr}(p(s)) : p \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}_t}\}$.

(β) For $p \in \mathbb{P}_t$ and $s \in \text{Dom}(p)$ we interpret $p(s)$ as a $\mathbb{P}_s$-name $(\text{tr}(p(s)), \mathbf{B}_{p,s}(\ldots, \underset{\sim}{\eta}_{r_{p,s}(\zeta)}, \ldots)_{\zeta<\xi_{p,s}})$

(e) $\mathbb{P}_t \models$ "$p \leq q$" iff

(α) $p, q \in \mathbb{P}_t$

(β) $\text{Dom}(p) \subseteq \text{Dom}(q)$

(γ) if $t \in \text{Dom}(p)$ then $(q \restriction L_{<t}) \Vdash_{\mathbb{P}_{\mathbf{m}(<t)}}$ "$p(t) \leq_{\underset{\sim}{\mathbb{Q}}_{\bar{\theta}}} q(t)$".

{c7} **Definition 1.10.** 1) For $p \in \mathbb{P}_{\mathbf{m}}$ let

(a) fsupp(p), the full support of p be $\cup\{\{r_{p(s)}(\zeta) : \zeta < \xi_{p,s}\} \cup \{s\} : s \in \text{Dom}(p)\}$

(b) wsupp(p), the wide support of p be $\cup\{t/E_{\mathbf{m}} : t \in \text{fsupp}(p)\}$.

{c6} 2) For $\mathbf{m} \in \mathbf{M}$ let $\mathbb{P}^{\mathbf{m}}_t = \mathbb{P}_{\mathbf{m},t}$, etc., in Definition 1.9.
3) For $L \subseteq L_{\mathbf{m}}$ let $\mathbb{P}_{\mathbf{m}}(L) = \mathbb{P}_{\mathbf{m}} \restriction \{p \in \mathbb{P}_{\mathbf{m}} : \text{fsupp}(p) \subseteq L\}$.
4) For $\mathbf{m} \in \mathbf{M}$ and $t \in L_{\mathbf{m}}$ let[4] $\underset{\sim}{\mathbb{Q}}_t = \underset{\sim}{\mathbb{Q}}_{\mathbf{m},t}$ be the $\mathbb{P}_t$-name of $\mathbb{Q}_{\bar{\theta}} \restriction \{(\nu, f) \in \mathbb{Q}_{\bar{\theta}} : f = \sup\{f_\iota : \iota < \iota(*)\}$ where $\iota(*) < \theta_{\ell g(\nu)}$ and $f_\iota \in (\prod_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\underset{\sim}{\eta}_s : s \in u]}$ for some $u \in \mathscr{P}_{\mathbf{m},t}\}$.

{c8} **Claim 1.11.** *For $\mathbf{m} \in \mathbf{M}$ (so $\mathbb{P}_t = \mathbb{P}_{\mathbf{m},t}$, etc.)*

(a) *the iteration $\mathbf{q}_{\mathbf{m}}$ is well defined, i.e. exist and is unique*

(b) (α) *if $t \in L^+_{\mathbf{m}}$ then $\mathbb{P}_t$ is indeed a forcing notion and is equal to $\mathbb{P}_{\mathbf{m}_{<t}}$,*

(β) *the $\mathbb{P}_t$-name $\underset{\sim}{\eta}_s$ does not depend on t as long as $s < t \in L^+_{\mathbf{m}}$,*

(γ) *$\underset{\sim}{\eta}_t$ is a $\mathbb{P}_{\mathbf{m}(\leq t)}$-name*

(c) *if $s <_L t$ are from $L^+_{\mathbf{m}}$ then*

(α) $p \in \mathbb{P}_s \Rightarrow p \in \mathbb{P}_t \wedge p \restriction L_{<s} = p$,

(β) *if $p, q \in \mathbb{P}_s$ then $\mathbb{P}_t \models$ "$p \leq q$" $\Leftrightarrow \mathbb{P}_s \models$ "$p \leq q$",*

(γ) *if $p \in \mathbb{P}_t$ then $p \restriction L_{<s} \in \mathbb{P}_s$ and $\mathbb{P}_t \models$ "$(p \restriction L_{<s}) \leq p$",*

(δ) $\mathbb{P}_t \models$ "$p \leq q$" $\Rightarrow \mathbb{P}_s \models$ "$p \restriction L_{<s} \leq q \restriction L_{<s}$",

(ε) $\mathbb{P}_s \lessdot \mathbb{P}_t$ *moreover*

(ζ) $p \in \mathbb{P}_t \wedge (p \restriction L_{<s}) \leq q \in \mathbb{P}_s \Rightarrow q \cup (p \restriction (L_{<t} \backslash L_{<s}) \in \mathbb{P}_t$ *is a $\leq$-lub of p, q*

(θ) $\mathbb{P}_{\mathbf{m},t} = \mathbb{P}_{\mathbf{m} \restriction L_{<t}}$

(d) *if L is an initial segment of $L_{\mathbf{m}}$ then $\mathbb{P}_{\mathbf{m} \restriction L} = \mathbb{P}_{\mathbf{m}} \restriction \{p \in \mathbb{P}_m : \text{dom}(p) \subseteq L$, equivalently $\text{fsupp}(p) \subseteq L\}$; this holds in particular for $L_{\leq t}, \mathbf{m}_{\leq t}$*

(e) *if $L_1 \subseteq L_2$ are initial segments of $L_{\mathbf{m}}$, then the parallel of clause (c) holds replacing $\mathbb{P}_{\mathbf{m},s}, \mathbb{P}_{\mathbf{m},t}$ by $\mathbb{P}_{\mathbf{m} \restriction L_1}, \mathbb{P}_{\mathbf{m} \restriction L_2}$, respectively.*

Proof. Straightforward. For $t \in L^+_{\mathbf{m}}$, by induction on $\text{dp}_{\mathbf{m}}(t)$, define $\mathbb{P}_t$ and prove the relevant parts of (a),(b),(c),(d),(e). $\square_{1.11}$

Note

{c13} [4]not used, could have used it in 1.15

{c10}

Observation 1.12. *If* $\mathbf{B}$ *is a* λ*-Borel function from* ${}^{\xi}(\Pi\bar{\theta})$ *to* $\mathscr{P}(\lambda)$ *or even* $\mathscr{H}(\lambda^+)$ *where* $\xi \le \lambda$ *<u>then</u> there is a* λ*-Borel function* $\mathbf{B}'$ *from* ${}^{\xi}(\Pi\bar{\theta})$ *to* $\mathbb{Q}_{\bar{\theta}}$ *(so absolutely to* $\mathbb{Q}_{\bar{\theta}}$*) such that for any* $\bar{\eta} \in {}^{\xi}(\Pi\bar{\theta})$ *we have, absolutely:*

- *if* $\mathbf{B}(\bar{\eta}) \in \mathbb{Q}_{\bar{\theta}}$ *then* $\mathbf{B}'(\bar{\eta}) = \mathbf{B}(\bar{\eta})$
- *if* $\mathbf{B}(\bar{\eta}) \notin \mathbb{Q}_{\bar{\theta}}$ *then* $\mathbf{B}'(\bar{\eta}) = (\emptyset, 0_\lambda)$*, the minimal member of* $\mathbb{Q}_{\bar{\theta}}$*.*

Proof. Easy. $\square_{1.12}$

{c11}

Claim 1.13. *Let* $\mathbf{m} \in \mathbf{M}$*.*
1) If $L^+_{\mathbf{m}} \models$ *"*$s < t$*" <u>then</u>*

(α) $\Vdash_{\mathbb{P}_{\mathbf{m},t}}$ "$\underset{\sim}{\eta}_s \in \prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$"

(β) *if* $\mathbf{G} \subseteq \mathbb{P}_t$ *is generic over* $\mathbf{V}$ *and* $\eta_r = \underset{\sim}{\eta}_r[\mathbf{G}]$ *for* $r \in L_{\mathbf{m},<t}$ *and* $u \in \mathscr{P}_{\mathbf{m},s}$ *and* $\nu \in \Pi\bar{\theta}$ *is from* $\mathbf{V}[\langle \eta_r : r \in u\rangle] \subseteq \mathbf{V}[\mathbf{G}]$ *<u>then</u>* $\nu <_{J^{\rm bd}_\lambda} \eta_s$*.*

2) $\mathbb{P}_{\mathbf{m}}$ *satisfies the* λ^+*-c.c.*
3) $\mathbb{P}_{\mathbf{m}}$ *is* $(<\lambda)$*-strategically complete (even* λ*-strategically complete but not used).*
4) If $\bar{p} = \langle p_i : i < \delta\rangle$ *is* $\le_{\mathbb{P}_{\mathbf{m}}}$*-increasing,* $\delta < \lambda$ *and* $i < j < \delta \wedge t \in \mathrm{Dom}(p_i) \Rightarrow \mathrm{tr}(p_i(t)) \lhd \mathrm{tr}(p_j(t))$ *<u>then</u>*[5] $\bar{p}$ *has a* $\le_{\mathbb{P}_{\mathbf{m}}}$*-upper bound* p*. Moreover,* $\mathrm{Dom}(p) = \cup\{\mathrm{Dom}(p_i) : i < \delta\}$ *and* $s \in \mathrm{Dom}(p_i) \Rightarrow \mathrm{tr}(p(s)) = \cup\{\mathrm{tr}(p_j(s)) : j \in [i,\delta)\}$*; in fact also* $\mathrm{fsupp}(p) = \cup\{\mathrm{fsupp}(p_i) : i < \delta\}$ *and* p *is a lub. Also, we can weaken the demand above to* $i < \delta \wedge s \in \mathrm{Dom}(p_i) \Rightarrow \delta < \theta_{\varepsilon(s)}$ *where we let* $\varepsilon(s) = \sup\{\ell g(\mathrm{tr}(p_j(s))) : j \in [i,\delta)\}$*.*
5A) If $\zeta < \lambda$ *and* $L^+_{\mathbf{m}} \models$ *"*$s < t$*", <u>then</u> the following is a dense open subset of* $\mathbb{P}_t$*:* $\mathscr{I}_{s,t,\zeta} = \{p \in \mathbb{P}_t : s \in \mathrm{Dom}(p)$ *and* $\mathrm{tr}(p(s))$ *has length* $\ge \zeta\}$*.*
5B) If $p \in \mathbb{P}_{\mathbf{m}}$ *and* $\zeta < \lambda$ *<u>then</u> for some* $q \in \mathbb{P}_{\mathbf{m}}$ *we have* $p \le q$ *and* $t \in \mathrm{Dom}(p) \Rightarrow \mathrm{tr}(p(t)) \lhd \mathrm{tr}(q(t))$ *and* $t \in \mathrm{Dom}(q) \Rightarrow \ell g(\mathrm{tr}(q(t))) > \zeta$*.*
6) If $\underset{\sim}{x}$ *is a* $\mathbb{P}_{\mathbf{m}}$*-name of a member of* $\mathscr{H}(\lambda^+)$*, e.g. of* $\mathbb{Q}_{\bar{\theta}}$ *(in* $\mathbf{V}[\mathbb{P}_{\mathbf{m}}]$*) <u>then</u> for some* $\xi \le \lambda$ *and* λ*-Borel function* $\mathbf{B} : {}^{\xi}(\Pi\bar{\theta}) \to \mathscr{H}(\lambda^+)$ *and a sequence* $\langle r_\zeta : \zeta < \xi\rangle$ *of members of* $L_{\mathbf{m}}$ *we have* $\Vdash_{\mathbb{P}_{\mathbf{m}}}$ "$\underset{\sim}{x} = \mathbf{B}(\ldots, \underset{\sim}{\eta}_{r_\zeta}, \ldots)_{\zeta<\xi}$"*.*
6A) If $t \in L^+_{\mathbf{m}}$ *and* $u \subseteq L_{<t}$ *and* $\Vdash_{\mathbb{P}_t}$ *"*$\underset{\sim}{y}$ *is a member of* $\mathbb{Q}_{\bar{\theta}}$ *from* $\mathbf{V}[\langle \underset{\sim}{\eta}_s : s \in u\rangle]$*", <u>then</u> for some* $\xi \le \lambda$ *and* λ*-Borel functions as in 1.9(6)(*γ*),* $\mathbf{B}_i : {}^{\xi}(\Pi\bar{\theta}) \to \mathbb{Q}_{\bar{\theta}}$ *for* {c6} $i < \xi$ *and sequence* $\langle r_\zeta : \zeta < \xi\rangle$ *of members of* u *we have* $\Vdash_{\mathbb{P}_t}$ *"for some* $i < \xi$ *we have* $\underset{\sim}{y} = \mathbf{B}_i(\ldots, \underset{\sim}{\eta}_{r_\zeta}, \ldots)_{\zeta<\xi}$*".*
7) If $\mathbf{m}, \mathbf{n}$ *are equivalent <u>then</u>* $\mathbb{P}_{\mathbf{m}} = \mathbb{P}_{\mathbf{n}}$ *and* $\mathbb{P}_{\mathbf{m},t} = \mathbb{P}_{\mathbf{n},t}$ *for* $t \in L^+_{\mathbf{m}} = L^+_{\mathbf{n}}$*.*
8) Assume that $p, q \in \mathbb{P}_{\mathbf{m}}$ *are incompatible <u>then</u> there are* r *and* s *such that:*

(a) $r \in \mathbb{P}_{\mathbf{m},s}$

(b) $s \in \mathrm{Dom}(p) \cap \mathrm{Dom}(q)$

(c) $(q{\restriction}L_{\mathbf{m},<s}) \le_{\mathbb{P}_{\mathbf{m}}} r$

(d) $p{\restriction}L_{\mathbf{m},<s} \le_{\mathbb{P}_{\mathbf{m}}} r$

(e) $r \Vdash_{\mathbb{P}_{\mathbf{m},<s}}$ *"*$p(s)$ *and* $q(s)$ *are incompatible in* $\mathbb{Q}_\theta$ *which means* $\mathrm{tr}(p(s)) \perp \mathrm{tr}(q(s))$*, i.e. are* $\trianglelefteq$*-incomparable <u>or</u>* $(\alpha) + (\beta) + (\gamma)$ *where:*

  (α) $\ell g(\mathrm{tr}(q(s))) \ne \ell g(\mathrm{tr}(p(s)))$

[5] But $\mathrm{tr}(p_i(t)) \trianglelefteq \mathrm{tr}(p_j(t))$ does not suffice.

(β) *if* $\ell g(\mathrm{tr}(q(s))) < \ell g(\mathrm{tr}(p(s)))$ *then for some ordinal* $\varepsilon, \ell g(\mathrm{tr}(q(s))) \le \varepsilon < \ell g(\mathrm{tr}(p(s)))$ *and* $r{\restriction}L_{\mathbf{m}(<s)} \Vdash_{\mathbb{P}_{\mathbf{m}(<s)}} \mathrm{tr}(p(s))(\varepsilon) < \underset{\sim}{f}_{q(s)}(\varepsilon)$"

(γ) *if* $\ell g(\mathrm{tr}(q(s))) > \ell g(\mathrm{tr}(p(s)))$ *then for some ordinal* $\varepsilon, \ell g(\mathrm{tr}(q(s))) > \varepsilon \ge \ell g(\mathrm{tr}(p(\alpha)))$ *and* $r{\restriction}L_{\mathbf{m}(<s)} \Vdash_{\mathbb{P}_{\mathbf{m}(<s)}}$ "$\mathrm{tr}(q(s))(\varepsilon) < \underset{\sim}{f}_{p(s)}(\varepsilon)$".

9) $\Vdash_{\mathbb{P}_\mathbf{m}}$ "$\mathbf{V}[\langle \underset{\sim}{\eta}_s : s \in L_\mathbf{m}\rangle] = \mathbf{V}[\underset{\sim}{\mathbf{G}}]$".

{c53} *Remark* 1.14. What is the use of e.g. (6),(6A)? See 2.11(A)(b) and 1.15.

Proof. We prove all parts by induction on $\mathrm{dp}_\mathbf{m}$.

{c8} 1) For clause (α) for each $\mathbf{m}$, using the induction hypothesis and 1.11(e), the problem is only when $\mathrm{dp}_\mathbf{m}(t) = \mathrm{dp}_\mathbf{m} - 1$ and use part (5A) proved below. For clause (β) use also part (6A) for $\mathbb{P}_{\mathbf{m}(<t)}$ proved below. In both cases the proof of the parts quoted does not rely on part (1).

2) If $p_\varepsilon \in \mathbb{P}_\mathbf{m}$ for $\varepsilon < \lambda^+$ then we by the Δ-system lemma can find u and unbounded $S \subseteq \lambda^+$ such that $\varepsilon \ne \zeta \in S \Rightarrow \mathrm{Dom}(p_\varepsilon) \cap \mathrm{Dom}(p_\zeta) = u$ and $\langle \mathrm{tr}(p_\varepsilon(\beta)) : \beta \in u\rangle$ is the same for all $\varepsilon \in S$. Now p_ε, p_ζ has a common upper bound for every $\varepsilon, \zeta \in u$, i.e. we define r by

- $\mathrm{Dom}(r) = \mathrm{Dom}(p_\varepsilon) \cup \mathrm{Dom}(p_\zeta)$
- $r(s) = p_\varepsilon(s)$ is $s \in \mathrm{Dom}(p_\varepsilon)\backslash\mathrm{Dom}(p_\zeta)$
- $r(s) = p_\zeta(s)$ if $s \in \mathrm{Dom}(p_\zeta)\backslash\mathrm{Dom}(p_\varepsilon)$
- if $s \in \mathrm{Dom}(p_\varepsilon) \cap \mathrm{Dom}(p_\zeta)$ <u>then</u> $r(s) = (\mathrm{tr}(p_\varepsilon(s)), \max\{\underset{\sim}{f}_{p_\varepsilon(s)}, \underset{\sim}{f}_{p_\zeta(s)}\})$.

3) By (4), the second sentence + (5B) below which use only the induction hypothesis.

4) We define p by:

- $\mathrm{Dom}(p) = \cup\{\mathrm{Dom}(p_i) : i < \delta\}$
- $\mathrm{tr}(p(s)) = \cup\{\mathrm{tr}(p_i(s)) : i < \delta$ satisfies $s \in \mathrm{Dom}(p_i)\}$
- $\underset{\sim}{f}_{p(s)} = \sup\{\underset{\sim}{f}_{p_i(s)} : i < \delta$ satisfies $s \in \mathrm{Dom}(p_i)\}$.

Note that here having to really start with $\langle \underset{\sim}{f}_{p_i(s),\iota} : \iota < \iota(p_i(s))\rangle$ and get $\langle \underset{\sim}{f}_{p(s),\iota} :$
{c6} $\iota < \iota(p(s))\rangle$, see 1.9(c)($\gamma$) causes no problem, similarly in the proof of part (2) - just take the union.

{c8} 5A) Obvious by the definition of $\mathbb{P}_\mathbf{m}$ and 1.11(c).

5B) The proof is split to cases.

<u>Case 1</u>: $\mathrm{dp}_\mathbf{m}$ is zero

So L_m is empty.

<u>Case 2</u>: $\mathrm{dp}_\mathbf{m} = \alpha + 1$

Hence $L_2 = \{s \in L : \mathrm{dp}_\mathbf{m}(s) = \alpha\}$ is non-empty and letting $L_1 = L_\mathbf{m}\backslash L_2$; clearly $s \in L_1 \Rightarrow \mathrm{dp}_\mathbf{m}(s) < \alpha$, so $\mathrm{dp}_{\mathbf{m}{\restriction}L_1} \le \alpha$. Let $\zeta_* = \sup(\{\ell g(\mathrm{tr}(p(s)) + 1 : s \in \mathrm{dom}(p)\} \cup \{\zeta + 1\})$. Hence applying (4) and (5B) to $\mathbf{m}{\restriction}L_1$, i.e. the induction hypothesis we can find q_1 such that $\mathbb{P}_{\mathbf{m}{\restriction}L_1} \models$ "$p{\restriction}L_1 \le q_1$" and $[s \in \mathrm{Dom}(q_1) \Rightarrow \ell g(\mathrm{tr}(q_1(s)) > \zeta_*]$ and q_1 forces a value to $\underset{\sim}{f}_{p(s),\iota}{\restriction}\zeta_*$, call it ρ_s for $s \in \mathrm{Dom}(p) \cap L_2$ and $\iota < \iota(p(s))$.

Define $q \in \mathbb{P}_\mathbf{m}$ by $\mathrm{Dom}(q) = \mathrm{Dom}(q_1) \cup (L_2 \cap \mathrm{dom}(p)), q{\restriction}L_1 = q_1$ and if $s \in$
{c8} $L_2 \cap \mathrm{Dom}(p)$ then $q(s) = (\rho_s, \rho_s{}^\frown(\underset{\sim}{f}_{p(s)}{\restriction}[\zeta_*, \lambda))$, recalling 1.11.

Easily q is as required.

Case 3: $\delta = \mathrm{dp}_{\mathbf{m}}$ is a limit ordinal of cofinality $\geq \lambda$

So $\alpha = \sup\{\mathrm{dp}_{\mathbf{m}}(s)+1 : s \in \mathrm{Dom}(p)\}$ is an ordinal $< \delta$ and let $L = \{s \in L_{\mathbf{m}} : \mathrm{dp}_{\mathbf{m}}(s) < \alpha\}$, so L is an initial segment of $L_{\mathbf{m}}$ and applying the induction hypothesis to $\mathbf{m}{\restriction}L, p$ we get q as required in $\mathbb{P}_{\mathbf{m}\restriction L}$ hence in $\mathbb{P}_{\mathbf{m}}$.

Case 4: $\delta = \mathrm{dp}_{\mathbf{m}}$ is a limit ordinal of cofinality $< \lambda$.

Let $\langle \alpha_i : i < \mathrm{cf}(\delta)\rangle$ be increasing continuous with limit δ, let $\alpha_{\mathrm{cf}(\delta)} = \delta$ and for $i \leq \mathrm{cf}(\delta)$ let $L_i := \{s \in L_{\mathbf{m}} : \mathrm{dp}_{\mathbf{m}}(s) < 1 + \alpha_i\}$.

Now we choose (p_i, ζ_i) by induction on $i < \mathrm{cf}(\delta)$ such that:

- (a) $p_i \in \mathbb{P}_{\mathbf{m}\restriction L_i}$
- (b) $\mathbb{P}_{\mathbf{m}\restriction L_i} \models$ "$p{\restriction}L_i \leq p_i$ and $p_j \leq p_i$" when $j < i$
- (c) if i is a limit ordinal then p_i is gotten from $\langle p_j : j < i\rangle$ as in part (4)
- (d) if $s \in \mathrm{Dom}(p_i)$ then $\ell g(\mathrm{tr}(p_i(s))) \geq \zeta_i$
- (e) $\langle \zeta_j : j < i\rangle$ is an increasing continuous sequence of ordinals $< \lambda$ and if i is non-limit then ζ_i is $> \zeta$ and $\geq \sum\limits_{j<i} |\mathrm{Dom}(p_j)| + |\mathrm{Dom}(p)|$ and $>$ $\sup(\{\ell g(\mathrm{tr}(p_j(s))) : j < i \text{ and } s \in p_j\} \cup \{\ell g(\mathrm{tr}(p(s))) : s \in \mathrm{Dom}(p)\})$.

Using 1.11 and the induction hypothesis this is easy. {c8}

6) For transparency assume $\Vdash$ "$\underset{\sim}{y} \in \prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$" or just $\in {}^{\lambda}\mathbf{V}$. By parts (4) + (5B), i.e. part (3), for each $\zeta < \lambda$ the following subset of $\mathbb{P}_{\mathbf{m},t}$ is open and dense: $\mathscr{I}_\zeta = \{p \in \mathbb{P}_{\mathbf{m},t}$: for some $\nu \in \prod\limits_{\varepsilon<\zeta} \theta_\varepsilon$ or $\in {}^{\zeta}\mathbf{V}$ (from $\mathbf{V}$!) we have $p \Vdash_{\mathbb{P}_{\mathbf{m},t}}$ "$\underset{\sim}{y}{\restriction}\zeta = \nu$"$\}$. Clearly there is a maximal antichain $\langle p_{\zeta,\varepsilon} : \varepsilon < \xi_\zeta\rangle$ of $\mathbb{P}_{\mathbf{m},t}$ included in $\mathscr{I}_\zeta$ and by part (2) without loss of generality $\xi_\zeta \leq \lambda$, the rest should be clear. In the general case we can code $\underset{\sim}{y}$ as a subset of λ, etc.

6A) This too should be clear as $\mathbb{P}_t$ satisfies the λ^+-c.c.

7) Look at the definitions.

8) Using parts (4) and (5B) and the definition this is easy.

9) Suppose toward contradiction that $\mathbf{G}_1 \neq \mathbf{G}_2$ are generic subsets of $\mathbb{P}_{\mathbf{m}}$ but $s \in L_{\mathbf{m}} \Rightarrow \underset{\sim}{\eta}_s[\mathbf{G}_1] = \eta_s = \underset{\sim}{\eta}_s[\mathbf{G}_2]$.

Let $p_1 \in \mathbf{G}_1 \backslash \mathbf{G}_2$ hence there is $p_2 \in \mathbf{G}_2$ such that $p_2 \Vdash_{\mathbb{P}_{\mathbf{m}}}$ "$p_1 \notin \underset{\sim}{\mathbf{G}}_2$" hence p_1, p_2 are incompatible. Let $L_* = \{s \in L_{\mathbf{m}} : \mathbf{G}_1 \cap \mathbb{P}_{\leq s} = \mathbf{G}_2 \cap \mathbb{P}_{\leq s}\}$ so L_* is an initial segment of $L_{\mathbf{m}}$. If $L_* = L_{\mathbf{m}}$ we can easily get a contradiction, so $L_* \neq L_{\mathbf{m}}$ and let $r \in L_{\mathbf{m}} \backslash L_*$ be such that $L_{<r} \subseteq L_*$. Now as in part (8) we can get a contradiction having found a common to upper bound to p_1, p_2.

Alternatively use part (6). $\square_{1.13}$ {c13}

Conclusion 1.15. *Let* $\mathbf{m} \in \mathbf{M}$ *and for notational transparency for some ordinal* $\beta(*), t \in L_{\mathbf{m}} \Leftrightarrow t \in \beta(*)$ *and* $s <_{\mathbf{m}} t \Rightarrow s < t$. *Then* $\mathbf{q}$ *is essentially a* $(<\lambda)$*-support iteration of length* $\beta(*)$ *with* $\underset{\sim}{\mathbb{Q}}_\alpha = \{(\nu, f) \in \mathbb{Q}_{\underset{\sim}{\theta}}^{\mathbf{V}[\langle \underset{\sim}{\eta}_\beta : \beta<\alpha\rangle]} : \nu \lhd f, f = \sup\{f_\iota : \iota < \iota(\alpha)\}, \iota(\alpha) < \lambda, \nu \lhd f_\iota$ *and* $\{f_\iota : \iota < \iota(\alpha)\} \subseteq \cup\{\mathbb{Q}_\theta^{\mathbf{V}[\langle \underset{\sim}{\eta}_\alpha : \alpha \in u\rangle]} : u \in \mathscr{P}_{\mathbf{m},\alpha}\}\}$ *with the natural order, i.e. the order of* $\mathbb{Q}_\theta^{\mathbf{V}[\mathbb{P}_\alpha]}$ *restricted to this set.*

Proof. Should be clear by 1.13. $\square_{1.15}$ {c11}

Till now $(E_{\mathbf{m}}, M_{\mathbf{m}})$ have played no role and we could have omitted them.

Definition 1.16. 1) We define the two-place relation $\leq = \leq_{\mathbf{M}}$ on $\mathbf{M}$ as follows: $\mathbf{m} \leq \mathbf{n}$ <u>iff</u> {c26}

- (a) $L_{\mathbf{m}} \subseteq L_{\mathbf{n}}$, as partial orders of course,
- (b) $M_{\mathbf{m}} = M_{\mathbf{n}}$, yes! equal,
- (c) $u_{\mathbf{m},t} = u_{\mathbf{n},t} \cap L_{\mathbf{m}}$ and[6] $\mathscr{P}_{\mathbf{m},t} = \{u \cap L_{\mathbf{m}} : u \in \mathscr{P}_{\mathbf{n},t}\}$ for $t \in M_{\mathbf{m}}$,
- (d) $u_{\mathbf{m},t} = u_{\mathbf{n},t}$ and $\mathscr{P}_{\mathbf{m},t} = \mathscr{P}_{\mathbf{n},t}$ for $t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$
- (e) if $t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ then $t/E'_{\mathbf{m}} = t/E'_{\mathbf{n}}$ hence $E'_{\mathbf{m}} = E'_{\mathbf{n}} \restriction L_{\mathbf{m}}$
- (f)
  - if $t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ then $\mathscr{P}_{\mathbf{m},t} = \mathscr{P}_{\mathbf{n},t}$
  - if $t \in M_{\mathbf{m}}$ and $s \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ then $\{u \in \mathscr{P}_{\mathbf{m},t} : u \subseteq s/E_{\mathbf{m}}\} = \{u \in \mathscr{P}_{\mathbf{n},t} : u \subseteq s/E_{\mathbf{n}}\}$
  - if $t \in M_{\mathbf{m}}$ then $\{u \in \mathscr{P}_{\mathbf{m},t} : u \subseteq M_{\mathbf{m}}\} = \{u \in \mathscr{P}_{\mathbf{n},t} : u \subseteq M_{\mathbf{n}}\}$.

2) We define the two-place relation $\leq_* = \leq^*_{\mathbf{M}}$ as in part (1) omitting clauses (b),(d),(e) and (f); natural but not used.

{c28}

Claim 1.17. *1) $\leq_{\mathbf{M}}$ is a partial order.*
2) If $\langle \mathbf{m}_\alpha : \alpha < \delta \rangle$ is $\leq_{\mathbf{M}}$-increasing, <u>then</u> its union $\mathbf{m}_\delta$ (naturally defined) is a $\leq_{\mathbf{M}}$-lub and $|L_{\mathbf{m}_\delta}| \leq \Sigma\{|L_{\mathbf{m}_\alpha}| : \alpha < \delta\}$.
3) If $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$ and $L \subseteq L_{\mathbf{m}}$ <u>then</u> $p \in \mathbb{P}_{\mathbf{m}}(L) \Leftrightarrow p \in \mathbb{P}_{\mathbf{n}}(L)$ for every p.
4) If $m \leq_M n$ and $\mathbb{P}_{\mathbf{m}} \lessdot \mathbb{P}_{\mathbf{n}}$ and $L \subseteq L_{\mathbf{m}}$ <u>then</u> $\mathbb{P}_{\mathbf{m}}(L) = \mathbb{P}_{\mathbf{n}}(L)$ as quasi orders.

Proof. Easy, e.g.
1) Why is $L_{\mathbf{m}_\delta} := \cup\{L_{\mathbf{m}_\alpha} \alpha < \delta\}$ well founded? Toward contradiction assume $\bar{t} = \langle t_n : n < \omega \rangle$ is $<_{L_{\mathbf{m}_\delta}}$-decreasing. We can replace $\bar{t}$ by any infinite subsequence. So without loss of generality

- $(*)$ either (α) or (β) where
  - (α) for every $n < m$ there is $s_{n,m} \in M_{\mathbf{m}_0}$ such that $t_m <_{L_\delta} s_{n,m} <_{L_\delta} t_n$
  - (β) for no $n < m$ this holds.

If clause (α) holds, then $\langle s_{n,n+1} : n < \omega \rangle$ is a $<_{M_0}$-decreasing sequence contradiction. If (β) holds for any $n < m$, let $\alpha(n) = \min\{\alpha : t_n \in L_{\mathbf{m}_\alpha}\}$; without loss of generality it is monotonically increasing or constant so as $M_{\mathbf{m}_{\alpha(n)}} = M_{\mathbf{m}_0}$; {c26} by 1.16(1)(e) we get $t_n/E_{\mathbf{m}_{\alpha(n+1)}} = t_{n+1}/E_{\mathbf{m}_{\alpha(n+1)}}$ hence $t_{n+1} \in L_{\mathbf{m}_{\alpha(n)}}$ hence $\alpha(n+1) \leq \alpha(n)$. As $L_{\mathbf{m}_{\alpha(n)}}$ is well founded we are done.
{c33s} 3) See the proof of $\boxplus_\alpha$ in the proof of 1.22. $\square_{1.17}$

{c31}

Claim 1.18. $(\mathbf{M}, \leq_{\mathbf{m}})$ *has amalgamation.*

That is, if $\mathbf{m}_0 \leq_{\mathbf{M}} \mathbf{m}_1, \mathbf{m}_0 \leq_{\mathbf{M}} \mathbf{m}_2$ and $L_{\mathbf{m}_1} \cap L_{\mathbf{m}_2} = L_{\mathbf{m}_0}$ <u>then</u> there is $\mathbf{m} \in \mathbf{M}$ such that $\mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}, \mathbf{m}_2 \leq_{\mathbf{M}} \mathbf{m}$ and $L_{\mathbf{m}} = L_{\mathbf{m}_1} \cup L_{\mathbf{m}_2}$.

{c24}{c26} *Proof.* Note that by clause $(e)(\gamma)$ of Definition 1.7 and clause (e) of Definition 1.16(1):

- $(*)$ assume $(s_1 \in L_{\mathbf{m}_1} \backslash L_{\mathbf{m}_0}) \cap (s_3 \in L_{\mathbf{m}_2} \backslash L_{\mathbf{m}_0})$ and $s_2 \in L_{\mathbf{m}_0}$;
  - if $s_1 <_{\mathbf{m}_1} s_2 \wedge s_2 <_{\mathbf{m}_2} s_3$ <u>then</u> for some $s'_2, s''_2 \in M_{\mathbf{m}_0}$ we have $s'_2 \leq_{\mathbf{m}_0} s_2 \leq_{\mathbf{m}_0} s''_2, s_1 <_{\mathbf{m}_1} s'_2 \wedge s''_2 <_{\mathbf{m}_2} s_3$

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[6] This is covered by clause (f) but see part (2).

- if $s_3 <_{\mathbf{m}_2} s_2 \wedge s_2 <_{\mathbf{m}_1} s_1$ then for some $s'_2, s''_2 \in M_{\mathbf{m}_0}$ we have $s'_2 \leq_{\mathbf{m}_0} s_2 \leq_{\mathbf{m}_0} s''_2$ and $s_3 <_{\mathbf{m}_2} s'_2 \wedge s''_2 <_{\mathbf{m}_1} s_1$.

We now define $\mathbf{m}$ by:

$(*)$ (a) (α) $t \in L_{\mathbf{m}}$ iff $t \in L_{\mathbf{m}_1} \vee t \in L_{\mathbf{m}_2}$
  (β) $M_{\mathbf{m}} = M_{\mathbf{m}_0}$
 (b) $s <_{\mathbf{m}} t$ iff one of the following occurs:
  (α) $s <_{\mathbf{m}_1} t$
  (β) $s <_{\mathbf{m}_2} t$
  (γ) $(s \in L_{\mathbf{m}_1} \backslash L_{\mathbf{m}_0})$ and $t \in L_{\mathbf{m}_2} \backslash L_{\mathbf{m}_0}$ and $(\exists r \in M_{\mathbf{m}_0})(s \leq_{\mathbf{m}_1} r \wedge r \leq_{\mathbf{m}_2} t)$
  (δ) $s \in L_{\mathbf{m}_2} \backslash L_{\mathbf{m}_0}$ and $t \in L_{\mathbf{m}_1} \backslash L_{\mathbf{m}_0}$ and for some $r \in M_{\mathbf{m}_0}, s \leq_{\mathbf{m}_2} r \wedge r \leq_{\mathbf{m}_1} t$
 (c) $u_{\mathbf{m},t}$ is
  (α) $u_{\mathbf{m}_1,t} \cup u_{\mathbf{m}_2,t}$ if[7] $t \in L_{\mathbf{m}_0}$
  (β) $u_{\mathbf{m}_1,t}$ if $t \in L_{\mathbf{m}_1} \backslash L_{\mathbf{m}_0}$
  (γ) $u_{\mathbf{m}_2,t}$ if $t \in L_{\mathbf{m}_2} \backslash L_{\mathbf{m}_0}$
 (d) $E'_{\mathbf{m}} = E'_{\mathbf{m}_1} \cup E'_{\mathbf{m}_2}$
 (e) $\mathscr{P}_{\mathbf{m},t}$ is
  (α) $\mathscr{P}_{\mathbf{m}_1,t} \cup \mathscr{P}_{\mathbf{m}_2,t}$ if $t \in L_{\mathbf{m}_0}$
  (β) $\mathscr{P}_{\mathbf{m}_1,t}$ if $t \in L_{\mathbf{m}_1} \backslash L_{\mathbf{m}_0}$
  (γ) $\mathscr{P}_{\mathbf{m}_2,t}$ if $t \in L_{\mathbf{m}_2} \backslash L_{\mathbf{m}_0}$.

Clearly

$\odot$ $\mathbf{m} \in \mathbf{M}$ and $\mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}$ and $\mathbf{m}_2 \leq_{\mathbf{M}} \mathbf{m}$.

So we are done. $\square_{1.18}$

{c32n}

Observation 1.19. *For $p, q \in \mathbb{P}_{\mathbf{m}}$ the following conditions are equivalent:*

(a) $q \Vdash$ "$p \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\mathbf{m}}}$"
(b) *if $s \in \mathrm{Dom}(p)$ then either $s \in \mathrm{Dom}(q)$ and $(q{\restriction}L_{\mathbf{m},<s}) \Vdash_{\mathbb{P}_{\mathbf{m},<s}}$ "$p(s) \leq q(s)$" or $s \notin \mathrm{Dom}(q), \mathrm{tr}(p(s)) = \emptyset$ and $q{\restriction}L_{\mathbf{m},<s} \Vdash_{\mathbb{P}_{\mathbf{m},<s}}$ "$p(s)$ is trivial, i.e. $\underset{\sim}{f}_{p(s)}$ is constantly zero"*
(c) $\mathbb{P}_{\mathbf{m}} \models$ "$p \leq q^+$" *where* $\mathrm{Dom}(q^+) = \mathrm{Dom}(q) \cup \mathrm{Dom}(p)$ *and* $q^+(s)$ *is*
 (α) $q(s)$ *if* $s \in \mathrm{Dom}(q)$
 (β) *the trivial condition if* $s \in \mathrm{dom}(p) \backslash \mathrm{dom}(q)$*; note that* $\mathrm{fsupp}(q^+) = \mathrm{fsupp}(q) \cup \mathrm{Dom}(p)$.

Remark 1.20. We shall use this freely. Maybe better to change the order.

Proof. Obvious recalling the properties of $\mathbb{Q}_{\bar{\theta}}$. $\square_{1.19}$

{c33n}
{c7}

Claim 1.21. *For $\mathbf{m} \in \mathbf{M}$, recalling 1.10(3), we have $\mathbb{P}_{\mathbf{m}}(L_1) \lessdot \mathbb{P}_{\mathbf{m}}(L_3)$ when:*

$(*)$ (a) $L_2 \subseteq L_3$ *are initial segments of* $L_{\mathbf{m}}$

[7] but recall that $t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}} \Rightarrow u_{\mathbf{m}_\ell,t} = u_{\mathbf{m}_0,t} \wedge \mathscr{P}_{\mathbf{m}_\ell,t} = \mathscr{P}_{\mathbf{m}_0,t}$

(b) $L_1 \subseteq L_3$ *and* $L_0 = L_1 \cap L_2$
(c) L_0 *is an initial segment of* L_1,
(d) $\mathbb{P}_{\mathbf{m}}(L_0) \lessdot \mathbb{P}_{\mathbf{m}}(L_2)$
(e) $L_1 \backslash L_0$ *is disjoint to* $M_{\mathbf{m}}$
(f) *if* $t \in L_1 \backslash L_0$ *then* $(t/E_{\mathbf{m}}) \cap L_{\mathbf{m},<t} \subseteq L_1$.

Proof. As $\mathrm{dp}_{\mathbf{m}}(L_1) < \infty$ it suffices to prove by induction on the ordinal γ that:

$\boxplus_\gamma$ if $\langle L_\ell : \ell \le 3\rangle$ satisfies $(*)$ of the claim and $\mathrm{dp}_{\mathbf{m}}(L_1) \le \gamma$ <u>then</u>
- (a) $\mathbb{P}_{\mathbf{m}}(L_1) \lessdot \mathbb{P}_{\mathbf{m}}(L_3)$
- (b) we have $p_1 \in \mathbb{P}_{\mathbf{m}}(L_1)$ and $p_1 \le q_1 \in \mathbb{P}_{\mathbf{m}}(L_1) \Rightarrow p_3, q_1$ are compatible in $\mathbb{P}_{\mathbf{m}}(L_3)$ <u>when</u>:
  - (α) $p_3 \in \mathbb{P}_{\mathbf{m}}(L_3)$
  - (β) $p_0 \in \mathbb{P}_{\mathbf{m}}(L_0)$
  - (γ) if $p_0 \le q_0 \in \mathbb{P}_{\mathbf{m}}(L_0)$ <u>then</u> $p_2 := p_3 \restriction L_2$ and q_0 are compatible in $P_{\mathbf{m}}(L_2)$
  - (δ) $p_1 = p_0 \cup (p_3 \restriction (L_1 \backslash L_0))$.

Why this holds? Assume we have arrived to γ.

<u>Clause (b)</u>: Recalling clause (f) of the assumption, indeed, $p_1 = p_0 \cup (p_3 \restriction (L_1 \backslash L_0)) \in \mathbb{P}_{\mathbf{m}}(L_1)$ by the definitions (clauses $(b)(\alpha),(\beta),(\delta)$ of $\boxplus_\gamma$), e.g. why $\mathrm{fsupp}(p_1) \subseteq L_1$? Note that if $s \in \mathrm{dom}(p_3 \restriction (L_1 \backslash L_0)$ then $s \in L_1 \backslash L_0 \subseteq L_1$ and $\{r_{p_3(s)}(\zeta) : \zeta < \xi_{p(s)}\}$
{c6} is included in L_3 because $p \in \mathbb{P}_{\mathbf{m}}(L_3)$ and in $L_{<s}$ by Definition 1.9. As $s \in L_1 \backslash L_0$
{c6} by $(*)(e)$ we have $s \notin M_{\mathbf{m}}$ hence by Definition 1.9 we have $\{r_{p_3(s)}(\zeta) : \zeta < \xi_{p(s)}\} \subseteq u_s \subseteq s/E_{\mathbf{m}}$. By $(*)(f)$ we have $(s/E_{\mathbf{m}}) \cap L_{\mathbf{m}<t} \subseteq L_1$ hence together $\{r_{p_3(s)}(\zeta) : \zeta < \xi_{p(s)}\} \subseteq L_1$, and we are done proving $\mathrm{fsupp}(p_1) \subseteq L_1$.

So the first statement in $\boxplus_\gamma(b)$ holds; what about the second? Toward contra-
{c11} diction assume q_1 contradicts the desired conclusion then by 1.13(8) there are s and p_3^+ such that:

$\oplus$
- (a) $s \in \mathrm{dom}(q_1) \cap \mathrm{dom}(p_3)$
- (b) $p_3^+ \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m},<s})$
- (c) p_3^+ is above $p_3 \restriction L_{\mathbf{m},<s}$ and above $q_1 \restriction L_{\mathbf{m},<s}$
- (d) $p_3^+ \Vdash_{\mathbb{P}_{\mathbf{m},<s}}$ "$p_3(s), q_1(s) \in \mathbb{Q}_{\bar\theta}$ are incompatible (in $\mathbb{Q}_{\bar\theta}$)".

So $s \in \mathrm{dom}(q_1) \subseteq L_1$ and as L_2 is an initial segment of $L_{\mathbf{m}}$ and clause (γ) of (b) (of $\boxplus_\gamma$), clearly $s \in L_0$ is impossible, so $s \in \mathrm{dom}(q_1) \backslash L_0 \subseteq L_1 \backslash L_0$. As $\mathbb{P}_{\mathbf{m}} \models$ "$p_1 \le q_1$", necessarily $q_1 \restriction L_{\mathbf{m},<s} \Vdash_{\mathbb{P}_{\mathbf{m},<s}}$ "$p_1(s) \le q_1(s)$", so as $q_1 \restriction L_{\mathbf{m},<s} \le p_3^+ \restriction L_{\mathbf{m},<s}$ (by $\oplus(c)$), also $p_3^+ \restriction L_{\mathbf{m},<s} \Vdash_{\mathbb{P}_{\mathbf{m},<s}}$ "$p_1(s) \le q_1(s)$". As $s \notin L_0$ clearly $p_1(s) = p_3(s)$ by clauses $\boxplus_\gamma(b)(\beta),(\delta)$, so $p_3^+ \restriction L_{\mathbf{m},<s} \Vdash_{\mathbb{P}_{\mathbf{m},<s}}$ "$p_3(s) \le q_1(s)$" and again easy contradiction to $\oplus(d)$.

<u>Clause (a)</u>:

Clearly $\mathbb{P}_{\mathbf{m}}(L_1) \subseteq \mathbb{P}_{\mathbf{m}}(L_3)$ as quasi orders. Next we shall prove $\mathbb{P}_{\mathbf{m}}(L_1) \le_{\mathrm{ic}} \mathbb{P}_{\mathbf{m}}(L_3)$, so assume $q_1, q_2 \in \mathbb{P}_{\mathbf{m}}(L_1)$ has a common upper bound p_3 in $\mathbb{P}_{\mathbf{m}}(L_3)$, and
{c6} we should find one in $\mathbb{P}_{\mathbf{m}}(L_1)$. Hence (see 1.9(e)$(\beta)$ we have $\mathrm{Dom}(q_1) \cup \mathrm{Dom}(q_2) \subseteq \mathrm{Dom}(p_3)$.

As $p_3 \restriction L_2 \in \mathbb{P}_{\mathbf{m}}(L_2)$ by $(*)(a)$ and we are assuming $\mathbb{P}_{\mathbf{m}}(L_0) \lessdot \mathbb{P}_{\mathbf{m}}(L_2)$, see $(*)(d)$ there is $p_0 \in \mathbb{P}_{\mathbf{m}}(L_0)$ such that $p_0 \le q \in \mathbb{P}_{\mathbf{m}}(L_0) \Rightarrow q, p_3 \restriction L_2$ are compatible in $\mathbb{P}_{\mathbf{m}}(L_2)$ and let $p_1 = p_0 \cup (p_3{\restriction}(L_1 \backslash L_0))$. By $\boxplus_\gamma(b)$, which we have proved noting that clauses $(\alpha) - (\delta)$ of $\boxplus_\gamma(b)$ holds, we know that $p_1 \in \mathbb{P}_{\mathbf{m}}(L_1)$ and $p_1 \le p'_1 \in \mathbb{P}_{\mathbf{m}}(L_1) \Rightarrow p_3, p'_1$ are compatible in $\mathbb{P}_{\mathbf{m}}(L_3)$. It suffices to prove that p_1 is a common upper bound of q_1, q_2.

We could have replaced p_0 by p'_0 whenever $p_0 \le p'_0 \in \mathbb{P}_{\mathbf{m}}(L_0)$. So without loss of generality for $\ell = 1, 2$ we have $\mathrm{dom}(q_\ell) \cap L_0 \subseteq \mathrm{dom}(p_0)$ hence $\subseteq \mathrm{dom}(p_1)$, also recall $\mathrm{dom}(q_\ell) \backslash L_0 \subseteq \mathrm{dom}(p_3) \cap L_1 \backslash L_0$ and by the choice of p_1 we have $\mathrm{dom}(p_3) \cap L_1 \backslash L_0 \subseteq \mathrm{dom}(p_1) \backslash L_0$.

So recalling $\mathrm{dom}(q_\ell) \subseteq L_1$ together $\mathrm{dom}(q_\ell) \subseteq \mathrm{dom}(p_1)$.

As we are assuming $\mathbb{P}_{\mathbf{m}}(L_0) \lessdot \mathbb{P}_{\mathbf{m}}(L_2)$ without loss of generality p_0 is above[8] $q_\ell{\restriction}L_0$. If toward contradiction we assume that $\ell \in \{1, 2\}$ and $q_\ell \nleq p_1$ then for some $s \in \mathrm{Dom}(q_\ell)$ we have $(q_\ell{\restriction}L_{\mathbf{m},<s}) \le (p_1{\restriction}L_{\mathbf{m},<s})$ but $p_1{\restriction}L_{\mathbf{m},<s} \nVdash_{\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m},<s})}$ "$q_\ell(s) \le p_1(s)$". Clearly, $s \in L_0$ is impossible so $s \in L_1 \backslash L_0$ hence $s \notin M_{\mathbf{m}}$ by clause $(*)(e)$.

Let $L'_0 = L_0, L'_1 = L_0 \cup (L_1 \cap L_{\mathbf{m},<s}), L'_2 = L_2, L'_3 = L_3$ so (L'_0, L'_1, L'_2, L'_3) satisfies the assumptions of the present claim and $\mathrm{dp}_{\mathbf{m}}(L'_1) < \gamma$, hence by the induction hypothesis, $\mathbb{P}_{\mathbf{m}}(L'_1) \lessdot \mathbb{P}_{\mathbf{m}}(L'_3)$.

Recall $s \in L_1 \backslash L_0$ hence $(s/E_{\mathbf{m}}) \cap L_{\mathbf{m},<s} \subseteq L_1$ by clause (f) of the assumption of the claim, so $\mathrm{fsupp}(p_1{\restriction}\{s\}) \backslash \{s\}, \mathrm{fsupp}(q_\ell{\restriction}\{s\}) \backslash \{s\}$ are $\subseteq L'_1$ hence $p_1(s), q_\ell(s)$ are $\mathbb{P}_{\mathbf{m}}(L'_1)$-names. So recalling $p_1{\restriction}L_{\mathbf{m},<s} \nVdash_{\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m},<s})}$ "$q_\ell(s) \le p_1(s)$" and $\mathbb{P}_{\mathbf{m}}(L'_1) \lessdot \mathbb{P}_{\mathbf{m}}(L'_3)$ and $L_{\mathbf{m},<s} \subseteq L_3 = L'_3$ we have $p_1{\restriction}L'_1 \nVdash_{\mathbb{P}_{\mathbf{m}}(L'_1)}$ "$q_\ell(s) \le p_1(s)$". Hence there is p_1^+ such that $p_1{\restriction}L'_1 \le p_1^+ \in \mathbb{P}_{\mathbf{m}}(L'_1)$ such that $p_1^+ \Vdash_{\mathbb{P}_{\mathbf{m}}(L'_1)}$ "$q_\ell(s) \nleq p_1(s)$" so recalling $\mathbb{P}_{\mathbf{m}}(L'_1) \lessdot \mathbb{P}(L'_3)$ we have $p_1^+ \Vdash_{\mathbb{P}_{\mathbf{m}}(L'_3)}$ "$q_\ell(s) \nleq p_1(s)$".

But by $\boxplus_{\gamma_1}(b)$ for $\gamma_1 = \mathrm{dp}_{\mathbf{m}}(L'_1)$, we know that p_1^+ and $p_3{\restriction}L_{\mathbf{m},<s}$ are compatible (in $\mathbb{P}_{\mathbf{m}}$, equivalently $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m},<s})$) so let $p_3^+ \in \mathbb{P}_m(L_{\mathbf{m},<s})$ be a common upper bound of $p_1^+, p_3{\restriction}L_{\mathbf{m},<s}$. Now $p_3^+ \Vdash_{\mathbb{P}_{\mathbf{m}}(L'_3)}$ "$q_\ell(s) \le p_1(s)$" because: $q_\ell \le p_3$ by the choice of p_3; $p_1(s) = p_3(s)$ by the choice of p_1 and $p_3 \le p_3^+$, see above. However, $p_3^+ \Vdash_{\mathbb{P}_{\mathbf{m}}(L'_3)}$ "$q_\ell(s) \nleq p_1(s)$" as $p_1^+ \le p_3^+$, see above.

So we have proved $\mathbb{P}_{\mathbf{m}}(L_1) \le_{\mathrm{ic}} \mathbb{P}_{\mathbf{m}}(L_3)$.

To finish proving clause $\boxplus_\gamma(a)$ that is $\mathbb{P}_{\mathbf{m}}(L_1) \lessdot \mathbb{P}_{\mathbf{m}}(L_3)$ note that clause $\boxplus_\gamma(b)$ does this as for every $p_3 \in \mathbb{P}_{\mathbf{m}}(L_3)$ there is p_0 as in $\boxplus_\gamma(\beta), (\gamma)$ by clause (d) of the claim's assumption and let p_1 be as defined in $\boxplus_\gamma(b)(\delta)$. $\square_{1.21}$

Claim 1.22. *We have* $\mathbb{P}_{\mathbf{m}_1}(L_1) = \mathbb{P}_{\mathbf{m}_2}(L_1)$ *(i.e. as quasi orders) and* $\mathbb{P}_{\mathbf{m}_\ell}(L_1) \lessdot \mathbb{P}_{\mathbf{m}_\ell}$ *for* $\ell = 1, 2$ *<u>when</u>:* {c33s}

$\boxdot$ (a) $\mathbf{m}_1 \le_{\mathbf{M}} \mathbf{m}_2$

(b) $L_0 \subseteq L_1 \subseteq L_{\mathbf{m}_1}$

(c) L_0 *is an initial segment of* L_1

(d) $\mathbb{P}_{\mathbf{m}_1}(L_0) = \mathbb{P}_{\mathbf{m}_2}(L_0)$

(e) $\mathbb{P}_{\mathbf{m}_\ell}(L_0) \lessdot \mathbb{P}_{\mathbf{m}_\ell}$ *for* $\ell = 1, 2$

[8] Why? It suffices to prove that there is $p'_0 \in \mathbb{P}_{\mathbf{m}}(L_0)$ above p_0 and above $q_\ell{\restriction}L_0$. So toward contradiction assume this fails hence there is $p_0^+ \in \mathbb{P}_{\mathbf{m}}(L_0)$ above p_0 incompatible with $q_\ell{\restriction}L_0$. By the choice of p_0 we know that $p_0^+, (p_3{\restriction}L_2)$ are compatible, so let $p_3^+ \in \mathbb{P}_{\mathbf{m}}(L_2)$ be a common upper bound. Now L_2 is an initial segment of $L_{\mathbf{m}}$ by $(*)(a)$ and p_3 is above q_ℓ hence $p_3{\restriction}L_2$ is above $q_\ell{\restriction}L_2$ and as $q_\ell \in \mathbb{P}_{\mathbf{m}}(L_1), L_0 = L_1 \cap L_2$ we have $q_\ell{\restriction}L_2 = q_\ell{\restriction}L_0, p_3{\restriction}L_2$ is above $q_\ell{\restriction}L_0$ but p_3^+ is above $p_3{\restriction}L_2$ hence p_3^+ is above $q_\ell{\restriction}L_2$. Also p_3^+ is above p_0^+ which forces $q_\ell{\restriction}L_0 \notin \mathop{\mathbf{G}}\limits_{\sim}{}_{\mathbb{P}_{\mathbf{m}}(L_0)}$, equivalently $q_\ell{\restriction}L_0 \notin \mathop{\mathbf{G}}\limits_{\sim}{}_{\mathbb{P}_{\mathbf{m}}(L_2)}$, contradiction.

(f) *if* $t \in L_1 \backslash L_0$ *then* $t \notin M_{\mathbf{m}_2}$ *(but see present* $\boxplus_\alpha$*) and* $L_{\mathbf{m}_1,<t} \cap (t/E_{\mathbf{m}_1}) = L_{\mathbf{m}_2,<t} \cap (t/E_{\mathbf{m}_2}) \subseteq L_1$.

{e32} *Remark* 1.23. Used only in the proof of $\boxplus_{4.4}$ inside the proof of 3.19, so we could have used $M_\beta, \mathscr{E}$ from there.

Proof. For $\ell \in \{1,2\}$ let $\bar{L}_\ell = \langle L_{\ell,i} : i < 4\rangle$ be defined by:

$\oplus_1$ (a) $L_{\ell,0} = L_0$
(b) $L_{\ell,1} = L_1$
(c) $L_{\ell,2} = \{s \in L_{\mathbf{m}_\ell} : s \le_{\mathbf{m}_\ell} t$ for some $t \in L_0\}$
(d) $L_{\ell,3} = L_{\mathbf{m}_\ell}$.

Clearly

{c33n} $\oplus_2$ (a) $(\mathbf{m}_\ell, \bar{L}_\ell)$ satisfies the assumptions of 1.21 hence
(b) $\mathbb{P}_{\mathbf{m}_\ell}(L_{\ell,1}) \lessdot \mathbb{P}_{\mathbf{m}_\ell}(L_{\ell,3})$ which means $\mathbb{P}_{\mathbf{m}_\ell}(L_1) \lessdot \mathbb{P}_{\mathbf{m}_\ell}$ for $\ell = 1,2$.

Why $\oplus_2$? Clearly it suffices to prove clause (a), so we just have to check clauses
{c33s} $(*)(a)-(f)$ of 1.22.

Clause $(*)(a)$:

By $\oplus_1(d), L_{\ell,3} = L_{\mathbf{m}_\ell}$ hence is an initial segment of $L_{\mathbf{m}_\ell}$ and by $\oplus(c), L_{\ell,2}$ is an initial segment of $L_{\mathbf{m}_\ell}$ which is $L_{\ell,3}$ so $L_{\ell,2} \subseteq L_{\ell,3}$.

Clause $(*)(b)$:

For the first statement, $L_{\ell,1} \subseteq L_{\ell,3}$ is trivial by $\oplus_1(d) + \oplus_1(b) + \boxdot(a),(b)$. The second statement says $L_{\ell,0} = L_{\ell,1} \cap L_{\ell,2}$. Now $L_{\ell,0} \subseteq L_{\ell,1}$ by $\boxdot(a),(b)$ of the claim and $\oplus_1(a),(b)$. Also $L_{\ell,0} \subseteq L_{\ell,2}$ holds by $\oplus_1(c)$ (and $\oplus_1(a)$). Together $L_{\ell,0} \subseteq L_{\ell,1} \cap L_{\ell,2}$; to prove the inverse inclusion assume $s \in L_{\ell,2} \cap L_{\ell,1}$, so as $s \in L_{\ell,2}$ by $\oplus_1(c)$ there is $t \in L_0$ such that $s \le_{\mathbf{m}_\ell} t$. But $s \in L_{\ell,1} = L_1$ so by $\boxdot(c)$ of the claim we have $s \in L_0$ as promised.

Clause $(*)(c)$:

Holds by $\boxdot(c)$ of the claim.

Clause $(*)(d)$:

By clause $\boxdot(f)$ of the claim and $\boxplus_2(c), L_{\ell,2}$ is an initial segment of $L_{\mathbf{m}_\ell}$, hence
{c8} by 1.11(e) we have $\mathbb{P}_{\mathbf{m}_\ell}(L_{\ell,2}) \lessdot \mathbb{P}_{\mathbf{m}_\ell} = \mathbb{P}_{\mathbf{m}_\ell}(L_{\ell,3})$. By $\boxdot(e)\,\mathbb{P}_{\mathbf{m}_0}(L_{\ell,0}) \lessdot \mathbb{P}_{\mathbf{m}_\ell}$; so together as $L_{\ell,0} \subseteq L_{\ell,2}$, we have $\mathbb{P}_{\mathbf{m}_\ell}(L_0) \lessdot \mathbb{P}_{\mathbf{m}_\ell}(L_{\ell,2})$.

Clause $(*)(e),(f)$:

Holds by $\boxdot(f)$ of the claim.

So $\oplus_2$ holds indeed. So now we deal with the other half.

Proof of: $\mathbb{P}_{\mathbf{m}_1}(L_1) = \mathbb{P}_{\mathbf{m}_2}(L_1)$.

Let $\langle s_\alpha : \alpha < \alpha(*)\rangle$ list $L_1 \backslash L_0$ such that $s_\alpha \le_{L_\mathbf{m}} s_\beta \Rightarrow \alpha \le \beta$. This is possible as $L_{\mathbf{m}_2}$ is well founded.

Now

$\oplus_3$ for $\ell = 1,2$ and $\alpha \le \alpha(*)$ let $\bar{L}^*_{\ell,\alpha} = \langle L^*_{\ell,\alpha,i} : i < 4\rangle$ be (so we can omit ℓ if $\ell = 0,1$)

(a) $L^*_{\ell,\alpha,0} = L_0$
(b) $L^*_{\ell,\alpha,1} = L_0 \cup \{s_\beta : \beta < \alpha\}$

(c) $L^*_{\ell,\alpha,2} = \{s \in L_{\mathbf{m}_\ell} : s \le_{\mathbf{m}_\ell} t \text{ for some } t \in L_0\}$

(d) $L^*_{\ell,\alpha,3} = L_{\mathbf{m}_\ell}$

$\oplus_4$ (a) $(\bar{\mathbf{m}}_\ell, \bar{L}^*_{\ell,\alpha})$ satisfies the assumption of 1.21 {c33n}

(b) $\mathbb{P}_{\mathbf{m}_\ell}(L^*_{\ell,\alpha,1}) \lessdot \mathbb{P}_{\mathbf{m}_\ell}(L^*_{\ell,\alpha,3})$.

[Why? Note the $\mathbf{m}_\ell, \langle L^*_{\ell,\alpha,i} : i < 4\rangle$ satisfies the assumptions of 1.22, hence $\oplus_2$ holds for $\mathbf{m}_\ell, \bar{L}_{\ell,\alpha}$ for $\alpha \le \alpha(*)$. Now by induction on $\alpha \le \alpha(*)$ we prove that: {c33s}

$\boxplus_\alpha$ $\mathbb{P}_{\mathbf{m}_1}(L^*_{\alpha,1}) = \mathbb{P}_{\mathbf{m}_2}(L^*_{\alpha,1})$.

Case 1: $\alpha = 0$

As $L^*_{1,\alpha,1} = L_0 = L^*_{2,\alpha,1}$, clause $\boxdot(d)$ of the assumption gives $\boxplus_\alpha$ as promised.

Case 2: α a limit ordinal

Easy by the definition of the iteration. That is, first we know $p \in \mathbb{P}_{\mathbf{m}_1}(L^*_{\alpha,1}) \Leftrightarrow \bigwedge\limits_{\beta<\alpha} [p{\restriction}L^*_{\beta,1} \in \mathbb{P}_{\mathbf{m}_1}(L^*_{\beta,1})] \Leftrightarrow \bigwedge\limits_{\beta<\alpha} [p{\restriction}L^*_{\beta,1} \in \mathbb{P}_{\mathbf{m}_2}(L^*_{\beta,1})] \Leftrightarrow p \in \mathbb{P}_{\mathbf{m}_2}(L^*_{\alpha,1})$; second, for $p, q \in \mathbb{P}_{\mathbf{m}_1}(L^*_{\alpha,1})$ by the definition of the order and the induction hypothesis, $\mathbb{P}_{\mathbf{m}_1}(L^*_{\alpha,1}) \models$ "$p \le q$" iff $\bigwedge\limits_{\beta<\alpha} [\mathbb{P}_{\mathbf{m}_1}(L^*_{\beta,1}) \models$ "$p{\restriction}L^*_{\beta,1} \le q{\restriction}L^*_{\beta,1}$"] iff $\bigwedge\limits_{\beta<\alpha} [\mathbb{P}_{\mathbf{m}_2}(L^*_{\beta,1}) \models$ "$p{\restriction}L^*_{\beta,1} \le q{\restriction}L^*_{\beta,1}$"] iff $\mathbb{P}_{\mathbf{m}_2}(L^*_{\alpha,1}) \models$ "$p \le q$".

So $\boxplus_\alpha$ holds.

Case 3: $\alpha = \beta + 1$

Clearly

$(*)_1$ $p \in \mathbb{P}_{\mathbf{m}_1}(L^*_{\alpha,1}) \Leftrightarrow p \in \mathbb{P}_{\mathbf{m}_2}(L^*_{\alpha,1})$.

Next

$(*)_2$ assume $p, q \in \mathbb{P}_{\mathbf{m}_1}(L^*_{\alpha,1})$ and we shall prove that $\mathbb{P}_{\mathbf{m}_1}(L^*_{\alpha,1}) \models$ "$p \le q$" implies $\mathbb{P}_{\mathbf{m}_2}(L^*_{\alpha,1}) \models$ "$p \le q$".

[Why? If $s_\beta \notin \mathrm{dom}(p)$ this is obvious by the induction hypothesis.

Hence we can assume $s_\beta \in \mathrm{dom}(p)$, so as we are assuming $\mathbb{P}_{\mathbf{m}_1}(L^*_{\alpha,1}) \models$ "$p \le q$", clearly $s_\beta \in \mathrm{dom}(q)$ hence $s_\beta \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$. First, similarly $\mathbb{P}_{\mathbf{m}_1}(L^*_{\beta,1}) \models$ "$(p{\restriction}L^*_{\beta,1}) \le (q{\restriction}L^*_{\beta,1})$" and $(q{\restriction}L^*_{\beta,1}) \Vdash_{\mathbb{P}_{\mathbf{m}_1,<s_\beta}}$ "$p(s_\beta) \le_{\mathbb{Q}_{\bar\theta}} q(s_\beta)$" by the definition of $\mathbb{P}_{\mathbf{m}_1}(L^*_{\beta,1})$. Second, as $q{\restriction}L^*_{\beta,1} \in \mathbb{P}_{\mathbf{m}_1}(L^*_{\beta,1}) = \mathbb{P}_{\mathbf{m}_2}(L^*_{\beta,1})$ and $\mathbb{P}_{\mathbf{m}_1}(L^*_{\beta,1}) \lessdot \mathbb{P}_{\mathbf{m}_2}$ by $\oplus_4$ and $\mathbb{P}_{\mathbf{m}_2}(L^*_{\beta,1}) \lessdot \mathbb{P}_{\mathbf{m}_2}$ by $\oplus_2$ and $p(s_\beta), q(s_\beta)$ are $\mathbb{P}_{\mathbf{m}_1}(L^*_{\beta,1})$-names (as $\mathrm{fsupp}(p(s_\beta), \mathrm{fsupp}(q(s_\beta)) \subseteq L^*_{\beta,1})$ necessarily we have $q{\restriction}L^*_{\beta,1} \Vdash_{\mathbb{P}_{\mathbf{m}_2}}$ "$p(s_\beta) \le_{\mathbb{Q}_{\bar\theta}} q(s_\beta)$". Third, as $\mathbb{P}_{\mathbf{m}_1}(L^*_{\beta,1}) \models$ "$p{\restriction}L^*_{\beta,1} \le q{\restriction}L^*_{\beta,1}$", by the induction hypothesis $\mathbb{P}_{\mathbf{m}_2}(L^*_{\beta,1}) \models$ "$p{\restriction}L^*_{\beta,1} \le q{\restriction}L^*_{\beta,1}$. Fourth, by the last two sentence and the definition of the order in $\mathbb{P}_{\mathbf{m}_2}$ we have $\mathbb{P}_{\mathbf{m}_2} \models$ "$p \le q$" so the conclusion of $(*)_2$ holds also in this case.

Note that if $s_\beta \in \mathrm{dom}(p) \backslash \mathrm{dom}(q)$ then $p \nleq q$, so we are done proving $(*)_2$.]

$(*)_3$ if $p, q \in \mathbb{P}_{\mathbf{m},1}(L^*_{\alpha,1})$ and $\mathbb{P}_{\mathbf{m}_2}(L^*_{\alpha,1}) \models$ "$p \le q$" then $\mathbb{P}_{\mathbf{m}_1}(L^*_{\alpha,1}) \models$ "$p \le q$".

[Why? Similar to the proof of $(*)_2$.]

By $(*)_1, (*)_2, (*)_3$ clearly $\boxplus_\alpha$ holds. So we carried the induction so $\boxplus_\alpha$ holds for every $\alpha \le \alpha(*)$ and for $\alpha = \alpha(*)$ we get $\mathbb{P}_{\mathbf{m}_1}(L_1) = \mathbb{P}_{\mathbf{m}_2}(L_2)$. Together with $\oplus_2(b)$ in the beginning of the proof we are done. $\square_{1.22}$

Definition 1.24. 0) For $L \subseteq L_{\mathbf{m}}, \mathbf{m} \in \mathbf{M}$ let {c34}

(a) $\mathrm{dp}^*_{\mathbf{m}}(L) = \cup\{\mathrm{dp}_{M_{\mathbf{m}}}(t) + 1 : t \in L \cap M_{\mathbf{m}}\}$

(b) $L^{\mathrm{dp}}_{\mathbf{m},\gamma} = \{t \in L_{\mathbf{m}}$: if $s \le_{L_{\mathbf{m}}} t \wedge s \in M_{\mathbf{m}}$ then $\mathrm{dp}_{M_{\mathbf{m}}}(s) < \gamma$; moreover, $\sup\{\mathrm{dp}_{M_{\mathbf{m}}}(s) : s \in M_{\mathbf{m}}$ and $s <_{L_{\mathbf{m}}} t\} < \gamma\}$.

{c7} 1) For an ordinal γ let $\mathbf{M}^{\mathrm{ec}}_\gamma$ be the class of $\mathbf{m} \in \mathbf{M}$ such that, recalling Definition 1.10(3):

(∗) if $\mathbf{m} \le_{\mathbf{M}} \mathbf{m}_1 \le_{\mathbf{M}} \mathbf{m}_2$ <u>then</u> $\mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma}) \lessdot \mathbb{P}_{\mathbf{m}_2}(L^{\mathrm{dp}}_{\mathbf{m}_2,\gamma})$ hence $L \subseteq L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma}$
{c28} implies $\mathbb{P}_{\mathbf{m}_1}(L) = \mathbb{P}_{\mathbf{m}_2}(L)$ (by 1.17(4)).

2) Let $\mathbf{M}_{\mathrm{ec}} = M^{\mathrm{ec}}_\infty$ be the class of $\mathbf{m}$ which $\in \mathbf{M}^{\mathrm{ec}}_\gamma$ for every γ.
3) Let $\mathbf{M}^{\mathrm{ec}}_{\chi,\gamma} = \{\mathbf{m} \in \mathbf{M}^{\mathrm{ec}}_\gamma : |L_{\mathbf{m}}| \le \chi\}$, similarly $\mathbf{M}^{\mathrm{ec}}_{\chi,\infty}$.

{c36d}

Observation 1.25. *1) Of course, $\mathbf{M}^{\mathrm{ec}}_{\gamma_2} \subseteq \mathbf{M}^{\mathrm{ec}}_{\gamma_1}$ and $L^{\mathrm{dp}}_{\mathbf{m},\gamma_1} \subseteq L^{\mathrm{dp}}_{\mathbf{m},\gamma_2}$ are initial segments of $L_{\mathbf{m}}$ when $\gamma_1 \le \gamma_2$.*
{c34} *2) In 1.24(1), the following are equivalent:*

(a) $\mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma}) \lessdot \mathbb{P}_{\mathbf{m}_2}(L^{\mathrm{dp}}_{\mathbf{m}_2,\gamma})$ *for every* γ

(b) $\mathbb{P}_{\mathbf{m}_1} \lessdot \mathbb{P}_{\mathbf{m}_2}$.

Proof. 1) Easy.
2) First, concerning (a) ⇒ (b), note that for γ large enough we have $\mathbb{P}_{\mathbf{m}_\ell}(L^{\mathrm{dp}}_{\mathbf{m}_\ell,\gamma}) = \mathbb{P}_{\mathbf{m}_\ell}$, so clear. Second, assume (b), note that $L^{\mathrm{dp}}_{\mathbf{m}_\ell,\gamma}$ is an initial segment of $L_{\mathbf{m}_\ell}$
{c8} hence $\mathbb{P}_{\mathbf{m}_\ell}(L^{\mathrm{dp}}_{\mathbf{m}_\ell,\gamma}) \lessdot \mathbb{P}_{\mathbf{m}_\ell}$ for $\ell = 1, 2$ by 1.11(c), hence we have $\mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma}) \lessdot \mathbb{P}_{\mathbf{m}_1} \lessdot \mathbb{P}_{\mathbf{m}_2}$, but $\lessdot$ is transitive, hence $\mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma}) \lessdot \mathbb{P}_{\mathbf{m}_2}$. Also $\mathbb{P}_{\mathbf{m}_2}(L^{\mathrm{dp}}_{\mathbf{m}_2,\gamma}) \lessdot \mathbb{P}_{\mathbf{m}_2}$ and $L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma} \subseteq L^{\mathrm{dp}}_{\mathbf{m}_2,\gamma}$ by the definition hence by the definition $p \in \mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma}) \Rightarrow p \in \mathbb{P}_{\mathbf{m}_2}(L^{\mathrm{dp}}_{\mathbf{m}_2,\gamma})$; but lastly $(\mathbb{Q}_1 \lessdot \mathbb{P} \wedge \mathbb{Q}_2 \lessdot \mathbb{P} \wedge (\forall p)(p \in \mathbb{Q}_1 \to p \in \mathbb{Q}_2) \Rightarrow \mathbb{Q}_1 \lessdot \mathbb{Q}_2)$ so we are done. $\square_{1.25}$

{c41}

Crucial Claim 1.26. *If $\chi \ge 2^{\lambda_2}$ and $\mathbf{m} \in \mathbf{M}_{\le\chi}$ <u>then</u> for some $\mathbf{n}$ we have $\mathbf{m} \le_{\mathbf{M}} \mathbf{n} \in \mathbf{M}_\chi$ and $\mathbf{n} \in \mathbf{M}_{\mathrm{ec}}$.*

Proof. Let $\mathscr{X} = \{\mathbf{n} : (\mathbf{m} \restriction M_{\mathbf{m}}) \le_{\mathbf{M}} \mathbf{n}$ and $L_{\mathbf{n}} \backslash M_{\mathbf{m}} = t/E''_{\mathbf{n}}$ for some t hence $\|L_{\mathbf{n}}\| \le \lambda_2\}$.

We define a two-place relation $\mathscr{E}$ on $\mathscr{X}$:

- $\mathbf{n}_1 \mathscr{E} \mathbf{n}_2$ <u>iff</u> ($\mathbf{n}_1, \mathbf{n}_2 \in \mathscr{X}$ and) there is an isomorphism h from $\mathbf{n}_1$ onto $\mathbf{n}_2$ over $\mathbf{m} \restriction M_{\mathbf{m}}$, that is: an isomorphism from $L_{\mathbf{n}_1}$ onto $L_{\mathbf{n}_2}$ over $M_{\mathbf{m}}$ such that $t \in L_{\mathbf{n}_1} \Rightarrow u_{\mathbf{n}_2,h(t)} = \{h(s) : s \in u_{\mathbf{n}_1,t}\}$ and $t \in L_{\mathbf{n}_1} \Rightarrow \mathscr{P}_{\mathbf{n}_2,h(t)} = \{\{h(s) : s \in u\} : u \in \mathscr{P}_{\mathbf{n}_1,t}\}$ and $s, t \in L_{\mathbf{n}_1} \Rightarrow (sE'_{\mathbf{n}_1} t \Leftrightarrow h(s) E'_{\mathbf{n}_2} h(t))$.

Clearly $\mathscr{E}$ is an equivalence relation.

By our assumptions $\chi \ge 2^{\lambda_2}$ and $\mathbf{n} \in \mathscr{X} \Rightarrow |L_{\mathbf{n}}| \le \lambda_2 \wedge (\forall t \in L_{\mathbf{n}})(\mathscr{P}_{\mathbf{n},t} \subseteq [L_{\mathbf{n},<t}]^{\le\lambda_2})$ hence recalling $\lambda_2 = (\lambda_2)^\lambda$ clearly $\mathscr{E}$ has $\le 2^{\lambda_2}$ equivalence classes and let $\langle \mathbf{n}_\alpha : \alpha < 2^{\lambda_2} \rangle$ be a set of representatives (not necessary, but no harm in allowing repetitions).

{c28} By 1.17(2) and 1.18 we can find $\mathbf{n}$ such that:

$(*)_1$ (a) $\mathbf{m} \le_{\mathbf{M}} \mathbf{n} \in \mathbf{M}_\chi$

(b) for every $\alpha < 2^{\lambda_2}$ we can find $\langle t_{\alpha,i} : i < \chi \rangle$ such that (α) $t_{\alpha,i} \in L_{\mathbf{n}} \backslash L_{\mathbf{m}}$

(β) $(\alpha \neq \beta) \vee (i \neq j) \Rightarrow t_{\alpha,i}/E_{\mathbf{n}} \neq t_{\beta,j}/E_{\mathbf{n}}$

(γ) $\mathbf{n} \restriction (t_{\alpha,i}/E_{\mathbf{n}})$ is $\mathscr{E}$-equivalent to $\mathbf{n}_\alpha$, see 1.8(0) on $t_{\alpha,i}/E_{\mathbf{n}}$. {c5}

Let us prove that $\mathbf{n}$ is as required. Let $\mathbf{n} \leq_{\mathbf{M}} \mathbf{n}_1 \leq_{\mathbf{M}} \mathbf{n}_2$ and define $\mathscr{F}$ as the set of functions f such that some L_1, L_2:

$(*)_2$ (a) $L_\ell \subseteq L_{\mathbf{n}_2}$

(b) $M_{\mathbf{m}} = M_{\mathbf{n}} \subseteq L_1 \cap L_2$

(c) $L_\ell \backslash M_{\mathbf{m}}$ has cardinality $\leq \lambda_2$

(d) L_ℓ is $E_{\mathbf{n}_2}$-closed, i.e. $t \in L_\ell \backslash M_{\mathbf{m}} \Rightarrow t/E_{\mathbf{n}_2} \subseteq L_\ell$

(e) f is an isomorphism from $\mathbf{n}_2 \restriction L_1$ onto $\mathbf{n}_2 \restriction L_2$ over $M_{\mathbf{m}}$, i.e.

- f is a one-to-one mapping from L_1 onto L_2
- $f \restriction M_{\mathbf{m}}$ is the identity
- f maps $\leq_{\mathbf{n}_2} \restriction L_1$ onto $\leq_{\mathbf{n}_2} \restriction L_2$
- for $s, t \in L_1$ we have $sE'_{\mathbf{n}_2}t \Leftrightarrow f(s)E'_{\mathbf{n}_2}f(t)$
- for $s, t \in L_1$ we have $s \in u_{\mathbf{n}_2,t} \Leftrightarrow f(s) \in u_{\mathbf{n}_2,f(t)}$
- for $t \in L_1$ we have $\mathscr{P}_{\mathbf{n}_2,f(t)} = \{\{f(s) : s \in u\} : u \in \mathscr{P}_{\mathbf{n}_2,t}\}$.

Clearly

$(*)_3$ if $f \in \mathscr{F}$ and $L' \subseteq L_{\mathbf{n}_1}, L'' \subseteq L_{\mathbf{n}_2}$ and $|L'| + |L''| \leq \lambda_2$ <u>then</u> for some $g \in \mathscr{F}$ extending f we have $L' \subseteq \mathrm{Dom}(g), L'' \subseteq \mathrm{Rang}(g)$ and $\mathrm{Rang}(g) \backslash (L'' \backslash \mathrm{Rang}(f)) \subseteq L_{\mathbf{n}_1}$.

We can finish as in the parallel of the Tarski-Vaught criterion for $\mathbb{L}_{\infty,\lambda_2^+}$. That is, first we can prove by induction on the ordinal $\gamma < |L_{\mathbf{n}_2}|^+$ and really just $\gamma < \|M_{\mathbf{n}_2}\|^+$ that:

$(*)_4$ letting $L_\gamma = L^{\mathrm{dp}}_{\mathbf{n}_2,\gamma}$, if $g \in \mathscr{F}$ <u>then</u>

(a) g maps $\mathrm{Dom}(g) \cap L_\gamma$ onto $\mathrm{Rang}(g) \cap L_\gamma$

(b) g induces an isomorphism $\hat{g}$ from $\mathbb{P}_{\mathbf{n}_2}(\mathrm{Dom}(g) \cap L_\gamma)$ onto $\mathbb{P}_{\mathbf{n}_2}(\mathrm{Rang}(g) \cap L_\gamma)$, that is: $\hat{g}(p) = q$ iff

(α) $p \in \mathbb{P}_{\mathbf{n}_2}(\mathrm{Dom}(g) \cap L_\gamma)$

(β) $q \in \mathbb{P}_{\mathbf{n}_2}(\mathrm{Rang}(g) \cap L_\gamma)$

(γ) g maps $\mathrm{dom}(p)$ onto $\mathrm{dom}(q)$ and $s \in \mathrm{dom}(p) \Rightarrow \mathrm{tr}(p(s)) = \mathrm{tr}(q(g(s)))$

(δ) if $s \in \mathrm{Dom}(g), g(s) = t \in \mathrm{Rang}(g)$ and $\underset{\sim}{f}_{p(s)} = \mathbf{B}_{p(s)}(\ldots, \underset{\sim}{\eta}_{r_{p(s)}(\zeta)}, \ldots)_{\zeta < \xi_{p(s)}}$ and $\underset{\sim}{f}_{q(t)} = \mathbf{B}_{q(t)}(\ldots, \underset{\sim}{\eta}_{r_{q(t)}(\zeta)}, \ldots)_{\zeta < \xi_{q(t)}}$ <u>then</u> $\xi_{q(t)} = \xi_{p(s)}, \mathbf{B}_{q(t)} = \mathbf{B}_{p(s)}$ and $\zeta < \xi_{p(s)} \Rightarrow r_{q(t)}(\zeta) = g(r_{p(s)}(\zeta))$

(ε) moreover in (δ) we have $\iota(s,p) = \iota(t,q)$ and if $\iota < \iota(s,p)$ then $w_{p,s,\iota} = w_{q,t,\iota}, \mathbf{B}_{p(s),\iota} = \mathbf{B}_{q(t),\iota}$.

Second,

$(*)_5$ $\mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1}) \lessdot \mathbb{P}_{\mathbf{n}_2}(L_\gamma)$.

[Why? By the definitions $\mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1}) \subseteq \mathbb{P}_{\mathbf{n}_2}(L_\gamma)$ as quasi orders.

Also if $p_1, p_2 \in \mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1})$ are compatible in $\mathbb{P}_{\mathbf{n}_2}(L_\gamma)$ let $q \in \mathbb{P}_{\mathbf{n}_2}(L_\gamma)$ be a common upper bound there. We can find an $E_{\mathbf{n}_2}$-closed $L' \subseteq L_\gamma \cap L_{\mathbf{n}_1}$ of cardinality $\leq \lambda_2$ (recalling $\mathbf{n} \in \mathscr{X} \Rightarrow |L_\mathbf{n}| \leq \lambda_2$) such that $p_1, p_2 \in \mathbb{P}_{\mathbf{n}_1}(L')$ and $E_{\mathbf{n}_2}$-closed $L'' \subseteq L_\gamma$ of cardinality $\leq \lambda_2$ such that $L' \subseteq L''$ and $q \in \mathbb{P}_{\mathbf{n}_2}(L'')$. Now we can find $f_1 \in \mathscr{F}$ such that $\mathrm{Dom}(f_1) = \cup\{t/E_{\mathbf{n}_2} : t \in L'\}$ recalling that $t/E_\mathbf{m} \supseteq M_\mathbf{m}$, see 1.8(0) and f_1 is the identity. Then by $(*)_3$ we can find $f_2 \in \mathscr{F}$ extending f_1 {c5}
with $\mathrm{Dom}(f_2) = \cup\{t/E_{\mathbf{n}_2} : t \in L''\}$ and $\mathrm{Rang}(f_2) \backslash \mathrm{Rang}(f_1) \subseteq L_{\mathbf{n}_1}$. So we have $\mathbb{P}_{\mathbf{n}_2} \models$ "$(p_1 \leq \hat{f}_2(q)) \wedge p_2 \leq \hat{f}_2(q)$" and $\hat{f}_2(q) \in \mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1})$ recalling $(*)_4$. So p_1, p_2 are compatible also in $\mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1})$. Obviously, if $p_1, p_2 \in \mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1})$ are compatible in $\mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_2})$ say q witness then q is a common upper bound of p_1, p_2 in $\mathbb{P}_{\mathbf{n}_1}(L_\gamma)$.

So every antichain of $\mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1})$ is an antichain of $\mathbb{P}_{\mathbf{n}_2}(L_\gamma)$. Similarly to the above every maximal antichain of $\mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1})$ is a maximal antichain of $\mathbb{P}_{\mathbf{n}_2}(L_\gamma)$; similarly for the other direction. So we are done.]

$(*)_6$ $\mathbb{P}_{\mathbf{n}_1}(L_\gamma \cap L_{\mathbf{n}_1}) = \mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1}) \lessdot \mathbb{P}_{\mathbf{n}_2}(L_\gamma)$.

[Why? We prove this by induction on γ, as in proving the Tarski-Vaught criterion
{e32} is sufficient (we shall later in the proof of 3.19, more specifically $\boxplus_4$ proves a similar statement in detail with weaker assumptions).]

Hence (using $\gamma = |L_{\mathbf{n}_2}|^+$)

$(*)_7$ $\mathbb{P}_{\mathbf{n}_1} \lessdot \mathbb{P}_{\mathbf{n}_2}$.

{c28} Hence for every $L \subseteq L_{\mathbf{n}_1}$ by 1.17(4) we have $\mathbb{P}_{\mathbf{n}_1}(L) = \mathbb{P}_{\mathbf{n}_2}(L)$ as required in
{c34} Definition 1.24. $\square_{1.26}$

§ 2. The Corrected $\mathbb{P}_{\mathbf{m}}$

{z19}

Definition 2.1. Let $\mathbb{P}$ be a forcing notion and $Y \subseteq \mathbb{P}$ and χ a regular cardinal.
1) Let $\mathbb{L}_\chi(Y)$ be the set of sentences formed from $\{p : p \in \mathbb{P}_Y\}$ closing under the operations $\neg p$ and $\bigwedge_{i<\alpha} p_i$, for $\alpha < \chi$; so propositional logic.
2) For $\mathbf{G} \subseteq \mathbb{P}$ and $\psi \in \mathbb{L}_\chi(Y)$ we define the truth value $\psi[\mathbf{G}]$ naturally (by induction on ψ starting with $p[\mathbf{G}] = \text{true} \Leftrightarrow p \in \mathbf{G}$).
3) Let $\mathbb{L}^+_\chi(Y, \mathbb{P})$, the $\mathbb{L}_\chi$-closure of Y for $\mathbb{P}$, ($Y \subseteq \mathbb{P}$; if $Y = \mathbb{P}$ we may omit Y) be the following partial order:

- set of elements $\{\psi \in \mathbb{L}_\chi(Y, \mathbb{P}) : \nVdash_{\mathbb{P}} \text{“}\psi[\underset{\sim}{\mathbf{G}}] = \text{false”}\}$
- the order $\psi_1 \le \psi_2$ iff $\Vdash_{\mathbb{P}}$ “if $\psi_2[\underset{\sim}{\mathbf{G}}] = \text{true}$ then $\psi_1[\underset{\sim}{\mathbf{G}}] = \text{true}$”.

4) The completion of $\mathbb{P}$ is the $\mathbb{L}_\chi$-closure of $\mathbb{P}$ for $\mathbb{P}, \mathbb{L}_\chi(\mathbb{P})$ where χ is minimal such that $\mathbb{P}$ satisfies the χ-c.c.

{z44}

Claim 2.2. *For a χ-c.c. forcing notion $\mathbb{P}$ and $Y \subseteq \mathbb{P}$ we have:*

(a) $\mathbb{L}^+_\chi(Y, \mathbb{P})$ *is a forcing notion*
(b) $\mathbb{P} \lessdot \mathbb{L}^+_\chi(\mathbb{P})$ *under the natural identification*[9]
(c) $\mathbb{L}^+_\chi(Y, \mathbb{P}) \lessdot \mathbb{L}^+_\chi(\mathbb{P})$
(d) $\mathbb{L}^+_{\chi_1}(Y, \mathbb{P}) \lessdot \mathbb{L}^+_{\chi_2}(Y, \mathbb{P})$ *when $\chi_1 \le \chi_2$ are regular (and $\chi_1 \ge \chi$)*
(e) *if $\mathbb{P}$ satisfies the χ_1-c.c. and $\chi_1 < \chi_2$ are regular* *__then__* $\mathbb{L}^+_{\chi_1}(Y, \mathbb{P})$ *is essentially equal to $\mathbb{L}^+_{\chi_2}(Y, P)$, i.e. up to the natural equivalence of elements in a quasi order*
(f) *if $Y = \mathbb{P}$ then $\mathbb{P}$ is a dense subset of $\mathbb{P}$.*

{z46}

Definition 2.3. Let $\mathbf{m} \in \mathbf{M}$.
1) For $t \in L_{\mathbf{m}}, \varepsilon < \lambda$ and $\eta \in \prod_{i<\varepsilon} \theta_i$ let $p = p^*_{t,\eta} \in \mathbb{P}_{\mathbf{m}}$ be the function with domain $\{t\}$ such that $p(t) = (\eta, \eta\hat{\ }0_\lambda)$, i.e. $f_{p(t)} \in \prod_{i<\lambda} \theta_i$ is defined by $f_{p(t)}(\varepsilon)$ is $\eta(\varepsilon)$ if $\varepsilon < \ell g(\eta)$ and is zero otherwise.
2) For $L \subseteq L_{\mathbf{m}}$ let $Y_L = Y_{\mathbf{m},L} = \{p^*_{t,\eta} : t \in L \text{ and } \eta \in \prod_{\varepsilon<\zeta} \theta_\varepsilon \text{ for some } \zeta < \lambda\}$.
3) For $L \subseteq L_{\mathbf{m}}$ let $\mathbb{P}_{\mathbf{m}}[L]$ be $\mathbb{L}_{\lambda^+}[Y_L, \mathbb{P}_{\mathbf{m}}]$, see Definition 2.1. {z19}
4) For $L \subseteq L_{\mathbf{m}}$ let $\mathbb{P}_{\mathbf{m}}(L) = \mathbb{P}_{\mathbf{m}} \restriction \{p \in \mathbb{P}_{\mathbf{m}} : \text{fsupp}(p) \subseteq L\}$, see Definition 1.10(1), {c7}
recalling 1.10(2),(3). {c7}
5) $\mathbb{P}'_{\mathbf{m}}$ is the partial order with the same set of elements as $\mathbb{P}_{\mathbf{m}}$ and $\le_{\mathbb{P}'_{\mathbf{m}}} = \{(p, q) : p, q \in \mathbb{P}_{\mathbf{m}}$ and no r above q is incompatible with $p\}$ and $\mathbb{P}'_{\mathbf{m}}(L) = \mathbb{P}'_{\mathbf{m}} \restriction \{p \in \mathbb{P}_{\mathbf{m}} : \text{fsupp}(p) \subseteq L\}$, we may “forget” the distinction[10].
6) For quasi orders $\mathbb{Q}_1, \mathbb{Q}_2$ let $\mathbb{Q}_1 \subseteq' \mathbb{Q}_2$ mean that:

(a) $s \in \mathbb{Q}_1 \Rightarrow s \in \mathbb{Q}_2$
(b) $s \le_{\mathbb{Q}_1} t \Rightarrow s \le_{\mathbb{Q}_2} t$.

7) For quasi orders $\mathbb{Q}_1, \mathbb{Q}_2$ let $\mathbb{Q}_1 \subseteq'_{\text{ic}} \mathbb{Q}_2$ means that $\mathbb{Q}_1 \subseteq' \mathbb{Q}_2$ and

(c) if $s, t \in \mathbb{Q}_1$ are incompatible in $\mathbb{Q}_1$ then they are incompatible in $\mathbb{Q}_2$.

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[9] Really $\mathbb{P} \lessdot' \mathbb{L}^+_\chi[\mathbb{P}]$ see 2.3, because $\mathbb{L}^+_\chi[\mathbb{P}] \models$ “$p \le q$” __iff__ $q \Vdash_{\mathbb{P}}$ “$p \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}}$”. {z46}
[10] Really the only difference is the possibility that $\text{dom}(p) \nsubseteq \text{dom}(q)$, see 1.19. {c32n}

8) We define $\lessdot'$ similarly.

{z48}

Claim 2.4. *Let $\mathbf{m} \in \mathbf{M}$ and $L \subseteq L_{\mathbf{m}}$.*
1) $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ is equivalent to $\mathbb{P}_{\mathbf{m}}$ as forcing notions, in fact, $\mathbb{P}_{\mathbf{m}} = \mathbb{P}_{\mathbf{m}}(L_m) \lessdot \mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$
{z19} *and is a dense subset of it under the natural identification (see 2.1(1)), but we should pedantically use $\mathbb{P}'_{\mathbf{m}}(L_{\mathbf{m}})$ or use $\lessdot'$.*
2) $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ is $(<\lambda)$-strategically complete and is λ^+-c.c.
3) $\mathbb{P}_{\mathbf{m}}(L) \subseteq \mathbb{P}_{\mathbf{m}}[L]$ as sets and $\mathbb{P}_{\mathbf{m}}[L] \lessdot \mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ and $\mathbb{P}_{\mathbf{m}}(L) \subseteq' \mathbb{P}_{\mathbf{m}}[L]$.
4) If $\mathbf{G} \subseteq \mathbb{P}_{\mathbf{m}}$ is generic over $\mathbf{V}$ and $\eta_t = \underset{\sim}{\eta}_t[\mathbf{G}]$ for $t \in L$ and $\mathbf{G}^+_L = \{\psi \in \mathbb{L}_{\lambda^+}(Y_L) :$
{z19} *$\psi[\mathbf{G}] = \text{true}\}$, see 2.1(3), <u>then</u> $\mathbf{V}[\mathbf{G}] = \mathbf{V}[\mathbf{G}^+] = \mathbf{V}[\langle \eta_t : t \in L_{\mathbf{m}}\rangle]$.*
5) In part (4), moreover $\mathbf{G}^+$ is a subset of $\mathbb{P}_{\mathbf{m}}[L]$ generic over $\mathbf{V}$.
6) $\mathbb{P}_{\mathbf{m}}(L_1) \subseteq \mathbb{P}_{\mathbf{m}}(L_2)$ and $\mathbb{P}_{\mathbf{m}}[L_1] \lessdot \mathbb{P}_{\mathbf{m}}[L_2]$ <u>when</u> $L_1 \subseteq L_2 \subseteq L_{\mathbf{m}}$.
7) If $\mathbf{m}, \mathbf{n} \in \mathbf{M}$ are equivalent <u>then</u> $\mathbb{P}_{\mathbf{m}}[L] = \mathbb{P}_{\mathbf{n}}[L]$ and $\mathbb{P}_{\mathbf{m}}(L) = \mathbb{P}_{\mathbf{n}}(L)$.
8) Assume I_* be a λ^+-directed partial order and $\bar{L} = \langle L_r : r \in I_*\rangle$ be such that $r \in I_* \Rightarrow L_r \subseteq L_{\mathbf{m}}$ and $r <_{I_*} s \Rightarrow L_r \subseteq L_s$ and $L = \cup\{L_r : r \in I_*\}$. <u>Then</u> $\mathbb{P}_{\mathbf{m}}[L] = \cup\{\mathbb{P}_{\mathbf{m}}[L_r] : r \in I_*\}$ and $\mathbb{P}_{\mathbf{m}}(L) = \cup\{\mathbb{P}_{\mathbf{m}}(L_r) : r \in I_*\}$.
9) If $\mathbf{m} \in \mathbf{M}_{\rm ec}$ and $\mathbf{m} \le_{\mathbf{M}} \mathbf{m}_1 \le_{\mathbf{M}} \mathbf{m}_2$ then $\mathbb{P}_{\mathbf{m}_1}[L_{\mathbf{m}}] \lessdot \mathbb{P}_{\mathbf{m}_2}[L_{\mathbf{m}}]$.

{z50}

Remark 2.5. What about $\mathbb{P}_{\mathbf{m}}(L) \subseteq'_{\rm ic} \mathbb{P}_{\mathbf{m}}[L]$ and "$\mathbb{P}_{\mathbf{m}}(L) \lessdot' \mathbb{P}_{\mathbf{m}}[L]$?

{e4} The problem is the mapping $p \mapsto p \restriction L$ defined in 3.1(3) does not have the required properties of preserving order as the forcing appears there.

Proof. 1) Easy.
{c11} 2) Follows by part (1) and 1.13.
3) The first statement by their definitions, the second statement by part (1).
{c11} 4), 5), 6) Should be clear recalling 1.13(9).
{c11} 7) Easy, recalling 1.13(7).
8),9) Easy. $\square_{2.4}$

{c43}

The Uniqueness Claim 2.6. *There is an isomorphism from $\mathbb{P}_{\mathbf{m}_1}[M_1]$ onto $\mathbb{P}_{\mathbf{m}_2}[M_2]$ which (recalling Definition 2.3(1)) maps $p^*_{t,\eta}$ to $p^*_{h(t),\eta}$ for $t \in M_1, \eta \in \cup\{\prod\limits_{\varepsilon<\zeta} \theta_\varepsilon :$*
{z46} *$\zeta < \lambda\}$ <u>when</u>:*

$\boxplus$ (a) $\mathbf{m}_\ell \in \mathbf{M}^{\rm ec}_\infty$ *for* $\ell = 1, 2$
(b) $M_\ell = M_{\mathbf{m}_\ell}$ *for* $\ell = 1, 2$
(c) *h is an isomorphism from $\mathbf{m}_1 \restriction M_1$ onto $\mathbf{m}_2 \restriction M_2$.*

Proof. By renaming without loss of generality $M_1 = M_2$ call it M and h is the identity and $L_{\mathbf{m}_1} \cap L_{\mathbf{m}_2} = M$. Let $\mathbf{m}_0 = \mathbf{m}_1 \restriction M = \mathbf{m}_2 \restriction M$ so $\mathbf{m}_0 \le_{\mathbf{M}} \mathbf{m}_\ell$ for $\ell = 1, 2$ and $L_{\mathbf{m}_0} = L_{\mathbf{m}_1} \cap L_{\mathbf{m}_2}$.

{c31} By 1.18, there is $\mathbf{m}$ such that $\mathbf{m}_1 \le_{\mathbf{M}} \mathbf{m}$ and $\mathbf{m}_2 \le_{\mathbf{M}} \mathbf{m}$. As $\mathbf{m}_1, \mathbf{m}_2 \in \mathbf{M}^{\rm ec}_\infty$
{z48} by 2.4(9) we have $\mathbb{P}_{\mathbf{m}_1}[M] = \mathbb{P}_{\mathbf{m}}[M]$ and $\mathbb{P}_{\mathbf{m}_2}[M] = \mathbb{P}_{\mathbf{m}}[M]$ so together we get the desired conclusion. $\square_{2.6}$

{c44}

Definition 2.7. 1) We call $\mathbf{m} \in \mathbf{M}$ reduced when $L_{\mathbf{m}} = M_{\mathbf{m}}$.
2) For $\mathbf{m} \in \mathbf{M}$ let $\mathbb{P}^{\rm cer}_{\mathbf{m}}$ be $\mathbb{P}_{\mathbf{n}}[L_{\mathbf{m}}]$ and $\mathbb{P}^{\rm cer}_{\mathbf{m}}[L]$ be $\mathbb{P}_{\mathbf{n}}[L]$ for $L \subseteq L_{\mathbf{m}}$ <u>when</u> $\mathbf{m} \le_{\mathbf{M}} \mathbf{n} \in \mathbf{M}_{\rm ec}$.

{c47}

Remark 2.8. 1) Why is $\mathbb{P}^{\rm cer}_{\mathbf{m}}[L]$ well defined? see below.
2) Here "cr" stands for corrected.

The interest in the definition is because

{c48}

Claim 2.9. *1) If* $\mathbf{m} \in \mathbf{M}$ *and* $L \subseteq L_{\mathbf{m}}$ *<u>then</u>* $\mathbb{P}^{\rm cer}_{\mathbf{m}}[L]$ *is well defined.*
2) $\mathbb{P}^{\rm cer}_{\mathbf{m}}[M_{\mathbf{m}}]$ *is well defined and depend only on* $\mathbf{m}{\restriction}M_{\mathbf{m}}$.
3) If $\mathbf{m} \le_{\mathbf{M}} \mathbf{n}$ *and* $L_1 \subseteq L_2 \subseteq L_{\mathbf{m}}$ *<u>then</u>* $\mathbb{P}^{\rm cer}_{\mathbf{m}}[L_1] = \mathbb{P}^{\rm cer}_{\mathbf{n}}[L_1] \lessdot \mathbb{P}^{\rm cer}_{\mathbf{n}}[L_2] \lessdot \mathbb{P}^{\rm cer}_{\mathbf{n}}$.

Proof. 1) By 1.26, $\mathbb{P}^{\rm cer}_{\mathbf{m}}[L]$ has at least one definition so it suffices to prove uniqueness. So assume $\mathbf{m} \le_{\mathbf{M}} \mathbf{m}_\ell \in \mathbf{M}_{\rm ec}$ for $\ell = 1, 2$ and we should prove that $\mathbb{P}_{\mathbf{m}_1}[L] = \mathbb{P}_{\mathbf{m}_2}[L]$. Without loss of generality $L_{\mathbf{m}_1} \cap L_{\mathbf{m}_2} = L_{\mathbf{m}}$. Now by 1.18 we can find $\mathbf{n} \in \mathbf{M}$ such that $\mathbf{m}_1 \le_{\mathbf{M}} \mathbf{n}$ and $\mathbf{m}_2 \le_{\mathbf{M}} \mathbf{n}$; as $\mathbf{m}_\ell \in \mathbf{M}_{\rm ec}$ see Definition 1.24 we have $\mathbb{P}_{\mathbf{m}_\ell} \lessdot \mathbb{P}_{\mathbf{n}}$ for $\ell = 1, 2$. As in the end of the proof of 2.6 we are done. {c41} {c31} {c43}
2) By 2.6. {c43}
3) Follows from Definition 1.24(2) and 2.7(2). $\square_{2.9}$ {c44} {c50}

Discussion 2.10. 1) But we like to prove for reduced $\mathbf{m} \in \mathbf{M}$ and $M \subseteq \mathbf{M}_{\mathbf{m}}$ that $\mathbb{P}^{\rm cer}_{\mathbf{m}{\restriction}M} \lessdot \mathbb{P}^{\rm cer}_{\mathbf{m}}$. This is delayed to 3.26. We now prove it suffices. {c80}
2) The reader may understand 2.11 without reading the rest of §2,§3 by ignoring clause (A)(d), <u>or</u> reading 2.1, 2.2. {c51} {z49} {c51}

Conclusion 2.11. *For every ordinal* δ_* *there is* $\mathbf{q} = \langle \mathbb{P}_\alpha, \underset{\sim}{\eta}_\alpha : \alpha \le \delta_* \rangle$ *such that:*

(A) (a) $\langle \mathbb{P}_\alpha : \alpha \le \delta_* \rangle$ *is* $\lessdot$*-increasing*
(b) $\underset{\sim}{\eta}_\alpha$ *is a* $\mathbb{P}_{\alpha+1}$*-name of a member of* $\prod_{\varepsilon<\lambda} \theta_\varepsilon$ *which dominates* $(\prod_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}_\alpha]}$
(c) $\underset{\sim}{\eta}_\alpha$ *is a generic for* $\mathbb{P}_{\alpha+1}/\mathbb{P}_\alpha$*, moreover* $\langle \underset{\sim}{\eta}_\beta : \beta < \alpha \rangle$ *is a generic for* $\mathbb{P}_\alpha$
(d) $p \in \mathbb{P}_\alpha$ *iff* $p \in \mathbb{L}_{\lambda^+}(Y_\alpha, \mathbb{P}_\alpha)$ *where* Y_α *is defined as in 2.3(2) with* α *here standing for* L *there and see 2.1* {z46} {z19}
(e) $\mathbb{P}_\alpha$ *is* $(< \lambda)$*-strategically complete and* λ^+*-c.c.*
(f) *if* $\delta \le \delta_*$ *has cofinality* $> \lambda$ *(actually* $\ge \lambda$ *suffice) <u>then</u>* $\mathbb{P}_\delta = \cup\{\mathbb{P}_\alpha : \alpha < \delta\}$
(g) $\mathbb{P}_{\delta_*}$ *has cardinality* $|\delta_*|^\lambda$

(B) *if* $\mathscr{U} \subseteq \delta_*$ *<u>then</u> the complete subforcing generated by* $\langle \underset{\sim}{\eta}_\alpha : \alpha \in \mathscr{U} \rangle$ *is isomorphic to* $\mathbb{P}_{{\rm otp}(\mathscr{U})}$

(C) *if* $\mathbf{G} \subseteq \mathbb{P}_{\delta_*}$ *is generic over* $\mathbf{V}$ *and* $\eta_\alpha = \underset{\sim}{\eta}_\alpha[\mathbf{G}]$ *for* $\alpha < \delta_*$ *and* $\eta'_\alpha \in \prod_{\varepsilon<\lambda} \theta_\varepsilon$ *for* $\alpha < \delta_*$ *and* $\{(\alpha, \varepsilon) : \alpha < \delta_*, \varepsilon < \lambda$ *and* $\eta'_\alpha(\varepsilon) \ne \eta_\alpha(\varepsilon)\}$ *has cardinality* $< \lambda$ *<u>then</u> also* $\langle \eta'_\alpha : \alpha < \delta_* \rangle$ *is a generic for* $\mathbb{P}_{\delta_*}$*, determining a different* $\mathbf{G}'$ *but* $\mathbf{V}[\mathbf{G}'] = \mathbf{V}[\mathbf{G}]$

(D) *in clause (C), moreover if* $\mathscr{U} \subseteq \delta$ *and* $\langle \alpha_i : i < {\rm otp}(\mathscr{U}) \rangle$ *list* $\mathscr{U}$ *in increasing order <u>then</u> for some unique* $\mathbf{G}'' \subseteq \mathbb{P}_{{\rm otp}(\mathscr{U})}$ *generic over* $\mathbf{V}, i < {\rm otp}(\mathscr{U}) \Rightarrow \eta'_{\alpha_i} = \underset{\sim}{\eta}_i[\mathbf{G}'']$.

Proof. Without loss of generality $\lambda_1 \ge |\delta_*|$.
We define $\mathbf{m} \in \mathbf{M}$ by:

(∗) (a) $L_{\mathbf{m}} = \delta_*$
(b) $M_{\mathbf{m}} = \delta_*$
(c) $u_{\mathbf{m},\alpha} = \alpha$ and $\mathscr{P}_{\mathbf{m},\alpha} = [\alpha]^{\le \lambda}$ for $\alpha < \delta_*$
(d) $E'_{\mathbf{m}} = \emptyset$.

It is easy to check that indeed $\mathbf{m} \in \mathbf{M}$ and let $\mathbf{n} \in \mathbf{M}_{\rm ec}$ be such that $\mathbf{m} \le_{\mathbf{M}} \mathbf{n}$, exists by the Crucial Claim 1.26 and let $\mathbb{P}_\alpha = \mathbb{P}_{\mathbf{n}}[\{i : i < \alpha\}]$ for $\alpha \le \delta_*$. {c41}

Now clearly clauses (A),(C) hold and $\mathbb{P}_\delta = \mathbb{P}^{\rm cer}_{\mathbf{m}}$ by 2.7(2), 2.9(1) and clause {c48} (A)(b) holds by 1.13(6A). As for clause (B), note that for every $L \subseteq \delta_*$, for $\mathbb{P}_{\mathbf{m}}[L]$ {c11} the sequence $\bar{\underset{\sim}{\eta}}_L = \langle \underset{\sim}{\eta}_\alpha : \alpha \in L\rangle$ is generic for $\mathbb{P}_{\mathbf{m}}[L]$ by Definition 2.3. {z46}

For $M \subseteq \delta_*$ let $\alpha = \mathrm{otp}(M)$ and $h : M \to \alpha$ be $h(i) = \mathrm{otp}(i \cap M)$ so h is an isomorphism from $\mathbf{m}{\restriction}M$ onto $\mathbf{m}{\restriction}\alpha$ hence by 3.26(2), with $\mathbf{m}, \mathbf{m}{\restriction}\alpha, M, \alpha$ here {c80} standing for $\mathbf{m}_1, \mathbf{m}_2, M_1, M_2$ there we have h induces an isomorphism from $\mathbb{P}^{\rm cer}_{\mathbf{m}}[M]$ onto $\mathbb{P}^{\rm cer}_{\mathbf{m}\restriction\alpha}[L_{\mathbf{m}\restriction\alpha}]$. Similarly, id_α induces an isomorphism from $\mathbb{P}^{\rm cer}_{\mathbf{m}\restriction\alpha}$ onto $\mathbb{P}^{\rm cer}_{\mathbf{m}}[\alpha]$.

Together we get clause (B). Also Clause (D) follows so we are done. $\square_{2.11}$

Similarly we can deal with such iterations with partial memory and spell out how $\mathbb{P}^{\rm cer}_{\mathbf{m}}[L]$ is defined from a $(<\lambda)$-support iteration with partial memory.

{c52} **Conclusion 2.12.** *Assume M is a well founded partial order and $\bar{u}' = \langle u'_t : t \in M\rangle, u_t \subseteq M_{<t}$ and $\bar{\mathscr{P}}' = \langle \mathscr{P}'_t : t \in M\rangle$ with $\mathscr{P}'_t \subseteq [u'_t]^{\le\lambda}$ is closed under subsets. Then we can find $\beta(*), h, \mathbb{P}_\beta = \mathbb{P}_{0,\beta}, \mathbb{P}_{1,\beta}, \underset{\sim}{\mathbb{Q}}_\alpha, \underset{\sim}{\eta}_\alpha, \underset{\sim}{\eta}'_s$ and $\mathbb{P}'_u$ (for $\beta \le \beta(*), \alpha < \beta(*), s \in M$ and $u \subseteq M$) and $\bar{u}, \bar{\mathscr{P}}$ such that:*

- (A) (a) $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\alpha : \beta \le \beta(*), \alpha < \beta(*)\rangle$ *is $(<\lambda)$-support iteration*
  - (b) (α) $\bar{u} = \langle u_\beta : \beta < \beta(*)\rangle$ *such that* $u_\beta \subseteq \beta$
    - (β) $\bar{\mathscr{P}} = \langle \mathscr{P}_\beta : \beta < \beta(*)\rangle$ *such that* $\mathscr{P}_\beta \subseteq [u_\beta]^{\le\lambda}$
  - (c) $\underset{\sim}{\eta}_\alpha$ *is a $\mathbb{P}_{\alpha+1}$-name of a member of* $\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$
  - (d) $\langle \underset{\sim}{\eta}_\alpha : \alpha < \beta\rangle$ *is generic for* $\mathbb{P}_\beta$
  - (e) {c7} $\underset{\sim}{\mathbb{Q}}_\alpha$ *is defined as in Definition 1.10(4)*
  - (f) $\Vdash_{\mathbb{P}_{\beta(*)}}$ "$\underset{\sim}{\eta}_\beta \in \prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$ *dominate every* $\nu \in \prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$ *from* $\mathbf{V}[\langle \underset{\sim}{\eta}_\alpha : \alpha \in u\rangle]$ *when* $u \in \mathscr{P}_\beta$
- (B) (a) *h is a one-to-one function from M into*[11] *$\beta(*)$; stipulate $h(\infty) = \beta(*)$*
  - (b) $s <_M t \Leftrightarrow h(s) < h(t)$
  - (c) $u_{h(t)} \cap \mathrm{Rang}(h) = \{h(s) : s \in u'_t\}$
  - (d) $\mathscr{P}_{h(t)} \cap [\mathrm{Rang}(h)]^{\le\lambda} = \{\{h(s) : s \in u\} : u \in \mathscr{P}'_t\}$
- (C) (a) $\mathbb{P}_{1,\beta} = \mathbb{L}^+_{\lambda^+}(Y_\beta, \mathbb{P}_\beta)$ *where we let* $Y_\beta = \{p^*_{\alpha,\nu} : \alpha < \beta, \nu \in \prod\limits_{\varepsilon<\zeta} \theta_\varepsilon$ *for some* $\zeta < \lambda\}$*, see 2.1, 2.3(1)* {z40}
  - (b) $\mathbb{P}_{1,u} = \mathbb{L}^+_{\lambda^+}(Y_u, \mathbb{P}_\beta)$*, where Y_u is defined similarly when $u \subseteq \beta(*)$*
  - (c) *$\mathbb{P}'_u$ is a forcing notion for $u \subseteq M$ and $\underset{\sim}{\eta}'_s$ is a $\mathbb{P}'_{\{s\}}$-name for $s \in M$ sn*
  - (d) *h induces an isomorphism from $\mathbb{P}'_u$ onto $\mathbb{P}_{1,\{h(s):s\in u\}}$ for $u \subseteq M$ and $\underset{\sim}{\eta}'_s$ to $\underset{\sim}{\eta}_{h(s)}$ for $s \in M$*
  - (e) *$\langle \underset{\sim}{\eta}_{h(s)} : s \in u\rangle$ is generic for $\mathbb{P}'_u$ for $u \subseteq M$*
- (D) (a) *$\mathbb{P}'_u \lessdot \mathbb{P}'_v$ when $u \subseteq v \subseteq M$*
  - (b) *$\mathbb{P}_\beta, \mathbb{P}_{1,u}, \mathbb{P}'_u$ are $(<\lambda)$-strategically complete and λ^+-c.c.*

[11] In general not onto!

(c) *if* $M_1, M_2 \subseteq M$ *and* f *is an isomorphism from* M_1 *onto* M_2 *as partial orders such that* $t \in M_1 \Rightarrow u'_{h(t)} \cap M_2 = \{f(s) : s \in u'_t \cap M_1\}$ *and* $t \in M_1 \Rightarrow \mathscr{P}'_{h(t)} \cap [M_2]^{\leq \lambda} = \{\{f(s) : s \in u \cap M_1\} : u \in \mathscr{P}'_t\}$ *then the mapping* $h(s) \mapsto h(f(s))$ *induce an isomorphism from the forcing notion* $\mathbb{P}'_{M_1}$ *onto* $\mathbb{P}'_{M_2}$.

Proof. Similarly. $\square_{2.12}$

§ 3. The main conclusion

{c50} We have a debt from §2, i.e. see discussion 2.10. Toward this we explicate what
{c41} appear in the proof of 1.26.

{e4} **Definition 3.1.** Let $\mathbf{m} \in \mathbf{M}$.
1) We say $\mathbf{m}$ is μ-wide <u>when</u> for every $t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ there are $t_\alpha \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ for $\alpha < \mu$ such that:

- (a) $\mathbf{m} \restriction (t_\alpha / E_{\mathbf{m}})$ is isomorphic to $\mathbf{m} \restriction (t/E_{\mathbf{m}})$ over $M_{\mathbf{m}}$
- (b) $\beta < \gamma < \mu \Rightarrow t_\beta / E''_{\mathbf{m}} \neq t_\gamma / E''_{\mathbf{m}}$.

1A) We say $\mathbf{m}$ is wide <u>when</u> it is λ^+-wide. We say $\mathbf{m}$ is very wide when it is $|L_{\mathbf{m}}|$-wide.
2) We say $\mathbf{m}$ is full <u>when</u>: if $\mathbf{m} \restriction M_{\mathbf{m}} \leq_{\mathbf{M}} \mathbf{n}$ and $E''_{\mathbf{n}}$ has exactly one equivalence class <u>then</u> for some $t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$, we have $\mathbf{n}$ is isomorphic to $\mathbf{m} \restriction (t/E_{\mathbf{m}})$ over $M_{\mathbf{m}}$.
3) For $L \subseteq L_{\mathbf{m}}$ we say $p \in \mathbb{P}_{\mathbf{m}}(L)$ is the projection (to L) of $q \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$ and write $p = q \upharpoonright L$ <u>when</u>:

- (a) $\mathrm{Dom}(p) = \mathrm{Dom}(q) \cap L$
- (b) if $s \in \mathrm{Dom}(p)$ then
  - (α) $\mathrm{tr}(p(s)) = \mathrm{tr}(q(s))$
  - (β) $\{\underset{\sim}{f}_{p(s),\iota} : \iota < \iota(p(s))\} = \{\underset{\sim}{f}_{q(s),\iota} : \iota < \iota(q(s))$ and $\bar{r}_{p(s),\iota}$ is a sequence of members of $L\}$.

4) Let $\mathscr{F}_{\mathbf{m}}$ be the set of the functions f such that for some L_1, L_2:

- (a) f is an isomorphism from $\mathbf{m} \restriction L_1$ onto $\mathbf{m} \restriction L_2$
- (b) L_ℓ is a subset of $L_{\mathbf{m}}$ for $\ell = 1, 2$
- (c) $M_{\mathbf{m}} \subseteq L_\ell$ for $\ell = 1, 2$ and $f \restriction M_{\mathbf{m}}$ is the identity
- (d) L_ℓ is $E_{\mathbf{m}}$-closed, i.e. $M_{\mathbf{m}} \subseteq L_\ell$ and if $t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ and $t \in L_\ell$ then $t/E_{\mathbf{m}} \subseteq L_\ell$ for $\ell = 1, 2$
- (e) $\{t/E''_{\mathbf{m}} : t \in L_\ell \backslash M_{\mathbf{m}}\}$ has cardinality $\leq \lambda$.

5) If $L_1, L_2 \subseteq L_{\mathbf{m}}$ and f is an isomorphism from $\mathbf{m} \restriction L_1$ onto $\mathbf{m} \restriction L_2$ <u>then</u> we let $\hat{f}$ be the one-to-one mapping[12] from $\mathbb{P}_{\mathbf{m}}(L_1)$ onto $\mathbb{P}_{\mathbf{m}}(L_2)$ as in $(*)_4(b)$ of the proof
{c41} of 1.26.
6) Let $\mathbb{P}^-_{\mathbf{m}}(L)$ be $\{p \in \mathbb{P}_{\mathbf{m}}(L) : \mathrm{fsupp}(p) \subseteq M_{\mathbf{m}}$ or $\mathrm{fsupp}(p) \subseteq t/E_{\mathbf{m}}$ for some $t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}\}$ with the order inherited from $\mathbb{P}_{\mathbf{m}}$.

{e5n} **Observation 3.2.** *Let $\mathbf{m} \in \mathbf{M}$ and $L \subseteq L_{\mathbf{m}}$.*
1) The projection of $q \in \mathbb{P}_{\mathbf{m}}$ to L is well defined and $\in \mathbb{P}_{\mathbf{m}}(L)$.
2) Moreover, it is unique.
3) If $p \in \mathbb{P}_{\mathbf{m}}(L)$ is the projection of $q \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$ <u>then</u> $p \leq q$.
4) For every $p \in \mathbb{P}_{\mathbf{m}}, p$ is equivalent to $\mathscr{S}_p := \{p \upharpoonright L : L = t/E_{\mathbf{m}}$ for some $t \in \mathrm{fsupp}(p)\} \cup \{p \upharpoonright M_{\mathbf{m}}$: if $\mathrm{fsupp}(p) \subseteq M_{\mathbf{m}}\}$, i.e. $\Vdash_{\mathbb{P}_{\mathbf{m}}}$ "$p \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\mathbf{m}}}$ <u>iff</u> $\mathscr{S}_p \subseteq \underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\mathbf{m}}}$".
5) For every $p \in \mathbb{P}_{\mathbf{m}}, p$ is equivalent to $\mathscr{S}'_p := \{p^{[t,\iota]} : t \in \mathrm{dom}(p)$ and $\iota < \iota(p(t))\}$ where $p^{[t,\iota]} \in \mathbb{P}_{\mathbf{m}}$ has domain $\{t\}$ and $p(t) = (\mathrm{tr}(p_t), \mathbf{B}_{p(t),\iota}(\langle \eta_{r_{p(t)}(\zeta)} : \zeta \in w_{p(t),\iota}\rangle)$;
{c6} *recall Definition 1.9 for the meaning of $\iota(p(t)), \mathbf{B}_{p(t),\iota}$, etc.*

[12] We have not said "order preserving"!

{e5p} {c6} {e8a} {e4} *Remark* 3.3. 1) Note that the choice in Definition 1.9$(c)(\gamma)$ to require such $\langle \underset{\sim}{f}_{p(t),\iota} : \iota < \iota(p_t)\rangle$ exists, is necessary for 3.2(4), which is crucial in the proof of 3.26.
2) In Definition 3.1(1A) we can choose "wide means λ-wide" as when applying it, if $X = \text{fsupp}(p)$ then for some $Y \subseteq L_{\mathbf{m}}$ of cardinality $< \lambda, X \subseteq \cup\{t/E_{\mathbf{m}} : t \in Y\}$.

Proof. Easy, e.g.
4) If $\text{fsupp}(p) \subseteq M_{\mathbf{m}}$ the statement says $\Vdash$ "$p \in \underset{\sim}{\mathbf{G}}$ iff $\{p\} \subseteq \underset{\sim}{\mathbf{G}}$", so trivial hence we assume $\text{fsupp}(p) \nsubseteq M_{\mathbf{m}}$. Now if $t \in \text{fsupp}(p)$ then trivially $p \upharpoonright (t/E_{\mathbf{m}}) \leq p$, hence $\Vdash$ "$p \in \underset{\sim}{\mathbf{G}}$ implies $\mathscr{S}_p \subseteq \underset{\sim}{\mathbf{G}}$".

For the other direction assume $q \in \mathbb{P}_{\mathbf{m}}$ forces $\mathscr{S}_p \subseteq \underset{\sim}{\mathbf{G}} \subseteq \mathbb{P}_{\mathbf{m}}$ and we shall prove that q is compatible with p, this suffices, so toward contradiction assume q, p are incompatible.

Without loss of generality $\text{Dom}(p) \subseteq \text{Dom}(q)$ and recalling $t \in \text{fsupp}(p) \Rightarrow q \Vdash$ "$p \upharpoonright (t/E_{\mathbf{m}}) \in \underset{\sim}{\mathbf{G}}$" clearly $s \in \text{dom}(p) \Rightarrow q \Vdash$ "$\text{tr}(p(s)) \subseteq \underset{\sim}{\eta}_s$" so necessarily $s \in \text{Dom}(p) \Rightarrow \text{tr}(p(s)) \subseteq \text{tr}(q(s))$. Recalling 1.13(8), as p, q are incompatible {c11}
there are $s \in \text{Dom}(p) \cap \text{Dom}(q)$ and q_1 such that $q{\upharpoonright}L_{\mathbf{m},<s} \leq q_1 \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m},<s})$ and $q_1 \Vdash$ "$q(s), p(s)$ are incompatible in $\mathbb{Q}_{\bar{\theta}}$".

As $\text{tr}(p(s)) \subseteq \text{tr}(q(s))$ this implies $q_1 \Vdash$ "$\text{tr}(q(s)), p(s)$ are incompatible, i.e. $f_{p(s)}{\upharpoonright}\ell g(\text{tr}(q(s))) \nleq \text{tr}(q(s))$". Recalling Definition 1.9$(c)(\gamma)$, $q_1 \Vdash$ "there is $\iota <$ {c6}
$\iota(s,p)$ such that $f_{p(s),\iota}, \text{tr}(q(s))$ are incompatible". Possibly increasing q_1, we can fix ι. But letting $t \in \text{fsupp}(p) \subseteq L_{\mathbf{m}}$ be such that $\bar{r}_{p(s),\iota} \subseteq t/E_{\mathbf{m}}$ this implies that $q_1 \Vdash$ "$p \upharpoonright (t/E_{\mathbf{m}}) \notin \underset{\sim}{\mathbf{G}}$ or $\text{tr}(q(s)) \nsubseteq \underset{\sim}{\eta}_s$". However, q_1, q are compatible and this contradicts the choice of q. $\square_{3.2}$

{e7}
Claim 3.4. *1) The* $\mathbf{n}$ *constructed in 1.26 satisfies: if* $\mathbf{n} \leq_{\mathbf{M}} \mathbf{n}_1$ *<u>then</u>* $\mathbf{n}_1$ *is wide,* {c41}
(if $\mathbf{n}_1 \in \mathbf{M}_\chi$ *even very wide) and full.*
2) If $\mathbf{n} \in \mathbf{M}_{\text{ec}}$ *and* $\mathbf{n} \leq_{\mathbf{M}} \mathbf{n}_1$ *then* $\mathbf{n}_1 \in \mathbf{M}_{\text{ec}}$.

Proof. 1) Holds by the proof of 1.26. {c41}
2) Holds by Definition 1.24(1),(2). $\square_{3.4}$ {c34} {e10}

Claim 3.5. *Assume* $\mathbf{m}$ *is wide.*
1) If $f \in \mathscr{F}_{\mathbf{m}}$ *and* $X \subseteq L_{\mathbf{m}}$ *has cardinality* $\leq \lambda$ *<u>then</u> there is* g *such that:*

- (a) $g \in \mathscr{F}_{\mathbf{m}}$
- (b) $f \subseteq g$
- (c) $\text{Dom}(g) = \text{Rang}(g)$
- (d) $X \subseteq \text{Dom}(g)$.

2) If $g \in \mathscr{F}_{\mathbf{m}}$ *and* $\text{Dom}(g) = \text{Rang}(g)$ *<u>then</u>* $g^{+\mathbf{m}} = g \cup \text{id}_{L_{\mathbf{m}} \setminus \text{Dom}(g)}$ *is an automorphism of* $\mathbf{m}$.
3) If f *is an automorphism of* $\mathbf{m}$ *<u>then</u> it naturally induces an automorphism* $\hat{f}$ *of* $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$ *similarly to* $\hat{f}$ *from* $(*)_4(b)$ *of the proof of 1.26.* {c41}
4) If $f \in \mathscr{F}_{\mathbf{m}}$ *<u>then</u> it induces an isomorphism* $\hat{f}$ *from* $\mathbb{P}_{\mathbf{m}}(\text{Dom}(f))$ *onto* $\mathbb{P}_{\mathbf{m}}(\text{Rang}(f))$.

Proof. 1) Easy by the definition of wide in 3.1(1) and of $\mathscr{F}_{\mathbf{m}}$ in 3.1(4). {e4}
2) Just read the definition of $\mathbf{m} \in \mathbf{M}$ and of $f \in \mathscr{F}_{\mathbf{m}}$, in particular:

- (a) if $t_1, t_2 \in L_{\mathbf{m}} \setminus M_{\mathbf{m}}$ are not $E'_{\mathbf{m}}$-equivalent then $(t_1/E_{\mathbf{m}}) \cap (t_2/E_{\mathbf{m}}) = M_{\mathbf{m}}$ and $\leq_{\mathbf{m}} {\upharpoonright}(t_1/E_{\mathbf{m}} \cup t_2/E_{\mathbf{m}})$ is determined by $\leq_{\mathbf{m}} {\upharpoonright}(t_1/E_{\mathbf{m}}), \leq_{\mathbf{m}} {\upharpoonright}(t_2/E_{\mathbf{m}})$
- (b) $g{\upharpoonright}M_{\mathbf{m}} = \text{id}_{M_{\mathbf{m}}}$.

3) Naturally by the definition.
4) Let $g \in \mathscr{F}$ be as in part (1) and let $h = g^{+\mathbf{m}}$ so an automorphism of $\mathbf{m}$ which extends g as in part (2). So $\hat{h}$ is an automorphism of $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$ and clearly $\hat{f} = \hat{h} \restriction \mathbb{P}_{\mathbf{m}}(\mathrm{Dom}(f))$ is as required. $\square_{3.5}$

{e16}

Claim 3.6. *Let $\mathbf{m} \in \mathbf{M}$ and $L \subseteq L_{\mathbf{m}}$.*
If $f_1, f_2 \in \mathscr{F}_{\mathbf{m}}$ then:

(a) $f_1 \subseteq f_2 \Rightarrow \hat{f}_1 \subseteq \hat{f}_2$
(b) $f_1 = f_2^{-1} \Rightarrow \hat{f}_1 = (\hat{f}_2)^{-1}$.

{c44} *Proof.* Just consider the definition, see 3.1(5) and $(*)_4(b)$ of the proof of 1.26. $\square_{3.6}$

{e19}
{e4}

Observation 3.7. *1) $\mathbb{P}^-_{\mathbf{m}}(L) \subseteq \mathbb{P}_{\mathbf{m}}(L)$, see Definition 3.1(6).*
2) For every $p \in \mathbb{P}_{\mathbf{m}}$ there is a sequence $\langle p_i : i < i(*) \rangle$ of $\leq \lambda$ members of $\mathbb{P}^-_{\mathbf{m}}$ such that $\Vdash_{\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]}$ "$p \in \underset{\sim}{\mathbf{G}}$ iff $\{p_i : i < i(*)\} \subseteq \underset{\sim}{\mathbf{G}}$".

Proof. 1) By their definitions.
{e5a} 2) Should be clear, see Definition 3.1(6) and 3.2(4). $\square_{3.7}$

{e21}
{e19}

Remark 3.8. 1) Observation 3.7 is not used.
2) Probably we can avoid using "wide" and prove the density of $\mathbf{M}_{\mathrm{ec}}$ with smaller cardinality but the present way seems more transparent.

{e24}

Definition 3.9. Assume $\mathbf{m} \in \mathbf{M}$.
1) Let $\mathscr{Y}_{\mathbf{m}}$ be the set of pairs $(t, \bar{s})$ such that $t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ and $\bar{s} \in {}^{\zeta}(t/E''_{\mathbf{m}})$ for some $\zeta < \lambda^+$; we may write $\bar{s}$ instead of $(t, \bar{s})$ as usually $\bar{s}$ determines t.
2) By induction on the ordinal γ we define when $(t_1, \bar{s}_1), (t_2, \bar{s}_2)$ are γ-equivalent in $\mathbf{m}$ or are $(\mathbf{m}, \gamma)$-equivalent:

(a) if $\gamma = 0$, letting $L_\ell = (M_{\mathbf{m}} \cup \mathrm{Rang}(\bar{s}_\ell))$ for $\ell = 1, 2$ there is h such that
   (α) h is an isomorphism from $\mathbf{m} \restriction L_1$ onto $\mathbf{m} \restriction L_2$
   (β) h maps $\bar{s}_1$ to $\bar{s}_2$
   (γ) $h \restriction M_{\mathbf{m}}$ is the identity
{c4}   (δ) h induces an isomorphism from $\mathbb{P}_{\mathbf{m}}(L_1)$ onto $\mathbb{P}_{\mathbf{m}}(L_2)$ (as defined in 1.7$(*)_4(b)$)
   (ε) moreover, h induces an isomorphism from $\mathbb{P}_{\mathbf{m}}[L_1]$ onto $\mathbb{P}_{\mathbf{m}}[L_2]$, as defined in 2.6, $p^*_{t,\eta} \mapsto p^*_{h(t),\eta}$, see 2.3(3)
{z46}
(b) if $\gamma = \beta + 1$ then for every $\varepsilon < \lambda^+$ and $\ell \in \{1, 2\}$ and $\bar{s}'_\ell \in {}^{\varepsilon}(t_\ell/E''_{\mathbf{m}})$ there is $\bar{s}'_{3-\ell} \in {}^{\varepsilon}(t_{3-\ell}/E''_{\mathbf{m}})$ such that $(t_1, \bar{s}_1 \hat{\ } \bar{s}'_1), (t_2, \bar{s}_2 \hat{\ } \bar{s}'_2)$ are β-equivalent
(c) if γ is a limit ordinal then $(t_1, \bar{s}_1), (t_2, \bar{s}_2)$ are β-equivalent for every $\beta < \gamma$.

{e26}

Remark 3.10. 1) Note above that if $\bar{s}_\ell$ is the empty sequence then t_ℓ would not be determined by $\bar{s}_\ell$, still in those cases the equivalence just means $\bar{s}_1 = \bar{s}_2$.
2) We can use $t/E_{\mathbf{m}}$ or $t/E'_{\mathbf{m}}$ instead of $t/E''_{\mathbf{m}}$ as everything is over $M_{\mathbf{m}}$.

{e27}

Claim 3.11. *For $\mathbf{m} \in \mathbf{M}$ and ordinal α the number of equivalence classes of "being $(\mathbf{m}, \alpha)$-equivalent" is $\leq \beth_{1+\alpha+1}(\lambda_1)$.*

Proof. By induction on α.

Case 1: $\alpha = 0$

Note that the set of elements of $\mathbb{P}_{\mathbf{m}}(M_{\mathbf{m}} \cup \mathrm{Rang}(\bar{s}))$ has cardinality $\le 2^{\lambda_1}$ (and even $\le (\lambda_1)^{\lambda}$) and depends just on $\mathbf{m} \restriction (M_{\mathbf{m}} \cup \mathrm{Rang}(\bar{s}))$ but there are $\beth_2(\lambda_1)$ possibilities for the quasi order on $\mathbb{P}_{\mathbf{m}}(L_1)$ and even for $\mathbb{P}_{\mathbf{m}}[L_1]$.

Case 2: α is a limit ordinal

By clause (c) of Definition 3.9, the number of α-equivalence classes is $\le \prod_{\beta<\alpha}$ (the number of β-equivalence classes) $\le \prod_{\beta<\alpha} \beth_{1+\beta+1}(\lambda_1) \le (\beth_{1+\alpha+1}(\lambda_1))^{\beth_{1+\alpha}} = \beth_{1+\alpha+1}(\lambda_1)$. {e24}

Case 3: $\alpha = \beta + 1$

Clearly every α-equivalence class can be coded as a set of β-equivalence classes hence the number of α-equivalence classes is $\le 2^{\beth_{1+\beta+1}(\lambda_1)} = \beth_{1+\beta+2}(\lambda_1) = \beth_{1+\alpha+1}(\lambda_1)$, as promised. $\square_{3.11}$

Definition 3.12. For an ordinal β, let $\mathscr{F}_{\mathbf{m},\beta}$ be the set of function f such that for some $t^\ell_i, \bar{s}^\ell_i$ for $i < i(*)$ and $\ell \in \{1,2\}$ we have: {e28}

- (a) $i(*) < \lambda^+$
- (b) $\langle t^\ell_i : i < i(*)\rangle$ is a sequence of pairwise non-$E''_{\mathbf{m}}$-equivalent members of $L_{\mathbf{m}} \backslash M_{\mathbf{m}}$
- (c) $\bar{s}^\ell_i \in {}^{\zeta(i)}(t^\ell_i/E''_{\mathbf{m}})$ where $\zeta(i) < \lambda^+$
- (d) $(t^1_i, \bar{s}^1_i), (t^2_i, \bar{s}^2_i)$ are β-equivalent (members of $\mathscr{Y}_m$)
- (e) f is an isomorphism from $\mathbf{m} \restriction L_1$ onto $\mathbf{m} \restriction L_2$ when $L_\ell = \cup\{\mathrm{Rang}(\bar{s}^\ell_i) : i < i(*)\} \cup M_{\mathbf{m}}$
- (f) $f \restriction M_{\mathbf{m}} =$ the identity
- (g) f maps $\bar{s}^1_i$ to $\bar{s}^2_i$ for $i < i(*)$.

2) For $f \in \mathscr{F}_{\mathbf{m},0}$ we define $\hat{f}$ as the mapping from $\mathbb{P}_{\mathbf{m}}(\mathrm{Dom}(f))$ onto $\mathbb{P}_{\mathbf{m}}(\mathrm{Rang}(f))$ induced by f; see clause 3.9(2)(a)(ε). {e24} {e29}

Claim 3.13. *Assume* $\mathbf{m}$ *is wide. The conditions* $p, q \in \mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ *are compatible* *when for some* ψ *the following condition holds:*

$(\text{suchthat})_{p,q,\psi}$

- (a) $\psi \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$
- (b) $\mathrm{wsupp}(p) \cap \mathrm{wsupp}(q) \subseteq M_{\mathbf{m}}$, *see Definition 1.10(1)(b), equivalently* $s \in \mathrm{fsupp}(p) \backslash M_{\mathbf{m}}, t \in \mathrm{fsupp}(q) \backslash M_{\mathbf{m}} \Rightarrow \neg(s E''_{\mathbf{m}} t)$ {c7}
- (c) *if* $\psi \le \varphi \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ *then* φ, p *are compatible in* $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$
- (d) ψ, q *are compatible in* $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$, *equivalently* $q \nVdash_{\mathbb{P}_{\mathbf{m}}}$ "$\psi[\underset{\sim}{\mathbf{G}}] = false$".

Remark 3.14. 1) We can use $(\text{suchthat})'_{p,q,\psi}$: omit clause (d) and add to clause (c): and φ, q are compatible in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$.
2) We use $\lambda > \aleph_0$ in the proof, to eliminate it we can immitate the completeness theorem for $\mathbb{L}_{\aleph_1,\aleph_0}$.

Proof. We choose (p_n, q_n, ψ_n) by induction on n such that:

$\boxplus_n$ $(a)(\alpha)$ $(\text{suchthat})_{p_n,q_n,\psi_n}$ holds if n is even

(β) $(\text{suchthat})_{q_n,p_n,\psi_n}$ holds if n is odd

(b) $(p_0, q_0, \psi_0) = (p, q, \psi)$

(c) if $n = 2m+1$ and $s \in \mathrm{dom}(p_{2m}) \cap M_{\mathbf{m}}$ <u>then</u> $s \in \mathrm{dom}(q_{2m+1})$ and $\mathrm{tr}(p_{2m}(s)) \subseteq \mathrm{tr}(q_{2m+1}(s))$

(d) if $n = 2m+2$ and $s \in \mathrm{dom}(q_{2m+1}) \cap M_{\mathbf{m}}$ <u>then</u> $s \in \mathrm{dom}(p_{2m+2})$ and $\mathrm{tr}(q_{2m+1}(s)) \subseteq \mathrm{tr}(p_{2m+2}(s))$

(e) if $n = m+1$ then $p_m \le p_n, q_m \le q_n$.

<u>Case 1</u>: For $n = 0$ use clause (b).

<u>Case 2</u>: $n = 2m+1$.

So the triple $(p_{2m}, q_{2m}, \psi_{2m})$ is well defined, let $u_{2m} = \mathrm{Dom}(p_{2m}) \cap M_{\mathbf{m}}$ and let $\bar{\nu} = \langle \nu_s : s \in u_{2m} \rangle$ be defined by $\nu_s = \mathrm{tr}(p_{2m}(s))$.

Clearly

$(*)_1$ $\psi_{2m} \vdash p^*_{s,\nu_s}$ for $s \in u_{2m}$.

[Why? Clearly $p_{2m} \vdash p^*_{s,\nu_s}$, i.e. $p^*_{s,\nu_s} \le p_{2m}$ in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$, hence if $\psi_{2m} \nvdash p^*_{s,\nu_s}$ then $\psi' = \psi_{2m} \wedge \neg p^*_{s,\nu_s} \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ is $\ge \psi_{2m}$ hence compatible with p_{2m}, contradiction, see clause (c) in $(\mathrm{suchthat})_{p,q,\psi}$.]

$(*)_2$ there is $q'_{2m} \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$ which is above q_{2m} and above ψ_{2m} hence $s \in u_{2m}$ implies $\nu_s \subseteq \mathrm{tr}(q'_{2m}(s))$ and $s \in \mathrm{Dom}(q'_{2m})$.

[Why? By clause (d) of $(\mathrm{suchthat})_{p_{2m},q_{2m},\psi_{2m}}$ which holds by $\boxplus_{2m}(a)(\alpha)$ recalling $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$ is dense $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$; the "hence" by $(*)_1$.]

$(*)_3$ there is $\psi'_{2m} \in \mathbb{P}_{\mathbf{m}}[M_m]$ such that:

(a) if $\psi'_{2m} \le \varphi \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ then φ, q'_{2m} are compatible in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$

(b) if $s \in u_{2m}$ then $\psi'_{2m} \vdash p^*_{s,\nu_s}$

(c) $\psi_{2m} \le \psi'_{2m}$.

[Why? Obvious using the λ^+-c.c., i.e. $\psi'_{2m} = \psi_{2m} \wedge \neg(\vee\{\varphi : \varphi \in \mathscr{I}\})$ where $\mathscr{I}$ is a max antichain of members $\psi \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ satisfying $\psi \perp q'_{2m}$ in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$.]

$(*)_4$ without loss of generality $\mathrm{wsupp}(q'_{2m}) \cap \mathrm{wsupp}(p_{2m}) \subseteq M_{\mathbf{m}}$.

{e10} [Why? As $\mathbf{m}$ is wide using automorphisms of $\mathbf{m}$, i.e. by 3.5.]

$(*)_5$ there is $p'_{2m} \in \mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ which is above p_{2m} and above ψ'_{2m}.

[Why? By the choice of ψ'_{2m} and clause (c) of $(\mathrm{suchthat})_{p_{2m},q_{2m},\psi_{2m}}$ which holds by $\boxplus_{2m}(a)(\alpha)$.]

$(*)_6$ without loss of generality $\mathrm{fsupp}(p'_{2m}) \cap \mathrm{fsupp}(q'_{2m}) \subseteq M_{\mathbf{m}}$.

{e10} [Why? As $\mathbf{m}$ is wide using 3.5.]

Lastly, let $p_n = p'_{2m}, q_n = q'_{2m}, \psi_n = \psi'_{2m}$ and check.

<u>Case 3</u>: $n = 2m+2$

Similar to case 2 the roles of the p's and the q's interchanged.

Having carried the induction we can find p_* the upper bound of $\{p_n : n < \omega\}$ as
{c11} in 1.13(4), in particular:

$(*)_7$ (a) $\mathrm{Dom}(p_*) = \bigcup_n \mathrm{Dom}(p_n)$; in fact, also $\mathrm{fsupp}(p_*) = \bigcup_n \mathrm{fsupp}(p_n)$

(b) if $s \in \mathrm{Dom}(p_n)$ then $\mathrm{tr}(p_*(s)) = \bigcup_{k \geq n} \mathrm{tr}(p_k(s))$.

Similarly let q_* be the upper bound of $\{q_n : n < \omega\}$ as in 1.13(4), so again: {c11}

$(*)_8$ (a) $\mathrm{Dom}(q_*) = \bigcup_n \mathrm{Dom}(q_n)$, in fact also $\mathrm{fsupp}(q_*) = \bigcup_n \mathrm{fsupp}(q_n)$

(b) if $s \in \mathrm{Dom}(q_n)$ then $\mathrm{tr}(q_*(s)) = \bigcup_{k \geq n} \mathrm{tr}(q_k(s))$.

Hence

$(*)_9$ (a) $p_*, q_* \in \mathbb{P}_{\mathbf{m}}$

(b) $\mathrm{Dom}(p_*) \cap \mathrm{Dom}(q_*) \subseteq M_{\mathbf{m}}$, in fact, $\mathrm{fsupp}(p_*) \cap \mathrm{Dom}(q_*) \subseteq M_{\mathbf{m}}$

(c) $\mathrm{Dom}(p_*) \cap M_{\mathbf{m}} = \mathrm{Dom}(q_*) \cap M_{\mathbf{m}}$

(d) if $s \in \mathrm{Dom}(p_*) \cap M_{\mathbf{m}}$, equivalently, $s \in \mathrm{Dom}(p_*) \cap \mathrm{Dom}(q_*)$ then $\mathrm{tr}(p_*(s)) = \mathrm{tr}(q_*(s))$.

[Why? Clause (a) by properties of $\mathbb{P}_{\mathbf{m}}$ and $p_n \leq p_{n+1}, q_n \leq q_{n+1}$ see above, clause (b) as $\mathrm{Dom}(p_{2m}) \cap \mathrm{Dom}(q_{2m}) \subseteq M_{\mathbf{m}}$ as $(\text{suchthat})_{p_{2m},q_{2m},\psi_{2m}}$, clause (c) by $\boxplus_n(c),(d)$, the first conclusion and clause (d) by $\boxplus_n(c),(d)$, the second conclusion.]

It follows that p_*, q_* are compatible in $\mathbb{P}_{\mathbf{m}}$ but $p = p_0 \leq p_*, q = q_0 \leq q_*$, so p, q are compatible as promised. $\square_{3.13}$

Claim 3.15. *The set $\{\psi_i : i < i(*)\} \cup \{\psi_*\}$ has a common upper bound in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ <u>when</u>:* {e30}

$(*)$ (a) *$\mathbf{m} \in \mathbf{M}$ is wide*

(b) *$i(*) < \lambda$*

(c) *$t_i \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ for $i < i(*)$*

(d) *t_i, t_j are not $E''_{\mathbf{m}}$-equivalence for $i < j < i(*)$*

(e) *$\psi_* \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$*

(f) *$X_i = t_i / E_{\mathbf{m}}$*

(g) *$\psi_i \in \mathbb{P}_{\mathbf{m}}[X_i]$*

(h) *if $\mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}] \models$ "$\psi_* \leq \varphi$" and $i < i(*)$ then ψ_i, φ are compatible in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ equivalently in $\mathbb{P}_{\mathbf{m}}[X_i]$.*

Remark 3.16. Note: λ-wide is enough.

Proof. As $\psi_* \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$, there is $p \in \mathbb{P}_{\mathbf{m}}$ such that $p \Vdash_{\mathbb{P}_{\mathbf{m}}}$ "$\psi_*[\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\mathbf{m}}}] = \text{true}$". As $\mathbf{m}$ is wide by 3.5 there is an automorphism f of $\mathbf{m}$ such that $i < i(*) \Rightarrow f''(\mathrm{wsupp}(p)) \cap X_i \subseteq M_{\mathbf{m}}$, hence without loss of generality $i < i(*) \Rightarrow \mathrm{wsupp}(p) \cap X_i \subseteq M_{\mathbf{m}}$. Now we choose p_i by induction on $i \leq i(*)$ such that: {e10}

$\boxplus$ (a) $p_i \in \mathbb{P}_{\mathbf{m}}$

(b) $\langle p_j : j \leq i \rangle$ is increasing

(c) if $s \in \mathrm{Dom}(p_i), i < i(*)$ <u>then</u> $\ell g(\mathrm{tr}(p_{i+1}(s))) > i(*)$

(d) $p_0 = p$

(e) if $i = j + 1$ then $p_i \Vdash$ "$\psi_j[\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\mathbf{m}}}] = \text{true}$

(f) $\mathrm{wsupp}(p_i)$ hence also $\mathrm{wsupp}(p_i)$ is disjoint to $\cup\{X_j \backslash M_{\mathbf{m}} : j \in [i, i(*))\}$.

This is sufficient for the claim as $p_{i(*)}$ is as required. So let us carry the induction. For $i=0$ use clause (d), for i limit by 1.12(4) we know that $\langle p_j : j < i\rangle$ has a $\le_{\mathbb{P}_{\mathbf{m}}}$-upper bound p_i with domain $= \cup\{\mathrm{Dom}(p_j) : j < i\}$ and $\mathrm{wsupp}(p_i) \subseteq \cup\{\mathrm{wsupp}(p_j) : j < i\}$ by 1.13(4), hence p_i is as required, in particular as in clause (f). {c10} {c11}

Lastly, assume $i = j+1$, now there is $\varphi_j \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ such that $\varphi_j \le \varphi \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}] \Rightarrow p_j, \varphi$ are compatible in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$. By an assumption $p_j \Vdash$ "$\psi_*[\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\mathbf{m}}}] =$ true" as p_0 forces this hence $\psi_* \le \varphi_j$. As $\varphi_j \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ by clause (h) of the assumption ψ_j, φ_j are compatible in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ hence have a common upper bound $\varphi^+_j \in \mathbb{P}_{\mathbf{m}}[X_j]$, so there is $q^0_j \in \mathbb{P}_{\mathbf{m}}$ above φ_j and ψ_j. As $\mathbf{m}$ is wide without loss of generality $\mathrm{wsupp}(q^0_j) \cap \mathrm{wsupp}(p_j) \subseteq M_{\mathbf{m}}$. Together (see 3.13) $(\mathrm{suchthat})_{p_j, q^0_j, \varphi_j}$ holds hence by 3.13 p_j, q^0_j has a common upper bound called p_i. As $\mathbf{m}$ is wide, without loss of generality $\mathrm{wsupp}(p_i) \cap X_i = M_{\mathbf{m}}$ for $j \in [i+1, i(*))$. {e29} {e29}

Clearly p_i is as required so we have finished the induction. So we are done. $\square_{3.15}$

{e31}

Conclusion 3.17. *If $\mathbf{m}$ is wide and $f \in \mathscr{F}_{\mathbf{m},\beta}$ and L_1, L_2 its domain and range respectively then f induces an isomorphism $\hat f$ from $\mathbb{P}_{\mathbf{m}}(L_1)$ onto $\mathbb{P}_{\mathbf{m}}(L_2)$.*

{e31a}

{e4} {e4} {e5n} {e5p} {e24} {e31} {e24}

Remark 3.18. 1) See Definition 3.1(5); note that this claim is not covered by Definition 3.1(4).
2) Here we use 3.2(4), so the choice in Definition 1.9$(c)(\gamma)$ is justified (see Remark 3.3(1)) used below in the proof.
3) We could have separated the definition of "analyze" and its properties.
4) Note that in Definition 3.9, we deal only with $L_1 \subseteq t/E_{\mathbf{m}}$ for some t.
5) How come even $\beta = 0$ is suitable for 3.17? The point is clause $(a)(\varepsilon)$ of Definition 3.9(2). But no real harm using larger β.

Proof. By the definitions, clearly $\hat f$ is a one-to-one function from $\mathbb{P}_{\mathbf{m}}(L_1)$ onto $\mathbb{P}_{\mathbf{m}}(L_2)$. Next assume $p_1, q_1 \in \mathbb{P}_{\mathbf{m}}(L_1), \mathrm{Dom}(p_1) \subseteq \mathrm{Dom}(q_1)$ and let $p_2 := \hat f(p_1), q_2 := \hat f(q_1)$; clearly they belong to $\mathbb{P}_{\mathbf{m}}(L_2)$. We shall prove that $\mathbb{P}_{\mathbf{m}} \models$ "$p_1 \le q_1$" iff $\mathbb{P}_{\mathbf{m}} \models$ "$p_2 \le q_2$".

Let $\langle t^1_i : i < i(*)\rangle$ be such that:

$\oplus_1$ (a) $t^1_i \in \mathrm{fsupp}(q_1) \backslash M_{\mathbf{m}} \subseteq L_1$ such that $\mathrm{fsupp}(q_1)$ is included in $\cup\{t^1_i/E_{\mathbf{m}} : i < i(*)\}$
(b) $\langle t^1_i : i < i(*)\rangle$ are pairwise non $E''_{\mathbf{m}}$-equivalent.

Next let

$\oplus_2$ (c) $t^2_i = f(t^1_i)$
(d) let $\bar t_\ell = \langle t^\ell_i : i < i(*)\rangle$ without loss of generality $\mathrm{fsupp}(p_\ell) \subseteq \cup\{t^\ell_i/E''_{\mathbf{m}} : i < j(*)\} \cup M_{\mathbf{m}}$, so $j(*) \le i(*)$.

For $i < i(*)$ let $\psi^*_{1,i} \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ be such that: $\vartheta \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ is compatible with $q_{1,i} := q_1 \restriction (t^1_i/E_{\mathbf{m}})$ (the projection!) iff $\vartheta \wedge \psi^*_{1,i} \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$; clearly exists as $\mathbb{P}_{\mathbf{m}}$ satisfies the λ^+-c.c. Let $\psi^*_1 = \wedge\{\psi^*_{1,i} : i < i(*)\}$.

Now $\psi^*_1 \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ as $q_1 \Vdash$ "$\psi^*_1[\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\mathbf{m}}}] =$ true". We will say "$\psi^*_1, \bar\psi^*_1 = \langle \psi^*_{1,i}, q_{1,i} : i < i(*)\rangle$ analyze q_1 or $(q_1, \bar t_1)$" when the above holds.

Next choose $\varphi^*_1, \langle \varphi^*_{1,i}, p_{1,i} : i < j(*)\rangle$ which analyze $p_1, \langle t^1_i : i < j(*)\rangle$. Why possible? As above.

Lastly, let $\psi^*_{2,i} = \check{f}(\psi^*_{1,i}), p_{2,i} = \check{f}(p_{1,i}), \psi^*_2 = \check{f}(\psi^*_1), \varphi^*_{2,i} = \check{f}(\varphi^*_{1,i}), q_{2,i} = \check{f}(q_{1,i}), \varphi^*_2 = \hat{f}(\varphi^*_1)$ where $\check{f}$ is the function from $\mathbb{L}_{\lambda^+}(Y_{L_1}, \mathbb{P}_{\mathbf{m}})$ onto $\mathbb{L}_{\lambda^+}(Y_{L_2}, \mathbb{P}_{\mathbf{m}})$ induced by f, i.e. where $\check{f}$ is the one-to-one function with domain $\mathbb{L}_{\lambda^+}[Y_{L_1}]$ defined by $p^*_{t,\eta} \mapsto p^*_{f(t),\eta}$

- $(*)$ for $\ell = 1,2$ the sequence $(p_\ell, q_\ell, \bar{\psi}^*_\ell, \bar{\psi}^*_\ell, \varphi^*_\ell, \bar{\varphi}^*_\ell)$ where $\bar{\psi}^*_\ell = \langle \psi^*_{\ell,i}, q_{\ell,i} : i < i_\ell(*)\rangle, \bar{\varphi}^*_\ell = \langle \varphi^*_{\ell,i}, p_{\ell,i} : i < i(*)\rangle$ satisfy the same demands as listed above for $\ell = 1,2$, that is
  - (a) $(\psi^*_\ell, \bar{\psi}^*_\ell)$ analyze $(q_\ell, \bar{t}_\ell)$ for $\ell = 1,2$
  - (b) $(\varphi^*_\ell, \bar{\varphi}^*_\ell)$ analyze $(p_\ell, \bar{t}_\ell \restriction j(*))$ for $\ell = 1,2$.

[Why? Think, recalling $f \restriction (t^1_i/E_{\mathbf{m}})$ is an isomorphism from $\mathbf{m} \restriction ((t^1_i/E_{\mathbf{m}}) \cap L_1)$ onto $\mathbf{m} \restriction ((t^2_i/E_{\mathbf{m}}) \cap L_2)$, etc.]

Next

- $\boxplus$ for $\ell = 1,2$ we have $(A)_\ell \Leftrightarrow (B)_\ell$ where
  - $(A)_\ell$ $\mathbb{P}_{\mathbf{m}} \models$ "$p_\ell \le q_\ell$"
  - $(B)_\ell$ for every $i < j(*)$ we have $\mathbb{P}_{\mathbf{m}}[t^\ell_i/E_{\mathbf{m}}] \models$ "$(\varphi^*_\ell \wedge p_{\ell,i}) \le (\psi^*_\ell \wedge q_{\ell,i})$".

Why? First, assume that the condition $(B)_\ell$ fails, say for i, hence there is $\vartheta \in \mathbb{P}_{\mathbf{m}}[t^\ell_i/E_{\mathbf{m}}]$ such that $\mathbb{P}_{\mathbf{m}}[t^\ell_i/E_{\mathbf{m}}] \models$ "$(\psi^*_\ell \wedge q_{\ell,i}) \le \vartheta$" and $\varphi^*_\ell \wedge p_{\ell,i} \wedge \vartheta \notin \mathbb{P}_{\mathbf{m}}[t^\ell_i/E_m]$. So by claim 3.15 there is $q^+_\ell \in \mathbb{P}_{\mathbf{m}}$ such that $q^+_\ell \in \mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ is above ϑ hence above {e30} ψ^*_ℓ and above $q_{\ell,j} = q_\ell \restriction (t^\ell_j/E_{\mathbf{m}})$ for $j < i(*)$. That is, first get $\psi \in \mathbb{P}_m[M_m]$ such that $\psi \ge \psi^*_\ell$ and $[\psi \le \psi' \in \mathbb{P}_m[M_m] \Rightarrow \psi', \vartheta$ are compatible] (using $\vartheta \ge \psi^*_\ell$). Then apply 3.15 to $(\{q_{\ell,j} : j < j(*)\} \cup \{\vartheta\}) \cup \{\psi\}$ to get q^+_ℓ. {e30}

Hence by 3.2(4) the condition q^+_ℓ is above q_ℓ but $q^+_\ell \Vdash$ "$\varphi^*_\ell \wedge p_{\ell,i}[\underset{\sim}{\mathbf{G}}] = \text{false}$" as q^+_ℓ {e5n} is above ϑ. However, $p_\ell \Vdash_{\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]}$ "$p_{\ell,i} \in \underset{\sim}{\mathbf{G}}$ and $\varphi^*_\ell \in \mathbf{G}$". By the last two sentences q^+_ℓ, p_ℓ are incompatible, in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ equivalently in $\mathbb{P}_{\mathbf{m}}$. So indeed $\neg(B)_\ell \Rightarrow \neg(A)_\ell$.

For the other direction assume condition $(B)_\ell$ holds, but condition $(A)_\ell$ fails and we shall get a contradiction. So there is $q^+_\ell \in \mathbb{P}_{\mathbf{m}}$ above q_ℓ incompatible with p_ℓ.

For each $i < i(*)$ as $(\psi^*_\ell, \langle \psi^*_{\ell,j}, q_{\ell,j} : j < i(*)\rangle)$ analyze q_ℓ, clearly $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}] \models$ "$(\psi^*_\ell \wedge q_{\ell,i}) \le q_\ell$" but $q_\ell \le q^+_\ell$ hence $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}] \models$ "$(\psi^*_\ell \wedge q_{\ell,i}) \le q^+_\ell$", and as we are assuming clause $(B)_\ell$ we have $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}] \models$ "$(\varphi^*_\ell \wedge p_{\ell,i}) \le q^+_\ell$". Hence by 3.2(4), q^+_ℓ {e5n} is above p_ℓ, contradiction. So indeed $(B)_\ell \Rightarrow (A)_\ell$.

Together, $\boxplus$ holds. Now clearly $(B)_1 \Leftrightarrow (B)_2$, see Definition 3.9, 3.12; so by $\boxplus$ {e28} we have $(A)_1 \Leftrightarrow (A)_2$ which is the desired conclusion. $\square_{3.17}$

{e32}

Claim 3.19. *We have $\mathbb{P}_{\mathbf{m}_1} \lessdot \mathbb{P}_{\mathbf{m}}$ <u>when</u>:*

- (a) $\mathbf{m}_1 \le_{\mathbf{M}} \mathbf{m}$
- (b) *if $t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}_1}$ and $\bar{s} \in {}^\zeta(t/E''_{\mathbf{m}}), \zeta < \lambda^+$ <u>then</u> we can find $t_i, \bar{s}_i$ for $i < \lambda^+$ such that:*
  - (α) $t_i \in L_{\mathbf{m}_1} \backslash M_{\mathbf{m}_1}$
  - (β) $t_i/E''_{\mathbf{m}_1} \ne t_j/E''_{\mathbf{m}_1}$ *when* $i \ne j < \lambda^+$
  - (γ) $\bar{s}_i \in {}^\zeta(t_i/E''_{\mathbf{m}_1})$
  - (δ) $(t_i, \bar{s}_i)$ *is ξ-equivalent to $(t, \bar{s})$ in $\mathbf{m}$ where*[13] $\xi = 1$.
- (c) $\mathbf{m}$ *is wide.*

[13]no real harm in using larger ξ

Remark 3.20. In the proof we use conclusion 3.17 but not clause $(a)(\varepsilon)$ of Definition 3.9(2). {e31} {e24}

Proof.

$\boxplus_1$ for $f \in \mathscr{F}_{\mathbf{m},\beta}$

(a) $\hat{f}$ preserves "p_2 is above p_1 in $\mathbb{P}_{\mathbf{m}}$", and its negations

(b) if $\beta > 0$ then $\hat{f}$ preserves also incompatibility in $\mathbb{P}_{\mathbf{m}}$.

{e34} {e28} [Why? Clause (a) holds by 3.17. For clause (b) use clause (a) and Definitions 3.9 and 3.12 or see the proof of $\boxplus_2$.]

$\boxplus_2$ if $p_i \in \mathbb{P}_{\mathbf{m}_1}$ for $i < i(*) < \lambda^+$ and $p \in \mathbb{P}_{\mathbf{m}}$ <u>then</u> there is p^* such that:

(a) $p^* \in \mathbb{P}_{\mathbf{m}_1}$, equivalently $p^* \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}_1})$

(b) $\mathbb{P}_{\mathbf{m}_1} \models$ "$p_i \le p^*$" iff $\mathbb{P}_{\mathbf{m}} \models$ "$p_i \le p$"

(c) $\mathbb{P}_{\mathbf{m}_1} \models$ "p_i, p^* are compatible" iff $\mathbb{P}_{\mathbf{m}} \models$ "p_i, p are compatible".

[Why? Let $q_i \in \mathbb{P}_{\mathbf{m}}$ be such that: if p_i, p are compatible in $\mathbb{P}_{\mathbf{m}}$ then $p_i \le q_i \wedge p \le q_i$. We can find $L_1 \subseteq L_2$ such that

- $M_{\mathbf{m}} \subseteq L_1 \subseteq L_{\mathbf{m}_1}, |L_1 \backslash M_{\mathbf{m}}| \le \lambda$
- $\{p_i : i < i(*)\} \subseteq \mathbb{P}_{\mathbf{m}}(L_1)$
- $L_1 \subseteq L_2 \subseteq L_{\mathbf{m}}, |L_2 \backslash M_{\mathbf{m}}| \le \lambda$ and $p, q_i \in \mathbb{P}_{\mathbf{m}}(L_2)$ for $i < i(*)$.

By the assumption of the claim there is $f \in \mathscr{F}_{\mathbf{m},1}$ such that:

- $\mathrm{Dom}(f) \subseteq \cup\{(t/E''_{\mathbf{m}}) \cap L_2 : t \in L_2\} \cup M_{\mathbf{m}}$
- $t \in L_1 \Rightarrow f{\restriction}(t/E_{\mathbf{m}} \cap L_2) = \mathrm{id}_{(t/E_{\mathbf{m}}) \cap L_2}$
- if $q \in \{q_i : i < i(*)\} \cup \{p\} \cup \{p_i : i < i(*)\}$ and $t \in \mathrm{Dom}(q) \backslash M_{\mathbf{m}}$ then $\mathrm{fsupp}(q(t)) \subseteq \mathrm{Dom}(f)$
- $\mathrm{Rang}(f) \subseteq L_{\mathbf{m}_1}$.

Let $p^* = \hat{f}(p)$: by $\boxplus_1(a)$ clearly clauses (a),(b) of $\boxplus_2$ holds; and the choice of the q_i's also the implication "if" of clause (c). The "only if" of clause (c) holds by $\boxplus_1(b)$ so we are done.]

$\boxplus_3$ if $p \in \mathbb{P}_{\mathbf{m}}$ then $p \in \mathbb{P}_{\mathbf{m}_1}$ <u>iff</u> $\mathrm{fsupp}(p) \subseteq L_{\mathbf{m}_1}$.

[Why? Obvious.]

{c34} Recalling Definition 1.24(0)(b)

$\boxplus_4$ for every ordinal γ, we have $\mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma}) \lessdot \mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m},\gamma})$.

[Why? We shall prove this by induction on γ using $\boxplus_2 + \boxplus_3$.

Note that

$\boxplus_{4.1}$ (a) $L^{\mathrm{dp}}_{\mathbf{m},\gamma} \cap L_{\mathbf{m}_1} = L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma}$

(b) if $f \in \mathscr{F}_{\mathbf{m},\beta}, s \in \mathrm{Dom}(f)$ and β is an ordinal then

- $s \in L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma} \Leftrightarrow f(s) \in L^{\mathrm{dp}}_{\mathbf{m},\gamma}$

(c) the parallel of $\boxplus_2$ holds for $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m},\gamma})$ so $p^* \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}_1,\gamma})$

(d) $L^{\mathrm{dp}}_{\mathbf{m},\gamma}$ is an initial segment of $L_{\mathbf{m}}$

(e) $L^{\rm dp}_{\mathbf{m}_1,\gamma}$ is an initial segment of $L_{\mathbf{m}_1}$

(f) $\mathbb{P}_{\mathbf{m}_1}(L^{\rm dp}_{\mathbf{m}_1,\gamma}) \lessdot \mathbb{P}_{\mathbf{m}_1}(L_{\mathbf{m}_1})$, similarly for $\mathbf{m}$.

We shall use this freely. The inductive proof on γ splits to three cases.

Case 1: $\gamma = 0$

So

- $E = E''_{\mathbf{m}} {\restriction} L^{\rm dp}_{\mathbf{m},\gamma}$ is an equivalence relation on $L^{\rm dp}_{\mathbf{m},\gamma}$
- $E {\restriction} L^{\rm dp}_{\mathbf{m}_1,\gamma} = E''_{\mathbf{m}_1} {\restriction} L^{\rm dp}_{\mathbf{m}_1,\gamma}$
- if $t \in L^{\rm dp}_{\mathbf{m}_1,\gamma}$ then $t \notin M_{\mathbf{m}_1}, t/E'_{\mathbf{m}_1} = t/E'_{\mathbf{m}}, (t/E'_{\mathbf{m}_1}) \cap L^{\rm dp}_{\mathbf{m}_1,\gamma} = (t/E_{\mathbf{m}_1}) \cap L^{\rm dp}_{\mathbf{m}_1,\gamma} = (t/E''_{\mathbf{m}}) \cap L^{\rm dp}_{\mathbf{m},\gamma}$ initial segment of $L_{\mathbf{m}_1}$ and of $L_{\mathbf{m}}$ and $\mathbb{P}_{\mathbf{m}}((t/E_{\mathbf{m}_1}) \cap L^{\rm dp}_{\mathbf{m}_1,\gamma}) = \mathbb{P}_{\mathbf{m}_1}((t/E_{\mathbf{m}_1}) \cap L^{\rm dp}_{\mathbf{m}_1,\gamma})$
- $\mathbb{P}_{\mathbf{m}}(L^{\rm dp}_{\mathbf{m},\gamma})$ is the product with $(< \lambda)$-support of $\{\mathbb{P}_{\mathbf{m}}((t/E_{\mathbf{m}_1}) \cap L^{\rm dp}_{\mathbf{m}_1,\gamma}) : t \in L^{\rm dp}_{\mathbf{m},\gamma}\}$
- similarly for $\mathbf{m}_1$.

So the result should be clear.

Case 2: $\gamma = \beta + 1$

Let $M_\beta = \{s \in M_{\mathbf{m}} : {\rm dp}_{\mathbf{m}}(s) = \beta\}$, clearly

$\boxplus_{4.2}$ (a) M_β is a set of pairwise incomparable elements

(b) (α) $s \in M_\beta \Rightarrow L_{\mathbf{m}_1,<s} \subseteq L^{\rm dp}_{\mathbf{m}_1,\beta} \wedge L_{\mathbf{m},<s} \subseteq L^{\rm dp}_{\mathbf{m}_1,\beta}$

(β) similarly for $\mathbf{m}$

(c) M_β is disjoint to $L^{\rm dp}_{\mathbf{m}_1,\beta}, L^{\rm dp}_{\mathbf{m},\beta}$

(d) $M_\beta \subseteq L^{\rm dp}_{\mathbf{m}_1,\gamma}$

(e) $L^{\rm dp}_{\mathbf{m},\beta} \cup M_\beta$ is an initial segment of $L_{\mathbf{m}}$

(f) $L^{\rm dp}_{\mathbf{m}_1,\beta} \cup M_\beta$ is an initial segment of $L_{\mathbf{m}_1}$.

As first half we prove

$\boxplus_{4.3}$ $\mathbb{P}_{\mathbf{m}_1}(L^{\rm dp}_{\mathbf{m}_1,\beta} \cup M_\beta) \lessdot \mathbb{P}_{\mathbf{m}}(L^{\rm dp}_{\mathbf{m},\beta} \cup M_\beta)$.

Why? Recalling $\boxplus_{4.1}(a)$, note

(a) for $p, q \in \mathbb{P}_{\mathbf{m}_1}(L^{\rm dp}_{\mathbf{m}_1,\beta} \cup M_\beta)$ we have $\mathbb{P}_{\mathbf{m}_1}(L^{\rm dp}_{\mathbf{m}_1,\gamma} \cup M_\beta) \models$ "$p \le q$" iff $\mathbb{P}_{\mathbf{m}}(L^{\rm dp}_{\mathbf{m}} \cup M_\beta) \models$ "$p \le q$".

[Why? Immediate by the definition of the order and the induction hypothesis.]

(b) for $p_1, p_2 \in \mathbb{P}_{\mathbf{m}_1}(L^{\rm dp}_{\mathbf{m}_1,\beta} \cup M_\beta)$ <u>then</u> p_1, p_2 are compatible in $\mathbb{P}_{\mathbf{m}_1}(L^{\rm dp}_{\mathbf{m}_1,\beta} \cup M_\beta)$ iff they are compatible in $\mathbb{P}_{\mathbf{m}}(L^{\rm dp}_{\mathbf{m},\beta} \cup M_\beta)$.

[Why? The implication $\Rightarrow$ holds by clause (a). So assume $p_3 \in \mathbb{P}_{\mathbf{m}}(L^{\rm dp}_{\mathbf{m},\beta} \cup M_\beta)$ is a common upper bound of p_1, p_2 in $\mathbb{P}_{\mathbf{m}}(L^{\rm dp}_{\mathbf{m},\beta} \cup M_\beta)$ equivalently in $\mathbb{P}_{\mathbf{m}}$.

Now there is $f \in \mathscr{F}_{\mathbf{m},1}$ such that

- $f {\restriction} ({\rm fsupp}(p_1) \cup {\rm fsupp}(p_2))$ is the identity, moreover
- $s \in {\rm wsupp}(p_1) \cup {\rm wsupp}(p_2) \wedge s \in {\rm dom}(f) \Rightarrow f(s) = s$,

- $\mathrm{Dom}(f) = \cup\{\mathrm{fsupp}(p_\ell) : \ell = 1,2,3\}$
- $\mathrm{Rang}(f) \subseteq L_{\mathbf{m}_1}$.

Hence clearly $f{\restriction}M_\beta = \mathrm{id}_{M_\beta}$ so by $\boxplus_{4.1}(b)$ we have $\mathrm{Rang}(f) \subseteq L^{\mathrm{dp}}_{\mathbf{m}_1,\beta} \cup M_\beta$ so $\hat{f}(p_3) \in \mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m}_1,\beta} \cup M_\beta)$.

By $\boxplus_1$ the condition $\hat{f}(p_3)$ is a common upper bound of p_1, p_2 in $\mathbb{P}_{\mathbf{m}}$ and by the previous sentence also in $\mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m}_1,\beta} \cup M_\beta)$, so by clause (a) the conclusion of (b) holds.]

(c) if $\mathscr{I}$ is a maximal antichain in $\mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\beta} \cup M_\beta)$ <u>then</u> $\mathscr{I}$ is a maximal antichain of $\mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m},\beta} \cup M_\beta)$.

[Why? As in the proof of (b) and of $\boxplus_2$.]

So we are done proving $\boxplus_{4.3}$.

Now we return to proving $\boxplus_4$, note

$\boxplus_{4.4}$ let $\mathscr{E} = \{(s_1,s_2) : s_1, s_2 \in L_*$ and $s_1/E_{\mathbf{m}} = s_2/E_{\mathbf{m}}\}$ where $L_* = L^{\mathrm{dp}}_{\mathbf{m},\gamma} \backslash (L^{\mathrm{dp}}_{\mathbf{m},\beta} \cup M_\beta)\}$ <u>then</u>

- (a) $\mathscr{E}$ is an equivalence relation on L_*
- (b) if $s_1, s_2 \in L_*$ and $s_1 \le_{L_{\mathbf{m}}} s_2$ then $s_1 \mathscr{E} s_2$
- (c) if $s_1, s_2 \in L_*$ and $s_1 \mathscr{E} s_2$ then $s_1 \in L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma} \Leftrightarrow s_2 \in L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma}$ (and both $\notin M_\beta$)
- (d) if $s \in L_*$ then $L_{\mathbf{m},<s} \subseteq L^{\mathrm{dp}}_{\mathbf{m},\beta} \cup M_\beta \cup (s/\mathscr{E})$
- (e) if $s \in L_* \cap L_{\mathbf{m}_1}$ then $L_{\mathbf{m}_1,<s} \subseteq L^{\mathrm{dp}}_{\mathbf{m}_1,\beta} \cup M_\beta \cup (s/\mathscr{E})$.

Hence let $L_0 = L^{\mathrm{dp}}_{\mathbf{m}_1,\beta} \cup M_\beta$ and $L_1 = L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma} = L^{\mathrm{dp}}_{\mathbf{m}_1} \cup M_\beta$ they satisfy all the
{c33s} assumptions of 1.22 hence its conclusion, so we are done easily proving Case 2 of $\boxplus_4$.

<u>Case 3</u>: γ is a limit ordinal

Clearly $p \in \mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma})$ iff $p \in \mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma})$; also each of them implies $p \in \mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m},\gamma})$ by the induction hypothesis. Also for $p, q \in \mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma})$ we have $\mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma}) \models$ "$p \le q$" <u>iff</u> $\mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m},\gamma}) \models$ "$p \le q$" by the definition of the order and the induction hypothesis. Together $\mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma}) \subseteq \mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m},\gamma})$, (as partial orders).

Next assume that $q_1, q_2 \in \mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma})$ and p_3 is a common upper bound of q_1, q_2 in $\mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m},\gamma})$.

We shall find $p_1 \in \mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma})$ such that:

$(*)_1$ (a) p_1 is above q_1, q_2 (in $\mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma})$ or equivalently in $\mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma})$),

(b) if $p_1 \le p'_1 \in \mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma})$ then p'_1, p_3 are compatible in $\mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m},\gamma})$.

This clearly suffices; why? e.g. if $\{r_i : i < i(*)\} \subseteq \mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma})$ is a maximal antichain of $\mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma})$ but not of $\mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m},\gamma})$, let $q_1 = q_2 = \emptyset$ and $p_3 \in \mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m},\gamma})$ be incompatible with every r_i; let p_1 be as in $(*)_1$, it gives a contradiction.

If $\mathrm{cf}(\gamma) \ge \lambda$ then for some $\gamma_1 < \gamma$ we have $q_1, q_2 \in \mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma_1})$ and $\mathrm{fsupp}(p_3) \cap L^{\mathrm{dp}}_{\mathbf{m},\gamma} \subseteq L^{\mathrm{dp}}_{\mathbf{m},\gamma_1}$ and use the induction hypothesis on γ_1 for clause (a) of $(*)_1$; for {c11} clause (b) of $(*)_1$ we also recall 1.13(8); (alternatively immitate the case $\mathrm{cf}(\gamma) < \lambda$, choosing "changing our minds" $\gamma_\varepsilon < \gamma$ with the induction). So assume $\aleph_0 \le \mathrm{cf}(\gamma) < \lambda$ and let $\langle \gamma_\varepsilon : \varepsilon < \mathrm{cf}(\gamma) \rangle$ be increasing continuous with limit γ.

Now we choose $p_{1,\varepsilon}$ by induction on $\varepsilon \le \mathrm{cf}(\gamma)$ such that:

$(*)_2$ (a) $p_{1,\varepsilon} \in \mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma_\varepsilon})$

(b) $(\gamma_\varepsilon, q_1 \restriction L^{\mathrm{dp}}_{\mathbf{m},\gamma_\varepsilon}, q_2 \restriction L^{\mathrm{dp}}_{\mathbf{m},\gamma_\varepsilon}, p_3 \restriction L^{\mathrm{dp}}_{\mathbf{m},\gamma_\varepsilon}, p_{1,\varepsilon})$ are like $(\gamma, q_1, q_2, p_3, p_1)$ in $(*)_1$

(c) $p_{1,\zeta} \le p_{1,\varepsilon}$ for $\zeta < \varepsilon$

(d) if $\varepsilon = \zeta + 1$ and $s \in \mathrm{dom}(p_{1,\zeta})$ then $\ell g(\mathrm{tr}(p_\varepsilon(s))) > \mathrm{cf}(\gamma)$.

So we are done proving $\boxplus_4$.]

$\boxplus_5$ $\mathbb{P}_{\mathbf{m}_1} \lessdot \mathbb{P}_{\mathbf{m}}$.

[Why? By $\boxplus_4$ for γ large enough.]

So we are done. $\square_{3.19}$

{c73}

Claim 3.21. *If $\mathbf{m} \in \mathbf{M}$ is reduced or just $L_{\mathbf{m}}$ has cardinality $\le \lambda_2$ <u>then</u> there is $\mathbf{n} \in \mathbf{M}_{\mathrm{ec}}$ of cardinality $\le \lambda_2$ such that $\mathbf{m} \le_{\mathbf{M}} \mathbf{n}$.*

{c76}

Remark 3.22. By this we may restrict ourselves to $\mathbf{M}_{\le \lambda_2}$ (but then similarly in the end of §2).

Proof. We choose χ large enough and $\mathbf{m}_* \in \mathbf{M}_\chi$ which is wide, belongs to $\mathbf{M}_{\mathrm{ec}}$ and $\mathbf{m} \le_{\mathbf{M}} \mathbf{m}_*$; moreover is full and very wide (as constructed in 1.26). {c41}

We can choose $\mathbf{n}$ such that:

$(*)$ (a) $\mathbf{n} \in \mathbf{M}$ and $\mathbf{n}$ is wide and $|L_{\mathbf{n}}| = \lambda_2$

(b) $\mathbf{m} \le \mathbf{n} \le_{\mathbf{M}} \mathbf{m}_*$

(c) $(\mathbf{n}, \mathbf{m}_*)$ satisfies the criterion from 3.19, with $\mathbf{m}_1, \mathbf{m}$ there standing for $\mathbf{n}, \mathbf{m}_*$ here. {e32}

[Why? Let $\xi = 1$ and recalling Definition 3.9(1) choose $\langle (t_\alpha, \bar{s}_\alpha) : \alpha < \lambda_2) \rangle$ such that $(t_\alpha, \bar{s}_\alpha) \in \mathscr{Y}_{\mathbf{m}_*}, t_\alpha \in L_{\mathbf{m}_*} \backslash M_{\mathbf{m}_*}, \langle t_\alpha / E_{\mathbf{m}} : \alpha < \lambda_2 \rangle$ are pairwise distinct and for every $(t, \bar{s}) \in \mathscr{Y}_{\mathbf{m}_*}$ there are λ^+ ordinals $\alpha < \lambda_2$ such that $(t, \bar{s}), (t_\alpha, \bar{s}_\alpha)$ are ξ-equivalent, possible by 3.11 recalling $\lambda_2 \ge \beth_3(\lambda_1)$. Let $L' = \cup \{ t_\alpha / E_{\mathbf{m}_*} : \alpha < \lambda_2 \} \cup L_{\mathbf{m}}$ and for each $t \in L' \backslash M_{\mathbf{m}_*}$ let $\langle s_{t,\alpha} : \alpha < \lambda^+ \rangle$ be such that $s_{t,\alpha} \in L_{\mathbf{m}_*} \backslash M_{\mathbf{m}_*}$ and $\mathbf{m}_* \restriction (s_{t,\alpha} / E_{\mathbf{m}_*})$ is isomorphic to $\mathbf{m}_* \restriction (t / E_{\mathbf{m}_*})$ over $M_{\mathbf{m}}$. Let $L = L' \cup \{ s_{t,\alpha} : \alpha < \lambda^+, t \in L' \backslash M_{\mathbf{m}_*} \}$ and $\mathbf{n} = \mathbf{m}_* \restriction L$. Now it is easy to check that $\mathbf{n}$ is as required.] {e24} {e27}

It suffices to prove that $\mathbf{n}$ belongs to $\mathbf{M}_{\mathrm{ec}}$, let $\mathbf{n} \le_{\mathbf{M}} \mathbf{n}_1 \le_{\mathbf{M}} \mathbf{n}_2$.

Without loss of generality $L_{\mathbf{n}_2}$ has cardinality $\le 2^{\lambda_2}$, by the LST argument (even $\le \lambda_2$, as we are assuming $\lambda_2 = (\lambda_2)^\lambda$), and as $\mathbf{m}_*$ is very wide and full without loss of generality $\mathbf{n}_2 \le_{\mathbf{M}} \mathbf{m}_*$. Now $(\mathbf{n}_1, \mathbf{m}_*)$ satisfies the criterion from 3.19 hence $\mathbb{P}_{\mathbf{n}_1} \lessdot \mathbb{P}_{\mathbf{m}_*}$. {e32}

Also the pair $(\mathbf{n}_2, \mathbf{m}_*)$ satisfies the criterion from 3.19 looking at the criterion. {e32} Hence by 3.19 we have $\mathbb{P}_{\mathbf{n}_2} \lessdot \mathbb{P}_{\mathbf{m}_*}$. {e32}

As $\mathbf{n}_1 \le_{\mathbf{M}} \mathbf{n}_2 \le_{\mathbf{M}} \mathbf{m}_*$ from the last two sentences it easily follows that $\mathbb{P}_{\mathbf{n}_1} \lessdot \mathbb{P}_{\mathbf{n}_2}$, so we are done. $\square_{3.21}$

Discussion 3.23. In what way does this proof help? Will it not be simpler to omit in Definition 1.9 clause (c) the $\iota_{p(s)}, \mathbf{B}_{p(s),\iota}$, etc.? {c6}

In this case in 3.1 we cannot define the projection directly hence we should look for projection as in general forcing, but then we run into problems of absoluteness. More specifically, 3.19 seems to be problematic; anyhow this does not matter. {e4} {e32}

{e37}

Definition 3.24. For $\mathbf{m} \in \mathbf{M}$ and $M \subseteq L_{\mathbf{m}}$ of cardinality $\le \lambda_1$ we define $\mathbf{n} := \mathbf{m}\langle M\rangle \in \mathbf{M}$ as follows:

- (a) $L_{\mathbf{n}} = L_{\mathbf{m}}$ even as a partial order
- (b) $\bar{u}_{\mathbf{n}} = \bar{u}_{\mathbf{m}}$ and $\bar{\mathscr{P}}_{\mathbf{n}} = \bar{\mathscr{P}}_{\mathbf{m}}$
- (c) $M_{\mathbf{n}} = M$; not $M_{\mathbf{m}}$!
- (d) $E'_{\mathbf{n}} = \{(s,t) : s,t \in L_{\mathbf{m}} \text{ and } \{s,t\} \nsubseteq M\}$.

{e39}

Claim 3.25. *Assume* $\mathbf{m} \in \mathbf{M}_{\le \lambda_2}$ *and* $M \subseteq M_{\mathbf{m}}$.
1) $\mathbf{n} := \mathbf{m}\langle M\rangle$ *indeed belongs to* $\mathbf{M}$ *and is equivalent to* $\mathbf{m}$ *hence* $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}}) = \mathbb{P}_{\mathbf{n}}(L_{\mathbf{m}})$.
2) If $\mathbf{n} \le_{\mathbf{M}} \mathbf{n}_1$ *then for some* $\mathbf{m}_1$ *we have* $\mathbf{m} \le_{\mathbf{M}} \mathbf{m}_1$ *and* $\mathbf{m}_1, \mathbf{n}_1$ *are equivalent.*
3) If $\mathbf{m} \in \mathbf{M}_{\rm ec}$ *and* $\mathbf{n} = \mathbf{m}\langle M\rangle$ *then* $\mathbf{n} \in \mathbf{M}_{\rm ec}$.

Proof. 1) Check, noting that $t \in L_{\mathbf{n}} \backslash M_{\mathbf{n}} \Rightarrow t \in L_{\mathbf{m}} \backslash M \Rightarrow |t/E'_{\mathbf{m}}| \le |L_{\mathbf{n}}| = |L_{\mathbf{m}}| \le \lambda_2$ and $|M_{\mathbf{m}}| = |M| \le |M_{\mathbf{m}}| \le \lambda_1$.
2) Given such $\mathbf{n}_1$ we now define $\mathbf{m}_1 \in \mathbf{M}$ by:

$(*)_1$
- (a) $L_{\mathbf{m}_1} = L_{\mathbf{n}_1}$,
- (b) $\bar{u}_{\mathbf{m}_1} = \bar{u}_{\mathbf{n}_1}$ and $\bar{\mathscr{P}}_{\mathbf{m}_1} = \bar{\mathscr{P}}_{\mathbf{n}_1}$
- (c) $M_{\mathbf{m}_1} = M_{\mathbf{m}}$,
- (d) $E'_{\mathbf{m}_1} = \{(s,t) : sE'_{\mathbf{m}}t \text{ \underline{or} } \{s,t\} \nsubseteq L_{\mathbf{m}} \backslash M \text{ and } sE'_{\mathbf{n}_1}t\}$.

Clearly

$(*)_2$
- (a) $\langle M_{\mathbf{m}}\rangle\hat{\ }\langle s/E''_{\mathbf{m}} : s \in L_{\mathbf{m}_1} \backslash M_{\mathbf{m}}\rangle\hat{\ }\langle t/E''_{\mathbf{n}_1} : t \in L_{\mathbf{n}_1} \backslash L_{\mathbf{n}}\rangle$ is a partition of $L_{\mathbf{m}_1} = L_{\mathbf{n}_1}$
- (b) $E''_{\mathbf{m}_1} = E'_{\mathbf{m}_1} {\restriction} \{(s,t) \in E'_{\mathbf{m}_1} \text{ and } s,t \notin M_{\mathbf{m}}\}$ is an equivalence relation, its equivalence classes being the sets listed in clause (a) except $M_{\mathbf{m}}$
- {c4} (c) $\mathbf{m}_1$ satisfies clasue $(e)(\gamma)$ of Definition 1.7

$(*)_3$
- (a) if $s \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ then
  - (α) $s \in L_{\mathbf{m}_1} \backslash M_{\mathbf{m}_1}$
  - (β) $s/E'_{\mathbf{m}_1} = s/E'_{\mathbf{m}}$
  - (γ) $u_{\mathbf{m}_1,s} = u_{\mathbf{n}_1,s} = u_{\mathbf{n},s} = u_{\mathbf{m},s}$
  - (δ) $\mathscr{P}_{\mathbf{m}_1,s} = \mathscr{P}_{\mathbf{m},s}$
- (b) if $s \in L_{\mathbf{m}_1} \backslash L_{\mathbf{m}}$ then
  - (α) $s \in L_{\mathbf{n}_1} \backslash L_{\mathbf{n}}$
  - (β) $s/E'_{\mathbf{m}_1} = s/E'_{\mathbf{n}_1}$
  - (γ) $u_{\mathbf{m}_1,s} = u_{\mathbf{n}_1}, s$
  - (δ) $\mathscr{P}_{\mathbf{m}_1,s} = \mathscr{P}_{\mathbf{n}_1,s}$
- (c) if $s \in M_{\mathbf{m}_1}$, i.e. $s \in M_{\mathbf{m}}$ then
  - (α) $u_{\mathbf{m}_1,s} = u_{\mathbf{n}_1,s}$

(β) $\mathscr{P}_{\mathbf{m}_1,s} = \mathscr{P}_{\mathbf{n}_1,s}$

and

$(*)_4$ (a) indeed $\mathbf{m}_1 \in \mathbf{M}$,
(b) $\mathbf{m} \le_{\mathbf{M}} \mathbf{m}_1$,
(c) $\mathbf{m}_1, \mathbf{n}_1$ are equivalent.

So we are done.
3) Assume $\mathbf{n} \le_{\mathbf{M}} \mathbf{n}_1 \le_{\mathbf{M}} \mathbf{n}_2$, as in the proof of part (2) there are $\mathbf{m}_1, \mathbf{m}_2$ such that $\mathbf{m} \le_{\mathbf{M}} \mathbf{m}_1 \le_{\mathbf{M}} \mathbf{m}_2$ and $\mathbf{m}_\ell, \mathbf{n}_\ell$ are equivalent for $\ell = 1,2$. As $\mathbf{m} \in \mathbf{M}_{\rm ec}$ we have $\mathbb{P}_{\mathbf{m}_1} \lessdot \mathbb{P}_{\mathbf{m}_2}$ but this means $\mathbb{P}_{\mathbf{n}_1} \lessdot \mathbb{P}_{\mathbf{n}_2}$, as required. $\square_{3.25}$

Conclusion 3.26. *1) If $\mathbf{m} \in \mathbf{M}, M \subseteq M_{\mathbf{m}}$ and $\mathbf{n} = \mathbf{m}{\restriction}M$ <u>then</u> $\mathbb{P}^{\rm cer}_{\mathbf{n}} \lessdot \mathbb{P}^{\rm cer}_{\mathbf{m}}$.* {c80}
2) If $\mathbf{m}_\ell \in \mathbf{M}$ and $M_\ell \subseteq M_{\mathbf{m}_\ell}$ for $\ell = 1,2$ and h is an isomorphism from $\mathbf{m}_1{\restriction}M_1$ onto $\mathbf{m}_2{\restriction}M_2$ <u>then</u> h induces an isomorphism from $\mathbb{P}^{\rm cer}_{\mathbf{m}_1}[M_1]$ onto $\mathbb{P}^{\rm cer}_{\mathbf{m}_2}[M_2]$.

Proof. 1) As in the proof of 3.21, without loss of generality $\mathbf{m}, \mathbf{n} \in \mathbf{M}_{\le\lambda_2}$. By 3.21 there is $\mathbf{m}_* \in \mathbf{M}^{\rm ec}_{\lambda_2}$ such that $\mathbf{m} \le_{\mathbf{M}} \mathbf{m}_*$ hence $\mathbb{P}^{\rm cer}_{\mathbf{m}} = \mathbb{P}_{\mathbf{m}_*}[L_{\mathbf{m}}]$. {c73}

Let $\mathbf{n}_* = \mathbf{m}_*\langle M\rangle$, see 3.24, so $\mathbf{n}_*{\restriction}M = \mathbf{n}$ and by 3.25(3) we have $\mathbf{n}_* \in \mathbf{M}_{\rm ec}$, hence $\mathbb{P}_{\mathbf{n}_*}[L_{\mathbf{n}}] = \mathbb{P}^{\rm cer}_{\mathbf{n}}$. But $\mathbf{n}_*, \mathbf{m}_*$ are equivalent, hence $\mathbb{P}_{\mathbf{n}_*} = \mathbb{P}_{\mathbf{m}_*}$ hence $\mathbb{P}_{\mathbf{n}_*}[L] = \mathbb{P}_{\mathbf{m}_*}[L]$ for every $L \subseteq L_{\mathbf{m}_*}$ hence by 2.9(3) $\mathbb{P}^{\rm cer}_{\mathbf{n}} = \mathbb{P}_{\mathbf{n}_*}[L_{\mathbf{n}}] \lessdot \mathbb{P}_{\mathbf{n}_*}[L_{\mathbf{m}}] = \mathbb{P}_{\mathbf{m}_*}[L_{\mathbf{m}}] = \mathbb{P}^{\rm cer}_{\mathbf{m}}$. So the conclusion holds. {e37} {c48}
2) Easy, too. $\square_{3.26}$

§ 4. Comment on [Sh:945]

How we connect to [Sh:945], i.e. justify $(*)_5$ in the proof of Lemma [Sh:945, 1.3(2)=La7].

Let M be the linear order $(\kappa,<), u'_\alpha = \alpha, \mathscr{P}'_\alpha = [\alpha]^{\le\lambda}$; we may replace it by an {c52} isomorphic copy. Now applying 2.12 there is $\mathbf{m}$ as promised in [Sh:945]. As $L_{\mathbf{m}}$ is a well founded partial order, we can let h be a one-to-one order preserving function from $L_{\mathbf{m}}$ onto some ordinal, call it $\beta(*)$.

So without loss of generality h is the identity, so let $L_{\mathbf{m}}$ be $(\beta(*),<_*), \mathscr{U}_* =$ {c52} $M_{\mathbf{m}}, \mathbb{P}_{1,\alpha} = \mathbb{P}_{\mathbf{m}}[\{\beta : \beta < \alpha\}, \mathbb{P}_{0,\alpha} = \mathbb{P}_\alpha$ in 2.12].

{z32}

Definition 4.1. For an ordinal $\alpha_* = \alpha(*)$ let $\mathbf{Q}_{\lambda,\bar\theta,\alpha(*)}$ be the class of objects $\mathbf{q}$ consisting of (omitting α_* means for some α_* and $\ell g(\mathbf{q}) = \alpha_{\mathbf{q}} = \alpha_*$):

(a) $\bar u = \langle u_\alpha : \alpha < \alpha_*\rangle$ and $\mathscr{P} = \langle \mathscr{P}_\alpha : \alpha < \alpha_*\rangle$ where $\mathscr{P}_\alpha \subseteq [u_\alpha]^{\le\lambda}, u_\alpha \subseteq \alpha$, without loss of generality $\mathscr{P}_\alpha$ is closed under subsets (but is not necessarily an ideal)

(b) $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \alpha_*, \beta < \alpha_*\rangle$ is a $(<\lambda)$-support iteration let $\mathbb{P}_{\mathbf{q}} = \mathbb{P}_{\mathbf{q},\alpha(\mathbf{q})}$ and $\mathbb{P}_{0,\alpha} = \mathbb{P}_\alpha, \mathbb{Q}_{0,\alpha} = \underset{\sim}{\mathbb{Q}}_\alpha$

(c) each of $\mathbb{P}_\alpha$ is strategically $(<\lambda)$-complete and λ^+-c.c.

(d) $\underset{\sim}{\eta}_\beta \in \Pi\bar\theta$ is the generic of $\mathbb{Q}_\beta$ where $\underset{\sim}{\eta}_\beta$, the generic of $\mathbb{Q}_p$ (defined in clause (e) below) is $\cup\{\eta_p : p \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}_\beta}\}$

(e) if $\mathbf{G} \subseteq \mathbb{P}_\beta$ is generic over $\mathbf{V}$ then $\underset{\sim}{\eta}_\alpha[\mathbf{G}]$ in $(\Pi\bar\theta, <_{J^{\rm bd}_\lambda})$ dominate every $\nu \in \mathbf{V}[\langle \underset{\sim}{\eta}_\gamma : \gamma \in u\rangle]$ when $u \in \mathscr{P}_\alpha$; moreover, in $\mathbf{V}[\mathbf{G}], \underset{\sim}{\mathbb{Q}}_\beta[\mathbf{G}]$ is the subforcing of $\mathbb{Q}_{\bar\theta}$ consisting of the $p \in \mathbb{Q}_\beta$ such that: for some $\bar s, \underset{\sim}{f}, \eta_p$ (so $\eta_p = \eta$, etc.) we have

  (α) $p = (\eta, f) = (\eta_p, f_p)$ so $\eta \in \prod\limits_{\varepsilon<\zeta} \theta_\varepsilon$ for some $\zeta < \lambda$

  (β) $\bar s = \langle (u_i, f_i) : i < i_*\rangle$

  (γ) $i_* < \lambda, u_i \in \mathscr{P}_\beta, \eta \triangleleft f_i \in \Pi\bar\theta$ and $f_i \in \mathbf{V}[\langle \underset{\sim}{\eta}_\gamma[\mathbf{G}] : \gamma \in u_i\rangle]$

  (δ) $f = \sup\{f_i : i < i_*\}$, i.e. $\varepsilon < \lambda \Rightarrow f(\varepsilon) = \cup\{f_i(\varepsilon) : i < i_*\}$

(f) for $\alpha \le \alpha_*, \mathbb{P}_{2,\alpha}$ is the completion of $\mathbb{P}_\alpha$; we can express it via transforming $\mathbb{P}_\alpha$ to a complete Boolean Algebra, or say:

  $(*)_1$ the elements of $\mathbb{P}_{2,\alpha}$ are of the form $\mathbf{B}(\ldots, \underset{\sim}{\eta}_\gamma, \ldots)_{i<\ell(*)}$ where:

    (α) $i(*) \le \lambda$

    (β) $\gamma_i \in \mathscr{U}$ for $i < i_*$

    (γ) $\mathbf{B}$ is a λ-Borel function from ${}^{i(*)}(\Pi\bar\theta)$ into $\{0,1\} = \{\text{false}, \text{true}\}$; $\mathbf{B}$ is from $\mathbf{V}$, of course, such that $\nVdash_{\mathbb{P}_{\mathbf{q}}}$ "$\mathbf{B}(\ldots, \underset{\sim}{\eta}_{\gamma_i}, \ldots)_{i<i(*)} = 0$"

  $(*)_2$ the order is natural: $\mathbb{P}_{2,\alpha} \models$ "$\mathbf{B}_1(\ldots, \underset{\sim}{\eta}_{\gamma(i,1)}, \ldots)_{i<i(*)} \le \mathbf{B}_2(\ldots, \underset{\sim}{\eta}_{\gamma(i,2)}, \ldots)_{i<i(2)}$" iff $\Vdash_{\mathbb{P}_\alpha}$ "if $\mathbf{B}_2(\ldots, \underset{\sim}{\eta}_{\gamma(i,2)}[\underset{\sim}{\mathbf{G}}, \ldots)_{i<i(1)}$ is equal to 1 then so is $\mathbf{B}_1(\ldots, \underset{\sim}{\eta}_{\gamma(i,1)}, \ldots)_{i<i(1)}$"

(g) for $\mathscr{U} \subseteq \alpha_*$ let $\mathbb{P}_{\mathscr{U}}$ be the subforcing of $\mathbb{P}_{2,\alpha(\mathbf{q})}$ consist of $\{\mathbf{B}(\ldots, \underset{\sim}{\eta}_{\gamma(i)}, \ldots)_{i<i(*)} \in \mathbb{P}_{\alpha(\mathbf{q})} : i(*) \le \lambda$ and $\gamma_i \in \mathscr{U}$ for every $i < i(*)\}$.

{z35}

{z32} **Claim 4.2.** *1) For any sequence $\langle u_\alpha, \mathscr{P}_\alpha : \alpha < \alpha_*\rangle$ as above, i.e. as in clause (a) of Definition 4.1, there is one and only one $\mathbf{q}$ as above and the $\mathbb{P}, \mathbb{P}^\bullet_{\mathbf{q},\mathscr{U}}$'s are as demanded.*

2) For every $\alpha \le \alpha_*$ the set $\mathbb{P}^\bullet_\alpha$ of $p \in \mathbb{P}_\alpha$ satisfying the following is dense:

(a) $\eta_p, i_p, \langle u_{p,i} : i < i_p \rangle$ *are objects (not just* $\mathbb{P}_\alpha$*-names)*

(b) *each* $\underset{\sim}{f}_i$ *has the form* $\mathbf{B}(\ldots, \underset{\sim}{\eta}_{\gamma)i,1)}, \ldots)_{j<j(*)\leq\lambda}$ *where* $\{\gamma(i,j) : j < j(i)\} \subseteq u_{p,i}$.

3) Above for every $v \subseteq \alpha$ *and* $j_* < \lambda$ *the set of* $p \in \mathbb{P}^\bullet_\alpha$ *such that* $v \subseteq \mathrm{dom}(p) \wedge (\forall \beta \in \mathrm{dom}(p))(\ell g(\eta_{p(\beta)}) > j_*)$ *is dense.*

4) $\mathbb{P}_{\mathbf{q},1,\alpha} \lessdot \mathbb{P}_{\mathbf{q},2,\alpha}$ *moreover* $\mathbb{P}_{\mathbf{q},1,\alpha}$ *is dense in* $\mathbb{P}^\bullet_{\mathbf{q},2,\alpha}$ *and* $\mathscr{U}_1 \subseteq \mathscr{U}_2 \subseteq \alpha_{\mathbf{q}} \Rightarrow \mathbb{P}_{\mathbf{q},\mathscr{U}_1} \lessdot \mathbb{P}_{\mathbf{q},\mathscr{U}_2} \lessdot \mathbb{P}_{\mathbf{q},\alpha}$ *so* $\mathbb{P}_{\mathbf{q},\{\beta:\beta<\alpha\}} = \mathbb{P}_{\mathbf{q},2,\alpha}$ *and* $|\mathbb{P}_{\mathbf{q},\mathscr{U}}| \leq |\mathscr{U}|^\lambda$.

5) If $\alpha < \alpha_*$ *and* $u \in \mathscr{P}_\alpha$ *then* $\underset{\sim}{\eta}_\alpha \in \Pi\bar{\theta}$ *dominate every* $\nu \in (\Pi\bar{\theta})^{\mathbf{V}[\underset{\sim}{\bar{\eta}}\restriction u]}$.

6) Assume $\mathbf{G} \subseteq \mathbb{P}_{\mathbf{q}}$ *is generic over* $\mathbf{V}, \eta_\alpha = \underset{\sim}{\eta}_\alpha[\mathbf{G}]$ *and* $\eta'_\alpha \in (\Pi\bar{\theta})^{\mathbf{V}[\mathbf{G}]}$ *for* $\alpha < \alpha_*$ *and* $\{(\alpha, \varepsilon) : \alpha < \alpha_*, \varepsilon < \alpha$ *and* $\eta_\alpha(\varepsilon) \neq \eta'_\alpha(\varepsilon)\}$ *has cardinality* $< \lambda$. *__Then__ for some (really unique)* $\mathbf{G}'$ *we have* $\mathbf{G}' \subseteq \mathbb{P}_{\mathbf{q}}$ *is generic over* $\mathbf{V}$ *and* α.

Proof. We prove this claim by induction on α_*.
1) With iteration $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \alpha_*, \beta < \alpha_* \rangle, \langle \underset{\sim}{\eta}_\beta : \beta < \alpha_* \rangle$, are defined in clause (b),(d),(e) of Definition 4.1. Now for clause (c) $\mathbb{P}_\beta$ are strategically $(< \lambda)$-complete and λ^+-c.c. follows by the iteration being $(< \kappa)$-support and the choice of the $\underset{\sim}{\mathbb{Q}}_\beta$'s. {z32}

Note that we do not claim $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\mathbb{Q}}_\alpha$ is strategically $(< \lambda)$-complete because the memory is partial; however (recalling the induction hypothesis on α_*):

$(*)$ the set $\{p \in \mathbb{P}_\alpha$: if $\beta \in \mathrm{dom}(p)$ then for $p(\beta)$ there are $\underset{\sim}{\bar{s}}, \underset{\sim}{f}, \underset{\sim}{\eta}$ as in 4.1(e) such that: {z32}

(a) $\underset{\sim}{\eta} \in \bigcup_\zeta \prod_{\varepsilon+\zeta} \theta_\varepsilon$ is an object, not just a $\mathbb{P}_\alpha$-name

(b) $\underset{\sim}{\bar{s}} = \langle (u_i, \underset{\sim}{f}_1) : i < i_* \rangle$, so $\langle u_i : i < i_* \rangle$ is an object

(c) $\underset{\sim}{f}_i$ is a $\mathbb{P}_\alpha$-name of the form $\mathbf{B}_i(\ldots, \underset{\sim}{\eta}_{\gamma(i,\varepsilon)}, \ldots)_{\varepsilon<\varepsilon(i)\leq\lambda}, \mathbf{B}_i$ an object as well as $\gamma(i, \varepsilon) \in u_i$.

For clause (c) we use "$\mathbb{P}_\alpha$ satisfies the λ^+-c.c."

Also the rest is easy. $\square_{4.2}$

{z38}

Theorem 4.3. *For any ordinal* α_* *there is a quadruple* $(\mathbf{q}, \delta_*, \mathscr{U}, h)$ *such that:*

(A) (a) $\mathbf{q} \in \mathbf{Q}_{\lambda,\bar{\theta}}$ *and let* $\delta_* = \ell g(\mathbf{q})$

(b) $\mathscr{U} \subseteq \delta_*$ *has order type* α_*

(c) h *is the order preserving function from* α_* *onto* $\mathscr{U}$

(d) *if* $\alpha \in \mathscr{U}$ *then* $\mathscr{U} \cap \alpha \in \mathscr{P}_{\mathbf{Q},\alpha}$

(B) *if* $\mathscr{U}_1 \subseteq \mathscr{U}, \mathscr{U}_2 \subseteq \mathscr{U}$ *is an initial segment of* $\mathscr{U}, \mathrm{otp}(\mathscr{U}_1) = \mathrm{otp}(\mathscr{U}_2)$ *and* g *is the order preserving function from* $\mathscr{U}_1$ *onto* $\mathscr{U}_2$, *__then__* g *induces an isomorphism* $\hat{g}$ *from* $\mathbb{P}_{\mathbf{q},\mathscr{U}_2}$ *onto* $\mathbb{P}_{\mathbf{q},\mathscr{U}_2}$ *mapping* $\underset{\sim}{\eta}_\beta$ *to* $\underset{\sim}{\eta}_{g(\beta)}$ *for* $\beta \in \mathscr{U}_2$.

Proof. By 2.12. $\square_{4.3}$ {c52}

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