ANOTHER ORDERING OF THE TEN CARDINAL CHARACTERISTICS IN CICHOŃ'S DIAGRAM

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ABSTRACT. It is consistent that

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{add}(\mathcal{M}) = \mathfrak{b} < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}.$$

Assuming four strongly compact cardinals, it is consistent that

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{add}(\mathcal{M}) = \mathfrak{b} < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \\ < \mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{M}) = \mathfrak{d} < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0}.$$

INTRODUCTION

We assume that the reader is familiar with basic properties of Amoeba, Hechler, random and Cohen forcing, and with the cardinal characteristics in Cichoń's diagram, Figure 1.

An arrow between $\mathfrak{x}$ and $\mathfrak{y}$ indicates that ZFC proves $\mathfrak{x} \leq \mathfrak{y}$. Moreover, $\max(\mathfrak{d}, \mathrm{non}(\mathcal{M})) = \mathrm{cof}(\mathcal{M})$ and $\min(\mathfrak{b}, \mathrm{cov}(\mathcal{M})) = \mathrm{add}(\mathcal{M})$. These (in)equalities are the only one provable. More precisely, all assignments of the values $\aleph_1$ and $\aleph_2$ to the characteristics in Cichoń's Diagram are consistent, provided they do not contradict the above (in)equalities. (A complete proof can be found in [BJ95, ch. 7].)

In the following, we will only deal with the ten "independent" characteristics listed in Figure 2 (they determine $\mathrm{cof}(\mathcal{M})$ and $\mathrm{add}(\mathcal{M})$).

Date: December 3, 2017.

2010 *Mathematics Subject Classification.* 03E17.

Key words and phrases. Set theory of the reals, Cichoń's diagram, Forcing, Compact cardinals.

Supported by Austrian Science Fund (FWF): P29575 & I3081 (first author), European Research Council grant ERC-2013-ADG 338821 (second author). The third author is recipient of a DOC Fellowship of the Austrian Academy of Sciences at the Institute of Discrete Mathematics and Geometry, TU Wien. This is publication number 1131 of the second author.

$$\begin{tikzcd}
& \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \arrow[r] & 2^{\aleph_0} \\
& & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[r] \arrow[uu] & \mathrm{add}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{cov}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}$$

FIGURE 1. Cichoń's diagram

$$\begin{tikzcd}
& \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \boxed{} \arrow[r] & \mathrm{cof}(\mathcal{N}) \arrow[r] & 2^{\aleph_0} \\
& & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[uu] \arrow[r] & \boxed{} \arrow[u] \arrow[r] & \mathrm{cov}(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}$$

FIGURE 2. The ten "independent" characteristics.

$$\begin{tikzcd}
& \lambda_2 \arrow[r] & \lambda_4 \arrow[r] & \boxed{} \arrow[r] & \lambda_8 \arrow[r] & \lambda_9 \\
& & \lambda_3 \arrow[u] \arrow[r] & \lambda_6 \arrow[u] & & \\
\aleph_1 \arrow[r] & \lambda_1 \arrow[uu] \arrow[r] & \boxed{} \arrow[u] \arrow[r] & \lambda_5 \arrow[u] \arrow[r] & \lambda_7 \arrow[uu] &
\end{tikzcd}$$

FIGURE 3. The old order.

$$\begin{tikzcd}
& \lambda_3 \arrow[r] & \lambda_4 \arrow[r] & \boxed{} \arrow[r] & \lambda_8 \arrow[r] & \lambda_9 \\
& & \lambda_2 \arrow[u] \arrow[r] & \lambda_7 \arrow[u] & & \\
\aleph_1 \arrow[r] & \lambda_1 \arrow[uu] \arrow[r] & \boxed{} \arrow[u] \arrow[r] & \lambda_5 \arrow[u] \arrow[r] & \lambda_6 \arrow[uu] &
\end{tikzcd}$$

FIGURE 4. The new order.

Regarding the left hand side, it was shown in [GMS16] that consistently

$$(\mathrm{left}_{\mathrm{old}}) \qquad \aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathrm{add}(\mathcal{M}) = \mathfrak{b} < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}.$$

(This corresponds to λ_1 to λ_5 in Figure 3.) The proof is repeated in [GKS], in a slightly different form which is more convenient for our purpose. Let us call this construction the "old construction".

In this paper, building on [She00], we give a construction to get a different order for these characteristics, where we swap $\mathrm{cov}(\mathcal{N})$ and $\mathfrak{b}$:

$$(\mathrm{left}_{\mathrm{new}}) \qquad \aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{add}(\mathcal{M}) = \mathfrak{b} < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}.$$

(This corresponds to λ_1 to λ_5 in Figure 4.)

This construction is more complicated than the old one. Let us briefly describe the reason: In both constructions, we assign to each of the cardinal characteristics of the left hand side a relation R. E.g., we use the "eventually different" relation $\mathrm{R}_4 \subseteq \omega^\omega \times \omega^\omega$ for $\mathrm{non}(\mathcal{M})$. We can then show that the characteristic remains "small" (i.e., is at most the intended value λ in the final model), because all single forcings we use in the iterations are either small (i.e., smaller than λ) or are "R-good". However, $\mathfrak{b}$ (with the "eventually dominating" relation $\mathrm{R}_2 \subseteq \omega^\omega \times \omega^\omega$) is an exception: We do not know any variant of an eventually different forcing (which we need to increase $\mathrm{non}(\mathcal{M})$) which satisfies that all of its subalgebras are R_2-good. Accordingly, the main effort (in both constructions) is to show that $\mathfrak{b}$ remains small.

In the old construction, each non-small forcing is a (σ-centered) subalgebra of the eventually different forcing $\mathbb{E}$. To deal with such forcings, ultrafilter limits of sequences of $\mathbb{E}$-conditions are introduced and used (and we require that all $\mathbb{E}$-subforcings are basically $\mathbb{E}$ intersected with some elementary model, and thus closed under limits of sequences in the model). In the new construction, we have to deal with an additional kind of "large" forcing: (subforcings of) random forcing. Ultrafilter limits do not work any more, but, similarly to [She00], we can use finite additive measures (FAMs) and interval-FAM-limits of random conditions. But now $\mathbb{E}$ doesn't seem wot work with interval-FAM-limits any more, so we replace it with a creature forcing notion $\tilde{\mathbb{E}}$.

We also have to show that $\mathrm{cov}(\mathcal{N})$ remains small. In the old construction, we could use a rather simple (and well understood) relation $\mathrm{R}^{\mathrm{old}}$ and use the fact that all σ-centered forcings are $\mathrm{R}^{\mathrm{old}}$-good: As all large forcings are subalgebras of either eventually different forcing or of Hechler forcing, they are all σ-centered. In the new construction, the large forcings we have to deal with are subforcings of $\tilde{\mathbb{E}}$. But $\tilde{\mathbb{E}}$ is not σ-centered, just (ρ,π)-linked. So we use a different (and more cumbersome) relation R_3, introduced in [OK14], where it is also shown that (ρ,π)-linked forcings are R_3-good.

Regarding the whole diagram, one can build on the construction for ($\mathrm{left}_{\mathrm{old}}$) to get simultaneously different values for all characteristics: Assuming four strongly compact cardinals, the following is consistent (cf. Figure 3):

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0}.$$

This is done in [GKS]; the essential ingredient for expanding the result to the right hand side is the concept of the Boolean ultrapower of a forcing notion. In exactly the same way we can expand our new version ($\mathrm{left}_{\mathrm{new}}$) to the right hand side, where also the characteristics dual to $\mathfrak{b}$ and $\mathrm{cov}(\mathcal{N})$ are swapped. So we get: If four strongly compact cardinals are consistent, then so is the following (cf. Figure 4):

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathfrak{b} < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{N}) < \mathfrak{d} < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0},$$

We closely follow the presentation of [GKS]. Several times, we refer to [GKS] and to [She00] for details in definitions or proofs. We thank Martin Goldstern and Diego Mejía for valuable discussions.

1. Finitely additive measure limits and the $\tilde{\mathbb{E}}$-forcing.

1.1. FAM-limits and random forcing. We briefly list some basic notation and facts around finite additive measures. (A bit more details can be found in Section 1 of [She00].)

Definition 1.1.
- A "partial FAM" (finitely additive measure) Ξ' is a finitely additive probability measure on a sub-Boolean algebra $\mathcal{B}$ of $\mathcal{P}(\omega)$, the power set of ω, such that $\{n\} \in \mathcal{B}$ and $\Xi'(\{n\}) = 0$ for all $n \in \omega$. We set $\mathrm{dom}(\Xi') = \mathcal{B}$.
- Ξ is a FAM if it is a partial FAM with $\mathrm{dom}(\Xi) = \mathcal{P}(\omega)$.
- For every FAM Ξ and bounded sequence of non-negative reals $\bar{a} = (a_n)_{n\in\omega}$ we can define in the natural way the average (or: integral) $\mathrm{Av}_{\Xi}(\bar{a})$, a non-negative real number.

[She00, 1.2] lists several results that informally say:

(∗) There is a FAM Ξ that assigns the values a_i to the sets A_i (for all i in some index set I) iff for each $I' \subseteq I$ finite and $\epsilon > 0$ there is a large finite $u \subseteq \omega$ such that the counting measure on u for A_i approximates a_i with an error of at most ϵ.

For the size of this u we can give a bound only depending on $|I'|$ and ϵ:

Lemma 1.2. *Given $N, k^* \in \omega$ and $\epsilon > 0$. Then there is an $M \in \omega$ such that: For all FAMs Ξ and $(A_n)_{n<N}$ there is a nonempty $u \subseteq \omega$ of size $\leq M$ such that* $\min(u) > k^*$ *and* $\Xi(A_n) - \epsilon < \frac{|A_n \cap u|}{|u|} < \Xi(A_n) + \epsilon$ *for all $n < N$.*

Proof. We can assume that $\epsilon = \frac{1}{L}$ for an integer L. $\{A_n : n \in N\}$ generates the set algebra $\mathfrak{B} \subseteq \mathcal{P}(\omega)$. Let $\mathcal{X}$ be the set of atoms of $\mathfrak{B}$. So $\mathcal{X}$ is a partition of ω of size $\leq 2^N$. Set $\mathcal{X}' = \{x \in \mathcal{X} : \Xi(x) > 0\}$. Every $x \in \mathcal{X}'$ is infinite, and $\sum_{x\in\mathcal{X}'} \Xi(x) = 1$.

Round $\Xi(x)$ to some number $\Xi^\epsilon(x) = \ell_x \cdot \frac{1}{L\cdot 2^N}$ for some integer $0 \leq \ell_x \leq L \cdot 2^N$, such that $|\Xi(x) - \Xi^\epsilon(x)| < \frac{1}{L\cdot 2^N}$ and $\sum_{x\in\mathcal{X}'} \Xi^\epsilon(x)$ is still 1. So $\sum_{x\in\mathcal{X}'} \ell_x = L \cdot 2^N$, and we construct u consisting of ℓ_x many points in x (for each $x \in \mathcal{X}'$). □

We will use the following variants of (∗), regarding the possibility to extend a partial FAM Ξ' to a FAM Ξ. The proofs are straightforward, if somewhat tedious (cf. [She00, 1.2(G) and 1.7]).

Fact 1.3. Let Ξ' be a partial FAM, and I some index set.

(a) Fix for each $i \in I$ some $A_i \subseteq \omega$.
If $A \cap \bigcap_{i\in I'} A_i \neq \emptyset$ for all $I' \subseteq I$ finite and $A \in \mathrm{dom}(\Xi')$ with and $\Xi'(A) > 0$,
then Ξ' can be extended to a FAM Ξ such that $\Xi(A_i) = 1$ for all $i \in I$.

(b) Fix for each $i \in I$ some real b^i and some bounded sequence of non-negative reals $\bar{a}^i = (a^i_k)_{k\in\omega}$.
If for each finite partition $(B_m)_{m<m^*}$ of ω into elements of $\mathrm{dom}(\Xi')$, for each $\epsilon > 0$, $k^* \in \omega$, and $I' \subseteq I$ finite there is a finite $u \subseteq \omega \setminus k^*$ such that
- for all $m < m^*$, $\Xi'(B_m) - \epsilon \leq \frac{|B_m\cap u|}{|u|} \leq \Xi'(B_m) + \epsilon$, and
- for all $i \in I'$, $\frac{1}{|u|} \sum_{k\in u} a^i_k \geq b^i - \epsilon$,

then Ξ' can be extended to a FAM Ξ such that $\mathrm{Av}_\Xi(\bar{a}^i) \geq b^i$ for all $i \in I$.

We first define what it means for a forcing Q to have FAM limits.

Remark 1.4. Intuitively, this means (in the simplest version): Fix a FAM Ξ. We can define for each sequence q_k of conditions that are all "similar" (e.g., have the same stem and measure) a limit $\lim_\Xi \bar{q}$. And we find in the Q-extension a FAM Ξ' extending Ξ, such that $\lim_\Xi(\bar{q})$ forces that the set of k satisfying $P(k) \equiv q_k \in G$ has "large" Ξ'-measure. Up to here, we get the notion used in [GMS16] and [GKS] (but there we use ultrafilters instead of FAMs, and "large" means being in the ultrafilter). However, we need a modification: Instead of single conditions q_k we use a finite sequences $(p_\ell)_{\ell\in I_k}$ (where I_k is a fixed, finite interval); and the condition $P(k)$, which we want to satisfy on a large set, now is $\frac{|\{\ell\in I_k:\, p_\ell\in G\}|}{|I_k|} > b$ for some suitable b. This is the notion used implicitly in [She00].

Notation. Let T^* be a compact subtree of $\omega^{<\omega}$, for example $T^* = 2^{<\omega}$. Let $s, t \in T^*$.
- $t \rhd s$ means "t is immediate successor of s".
- $|s|$ is the length of s (i.e.: the height, or level, of s).
- $[t]$ is the set of nodes in T^* comparable with t.
- For a subtree $T \subseteq T^*$ we set $\lim(T) = \{x \in \omega^\omega : (\forall n \in \omega) x \restriction n \in T\}$.
- Leb is the canonical measure on the Borel subsets of $\lim(T^*)$. For a subtree T of T^*, we also write $\mathrm{Leb}(T)$ instead of $\mathrm{Leb}(\lim(T))$.[1]

We will assume the following setup:

Assumption 1.5.
- Q is a forcing notion,
- $Q' \subseteq Q$ is dense and the domain of functions $\mathtt{trunk}$ and loss, where $\mathtt{trunk}(q) \in H(\aleph_0)$ and $\mathrm{loss}(q)$ is a rational.
- For each $\epsilon > 0$ the set $\{q \in Q' : \mathrm{loss}(q) < \epsilon\}$ is dense (in Q' and thus in Q).

[1] I.e., we define $\mathrm{Leb}([s])$ by induction on the height of $s \in T^*$ as follows: $\mathrm{Leb}(T^*) = 1$, and if s has n many immediate successors in T^*, then $\mathrm{Leb}([t]) = \frac{1}{n\cdot\mathrm{Leb}([s])}$ for any such successor. This defines a measure on each basic clopen set, which in turn defines a (probability) measure on the Borel subsets of $\lim(T^*)$ (a closed subset of ω^ω).

- $\{p \in Q' : (\mathrm{trunk}(p), \mathrm{loss}(p)) = (\mathrm{trunk}^*, \mathrm{loss}^*)\}$ is $\lfloor \frac{1}{\mathrm{loss}^*} \rfloor$-linked (i.e., each $\lfloor \frac{1}{\mathrm{loss}^*} \rfloor$ many such conditions are compatible).
- $\bar{I} = (I_k)_{k\in\omega}$ is an interval partition of ω such that $|I_k|$ converges to infinity.[2]

In this paper, Q will be one of the following two forcing notions: random forcing, or $\tilde{\mathbb{E}}$ (as defined in Definition 1.11). We will now specify the instance of random forcing that we will use:

Definition 1.6.
- A random condition is a tree $T \subseteq 2^{<\omega}$ such that $\mathrm{Leb}(T \cap [t]) > 0$ for all $t \in T$.
- $\mathrm{trunk}(T)$ is the stem of T (i.e., the shortest splitting node).
- If $\mathrm{Leb}(T) = \mathrm{Leb}([\mathrm{trunk}(T)])$, we set $\mathrm{loss}(T) = 0$. Otherwise, let m be the maximal natural number such that
$$\mathrm{Leb}(T) \geq \mathrm{Leb}([\mathrm{trunk}(T)])(1 - \frac{1}{m})$$
and set[3] $\mathrm{loss}(T) = \frac{1}{m}$. In any case, $\mathrm{Leb}(T) \geq 2^{-|\mathrm{trunk}(T)|}(1 - \mathrm{loss}(T))$.

Definition 1.7. Fix Q and functions (trunk, loss) as in Assumption 1.5, a FAM Ξ and a function $\lim_\Xi : Q^\omega \to Q$. Let us call the objects mentioned so far a "limit setup". Let a $(\mathrm{trunk}^*, \mathrm{loss}^*)$-sequence be a sequence $(q_\ell)_{\ell\in\omega}$ of Q-conditions such that $\mathrm{trunk}(q_\ell) = \mathrm{trunk}^*$ and $\mathrm{loss}(q_\ell)) = \mathrm{loss}^*$ for all $\ell \in \omega$.

We say "$\lim_\Xi$ is a strong FAM limit for intervals", if the following is satisfied: Given
- a pair $(\mathrm{trunk}^*, \mathrm{loss}^*)$, $j^* \in \omega$, and $(\mathrm{trunk}^*, \mathrm{loss}^*)$-sequences $\bar{q}^j$ for $j < j^*$,
- $\epsilon > 0$, $k^* \in \omega$,
- $m^* \in \omega$ and a partition of ω into sets B_m ($m \in m^*$), and
- a condition q stronger than all $\lim_\Xi(\bar{q}^j)$ for all $j < j^*$,

there is a finite $u \subseteq \omega \setminus k^*$ and a q' stronger than q such that
- $\Xi(B_m) - \epsilon < \frac{|u\cap B_m|}{|u|} < \Xi(B_m) + \epsilon$ for $m < m^*$,
- $\frac{1}{|u|} \sum_{k\in u} \frac{|\{\ell\in I_k : q' \leq q^j_\ell\}|}{|I_k|} \geq 1 - \mathrm{loss}^* - \epsilon$ for $j < j^*$

The motivation for this definition is the following:

Lemma 1.8. *Assume that* $\lim_\Xi$ *is such a limit. Then there is a* Q*-name* Ξ^+ *such that for every* $(\mathrm{trunk}^*, \mathrm{loss}^*)$*-sequence* $\bar{q}$ *the limit* $\lim_\Xi(\bar{q})$ *forces* $\Xi^+(A_{\bar{q}}) \geq 1 - \sqrt{\mathrm{loss}^*}$*, where*
$$A_{\bar{q}} = \{k \in \omega : |\{\ell \in I_k : q_\ell \in G\}| \geq |I_k| \cdot (1 - \sqrt{\mathrm{loss}^*})\}$$

Proof. Work in the Q-extension. Now Ξ is a partial FAM. Let J enumerate all suitable sequences $\bar{q} \in V$ with $\lim_\Xi(\bar{q}) \in G$, and for such a sequence $\bar{q}^j$ set $a^j_k = \frac{|\{\ell\in I_k : q^j_\ell \in G\}|}{|I_k|}$, and $b^j = 1 - \mathrm{loss}^*$. Using that Ξ satisfies Definition 1.7, we can apply Fact 1.3(2), we can extend Ξ to some FAM Ξ^+ such that $\mathrm{Av}_{\Xi^+}(\bar{a}^j) \geq 1 - \mathrm{loss}^*$ for $j < j^*$. So $\Xi^+(A^{\bar{q}^j}) + (1 - \Xi^+(A^{\bar{q}^j})) \cdot (1 - \sqrt{\mathrm{loss}^*}) \geq \mathrm{Av}_{\Xi^+}(a^j_k) \geq 1 - \mathrm{loss}^*$, and thus $\Xi^+(A^{\bar{q}^j}) \geq 1 - \sqrt{\mathrm{loss}^*}$. $\square$

[2]In [She00], trunk and loss are called h^2 and h^1; and instead of I_k the interval is called $[n^*_k, n^*_{k+1}-1]$. Moreover, in [She00] the sequence $(n^*_k)_{k\in\omega}$ is one of the parameters of a "blueprint", whereas we assume that the I_k are fixed.

[3]In [She00], this is implicit in 2.11(f).

Definition 1.9. $(Q, \mathtt{trunk}, \mathrm{loss})$ as in Assumption 1.5 "has strong FAM limits for intervals", if for every FAM Ξ and pairs $(\mathtt{trunk}^*, \mathrm{loss}^*)$ there is a function $\lim_\Xi$ that is a strong FAM limit for intervals.

Lemma 1.10. *Random forcing has strong FAM-limits for intervals.*

Proof. $\lim_\Xi$ is implicitly defined in [She00, 2.18], in the following way: Given a sequence r_ℓ with $(\mathtt{trunk}(p_\ell), \mathrm{loss}(p_\ell)) = (\mathtt{trunk}^*, \mathrm{loss}^*)$, we can set $r^* = [\mathtt{trunk}^*]$ and $b = 1 - \mathrm{loss}^*$; and we set n^*_k such that $I_k = [n^*_k, n^*_{k+1} - 1]$. We now use these objects to apply [She00, 2.18] (note that (c)($*$) is satisfied). This gives $r^\otimes$, and we define $\lim_\Xi(\bar r)$ to be $r^\otimes$.

In [She00, 2.17], it is shown that this $r^\otimes$ satisfies Definition 1.7, i.e., is a limit: If r is stronger than all limits $r^{\otimes i}$, then r satisfies [She00, 2.17($*$)]. □

1.2. **The forcing $\tilde{\mathbb{E}}$.** We now define $\tilde{\mathbb{E}}$, a variant of the forcing notion Q^2 defined in [HS]:

Definition 1.11. By induction on the height $h \geq 0$, we define a compact homogeneous tree $T^* \subset \omega^{<\omega}$, and set

$$\rho(h) := \max(|T^* \cap \omega^h|, h+2) \quad \text{and} \quad \pi(h) := (h^2 \rho(h))^{\rho(h)^h}, \tag{1.12}$$

we set Ω_s to be the set $\{t \rhd s \ :\ t \in T^*\}$, i.e., the set of immediate successors of s, and define for each s a measure μ_s on the subsets of Ω_s. In more detail:

- The unique element of T^* of height 0 is $\langle\rangle$, i.e., $T^* \cap \omega^0 = \{\langle\rangle\}$.
- We set
  $$a(h) = \pi(h)^{h+2}, \quad M(h) = a(h)^2, \quad \text{and} \quad \mu_h(n) = \log_{a(h)}\left(\frac{M(h)}{M(h)-n}\right)$$
  for natural numbers $0 \leq n \leq M(h)$.
- For any $s \in T^* \cap \omega^h$, we set $\Omega_s = \{s^\frown \ell \ :\ \ell \in M(h)\}$ (which defines $T^* \cap \omega^{h+1}$). For $A \subset \Omega_s$, we set $\mu_s(A) := \mu_h(|A|)$. So $|\Omega_s| = M(h)$, $\mu_s(\emptyset) = 0$ and $\mu_s(\Omega_s) = \infty$. Note that $|A| = |\Omega_s| \cdot \left(1 - a(h)^{-\mu_s(A)}\right)$.

We can now define $\tilde{\mathbb{E}}$:

Definition 1.13.
- For a subtree $p \subseteq T^*$, the stem of p is the smallest splitting node. For $s \in p$, we set $\mu_s(p) = \mu_s(\{t \in p \ :\ t \rhd s\})$.

  $\tilde{\mathbb{E}}$ consists of subtrees p with some stem s^* of height h^* such that $\mu_t(p) \geq 1 + \frac{1}{h^*}$ for all $t \in p$ above s^*. (So the only condition with $h^* = 0$ is the full condition, where all norms are ∞.)

  $\tilde{\mathbb{E}}$ is ordered by inclusion.
- $\mathtt{trunk}(p)$ is the stem of p.

  $\mathrm{loss}(p)$ is defined if there is an $m \geq 2$ satisfying the following, and in that case $\mathrm{loss}(p) = \frac{1}{m}$ for the maximal such m:
  - p has stem s^* of height $h^* > m + 1$,
  - $\mu_s(p) \geq 1 + \frac{1}{m}$ for all $s \in p$ of height $\geq h^*$,
  - h^* is large enough so that $\prod_{h=h^*}^{\infty}(1 - \frac{1}{h^2}) > 1 - \frac{1}{2m}$.

  (Note that $\mathrm{loss}(p)$, whenever defined, is < 1.)

By simply extending the stem, we can find for any $p \in \tilde{\mathbb{E}}$ and $\epsilon > 0$ some $q \leq p$ with $\mathrm{loss}(q) < \epsilon$. (That shows that one of the assumptions in 1.5 is satisfied; the other one is dealt with in Lemma 1.17(a).)

Lemma 1.14. *Let $s \in T^*$ be of height h. Let A and all A_i be proper subsets of Ω_s.*

(a) *If* $\mu(A) \geq 1$*, then* $|A| \geq |\Omega_s| \cdot (1 - \frac{1}{h^2})$*.*

(b) $\mu_s(A \setminus \{t\}) \geq \mu_s(A) - \frac{1}{h}$ *for* $t \in A$*.*

(c) *For* $i < \pi(h)$*, assume that* $A_i \subseteq \Omega_s$ *satisfies* $\mu(A_i) \geq x$*. Then* $\mu(\bigcap_{i\in\pi(h)} A_i) \geq x - \frac{1}{h}$*.*

(d) *For* $i < I$ *(an arbitrary finite index set) pick proper* $A_i \subsetneq \Omega_s$ *such that* $\mu(A_i) \geq x$*, and assign weighs* a_i *to* A_i *such that* $\sum_{i\in I} a_i = 1$*. Then*

$$\mu_s(B) \geq x - \frac{2}{h} \quad \textit{for} \quad B \coloneqq \Big\{ t \in \Omega : \sum_{t \in A_i} a_i > 1 - \frac{1}{h^2} \Big\}. \tag{1.15}$$

Proof. (a) Trivial, as $a(h)^{-\mu_s(A)} \leq \frac{1}{a(h)} < \frac{1}{h^2}$.

(b) $\mu_s(A \setminus \{t\}) = \log_{a(h)}(|\Omega_s|) - \log_{a(h)}(|\Omega_s| - |A| - 1) \geq$

$$\geq \log_{a(h)}(|\Omega_s|) - \log_{a(h)}(2(|\Omega_s| - |A|)) \geq \mu_s(A) - \log_{a(h)}(2) \geq \mu_s(A) - \frac{1}{h}.$$

(c) $\mu(\bigcap_{i\in\pi(h)} A_i) = \log_{a(h)}(|\Omega_s|) - \log_{a(h)}(|\Omega_s| - |\bigcap_{i\in\pi(h)} A_i|) =$

$$= \log_{a(h)}(|\Omega_s|) - \log_{a(h)}(\textstyle\bigcup_{i\in\pi(h)} |\Omega_s - A_i|) \geq$$

$$\geq \log_{a(h)}(|\Omega_s|) - \log_{a(h)}(\pi(h) \cdot \max_{i\in\pi(h)} |\Omega_s - A_i|) \geq x - \log_{a(h)}(\pi(h)) \geq x - \frac{1}{h}.$$

(d) Set $y = \sum_{i\in I} a_i \cdot |A_i|$. On the one hand, $y \geq |\Omega_s| \cdot (1 - a(h)^{-x})$. On the other hand, $y \leq \sum_{t\in\Omega_s} \sum_{t\in A_i} a_i \leq |B| + (|\Omega_s \setminus B|) \cdot (1 - \frac{1}{h^2})$.

So $|B| \geq |\Omega_s|(1 - h^2 a(h)^{-x}) > |\Omega_s|(1 - a(h)^{-(x-\frac{2}{h})})$, as $a(h)^{\frac{2}{h}} > \pi(h)^{h\frac{2}{h}} \geq h^2$. □

$\tilde{\mathbb{E}}$ is not σ-centered, but it satisfied a property, first defined in [OK14], which is between σ-centered and σ-linked:

Definition 1.16. Fix π, ρ functions from ω to ω converging to infinity. Q is (ρ, π)-linked if there are $\pi(i)$-linked $Q^i_j \subseteq Q$ for $i < \omega, j < \rho(i)$ such that each $q \in Q$ is every $\bigcup_{j<\rho(i)} Q^i_j$ for sufficiently large i.

Lemma 1.17. *Recall that* ρ *and* π *are defined in* (1.12)*.*

(a) *If* $\pi(h)$ *many conditions* $(p_i)_{i\in\pi(h)}$ *have a common node* s *of height* h*, then there is a* q *stronger than each* p_i*.*

(b) $\tilde{\mathbb{E}}$ *is* (ρ, π)*-linked (So in particular it is ccc).*

(c) *The* $\tilde{\mathbb{E}}$*-generic real* η *is eventually different (from every real in* $\lim(T^*)$*, and therefore from every real in* ω^ω *as well).*

(d) $\mathrm{Leb}(p) \geq \mathrm{Leb}([\mathrm{trunk}^*]) \cdot (1 - \frac{1}{2}\mathrm{loss}(p))$*; more explicitly: for any* $h > |\mathrm{trunk}(p)|$*,*

$$\frac{|p \cap \omega^h|}{|T^* \cap \omega^h \cap [\mathrm{trunk}(p)]|} \geq 1 - \frac{1}{2}\mathrm{loss}(p).$$

(e) *Set* $Q' = \mathrm{dom}(\mathrm{loss})$*, i.e.,* $Q' \subseteq \tilde{\mathbb{E}}$ *is dense.* Q' *is an incompatibility-preserving subforcing of random forcing, where we use the variant of random forcing on* $\lim(T^*)$ *instead of* 2^ω*. Let* B' *be the the sub-Boolean-algebra of* $\mathrm{Borel}/\mathrm{Null}$ *generated by* Q' *(where we identify* q *and* $\lim(q)$*). Then* Q' *is dense in* B'*. (Again, here we mean the Borel subsets of* $\lim(T^*)$*.)*

Proof. (a) Use 1.14(c).

(b) For each $h \in \omega$, enumerate $T^* \cap \omega^h$ as $\{s^h_1, \ldots, s^h_{\rho(h)}\}$, and set $Q^h_i = \{p \in \tilde{\mathbb{E}} : s^h_i \in p\}$. So for all h, Q^h_i is $\pi(h)$-linked, and $\bigcup_{i<\rho(h)} Q^h_i = Q$.

(c) Use 1.14(b).

(d) Use 1.14(a) and the definition of loss.

(e) As in the previous item, we get that $\mathrm{Leb}(p \cap [t]) > 0$ whenever $p \in Q'$ and $t \in p$. So Q' is a subset of random forcing. As both sets are ordered by inclusion, Q' is a subforcing. If $q_1, q_2 \in Q'$ and q_1, q_2 are compatible as a random condition, then $q_1 \cap q_2$ has arbitrary high nodes, in particular a node above both stems, which implies that q_1 is compatible with q_2 in $\tilde{\mathbb{E}}$ and therefore in Q'. It remains to show that Q' is dense in B'. It is enough to show: If $x \neq 0$ in B' has the form $x = \bigwedge_{i<i^*}[\lim(q_i)] \wedge \bigwedge_{j<j^*}[\lim(T^*) \setminus \lim(q^j)]$ then there is some $q \in Q'$ with $[\lim(q)] < x'$. Note that $0 \neq x = [A]$ for $A = \lim\left(\bigcap_{i<i^*} q_i\right) \setminus \bigcup_{j<j^*} \lim(q^j)$, so pick some $r \in A$ and pick $h > i^*$ large enough such that $s = r \restriction h$ is not in any q^j. Then $\bigcap_{i<i^*} q_i \cap [s]$ is as required. □

Lemma 1.18. *$\tilde{\mathbb{E}}$ has strong FAM-limits for intervals.*

Proof. Let $(p_\ell)_{\ell\in\omega}$ be a (s^*, loss^*)-sequence, s^* of height h^*. In particular, all norms in all conditions of the sequence are at least $1 + \mathrm{loss}^*$.

We will first construct $(q_k)_{k\in\omega}$ such that q_k forces $\frac{|\{\ell\in I_k:\, p_\ell\in G\}|}{|I_k|} \geq 1 - \frac{1}{2}\mathrm{loss}^*$. We will then use $\bar{q}$ to define $\lim_\Xi(\bar{p})$, and in the third step show that it is as required.

Step 1: So let us define q_k. Fix $k \in \omega$.

- Set $\tilde{\zeta}^h = 1$ for $h \leq h^*$, and $\tilde{\zeta}^h \coloneqq 1 - \prod_{m=h^*}^{h-1}(1 - \frac{1}{m^2})$. This is an increasing sequence converging to some $\tilde{\zeta}^\infty \leq \frac{1}{2}\mathrm{loss}^*$ (by the definition of loss).
- Set $X_t = \{\ell \in I_k \,:\, t \in p_\ell\}$ and $Y_h = \left\{t \in [s^*] \cap \omega^h \,:\, |X_t| \geq |I_k| \cdot (1 - \tilde{\zeta}^h)\right\}$.
- We define q_k by induction on the level, such that $q_k \cap \omega^h \subseteq Y_h$. The stem is s^*. (Note that $X_{s^*} = I_k$ and so $s^* \in Y_{h^*}$.) For $s \in q_k \cap \omega^h$ (and thus, by induction hypothesis, in Y_h), we set $q_k \cap [s] \cap \omega^{h+1} = [s] \cap Y_{h+1}$, i.e., a successor t of s is in q_k iff it is Y_{h+1}. Then $\mu_s(q_k) \geq 1 + \mathrm{loss}^* - \frac{2}{h}$.

  Proof: Set $I = X_s$. By induction, $|X_s| \geq |I_k| \cdot (1 - \tilde{\zeta}^h)$. For $\ell \in I$, set $A_\ell = p_\ell \cap [s] \cap \omega^{h+1}$, i.e., the immediate successors of s in p_ℓ. Obviously $\mu_s(A_\ell) \geq 1 + \mathrm{loss}^*$. We give each A_ℓ equal weight $a_\ell = \frac{1}{|I|}$. According to (1.15), the set $B = \{t \rhd s \,:\, |\{\ell \in X_s \,:\, t \in A_\ell\}| \geq |I| \cdot (1 - \frac{1}{h^2})\}$ has norm at least $1 + \mathrm{loss}^* - \frac{2}{h}$.
- q_k forces that $p_\ell \in G$ for $\geq |I_k| \cdot (1 - \frac{1}{2}\mathrm{loss}^*)$ many $\ell \in I_k$.

  Proof: Let $r < q_k$ have stem s' of length h', without loss of generality $h' > |I_k| + 1$. As $s' \in Y_{h'}$, there are $|I_k| \cdot (1 - \frac{1}{2}\mathrm{loss}^*)$ many $\ell \in I_k$ such that $s' \in p_\ell$. We can intersect these $\leq |I_k|$ many conditions, and r; the resulting condition is stronger than r and than $|I_k| \cdot (1 - \frac{1}{2}\mathrm{loss}^*)$ many of the p_ℓ.

Step 2: Now we use $(q_k)_{k\in\omega}$ to construct by induction on the height $q^* = \lim_\Xi(\bar{p})$, a condition with stem s^* and all norms $\geq 1 + \mathrm{loss}^* - \frac{4}{h}$ such that for all $s \in q^*$,

$$\Xi(Z_s) \geq \prod_{m=h^*}^{h-1}(1 - \frac{1}{m^2}) \geq 1 - \frac{1}{2}\mathrm{loss}^*, \quad \text{for} \quad Z_s = \{k \in \omega \,:\, s \in q_k\}. \tag{1.19}$$

Given an $s \geq s^*$ satisfying this, set $A(k)$ to be the s-successors in q_k for each $k \in Z_s$. Enumerate the (finitely many) $A(k)$ as $(A_i)_{i\in I}$. Clearly $\mu_s(A_i) \geq 1 + \mathrm{loss}^* - \frac{2}{h}$. Assign to A_i the weight

$$a_i = \frac{\Xi(\{k \in Z_s \,:\, A(k) = A_i\})}{\Xi(Z_s)}.$$

Again using (1.15), $\mu_s(B) \geq 1 + \text{loss}^* - \frac{4}{h}$, where B consists of those successors t of s such that

$$1 - \frac{1}{h^2} < \sum_{t \in A_i} a_i = \frac{1}{\Xi(Z_s)} \Xi(\{k \in Z_s \,:\, t \in q_k\}) \leq \frac{1}{\Xi(Z_s)} \Xi(Z_t).$$

We use B as the set of s-successors in q^*.

Step 3: We now show that this limit works: As in Definition 1.7, fix m^*, $(B_m)_{m<m^*}$, ϵ, k^*, i^* and sequences $(p^i_\ell)_{\ell<\omega}$ for $i < i^*$, such that $(\text{trunk}(p^i_\ell), \text{loss}(p^i_\ell)) = (\text{trunk}^*, \text{loss}^*)$.

For each $i < i^*$, $\bar{q}^i = (q^i_k)_{k\in\omega}$ is defined from $\bar{p}^i = (p^i_\ell)_{\ell\in\omega}$, and in turn defines the limit $\lim_\Xi(\bar{p}^i)$. Let q be stronger than all $\lim_\Xi(\bar{p}^i)$.

Let M be as in Lemma 1.2, for $N = m^* + i^*$. So for N many sets there is a u of size M (above k^*) which approximates the measure well. We using the following sets:

- B_m (for $m < m^*$).
- Fix a $s \in q$ of height $h > M \cdot i^*$; and use the i^* many sets $Z^i_s \subseteq \omega$ of (1.19).

Accordingly, there is an u (starting above k^*) of size $\leq M$ with

- $\Xi(B_m) - \epsilon \leq \frac{|B_m \cap u|}{|u|} \leq \Xi(B_m) + \epsilon$ for each $m < m^*$, and
- $\frac{|Z^i_s \cap u|}{|u|} \geq 1 - \frac{1}{2}\,\text{loss} - \epsilon$ for each $i < i^*$.

So for each $i \in i^*$ there are at least $|u| \cdot (1 - \frac{1}{2}\,\text{loss} - \epsilon)$ many $k \in u$ with $s \in q^i_k$. There is a condition r stronger than all those q^i_k as well as q ($\leq M \cdot i^* + 1$ many conditions of height $h > M \cdot i^*$ with common node s). So r forces, for all $i < i^*$ and $k \in u \cap Z^i_s$, that $q^i_k \in G$ and therefore that $|\{\ell \in I_k \,:\, p^i_\ell \in G\}| \geq |I_k|(1 - \frac{1}{2}\,\text{loss}^*)$. By increasing r to some q', we can assume that r decides which p^i_ℓ are in G and that r is actually stronger than each p^i_ℓ decided to be in G. So all in all we get $q' \leq q$ such that

$$\frac{1}{|u|} \sum_{k\in u} \frac{|\{\ell \in I_k \,:\, q' \leq p^j_\ell\}|}{|I_k|} \;\geq\; \frac{1}{|u|} |\{k \in u \,:\, k \in Z^j_s\}|(1 - \frac{1}{2}\,\text{loss}^*) \;\geq\; 1 - \text{loss}^* - \epsilon,$$

as required. □

2. The left hand side of Cichoń's diagram

We write $\mathfrak{x}_1$ for $\text{add}(\mathcal{N})$, $\mathfrak{x}_2$ for $\mathfrak{b}$ (which will also be $\text{add}(\mathcal{M})$), $\mathfrak{x}_3$ for $\text{cov}(\mathcal{N})$ and $\mathfrak{x}_4$ for $\text{non}(\mathcal{M})$.

2.1. Good iterations and the LCU property.

We want to show that some forcing $\mathbb{P}^5$ results in $\mathfrak{x}_i = \lambda_i$ (for $i = 1 \ldots 4$). So we have to show two "directions", $\mathfrak{x}_i \leq \lambda_i$ and $\mathfrak{x}_i \geq \lambda_i$.

For $i = 1, 3, 4$ (i.e., for all the characteristics on the left hand side apart from $\mathfrak{b} = \text{add}(\mathcal{M})$), the direction $\mathfrak{x}_i \leq \lambda_i$ will be given by the fact that $\mathbb{P}^5$ is (R_i, λ_i)-good for a suitable relation R_i. (For $i = 2$, i.e., the unbounding number, we will have to work more.)

We will use the following relations:

Definition 2.1.
1. Let C be the set of strictly positive rational sequences $(q_n)_{n\in\omega}$ such that $\sum_{n\in\omega} q_n \leq 1$.[4] Let $\mathrm{R}_1 \subseteq C^2$ be defined by: $f \,\mathrm{R}_1\, g$ if $(\forall^* n \in \omega)\, f(n) \leq g(n)$.
2. $\mathrm{R}_2 \subseteq (\omega^\omega)^2$ is defined by: $f \,\mathrm{R}_2\, g$ if $(\forall^* n \in \omega)\, f(n) \leq g(n)$.
4. $\mathrm{R}_4 \subseteq (\omega^\omega)^2$ is defined by: $f \,\mathrm{R}_4\, g$ if $(\forall^* n \in \omega)\, f(n) \neq g(n)$.

[4] It is easy to see that C is homeomorphic to ω^ω, when we equip the rationals with the discrete topology and use the product topology.

So far, these relations fit the usual framework of goodness, as introduced in [JS90] and [Bre91] and summarized, e.g., in [BJ95, 6.4] or [GMS16, Sec. 3] or [Mej13, Sec. 2]. For $\mathfrak{x}_3$, i.e., $\mathrm{cov}(\mathcal{N})$, we will use a relation R_3 that does not fit this framework (as the range of the relation is not a Polish space). Nevertheless, the property "(R_3, λ)-good" behaves just as in the usual framework (e.g., finite support limits of good forcings are good, etc.). The relation R_3 was implicitly used by Kamo and Osuga [OK14], who investigated (R_3, λ)-goodness.[5] It was also used in [BM14]; a unifying notation for goodness (which works for the usual cases as well as relations such as R_3) is given in [MC, §4].

Definition 2.2. We call a set $\mathcal{E} \subset \omega^\omega$ an R_3-parameter, if for all $e \in \mathcal{E}$

- $\lim e(n) = \infty$, $e(n) \leq n$, $\lim(n - e(n)) = \infty$,
- there is some $e' \in \mathcal{E}$ such that $(\forall^* n)\, e(n) + 1 \leq e'(n)$, and
- for all countable $\mathcal{E}' \subseteq \mathcal{E}$ there is some $e \in \mathcal{E}$ such that for all $e' \in \mathcal{E}'$ $(\forall^* n)\, e(n) \geq e'(n)$.

Fix such an R_3-parameter $\mathcal{E}$ of size $\aleph_1$ (it is easy to see that such a parameter exists.) Recall that ρ and π were defined in (1.12). Set

$$b(h) = h^2 \rho(h)^h, \quad \mathcal{S} = \left\{\psi \in \prod_{h\in\omega} P(b(h)) : (\forall h \in \omega)\, |\psi(h)| \leq \rho(h)^h\right\},$$
$$\mathcal{S}_e = \left\{\phi \in \prod_{h\in\omega} P(b(h)) : (\forall h \in \omega)\, |\phi(h)| \leq \rho(h)^{e(h)}\right\} \quad \text{and} \quad \hat{\mathcal{S}} = \bigcup_{e\in\mathcal{E}} \mathcal{S}_e.$$

We can now define the relation for $\mathrm{cov}(\mathcal{N})$:

3. $\mathrm{R}_3 \subseteq \mathcal{S} \times \hat{\mathcal{S}}$ is defined by: $\psi\, \mathrm{R}_3\, \phi$ iff $(\forall^* n \in \omega)\, \phi(n) \not\subseteq \psi(n)$.

Note that $\mathcal{S}_e \subset \hat{\mathcal{S}} \subset \mathcal{S}$ and that $\mathcal{S}_e$ and $\mathcal{S}$ are Polish spaces. So we can evaluate in any model M containing e the according $\mathcal{S}_e^M$ and $\mathcal{S}^M$, and if the model contains $\mathcal{E}$ (in particular, in any forcing extension) we can define the set $\hat{\mathcal{S}}^M = \bigcup_{e\in\mathcal{E}} \mathcal{S}_e^M$. (Note that in any ccc forcing extension, $\mathcal{E}$ remains a suitable R_3-parameter.) Of course, we will omit the superscript M in the following.

Definition 2.3. Fix one of these relations $R \subseteq X \times Y$.

- We say "f is bounded by g" if $f\, \mathrm{R}\, g$; and, for $\mathcal{Y} \subseteq \omega^\omega$, "$f$ is bounded by $\mathcal{Y}$" if $(\exists y \in \mathcal{Y})\, f\, \mathrm{R}\, y$. We say "unbounded" for "not bounded". (I.e., f is unbounded by $\mathcal{Y}$ if $(\forall y \in \mathcal{Y})\, \neg f\, \mathrm{R}\, y$.)
- We call $\mathcal{X}$ an R-unbounded family, if $\neg(\exists g)\, (\forall x \in \mathcal{X}) x\, \mathrm{R}\, g$, and an R-dominating family if $(\forall f)\, (\exists x \in \mathcal{X})\, f\, \mathrm{R}\, x$.
- Let $\mathfrak{b}_i$ be the minimal size of an R_i-unbounded family,
- and let $\mathfrak{d}_i$ be the minimal size of an R_i-dominating family.

We only need the following connection between R_i and the cardinal characteristics:

Lemma 2.4.
1. $\mathrm{add}(\mathcal{N}) = \mathfrak{b}_1$ *and* $\mathrm{cof}(\mathcal{N}) = \mathfrak{d}_1$.
2. $\mathfrak{b} = \mathfrak{b}_2$ *and* $\mathfrak{d} = \mathfrak{d}_2$.
3. $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{b}_3$ *and* $\mathrm{non}(\mathcal{N}) \geq \mathfrak{d}_3$.
4. $\mathrm{non}(\mathcal{M}) = \mathfrak{b}_4$ *and* $\mathrm{cov}(\mathcal{M}) = \mathfrak{d}_4$.

Proof. (2) holds by definition. (1) can be found in [BJ95, 6.5.B]. (4) is a result of [Mil82; Bar87], cf. [BJ95, 2.4.1 and 2.4.7].

To see (3), we work in the space $\Omega = \prod_{h\in\omega} b(h)$ with the usual (uniform) measure. It is well known that we get the same values for the characteristics $\mathrm{cov}(\mathcal{N})$ and $\mathrm{non}(\mathcal{N})$ whether

[5] They use the notation $(*^{<\lambda}_{c,h})$, cf. [OK14, Def. 6].

we define them using Ω or, as usual, 2^ω (or $[0,1]$ for that matter, etc). Given $\psi \in \mathcal{S}$, note that

$$N_\psi = \{\eta \in \Omega : (\exists^\infty h)\, \eta(h) \in \psi(h)\}$$

is a Null set, as $\{\eta \in \Omega : (\forall h > k)\, \eta(h) \in \psi(h)\}$ has measure $\prod_{h>k}(1 - \frac{|\psi(h)|}{b(h)}) \geq \prod_{h>k}(1 - \frac{1}{h^2})$, which converges to 1 for $k \to \infty$.

Let $\mathcal{A} \subseteq \mathcal{S}$ be an R_3-unbounded family. So for every $\phi \in \hat{\mathcal{S}}$ there is some $\psi \in A$ such that $(\exists^\infty h)\, \psi(h) \supseteq \phi(h)$. In particular, for each $\eta \in \Omega$, there is a $\psi \in A$ with $\eta \in N_\psi$; i.e., $\mathrm{cov}(\mathcal{N}) \leq |\mathcal{A}|$.

Analogously, let X be a non-null set (in Ω). For each ψ there is an $x \in X \setminus N_\psi$, so $\phi_x(n) = \{x(n)\}$ satisfies $\psi\, \mathrm{R}_3\, \phi_x$. □

Remark 2.5. As shown implicitly in [OK14], and explicitly in [MC, 4.19], we actually get $\mathrm{cov}(\mathcal{N}) \leq c^\exists_{b,h^{\mathrm{Id}}} \leq \mathfrak{b}(\mathrm{R}_3)$.

Definition 2.6. Let P be a ccc forcing, λ an uncountable regular cardinal, and $\mathrm{R}_i \subseteq X \times Y$ one of the relations above (so for $i = 1, 2, 4$, $Y = X$, and for $i = 3$ $Y = \bigcup_{e \in \mathcal{E}} \mathcal{S}_e$). P is (R_i, λ)-good, if for each P-name r for an element of Y there is (in V) a nonempty set $\mathcal{Y} \subseteq Y$ of size $<\lambda$ such that every $f \in X$ (in V) that is R_i-unbounded by $\mathcal{Y}$ is forced to be R_i-unbounded by r as well.

Note that λ-good trivially implies μ-good if $\mu \geq \lambda$ are regular.

Lemma 2.7. *Let λ be uncountable regular.*

a. *Forcings of size $<\lambda$ are (R_i, λ)-good. In particular, Cohen forcing is $(\mathrm{R}_i, \aleph_1)$-good.*
b. *A FS ccc iteration of (R_i, λ)-good forcings (and in particular, a composition of two such forcings) is (R_i, λ)-good.*
1. *A sub-Boolean-algebra of the random algebra is $(\mathrm{R}_1, \aleph_1)$-good. Any σ-centered forcing notion is $(\mathrm{R}_1, \aleph_1)$-good.*
3. *A (ρ, π)-linked forcing is $(\mathrm{R}_3, \aleph_1)$-good, provided $\pi(h) \geq b^{\rho(h)^h} = (h^2\rho(h))^{\rho(h)^h}$.*

Proof. For $i = 1, 2, 4$, a. and b. are proven in [JS90], cf. [BJ95, 6.4]. The same proof works for $i = 3$, as shown in [OK14, Lem. 12,13]. The proof for the uniform framework can be found in [MC, 4.10,4.13].

1. follows from [JS90] and [Kam89], cf. [BJ95, 6.5.17–18].
3. is shown in [OK14, Lem. 10], cf. [MC, Lem. 4.21] □

Remark 2.8. (R_1, ω_1) goodness also follows from (ρ, π)-linkedness, cf. [BM14, 5.11–13], cf. [MC, 4.17].

Each relation R_i is a subset of some $X \times Y$, where X is either 2^ω, ω^ω (or homeomorphic to it) or $\mathcal{S}$, and Y is the range of R_i.

Lemma 2.9. *For each i and each $g \in Y$, the set $\{f \in X : f\, \mathrm{R}_i\, g\} \subseteq X$ is meager.*

Proof. We have explicitly defined each $f\, \mathrm{R}_i\, g$ as $\forall^* n R^n_i(f, g)$ for some R^n_i. The lemma follows easily from the fact that for each $n \in \omega$, the set $\{f \in X : R^n_i(f, g)\}$ is closed nowhere dense. □

Lemma 2.10. *Let $\lambda \leq \kappa \leq \mu$ be uncountable regular cardinals. Force with μ many Cohen reals $(c_\alpha)_{\alpha \in \mu}$, followed by an (R_i, λ)-good forcing. Note that each Cohen real c_β can be interpreted as element of the polish space X where $\mathrm{R}_i \subseteq X \times Y$. Then we get: For every real r in the final extension's Y, the set $\{\alpha \in \kappa : c_\alpha$ is R_i-unbounded by $r\}$ is cobounded in κ. I.e., $(\exists \alpha \in \kappa)\, (\forall \beta \in \kappa \setminus \alpha)\, \neg c_\alpha\, \mathrm{R}_i\, r$.*

Proof. Work in the intermediate extension after κ many Cohen reals, let us call it V_κ. The remaining forcing (i.e., $\mu \setminus \kappa$ many Cohens composed with the good forcing) is good; so applying the definition we get (in V_κ) a set $\mathcal{Y} \subseteq Y$ of size $<\lambda$.

As the initial Cohen extension is ccc, and $\kappa \geq \lambda$ is regular, we get some $\alpha \in \kappa$ such that each element y of $\mathcal{Y}$ already exists in the extension by the first α many Cohens, call it V_α.

Fix some $\beta \in \kappa \setminus \alpha$ and $y \in Y$. As the $\{x \in X : x \mathrel{\mathrm{R}_i} y\}$ is a meager set already defined in V_α, we get $\neg c_\beta \mathrel{\mathrm{R}_i} y$. Accordingly, c_β is unbounded by $\mathcal{Y}$; and, by the definition of good, unbounded by r as well. □

In the light of this result, let us revisit Lemma 2.4 with some new notation, the "linearly cofinally unbounded" property LCU:

Definition 2.11. For $i = 1, 2, 3, 4$, γ a limit ordinal, and P a ccc forcing notion, let $\mathsf{LCU}_i(P, \gamma)$ stand for:

> There is a sequence $(x_\alpha)_{\alpha\in\gamma}$ of P-names such that for every P-name y
> $(\exists \alpha \in \gamma)\,(\forall \beta \in \gamma \setminus \alpha)\ P \Vdash \neg x_\beta \mathrel{\mathrm{R}_i} y$).

Lemma 2.12.
- *$\mathsf{LCU}_i(P, \delta)$ is equivalent to $\mathsf{LCU}_i(P, \mathrm{cf}(\delta))$.*
- *If λ is regular, then $\mathsf{LCU}_i(P, \lambda)$ implies $\mathfrak{b}_i \leq \lambda$ and $\mathfrak{d}_i \geq \lambda$.*

In particular:

1. $\mathsf{LCU}_1(P, \lambda)$ *implies* $P \Vdash (\,\mathrm{add}(\mathcal{N}) \leq \lambda \,\&\, \mathrm{cof}(\mathcal{N}) \geq \lambda\,)$.
2. $\mathsf{LCU}_2(P, \lambda)$ *implies* $P \Vdash (\,\mathfrak{b} \leq \lambda \,\&\, \mathfrak{d} \geq \lambda\,)$.
3. $\mathsf{LCU}_3(P, \lambda)$ *implies* $P \Vdash (\,\mathrm{cov}(\mathcal{N}) \leq \lambda \,\&\, \mathrm{non}(\mathcal{N}) \geq \lambda\,)$.
4. $\mathsf{LCU}_4(P, \lambda)$ *implies* $P \Vdash (\,\mathrm{non}(\mathcal{M}) \leq \lambda \,\&\, \mathrm{cov}(\mathcal{M}) \geq \lambda\,)$.

Proof. Assume that $(\alpha_\beta)_{\beta\in\mathrm{cf}(\delta)}$ is increasing continuous and cofinal in δ. If $(x_\alpha)_{\alpha\in\delta}$ witnesses $\mathsf{LCU}_i(P, \delta)$, then $(x_{\alpha_\beta})_{\beta\in\mathrm{cf}(\delta)}$ witnesses $\mathsf{LCU}_i(P, \mathrm{cf}(\delta))$. And if $(x_\beta)_{\beta\in\mathrm{cf}(\delta)}$ witnesses $\mathsf{LCU}_i(P, \mathrm{cf}(\delta))$, then $(y_\alpha)_{\alpha\in\delta}$ witnesses $\mathsf{LCU}_i(P, \mathrm{cf}(\delta))$, where $y_\alpha \coloneqq x_\beta$ for $\alpha \in [\alpha_\beta, \alpha_{\beta+1})$.

The set $\{x_\alpha : \alpha \in \lambda\}$ is certainly forced to be R_i-unbounded; and given a set $Y = \{y_j : j < \theta\}$ of $\theta < \lambda$ many P-names, each has a bound $\alpha_j \in \lambda$ so that $(\forall \beta \in \lambda \setminus \alpha_j)\ P \Vdash \neg x_\beta \mathrel{\mathrm{R}_i} y_j$), so for any $\beta \in \lambda$ above all α_j we get $P \Vdash \neg x_\beta \mathrel{\mathrm{R}_i} y_j$ for all j; i.e., Y cannot be dominating. □

2.2. **The inital forcing $\mathbb{P}^5$: Partial forcings and the COB property.** Assume we have a forcing iteration $(P_\beta, Q_\beta)_{\beta<\alpha}$ with limit P_α, where each Q_β is a set of reals such that the generic filter of Q_β is determined (in a Borel way) from some generic real η_β. Fix some $w \subseteq \alpha$. We define the P_α-name $\mathbb{R}^w$ to be the set of reals (called w-reals) that can be Borel-calculated from generics at w alone, and let Q_α be the set of all random forcing conditions in $\mathbb{R}^w$. Clearly Q_α is a subforcing (not necessarily a complete one) of the full random forcing. Actually, Q_α is dense in $\mathbb{R}^w \cap \mathrm{Borel}/\mathrm{Null}$, which in turn is a sub-Boolean-algebra of $\mathrm{Borel}/\mathrm{Null}$. So Q_α is ccc and furthermore $(\mathrm{R}_1, \aleph_1)$-good.

We call this forcing "partial random forcing defined from w". (See [KTT, Sec. 1.2] for a more formal definition and more details.) Note that this partial random forcing Q_α has strong FAM-limits for intervals, cf. 1.10, in the following sense:

Fact 2.13. Assume that for each sequence $(\bar{p})_{\ell\in\omega}$ in Q_α, the random condition $\lim_\Xi(\bar{p})$ happens to be in Q_α. Then $\lim_\Xi(\bar{p})$ is a strong FAM limit for Q_α.

Analogously, we define the "partial amoeba", "partial Hechler", and "partial $\tilde{\mathbb{E}}$" forcings. Note that a partial Hechler forcing is σ-linked, and that a partial $\tilde{\mathbb{E}}$ forcing is (ρ, π)-linked and that it has strong FAM-limits for intervals, provided that the limits of partial $\tilde{\mathbb{E}}$-conditions are again in the partial forcing (as in the random case).

Assume that λ is regular uncountable and $\mu < \lambda$ implies $\mu^{\aleph_0} < \lambda$. Then $|w| < \lambda$ implies that the sizes of the partial forcings defined by w are $<\lambda$.

We will assume the following throughout the paper:

Assumption 2.14. $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < \lambda_5$ are regular cardinals such that $\mu < \lambda_i$ implies $\mu^{\aleph_0} < \lambda_i$.

We set $\delta_5 = \lambda_5 + \lambda_5$, and partition $\delta_5 \setminus \lambda_5$ into unbounded sets S^1, S^2, S^3 and S^4. Fix for each $\alpha \in \delta_5 \setminus \lambda_5$ some $w_\alpha \subseteq \alpha$ such that each $\{w_\alpha : \alpha \in S^i\}$ is cofinal[6] in $[\delta]^{<\lambda_i}$.

The reader can assume that $(\lambda_i)_{i=1,\dots,5}$, $(S^i)_{i=1,\dots,4}$ as well as $(w_\alpha)_{\alpha\in S^i}$ for $i = 1, 2$ have been fixed once and for all (let us call them "fixed parameters"), whereas we will investigate various possibilities for $\bar{w} = (w_\alpha)_{\alpha\in S^3\cup S^4}$ in the following two sections. (We will call such $\bar{w}$ that satisfy the assumption "cofinal parameters".)

Definition 2.15. Let $\mathbb{P}^5 = (P_\alpha, Q_\alpha)_{\alpha\in\delta_5}$ be the FS iteration where Q_α is Cohen forcing for $\alpha \in \lambda_5$, and

$$Q_\alpha \text{ is the partial} \left\{\begin{array}{c} \text{amoeba} \\ \text{Hechler} \\ \text{random} \\ \tilde{\mathbb{E}} \end{array}\right\} \text{forcing defined from } w_\alpha \text{ if } \alpha \text{ is in} \left\{\begin{array}{c} S^1 \\ S^2 \\ S^3 \\ S^4 \end{array}\right.$$

Lemma 2.16. $\mathsf{LCU}_i(\mathbb{P}^5, \kappa)$ *holds for* $i = 1, 3, 4$ *and each regular cardinal* κ *in* $[\lambda_i, \lambda_5]$.

Proof. For $i = 1$: $\tilde{\mathbb{E}}$ forcing is (equivalent to) a sub-Boolean-algebra of the random algebra, according to Lemma 1.17(e). The same holds for partial $\tilde{\mathbb{E}}$ forcing as well (if the q_i and q^j in the proof of the lemma are Borel in w, then so is $\bigcap q_i \cap [s]$). Of course the same holds for partial random, whereas partial Hechler is σ-centered and the partial amoeba forcings are small, i.e., have size $<\lambda_1$. So according to Lemma 2.7, all iterands Q_α (and therefore the limits as well) are $(\mathrm{R}_1, \lambda_1)$-good.

For $i = 3$, note that partial $\tilde{\mathbb{E}}$ forcing is (ρ, π)-linked (according to Lemma 1.17(b), which also applies to the partial version). All other iterands have size $<\lambda_3$, so the forcing is $(\mathrm{R}_3, \lambda_3)$-good.

For $i = 4$ it is enough to note that *all* iterands are small, i.e., of size $<\lambda_4$.

We can now apply Lemma 2.10. □

So in particular, $\mathbb{P}^5$ forces $\mathrm{add}(\mathcal{N}) \le \lambda_1$, $\mathrm{cov}(\mathcal{N}) \le \lambda_3$, $\mathrm{non}(\mathcal{M}) \le \lambda_4$ and $\mathrm{cov}(\mathcal{M}) = \mathrm{non}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \lambda_5 = 2^{\aleph_0}$; i.e., the respective left hand characteristics are small. We now show that they are also large, using the "cone of bounds" property COB:

Definition 2.17. For a ccc forcing notion P, regular uncountable cardinals λ, μ and $i = 1, 2, 4$, let $\mathsf{COB}_i(P, \lambda, \mu)$ stand for:

> There is a $<\lambda$-directed partial order $(S, \prec)$ of size μ and a sequence $(g_s)_{s\in S}$ of P-names for reals such that for each P-name f of a real
> $(\exists s \in S)\,(\forall t \succ s)\, P \Vdash f\ \mathrm{R}_i\ g_t$.

For $i = 3$, let $\mathsf{COB}_3(P, \lambda, \mu)$ stand for:

> There is a $<\lambda$-directed partial order $(S, \prec)$ of size μ and a sequence $(g_s)_{s\in S}$ of P-names for reals such that for each P-name f of a null-set
> $(\exists s \in S)\,(\forall t \succ s)\, P \Vdash g_t \notin f$.

[6]i.e., if $\alpha \in S^i$ then $|w_\alpha| < \lambda_i$, and for all $u \subseteq \delta_5$, $|u| < \lambda_i$ there is some $\alpha \in S^i$ with $w_\alpha \supseteq u$.

So s is the tip of a cone that consists of elements bounding f, where in case $i = 3$ we implicitly use an additional relation $N \, \mathrm{R}_3{}' \, r$ expressing that the null-set N doesn't contain the real r. Note that $\mathrm{cov}(\mathcal{N})$ is the bounding number $\mathfrak{b}'_3$ of $\mathrm{R}_3{}'$, and $\mathrm{non}(\mathcal{N})$ the dominating number $\mathfrak{d}'_3$. So $\mathrm{add}(\mathcal{N}) = \mathfrak{b}'_3 \leq \mathfrak{b}_3$ and $\mathrm{non}(\mathcal{N}) = \mathfrak{d}'_3 \geq \mathfrak{d}_3$ (as defined in Lemma 2.4).

$\mathsf{COB}_i(P, \lambda, \mu)$ implies that P forces that $\mathfrak{b}_i \geq \lambda$ and that $\mathfrak{d}_i \leq \mu$ (for $i = 1, 2, 4$, and the same for $i = 3$ and $\mathfrak{b}'_3$, $\mathfrak{d}'_3$): Clearly P forces that $\{g_s : s \in S\}$ is dominating. And if A is set of names of size $\kappa < \lambda$, then for each $f \in A$ the definition gives a bound $s(f)$ and directedness some $t \succ s(f)$ for all f, i.e., g_t bounds all elements of A. So we get:

Lemma 2.18.
1. $\mathsf{COB}_1(P, \lambda, \mu)$ *implies* $P \Vdash (\,\mathrm{add}(\mathcal{N}) \geq \lambda \,\&\, \mathrm{cof}(\mathcal{N}) \leq \mu\,)$.
2. $\mathsf{COB}_2(P, \lambda, \mu)$ *implies* $P \Vdash (\,\mathfrak{b} \geq \lambda \,\&\, \mathfrak{d} \leq \mu\,)$.
3. $\mathsf{COB}_3(P, \lambda, \mu)$ *implies* $P \Vdash (\,\mathrm{cov}(\mathcal{N}) \geq \lambda \,\&\, \mathrm{non}(\mathcal{N}) \leq \mu\,)$.
4. $\mathsf{COB}_4(P, \lambda, \mu)$ *implies* $P \Vdash (\,\mathrm{non}(\mathcal{M}) \geq \lambda \,\&\, \mathrm{cov}(\mathcal{M}) \leq \mu\,)$.

Lemma 2.19. $\mathsf{COB}_i(\mathbb{P}^5, \lambda_i, \lambda_5)$ *holds (for $i = 1, 2, 3, 4$).*

Proof. We use the following facts:

1. Partial amoeba forcing adds a sequence $\bar{b}$ R_1-dominating all w-reals in C. I.e., a partial amoeba forces $\bar{q} \, \mathrm{R}_1 \, \bar{b}$ for all $\bar{q}$ in $C \cap \mathbb{R}^{\omega}$.
   (The simple proof can be found in [GKS, Lem. 1.4], a slight variation in [BJ95].)
2. Partial Hechler forcing adds a real R_2-dominating all w-reals.
3. Partial random forcing adds the random real r which is not in any nullset (Borel-code) $N \in \mathbb{R}^w$. So $r \, \mathrm{R}_3{}'$-dominates all w-reals.
4. The generic branch $\eta \in \lim(T^*)$ added by a partial $\tilde{\mathbb{E}}$ forcings is eventually different each all w-reals, i.e., R_4-dominates all w-reals.
   (Proof: Lemma 1.17(c) also works for the partial version: Removing all $r \restriction n$ for an w-real r from a partial-$\tilde{\mathbb{E}}$-condition p results in a partial-$\tilde{\mathbb{E}}$-condition.)

Fix $i \in \{1, 2, 3, 4\}$, and set $S = S^i$ and $s \prec t$ if $w_s \subsetneq w_t$, and let g_s be the generic added at s (e.g., the partial random real in case of $i = 3$, etc). A $\mathbb{P}^5$-name f depends (in a Borel way) on a countable index set $w^* \subseteq \delta$. Fix some $s \in S^i$ such that $w_s \supseteq w^*$. Pick any $t \succ s$. Then $w_t \supseteq w_s$, so w_t contains all information to calculate f, so as we have seen $\mathbb{P}^5 \Vdash f \, \mathrm{R}_i \, g_t$ (or: $\mathbb{P}^5 \Vdash f \, \mathrm{R}_3{}' \, g_t$ for $i = 3$). □

So to summarize what we know so far about $\mathbb{P}^5$:

Fact 2.20.
- COB_i holds for $i = 1, 2, 3, 4$. So the left hand side characteristics are large.
- LCU_i holds for $i = 1, 3, 4$. So the left hand side characteristics other than $\mathfrak{b}$ are small.

What is missing is "$\mathfrak{b}$ small". We will deal with this now.

2.3. **Dealing with $\mathfrak{b}$ (without GCH).** In this section, we follow [GKS, 1.3], additionally using techniques inspired by [She00].

We assume the following (in addition to Assumption 2.14):

Assumption 2.21. (This section only.) $\chi < \lambda_3$ is regular such that $\chi^+ \geq \lambda_2$ and $2^\chi = |\delta_5| = \lambda_5$.

Set $S^0 = \lambda_5 \cup S^1 \cup S^2$. So $\delta_5 = S^0 \cup S^3 \cup S^4$, and $\mathbb{P}^5$ is a FS ccc iteration along δ_5 such that $\alpha \in S^0$ implies $|Q_\alpha| < \lambda_2$, i.e., $|Q_\alpha| \leq \chi$ (and Q_α is a partial random forcing for $\alpha \in S^3$ and a partial $\tilde{\mathbb{E}}$-forcing for $\alpha \in S^4$).

Let us fix, for each $\alpha \in S^0$, a P_α-name

$$i_\alpha : Q_\alpha \to \chi \text{ injective.} \tag{2.22}$$

Definition 2.23.
- A "partial guardrail" is a function h defined on a subset of δ_5 such that, for $\alpha \in \mathrm{dom}(h)$: $h(\alpha) \in \chi$ if $\alpha \in S^0$; and $h(\alpha)$ is a pair (x, y) with $x \in H(\aleph_0)$ and y a rational number otherwise. (Any $(\mathtt{trunk}, \mathrm{loss})$-pair is of this form.)
- A "countable guardrail" is a partial guardrail with countable domain. A "full guardrail" is a partial guardrail with domain δ_5.

We will use the following lemma, which is a consequence of the Engelking-Karlowicz theorem [EK65] on the density of box products (cf. [GMS16, 5.1]):

Lemma 2.24. *(As $|\delta_5| \leq 2^\chi$.) There is a family H^* of full guardrails of cardinality χ such that each countable guardrail is extended by some $h \in H^*$. We will fix such an H^*.*

Note that the notion of guardrail (and the density property required in Lemma 2.24) only depends on χ, δ_5, S^0, S^3 and S^4, i.e., on fixed parameters; so we can fix an H^* that will work for all cofinal parameters $(w_\alpha)_{\alpha \in S^3 \cup S^4}$.

Once we have decided on $\bar{w}$, and thus have defined $\mathbb{P}^5$, we can define the following:

Definition 2.25. $D^* \subseteq \mathbb{P}^5$ consists of p such that there is a partial guardrail h (and we say: "p follows h") with $\mathrm{dom}(h) \supseteq \mathrm{supp}(p)$ and, for all $\alpha \in \mathrm{dom}(p)$,
- If $\alpha \in S^0$, then $p \restriction \alpha \Vdash i_\alpha(p(\alpha)) = h_\alpha$.
- If $\alpha \in S^3 \cup S^4$, the empty condition of P_α forces

$$p(\alpha) \in Q_\alpha \text{ and } (\mathtt{trunk}(p(\alpha)), \mathrm{loss}(p(\alpha))) = h(\alpha).$$

- Furthermore, $\sum_{\alpha \in \mathrm{dom}(p) \cap (S^3 \cup S^4)} \sqrt{\mathrm{loss}(p(\alpha))} < \frac{1}{2}$.

Lemma 2.26. *$D^* \subseteq \mathbb{P}^5$ is dense.*

Proof. By induction we show that for any sequence $(\epsilon_i)_{i \in \omega}$ of positive numbers the following set of p is dense: If $\mathrm{dom}(p) = \{\alpha_0, \dots, \alpha_m\}$, where $\alpha_0 > \alpha_1 >, \dots$ (i.e., we enumerate downwards), $\mathrm{loss}^p_{\alpha_n} < \epsilon_n$ whenever $\alpha_n \in S^3 \cup S^4$. For the successor step, we use that the set of $q \in Q_\alpha$ such that $\mathrm{loss}(q) < \epsilon_0$ is forced to be dense. □

Remark 2.27. So the set of conditions following *some* guardrail is dense. For each *fixed* guardrail h, the set of all conditions p following h is n-linked, provided that each loss in the domain of h is $< \frac{1}{n}$ (cf. Assumption 1.5).

Definition 2.28.
- A "Δ-system with heart ∇ following the guardrail h" is a family $\bar{p} = (p_i)_{i \in I}$ of conditions such that
  - all p_i are in D^* and follow h,
  - $(\mathrm{dom}(p_i))_{i \in I}$ is a Δ system with heart ∇ in the usual sense (so $\nabla \subseteq \delta_5$ is finite)
  - the following is independent of $i \in I$:
    * $|\mathrm{dom}(p_i)|$, which we call $m^{\bar{p}}$.
      Let $(\alpha_i^{\bar{p},n})_{n < m^{\bar{p}}}$ increasingly enumerate $\mathrm{dom}(p_i)$.
    * Whether $\alpha_i^{\bar{p},n}$ is less than, equal to or bigger than the k-th element of ∇.
      In particular it is independent of i whether $\alpha_i^{\bar{p},n} \in \nabla$, in which case we call n a "heart position".
    * Whether $\alpha_i^{\bar{p},n}$ is in S^0, in S^3 or in S^4.
      If $\alpha_i^{\bar{p},n} \in S^j$, we call n an "S^j-position".

∗ If n is not an S^0-position:[7] The value of $h(\alpha_i^{\bar p,n}) =: (\mathrm{trunk}^{\bar p,n}, \mathrm{loss}^{\bar p,n})$. If n is an S^0-position, we set $\mathrm{loss}^{\bar p,n} := 0$.

- A "countable Δ-system" $\bar p = (p_\ell : \ell \in \omega)$ is a Δ system that additionally satisfies:
  - For each non-heart position[8] $n < m^{\bar p}$, the sequence $(\alpha_\ell^{\bar p,n})_{\ell\in\omega}$ is strictly increasing.
- Let $\bar p$ be a countable Δ-system, and assume that $\bar\Xi = (\Xi_\alpha : \alpha \in u)$ is a sequence such that $u \supseteq \nabla \cap (S^3 \cup S^4)$ and each Ξ_α is a P_α-name of a FAM. Then we define $\lim_{\bar\Xi}(\bar p)$ to be the following function with domain ∇:
  - If $\alpha \in \nabla \cap S^0$, then $\lim_{\bar\Xi}(\bar p)(\alpha)$ is the common value of all $p_n(\alpha)$. (Recall that this value is already determined by the guardrail h.)
  - If $\alpha \in \nabla \cap (S^3 \cup S^4)$, then $\lim_{\bar\Xi}(\bar p)(\alpha)$ is (forced by $\mathbb{P}^5_\alpha$ to be) $\lim_{\Xi_\alpha}(p_\ell(\alpha))_{\ell\in\omega}$ (a random condition if $\alpha \in S^3$, and a $\tilde{\mathbb{E}}$-condition if $\alpha \in S^4$).

(Recall that we have fixed a limit notion $\lim_\Xi$ for random and $\tilde{\mathbb{E}}$, cf. Lemmata 1.10 and 1.18.)

Note that in general $\lim_{\bar\Xi}(\bar p)$ will not be a condition in $\mathbb{P}^5$: For $\alpha \in S^3$, the object $\lim_{\bar\Xi} \bar p(\alpha)$ will be forced to be a random condition, but not necessarily in the *partial* random forcing Q_α; and analogously for $\alpha \in S^4$ and $\tilde{\mathbb{E}}$.

Fact 2.29.
- Each infinite Δ-system $(p_i)_{i\in I}$ contains a countable Δ-system. I.e., there is a sequence i_ℓ in I such that $(p_{i_\ell})_{\ell\in\omega}$ is a countable Δ-system..
- If $\bar p$ is a (countable) Δ-system following h with heart ∇, and $\beta \in \nabla \cup (\max(\nabla + 1))$, then $\bar p \restriction \beta = (p_i \restriction \beta)_{i\in I}$ is again a (countable) Δ-system following h, now with heart $\nabla \cap \beta$.

Recall that we assume all of the parameters defining $\mathbb{P}^5 = (P_\alpha, Q_\alpha)_{\alpha\in\delta_5}$ to be fixed, apart from $\bar w = (w_\alpha)_{\alpha\in S^3\cup S^4}$. Once we fix $\bar w$, we know P_β. We now give a specific way to construct such $\bar w$, which allows to keep $\mathfrak{b}$ small.

Lemma/Construction 2.30. *We can construct by induction on $\alpha \in \delta_5$ for each $h \in H^*$ some Ξ^h_α, and, if $\alpha \in S^3 \cup S^4$, also w_α, such that:*

(a) Each Ξ^h_α is a P_α-name of a FAM extending $\bigcup_{\beta<\alpha} \Xi^h_\beta$.

(b) If α is a limit of countable cofinality: Assume $\bar p$ is a countable Δ-system in P_α following h, and $n < m^{\bar p}$ such that $(\alpha_\ell^{\bar p,n})_{\ell\in\omega}$ has supremum α. Then $A_{\bar p,n}$ is forced to have Ξ^h_α-measure 1*, where*

$$A_{\bar p,n} := \left\{ k \in \omega :\ \left|\{\ell \in I_k : p_\ell(\alpha_\ell^{\bar p,n}) \in G(\alpha_\ell^{\bar p,n})\}\right| \ge |I_k| \cdot \left(1 - \sqrt{\mathrm{loss}^{\bar p,n}}\right)\right\}$$

(c) For each countable Δ-system $\bar p$ in P_α following h the object $\lim_{(\Xi^h_\beta)_{\beta<\alpha}}(\bar p)$ *is a condition in P_α and forces*

$$\Xi^h_\alpha(A_{\bar p}) \ge 1 - \sum_{n<m^{\bar p}} \sqrt{\mathrm{loss}^{\bar p,n}},\ \textit{where}$$

$$A_{\bar p} := \left\{ k \in \omega :\ \left|\{\ell \in I_k : p_\ell \in G_\alpha\}\right| \ge |I_k| \cdot \left(1 - \sum_{n<m^{\bar p}} \sqrt{\mathrm{loss}^{\bar p,n}}\right)\right\}.$$

[7] If n is a S^0-position, $h(\alpha_i^{\bar p,n})$ will generally not be be independent of i; unless of course n is a heart position.

[8] For a heart position n, $(\alpha_\ell^{\bar p,n})_{\ell\in\omega}$ is of course constant.

(d) *For $\alpha \in S^3$: $w_\alpha \subseteq \alpha$, $|w_\alpha| < \lambda_3$, and P_α forces that Q_α is closed under Ξ^h_α-limits for all $h \in H^*$.*
Actually, the set of w_α satisfying this is an ω_1-club set[9] *in $[\alpha]^{<\lambda_3}$.*
Analogously for S^4 and λ_4.

Proof. **(a&c) for $\mathbf{cf}(\alpha) > \omega$:** We set $\Xi^h_\alpha = \bigcup_{\beta<\alpha} \Xi^h_\beta$. As there are no new reals at uncountable confinalities, this is a FAM. Each countable Δ-system is bounded by some $\beta < \alpha$, and, by induction, (c) holds for β; so (c) holds for α as well.

(a&b) for $\mathbf{cf}(\alpha) = \omega$: Fix h. We will show that P_α forces $A \cap \bigcap_{j<j^*} A_{\bar p^j, n^j} \neq \emptyset$, where A is a Ξ^h_β-positive set for some $\beta < \alpha$, and each $(\bar p^j, n^j)$ is as in (b).

Then we can work in the P_α-extension and apply Lemma 1.3(a), using $\bigcup_{\beta<\alpha} \Xi^h_\beta$ as the partial FAM Ξ'. This gives an extension of Ξ' to a FAM that assigns measure one to all $A_{\bar p,n}$. Let Ξ_α be the P_α-name of this extension. So this takes care of (a) and (b).

So assume towards a contradiction that some $p \in P_\alpha$ forces

$$A \cap \bigcap_{j<j^*} A_{\bar p^j, n^j} = \emptyset.$$

We can assume that p decides the β such that $A \in V_\beta$, that β is above the hearts of all Δ-sequences $\bar p^j$ involved, and that $\mathrm{supp}(p) \subseteq \beta$. We can extend p to some $p^* \in P_\beta$ to decide $k \in A$ for some "large" k: By large, we mean:

- Let $F(l; n, p)$ (the cumulative binomial probability distribution) be the probability that n independent experiments, each with success probability p, will have at most l successful outcomes. As $\lim_{n\to\infty} F(n \cdot p'; n, p) = 0$ for all $p' < p$, and as $\lim_{k\to\infty} |I_k| = \infty$, we can find some k such that

$$F(|I_k| p'_j; |I_k|, p_j) < \frac{1}{2 \cdot j^*} \tag{2.31}$$

  for all $j < j^*$, where we set $p'_j \coloneqq 1 - \sqrt{\mathrm{loss}^{\bar p^j, n^j}}$ and $p_j \coloneqq 1 - \frac{1+\sqrt{2}}{2} \cdot \mathrm{loss}^{\bar p^j, n^j}$. (Note that $p'_j < p_j$, as $\mathrm{loss}^{\bar p^j, n^j} \leq \frac{1}{2}$.)
- All elements of $Y = \{\alpha^{\bar p^j, n^j}_\ell : j < j^* \text{ and } \ell \in I_k\}$ are larger than β. (This is possible as each sequence $(\alpha^{\bar p^j, n^j}_\ell)_{\ell<\omega}$) has supremum α.) We enumerate Y by the increasing sequence $(\beta_i)_{i\in M}$, and set $\beta_{-1} = \beta$.

We will find $q \leq p^*$ forcing that $k \in \bigcap_{j<j^*} A_{\bar p^j, n^j}$. To this end, we define a finite tree $\mathcal{T}$ of height M, and assign to each $s \in \mathcal{T}$ of height i a condition $q_s \in P_{\beta_{i-1}+1}$ (decreasing along each branch) and a probability $\mathrm{pr}_s \in [0,1]$, such that $\sum_{t \rhd s} \mathrm{pr}_t = 1$ for all non-terminal nodes $s \in \mathcal{T}$. For s the root of $\mathcal{T}$, i.e., for the unique s of height 0, we set $q_s = p^* \in P_{\beta_{-1}}$ and $\mathrm{pr}_s = 1$.

So assume we have already constructed $q_s \in P_{\beta_{i-1}+1}$ for some s of height $i < M$. We will now take care of index β_i and construct the set of successors of s, and for each successor t, a $q_t \leq q_s$ in P_{β_i+1}.

- If $\beta_i \in S^0$, the guardrail guarantees that $\beta_i \in \mathrm{dom}(p^j_\ell)$ implies $p^j_\ell \restriction \beta_i \Vdash i_{\beta_i}(p^j_\ell(\beta_i)) = h(\beta_i)$. In that case we use a unique $\mathcal{T}$-successor t of s, and we set $q_t = q_s^\frown(\beta_i, i^{-1}_{\beta_i} h(\beta_i))$, and $\mathrm{pr}_t = 1$.

[9] I.e., for each $w^* \in [\alpha]^{<\lambda_4}$ there is a $w_\alpha \supseteq w^*$ satisfying (c), and if $(w^i)_{i\in\omega_1}$ is an increasing sequence of sets satisfying (c), then the limit $w_\alpha \coloneqq \bigcup_{i\in\omega_1} w^i$ satisfies (c) as well.

In the following we assume $\beta_i \notin S^0$.

- Let J_i be the set of $j < j^*$ such that there is an $\ell \in I_k$ with $\alpha_\ell^{\bar p^j, n^j} = \beta_i$ (there is at most one such ℓ). For $j \in J_i$, set $r_i^j = p_\ell^j(\beta_i)$ for the according ℓ. So each r_i^j is a P_{β_i}-name for an element of Q_{β_i}.

  The guardrail gives us the constant value $(\mathtt{trunk}_i^*, \mathrm{loss}_i^*) \coloneqq h(\beta_i)$ (which is equal to $(\mathtt{trunk}^{\bar p^j, n^j}, \mathrm{loss}^{\bar p^j, n^j})$ for all $j \in J_i$).
- The case $\beta_i \in S^3$, i.e., the case of random forcing, is basically [She00, 2.14]:

  For $x \subseteq [\mathtt{trunk}_i^*]$, set $\mathrm{Leb}^{\mathrm{rel}}(x) = \frac{\mathrm{Leb}(x)}{\mathrm{Leb}([\mathtt{trunk}_i^*])}$. Note that the r_i^j are closed subsets of $[\mathtt{trunk}_i^*]$ and $\mathrm{Leb}^{\mathrm{rel}}(r_i^j) \geq 1 - \mathrm{loss}_i^*$.

  Let $\mathcal{B}^*$ be the power set of $[\mathtt{trunk}_i^*]$; and let $\mathcal{B}$ be the sub-Boolean-algebra generated by by r_i^j $(j \in J_i)$, let $\mathcal{X}$ be the set of atoms and $\mathcal{X}' = \{x \in \mathcal{X} : \mathrm{Leb}^{\mathrm{rel}}(x) > 0\}$. So $|\mathcal{X}'| \leq 2^{J_i} \leq 2^{j^*}$, $\sum_{x\in\mathcal{X}'} \mathrm{Leb}^{\mathrm{rel}}(x) = 1$, and $\sum_{x\in X', x\subseteq r_i^j} \mathrm{Leb}^{\mathrm{rel}}(x) = \mathrm{Leb}^{\mathrm{rel}}(r_i^j)$.

  So far, $\mathcal{X}'$ is a P_{β_i}-name. Now we increase q_s inside P_{β_i} to some q^+ deciding which of the (finitely many) Boolean combinations result in elements of $\mathcal{X}'$, and also deciding rational numbers y_x $(x \in \mathcal{X}')$ with sum 1 such that $|\,\mathrm{Leb}^{\mathrm{rel}}(x) - y_x| < \frac{\sqrt{2}-1}{2} \cdot \mathrm{loss}_i^* \cdot 2^{-j^*}$.

  We can now define the immediate successors of s in $\mathcal{T}$: For each $x \in X'$, add an immediate successor t_x and assign to it the probability $\mathrm{pr}_{t_x} = y_x$ and the condition $q_{t_x} = q^{+\frown}(\beta_i, r_x)$, where r_x is a (name for a) partial random condition below x (such a condition exists, as the Lebesgue positive intersection of finitely many partial random condition contains a partial random condition).

  Note that when we choose a successors t randomly (according to the assigned probabilities pr_t), then for each $j \in J$ the probability of $q^+ \Vdash q_t(\beta_i) \leq r_i^j$ is at least $\sum_{x\in X', x\subseteq r_i^j} \mathrm{pr}_x \geq \sum_{x\in X', x\subseteq r_i^j} (\mathrm{Leb}^{\mathrm{rel}}(x) - \frac{\sqrt{2}-1}{2} \cdot \mathrm{loss}_i^* \cdot 2^{-j^*})$
  $\geq \Big(\sum_{x\in X', x\subseteq r_i^j} \mathrm{Leb}^{\mathrm{rel}}(x) \Big) - \frac{\sqrt{2}-1}{2} \cdot \mathrm{loss}_i^* = \mathrm{Leb}^{\mathrm{rel}}(r_i^j) - \frac{\sqrt{2}-1}{2} \cdot \mathrm{loss}_i^*$
  $\geq 1 - \mathrm{loss}_i^* - \frac{\sqrt{2}-1}{2} \cdot \mathrm{loss}_i^* = 1 - \frac{1+\sqrt{2}}{2} \cdot \mathrm{loss}_i^*$.
- The case $\beta_i \in S^4$, i.e., the case of $\tilde{\mathbb{E}}$:

  Recall that $\tilde{\mathbb{E}}$-conditions are subtrees of some basic compact tree T^*, and there is a h such that: if $|I_k|$ many conditions share a common node at height h, then they are compatible.

  All conditions r_i^j have the same stem $s^* = \mathtt{trunk}_i^*$. For each $j \in J_i$, set $d(j) = r_i^j \cap \omega^h$. Note that ($P_{\beta_i}$ forces that) $d(j)$ is a subset of $T^* \cap [s^*] \cap \omega^h$ of relative size $\geq 1 - \frac{1}{2} \mathrm{loss}_i^*$ (according to Lemma 1.17(d)). First find $q^+ \leq q_s$ in P_{β_i} deciding all $d(j)$.

  We can now define the immediate successors of s in $\mathcal{T}$: For each $x \in T^* \cap [s] \cap \omega^h$ add an immediate successor t_x, and assign to it the uniform probability (i.e., $\mathrm{pr}_{t_x} = \frac{1}{|T^*\cap[s]\cap\omega^h|}$) and the condition $q_{t_x} = q^{+\frown}(\beta_i, r_x)$, where r_x is a partial $\tilde{\mathbb{E}}$-condition stronger than all r_i^j that satisfy $x \in d(j)$. (Such a condition exists, as we can intersect $\leq |I_k|$ many conditions of height x.)

  Tf we chose t randomly, then for each $j \in J$ the probability of $q^+ \Vdash q_t \leq r_i^j$ is at least $1 - \frac{1}{2} \mathrm{loss}_i^* \geq 1 - \frac{1+\sqrt{2}}{2} \cdot \mathrm{loss}_i^*$.

In the end, we get a tree $\mathcal{T}$ of height M, and we can chose a random branch through $\mathcal{T}$, according to the assigned probabilities. We can identify the branch with its terminal node t^*, so in this notation the branch t^* has probability $\prod_{n\leq M} \mathrm{pr}_{t\restriction n}$.

Fix $j < j^*$. There are $|I_k|$ many levels $i < M$ such that at β_i we deal with the $(\bar{p}^j, n^j)$-case. Let M^j be the set of these levels. For each $i \in M^j$, we perform an experiment, by asking whether the next step $t \in \mathcal{T}$ (from the current s at level i) will satisfy $q_t \restriction \beta_i \Vdash q_t(\beta_i) \leq r^j_i$. While the exact probability for success will depend on which s at level i we start from, a lower bound is given by $1 - \frac{1+\sqrt{2}}{2} \cdot \mathrm{loss}^*_i$. Recall that $\mathrm{loss}^*_i = \mathrm{loss}^{\bar{p}^j,n^j}$, and that we set $p_j := 1 - \frac{1+\sqrt{2}}{2} \cdot \mathrm{loss}^*_i$ and $p'_j := 1 - \sqrt{\mathrm{loss}^{\bar{p}^j,n^j}}$ in (2.31). So the chance of our branch t^* having success fewer than $|I_k| \cdot (1 - \sqrt{\mathrm{loss}^{\bar{p}^j,n^j}})$ many times, out of the the $|I_k|$ many tries, (let us call such a t^* "bad for j") is at most $F(|I_k|p'; |I_k|, p) \leq \frac{1}{2j^*}$.

Accordingly, the measure of branches that are not bad for *any* $j < j^*$ is at least $\frac{1}{2}$. Fix such a branch t^*. Then for each $j < j^*$,

$$\left|\{i \in M^j \,:\, q_{t^*} \restriction \beta_i \Vdash q_{t^*}(\beta_i) \leq r^j_i\}\right| \geq |I_k| \cdot \left(1 - \sqrt{\mathrm{loss}^{\bar{p}^j,n^j}}\right),$$

and thus q_{t^*} forces that

$$\left|\{\ell \in I_k \,:\, p_\ell(\alpha^{\bar{p}^j,n^j}_\ell) \in G(\alpha^{\bar{p}^j,n^j}_\ell)\}\right| \geq |I_k| \cdot \left(1 - \sqrt{\mathrm{loss}^{\bar{p}^j,n^j}}\right).$$

(c) for $\mathrm{cf}(\alpha) = \omega$:

Fix $\bar{p}$ as in the assumption of (c). To simplify notation, let us assume that $\nabla \neq \emptyset$ and that $\sup(\nabla) < \sup(\mathrm{dom}(p_\ell))$ (for some, or equivalently: all, $\ell \in \omega$). Let $0 < n_0 < m^{\bar{p}}$ be such that $\sup(\nabla)$ is at position $n_0 - 1$ in $\mathrm{dom}(p_\ell)$, i.e., $\sup(\nabla) = \alpha^{\bar{p},n_0-1}_\ell$ (independent of ℓ), and set $\beta := \sup(\nabla) + 1$.

$\bar{p} \restriction \beta$ is again a countable Δ-system following the same h, and $\lim_{(\Xi^h_\gamma)_{\gamma<\alpha}}(\bar{p})$ is by definition identical to $\lim_{(\Xi^h_\gamma)_{\gamma<\beta}}(\bar{p} \restriction \beta)$, which by induction is a valid condition and forces (c) for $\bar{p} \restriction \beta + 1$. This gives us the set $A_{\bar{p}\restriction\beta}$ of measure at least $1 - \sum_{n<n_0} \sqrt{\mathrm{loss}^{\bar{p},n}}$.

For the positions $n_0 \leq n < m^{\bar{p}}$, all $(\alpha^{\bar{p},n}_\ell)_{\ell\in\omega}$ are strictly increasing sequences above β with some limit $\alpha_j \leq \alpha$. Then (b) (applied to α_j) gives us an according measure-1-set $A_{\bar{p},n}$.

So $\lim_{(\Xi^h_\gamma)_{\gamma<\alpha}}(\bar{p})$ forces that $A' = A_{\bar{p}\restriction\beta} \cap \bigcap_{n_0\leq n<m^{\bar{p}}} A_{\bar{p},n}$ has measure $\Xi^h_\alpha(A') \geq 1 - \sum_{n<n_0} \sqrt{\mathrm{loss}^{\bar{p},n}} \geq 1 - \sum_{n<m^{\bar{p}}} \sqrt{\mathrm{loss}^{\bar{p},n}}$.

Note that $p_\ell \in G$ iff $p_\ell \restriction \beta \in G_\beta$ and $p_\ell(\alpha^{\bar{p},n}) \in G(\alpha^{\bar{p},n})$ for all $n_0 \leq n < m^{\bar{p}}$.

Fix $k \in A'$. As $k \in A_{\bar{p}\restriction\beta}$, the relative frequency for $\ell \in I_k$ to *not* satisfy $p_\ell \restriction \beta \in G_\beta$ is at most $\sum_{n<n_0} \sqrt{\mathrm{loss}^{\bar{p},n}}$. For any $n_0 \leq n < m^{\bar{p}}$, as $k \in A_{\bar{p},n}$, the relative frequency for *not* $p_\ell(\alpha^{\bar{p},n}) \in G(\alpha^{\bar{p},n})$ is at most $\sqrt{\mathrm{loss}^{\bar{p},n}}$. So the relative frequency for $p_\ell \in G$ to fail is at most $\sum_{n<n_0} \sqrt{\mathrm{loss}^{\bar{p},n}} + \sum_{n_0\leq n<m^{\bar{p}}} \sqrt{\mathrm{loss}^{\bar{p},n}}$, as required.

(a&c) for $\alpha = \gamma + 1$ successor:

For $\alpha \in S^0$ this is clear: Let Ξ^h_α be the name of some FAM extending Ξ^h_γ. Let $\bar{p}$ be as in (c), without loss of generality $\gamma \in \nabla$. Then $q^+ := \lim_{(\Xi^h_\beta)_{\beta<\alpha}}(\bar{p}) = q^\frown(\gamma, r)$, where $q := \lim_{(\Xi^h_\beta)_{\beta<\gamma}}(\bar{p} \restriction \gamma)$ and r is the condition determined by $h(\gamma)$, i.e., each $p_\ell \restriction \gamma$ forces $p_\ell(\gamma) = r$. In particular, q^+ forces that $p_\ell \in G_\alpha$ iff $p_\ell \restriction \gamma \in G_\alpha$. By induction, (c) holds for γ, and therefore we get (c) for α.

Assume $\alpha \in S^3$. According to 1.10, random forcing has FAM limits; and as argued in Fact 2.13, this also applies to Q_γ, as we already know (d) for γ. So we can apply in the P_γ-extension Lemma 1.8, which gives us the required $\Xi^h_\alpha := \Xi^+$. Given $\bar{p}$ as in (c), again without loss of generality $\gamma \in \nabla$, we set $q^+ := \lim_{(\Xi^h_\beta)_{\beta<\alpha}}(\bar{p}) = q^\frown(\gamma, r)$, where $q := \lim_{(\Xi^h_\beta)_{\beta<\gamma}}(\bar{p} \restriction \gamma)$ and r is $\lim_{\Xi^+}(p_\ell(\gamma))_{\ell\in\omega}$. Then by induction q forces that $\Xi^h_\gamma(A_{\bar{p}\restriction\gamma}) \geq 1 - \sum_{n<m^p-1}\sqrt{\mathrm{loss}^{\bar{p},n}}$. In the P_γ-extention, Lemma 1.8 gives us that r forces that $\Xi^+(A_{(p_\ell(\gamma))_{\ell\in\omega}}) \geq 1 - \sqrt{\mathrm{loss}^{\bar{p},m^p-1}}$. So $q^\frown r$ forces that $\Xi^h_\alpha(A_{\bar{p}}) \geq 1 - \sum_{n<m^p}\sqrt{\mathrm{loss}^{\bar{p},n}}$.

The argument for $\alpha \in S^4$ is the same.

(d) If $\alpha \in S^3$, let $\lambda := \lambda_3$ and let Q be random forcing. If $\alpha \in S^4$, let $\lambda := \lambda_4$ and let Q be $\tilde{\mathbb{E}}$.

For any $w \subseteq \alpha$, let Q^w be the (P_α-name for) the partial Q forcing defined using w. Start with some $w^0 \subseteq \alpha$ of size $<\lambda$. There are $|w^0|^{\aleph_0}$ many sequences in Q^{w^0}. For any $h \in H^*$ and any such sequence, the Ξ^h_α-limit is a real; so we can extend w^0 by a countable set to some w' such that $Q^{w'}$ contains this specific limit. We can do that for all $h \in H^*$ and all sequences, resulting in some $w^1 \supseteq w^0$ still of size $<\lambda$. We iterate this construction and get w^i for $i \leq \omega_1$, taking the unions at limits. Then $w_\alpha := w^{\omega_1}$ is as required, as $Q_\alpha := Q^{w_\alpha} = \bigcup_{i<\omega_1} Q^{w_i}$.

So this proof actually shows that the set of w_α with the desired property is an ω_1-club. $\square$

Note that in (c), the condition $\lim_{\bar{\Xi}^h}\bar{p}$ in particular forces infinitely many p_ℓ to be in G. So after carrying out the construction as above, we get a forcing notion $\mathbb{P}^5$ satisfying:

(2.32) For every countable Δ-system $\bar{p}$ there is some q forcing that infinitely many p_ℓ are in the generic filter.

Lemma 2.33. $\mathsf{LCU}_2(\mathbb{P}^5, \kappa)$ *for* $\kappa \in [\lambda_2, \lambda_5]$, *witnessed by the sequence* $(c_\alpha)_{\alpha<\kappa}$ *of the first* κ *many Cohen reals.*

Proof. Fix a $\mathbb{P}^5$-name $y \in \omega^\omega$. We have to show that $(\exists\alpha \in \kappa)\,(\forall\beta \in \kappa \setminus \alpha)\,\mathbb{P}^5 \Vdash \neg c_\beta \leq^* y)$.

Assume towards a contradiction that p^* forces that there are unboundedly many $\alpha \in \kappa$ with $c_\alpha \leq^* y$, and enumerate them as $(\alpha_i)_{i\in\kappa}$. Pick $p^i \leq p^*$ deciding α_i to be some β^i, and also deciding n_i such that $(\forall m \geq n_i)\, c_{\alpha_i}(m) \leq y(m)$. We can assume that $\beta^i \in \mathrm{dom}(p^i)$. Note that β^i is a Cohen position (as $\beta^i < \kappa \leq \lambda_5$), and we can assume that $p^i(\beta^i)$ is a Cohen condition in V (and not just a P_{β_i}-name for such a condition). By strengthening and thinning out, we may assume:

- $(p^i)_{i\in\kappa}$ forms a Δ system with heart ∇.
- All n_i are equal to some n^*.
- $p^i(\beta_i)$ is always the same Cohen condition $s \in \omega^{<\omega}$, without loss of generality of length $|s| = n^{**} \geq n^*$.
- For some position $n < m^{\bar{p}}$, β^i is the n-th element of $\mathrm{supp}(p^i)$.

Note that this n cannot be a heart condition: For any $\beta \in \kappa$, at most $|\beta|$ many p^i can force $\alpha_i = \beta$, as p^i forces that $\alpha_i \geq i$ for all i.

Pick a countable subset of this Δ-system which forms a countable Δ-system $(p_\ell)_{\ell\in\omega}$. So $p_\ell = p^{i_\ell}$ for some $i_\ell \in I$, and we set $\beta_\ell = \beta^{i_\ell}$. In particular all β_ℓ are distinct. Now extend each p_ℓ to p'_ℓ by extending the Cohen condition $p_\ell(\beta_\ell) = s$ to $s^\frown\ell$ (i.e., forcing $c_{\beta_\ell}(n^{**}) = \ell$). Note that $(p'_i)_{i\in\omega}$ is still a countable Δ-system (following some new guardrail); accordingly there is a q forcing that infinitely many of the p'_ℓ are in the generic filter. But each such p'_ℓ forces that $c_{\beta_\ell}(n^{**}) = \ell \leq y(n^{**})$, a contradiction. □

2.4. **The left hand side.** We have now finished the consistency proof for the left hand side:

Theorem 2.34. *Assume GCH and let λ_i be an increasing sequence of regular cardinals for $i = 1, \dots, 5$. Then there is a cofinalities-preserving forcing P resulting in*

$$\mathrm{add}(\mathcal{N}) = \lambda_1 < \mathrm{add}(\mathcal{M}) = \mathfrak{b} = \lambda_2 < \mathrm{cov}(\mathcal{N}) = \lambda_3 < \mathrm{non}(\mathcal{M}) = \lambda_4 < \\ < \mathrm{cov}(\mathcal{M}) = 2^{\aleph_0} = \lambda_5.$$

Proof. Set $\chi = \lambda_2$, and let R be a $<\chi$-closed, χ^+-cc forcing which forces $2^\chi = \lambda_5$. So in the R-extension, Assumption 2.21 is satisfied, and we can construct $\mathbb{P}^5$ according to Assumption 2.14 and Construction 2.30. Fact 2.20 gives us all inequalities for the left hand side, apart from $\mathfrak{b} \leq \lambda_2$, which we get from 2.33.

In the R-extension, CH holds and P is a FS ccc iteration of length δ_5, $|\delta_5| = \lambda_5$, and each iterand is a set of reals; so $2^{\aleph_0} \leq \lambda_5$ is forced. Also, any FS ccc iteration of length δ (of nontrivial iterands) forces $\mathrm{cov}(\mathcal{M}) \geq \mathrm{cf}(\delta)$: Without loss of generality $\mathrm{cf}(\delta) = \lambda$ is uncountable. Any set A of (Borel codes for) meager sets that has size $<\lambda$ already appears at some stage $\alpha < \delta$, and the iteration at state $\alpha + \omega$ adds a Cohen real over the V_α, so A will not cover all reals. □

Remark 2.35. So this consistency result is reasonably general, we can, e.g., use the values $\lambda_i = \aleph_{i+1}$, or $\lambda_i = \aleph_{\omega \cdot i+1}$. This is in contrast to the result for the whole diagram, where λ_2 has to be the successor of a regular, and the small λ_i have to be separated by strongly compact cardinals.

3. Ten different values in Cichoń's diagram

We can now apply, with hardly any change, the technique of [GKS] to get the following:

Theorem 3.1. *Assume GCH and that $\aleph_1 < \kappa_9 < \lambda_1 < \kappa_8 < \lambda_2 < \kappa_7 < \lambda_3 < \kappa_6 < \lambda_4 < \lambda_5 < \lambda_6 < \lambda_7 < \lambda_8 < \lambda_9$ are regular, $\lambda_2 = \chi^+$ with χ regular, and κ_i strongly compact for $i = 6, 7, 8, 9$. Then there is a ccc forcing notion $\mathbb{P}^9$ resulting in:*

$$\mathrm{add}(\mathcal{N}) = \lambda_1 < \mathfrak{b} = \mathrm{add}(\mathcal{M}) = \lambda_2 < \mathrm{cov}(\mathcal{N}) = \lambda_3 < \mathrm{non}(\mathcal{M}) = \lambda_4 < \\ < \mathrm{cov}(\mathcal{M}) = \lambda_5 < \mathrm{non}(\mathcal{N}) = \lambda_6 < \mathfrak{d} = \mathrm{cof}(\mathcal{M}) = \lambda_7 < \mathrm{cof}(\mathcal{N}) = \lambda_8 < 2^{\aleph_0} = \lambda_9.$$

To do this, we first have to show that we can achieve the order for the left hand side, i.e., Theorem 2.34, starting with GCH and using a FS ccc iteration $\mathbb{P}^5$ alone (instead of using $P = R * \mathbb{P}^5$, where R is not ccc). This is the argument that requires $\lambda_2 = \chi^+$. We will just briefly sketch it here, as it can be found with all details in [GKS, 1.4]:

- We already know that in the R-extension, (where R is $<\chi$-closed, χ^+-cc and forces $2^\chi = \lambda_5$) we can find by the inductive construction 2.30 suitable w_α such that $R * \mathbb{P}^5$ works.
- We now perform a similar inductive construction in the ground model: At stage α, we know that there is an R-name for a suitable w^1_α of size $< \lambda_i$ (where i is 3 in the random and 4 in the $\tilde{\mathbb{E}}$-case). This name can be covered by some set $\tilde{w}^1_\alpha$ in V, still of size $< \lambda_i$, as R is χ^+-cc. Moreover, in the R-extension, the suitable parameters form an ω_1-club; so there is a suitable $w^2_\alpha \supseteq \tilde{w}^1_\alpha$, etc. Iterating ω_1 many times and taking the union at the end leads to w_α in V which is forced by R to be suitable.

- Not only w_α is in V, but the construction for w_α is performed in V, so we can construct the whole sequence $\bar{w} = (w_\alpha)_{\alpha\in\delta_5}$ in V.
- We now know that in the R-extension, the forcing $\mathbb{P}^5$ defined from $\bar{w}$ will satisfy $\mathsf{LCU}_2(\mathbb{P}^5, \kappa)$ in the form of Lemma 2.33.
- By an absoluteness argument, we can show that actually in V the forcing $\mathbb{P}^5$ defined form $\bar{w}$ will satisfy Lemma 2.33 as well.

The rest of the proof is the same as in [GKS, Sec. 2], where we interchange $\mathfrak{b}$ and $\mathrm{cov}(\mathcal{N})$ as well as $\mathfrak{d}$ and $\mathrm{non}(\mathcal{N})$.

We cite the following facts from [GKS, 2.2–2.5]:

Facts 3.2. (a) If κ is a strongly compact cardinal and $\theta > \kappa$ regular, then there is an elementary embedding $j_{\kappa,\theta} : V \to M$ (in the following just called j) such that
- the critical point of j is κ, $\mathrm{cf}(j(\kappa)) = |j(\kappa)| = \theta$,
- $\max(\theta, \lambda) \leq j(\lambda) < \max(\theta, \lambda)^+$ for all $\lambda \geq \kappa$ regular, and
- $\mathrm{cf}(j(\lambda)) = \lambda$ for $\lambda \neq \kappa$ regular,

and such that the following is satisfied:

(b) If P is a FS ccc iteration along δ, then $j(P)$ is a FS ccc iteration along $j(\delta)$.

(c) $\mathsf{LCU}_i(P, \lambda)$ implies $\mathsf{LCU}_i(j(P), \mathrm{cf}(j(\lambda)))$, and thus $\mathsf{LCU}_i(j(P), \lambda)$ if $\lambda \neq \kappa$ regular.[10]

(d) If $\mathsf{COB}_i(P, \lambda, \mu)$, then $\mathsf{COB}_i(j(P), \lambda, \mu')$, for $\mu' = \begin{cases} |j(\mu)| & \text{if } \kappa > \lambda \\ \mu & \text{if } \kappa < \lambda. \end{cases}$

Using these facts, it is easy to finish the proof:[11]

Proof of Theorem 3.1.
- Start with $\mathbb{P}^5$. This is an iteration of length δ_5 with $\mathrm{cf}(\delta_5) = |\delta_5| = \lambda_5$, satisfying:

(3.3) For all i: $\mathsf{LCU}_i(\mathbb{P}^5, \mu)$ for all $\mu \in [\lambda_i, \lambda_5]$, and $\mathsf{COB}_i(\mathbb{P}^5, \lambda_i, \lambda_5)$.

As a consequence,the characteristics

$$\begin{pmatrix} \mathrm{cov}(\mathcal{N}) & \mathrm{non}(\mathcal{M}) & \mathfrak{d} & \mathrm{cof}(\mathcal{N}) \\ \mathrm{add}(\mathcal{N}) & \mathfrak{b} & \mathrm{cov}(\mathcal{M}) & \mathrm{non}(\mathcal{N}) \end{pmatrix}$$

are forced by $\mathbb{P}^5$ to have values[12]

$$\begin{pmatrix} \lambda_3 & \lambda_4 & \lambda_5 & \lambda_5 \\ \lambda_1 & \lambda_2 & \lambda_5 & \lambda_5 \end{pmatrix}$$

- Consider the embedding $j_5 := j_{\kappa_6,\lambda_6}$. According to 3.2(b), $\mathbb{P}^6 := j_5(\mathbb{P}^5)$ is a FS ccc iteration of length $\delta_6 := j_5(\delta_5)$. As $|\delta_6| = \lambda_6$, the continuum is forced to have size λ_6.

  For $i = 1$, we have $\mathsf{LCU}_1(\mathbb{P}^5, \mu)$ for all regular $\mu \in [\lambda_1, \lambda_5]$, so using 3.2(c) we get $\mathsf{LCU}_1(\mathbb{P}^6, \mu)$ for all regular $\mu \in [\lambda_1, \lambda_5]$ different to κ_6; as well as $\mathsf{LCU}_1(\mathbb{P}^6, \lambda_6)$ (as $\mathrm{cf}(j(\kappa_6)) = \lambda_6$). For $\mu = \lambda_1$ the former implies $\mathbb{P}^6 \Vdash \mathrm{add}(\mathcal{N}) \leq \lambda_1$, and the latter $\mathbb{P}^6 \Vdash \mathrm{cof}(\mathcal{N}) \geq \lambda_6 = 2^{\aleph_0}$.

[10]In [GKS], we only used "classical" relations R_3 that are defined on a polish space in an absolute way. In this paper, we use the relation R_3 which is not of this kind. However, the proof still works without any change: The parameter $\mathcal{E}$ used to define the relation R_3, cf. Definition 2.2, is a set of reals. So $j(\mathcal{E}) = \mathcal{E}$, and we can still use the usual absoluteness arguments between M and V. (A parameter not element of $H(\kappa_9)$ might be a problem.)

[11]This is identical to the argument in [GKS], with the roles of $\mathfrak{b}$ and $\mathrm{cov}(\mathcal{N})$, as well as their duals, switched.

[12]These values, and the ones forced by the "intermediate forcings" $\mathbb{P}^6$ to $\mathbb{P}^8$, are not required for the argument; they should just illustrate what is going on.

More generally, we get from (3.3) and 3.2(c)

(3.4) For all i: $\mathsf{LCU}_i(\mathbb{P}^6, \mu)$ for all regular $\mu \in [\lambda_i, \lambda_5] \setminus \{\kappa_6\}$.
For $i < 4$: $\mathsf{LCU}_i(\mathbb{P}^6, \lambda_6)$.

So in particular for $\mu = \lambda_i$, we see that the characteristics on the left do not increase; for $\mu = \lambda_5$ that the ones on the right are still at least λ_5; and for $i < 4$ an $\mu = \lambda_6$ that the according characteristics on the right will have size continuum. (But not for $i = 4$, as $\kappa_4 < \lambda_4$. And we will see that non$(\mathcal{M})$ will be below the continuum.)

Dually, because $\lambda_3 < \kappa_6 < \lambda_4$, we get from (3.3) and 3.2(d)

(3.5) For $i < 4$: $\mathsf{COB}_i(\mathbb{P}^6, \lambda_i, \lambda_6)$. For $i = 4$: $\mathsf{COB}_4(\mathbb{P}^6, \lambda_4, \lambda_5)$.

(The former because $|j_6(\lambda_5)| = \max(\lambda_6, \lambda_5) = \lambda_6$.) So the characteristics on the left do not decrease, and $\mathbb{P}^6 \Vdash \text{non}(\mathcal{M}) \leq \lambda_5$.

Accordingly, $\mathbb{P}^6$ forces the following values:

$$\begin{pmatrix} \lambda_3 & \lambda_4 & \lambda_6 & \lambda_6 \\ \lambda_1 & \lambda_2 & \lambda_5 & \lambda_6 \end{pmatrix}$$

- We now apply a new embedding, $j_7 \coloneqq j_{\kappa_7,\lambda_7}$, to the forcing $\mathbb{P}^6$ that we just constructed. (We always work in V, not in any inner model M or any forcing extention.) As before, set $\mathbb{P}^7 \coloneqq j_7(\mathbb{P}^6)$, a FS ccc iteration of length $\delta_7 = j_7(\delta_6)$, forcing the continuum to have size λ_7.

  Now $\kappa_7 \in (\lambda_2, \lambda_3)$, so arguing as before, we get from (3.4)

(3.6) For all i: $\mathsf{LCU}_i(\mathbb{P}^7, \mu)$ for all regular $\mu \in [\lambda_i, \lambda_5] \setminus \{\kappa_6, \kappa_7\}$.
For $i < 4$: $\mathsf{LCU}_i(\mathbb{P}^7, \lambda_6)$. For $i < 3$: $\mathsf{LCU}_i(\mathbb{P}^7, \lambda_7)$.

and from (3.5)

(3.7) For $i < 3$: $\mathsf{COB}_i(\mathbb{P}^7, \lambda_i, \lambda_7)$.
For $i = 3$: $\mathsf{COB}_3(\mathbb{P}^7, \lambda_3, \lambda_6)$. For $i = 4$: $\mathsf{COB}_4(\mathbb{P}^7, \lambda_4, \lambda_5)$.

Accordingly, $\mathbb{P}^7$ forces the following values:

$$\begin{pmatrix} \lambda_3 & \lambda_4 & \lambda_7 & \lambda_7 \\ \lambda_1 & \lambda_2 & \lambda_5 & \lambda_6 \end{pmatrix}$$

- Now we set $\mathbb{P}^8 \coloneqq j_{\kappa_8,\lambda_8}(\mathbb{P}^7)$, a FS ccc iteration of length δ_8. Now $\kappa_8 \in (\lambda_1, \lambda_2)$, and as before, we get from (3.6)

(3.8) For all i: $\mathsf{LCU}_i(\mathbb{P}^8, \mu)$ for all regular $\mu \in [\lambda_i, \lambda_5] \setminus \{\kappa_6, \kappa_7, \kappa_8\}$.
For $i < 4$: $\mathsf{LCU}_i(\mathbb{P}^8, \lambda_6)$. For $i < 3$: $\mathsf{LCU}_i(\mathbb{P}^8, \lambda_7)$.
For $i < 2$ (i.e., $i = 1$): $\mathsf{LCU}_1(\mathbb{P}^8, \lambda_8)$.

and from (3.7)

(3.9) For $i = 1$: $\mathsf{COB}_1(\mathbb{P}^8, \lambda_1, \lambda_8)$. For $i = 2$: $\mathsf{COB}_2(\mathbb{P}^8, \lambda_2, \lambda_7)$.
For $i = 3$: $\mathsf{COB}_3(\mathbb{P}^8, \lambda_3, \lambda_6)$. For $i = 4$: $\mathsf{COB}_4(\mathbb{P}^8, \lambda_4, \lambda_5)$.

Accordingly, $\mathbb{P}^8$ forces the following values:

$$\begin{pmatrix} \lambda_3 & \lambda_4 & \lambda_7 & \lambda_8 \\ \lambda_1 & \lambda_2 & \lambda_5 & \lambda_6 \end{pmatrix}$$

- Finally we set $\mathbb{P}^9 \coloneqq j_{\kappa_9,\lambda_9}(\mathbb{P}^8)$, a FS ccc iteration of length δ_9 with $|\delta_9| = \lambda_9$, i.e., the continuum will have size λ_9. As $\kappa_9 < \lambda_1$, (3.8) and (3.9) also hold for $\mathbb{P}^9$ instead of $\mathbb{P}^8$. Accordingly, we get the same values for the diagram as for $\mathbb{P}^8$, and additionally continuum of size λ_9. □

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