TWO RESULTS ON CARDINAL INVARIANTS AT UNCOUNTABLE CARDINALS

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Abstract. We prove two ZFC theorems about cardinal invariants above the continuum which are in sharp contrast to well-known facts about these same invariants at the continuum. It is shown that for an uncountable regular cardinal κ, $\mathfrak{b}(\kappa) = \kappa^+$ implies $\mathfrak{a}(\kappa) = \kappa^+$. This improves an earlier result of Blass, Hyttinen, and Zhang [3]. It is also shown that if $\kappa \geq \beth_\omega$ is an uncountable regular cardinal, then $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$. This result partially dualizes an earlier theorem of the authors [6].

1. Introduction

The theory of cardinal invariants at uncountable regular cardinals remains less developed than the theory at ω. One of the first papers to explore the situation above ω was by Cummings and Shelah [4]. In that paper, they considered the direct analogues of the bounding and dominating numbers. They also considered bounding and domination modulo the club filter, a notion which has no counterpart at ω but which becomes very natural at uncountable regular cardinals. Recall the following definitions.

Definition 1. Let $\kappa > \omega$ be a regular cardinal. Let $f, g \in \kappa^\kappa$. $f \leq^* g$ means that $|\{\alpha < \kappa : g(\alpha) < f(\alpha)\}| < \kappa$ and $f \leq_{\mathrm{cl}} g$ means that $\{\alpha < \kappa : g(\alpha) < f(\alpha)\}$ is non-stationary. We say that $F \subset \kappa^\kappa$ is $*$-*unbounded* if $\neg \exists g \in \kappa^\kappa \forall f \in F\, [f \leq^* g]$ and we say that F is cl-*unbounded* if $\neg \exists g \in \kappa^\kappa \forall f \in F\, [f \leq_{\mathrm{cl}} g]$. Define

$$\mathfrak{b}(\kappa) = \min\{|F| : F \subset \kappa^\kappa \wedge F \text{ is } *\text{-unbounded}\},$$
$$\mathfrak{b}_{\mathrm{cl}}(\kappa) = \min\{|F| : F \subset \kappa^\kappa \wedge F \text{ is cl-unbounded}\}.$$

We say that $F \subset \kappa^\kappa$ is $*$-*dominating* if $\forall g \in \kappa^\kappa \exists f \in F\, [g \leq^* f]$ and we say that F is cl-*dominating* if $\forall g \in \kappa^\kappa \exists f \in F\, [g \leq_{\mathrm{cl}} f]$. Define

$$\mathfrak{d}(\kappa) = \min\{|F| : F \subset \kappa^\kappa \text{ and } F \text{ is } *\text{-dominating}\}.$$
$$\mathfrak{d}_{\mathrm{cl}}(\kappa) = \min\{|F| : F \subset \kappa^\kappa \text{ and } F \text{ is cl-dominating}\}.$$

Cummings and Shelah [4] proved that for any regular κ, $\kappa^+ \leq \mathrm{cf}(\mathfrak{b}(\kappa)) = \mathfrak{b}(\kappa) \leq \mathrm{cf}(\mathfrak{d}(\kappa)) \leq \mathfrak{d}(\kappa) \leq 2^\kappa$, and that these are the only relations between $\mathfrak{b}(\kappa)$ and $\mathfrak{d}(\kappa)$ that are provable in ZFC, thereby generalizing a classical result of Hechler from the case $\kappa = \omega$. Quite remarkably, they also showed that for every regular $\kappa > \omega$, $\mathfrak{b}(\kappa) = \mathfrak{b}_{\mathrm{cl}}(\kappa)$, and that if $\kappa \geq \beth_\omega$ is regular, then $\mathfrak{d}(\kappa) = \mathfrak{d}_{\mathrm{cl}}(\kappa)$. The question of whether $\mathfrak{d}_{\mathrm{cl}}(\kappa) < \mathfrak{d}(\kappa)$ is consistent for any κ was left open; as far as we are aware, it remains open.

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Other early papers which studied the splitting number at uncountable cardinals revealed interesting differences with the situation at ω. Recall the following definitions.

Definition 2. Let $\kappa > \omega$ be a regular cardinal. For $A, B \in \mathcal{P}(\kappa)$, $A \subset^* B$ means $|A \setminus B| < \kappa$. For a family $F \subset [\kappa]^\kappa$ and a set $B \in \mathcal{P}(\kappa)$, B is said to *reap* F if for every $A \in F$, $|A \cap B| = |A \cap (\kappa \setminus B)| = \kappa$. We say that $F \subset [\kappa]^\kappa$ is *unreaped* if there is no $B \in \mathcal{P}(\kappa)$ that reaps F.

$$\mathfrak{r}(\kappa) = \min\left\{|F| : F \subset [\kappa]^\kappa \text{ and } F \text{ is unreaped}\right\}.$$

A family $F \subset \mathcal{P}(\kappa)$ is called a *splitting family* if

$$\forall B \in [\kappa]^\kappa \exists A \in F\left[|B \cap A| = |B \cap (\kappa \setminus A)| = \kappa\right].$$

$$\mathfrak{s}(\kappa) = \min\left\{|F| : F \subset \mathcal{P}(\kappa) \text{ and } F \text{ is a splitting family}\right\}.$$

For instance, Suzuki [10] showed that for a regular cardinal $\kappa > \omega$, $\mathfrak{s}(\kappa) \geq \kappa$ iff κ is strongly inaccessible and $\mathfrak{s}(\kappa) \geq \kappa^+$ iff κ is weakly compact. Zapletal [11] additionally showed that the statement that there exists some regular uncountable cardinal κ for which $\mathfrak{s}(\kappa) \geq \kappa^{++}$ has large consistency strength, significantly more than a measurable cardinal. More recently, the authors proved in [6] that $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$ for all regular $\kappa > \omega$. This is in marked contrast to the situation at ω, where it is known that $\mathfrak{s}(\omega)$ and $\mathfrak{b}(\omega)$ are independent. More information about cardinal invariants at ω can be found in [2].

Blass, Hyttinen, and Zhang [3] is a work about the almost disjointness number at regular uncountable cardinals. Let us recall the definition of maximal almost disjoint families.

Definition 3. Let $\kappa > \omega$ be a regular cardinal. $A, B \in [\kappa]^\kappa$ are said to be *almost disjoint* or *a.d.* if $|A \cap B| < \kappa$. A family $\mathscr{A} \subset [\kappa]^\kappa$ is said to be *almost disjoint* or *a.d.* if the members of $\mathscr{A}$ are pairwise a.d. Finally $\mathscr{A} \subset [\kappa]^\kappa$ is called *maximal almost disjoint* or *m.a.d.* if $\mathscr{A}$ is an a.d. family, $|\mathscr{A}| \geq \kappa$, and $\mathscr{A}$ cannot be extended to a larger a.d. family in $[\kappa]^\kappa$.

$$\mathfrak{a}(\kappa) = \min\left\{|\mathscr{A}| : \mathscr{A} \subset [\kappa]^\kappa \text{ and } \mathscr{A} \text{ is m.a.d.}\right\}.$$

Blass, Hyttinen, and Zhang [3] proved that if $\kappa > \omega$ is regular, then $\mathfrak{d}(\kappa) = \kappa^+$ implies $\mathfrak{a}(\kappa) = \kappa^+$. This is potentially different from the situation at ω: it remains an open problem whether $\mathfrak{d}(\omega) = \aleph_1$ implies $\mathfrak{a}(\omega) = \aleph_1$, while Shelah [8] showed the consistency of $\mathfrak{d}(\omega) = \aleph_2 < \aleph_3 = \mathfrak{a}(\omega)$ (see also Question 14).

There is also a well-developed theory of duality for cardinal invariants at ω. Thus, for example, $\mathfrak{b}(\omega)$ and $\mathfrak{d}(\omega)$ are dual to each other, while $\mathfrak{s}(\omega)$ and $\mathfrak{r}(\omega)$ are duals. The ZFC inequality $\mathfrak{s}(\omega) \leq \mathfrak{d}(\omega)$ dualizes to the inequality $\mathfrak{b}(\omega) \leq \mathfrak{r}(\omega)$, and indeed even the proof of $\mathfrak{s}(\omega) \leq \mathfrak{d}(\omega)$ dualizes to the proof of $\mathfrak{b}(\omega) \leq \mathfrak{r}(\omega)$. It is possible to make this notion of duality precise using Galois-Tukey connections. We refer the reader to [2] for further details about duality of cardinal invariants at ω. It is unclear at present if there can be a smooth theory of duality for cardinal invariants at uncountable cardinals too. For example, if we try to naïvely dualize Suzuki's result mentioned above that $\mathfrak{s}(\kappa)$ is small for most κ, then we would be trying to show that $\mathfrak{r}(\kappa)$ is large for most κ. In other words, we might expect to show that if κ is not weakly compact, then $\mathfrak{r}(\kappa) = 2^\kappa$. However it is still an open problem whether the inequality $\mathfrak{r}(\aleph_1) < 2^{\aleph_1}$ is consistent (see Question 16). Nevertheless, it is of interest to ask whether for all regular $\kappa > \omega$ the result from [6] that $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$ can be dualized to the result that $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$.

We present two further ZFC theorems on cardinal invariants at uncountable regular cardinals in the paper. Our first result, Theorem 5, says that if $\kappa > \omega$ is regular, then $\mathfrak{b}(\kappa) = \kappa^+$ implies $\mathfrak{a}(\kappa) = \kappa^+$. This improves the above mentioned result of Blass, Hyttinen, and Zhang [3]. It also shows that ω is unique among regular cardinals in that it is the only such κ where $\mathfrak{b}(\kappa) = \kappa^+ < \kappa^{++} = \mathfrak{a}(\kappa)$ is consistent. Our next result, Theorem 13, is a partial dual to our earlier result from [6]. It says that for all regular cardinals $\kappa \geq \beth_\omega$, $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$. Thus for sufficiently large κ, the invariants $\mathfrak{s}(\kappa), \mathfrak{b}(\kappa), \mathfrak{d}(\kappa)$, and $\mathfrak{r}(\kappa)$ are provably comparable and ordered as $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa) \leq \mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$. The proof of our first theorem makes use of the equality $\mathfrak{b}(\kappa) = \mathfrak{b}_{\mathrm{cl}}(\kappa)$ of Cummings and Shelah [4] discussed before. Their theorem that $\mathfrak{d}(\kappa) = \mathfrak{d}_{\mathrm{cl}}(\kappa)$ for all regular $\kappa \geq \beth_\omega$ is not directly used. However the main idea of the proof of our Theorem 13 is similar to the main idea in the proof of $\mathfrak{d}(\kappa) = \mathfrak{d}_{\mathrm{cl}}(\kappa)$ – both results use the revised GCH of Shelah, which is a striking application of PCF theory exposed in [7].

Finally one word about our notation, which is standard. $X \subset Y$ means that $\forall x\,[x \in X \implies x \in Y]$. So the symbol "$\subset$" does not mean "proper subset". If f is a function and $X \subset \mathrm{dom}(f)$, then $f''X$ is the image of X under f, that is $f''X = \{f(x) : x \in X\}$.

2. The bounding and almost disjointness numbers: A ZFC result

We will quote the following well-known result of Cummings and Shelah [4].

Theorem 4 (see Theorem 6 of [4])**.** *For every regular cardinal $k > \omega$, $\mathfrak{b}(\kappa) = \mathfrak{b}_{\mathrm{cl}}(\kappa)$.*

Theorem 5. *Let $\kappa > \omega$ be a regular cardinal. If $\mathfrak{b}(\kappa) = \kappa^+$, then $\mathfrak{a}(\kappa) = \kappa^+$.*

Proof. The hypothesis and Theorem 4 imply that there exists a sequence $\langle f_\delta : \delta < \kappa^+\rangle$ of functions in κ^κ with the property that for any $g \in \kappa^\kappa$, there is a $\delta < \kappa^+$ such that $\{\alpha < \kappa : g(\alpha) < f_\delta(\alpha)\}$ is stationary in κ. For any $E \subset \kappa$, if $\mathrm{otp}(E) = \kappa$, then let $\langle \mu_{E,\xi} : \xi < \kappa\rangle$ be the increasing enumeration of E. For each $\delta < \kappa^+$, let $C_\delta = \{\alpha < \kappa : \alpha \text{ is closed under } f_\delta\}$. Recall that C_δ is a club in κ. Also, fix a sequence $\langle e_\delta : \kappa \leq \delta < \kappa^+\rangle$ of bijections $e_\delta : \kappa \to \delta$. We will construct a sequence $\langle\langle A_\delta, E_\delta\rangle : \delta < \kappa^+\rangle$ satisfying the following conditions for each $\delta < \kappa^+$:

(1) $A_\delta \in [\kappa]^\kappa$ and $E_\delta \subset C_\delta$ is a club in κ;
(2) $\forall \gamma < \delta\,[|A_\gamma \cap A_\delta| < \kappa]$;
(3) if $\kappa \leq \delta$, then $A_\delta = \bigcup_{\xi<\kappa} B_{\delta,\xi}$, where for each $\xi < \kappa$, $B_{\delta,\xi}$ is defined to be

$$\left\{\mu_{E_\delta,\xi} \leq \alpha < \mu_{E_\delta,\xi+1} : \forall \nu < \mu_{E_\delta,\xi}\left[\alpha \notin A_{e_\delta(\nu)}\right]\right\}.$$

Suppose for a moment that such a sequence can be constructed. Let $\mathscr{A} = \{A_\delta : \delta < \kappa^+\}$. By (1) and (2), $\mathscr{A}$ is an a.d. family in $[\kappa]^\kappa$ of size κ^+. We claim that it is maximal. To see this, fix $B \in [\kappa]^\kappa$. Define a function $g : \kappa \to \kappa$ by stipulating that for each $\mu \in \kappa$, $g(\mu) = \sup\left(\{\min(B \setminus (\mu+1))\} \cup \{f_\nu(\mu) : \nu \leq \mu\}\right)$. Find $\delta < \kappa^+$ such that $S = \{\mu \in \kappa : g(\mu) < f_\delta(\mu)\}$ is stationary in κ. Note that $\kappa \leq \delta$. Therefore the consequent of (3) applies to δ. Let $I = \{\xi < \kappa : B_{\delta,\xi} \cap B \neq 0\}$. If $|I| = \kappa$, then $|A_\delta \cap B| = \kappa$, and we are done. So assume that $|I| < \kappa$. Then $\{\mu_{E_\delta,\xi} : \xi \in I\} \subset E_\delta \subset \kappa$ and $|\{\mu_{E_\delta,\xi} : \xi \in I\}| \leq |I| < \kappa$. Therefore $\sup\left(\{\mu_{E_\delta,\xi} : \xi \in I\}\right) = \nu_0 < \kappa$. Now $\{\mu \in E_\delta : \mu > \nu_0\}$ is a club in κ and $T = S \cap \{\mu \in E_\delta : \mu > \nu_0\}$ is stationary in κ. Consider any $\mu \in T$. There exists $\xi \in \kappa \setminus I$ with $\mu = \mu_{E_\delta,\xi}$. Note that $B_{\delta,\xi} \cap B = 0$ because $\xi \notin I$. On the other hand, $\mu_{E_\delta,\xi} = \mu < \min(B \setminus (\mu+1)) \leq g(\mu) < f_\delta(\mu) < \mu_{E_\delta,\xi+1}$ because $\mu \in S$ and because $\mu_{E_\delta,\xi+1} \in C_\delta$. Since $\min(B \setminus (\mu+1)) \notin B_{\delta,\xi}$, it follows from the definition of $B_{\delta,\xi}$ that $\exists \nu < \mu\left[\min(B \setminus (\mu+1)) \in A_{e_\delta(\nu)}\right]$. Thus we have proved that for each $\mu \in T$, $\exists \nu < \mu \exists \beta \in B\left[\mu < \beta \wedge \beta \in A_{e_\delta(\nu)}\right]$. Since T is stationary in κ, there exist $T^* \subset T$ and ν such that T^* is stationary in κ and for each $\mu \in T^*$, $\nu < \mu$ and $\exists \beta \in B\left[\mu < \beta \wedge \beta \in A_{e_\delta(\nu)}\right]$. It now easily follows that $\left|A_{e_\delta(\nu)} \cap B\right| = \kappa$. This proves the maximality of $\mathscr{A}$. Since $|\mathscr{A}| = \kappa^+$, we have $\mathfrak{a}(\kappa) \leq \kappa^+$, while standard arguments (see Theorem 1.2 of [5]) show that $\kappa^+ \leq \mathfrak{a}(\kappa)$. Hence we have $\mathfrak{a}(\kappa) = \kappa^+$.

Thus it suffices to construct a sequence satisfying (1)–(3) above. Let $\langle A_\gamma : \gamma \in \kappa \rangle$ be any partition of κ into κ many pairwise disjoint pieces of size κ. For each $\gamma < \kappa$, let $E_\gamma = C_\gamma$. It is clear that the sequence $\langle \langle A_\gamma, E_\gamma \rangle : \gamma < \kappa \rangle$ satisfies (1)–(3). Now fix $\kappa^+ > \delta \geq \kappa$ and assume that $\langle \langle A_\gamma, E_\gamma \rangle : \gamma < \delta \rangle$ satisfying (1)–(3) is given. We construct A_δ and E_δ as follows. Let θ be a sufficiently large regular cardinal. Let $x = \{\kappa, \langle f_\delta : \delta < \kappa^+ \rangle, \langle C_\delta : \delta < \kappa^+ \rangle, \langle e_\delta : \kappa \leq \delta < \kappa^+ \rangle, \delta, \langle \langle A_\gamma, E_\gamma \rangle : \gamma < \delta \rangle\}$. Let $\langle N_\xi : \xi < \kappa \rangle$ be such that

(4) $\forall \xi < \kappa \, [N_\xi \prec H(\theta) \wedge x \in N_\xi]$;
(5) $\forall \xi < \kappa \, [|N_\xi| < \kappa \wedge \mu_\xi = N_\xi \cap \kappa \in \kappa]$;
(6) $\forall \xi < \xi + 1 < \kappa \, [\langle N_\zeta : \zeta \leq \xi \rangle \in N_{\xi+1}]$;
(7) $\forall \xi < \kappa \left[\xi \text{ is a limit ordinal} \implies N_\xi = \bigcup_{\zeta < \xi} N_\zeta\right]$.

Observe that these conditions imply that $\forall \zeta < \xi < \kappa \, [N_\zeta \in N_\xi \wedge N_\zeta \subset N_\xi]$. Observe also that $E_\delta = \{\mu_\xi : \xi < \kappa\}$ is a club in κ and that $\mu_{E_\delta, \xi} = \mu_\xi$, for all $\xi < \kappa$. Next for each $\xi < \kappa$, $C_\delta \in N_\xi$. It follows that $\mu_\xi \in C_\delta$ because C_δ is a club in κ. So $E_\delta \subset C_\delta$. Now define $A_\delta = \bigcup_{\xi < \kappa} B_{\delta, \xi}$, where for each $\xi < \kappa$, $B_{\delta, \xi}$ is

$$\left\{\mu_\xi \leq \alpha < \mu_{\xi+1} : \forall \nu < \mu_\xi \left[\alpha \notin A_{e_\delta(\nu)}\right]\right\}.$$

It is clear that (3) is satisfied by definition and that $A_\delta \subset \kappa$. So to complete the proof, it suffices to check that $|A_\delta| = \kappa$ and that $\forall \gamma < \delta \, [|A_\gamma \cap A_\delta| < \kappa]$. To see the second statement, fix any $\gamma < \delta$. Since $e_\delta : \kappa \to \delta$ is a bijection, we can find $\nu \in \kappa$ with $e_\delta(\nu) = \gamma$. Find $\zeta < \kappa$ with $\nu < \mu_\zeta$. Consider any $\xi < \kappa$ so that $\zeta \leq \xi$. Then $\nu < \mu_\zeta \leq \mu_\xi$. It follows that $A_\gamma \cap B_{\delta, \xi} = A_{e_\delta(\nu)} \cap B_{\delta, \xi} = 0$. Therefore, $A_\gamma \cap A_\delta = \bigcup_{\xi < \kappa} (A_\gamma \cap B_{\delta, \xi}) = \bigcup_{\xi < \zeta} (A_\gamma \cap B_{\delta, \xi}) \subset \bigcup_{\xi < \zeta} B_{\delta, \xi}$. For each $\xi < \zeta$, $|B_{\delta, \xi}| < \kappa$. So $\bigcup_{\xi < \zeta} B_{\delta, \xi}$ is the union of $\leq |\zeta| \leq \zeta < \kappa$ many sets each of size $< \kappa$. Since κ is regular, we conclude that $\left|\bigcup_{\xi < \zeta} B_{\delta, \xi}\right| < \kappa$. So $|A_\gamma \cap A_\delta| < \kappa$, as needed.

Finally we check that for each $\xi < \kappa$, $B_{\delta, \xi} \neq 0$. This will imply that $|A_\delta| = \kappa$. Fix any $\xi < \kappa$. Note that for each $\nu < \mu_\xi$, $\left|A_{e_\delta(\mu_\xi)} \cap A_{e_\delta(\nu)}\right| < \kappa$. Therefore $R_\xi = \bigcup_{\nu < \mu_\xi} \left(A_{e_\delta(\mu_\xi)} \cap A_{e_\delta(\nu)}\right)$ is the union of at most $|\mu_\xi| \leq \mu_\xi < \kappa$ many sets each having size $< \kappa$. Since κ is regular, it follows that $|R_\xi| < \kappa$. Hence there is an $\alpha \in A_{e_\delta(\mu_\xi)} \setminus R_\xi$ with $\mu_\xi \leq \alpha$ because $\left|A_{e_\delta(\mu_\xi)}\right| = \kappa$. Since $N_{\xi+1} \prec H(\theta)$ and since all the relevant parameters belong to $N_{\xi+1}$, we conclude that there exists $\alpha \in N_{\xi+1}$ such that $\alpha \in \kappa$, $\mu_\xi \leq \alpha$, and $\forall \nu \in \mu_\xi \left[\alpha \notin A_{e_\delta(\nu)}\right]$. Now we have that $\mu_\xi \leq \alpha < \mu_{\xi+1}$ and so $\alpha \in B_{\delta, \xi}$. This shows that $B_{\delta, \xi} \neq 0$ and concludes the proof. $\dashv$

3. The reaping and dominating numbers: an application of PCF theory

We begin with a well-known fact, whose proof we include for completeness.

Definition 6. Let $\kappa > \omega$ be a regular cardinal. If $A \in [\kappa]^\kappa$, then we let $e_A : \kappa \to A$ be the order isomorphism from $\langle \kappa, \in \rangle$ to $\langle A, \in \rangle$. We also define a function $s_A : \kappa \to A$ by setting $s_A(\alpha) = \min(A \setminus (\alpha + 1))$, for each $\alpha \in \kappa$. We also write $\lim(\kappa) = \{\alpha < \kappa : \alpha \text{ is a limit ordinal}\}$ and $\operatorname{succ}(\kappa) = \{\alpha < \kappa : \alpha \text{ is a successor ordinal}\}$.

Lemma 7 (Folklore)**.** *If $\kappa > \omega$ is a regular cardinal, then $\mathfrak{r}(\kappa) \geq \kappa^+$.*

Proof. Let $F \subset [\kappa]^\kappa$ be a family with $|F| \leq \kappa$. We must find a $B \in \mathcal{P}(\kappa)$ which reaps F. If F is empty, then $B = \kappa$ will work. So assume F is non-empty. Let $\{A_\alpha : \alpha < \kappa\}$ enumerate F, possibly with repetitions. For each $\alpha < \kappa$, let $C_\alpha = \{\delta < \kappa : \delta \text{ is closed under } s_{A_\alpha}\}$. Then $C = \{\delta < \kappa : \forall \alpha < \delta \, [\delta \in C_\alpha]\}$ is a club in κ. For each $\xi \in \kappa$, let $B_\xi = \{\zeta < e_C(\xi+1) : e_C(\xi) \le \zeta\}$. Note that for all $\alpha < e_C(\xi+1)$, $A_\alpha \cap B_\xi \ne 0$. Also for any distinct $\xi, \xi' \in \kappa$, $B_\xi \cap B_{\xi'} = 0$. Put $B = \bigcup\{B_\xi : \xi \in \lim(\kappa)\}$. Then $B \in \mathcal{P}(\kappa)$ and since for each $\alpha < \kappa$ and each $\xi \in \lim(\kappa) \setminus \alpha$, $A_\alpha \cap B_\xi \ne 0$, $|A_\alpha \cap B| = \kappa$, for all $\alpha < \kappa$. Furthermore, $\bigcup\{B_{\xi'} : \xi' \in \mathrm{succ}(\kappa)\} \subset \kappa \setminus B$, and since for each $\alpha < \kappa$ and for each $\xi' \in \mathrm{succ}(\kappa) \setminus \alpha$, $A_\alpha \cap B_{\xi'} \ne 0$, $|A_\alpha \cap (\kappa \setminus B)| = \kappa$, for all $\alpha < \kappa$. Thus B reaps F. ⊣

The above proof really shows that $\mathfrak{r}(\kappa) \ge \mathfrak{b}(\kappa)$. However we will not need this in what follows. The proof of the main theorem is broken into two cases. For the remainder of this section, let $\kappa > \omega$ be a fixed regular cardinal. The crucial definition is the following.

Definition 8. Let $E_2 \subset E_1$ both be clubs in κ. For each $\xi \in \kappa$, define $\mathrm{set}(E_1, \xi) = \{\zeta < s_{E_1}(\xi) : \xi \le \zeta\}$. Define $\mathrm{set}(E_2, E_1) = \bigcup\{\mathrm{set}(E_1, \xi) : \xi \in E_2\}$.

Lemma 9. *Suppose that $F \subset [\kappa]^\kappa$ is an unreaped family with $|F| = \mathfrak{r}(\kappa)$. Assume there is a club $E_1 \subset \kappa$ such that for each club $E \subset E_1$, there exists $A \in F$ with $A \subset^* \mathrm{set}(E, E_1)$. Then $\mathfrak{d}(\kappa) \le \mathfrak{r}(\kappa)$.*

Proof. For each $A \in F$ define a function $g_A : \kappa \to \kappa$ as follows. Given $\beta \in \kappa$, $g_A(\beta) = s_A(s_{E_1}(\beta))$. Then $|\{g_A : A \in F\}| \le |F| = \mathfrak{r}(\kappa)$, and we will check that this is a dominating family of functions. To this end, fix any $f \in \kappa^\kappa$. Put

$$E_f = \{\xi \in E_1 : \xi \text{ is closed under } f\}.$$

Then $E_f \subset E_1$ and it is a club in κ. By hypothesis there exist $A \in F$ and $\delta \in \kappa$ with $A \setminus \delta \subset \mathrm{set}(E_f, E_1)$. We claim that for any $\zeta \in \kappa$, if $\zeta \ge \delta$, then $f(\zeta) < g_A(\zeta)$. Indeed suppose $\delta \le \zeta < \kappa$ is given. Let $\gamma = s_{E_1}(\zeta) > \zeta$ and let $g_A(\zeta) = \beta = s_A(s_{E_1}(\zeta))$. Then $\beta \in A$ and $\delta \le \zeta < s_{E_1}(\zeta) < \beta$. Thus $\beta \in \mathrm{set}(E_f, E_1)$. Let $\zeta' \in E_f$ be such that $\zeta' \le \beta < s_{E_1}(\zeta')$. It could not be the case that $\zeta' < \gamma$, for if that were the case, then the inequality $\beta < s_{E_1}(\zeta') \le \gamma = s_{E_1}(\zeta) < \beta$ would be true, which is impossible. Therefore $\gamma \le \zeta'$ and since $\zeta < \gamma \le \zeta'$ and ζ' is closed under f, we have $f(\zeta) < \zeta' \le \beta = g_A(\zeta)$, as claimed. Hence $f \le^* g_A$. As $f \in \kappa^\kappa$ was arbitrary, this proves that $\{g_A : A \in F\}$ is dominating, and so $\mathfrak{d}(\kappa) \le |\{g_A : A \in F\}| \le \mathfrak{r}(\kappa)$. ⊣

The proof in the case when the hypothesis of Lemma 9 fails will make use of Shelah's Revised GCH, which is a theorem of ZFC. Let us recall the definition of various notions that are relevant to the revised GCH.

Definition 10. Let κ and λ be cardinals. Define $\lambda^{[\kappa]}$ to be

$$\min\left\{|\mathcal{P}| : \mathcal{P} \subset [\lambda]^{\le\kappa} \text{ and } \forall u \in [\lambda]^\kappa \exists \mathcal{P}_0 \subset \mathcal{P}\left[|\mathcal{P}_0| < \kappa \text{ and } u = \bigcup \mathcal{P}_0\right]\right\}.$$

The operation $\lambda^{[\kappa]}$ is sometimes referred to as the *weak power*.

The following remarkable ZFC result was obtained by Shelah in [7] as one of the many fruits of his PCF theory. A nice exposition of its proof may be also be found in Abraham and Magidor [1]. Another relevant reference is Shelah [9].

Theorem 11 (The Revised GCH)**.** *If θ is a strong limit uncountable cardinal, then for every $\lambda \ge \theta$, there exists $\sigma < \theta$ such that for every $\sigma \le \kappa < \theta$, $\lambda^{[\kappa]} = \lambda$.*

Corollary 12. *Let $\mu \ge \beth_\omega$ be any cardinal. There exists an uncountable regular cardinal $\theta < \beth_\omega$ and a family $\mathcal{P} \subset [\mu]^{\le\theta}$ such that $|\mathcal{P}| \le \mu$ and for each $u \in [\mu]^\theta$, there exists $v \in \mathcal{P}$ with the property that $v \subset u$ and $|v| \ge \aleph_0$.*

Proof. $\beth_\omega$ is a strong limit uncountable cardinal. Therefore Theorem 11 applies and implies that there exists $\sigma < \beth_\omega$ such that for every $\sigma \le \theta < \beth_\omega$, $\mu^{[\theta]} = \mu$. It is possible to choose an uncountable regular cardinal θ satisfying $\sigma \le \theta < \beth_\omega$. Since $\mu^{[\theta]} = \mu$, there exists $\mathcal{P} \subset [\mu]^{\le\theta}$ such that $|\mathcal{P}| = \mu$ and for each $u \in [\mu]^\theta$, there exists $\mathcal{P}_0 \subset \mathcal{P}$ with the property that $|\mathcal{P}_0| < \theta$ and $u = \bigcup \mathcal{P}_0$. Now suppose that $u \in [\mu]^\theta$ is given. Let $\mathcal{P}_0 \subset \mathcal{P}$ be such that $|\mathcal{P}_0| < \theta$ and $u = \bigcup \mathcal{P}_0$. Since θ is a regular cardinal and $|u| = \theta$, it follows that $|v| = \theta \geq \aleph_0$, for some $v \in \mathcal{P}_0$. This is as required because $v \in \mathcal{P}$ and $v \subset u$. ⊣

The proof of the following theorem is similar to the proof of Cummings and Shelah's theorem from [4] that if $\kappa \geq \beth_\omega$, then $\mathfrak{d}(\kappa) = \mathfrak{d}_{\mathrm{cl}}(\kappa)$.

Theorem 13. *If $\kappa \geq \beth_\omega$, then $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$.*

Proof. Write $\mu = \mathfrak{r}(\kappa)$. Let $F \subset [\kappa]^\kappa$ be such that F is unreaped and $|F| = \mu$. Then $\beth_\omega \leq \kappa < \kappa^+ \leq \mathfrak{r}(\kappa) = \mu$. So applying Corollary 12, fix an uncountable regular cardinal $\theta < \beth_\omega$ satisfying the conclusion of Corollary 12. Note that $|\theta \times \mu| = \mu$ because $\theta < \beth_\omega < \mu$. So $|\theta \times F| = \mu$. Therefore applying Corollary 12, find a family $\mathcal{P} \subset [\theta \times F]^{\leq \theta}$ such that $|\mathcal{P}| \leq \mu$ and $\mathcal{P}$ has the property that for each $u \in [\theta \times F]^\theta$, there exists $v \in \mathcal{P}$ satisfying $v \subset u$ and $|v| \geq \aleph_0$. Put $X = F \cup \mu \cup \mathcal{P} \cup \{\theta, \mu, \kappa, \kappa^\kappa, \mathcal{P}(\kappa)\}$. Then $|X| = \mu$, and so if χ is a sufficiently large regular cardinal, then there exists $M \prec H(\chi)$ with $|M| = \mu$ and $X \subset M$. We will aim to prove that $M \cap \kappa^\kappa$ is a dominating family.

In view of Lemma 9 it may be assumed that for any club $E_1 \subset \kappa$, there exists a club $E_2 \subset E_1$ such that for all $B \in F$, $B \not\subset^* \mathrm{set}(E_2, E_1)$. Since F is an unreaped family and since $\mathrm{set}(E_2, E_1) \in \mathcal{P}(\kappa)$ whenever $E_2 \subset E_1$ are both clubs in κ, it follows that for each club $E_1 \subset \kappa$, there exist a club $E_2 \subset E_1$ and a $B \in F$ such that $B \subset^* \kappa \setminus \mathrm{set}(E_2, E_1)$. Let $f \in \kappa^\kappa$ be a fixed function. Construct a sequence $\langle \langle E_i, E_i^1, B_i \rangle : i < \theta \rangle$ by induction on $i < \theta$ so that the following conditions are satisfied at each $i < \theta$:

1. E_i and E_i^1 are both clubs in κ, $E_i^1 \subset E_i$, and $\forall j < i \left[E_i \subset E_j^1 \right]$;
2. $B_i \in F$ and $B_i \subset^* \kappa \setminus \mathrm{set}(E_i^1, E_i)$;
3. if $i = 0$, then $E_i = \{\alpha < \kappa : \alpha \text{ is closed under } f\}$.

We first show how to construct such a sequence. When $i = 0$, put $E_i = \{\alpha < \kappa : \alpha \text{ is closed under } f\}$. Then E_i is a club in κ, and so there exist a club $E_i^1 \subset E_i$ and a $B_i \in F$ with $B_i \subset^* \kappa \setminus \mathrm{set}(E_i^1, E_i)$. Next suppose that $\theta > i > 0$ and that $\langle \langle E_j, E_j^1, B_j \rangle : j < i \rangle$ satisfying (1)–(3) is given. Then $\{E_j^1 : j < i\}$ is a collection of $\leq |i| \leq i < \theta < \beth_\omega \leq \kappa$ many clubs in κ. Therefore $E_i = \bigcap_{j<i} E_j^1$ is a club in κ. We have $\forall j < i \left[E_i \subset E_j^1 \right]$ and moreover there exist a club $E_i^1 \subset E_i$ and a $B_i \in F$ such that $B_i \subset^* \kappa \setminus \mathrm{set}(E_i^1, E_i)$. It is clear that E_i, E_i^1, and B_i are as required. This completes the construction of the sequence $\langle \langle E_i, E_i^1, B_i \rangle : i < \theta \rangle$.

Now define a function $u : \theta \to F$ by setting $u(i) = B_i$ for all $i \in \theta$. Then $u \subset \theta \times F$ and $|u| = |\mathrm{dom}(u)| = \theta$. Hence by the choice of $\mathcal{P}$ and M, there exists $v \in \mathcal{P} \subset X \subset M$ such that $v \subset u$ and $|v| \geq \aleph_0$. v is a function and $c = \mathrm{dom}(v) \subset \mathrm{dom}(u) = \theta$. Moreover, $\aleph_0 \leq |v| = |c|$ and $c \in M$. Hence we can find $d \in M$ so that $d \subset c$ and $\mathrm{otp}(d) = \omega$. Let $w = v \restriction d \in M$. Since $\kappa > \omega$ is regular, there exists a function $g \in \kappa^\kappa$ with the property that for each $\alpha \in \kappa$, $\forall i \in d \exists \beta \in w(i) = B_i \left[\alpha < \beta < g(\alpha) \right]$. We may further assume that $g \in M$ because all of the relevant parameters belong to M. Let $\langle i_n : n \in \omega \rangle$ be the strictly increasing enumeration of d. Recall that for each $n \in \omega$, $E_{i_n}^1 \subset E_{i_n} \subset \kappa$ are both clubs in κ and that $B_{i_n} \subset^* \kappa \setminus \mathrm{set}(E_{i_n}^1, E_{i_n})$. In particular, for each $n \in \omega$, there exists $\delta_n \in \kappa$ so that $B_{i_n} \setminus \delta_n \subset \kappa \setminus \mathrm{set}(E_{i_n}^1, E_{i_n})$, and also $\min(E_{i_n}) \in \kappa$. Hence $\{\delta_n : n \in \omega\} \cup \{\min(E_{i_n}) : n \in \omega\}$ is a countable subset of κ, whence $\{\delta_n : n \in \omega\} \cup \{\min(E_{i_n}) : n \in \omega\} \subset \delta$, for some $\delta \in \kappa$. We will argue that for each $\alpha \in \kappa$, if $\alpha \geq \delta$, then $f(\alpha) < g(\alpha)$. To this end, let $\alpha \in \kappa$ be fixed, and assume that $\delta \leq \alpha$. For each $n \in \omega$, since $E_{i_n} \subset \kappa$ is a club in κ and since $\min(E_{i_n}) < \delta \leq \alpha < \alpha + 1 < \kappa$, it follows that $\xi_n = \sup(E_{i_n} \cap (\alpha + 1)) \in E_{i_n}$. Also $\forall n \in \omega \left[\xi_{n+1} \leq \xi_n \right]$

because $\forall n \in \omega \left[E_{i_{n+1}} \subset E_{i_n}\right]$. It follows that there exist ξ and $N \in \omega$ such that $\forall n \geq N \left[\xi_n = \xi\right]$. Note that $\xi \in E_{i_{N+1}} \subset E^1_{i_N}$. Consider $s_{E_{i_N}}(\xi)$. $s_{E_{i_N}}(\xi) \in E_{i_N}$ and $s_{E_{i_N}}(\xi) > \xi = \xi_N = \sup(E_{i_N} \cap (\alpha+1))$. Therefore $s_{E_{i_N}}(\xi) \geq \alpha + 1 > \alpha$. Since $s_{E_{i_N}}(\xi) \in E_{i_N} \subset E_0$, $s_{E_{i_N}}(\xi)$ is closed under f. Therefore $f(\alpha) < s_{E_{i_N}}(\xi)$. Next by the choice of g, there exists $\beta \in B_{i_N}$ with $\alpha < \beta < g(\alpha)$. Note that $\delta_N < \delta \leq \alpha < \beta$. Hence $\beta \in B_{i_N} \setminus \delta_N \subset \kappa \setminus \mathrm{set}(E^1_{i_N}, E_{i_N})$, in other words, $\beta \notin \mathrm{set}(E^1_{i_N}, E_{i_N})$. Note that $\xi = \sup(E_{i_N} \cap (\alpha+1)) \leq \alpha < \beta$. Since $\xi \in E^1_{i_N}$, $\beta \geq s_{E_{i_N}}(\xi)$. Putting all this information together, we have $f(\alpha) < s_{E_{i_N}}(\xi) \leq \beta < g(\alpha)$, as required.

Thus we have proved that $f \leq^* g$. Since $f \in \kappa^\kappa$ was arbitrary and since $g \in M \cap \kappa^\kappa$, we have proved that $M \cap \kappa^\kappa$ is a dominating family. Therefore $\mathfrak{d}(\kappa) \leq |M \cap \kappa^\kappa| \leq |M| = \mu = \mathfrak{r}(\kappa)$. ⊣

4. Questions

It is unknown how large $\mathfrak{b}(\kappa)$ needs to be for the configuration $\mathfrak{b}(\kappa) < \mathfrak{a}(\kappa)$ to be consistent. So we ask

Question 14. *Does $\mathfrak{b}(\kappa) = \kappa^{++}$ imply that $\mathfrak{a}(\kappa) = \kappa^{++}$, for every regular cardinal $\kappa > \omega$?*

It is not possible to step-up the proof of Theorem 5 in any straightforward way. If Question 14 has a positive answer, then the proof is likely to involve quite a different argument.

Theorem 13 of course gives no information about the relationship between $\mathfrak{d}(\kappa)$ and $\mathfrak{r}(\kappa)$ when $\kappa < \beth_\omega$.

Question 15. *If $\omega < \kappa < \beth_\omega$ is a regular cardinal, then does $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$ hold? In particular, is $\mathfrak{d}(\aleph_n) \leq \mathfrak{r}(\aleph_n)$, for all $1 \leq n < \omega$?*

In trying to tackle this problem, it may seem reasonable to first try to produce a model where $\mathfrak{r}(\aleph_n) < 2^{\aleph_n}$, for if $\mathfrak{r}(\aleph_n)$ is provably equal to $2^{\aleph_n}$, then of course $\mathfrak{d}(\aleph_n) \leq \mathfrak{r}(\aleph_n)$. This is closely related to a well-known question of Kunen about the minimal size of a base for a uniform ultrafilter on $\aleph_1$.

Question 16. *Is $\mathfrak{r}(\aleph_1) < 2^{\aleph_1}$ consistent? Is $\mathfrak{u}(\aleph_1) < 2^{\aleph_1}$ consistent?*

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