INFINITE MONOCHROMATIC SUMSETS FOR COLOURINGS OF THE REALS

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Abstract. N. Hindman, I. Leader and D. Strauss proved that it is consistent that there is a colouring of $\mathbb{R}$ with finitely many colours so that no infinite sumset $X+X$ is monochromatic. Our aim in this paper is to prove a consistency result in the opposite direction: we show that, under certain set-theoretic assumptions, for any colouring c of $\mathbb{R}$ with finitely many colours there is an infinite $X \subseteq \mathbb{R}$ so that $c \restriction X+X$ is constant.

1. Introduction

Neil Hindman's famous sumset theorem states that for any finite colouring of the natural numbers $\mathbb{N}$, there is an infinite set X so that all sums of distinct elements of X are coloured the same. There is a striking difference however, if one allows repetitions in the sumsets; let us begin by recalling a famous open problem of J. Owings:

Problem 1.1 ([12]). *Is there a colouring of $\mathbb{N}$ with 2 colours which is not constant on sets of the form $X+X(=\{x+y : x, y \in X\})$ whenever $X \subseteq \mathbb{N}$ is infinite?*

The answer is yes if one is allowed to use 3 colours [6], but, surprisingly, Problem 1.1 is still unsolved. Hence, it is very natural to ask, what happens if instead of the natural numbers we colour the real numbers. The following theorem is a recent result of N. Hindman, I. Leader and D. Strauss:

Theorem 1.2 ([7]). *It is consistent (namely, it follows from $2^{\aleph_0} = |\mathbb{R}| < \aleph_\omega$) that there is a finite colouring c of $\mathbb{R}$ such that c is not constant on any set of the form $X+X$ where $X \subseteq \mathbb{R}$ is infinite.*

It was asked in [7, Question 2.9] if the conclusion of Theorem 1.2 is true in ZFC. The purpose of this paper is to show that the conclusion of Theorem 1.2 can fail in some models of ZFC constructed with the use of an ω_1-Erdős cardinal.

The group $(\mathbb{R}, +)$ is isomorphic to the direct sum of continuum many copies of $(\mathbb{Q}, +)$, and so we have to consider colourings of this direct sum. Thus, let $Q(\kappa)$ denote the group $\bigoplus_\kappa \mathbb{Q}$, i.e. the set of functions with finite support from κ to $\mathbb{Q}$ with the operation of coordinatewise addition, and let the semigroup $N(\kappa) = \bigoplus_\kappa \mathbb{N}$ be defined similarly. As before, if X is a subset of one of the above (semi)groups, let $X+X = \{x+y : x, y \in X\}$, and note that repetitions are allowed in the sumsets.

In order to prove Theorem 1.2, the authors of [7] showed the following:

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Theorem 1.3 ([7]). *For each $n \in \omega$, there is a map $c : Q(\aleph_n) \to 2^{4+n} \cdot 9$ such that $c \restriction X + X$ is not constant for any infinite $X \subseteq Q(\aleph_n)$.*

Now, our main result is the following:

Main Theorem. *Consistently relative to an ω_1-Erdős cardinal, for any $c : N(2^{\aleph_0}) \to r$ with r finite there is an infinite $X \subseteq N(2^{\aleph_0})$ so that $c \restriction X + X$ is constant.*

Since $N(2^{\aleph_0})$ embeds into $\mathbb{R}$, we immediately have the desired conclusion.

Corollary. *Consistently relative to an ω_1-Erdős cardinal, for any $c : \mathbb{R} \to r$ with r finite there is an infinite $X \subseteq \mathbb{R}$ so that $c \restriction X + X$ is constant.*

It would be interesting to know if there is a proof of consistency relative to ZFC without any extra large cardinal assumptions. We also mention, that it is possible that the conclusion of our main theorem with $r = 2$ follows from ZFC alone, and our paper concludes with further open problems.

Let us briefly mention the related problem of studying *uncountable* sumsets. Starting with [7] a sequence of papers [9, 14, 3, 4, 10, 2] elaborates on this question and its relatives; we refer the reader to the introduction of [4] for an excellent summary.

In particular, D. Fernández-Bretón and A. Rinot [4] proved a very strong failure of Hindman's theorem in the uncountable: one can colour $\mathbb{R}$ with $\aleph_0$ many colours such that all colours appear on all sumsets of size $2^{\aleph_0}$ (even when repetitions are not allowed in the sumsets); there is also a 2-colouring of $\mathbb{R}$ without uncountable monochromatic sumsets [9, 14]. We emphasize that both of these results are proved in ZFC.

The proof of our Main Theorem is a combination of various ideas from the six authors from between 2015 and 2017. I. Leader and P. A. Russell [11], and independently P. Komjáth[1] proved that if a colouring $c : N(\kappa) \to r$ is *canonical* in some sense[2] on a large set then infinite monochromatic sumsets can be found. Hence, the conclusion of the Main Theorem was known for $N(\kappa)$ instead of $N(2^{\aleph_0})$ where κ is large enough relative to the number of colours r. For example, $\kappa \geq \beth_\omega$ suffices for any finite r, since it allows the application of the Erdős-Rado partition theorem with high exponents; the interested reader can consult [11], or see Step 1 of our proof.

Since $2^{\aleph_0}$ badly fails such strong positive partition relations, other ideas were required. D. Soukup and Z. Vidnyánszky analysed the situation further to see what alternative partition relation might suffice to carry out some form of such a canonization argument. They could prove that $N(2^{\aleph_1})$ satisfies the Main Theorem consistently (using a measurable cardinal and forcing again), or $N((2^{\aleph_0})^+)$ with $r = 2$ in ZFC, but a way to descend to $2^{\aleph_0}$ had eluded them until now.

It was recently at the 6th European Set Theory Conference, that S. Shelah recommended that we look at [13] in response to our enquiry about monochromatic sumsets for colourings of $\mathbb{R}$ and positive partition relations for the continuum. Soon afterwards, the last two authors found a way to combine [13, Theorem 3.1] with all the aforementioned machinery resulting in the Main Theorem.

Throughout the paper, we will use standard notations and facts which can be found in [8]. Our proof is elementary; it only uses independence results as black boxes, so it can be understood without any familiarity with the techniques of forcing, large cardinals etc.

[1] Personal communications.

[2] That is, $c(x)$ only depends on the values of x in its 'support' but not the support itself.

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2. The proof of the Main Theorem

To prove the Main Theorem, we will use a model described in [13, Theorem 3.1 (2)]. Roughly speaking, in this model every finite colouring of finite n-tuples of the continuum can be restricted to an uncountable set W, where the behaviour of the colouring is highly controlled (actually, this can be done simultaneously, for finitely many colourings with different n's). Our strategy in the proof is to find such a set W for an appropriate collection of colourings and, after further thinnings of W, define the desired infinite set $X \subseteq N(2^{\aleph_0})$ with supports in W.

In order to formulate the precise statement of [13, Theorem 3.1 (2)], we need a couple of definitions. As usual, for distinct $s, t \in 2^\omega$ let $\Delta(s,t)$ stand for $\min\{n : s(n) \neq t(n)\}$.

Definition 2.1. *We say that $\bar{t}, \bar{s} \in (2^\omega)^i$ are* similar *iff for all $l_1, l_2, l_3, l_4 < i$:*

(1) $\Delta(\bar{t}(l_1), \bar{t}(l_2)) < \Delta(\bar{t}(l_3), \bar{t}(l_4))$ *iff* $\Delta(\bar{s}(l_1), \bar{s}(l_2)) < \Delta(\bar{s}(l_3), \bar{s}(l_4))$,

(2)
$$\bar{t}(l_3) \restriction n <_{\text{lex}} \bar{t}(l_4) \restriction n \text{ for } n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$$
iff
$$\bar{s}(l_3) \restriction m <_{\text{lex}} \bar{s}(l_4) \restriction m \text{ for } m = \Delta(\bar{s}(l_1), \bar{s}(l_2)),$$

(3)
$$\bar{t}(l_3)(n) = 0 \text{ for } n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$$
iff
$$\bar{s}(l_3)(m) = 0 \text{ for } m = \Delta(\bar{s}(l_1), \bar{s}(l_2)).$$

Now, given $c_i : [W]^i \to r$ and $F : W \to 2^\omega$, we say that c_i is F-canonical *if $c_i(\bar{\alpha}) = c_i(\bar{\beta})$ for all increasing $\bar{\alpha} = \{\alpha_0 < \cdots < \alpha_{i-1}\}, \bar{\beta} = \{\beta_0 < \cdots < \beta_{i-1}\} \subset W$ so that $(F(\alpha_l))_{l<i}$ and $(F(\beta_l))_{l<i}$ are similar.*

Theorem 2.2. [13, Theorem 3.1 (2)] *Suppose that λ is an ω_1-Erdős cardinal in V. Then there is a forcing notion $\mathbb{P}$ so that $V^{\mathbb{P}}$ satisfies the following:*

(1) $2^{\aleph_0} = \lambda$,
(2) $MA_{\aleph_1}$*(Knaster), and*
(3) for all $(c_i)_{i<k}$ so that $c_i : [\lambda]^i \to r$ (with $r, k \in \omega$) there is $W \in [\lambda]^{\aleph_1}$ and injective $F : W \to 2^\omega$ so that $c_i \restriction [W]^i$ is F-canonical.

We mention that $\mathbb{P}$ is of the form $\mathbb{P}_0 * \mathbb{P}_1$ where $\mathbb{P}_0$ is $< \lambda$-closed of size λ, and $\mathbb{P}_1 \in V^{\mathbb{P}_0}$ is ccc. $\mathbb{P}$ collapses no cardinals $\leq \lambda$ so the continuum will be very large. In fact, we do not know if Theorem 2.2 could hold for say $2^{\aleph_0} = \aleph_{\omega+1}$, or how to remove the use of large cardinals.

We will not define $\mathrm{MA}_{\aleph_1}(\mathrm{Knaster})$, because we only need a particular corollary of this axiom.[3]

Theorem 2.3. *Suppose $MA_{\aleph_1}(Knaster)$. Then for any $r<\omega$, and $g:\omega\times\omega_1\to r$ there are $A\in[\omega]^\omega$, $B\in[\omega_1]^{\omega_1}$ so that $g\restriction A\times B$ is constant.*

The conclusion above is abbreviated as $\binom{\omega_1}{\omega}\to\binom{\omega_1}{\omega}^{1,1}_r$. For the interested reader, we mention that $\mathrm{MA}_{\aleph_1}(\mathrm{Knaster})$ implies $\aleph_1<\mathfrak{s}$ by [1, Theorem 7.7] and the discussion there, and $\aleph_1<\mathfrak{s}$ implies $\binom{\omega_1}{\omega}\to\binom{\omega_1}{\omega}^{1,1}_r$ by [5, Claim 2.4]. The only important thing for us is that $\binom{\omega_1}{\omega}\to\binom{\omega_1}{\omega}^{1,1}_r$ holds in our model.

Finally, we introduce some notation. Given $s\in\mathbb{N}^{<\omega}$ and $a\in[2^{\aleph_0}]^{|s|}$, we define $x=s*a\in N(2^{\aleph_0})$ by $\mathrm{supp}(x)=a$ and $x(a(i))=s(i)$ where $\{a(i):i<|s|\}$ is the increasing enumeration of a.

Given $c:N(2^{\aleph_0})\to r$ and $s\in\mathbb{N}^{<\omega}$, we define $c_s:[2^{\aleph_0}]^{|s|}\to r$ by

$$c_s(a)=c(s*a).$$

Proof of Main Theorem. We prove that the conclusion of our Main Theorem holds in any model that satisfies conditions (1)-(3) of Theorem 2.2. So, let us fix a colouring $c:N(2^{\aleph_0})\to r$ with some $r<\omega$.

Step 1. We define $s_l\in\mathbb{N}^{<\omega}$ by $s_l=(2,2,\dots,2,2,4,\dots,4)$ where $2l$-many 2 are followed by $(r-l)$-many 4 for each $l\le r$; see Figure 1. So, for a fixed r, we defined $(r+1)$-many patterns.

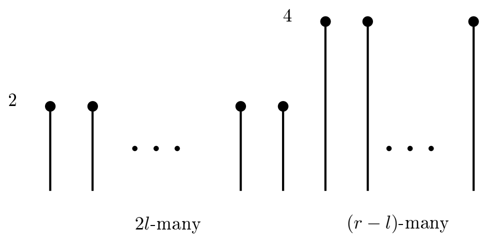

FIGURE 1. The pattern s_l.

E.g. for $r=2$ we let $s_0=(4,4)$, $s_1=(2,2,4)$ and $s_2=(2,2,2,2)$.

Now $c_{s_l}:[2^{\aleph_0}]^{r+l}\to r$ so there is some $W\in[2^{\aleph_0}]^{\aleph_1}$ and injective $F:W\to 2^\omega$ so that $c_{s_l}\restriction[W]^{r+l}$ is F-canonical for all $l\le r$.

Remark 2.4. *If c_{s_l} would be constant on W for all $l\le r$ then we could finish the proof rather fast, as noticed by Komjáth, Leader and Russell. Indeed, say we are in the case of $r=2$. Then two out of the colourings c_{s_0}, c_{s_1} and c_{s_2} will have the same constant value, say both c_{s_1} and c_{s_2} are constant 0. Then pick $\alpha<\beta<\gamma_0<\gamma_1<\dots\in W$ and let $x_i=\{(\alpha,1),(\beta,1),(\gamma_i,2)\}$. It is easy to see that*

$$c(2x_i)=c_{s_1}(\alpha,\beta,\gamma_i)=0=c_{s_2}(\alpha,\beta,\gamma_i,\gamma_j)=c(x_i+x_j).$$

[3] The Knaster property and Martin's axiom are covered by [8] in detail.

So $c \restriction X + X = 0$ for $X = \{x_i : i < \omega\}$. Unfortunately, being F-canonical is rather far from being constant, so we are far from done.

Notation: we let $\Delta_F(\alpha, \beta) = \Delta(F(\alpha), F(\beta))$ for $\alpha \neq \beta \in W$. Also, let $\mathrm{stp}_F(\alpha_0, \dots, \alpha_k)$ denote the similarity type (see Definition 2.1) of $(F(\alpha_0), \dots, F(\alpha_k))$ for some $\alpha_0 < \alpha_1 < \dots < \alpha_k$.

Step 2. Select $A_0, \dots, A_{r-1} \subset W$ so that

(1) $F''A_l < F''A_k$ (in 2^ω) for all $l < k < r$,
(2) $|A_l| = \aleph_1$ for $l \leq r$, and
(3) there is an increasing sequence $\nu_l < \omega$ (for $l < r - 1$) so that
- $\Delta_F(\alpha, \alpha') = \nu_l$ for all $\alpha \in A_l, \alpha \in A_k$ with $l < k < r$, and
- $\Delta_F(\alpha, \alpha') > \nu_{r-2}$ for all $\alpha, \alpha' \in A_l$ with $l < r$.

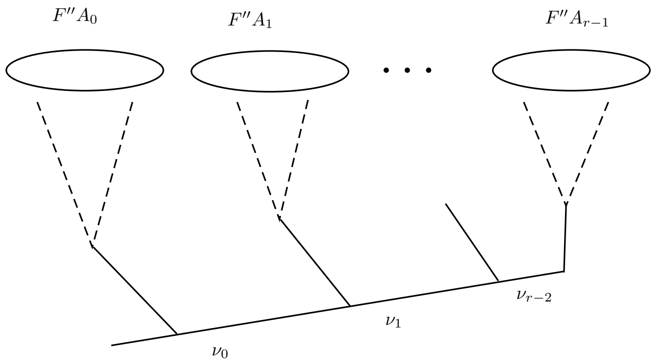

FIGURE 2. The r uncountable blocks from W with nicely ordered images.

See Figure 2. Now, this ensures that there is a single similarity type ρ_0 so that $\mathrm{stp}_F(\alpha_0, \dots, \alpha_{r-1}) = \rho_0$ for all $(\alpha_0, \dots, \alpha_{r-1})_< \in A_0 \times \dots \times A_{r-1}$, where we use the notation $\{\alpha_i\}_<$ or $(\alpha_i)_<$ to denote $\alpha_0 < \alpha_1 < \dots$, in other words, the fact that the sequence of α_i's is increasing in the ordering of ordinals.

Step 3. Our strategy is similar to Remark 2.4, and we will be interested in the types of tuples of the following special form:

Definition 2.5. *An ℓ-*candidate with respect to $\mathbf{A} = (A_l)_{l \leq r}$ *is an $(r+\ell)$-tuple of elements of W where we take two elements from each of $A_0, \dots, A_{\ell-1}$, and a single point from each A_k for $\ell \leq k \leq r$.*

In Step 2, we made sure that all 0-candidates with respect to $\mathbf{A}$ have the same similarity type ρ_0. Next, we will work towards making ℓ-candidates more similar by shrinking each set A_l appropriately. In particular, we need to look at the new splitting levels δ (coming from the first ℓ-many pairs), and make sure that the same values appear when we evaluate branches (in the same position) at δ.

Claim 2.6. *There is $A'_l = \{\alpha^l_i : i \leq \omega\}_< \subseteq A_l$ for each $l < r$ so that*

(1) $F \restriction \bigcup_{l<r} A'_l$ is order preserving and $\sup_{i \in \omega} F(\alpha^l_i) = F(\alpha^l_\omega)$ in 2^ω,
(2) there are strictly increasing $(\delta^l_i)_{i<\omega}$ so that

(a) $\Delta_F(\alpha^l_i, \alpha^l_j) = \delta^l_i$ *for all* $i < j \leq \omega$, *and*
(b) $\delta^l_i < \delta^k_i < \delta^l_{i+1}$ *for* $l < k \leq r$ *and* $i < \omega$,

(3) there are $u_l : \{l+1, \ldots, r-1\} \to 2$ *for* $l < r$ *such that* $u_l(k) = F(\alpha)(\delta^l_i)$ *for any* $\alpha \in A_k$ *and* $l < k < r$.

See Figure 3 for a picture; we have marked the important splitting levels on later branches, with the same symbols marking the same values.

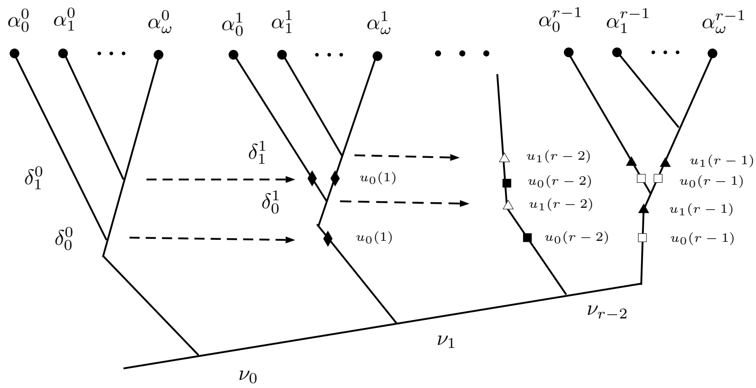

FIGURE 3. Selecting subsequences with the important positions marked.

Proof. First, shrink A_0 to some A''_0 of type $\omega+1$ so that $F \restriction A''_0$ is increasing, continuous, and (2)(a) is satisfied; next, replace $A_1, \ldots, A_{r-1}$ by their uncountable subsets so that every element of their union is greater than every element of A''_0 (we use the notation $A_1, \ldots, A_{r-1}$ for the shrinked sets as well). Note that (2)(a) will still hold no matter how we shrink A''_0 further (just the splitting sequence is redefined).

Define

$$g_{0k} : (A''_0 \setminus \{\max A''_0\}) \times A_k \to 2$$

for $0 < k < r$ by $g_{0k}(\alpha^0_i, \beta) = F(\beta)(\delta^0_i) \in 2$. Now, apply $\binom{\omega_1}{\omega} \to \binom{\omega_1}{\omega}^{1,1}_2$ to each $g_{01}, g_{02}, \ldots, g_{0r-1}$, successively shrinking $A''_0 \setminus \{\max A''_0\}$ $(r-1)$-many times and each A_k once. The constant values $u_0(k)$ for g_k will define $u_0 : r \setminus 1 \to 2$. Note that we made sure that (3) is satisfied no matter how we shrink A''_0 or A_l for $0 < l < r-1$ in later steps.

We move onto A_1 and proceed similarly: first, shrink A_1 to a type $\omega+1$ sequence so that the restriction of F is increasing, continuous, and (2)(a) is satisfied. Then shrink the sets $A_2, \ldots, A_{r-1}$ so that every element of their union is greater than the elements of A_1. Next, shrink each set further so that (3) holds for $l = 1$ using $\binom{\omega_1}{\omega} \to \binom{\omega_1}{\omega}^{1,1}_2$; this defines u_1. Then we move to A_2 etc.

Notice that this process yields $A'_l \subseteq A_l$ so that (1), (2)(a) and (3) holds. Finally, we do a final (simultaneous) shrinking to ensure (2)(b).

□

Again, to make notation lighter, we drop the primes i.e., we forget about the original uncountable A_l defined in Step 2, and call our new $\omega+1$-sequences from Claim 2.6 A_l (instead of A'_l).

Now, what decides the similarity type of an ℓ-candidate with respect to $\mathbf{A} = (A_l)_{l \leq r}$? We will focus on ℓ-candidates of a certain special form:

Definition 2.7. *A* canonical ℓ-candidate *is an ℓ-candidate with respect to $\boldsymbol{A}$ of the form*

$$\bar{\alpha} = (\alpha^0_{i_0}, \alpha^0_{j_0}, \dots, \alpha^{\ell-1}_{i_{\ell-1}}, \alpha^{\ell-1}_{j_{\ell-1}}, \alpha^{\ell}_{i_\ell}, \dots, \alpha^{r-1}_{i_{r-1}})$$

so that

(1) $i_k < j_k \leq \omega$ for $k < \ell$ and $i_0 \leq i_1 \leq \dots \leq i_{r-1} \leq \omega$, and
(2) if $j_k \neq \omega$ then $\delta^k_{j_k} > \delta^{\ell-1}_{i_{\ell-1}}$.

For example, if we fix $\ell < k < r$ and $i < \omega$ then

$$\{\alpha^l_0, \alpha^l_\omega : l < \ell\} \cup \{\alpha^l_i : \ell \leq l < k\} \cup \{\alpha^l_\omega : k \leq l < r\}$$

is a canonical ℓ-candidate. The corresponding indeces are $i_l = 0 < j_l = \omega$ for $l < \ell$, $i_l = i$ for $\ell \leq l < k$ and $i_k = \omega$ for $k \leq l < r$. Condition (2) is vacuously satisfied. These particular canonical ℓ-candidates will have an important role later.

Now, note that any 0-candidate is a canonical 0-candidate and the type of these are fixed already. Our next claim is that for a fixed sequence of i's the similarity type of a canonical ℓ-candidate does not depend on the choice of the j's:

Claim 2.8. *Let $\boldsymbol{i} = \{i_0 \leq i_1 \leq \dots \leq i_{r-1}\}$ be a sequence of ordinals $\leq \omega$ and $\bar{\alpha} = (\alpha^0_{i_0}, \alpha^0_{j_0}, \dots, \alpha^{\ell-1}_{i_{\ell-1}}, \alpha^{\ell-1}_{j_{\ell-1}}, \alpha^{\ell}_{i_\ell}, \dots, \alpha^{r-1}_{i_{r-1}})$ and $\bar{\alpha}' = (\alpha^0_{i_0}, \alpha^0_{j'_0}, \dots, \alpha^{\ell-1}_{i_{\ell-1}}, \alpha^{\ell-1}_{j'_{\ell-1}}, \alpha^{\ell}_{i_\ell}, \dots, \alpha^{r-1}_{i_{r-1}})$ be two canonical ℓ-candidates. Then* $\mathrm{stp}(\bar{\alpha}) = \mathrm{stp}(\bar{\alpha}')$.

Proof. Observe first that for a canonical ℓ-candidate $\bar{\alpha}$, $\mathrm{stp}_F(\bar{\alpha})$ is decided by the values of the $2k$-sequences

$$(F(\alpha^0_{i_0})(\delta^k_{i_k}), F(\alpha^0_{j_0})(\delta^k_{i_k}), \dots, F(\alpha^{k-1}_{i_{k-1}})(\delta^k_{i_k}), F(\alpha^{k-1}_{j_{k-1}})(\delta^k_{i_k}))$$

for each $k < \ell$ (see Figure 4): indeed, by the definition of similarity we only need to consider the values of the reals in $F''\bar{\alpha}$ at splitting levels and (3) of Claim 2.6 guarantees that for the splitting at $\delta^k_{i_k}$ we only need to check the values of reals below $F''A_k$. Also, as $\Delta_F(\alpha^m_{j_m}, \alpha^m_\omega) = \delta^m_{j_m} > \delta^k_{i_k}$ whenever $m < k$ and $\alpha^m_{j_m} \neq \alpha^m_\omega$, we get that

$$(F(\alpha^0_{i_0})(\delta^k_{i_k}), F(\alpha^0_{\omega})(\delta^k_{i_k}), \dots, F(\alpha^{k-1}_{i_{k-1}})(\delta^k_{i_k}), F(\alpha^{k-1}_{\omega})(\delta^k_{i_k})) =$$
$$= (F(\alpha^0_{i_0})(\delta^k_{i_k}), F(\alpha^0_{j_0})(\delta^k_{i_k}), \dots, F(\alpha^{k-1}_{i_{k-1}})(\delta^k_{i_k}), F(\alpha^{k-1}_{j_{k-1}})(\delta^k_{i_k})).$$

The first part of the equation does not depend on the sequence $j_0, \dots, j_{l-1}$ so the second part neither. This, and our first observation, finishes the proof of the claim.

□

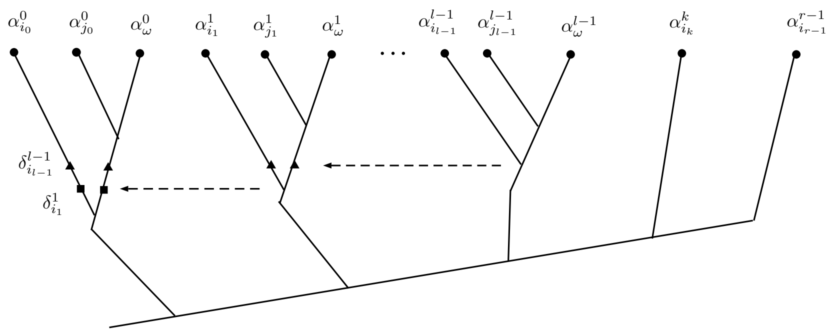

FIGURE 4. The important positions marked for deciding similarity types.

For a sequence $\mathbf{i} = \{i_0 \le i_1 \le \cdots \le i_{r-1}\}$, let us define $\mathfrak{e}(\mathbf{i}, k) : 2k \to 2$ by

$$\mathfrak{e}(\mathbf{i}, k)(2m) = F(\alpha^m_{i_m})(\delta^k_{i_k}) \text{ and } \mathfrak{e}(\mathbf{i}, k)(2m+1) = F(\alpha^m_\omega)(\delta^k_{i_k})$$

for $m < k$ and $i_k \neq \omega$; if $i_k = 0$, we declare $\mathfrak{e}(\mathbf{i}, k) = \emptyset$.

If we work with multiple $\mathbf{A}$ sequences at the same time then we mark the above functions as $\mathfrak{e}_{\mathbf{A}}(\mathbf{i}, k)$.

Now, Claim 2.8 implies that the similarity type of a canonical ℓ-candidate with respect to $\mathbf{A}$ is decided solely by the sequence $(\mathfrak{e}_{\mathbf{A}}(\mathbf{i}, k))_{k<r}$.

Step 4. Next, we thin out each A_l further so that the function $\mathfrak{e}(\cdot, k)$, at a fixed k, gives the same values for *any choice* of $\mathbf{i}$.

Claim 2.9. *There are $\tilde{A}_l = \{\tilde{\alpha}^l_i : i \le \omega\}_< \subseteq A_l$ for $l < r$ so that $\tilde{\mathbf{A}} = (\tilde{A}_l)_{l<r}$ still satisfies the conditions of Claim 2.6, and there are $\mathfrak{e}(k) : 2k \to 2$ so that for any $\mathbf{i} = \{i_0 \le i_1 \le \cdots \le i_{r-1}\}$*

$$\mathfrak{e}_{\tilde{\mathbf{A}}}(\mathbf{i}, k) = \mathfrak{e}(k),$$

or in other words, if $\delta = \Delta_F(\tilde{\alpha}^k_{i_k}, \tilde{\alpha}^k_\omega)$, we have

$$\mathfrak{e}(k)(2m) = F(\tilde{\alpha}^m_{i_m})(\delta) \text{ and } \mathfrak{e}(k)(2m+1) = F(\tilde{\alpha}^m_\omega)(\delta)$$

for $m < k$.

Proof. We can do this by a simple application of the classical (r-dimensional) Ramsey theorem. Note that we associated with any choice of $\mathbf{i} = \{i_0 \le i_1 \le \cdots \le i_{r-1}\} \in [\omega]^r$ a sequence $\psi(\mathbf{i}) = (\mathfrak{e}(\mathbf{i}, k))_{k<r}$. So, we defined

$$\psi : [\omega]^r \to \prod_{k<r} (2^{2k} \cup \{\emptyset\}),$$

and hence there is an $I \in [\omega]^\omega$ and $(\mathfrak{e}(k))_{k<r}$ so that $\psi \restriction [I]^r$ is constant $(\mathfrak{e}(k))_{k<r}$.

Now, we simply let $\tilde{A}_l = \{\alpha^l_i : i \in I \cup \{\omega\}\}$ for $l < r$. It is easily checked that the conclusions of Claim 2.6 are still satisfied.

□

Again, we drop the tilde notation and assume that $\mathbf{A}$ satisfies the conclusions of Claim 2.9. It follows from these conclusions that any canonical ℓ-candidate with respect to $\mathbf{A}$ has a single type ρ_ℓ, no matter how we choose the defining sequence $\mathbf{i}$.

Step 5. We are ready to find our infinite monochromatic sumset. Recall that, as the colourings c_{s_l} are F-canonical on $[\bigcup_{i<r} A_i]^{r+l}$ the colour of a sequence is determined solely by its similarity type, hence, each similarity type ρ_ℓ is assigned a colour by c from r and we have $r+1$ such types. So, there is $\ell < k \le r$ so that ρ_ℓ and ρ_k are coloured the same.

Claim 2.10. *There is an $X = \{x_i : i < \omega\} \subseteq N(2^{\aleph_0})$ with $\mathrm{supp}(x_i) \subset W$ so that*

(1) *$\mathrm{supp}(2x_i)$ has similarity type ρ_ℓ and $2x_i = s_\ell * \mathrm{supp}(2x_i)$, and*
(2) *$\mathrm{supp}(x_i + x_j)$ has similarity type ρ_k and $x_i + x_j = s_k * \mathrm{supp}(x_i + x_j)$*

for all $i < j < \omega$.

In particular, $c(2x_i) = c_{s_\ell}(\rho_\ell) = c_{s_k}(\rho_k) = c(x_i + x_j)$ so $c \restriction X + X$ is constant as desired.

Proof. We will make sure that $\mathrm{supp}(2x_i)$ is a canonical ℓ-candidate and $\mathrm{supp}(x_i + x_j)$ is a canonical k-candidate with respect to $\mathbf{A}$. We let

$$\mathrm{supp}(x_i) = \{\alpha^l_0, \alpha^l_\omega : l < \ell\} \cup \{\alpha^l_i : \ell \le l < k\} \cup \{\alpha^l_\omega : k \le l < r\}$$

and define x_i as follows:

$$x_i(\alpha) = \begin{cases} 1, & \text{for } \alpha \in \{\alpha_0^l, \alpha_\omega^l : l < \ell\}, \\ 2, & \text{for } \alpha \in \{\alpha_i^l : \ell \leq l < k\} \cup \{\alpha_\omega^l : k \leq l < r\}. \end{cases}$$

It is clear that $\mathrm{supp}(2x_i) = \mathrm{supp}(x_i)$ is a canonical ℓ-candidate with respect to $\mathbf{A}$ so has type ρ_ℓ. $2x_i = s_\ell * \mathrm{supp}(2x_i)$ by the definition of x_i.

Now

$$\mathrm{supp}(x_i + x_j) = \{\alpha_0^l, \alpha_\omega^l : l < \ell\} \cup \{\alpha_i^l, \alpha_j^l : \ell \leq l < k\} \cup \{\alpha_\omega^l : k \leq l < r\}$$

for any $i < j < \omega$. This is a canonical k-candidate; indeed, we need to check (2) i.e. that $\delta_j^l > \delta_i^{k-1}$ for all $\ell \leq l < k$. This is clear from Claim 2.6 however.

□

This finishes the proof of the Main Theorem.

□

3. Open problems

There are various problems that remain open at this point. In order to state these questions concisely, we introduce the following notation: given some additive structure $(A, +)$, let $h(A)$ denote the minimal r so that there is an r-colouring of A with no monochromatic set of the form $X + X$ for some infinite $X \subseteq A$. Note that the larger $h(A)$ is, the stronger partition property A satisfies.

Now, with this new notation, Owings' problem simply asks if $h(\mathbb{N}) = 2$. It would be interesting to see whether $h(\mathbb{Q}) > 2$, or if $h(\mathbb{R}) > 2$ is provable without extra set theoretic assumptions.

It is easy to see that $h(Q(\kappa)) \leq \aleph_0$, and so the conclusion of our main theorem can be rephrased as $h(\mathbb{R}) = \aleph_0$. Now, we ask the following:

Problem 3.1. *Is the large cardinal assumption necessary in proving the consistency of* $h(\mathbb{R}) = \aleph_0$*?*

Problem 3.2. *Does* $h(\mathbb{R}) = \aleph_0$ *hold if* $2^{\aleph_0}$ *is real valued measurable?*

Problem 3.3. *What is the asymptotic behaviour of the function* $n \mapsto h(Q(\aleph_n))$*? Is it truly exponential (cf. Theorem 1.3)?*

Probably the most intriguing question about large direct sums is the following:

Problem 3.4. *Is* $h(Q(\aleph_\omega)) = \aleph_0$ *provable in ZFC?*

GCH implies $h(Q(\aleph_\omega)) = \aleph_0$ [11], and a positive answer in ZFC would strengthen our main theorem and answer the first two problems above.

The study of Abelian semigroups, and finding larger, uncountable monochromatic sumsets has been already started in [4, 11]. It would also be interesting to see what one can say about unbalanced sumsets i.e., sets of the form $X + Y$ with $X, Y \subseteq A$. More precisely:

Problem 3.5. *Given an additive structure* $(A, +)$ *and* r*, characterise those* (κ, λ) *such that whenever* $c : A \to r$ *then there is* $X, Y \subseteq A$ *with* $|X| = \kappa, |Y| = \lambda$ *and* $c \restriction X + Y$ *constant.*

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