

BOREL SETS WITHOUT PERFECTLY MANY OVERLAPPING TRANSLATIONS

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ABSTRACT. For a cardinal $\lambda < \lambda_{\omega_1}$ we give a ccc forcing notion \mathbb{P} such that

$\Vdash_{\mathbb{P}}$ “ some Σ_2^0 set $B \subseteq {}^\omega 2$ admits a sequence $\langle \eta_\alpha : \alpha < \lambda \rangle$ of distinct elements of ${}^\omega 2$ such that $|(\eta_\alpha + B) \cap (\eta_\beta + B)| \geq 6$ for all $\alpha, \beta < \lambda$ but does not have a perfect set of such η 's ”.

The construction closely follows the one from Shelah [6, Section 1].

1. INTRODUCTION

Shelah [6] analyzed when there are Borel in the plane which contain large squares but no perfect squares. A rank on models with a countable vocabulary was introduced and a used to define a cardinal λ_{ω_1} (the first λ such that there is no model with universe λ , countable vocabulary and rank $< \omega_1$). It was shown in [6, Claim 1.12] that every Borel set $B \subseteq {}^\omega 2 \times {}^\omega 2$ which contains a λ_{ω_1} -square must contain a perfect square. On the other hand, by [6, Theorem 1.13], if $\mu = \mu^{\aleph_0} < \lambda_{\omega_1}$ then some ccc forcing notion forces that (the continuum is arbitrarily large and) some Borel set contains a μ -square but no μ^+ -square.

We would like to understand what the results mentioned above mean for general relations. Natural first step is to ask about Borel sets with $\mu \geq \aleph_1$ pairwise disjoint translations but without any perfect set of such translations, as motivated e.g. by Balcerzak, Rosłanowski and Shelah [1] (were we studied the σ -ideal of subsets of ${}^\omega 2$ generated by Borel sets with a perfect set of pairwise disjoint translations). A generalization of this direction could follow Zakrzewski [7] who introduced perfectly k -small sets.

However, preliminary analysis of the problem revealed that another, somewhat orthogonal to the one described above, direction is more natural in the setting of [6]. Thus we investigate Borel sets with many, but not too many, pairwise overlapping intersections.

Easily, every uncountable Borel subset B of ${}^\omega 2$ has a perfect set of pairwise non-disjoint translations (just consider a perfect set $P \subseteq B$ and note that for $x, y \in P$ we have $\mathbf{0}, x+y \in (B+x) \cap (B+y)$). The problem of many non-disjoint translations becomes more interesting if we demand that the intersections have more elements. Note that in ${}^\omega 2$, if $x + b_0 = y + b_1$ then also $x + b_1 = y + b_0$, so $x \neq y$ and $|(B+x) \cap (B+y)| < \omega$ imply that $|(B+x) \cap (B+y)|$ is even.

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In the present paper we study the case when the intersections $(B+x) \cap (B+y)$ have at least 6 elements. We show that for $\lambda < \lambda_{\omega_1}$ there is a ccc forcing notion \mathbb{P} adding a Σ_2^0 subset B of the Cantor space ${}^\omega 2$ such that

- for some $H \subseteq {}^\omega 2$ of size λ , $|(B+h) \cap (B+h')| \geq 6$ for all $h, h' \in H$, but
- for every perfect set $P \subseteq {}^\omega 2$ there are $x, x' \in P$ with $|(B+x) \cap (B+x')| < 6$.

We fully utilize the algebraic properties of $({}^\omega 2, +)$, in particular the fact that all elements of ${}^\omega 2$ are self-inverse. The general case of Polish groups will be investigated in the subsequent work [5].

In Section 2 of the paper we recall the rank from [6]. We give the relevant definitions, state and prove all the properties needed for our results later. In the third section we analyze when a Σ_2^0 subset of ${}^\omega 2$ has a perfect set of pairwise overlapping translations. The main consistency result concerning adding a Borel set with no perfect set of overlapping translations is given in the fourth section.

Notation: Our notation is rather standard and compatible with that of classical textbooks (like Jech [3] or Bartoszyński and Judah [2]). However, in forcing we keep the older convention that *a stronger condition is the larger one*.

- (1) For a set u we let

$$u^{(2)} = \{(x, y) \in u \times u : x \neq y\}.$$

- (2) The Cantor space ${}^\omega 2$ of all infinite sequences with values 0 and 1 is equipped with the natural product topology and the group operation of coordinate-wise addition $+$ modulo 2.
- (3) Ordinal numbers will be denoted by the lower case initial letters of the Greek alphabet $\alpha, \beta, \gamma, \delta$. Finite ordinals (non-negative integers) will be denoted by letters $a, b, c, d, i, j, k, \ell, m, n, M$ and ι .
- (4) The Greek letters κ, λ will stand for uncountable cardinals.
- (5) For a forcing notion \mathbb{P} , all \mathbb{P} -names for objects in the extension via \mathbb{P} will be denoted with a tilde below (e.g., $\tilde{\tau}, \tilde{X}$), and $G_{\mathbb{P}}$ will stand for the canonical \mathbb{P} -name for the generic filter in \mathbb{P} .

2. THE RANK

We will remind some basic facts from [6, Section 1] concerning a rank (on models with countable vocabulary) which will be used in the construction of a forcing notion in the next section. For the convenience of the reader we provide proofs for most of the claims, even though they were given in [6]. Our rank rk is the rk^0 of [6] and rk^* is the rk^2 there.

Let λ be a cardinal and \mathbb{M} be a model with the universe λ and a countable vocabulary τ .

Definition 2.1. (1) By induction on ordinals α , for finite non-empty sets $w \subseteq \lambda$ we define when $\text{rk}(w, \mathbb{M}) \geq \alpha$. Let $w = \{a_0, \dots, a_n\} \subseteq \lambda$, $|w| = n + 1$.

- (a) $\text{rk}(w) \geq 0$ if and only if for every quantifier free formula $\varphi \in \mathcal{L}(\tau)$ and each $k \leq n$, if $\mathbb{M} \models \varphi[a_0, \dots, a_k, \dots, a_n]$ then the set

$$\{a \in \lambda : \mathbb{M} \models \varphi[a_0, \dots, a_{k-1}, a, a_{k+1}, \dots, a_n]\}$$

is uncountable;

- (b) if α is limit, then $\text{rk}(w, \mathbb{M}) \geq \alpha$ if and only if $\text{rk}(w, \mathbb{M}) \geq \beta$ for all $\beta < \alpha$;

- (c) $\text{rk}(w, \mathbb{M}) \geq \alpha + 1$ if and only if for every quantifier free formula $\varphi \in \mathcal{L}(\tau)$ and each $k \leq n$, if $\mathbb{M} \models \varphi[a_0, \dots, a_k, \dots, a_n]$ then there is $a^* \in \lambda \setminus w$ such that

$$\text{rk}(w \cup \{a^*\}, \mathbb{M}) \geq \alpha \quad \text{and} \quad \mathbb{M} \models \varphi[a_0, \dots, a_{k-1}, a^*, a_{k+1}, \dots, a_n].$$

- (2) Similarly, for finite non-empty sets $w \subseteq \lambda$ we define when $\text{rk}^*(w, \mathbb{M}) \geq \alpha$ (by induction on ordinals α). Let $w = \{a_0, \dots, a_n\} \subseteq \lambda$. We take clauses (a) and (b) above and

- (c)* $\text{rk}^*(w, \mathbb{M}) \geq \alpha + 1$ if and only if for every quantifier free formula $\varphi \in \mathcal{L}(\tau)$ and each $k \leq n$, if $\mathbb{M} \models \varphi[a_0, \dots, a_k, \dots, a_n]$ then there are pairwise distinct $\langle a_i^* : i < \omega_1 \rangle \subseteq \lambda \setminus (w \setminus \{a_k\})$ such that $a_0^* = a_k$ and for all $i < j < \omega_1$ we have

$$\text{rk}^*(w \setminus \{a_k\} \cup \{a_i^*, a_j^*\}, \mathbb{M}) \geq \alpha \quad \text{and} \quad \mathbb{M} \models \varphi[a_0, \dots, a_{k-1}, a_i^*, a_{k+1}, \dots, a_n].$$

By a straightforward induction on α one easily shows the following observation.

Observation 2.2. *If $\emptyset \neq v \subseteq w$ then*

- $\text{rk}(w, \mathbb{M}) \geq \alpha \geq \beta$ implies $\text{rk}(v, \mathbb{M}) \geq \beta$, and
- $\text{rk}^*(w, \mathbb{M}) \geq \alpha \geq \beta$ implies $\text{rk}^*(v, \mathbb{M}) \geq \beta$.

Hence we may define the rank functions on finite non-empty subsets of λ .

Definition 2.3. The ranks $\text{rk}(w, \mathbb{M})$ and $\text{rk}^*(w, \mathbb{M})$ of a finite non-empty set $w \subseteq \lambda$ are defined as:

- $\text{rk}(w, \mathbb{M}) = -1$ if $\neg(\text{rk}(w, \mathbb{M}) \geq 0)$, and
 $\text{rk}^*(w, \mathbb{M}) = -1$ if $\neg(\text{rk}^*(w, \mathbb{M}) \geq 0)$,
- $\text{rk}(w, \mathbb{M}) = \infty$ if $\text{rk}(w, \mathbb{M}) \geq \alpha$ for all ordinals α , and
 $\text{rk}^*(w, \mathbb{M}) = \infty$ if $\text{rk}^*(w, \mathbb{M}) \geq \alpha$ for all ordinals α ,
- for an ordinal α : $\text{rk}(w, \mathbb{M}) = \alpha$ if $\text{rk}(w, \mathbb{M}) \geq \alpha$ but $\neg(\text{rk}(w, \mathbb{M}) \geq \alpha + 1)$,
and $\text{rk}^*(w, \mathbb{M}) = \alpha$ if $\text{rk}^*(w, \mathbb{M}) \geq \alpha$ but $\neg(\text{rk}^*(w, \mathbb{M}) \geq \alpha + 1)$.

Definition 2.4. (1) For an ordinal ε and a cardinal λ let $\text{NPr}_\varepsilon(\lambda)$ be the following statement: “there is a model \mathbb{M}^* with the universe λ and a countable vocabulary τ^* such that $\sup\{\text{rk}(w, \mathbb{M}^*) : \emptyset \neq w \in [\lambda]^{<\omega}\} < \varepsilon$.”

- (2) The statement $\text{NPr}_\varepsilon^*(\lambda)$ is defined similarly but using the rank rk^* .
(3) $\text{Pr}_\varepsilon(\lambda)$ and $\text{Pr}_\varepsilon^*(\lambda)$ are the negations of $\text{NPr}_\varepsilon(\lambda)$ and $\text{NPr}_\varepsilon^*(\lambda)$, respectively.

Observation 2.5. (1) *If a model \mathbb{M}^+ (on λ) is an expansion of the model \mathbb{M} , then $\text{rk}^*(w, \mathbb{M}^+) \leq \text{rk}(w, \mathbb{M}^+) \leq \text{rk}(w, \mathbb{M})$.*

- (2) *If λ is uncountable and $\text{NPr}_\varepsilon(\lambda)$, then there is a model \mathbb{M}^* with the universe λ and a countable vocabulary τ^* such that*
- $\text{rk}(\{a\}, \mathbb{M}^*) \geq 0$ for all $a \in \lambda$ and
 - $\text{rk}(w, \mathbb{M}^*) < \varepsilon$ for every finite non-empty set $w \subseteq \lambda$.

Proposition 2.6 (See [6, Claim 1.7]). (1) $\text{NPr}_1(\omega_1)$.

- (2) *If $\text{NPr}_\varepsilon(\lambda)$, then $\text{NPr}_{\varepsilon+1}(\lambda^+)$.*
(3) *If $\text{NPr}_\varepsilon(\mu)$ for $\mu < \lambda$ and $\text{cf}(\lambda) = \omega$, then $\text{NPr}_{\varepsilon+1}(\lambda)$.*
(4) $\text{NPr}_\varepsilon(\lambda)$ implies $\text{NPr}_\varepsilon^*(\lambda)$.

Proof. (1) Let Q be a binary relational symbol and let \mathbb{M}_1 be a model with the universe ω_1 , the vocabulary $\tau(\mathbb{M}_1) = \{Q\}$ and such that $Q^{\mathbb{M}_1} = \{(\alpha, \beta) \in \omega_1 \times \omega_1 : \alpha < \beta\}$. Then for each $\alpha_0 < \alpha_1 < \omega_1$ we have $\mathbb{M}_1 \models Q[\alpha_0, \alpha_1]$ but the set

$\{\alpha < \omega_1 : \mathbb{M}_1 \models Q[\alpha, \alpha_1]\}$ is countable. Hence $\text{rk}(w, \mathbb{M}_1) = -1$ whenever $|w| \geq 2$ and $\text{rk}(\{\alpha\}, \mathbb{M}_1) = 0$ for $\alpha \in \omega_1$. Consequently, \mathbb{M}_1 witnesses $\text{NPr}_1(\omega_1)$.

(2) Assume $\text{NPr}_\varepsilon(\lambda)$ holds true as witnessed by a model \mathbb{M} with the universe λ and a countable vocabulary τ . We may assume that $\tau = \{R_i : i < \omega\}$, where each R_i is a relational symbol of arity $n(i)$. Let S be a new binary relational symbol, T be a new unary relational symbol, and Q_i be a new $(n(i) + 1)$ -ary relational symbol (for $i < \omega$). Let $\tau^+ = \{R_i, Q_i : i < \omega\} \cup \{S, T\}$.

For each $\gamma \in [\lambda, \lambda^+)$ fix a bijection $f_\gamma : \gamma \xrightarrow{1-1} \lambda$ with f_λ being the identity. We define a model \mathbb{M}^+ :

- the vocabulary of \mathbb{M}^+ is τ^+ and the universe of \mathbb{M}^+ is λ^+ ,
- $R_i^{\mathbb{M}^+} = R_i^{\mathbb{M}} \subseteq \lambda^{n(i)}$,
- $Q_i^{\mathbb{M}^+} = \{(a_0, \dots, a_{n(i)-1}, a_{n(i)}) : \lambda \leq a_{n(i)} < \lambda^+ \text{ \& } (\forall \ell < n(i))(a_\ell < a_{n(i)}) \text{ \& } (f_{a_{n(i)}}(a_0), \dots, f_{a_{n(i)}}(a_{n(i)-1})) \in R_i^{\mathbb{M}}\}$,
- $S^{\mathbb{M}^+} = \{(a_0, a_1) \in \lambda^+ \times \lambda^+ : a_0 < a_1\}$ and $T^{\mathbb{M}^+} = [\lambda, \lambda^+)$.

Claim 2.6.1. (i) If $\lambda \leq \gamma < \lambda^+$, $\emptyset \neq w \subseteq \gamma$, then $\text{rk}(w \cup \{\gamma\}, \mathbb{M}^+) \leq \text{rk}(f_\gamma[w], \mathbb{M})$ and thus $\text{rk}(w \cup \{\gamma\}, \mathbb{M}^+) < \varepsilon$.
(ii) If $\emptyset \neq w \subseteq \lambda$, then $\text{rk}(w, \mathbb{M}^+) \leq \text{rk}(w, \mathbb{M})$ and thus $\text{rk}(w, \mathbb{M}^+) < \varepsilon$.
(iii) If $\lambda \leq \gamma < \lambda^+$, then $\text{rk}(\{\gamma\}, \mathbb{M}^+) \leq \varepsilon$.

Proof of the Claim. (i) By induction on α we show that $\alpha \leq \text{rk}(w \cup \{\gamma\}, \mathbb{M}^+)$ implies $\alpha \leq \text{rk}(f_\gamma[w], \mathbb{M})$ (for all sets $w \subseteq \gamma$ with fixed $\gamma \in [\lambda, \lambda^+)$).

(*)₀ Assume $\text{rk}(w \cup \{\gamma\}, \mathbb{M}^+) \geq 0$, $w = \{a_0, \dots, a_n\}$ and $k \leq n$. Let $\varphi(x_0, \dots, x_n)$ be a quantifier free formula in the vocabulary τ such that

$$\mathbb{M} \models \varphi[f_\gamma(a_0), \dots, f_\gamma(a_k), \dots, f_\gamma(a_n)].$$

Let $\varphi^*(x_0, \dots, x_n, x_{n+1})$ be a quantifier free formula in the vocabulary τ^+ obtained from φ by replacing each $R_i(y_0, \dots, y_{n(i)-1})$ (where $\{y_0, \dots, y_{n(i)-1}\} \subseteq \{x_0, \dots, x_n\}$) with $Q_i(y_0, \dots, y_{n(i)-1}, x_{n+1})$ and let φ^+ be

$$\varphi^*(x_0, \dots, x_n, x_{n+1}) \wedge S(x_0, x_{n+1}) \wedge \dots \wedge S(x_n, x_{n+1}).$$

Then $\mathbb{M}^+ \models \varphi^+[a_0, \dots, a_k, \dots, a_n, \gamma]$. By our assumption on $w \cup \{\gamma\}$ we know that the set $A = \{b < \lambda^+ : \mathbb{M}^+ \models \varphi^+[a_0, \dots, a_{k-1}, b, a_{k+1}, \dots, a_n, \gamma]\}$ is uncountable. Clearly $A \subseteq \gamma$ (note $S(x_k, x_{n+1})$ in φ^+) and thus the set $f_\gamma[A]$ is an uncountable subset of λ . For each $b \in A$ we have $\mathbb{M} \models \varphi[f_\gamma(a_0), \dots, f_\gamma(b), \dots, f_\gamma(a_n)]$, so now we may conclude that $\text{rk}(f_\gamma[w], \mathbb{M}) \geq 0$.

(*)₁ Assume $\text{rk}(w \cup \{\gamma\}, \mathbb{M}^+) \geq \alpha + 1$. Let $\varphi(x_0, \dots, x_n)$ be a quantifier free formula in the vocabulary τ , $k \leq n$ and $w = \{a_0, \dots, a_n\}$, and suppose that $\mathbb{M} \models \varphi[f_\gamma(a_0), \dots, f_\gamma(a_k), \dots, f_\gamma(a_n)]$. Let φ^* and φ^+ be defined exactly as in (*₀). Then $\mathbb{M}^+ \models \varphi^+[a_0, \dots, a_k, \dots, a_n, \gamma]$. By our assumption there is $a^* \in \lambda^+ \setminus (w \cup \{\gamma\})$ such that $\mathbb{M}^+ \models \varphi^+[a_0, \dots, a^*, \dots, a_n, \gamma]$ and $\text{rk}(w \cup \{\gamma, a^*\}, \mathbb{M}^+) \geq \alpha$. Necessarily $a^* < \gamma$, and by the inductive hypothesis $\text{rk}(f_\gamma[w \cup \{a^*\}], \mathbb{M}) \geq \alpha$. Clearly $\mathbb{M} \models \varphi[f_\gamma(a_0), \dots, f_\gamma(a^*), \dots, f_\gamma(a_n)]$ and we may conclude $\text{rk}(f_\gamma[w], \mathbb{M}) \geq \alpha + 1$.

(*)₂ If α is limit and $\text{rk}(w \cup \{\gamma\}, \mathbb{M}^+) \geq \alpha$ then, by the inductive hypothesis, for each $\beta < \alpha$ we have $\beta \leq \text{rk}(w \cup \{\gamma\}, \mathbb{M}^+) \leq \text{rk}(f_\gamma[w], \mathbb{M})$. Hence $\alpha \leq \text{rk}(f_\gamma[w], \mathbb{M})$.

(ii) Induction similar to part (i). For a quantifier free formula $\varphi(x_0, \dots, x_n)$ in the vocabulary τ , let φ^* be the formula $\varphi(x_0, \dots, x_n) \wedge \neg T(x_0) \wedge \dots \wedge \neg T(x_n)$ (so φ^* is a

quantifier free formula in the vocabulary τ^+). If φ witnesses that $\neg(\text{rk}(w, \mathbb{M}) \geq 0)$, then φ^* witnesses $\neg(\text{rk}(w, \mathbb{M}^+) \geq 0)$, and similarly with $\alpha + 1$ in place of 0.

(iii) Suppose towards contradiction that $\varepsilon + 1 \leq \text{rk}(\{\gamma\}, \mathbb{M}^+)$. Since $\mathbb{M}^+ \models T[\gamma]$, we may find $\gamma' \neq \gamma$ such that $\text{rk}(\{\gamma, \gamma'\}, \mathbb{M}^+) \geq \varepsilon$ and $\mathbb{M}^+ \models T[\gamma']$. Let $\{\gamma, \gamma'\} = \{\gamma_0, \gamma_1\}$ where $\gamma_0 < \gamma_1$. It follows from part (i) that $\text{rk}(\{\gamma_0, \gamma_1\}, \mathbb{M}^+) < \varepsilon$, a contradiction. \square

It follows from Claim 2.6.1 (and Observation 2.2) that $\text{rk}(w, \mathbb{M}^+) \leq \varepsilon$ for every non-empty set $w \subseteq \lambda^+$. Consequently, the model \mathbb{M}^+ witnesses $\text{NPr}_{\varepsilon+1}(\lambda^+)$.

(3) Let $\langle \mu_n : n < \omega \rangle$ be an increasing sequence cofinal in λ . For each n fix a model \mathbb{M}_n with a countable vocabulary $\tau(\mathbb{M}_n)$ consisting of relational symbols only and with the universe μ_n and such that $\text{rk}(w, \mathbb{M}_n) < \varepsilon$ for nonempty finite $w \subseteq \mu_n$. We also assume that $\tau(\mathbb{M}_n) \cap \tau(\mathbb{M}_m) = \emptyset$ for $n < m < \omega$. Let P_n (for $n < \omega$) be new unary relational symbols and let $\tau^+ = \bigcup \{ \tau(\mathbb{M}_n) : n < \omega \} \cup \{ P_n : n < \omega \}$. Consider a model \mathbb{M}^+ in vocabulary τ^+ with the universe λ and such that

- $P_n^{\mathbb{M}} = \mu_n$ for $n < \omega$, and
- for each $n < \omega$ and $S \in \tau(\mathbb{M}_n)$ we have $S^{\mathbb{M}} = S^{\mathbb{M}_n}$.

Claim 2.6.2. *If w is a finite non-empty subset of μ_n , $n < \omega$, then $\text{rk}(w, \mathbb{M}) \leq \text{rk}(w, \mathbb{M}_n) < \varepsilon$.*

Proof of the Claim. Similar to the proofs in Claim 2.6.1. \square

(4) Follows from Observation 2.5(1). \square

Proposition 2.7 (See [6, Conclusion 1.8]). (1) $\text{Pr}_{\omega_1}^*(\beth_{\omega_1})$ holds and hence also $\text{Pr}_{\omega_1}(\beth_{\omega_1})$.

(2) Assume $\beta < \alpha < \omega_1$, \mathbb{M} is a model with a countable vocabulary τ and the universe μ , $m, n < \omega$, $n > 0$, $A \subseteq \mu$ and $|A| \geq \beth_{\omega \cdot \alpha}$. Then there is $w \subseteq A$ with $|w| = n$ and $\text{rk}^*(w, \mathbb{M}) \geq \omega \cdot \beta + m$ ¹.

Proof. (1) Follows from part (2) (and 2.6(4)).

(2) Induction on $\alpha < \omega_1$.

STEP $\alpha = 1$ (AND $\beta = 0$): Let \mathbb{M}, μ, n, m be as in the assumptions, $A \subseteq \mu$ and $|A| \geq \beth_{\omega}$. Using the Erdős–Rado theorem we may choose a sequence $\langle a_i : i < \omega_2 \rangle$ of distinct elements of A such that:

- (a) the quantifier free type of $\langle a_{i_0}, \dots, a_{i_{m+n}} \rangle$ in \mathbb{M} is constant for $i_0 < \dots < i_{m+n} < \omega_2$, and
- (b) for each $k \leq m + n$ the value of $\min\{\omega, \text{rk}^*(\{a_{i_0}, \dots, a_{i_{m+n-k}}\}, \mathbb{M})\}$ is constant for $i_0 < \dots < i_{m+n-k} < \omega_2$.

Let $i_\ell = \omega_1 \cdot (\ell + 1)$ (for $\ell = -1, 0, \dots, n + m$). Suppose $\phi(x_0, \dots, x_{n+m}) \in \mathcal{L}(\tau)$ is a quantifier free formula, $k \leq n + m$ and $\mathbb{M} \models \phi[a_{i_0}, \dots, a_{i_k}, \dots, a_{i_{n+m}}]$. It follows from the property stated in (a) above that for every i in the (uncountable) interval (i_{k-1}, i_k) we have $\mathbb{M} \models \varphi[a_{i_0}, \dots, a_{i_{k-1}}, a_i, a_{i_{k+1}}, \dots, a_{i_{n+m}}]$. Consequently, $\text{rk}^*(\{a_{i_0}, \dots, a_{i_{n+m}}\}, \mathbb{M}) \geq 0$, and the homogeneity stated in (b) implies that for every nonempty set $w \subseteq \omega_2$ with at most $n + m + 1$ elements we have $\text{rk}^*(\{a_i : i \in w\}, \mathbb{M}) \geq 0$. Now, by induction on $k \leq m + n$ we will argue that

¹“.” stands for the ordinal multiplication

$(*)_k$ for every nonempty set $w \subseteq \omega_2$ with at most $n + m + 1 - k$ elements we have $\text{rk}^*(\{a_i : i \in w\}, \mathbb{M}) \geq k$.

We have already justified $(*)_0$. For the inductive step assume $(*)_k$ and $k < m + n$. Let $i_\ell = \omega_1 \cdot (\ell + 1)$ and suppose that $\varphi(x_0, \dots, x_{m+n-k-1})$ is a quantifier free formula, $\mathbb{M} \models \varphi[a_{i_0}, \dots, a_{i_z}, \dots, a_{i_{m+n-k-1}}]$ and $0 \leq z \leq n + m - k - 1$. By the homogeneity stated in (a), for every i in the uncountable interval (i_{z-1}, i_z) we have $\mathbb{M} \models \varphi[a_{i_0}, \dots, a_{i_{z-1}}, a_i, a_{i_{z+1}}, \dots, a_{i_{m+n-k-1}}]$. The inductive hypothesis $(*)_k$ implies that $\text{rk}^*(\{a_{i_0}, \dots, a_{i_{z-1}}, a_i, a_j, a_{i_{z+1}}, \dots, a_{i_{m+n-k-1}}\}, \mathbb{M}) \geq k$ (for any $i_{z-1} < i < j \leq i_z$). Now we easily conclude that $k + 1 \leq \text{rk}^*(\{a_{i_0}, \dots, a_{i_{m+n-k-1}}\}, \mathbb{M})$ and $(*)_{k+1}$ follows by the homogeneity given by (b).

Finally note that $(*)_{m+1}$ gives the desired conclusion: taking any $i_0 < \dots < i_{n-1} < \omega_2$ we will have $m + 1 \leq \text{rk}^*(\{a_{i_0}, \dots, a_{i_{n-1}}\}, \mathbb{M})$.

STEP $\alpha = \gamma + 1$: Let \mathbb{M}, μ, n, m be as in the assumptions, $A \subseteq \mu$ and $|A| \geq \beth_{\omega \cdot \gamma + \omega}$. By the Erdős–Rado theorem we may choose a sequence $\langle a_i : i < \beth_{\omega \cdot \gamma} \rangle$ of distinct elements of A such that the following two demands are satisfied.

- (c) The quantifier free type of $\langle a_{i_0}, \dots, a_{i_{m+n}} \rangle$ in \mathbb{M} is constant for $i_0 < \dots < i_{m+n} < \beth_{\omega \cdot \gamma}$.
- (d) For each $k \leq m + n$ the value of $\min\{\omega \cdot \alpha, \text{rk}^*(\{a_{i_0}, \dots, a_{i_{m+n-k}}\}, \mathbb{M})\}$ is constant for $i_0 < \dots < i_{m+n-k} < \beth_{\omega \cdot \gamma}$.

For any $\ell < \omega$ and $\gamma' < \gamma$, we may apply the inductive hypothesis to $\{a_i : i < \beth_{\omega \cdot \gamma'}\}$, $\ell, m + n + 1$ and γ' to find $i_0 < \dots < i_{m+n} < \beth_{\omega \cdot \gamma}$ such that $\text{rk}^*(\{a_{i_0}, \dots, a_{i_{m+n}}\}, \mathbb{M}) \geq \omega \cdot \gamma' + \ell$. By the homogeneity in (d) this implies that

$(**)_{0}$ for all $i_0 < \dots < i_{m+n} < \beth_{\omega \cdot \gamma}$ we have $\text{rk}^*(\{a_{i_0}, \dots, a_{i_{m+n}}\}, \mathbb{M}) \geq \omega \cdot \gamma$.

Now, by induction on $k \leq m + n$ we argue that

$(**)_{k}$ for each $i_0 < \dots < i_{m+n-k} < (\beth_{\omega \cdot \gamma})^+$ we have

$$\omega \cdot \gamma + k \leq \text{rk}^*(\{a_{i_0}, \dots, a_{i_{m+n-k}}\}, \mathbb{M}).$$

So assume $(**)_{k}$, $k < m + n$ and let $i_\ell = \omega_1 \cdot (\ell + 1)$ (for $\ell = -1, 0, \dots, m + n$) and $0 \leq z \leq n + m - k - 1$. Suppose that $\mathbb{M} \models \varphi[a_{i_0}, \dots, a_{i_z}, \dots, a_{i_{m+n-k-1}}]$. Then by the homogeneity in (c), for every i in the uncountable interval (i_{z-1}, i_z) we have $\mathbb{M} \models \varphi[a_{i_0}, \dots, a_{i_{z-1}}, a_i, a_{i_{z+1}}, \dots, a_{i_{m+n-k-1}}]$. By the inductive hypothesis $(**)_{k}$ we know $\omega \cdot \gamma + k \leq \text{rk}^*(\{a_{i_0}, \dots, a_{i_{z-1}}, a_i, a_j, a_{i_{z+1}}, \dots, a_{i_{m+n-k-1}}\}, \mathbb{M})$ (for $i_{z-1} < i < j \leq i_z$). Now we easily conclude that $\omega \cdot \gamma + k + 1 \leq \text{rk}^*(\{a_{i_0}, \dots, a_{i_{m+n-k-1}}\}, \mathbb{M})$, and $(**)_{k+1}$ follows by the homogeneity in (d).

Finally note that $(**)_{m+1}$ gives the desired conclusion: taking any $i_0 < \dots < i_{n-1} < \beth_{\omega \cdot \gamma}$ we will have $\text{rk}^*(\{a_{i_0}, \dots, a_{i_{n-1}}\}, \mathbb{M}) \geq \omega \cdot \gamma + m + 1$.

STEP α IS LIMIT: should be clear. \square

Definition 2.8. Let λ_{ω_1} be the smallest cardinal κ such that $\text{Pr}_{\omega_1}(\kappa)$ and $\lambda_{\omega_1}^*$ be the smallest cardinal κ such that $\text{Pr}_{\omega_1}^*(\kappa)$.

By Propositions 2.6(4) and 2.7 we have $\lambda_{\omega_1} \leq \lambda_{\omega_1}^* \leq \beth_{\omega_1}$.

Proposition 2.9 (See [6, Claim 1.10(1)]). *If \mathbb{P} is a ccc forcing notion and λ is a cardinal such that $\text{Pr}_{\omega_1}^*(\lambda)$ holds, then $\Vdash_{\mathbb{P}} \text{Pr}_{\omega_1}^*(\lambda)$ and hence also $\text{Pr}_{\omega_1}(\lambda)$.*

Proof. Suppose towards contradiction that for some $p \in \mathbb{P}$ we have $p \Vdash_{\mathbb{P}} \text{NPr}_{\omega_1}^*(\lambda)$. Let $\tau = \{R_{n,\zeta} : n, \zeta < \omega\}$ where $R_{n,\zeta}$ is an n -ary relation symbol (for $n, \zeta < \omega$).

Then we may pick a name $\underline{\mathbb{M}}$ for a model on λ in vocabulary τ and an ordinal $\alpha_0 < \omega_1$ such that

- $p \Vdash$ “ $\underline{\mathbb{M}} = (\lambda, \{R_{n,\zeta}^{\underline{\mathbb{M}}}\}_{n,\zeta < \omega})$ is a model such that
- (a) for every n and a quantifier free formula $\varphi(x_0, \dots, x_{n-1}) \in \mathcal{L}(\tau)$ there is $\zeta < \omega$ such that for all a_0, \dots, a_{n-1}

$$\underline{\mathbb{M}} \models \varphi[a_0, \dots, a_{n-1}] \Leftrightarrow R_{n,\zeta}[a_0, \dots, a_{n-1}]$$
 - (b) $\sup\{\text{rk}(w, \underline{\mathbb{M}}) : \emptyset \neq w \in [\lambda]^{<\omega}\} < \alpha_0$ ”.

Now, let $S_{n,\zeta,\beta,k}$ be an n -ary predicate (for $k < n, \zeta < \omega$ and $-1 \leq \beta < \alpha_0$) and let $\tau^* = \{S_{n,\zeta,\beta,k} : k < n < \omega, \zeta < \omega \text{ and } -1 \leq \beta < \alpha_0\}$. (So τ^* is a countable vocabulary.) We define a model \mathbb{M}^* in the vocabulary τ^* . The universe of \mathbb{M}^* is λ and for $k < n, \zeta < \omega$ and $-1 \leq \beta < \alpha_0$:

$$S_{n,\zeta,\beta,k}^{\mathbb{M}^*} = \left\{ (a_0, \dots, a_{n-1}) \in {}^n\lambda : a_0 < \dots < a_{n-1} \text{ and some condition } q \geq p \text{ forces that } \right. \\ \left. \begin{array}{l} \text{“}\underline{\mathbb{M}} \models R_{n,\zeta}[a_0, \dots, a_{n-1}] \text{ and } \text{rk}^*({a_0, \dots, a_{n-1}}, \underline{\mathbb{M}}) = \beta \text{ and} \\ R_{n,\zeta, k} \text{ witness that } \neg(\text{rk}^*({a_0, \dots, a_{n-1}}, \underline{\mathbb{M}}) \geq \beta + 1) \text{”} \end{array} \right\}.$$

Claim 2.9.1. *For every n and every increasing tuple $(a_0, \dots, a_{n-1}) \in {}^n\lambda$ there are $\zeta < \omega$ and $-1 \leq \beta < \alpha_0$ and $k < n$ such that $\mathbb{M}^* \models S_{n,\zeta,\beta,k}[a_0, \dots, a_{n-1}]$.*

Proof of the Claim. Should be clear. \square

Claim 2.9.2. *If $(a_0, \dots, a_{n-1}) \in {}^n\lambda$ and $\mathbb{M}^* \models S_{n,\zeta,\beta,k}[a_0, \dots, a_{n-1}]$, then*

$$\text{rk}^*({a_0, \dots, a_{n-1}}, \mathbb{M}^*) \leq \beta.$$

Proof of the Claim. First let us deal with the case of $\beta = -1$. Assume towards contradiction that $\mathbb{M}^* \models S_{n,\zeta,-1,k}[a_0, \dots, a_{n-1}]$, but $\text{rk}^*({a_0, \dots, a_{n-1}}, \mathbb{M}^*) \geq 0$. Then we may find distinct $\langle b_i : i < \omega_1 \rangle \subseteq \lambda \setminus \{a_0, \dots, a_{n-1}\}$ such that

$$(\otimes)_1 \mathbb{M}^* \models S_{n,\zeta,-1,k}[a_0, \dots, a_{k-1}, b_i, a_{k+1}, \dots, a_{n-1}] \text{ for all } i < \omega_1.$$

For $i < \omega_1$ let $p_i \in \mathbb{P}$ be such that $p_i \geq p$ and

$$p_i \Vdash \text{“} \underline{\mathbb{M}} \models R_{n,\zeta}[a_0, \dots, b_i, \dots, a_{n-1}] \text{ and } \text{rk}^*({a_0, \dots, b_i, \dots, a_{n-1}}, \underline{\mathbb{M}}) = -1 \text{ and} \\ R_{n,\zeta, k} \text{ witness that } \neg(\text{rk}^*({a_0, \dots, a_{k-1}, b_i, a_{k+1}, \dots, a_{n-1}}, \underline{\mathbb{M}}) \geq 0) \text{”}$$

Let \underline{Y} be a name \mathbb{P} -name such that $p \Vdash \underline{Y} = \{i < \omega_1 : p_i \in \mathcal{G}_{\mathbb{P}}\}$. Since \mathbb{P} satisfies ccc, we may pick $p^* \geq p$ such that $p^* \Vdash$ “ \underline{Y} is uncountable”.

$$p^* \Vdash (\forall i \in \underline{Y}) (\underline{\mathbb{M}} \models R_{n,\zeta}[a_0, \dots, a_{k-1}, b_i, a_{k+1}, \dots, a_{n-1}]),$$

so also

$$p^* \Vdash \{b < \lambda : \underline{\mathbb{M}} \models R_{n,\zeta}[a_0, \dots, a_{k-1}, b, a_{k+1}, \dots, a_{n-1}]\} \text{ is uncountable.}$$

But

$$p^* \Vdash (\forall i \in \underline{Y}) (R_{n,\zeta, k} \text{ witness } \neg(\text{rk}^*({a_0, \dots, a_{k-1}, b_i, a_{k+1}, \dots, a_{n-1}}, \underline{\mathbb{M}}) \geq 0)),$$

and hence

$$p^* \Vdash \{b < \lambda : \underline{\mathbb{M}} \models R_{n,\zeta}[a_0, \dots, a_{k-1}, b, a_{k+1}, \dots, a_{n-1}]\} \text{ is countable,}$$

a contradiction.

Next we continue the proof of the Claim by induction on $\beta < \alpha_0$, so we assume that $0 \leq \beta$ and for $\beta' < \beta$ our claim holds true (for any n, ζ, k). Assume towards contradiction that $\mathbb{M}^* \models S_{n,\zeta,\beta,k}[a_0, \dots, a_{n-1}]$, but $\text{rk}^*({a_0, \dots, a_{n-1}}, \mathbb{M}^*) \geq \beta + 1$. Then we may find distinct $\langle b_i : i < \omega_1 \rangle \subseteq \lambda \setminus (w \setminus \{a_k\})$ such that

- (\oplus)₁ $\mathbb{M}^* \models S_{n,\zeta,\beta,k}[a_0, \dots, a_{k-1}, b_i, a_{k+1}, \dots, a_{n-1}]$ for all $i < \omega_1$, $b_0 = a_k$ and
 (\oplus)₂ $\text{rk}^*(\{a_0, \dots, a_{k-1}, b_i, b_j, a_{k+1}, \dots, a_{n-1}\}, \mathbb{M}^*) \geq \beta$ for all $i < j < \omega_1$.

For $i < \omega_1$ let $p_i \in \mathbb{P}$ be such that $p_i \geq p$ and

- $p_i \Vdash$ “ $\mathbb{M} \models R_{n,\zeta}[a_0, \dots, b_i, \dots, a_{n-1}]$ and $\text{rk}^*(\{a_0, \dots, b_i, \dots, a_{n-1}\}, \mathbb{M}) = \beta$ and
 $R_{n,\zeta}, k$ witness that $\neg(\text{rk}^*(\{a_0, \dots, a_{k-1}, b_i, a_{k+1}, \dots, a_{n-1}\}, \mathbb{M}) \geq \beta + 1)$ ”

Take $p^* \geq p$ such that

$$p^* \Vdash \text{“}\underline{Y} \stackrel{\text{def}}{=} \{i < \omega_1 : p_i \in \mathcal{G}_{\mathbb{P}}\} \text{ is uncountable”}.$$

Since

$$p^* \Vdash (\forall i \in \underline{Y}) \left(\mathbb{M} \models R_{n,\zeta}[a_0, \dots, a_{k-1}, b_i, a_{k+1}, \dots, a_{n-1}] \wedge \right. \\ \left. R_{n,\zeta}, k \text{ witness that } \neg(\text{rk}^*(\{a_0, \dots, b_i, \dots, a_{n-1}\}, \mathbb{M}) \geq \beta + 1) \right),$$

we see that

$$p^* \not\Vdash (\forall i, j \in \underline{Y}) (i \neq j \Rightarrow \text{rk}^*(\{a_0, \dots, a_{k-1}, b_i, b_j, a_{k+1}, \dots, a_{n-1}\}, \mathbb{M}) \geq \beta).$$

Consequently we may pick $q \geq p^*$, $i_0, j_0 < \omega_1$ and $\gamma < \beta$ and $\xi < \omega$ and $\ell \leq n$ such that $b_{i_0} < b_{j_0}$ and

$$q \Vdash \text{“}p_{i_0}, p_{j_0} \in \mathcal{G}_{\mathbb{P}} \text{ and } \text{rk}^*(\{a_0, \dots, a_{k-1}, b_{i_0}, b_{j_0}, a_{k+1}, \dots, a_{n-1}\}, \mathbb{M}) = \gamma \text{ and } \\ R_{n+1,\xi} \text{ and } \ell \text{ witness that } \\ \neg(\text{rk}^*(\{a_0, \dots, a_{k-1}, b_{i_0}, b_{j_0}, a_{k+1}, \dots, a_{n-1}\}, \mathbb{M}) \geq \gamma + 1)\text{”}.$$

Then $\mathbb{M}^* \models S_{n+1,\xi,\ell,\gamma}[a_0, \dots, a_{k-1}, b_{i_0}, b_{j_0}, a_{k+1}, \dots, a_{n-1}]$ and by the inductive hypothesis $\text{rk}^*(\{a_0, \dots, a_{k-1}, b_{i_0}, b_{j_0}, a_{k+1}, \dots, a_{n-1}\}, \mathbb{M}) \leq \gamma$, contradicting clause (\oplus)₂ above. \square

Corollary 2.10. *Let $\mu = \beth_{\omega_1} \leq \kappa$ and \mathbb{C}_{κ} be the forcing notion adding κ Cohen reals. Then $\Vdash_{\mathbb{C}_{\kappa}} \lambda_{\omega_1} \leq \mu \leq \mathfrak{c}$.*

3. SPECTRUM OF TRANSLATION NON-DISJOINTNESS

Definition 3.1. Let $B \subseteq \omega^2$ and $1 \leq k \leq \mathfrak{c}$.

- (1) We say that B is *perfectly orthogonal to k -small* (or a k -**pots**-set) if there is a perfect set $P \subseteq \omega^2$ such that $|(B+x) \cap (B+y)| \geq k$ for all $x, y \in P$. The set B is a k -**npots**-set if it is not k -**pots**.
- (2) We say that B has λ *many pairwise k -non-disjoint translations* if for some set $X \subseteq \omega^2$ of cardinality λ , for all $x, y \in X$ we have $|(B+x) \cap (B+y)| \geq k$.
- (3) We define the *spectrum of translation k -non-disjointness* of B as

$$\text{stnd}_k(B) = \{(x, y) \in \omega^2 \times \omega^2 : |(B+x) \cap (B+y)| \geq k\}.$$

Remark 3.2. (1) Note that if $B \subseteq \omega^2$ is an uncountable Borel set, then there is a perfect set $P \subseteq B$. For B, P as above for every $x, y \in P$ we have $0 = x+x = y+y \in (B+x) \cap (B+y)$ and $x+y \in (B+x) \cap (B+y)$. Consequently every uncountable Borel subset of ω^2 is a 2-**pots**-set.

- (2) Assume $B \subseteq \omega^2$ and $x, y \in \omega^2$. If $b_x, b_y \in B$ and $b_x + x = b_y + y \in (B+x) \cap (B+y)$, then also $b_x + y = b_y + x \in (B+x) \cap (B+y)$. Consequently, if $(B+x) \cap (B+y) \neq \emptyset$ is finite, then it has an even number of elements.

- Proposition 3.3.** (1) Let $1 \leq k \leq \mathfrak{c}$. A set $B \subseteq {}^\omega 2$ is a k -**pots**-set if and only if there is a perfect set $P \subseteq {}^\omega 2$ such that $P \times P \subseteq \text{stnd}_k(B)$.
- (2) Assume $k < \omega$. If B is Σ_2^0 , then $\text{stnd}_k(B)$ is Σ_2^0 as well. If B is Borel, then $\text{stnd}_k(B)$ and $\text{stnd}_\omega(B)$ are Σ_1^1 and $\text{stnd}_\mathfrak{c}(B)$ is Δ_2^1 .
- (3) Let $\mathfrak{c} < \kappa \leq \mu$ and let \mathbb{C}_μ be the forcing notion adding μ Cohen reals. Then, remembering Definition 3.1(2),
 $\Vdash_{\mathbb{C}_\mu}$ “if a Borel set $B \subseteq {}^\omega 2$ has κ many pairwise k -non-disjoint translates, then B is an k -**pots**-set”.
- (4) If $k < \omega$, B is a (code for) Σ_2^0 k -**npots**-set and \mathbb{P} is a forcing notion, then $\Vdash_{\mathbb{P}}$ “ B is a (code for) k -**npots**-set”.
- (5) Assume $\text{Pr}_{\omega_1}(\lambda)$. If $k \leq \omega$ and a Borel set $B \subseteq {}^\omega 2$ has λ many pairwise k -nondisjoint translates, then it is a k -**pots**-set.

Proof. (2) Let $B = \bigcup_{n < \omega} F_n$, where each F_n is a closed subset of ${}^\omega 2$. Then

$$(x, y) \in \text{stnd}_k(B) \Leftrightarrow (\exists n_0, \dots, n_{k-1}, m_0, \dots, m_{k-1}, N < \omega) (\exists z_0, \dots, z_{k-1} \in {}^\omega 2) (\forall i, j < k) \left(\begin{aligned} & (i \neq j \Rightarrow z_i \upharpoonright N \neq z_j \upharpoonright N) \wedge z_i + x \in F_{n_i} \wedge z_i + y \in F_{m_i} \end{aligned} \right)$$

The formula $(\forall i, j < k) ((i \neq j \Rightarrow z_i \upharpoonright N \neq z_j \upharpoonright N) \wedge z_i + x \in F_{n_i} \wedge z_i + y \in F_{m_i})$ represents a compact subset of $({}^\omega 2)^{k+2}$ and hence easily the assertion follows.

- (3) This is a consequence of (1,2) above and Shelah [6, Fact 1.16].
- (4) If B is a Σ_2^0 set then the formula “there is a perfect set $P \subseteq {}^\omega 2$ such that for all $x, y \in P$ we have $(x, y) \in \text{stnd}_k(B)$ ” is Σ_2^1 (remember (2) above).
- (5) By [6, Claim 1.12(1)]. □

We want to analyze k -**pots**-sets in more detail, restricting ourselves to Σ_2^0 subsets of ${}^\omega 2$. For the rest of this section we assume the following Hypothesis.

- Hypothesis 3.4.** (1) $T_n \subseteq {}^{\omega > 2}$ is a tree with no maximal nodes (for $n < \omega$);
- (2) $B = \bigcup_{n < \omega} \lim(T_n)$, $\bar{T} = \langle T_n : n < \omega \rangle$;
- (3) $2 \leq \iota < \omega$, $k = 2\iota$.

Definition 3.5. Let $\mathbf{M}_{\bar{T}, k}$ consist of all tuples

$$\mathbf{m} = (\ell_{\mathbf{m}}, u_{\mathbf{m}}, \bar{h}_{\mathbf{m}}, \bar{g}_{\mathbf{m}}) = (\ell, u, \bar{h}, \bar{g})$$

such that:

- (a) $0 < \ell < \omega$, $u \subseteq {}^{\ell 2}$ and $2 \leq |u|$;
- (b) $\bar{h} = \langle h_i : i < \iota \rangle$, $\bar{g} = \langle g_i : i < \iota \rangle$ and for each $i < \iota$ we have
 $h_i : u^{(2)} \rightarrow \omega$ and $g_i : u^{(2)} \rightarrow \bigcup_{n < \omega} (T_n \cap {}^{\ell 2})$;
- (c) $g_i(\eta, \nu) \in T_{h_i(\eta, \nu)} \cap {}^{\ell 2}$ for all $(\eta, \nu) \in u^{(2)}$, $i < \iota$;
- (d) if $(\eta, \nu) \in u^{(2)}$ and $i < \iota$, then $\eta + g_i(\eta, \nu) = \nu + g_i(\nu, \eta)$;
- (e) for any $(\eta, \nu) \in u^{(2)}$, there are no repetitions in the sequence $\langle g_i(\eta, \nu), g_i(\nu, \eta) : i < \iota \rangle$.

Definition 3.6. Assume $\mathbf{m} = (\ell, u, \bar{h}, \bar{g}) \in \mathbf{M}_{\bar{T},k}$ and $\rho \in {}^{\ell}2$. We define $\mathbf{m} + \rho = (\ell', u', \bar{h}', \bar{g}')$ by

- $\ell' = \ell$, $u' = \{\eta + \rho : \eta \in u\}$,
- $\bar{h}' = \langle h'_i : i < \iota \rangle$ where $h'_i : (u')^{(2)} \rightarrow \omega$ are such that $h'_i(\eta + \rho, \nu + \rho) = h_i(\eta, \nu)$ for $(\eta, \nu) \in u^{(2)}$,
- $\bar{g}' = \langle g'_i : i < \iota \rangle$ where $g'_i : (u')^{(2)} \rightarrow \bigcup_{n < \omega} (T_n \cap {}^{\ell}2)$ are such that $g'_i(\eta + \rho, \nu + \rho) = g_i(\eta, \nu)$ for $(\eta, \nu) \in u^{(2)}$.

Also if $\rho \in {}^{\omega}2$, then we set $\mathbf{m} + \rho = \mathbf{m} + (\rho \upharpoonright \ell)$.

Observation 3.7. (1) If $\mathbf{m} \in \mathbf{M}_{\bar{T},k}$ and $\rho \in {}^{\ell}2$, then $\mathbf{m} + \rho \in \mathbf{M}_{\bar{T},k}$.
(2) For each $\rho \in {}^{\omega}2$ the mapping

$$\mathbf{M}_{\bar{T},k} \longrightarrow \mathbf{M}_{\bar{T},k} : \mathbf{m} \mapsto \mathbf{m} + \rho$$

is a bijection.

Definition 3.8. Assume $\mathbf{m}, \mathbf{n} \in \mathbf{M}_{\bar{T},k}$. We say that \mathbf{n} extends \mathbf{m} ($\mathbf{m} \sqsubseteq \mathbf{n}$ in short) if and only if:

- $\ell_{\mathbf{m}} \leq \ell_{\mathbf{n}}$, $u_{\mathbf{m}} = \{\eta \upharpoonright \ell_{\mathbf{m}} : \eta \in u_{\mathbf{n}}\}$, and
- for every $(\eta, \nu) \in (u_{\mathbf{n}})^{(2)}$ such that $\eta \upharpoonright \ell_{\mathbf{m}} \neq \nu \upharpoonright \ell_{\mathbf{m}}$ and each $i < \iota$ we have $h_i^{\mathbf{m}}(\eta \upharpoonright \ell_{\mathbf{m}}, \nu \upharpoonright \ell_{\mathbf{m}}) = h_i^{\mathbf{n}}(\eta, \nu)$ and $g_i^{\mathbf{m}}(\eta \upharpoonright \ell_{\mathbf{m}}, \nu \upharpoonright \ell_{\mathbf{m}}) = g_i^{\mathbf{n}}(\eta, \nu) \upharpoonright \ell_{\mathbf{m}}$.

Definition 3.9. We define a function $\text{ndrk} : \mathbf{M}_{\bar{T},k} \rightarrow \text{ON} \cup \{\infty\}$ declaring inductively when $\text{ndrk}(\mathbf{m}) \geq \alpha$ (for an ordinal α).

- $\text{ndrk}(\mathbf{m}) \geq 0$ always;
- if α is a limit ordinal, then

$$\text{ndrk}(\mathbf{m}) \geq \alpha \Leftrightarrow (\forall \beta < \alpha)(\text{ndrk}(\mathbf{m}) \geq \beta);$$

- if $\alpha = \beta + 1$, then $\text{ndrk}(\mathbf{m}) \geq \alpha$ if and only if for every $\nu \in u_{\mathbf{m}}$ there is $\mathbf{n} \in \mathbf{M}_{\bar{T},k}$ such that $\ell_{\mathbf{n}} > \ell_{\mathbf{m}}$, $\mathbf{m} \sqsubseteq \mathbf{n}$ and $\text{ndrk}(\mathbf{n}) \geq \beta$ and

$$|\{\eta \in u_{\mathbf{n}} : \nu \triangleleft \eta\}| \geq 2;$$

- $\text{ndrk}(\mathbf{m}) = \infty$ if and only if $\text{ndrk}(\mathbf{m}) \geq \alpha$ for all ordinals α .

We also define

$$\text{NDRK}(\bar{T}) = \sup\{\text{ndrk}(\mathbf{m}) + 1 : \mathbf{m} \in \mathbf{M}_{\bar{T},k}\}.$$

Lemma 3.10. (1) The relation \sqsubseteq is a partial order on $\mathbf{M}_{\bar{T},k}$.

(2) If $\mathbf{m}, \mathbf{n} \in \mathbf{M}_{\bar{T},k}$ and $\mathbf{m} \sqsubseteq \mathbf{n}$ and $\alpha \leq \text{ndrk}(\mathbf{n})$, then $\alpha \leq \text{ndrk}(\mathbf{m})$.

(3) The function ndrk is well defined.

(4) If $\mathbf{m} \in \mathbf{M}_{\bar{T},k}$ and $\rho \in {}^{\omega}2$ then $\text{ndrk}(\mathbf{m}) = \text{ndrk}(\mathbf{m} + \rho)$.

(5) If $\mathbf{m} \in \mathbf{M}_{\bar{T},k}$, $\nu \in u_{\mathbf{m}}$ and $\text{ndrk}(\mathbf{m}) \geq \omega_1$, then there is an $\mathbf{n} \in \mathbf{M}_{\bar{T},k}$ such that $\mathbf{m} \sqsubseteq \mathbf{n}$, $\text{ndrk}(\mathbf{n}) \geq \omega_1$, and

$$|\{\eta \in u_{\mathbf{n}} : \nu \triangleleft \eta\}| \geq 2.$$

(6) If $\mathbf{m} \in \mathbf{M}_{\bar{T},k}$ and $\infty > \text{ndrk}(\mathbf{m}) = \beta > \alpha$, then there is $\mathbf{n} \in \mathbf{M}_{\bar{T},k}$ such that $\mathbf{m} \sqsubseteq \mathbf{n}$ and $\text{ndrk}(\mathbf{n}) = \alpha$.

(7) If $\text{NDRK}(\bar{T}) \geq \omega_1$, then $\text{NDRK}(\bar{T}) = \infty$.

(8) Assume $\mathbf{m} \in \mathbf{M}_{\bar{T},k}$ and $u' \subseteq u_{\mathbf{m}}$, $|u'| \geq 2$. Put $\ell' = \ell_{\mathbf{m}}$, $h'_i = h_i^{\mathbf{m}} \upharpoonright u'^{(2)}$ and $g'_i = g_i^{\mathbf{m}} \upharpoonright u'^{(2)}$ (for $i < \iota$), and let $\mathbf{m} \upharpoonright u' = (\ell', u', \bar{h}', \bar{g}')$. Then $\mathbf{m} \upharpoonright u' \in \mathbf{M}_{\bar{T},k}$ and $\text{ndrk}(\mathbf{m}) \leq \text{ndrk}(\mathbf{m} \upharpoonright u')$.

Proof. (1) Should be clear.

(2) Induction on α . If $\alpha = \alpha_0 + 1$ and $\mathbf{n}' \sqsupseteq \mathbf{n}$ is one of the witnesses used to claim that $\text{ndrk}(\mathbf{n}) \geq \alpha_0 + 1$, then this \mathbf{n}' can also be used for \mathbf{m} . Hence we can argue the successor step of the induction. The limit steps are even easier.

(3) One has to show that if $\beta < \alpha$ and $\text{ndrk}(\mathbf{m}) \geq \alpha$, then $\text{ndrk}(\mathbf{m}) \geq \beta$. This can be shown by induction on α : at the successor stage if \mathbf{n} is one of the witnesses used to claim that $\text{ndrk}(\mathbf{m}) \geq \alpha + 1$, then $\text{ndrk}(\mathbf{n}) \geq \alpha$. By (2) we get $\text{ndrk}(\mathbf{m}) \geq \alpha$ and by the inductive hypothesis $\text{ndrk}(\mathbf{m}) \geq \gamma$ for $\gamma \leq \alpha$. Limit stages should be clear too.

(4) Should be clear.

(5) Let \mathcal{N} be the collection of all $\mathbf{n} \in \mathbf{M}_{\bar{T},k}$ such that $\mathbf{m} \sqsubseteq \mathbf{n}$ and $|\{\eta \in u_{\mathbf{n}} : \nu \triangleleft \eta\}| \geq 2$. If $\text{ndrk}(\mathbf{n}_0) \geq \omega_1$ for some $\mathbf{n}_0 \in \mathcal{N}$, then we are done. So suppose towards contradiction that there is no such \mathbf{n}_0 . Then, as \mathcal{N} is countable,

$$\alpha_0 \stackrel{\text{def}}{=} \sup\{\text{ndrk}(\mathbf{n}) + 1 : \mathbf{n} \in \mathcal{N}\} < \omega_1.$$

But $\text{ndrk}(\mathbf{m}) \geq \alpha_0 + 1$ implies that $\text{ndrk}(\mathbf{n}_1) \geq \alpha_0$ for some $\mathbf{n}_1 \in \mathcal{N}$, a contradiction.

(6) Induction on ordinals β (for all $\alpha < \beta$). The main point is that if $\text{ndrk}(\mathbf{m}) = \beta$, then for some $\nu \in u_{\mathbf{m}}$ we cannot find \mathbf{n} as needed for witnessing $\text{ndrk}(\mathbf{m}) \geq \beta + 1$, but for each $\gamma < \beta$ we can find \mathbf{n} needed for $\text{ndrk}(\mathbf{m}) \geq \gamma + 1$. Therefore for each $\gamma < \beta$ we may find $\mathbf{n} \sqsupseteq \mathbf{m}$ such that $\gamma \leq \text{ndrk}(\mathbf{n}) < \beta$.

(7) Follows from (6) above.

(8) It should be clear that $(\ell', u', \bar{h}', \bar{g}') \in \mathbf{M}_{\bar{T},k}$. Also, by a straightforward induction on α for all \mathbf{m} and restrictions $\mathbf{m} \upharpoonright u'$, one shows that

$$\alpha \leq \text{ndrk}(\mathbf{m}) \Rightarrow \alpha \leq \text{ndrk}(\mathbf{m} \upharpoonright u').$$

□

Proposition 3.11. *The following conditions are equivalent.*

- (a) $\text{NDRK}(\bar{T}) \geq \omega_1$.
- (b) $\text{NDRK}(\bar{T}) = \infty$.
- (c) *There is a perfect set $P \subseteq {}^\omega 2$ such that*

$$(\forall \eta, \nu \in P)(|(B + \eta) \cap (B + \nu)| \geq k).$$

- (d) *In some ccc forcing extension, there is $A \subseteq {}^\omega 2$ of cardinality λ_{ω_1} such that*

$$(\forall \eta, \nu \in A)(|(B + \eta) \cap (B + \nu)| \geq k).$$

Proof. (a) \Rightarrow (b) This is Lemma 3.10(7).

(b) \Rightarrow (c) If $\text{NDRK}(\bar{T}) = \infty$ then there is $\mathbf{m}_0 \in \mathbf{M}_{\bar{T},k}$ with $\text{ndrk}(\mathbf{m}_0) \geq \omega_1$. Using Lemma 3.10(5) we may now choose a sequence $\langle \mathbf{m}_j : j < \omega \rangle \subseteq \mathbf{M}_{\bar{T},k}$ such that for each $j < \omega$:

- (i) $\mathbf{m}_j \sqsubseteq \mathbf{m}_{j+1}$,
- (ii) $\text{ndrk}(\mathbf{m}_j) \geq \omega_1$,
- (iii) $|\{\eta \in u_{\mathbf{m}_{j+1}} : \nu \triangleleft \eta\}| \geq 2$ for each $\nu \in u_{\mathbf{m}_j}$.

Let $P = \{\rho \in {}^\omega 2 : (\forall j < \omega)(\rho \upharpoonright \ell_{\mathbf{m}_j} \in u_{\mathbf{m}_j})\}$. Clearly, P is a perfect set. For $\eta, \nu \in P$, $\eta \neq \nu$, let j_0 be the smallest such that $\eta \upharpoonright \ell_{\mathbf{m}_{j_0}} \neq \nu \upharpoonright \ell_{\mathbf{m}_{j_0}}$ and let

$$G_i(\eta, \nu) = \bigcup \{g_i^{\mathbf{m}_j}(\eta \upharpoonright \ell_{\mathbf{m}_j}, \nu \upharpoonright \ell_{\mathbf{m}_j}) : j \geq j_0\} \in \lim \left(T_{h_i^{\mathbf{m}_{j_0}}(\eta \upharpoonright \ell_{\mathbf{m}_{j_0}}, \nu \upharpoonright \ell_{\mathbf{m}_{j_0}})} \right) \quad \text{for } i < \iota.$$

Then $G_i : P^{(2)} \rightarrow B$ and for $(\eta, \nu) \in P^{(2)}$ and $i < \iota$:

$$\eta + G_i(\eta, \nu) = \nu + G_i(\nu, \eta) \quad \text{and} \quad \eta + G_i(\nu, \eta) = \nu + G_i(\eta, \nu).$$

Moreover, there are no repetitions in the sequence $\langle G_i(\eta, \nu), G_i(\nu, \eta) : i < \iota \rangle$. Hence, for distinct $\eta, \nu \in P$ we have $|(B + \eta) \cap (B + \nu)| \geq 2\iota = k$.

(c) \Rightarrow (d) Assume (c). Let $\kappa = \beth_{\omega_1}$. By Corollary 2.10 we know that $\Vdash_{\mathbb{C}_\kappa} \lambda_{\omega_1} \leq \mathfrak{c}$. Remembering Proposition 3.3(1,2), we note that the formula “ $P \times P \subseteq \text{std}_k(B)$ ” is Π_1^1 , so it holds in the forcing extension by \mathbb{C}_κ . Now we easily conclude (d).

(d) \Rightarrow (a) Assume (d) and let \mathbb{P} be the ccc forcing notion witnessing this assumption, $G \subseteq \mathbb{P}$ be generic over \mathbf{V} . Let us work in $\mathbf{V}[G]$.

Let $\langle \eta_\alpha : \alpha < \lambda_{\omega_1} \rangle$ be a sequence of distinct elements of ${}^\omega 2$ such that

$$(\forall \alpha < \beta < \lambda_{\omega_1}) (|(B + \eta_\alpha) \cap (B + \eta_\beta)| \geq k).$$

Let $\tau = \{R_{\mathbf{m}} : \mathbf{m} \in \mathbf{M}_{\bar{T}, k}\}$ be a (countable) vocabulary where each $R_{\mathbf{m}}$ is a $|u_{\mathbf{m}}|$ -ary relational symbol. Let $\mathbb{M} = (\lambda_{\omega_1}, \{R_{\mathbf{m}}^{\mathbb{M}}\}_{\mathbf{m} \in \mathbf{M}_{\bar{T}, k}})$ be the model in the vocabulary τ , where for $\mathbf{m} = (\ell, u, \bar{h}, \bar{g}) \in \mathbf{M}_{\bar{T}, k}$ the relation $R_{\mathbf{m}}^{\mathbb{M}}$ is defined by

$$R_{\mathbf{m}}^{\mathbb{M}} = \left\{ (\alpha_0, \dots, \alpha_{|u|-1}) \in (\lambda_{\omega_1})^{|u|} : \{\eta_{\alpha_0} \upharpoonright \ell, \dots, \eta_{\alpha_{|u|-1}} \upharpoonright \ell\} = u \text{ and} \right. \\ \left. \begin{array}{l} \text{for distinct } j_1, j_2 < |u| \text{ there are } G_i(\alpha_{j_1}, \alpha_{j_2}) \text{ (for } i < \iota) \text{ such that} \\ g_i(\eta_{\alpha_{j_1}} \upharpoonright \ell, \eta_{\alpha_{j_2}} \upharpoonright \ell) \triangleleft G_i(\alpha_{j_1}, \alpha_{j_2}) \in \lim (T_{h_i(\eta_{\alpha_{j_1}} \upharpoonright \ell, \eta_{\alpha_{j_2}} \upharpoonright \ell)}) \text{ and} \\ \eta_{\alpha_{j_1}} + G_i(\alpha_{j_1}, \alpha_{j_2}) = \eta_{\alpha_{j_2}} + G_i(\alpha_{j_2}, \alpha_{j_1}) \end{array} \right\}.$$

Claim 3.11.1. (1) *If $\alpha_0, \alpha_1, \dots, \alpha_{j-1} < \lambda_{\omega_1}$ are distinct, $j \geq 2$, then for sufficiently large $\ell < \omega$ there is $\mathbf{m} \in \mathbf{M}_{\bar{T}, k}$ such that*

$$\ell_{\mathbf{m}} = \ell, \quad u_{\mathbf{m}} = \{\eta_{\alpha_0} \upharpoonright \ell, \dots, \eta_{\alpha_{j-1}} \upharpoonright \ell\} \quad \text{and} \quad \mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \dots, \alpha_{j-1}].$$

(2) *Assume that $\mathbf{m} \in \mathbf{M}_{\bar{T}, k}$, $j < |u_{\mathbf{m}}|$, $\alpha_0, \alpha_1, \dots, \alpha_{|u_{\mathbf{m}}|-1} < \lambda_{\omega_1}$ and $\alpha^* < \lambda_{\omega_1}$ are all pairwise distinct and such that $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \dots, \alpha_j, \dots, \alpha_{|u_{\mathbf{m}}|-1}]$ and $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \dots, \alpha_{j-1}, \alpha^*, \alpha_{j+1}, \dots, \alpha_{|u_{\mathbf{m}}|-1}]$. Then for every sufficiently large $\ell > \ell_{\mathbf{m}}$ there is $\mathbf{n} \in \mathbf{M}_{\bar{T}, k}$ such that $\mathbf{m} \sqsubseteq \mathbf{n}$ and*

$$\ell_{\mathbf{n}} = \ell, \quad u_{\mathbf{n}} = \{\eta_{\alpha_0} \upharpoonright \ell, \dots, \eta_{\alpha_{|u_{\mathbf{m}}|-1}} \upharpoonright \ell, \eta_{\alpha^*} \upharpoonright \ell\} \quad \text{and} \quad \mathbb{M} \models R_{\mathbf{n}}[\alpha_0, \dots, \alpha_{|u_{\mathbf{m}}|-1}, \alpha^*].$$

(3) *If $\mathbf{m} \in \mathbf{M}_{\bar{T}, k}$ and $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \dots, \alpha_{|u_{\mathbf{m}}|-1}]$, then*

$$\text{rk}(\{\alpha_0, \dots, \alpha_{|u_{\mathbf{m}}|-1}\}, \mathbb{M}) \leq \text{ndrk}(\mathbf{m}).$$

Proof of the Claim. (1) For distinct $j_1, j_2 < j$ let $G_i(\alpha_{j_1}, \alpha_{j_2}) \in B$ (for $i < \iota$) be such that

$$\eta_{\alpha_{j_1}} + G_i(\alpha_{j_1}, \alpha_{j_2}) = \eta_{\alpha_{j_2}} + G_i(\alpha_{j_2}, \alpha_{j_1})$$

and there are no repetitions in the sequence $\langle G_i(\alpha_{j_1}, \alpha_{j_2}), G_i(\alpha_{j_2}, \alpha_{j_1}) : i < \iota \rangle$. Suppose that $\ell < \omega$ is such that for any distinct $j_1, j_2 < j$ we have $\eta_{\alpha_{j_1}} \upharpoonright \ell \neq \eta_{\alpha_{j_2}} \upharpoonright \ell$ and there are no repetitions in the sequence $\langle G_i(\alpha_{j_1}, \alpha_{j_2}) \upharpoonright \ell, G_i(\alpha_{j_2}, \alpha_{j_1}) \upharpoonright \ell : i < \iota \rangle$. Now let $u = \{\eta_{\alpha_{j'}} \upharpoonright \ell : j' < j\}$, and for $i < \iota$ let $g_i(\eta_{\alpha_{j_1}} \upharpoonright \ell, \eta_{\alpha_{j_2}} \upharpoonright \ell) = G_i(\alpha_{j_1}, \alpha_{j_2}) \upharpoonright \ell$, and let $h_i(\eta_{\alpha_{j_1}} \upharpoonright \ell, \eta_{\alpha_{j_2}} \upharpoonright \ell) < \omega$ be such that $G_i(\alpha_{j_1}, \alpha_{j_2}) \in \lim (T_{h_i(\eta_{\alpha_{j_1}} \upharpoonright \ell, \eta_{\alpha_{j_2}} \upharpoonright \ell)})$.

It should be clear that this way we defined $\mathbf{m} = (\ell, u, \bar{h}, \bar{g}) \in \mathbf{M}_{\bar{T},k}$ and $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \dots, \alpha_{j-1}]$.

(2) An obvious modification of the argument above.

(3) By induction on β we show that for every $\mathbf{m} \in \mathbf{M}_{\bar{T},k}$ and all $\alpha_0, \dots, \alpha_{|u_{\mathbf{m}}|-1} < \lambda_{\omega_1}$ such that $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \dots, \alpha_{|u_{\mathbf{m}}|-1}]$:

$$\beta \leq \text{rk}(\{\alpha_0, \dots, \alpha_{|u_{\mathbf{m}}|-1}\}, \mathbb{M}) \text{ implies } \beta \leq \text{ndrk}(\mathbf{m}).$$

STEPS $\beta = 0$ AND β IS LIMIT: should be clear.

STEP $\beta = \gamma + 1$: Suppose $\mathbf{m} \in \mathbf{M}_{\bar{T},k}$ and $\alpha_0, \dots, \alpha_{|u_{\mathbf{m}}|-1} < \lambda_{\omega_1}$ are such that $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \dots, \alpha_{|u_{\mathbf{m}}|-1}]$ and $\gamma + 1 \leq \text{rk}(\{\alpha_0, \dots, \alpha_{|u_{\mathbf{m}}|-1}\}, \mathbb{M})$. Let $\nu \in u_{\mathbf{m}}$, so $\nu = \eta_{\alpha_j} \upharpoonright \ell_{\mathbf{m}}$ for some $j < |u_{\mathbf{m}}|$. Since $\gamma + 1 \leq \text{rk}(\{\alpha_0, \dots, \alpha_{|u_{\mathbf{m}}|-1}\}, \mathbb{M})$ we may find $\alpha^* \in \lambda_{\omega_1} \setminus \{\alpha_0, \dots, \alpha_{|u_{\mathbf{m}}|-1}\}$ such that $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \dots, \alpha_{j-1}, \alpha^*, \alpha_{j+1}, \dots, \alpha_{|u_{\mathbf{m}}|-1}]$ and $\text{rk}(\{\alpha_0, \dots, \alpha_{|u_{\mathbf{m}}|-1}, \alpha^*\}, \mathbb{M}) \geq \gamma$. Taking sufficiently large ℓ we may use clause (2) to find $\mathbf{n} \in \mathbf{M}_{\bar{T},k}$ such that $\mathbf{m} \sqsubseteq \mathbf{n}$, $\ell_{\mathbf{n}} = \ell$ and $\mathbb{M} \models R_{\mathbf{n}}[\alpha_0, \dots, \alpha_{|u_{\mathbf{m}}|-1}, \alpha^*]$ and $|\{\eta \in u_{\mathbf{n}} : \nu \triangleleft \eta\}| \geq 2$. By the inductive hypothesis we have also $\gamma \leq \text{ndrk}(\mathbf{n})$. Now we may easily conclude that $\gamma + 1 \leq \text{ndrk}(\mathbf{m})$. \square

By the definition of λ_{ω_1} ,

$$(\odot) \sup\{\text{rk}(w, \mathbb{M}) : \emptyset \neq w \in [\lambda_{\omega_1}]^{<\omega}\} \geq \omega_1$$

Now, suppose that $\beta < \omega_1$. By (\odot) , there are distinct $\alpha_0, \dots, \alpha_{j-1} < \lambda_{\omega_1}$, $j \geq 2$, such that $\text{rk}(\{\alpha_0, \dots, \alpha_{j-1}\}, \mathbb{M}) \geq \beta$. By Claim 3.11.1(1) we may find $\mathbf{m} \in \mathbf{M}_{\bar{T},k}$ such that $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \dots, \alpha_{j-1}]$. Then by Claim 3.11.1(3) we also have $\text{ndrk}(\mathbf{m}) \geq \beta$. Consequently, $\text{NDRK}(\bar{T}) \geq \omega_1$.

All the considerations above were carried out in $\mathbf{V}[G]$. However, the rank function ndrk is absolute, so we may also claim that in \mathbf{V} we have $\text{NDRK}(\bar{T}) \geq \omega_1$. \square

Corollary 3.12. *Assume that $\varepsilon \leq \omega_1$ and $\text{Pr}_{\varepsilon}(\lambda)$. If there is $A \subseteq {}^\omega 2$ of cardinality λ such that*

$$(\forall \eta, \nu \in A) (|(B + \eta) \cap (B + \nu)| \geq k),$$

then $\text{NDRK}(\bar{T}) \geq \varepsilon$.

Proof. This is essentially shown by the proof of the implication (d) \Rightarrow (a) of Proposition 3.11. \square

4. THE FORCING

In this section we construct a forcing notion adding a sequence \bar{T} of subtrees of ${}^{\omega > 2}$ such that $\text{NDRK}(\bar{T}) < \omega_1$. The sequence \bar{T} will be added by finite approximations, so it will be convenient to have finite version of Definition 3.5.

Definition 4.1. Assume that

- $2 \leq \iota < \omega$, $k = 2\iota$, and $0 < n, M < \omega$,
- $\bar{t} = \langle t_m : m < M \rangle$, and each t_m is a subtree of ${}^{n \geq 2}$ in which all terminal branches are of length n ,
- $T_j \subseteq {}^{\omega > 2}$ (for $j < \omega$) are trees with no maximal nodes, $\bar{T} = \langle T_j : j < \omega \rangle$ and $t_m = T_m \cap {}^{n \geq 2}$ for $m < M$,
- $\mathbf{M}_{\bar{T},k}$ is defined as in Definition 3.5.

- (1) Let $\mathbf{M}_{\bar{T},k}^n$ consist of all tuples $\mathbf{m} = (\ell_{\mathbf{m}}, u_{\mathbf{m}}, \bar{h}_{\mathbf{m}}, \bar{g}_{\mathbf{m}}) \in \mathbf{M}_{\bar{T},k}$ such that $\ell_{\mathbf{m}} \leq n$ and $\text{rng}(h_i^{\mathbf{m}}) \subseteq M$ for each $i < \iota$.

(2) Assume $\mathbf{m}, \mathbf{n} \in \mathbf{M}_{\bar{t},k}^n$. We say that \mathbf{m}, \mathbf{n} are essentially the same ($\mathbf{m} \dot{=} \mathbf{n}$ in short) if and only if:

- $\ell_{\mathbf{m}} = \ell_{\mathbf{n}}, u_{\mathbf{m}} = u_{\mathbf{n}}$ and
- for each $(\eta, \nu) \in (u_{\mathbf{m}})^{(2)}$ we have

$$\{\{g_i^{\mathbf{m}}(\eta, \nu), g_i^{\mathbf{m}}(\nu, \eta)\} : i < \iota\} = \{\{g_i^{\mathbf{n}}(\eta, \nu), g_i^{\mathbf{n}}(\nu, \eta)\} : i < \iota\},$$

and for $i, j < \iota$:

if $g_i^{\mathbf{m}}(\eta, \nu) = g_j^{\mathbf{n}}(\eta, \nu)$, then $h_i^{\mathbf{m}}(\eta, \nu) = h_j^{\mathbf{n}}(\eta, \nu)$,

if $g_i^{\mathbf{m}}(\eta, \nu) = g_j^{\mathbf{n}}(\nu, \eta)$, then $h_i^{\mathbf{m}}(\eta, \nu) = h_j^{\mathbf{n}}(\nu, \eta)$.

(3) Assume $\mathbf{m}, \mathbf{n} \in \mathbf{M}_{\bar{t},k}^n$. We say that \mathbf{n} essentially extends \mathbf{m} ($\mathbf{m} \sqsubseteq^* \mathbf{n}$ in short) if and only if:

- $\ell_{\mathbf{m}} \leq \ell_{\mathbf{n}}, u_{\mathbf{m}} = \{\eta \upharpoonright \ell_{\mathbf{m}} : \eta \in u_{\mathbf{n}}\}$, and
- for every $(\eta, \nu) \in (u_{\mathbf{n}})^{(2)}$ such that $\eta \upharpoonright \ell_{\mathbf{m}} \neq \nu \upharpoonright \ell_{\mathbf{m}}$ we have

$$\{\{g_i^{\mathbf{m}}(\eta \upharpoonright \ell_{\mathbf{m}}, \nu \upharpoonright \ell_{\mathbf{m}}), g_i^{\mathbf{m}}(\nu \upharpoonright \ell_{\mathbf{m}}, \eta \upharpoonright \ell_{\mathbf{m}})\} : i < \iota\} = \{\{g_i^{\mathbf{n}}(\eta, \nu) \upharpoonright \ell_{\mathbf{m}}, g_i^{\mathbf{n}}(\nu, \eta) \upharpoonright \ell_{\mathbf{m}}\} : i < \iota\},$$

and for $i, j < \iota$:

if $g_i^{\mathbf{m}}(\eta \upharpoonright \ell_{\mathbf{m}}, \nu \upharpoonright \ell_{\mathbf{m}}) = g_j^{\mathbf{n}}(\eta, \nu) \upharpoonright \ell_{\mathbf{m}}$, then $h_i^{\mathbf{m}}(\eta \upharpoonright \ell_{\mathbf{m}}, \nu \upharpoonright \ell_{\mathbf{m}}) = h_j^{\mathbf{n}}(\eta, \nu)$,

if $g_i^{\mathbf{m}}(\eta \upharpoonright \ell_{\mathbf{m}}, \nu \upharpoonright \ell_{\mathbf{m}}) = g_j^{\mathbf{n}}(\nu, \eta) \upharpoonright \ell_{\mathbf{m}}$, then $h_i^{\mathbf{m}}(\eta \upharpoonright \ell_{\mathbf{m}}, \nu \upharpoonright \ell_{\mathbf{m}}) = h_j^{\mathbf{n}}(\nu, \eta)$.

Observation 4.2. If $\mathbf{m} \in \mathbf{M}_{\bar{t},k}^n$ and $\rho \in {}^{\ell_{\mathbf{m}}}2$, then $\mathbf{m} + \rho \in \mathbf{M}_{\bar{t},k}^n$.

Lemma 4.3. Let $0 < \ell < \omega$ and let $\mathcal{B} \subseteq {}^{\ell}2$ be a linearly independent set of vectors (in $({}^{\ell}2, +)$ over $(2, +_2, \cdot_2)$).

- (1) If $\mathcal{A} \subseteq {}^{\ell}2$, $|\mathcal{A}| \geq 5$ and $\mathcal{A} + \mathcal{A} \subseteq \mathcal{B} + \mathcal{B}$, then for a unique $x \in {}^{\ell}2$ we have $\mathcal{A} + x \subseteq \mathcal{B}$.
- (2) Let $b^* \in \mathcal{B}$. Suppose that $\rho_i^0, \rho_i^1 \in (\mathcal{B} \cup (b^* + \mathcal{B})) \setminus \{\mathbf{0}, b^*\}$ (for $i < 3$) are such that
 - (a) there are no repetitions in $\langle \rho_i^0, \rho_i^1 : i < 3 \rangle$, and
 - (b) $\rho_i^0 + \rho_i^1 = \rho_j^0 + \rho_j^1$ for $i < j < 3$.
 Then $\{\{\rho_i^0, \rho_i^1\} : i < 3\} \subseteq \{\{b, b + b^*\} : b \in \mathcal{B}, b \neq b^*\}$.

Proof. Easy, for (1) see e.g. [4, Lemma 2.3]. □

Theorem 4.4. Assume $\text{NPr}_{\omega_1}(\lambda)$ and let $3 \leq \iota < \omega$. Then there is a ccc forcing notion \mathbb{P} of size λ such that

$$\begin{aligned} \Vdash_{\mathbb{P}} \text{ "for some } \Sigma_2^0 \text{ } 2\iota\text{-npts-set } B \subseteq {}^{\omega}2 \text{ there is a sequence } \langle \eta_{\alpha} : \alpha < \lambda \rangle \\ \text{of distinct elements of } {}^{\omega}2 \text{ such that} \\ |(\eta_{\alpha} + B) \cap (\eta_{\beta} + B)| \geq 2\iota \text{ for all } \alpha, \beta < \lambda \text{ "}. \end{aligned}$$

Proof. We may assume that λ is uncountable.

Fix a countable vocabulary $\tau = \{R_{n,\zeta} : n, \zeta < \omega\}$, where $R_{n,\zeta}$ is an n -ary relational symbol (for $n, \zeta < \omega$). By the assumption on λ , we may fix a model $\mathbb{M} = (\lambda, \{R_{n,\zeta}^{\mathbb{M}}\}_{n,\zeta < \omega})$ in the vocabulary τ with the universe λ and an ordinal $\alpha^* < \omega_1$ such that:

- (\otimes)_a for every n and a quantifier free formula $\varphi(x_0, \dots, x_{n-1}) \in \mathcal{L}(\tau)$ there is $\zeta < \omega$ such that for all $a_0, \dots, a_{n-1} \in \lambda$,

$$\mathbb{M} \models \varphi[a_0, \dots, a_{n-1}] \Leftrightarrow R_{n,\zeta}[a_0, \dots, a_{n-1}],$$

- (\otimes)_b $\sup\{\text{rk}(v, \mathbb{M}) : \emptyset \neq v \in [\lambda]^{<\omega}\} < \alpha^*$,

- (\otimes)_c the rank of every singleton is at least 0.

For a nonempty finite set $v \subseteq \lambda$ let $\text{rk}(v) = \text{rk}(v, \mathbb{M})$, and let $\zeta(v) < \omega$ and $k(v) < |v|$ be such that $R_{|v|, \zeta(v)}, k(v)$ witness the rank of v . Thus letting $\{a_0, \dots, a_k, \dots, a_{n-1}\}$ be the increasing enumeration of v and $k = k(v)$ and $\zeta = \zeta(v)$, we have

(*)_d if $\text{rk}(v) \geq 0$, then $\mathbb{M} \models R_{n, \zeta}[a_0, \dots, a_k, \dots, a_{n-1}]$ but there is no $a \in \lambda \setminus v$ such that

$$\text{rk}(v \cup \{a\}) \geq \text{rk}(v) \quad \text{and} \quad \mathbb{M} \models R_{n, \zeta}[a_0, \dots, a_{k-1}, a, a_{k+1}, \dots, a_{n-1}],$$

(*)_e if $\text{rk}(v) = -1$, then $\mathbb{M} \models R_{n, \zeta}[a_0, \dots, a_k, \dots, a_{n-1}]$ but the set

$$\{a \in \lambda : \mathbb{M} \models \varphi[a_0, \dots, a_{k-1}, a, a_{k+1}, \dots, a_{n-1}]\}$$

is countable.

Without loss of generality we may also require that (for $\zeta = \zeta(v)$, $n = |v|$)

(*)_f for every $b_0, \dots, b_{n-1} < \lambda$

$$\text{if } \mathbb{M} \models R_{n, \zeta}[b_0, \dots, b_{n-1}] \text{ then } b_0 < \dots < b_{n-1}.$$

Now we will define a forcing notion \mathbb{P} . A *condition* p in \mathbb{P} is a tuple

$$(w^p, n^p, M^p, \bar{\eta}^p, \bar{t}^p, \bar{r}^p, \bar{h}^p, \bar{g}^p, \mathcal{M}^p) = (w, n, M, \bar{\eta}, \bar{t}, \bar{r}, \bar{h}, \bar{g}, \mathcal{M})$$

such that the following demands (*)₁–(*)₁₁ are satisfied.

- (*)₁ $w \in [\lambda]^{<\omega}$, $|w| \geq 5$, $0 < n, M < \omega$.
- (*)₂ $\bar{\eta} = \langle \eta_\alpha : \alpha \in w \rangle$ is a sequence of linearly independent vectors in ${}^n 2$ (over the field \mathbb{Z}_2); so in particular $\eta_\alpha \in {}^n 2$ are pairwise distinct non-zero sequences (for $\alpha \in w$).
- (*)₃ $\bar{t} = \langle t_m : m < M \rangle$, where $\emptyset \neq t_m \subseteq {}^{n \geq 2}$ for $m < M$ is a tree in which all terminal branches are of length n and $t_m \cap t_{m'} \cap {}^{n \geq 2} = \emptyset$ for $m < m' < M$.
- (*)₄ $\bar{r} = \langle r_m : m < M \rangle$, where $0 < r_m \leq n$ for $m < M$.
- (*)₅ $\bar{h} = \langle h_i : i < \iota \rangle$, where $h_i : w^{(2)} \rightarrow M$.
- (*)₆ $\bar{g} = \langle g_i : i < \iota \rangle$, where $g_i : w^{(2)} \rightarrow \bigcup_{m < M} (t_m \cap {}^{n \geq 2})$, and $g_i(\alpha, \beta) \in t_{h_i(\alpha, \beta)}$ and $\eta_\alpha + g_i(\alpha, \beta) = \eta_\beta + g_i(\beta, \alpha)$ for $(\alpha, \beta) \in w^{(2)}$ and $i < \iota$.
- (*)₇ There are no repetitions in the sequence

$$\langle g_i(\alpha, \beta) : i < \iota, (\alpha, \beta) \in w^{(2)} \rangle.$$

(*)₈ \mathcal{M} consists of all those $\mathbf{m} \in \mathbf{M}_{\bar{t}, k}^n$ (see Definition 4.1) that for some ℓ_*, w_* we have

(*)₈^a $w_* \subseteq w$, $5 \leq |w_*|$, $0 < \ell_{\mathbf{m}} = \ell_* \leq n$, and for each $(\alpha, \beta) \in (w_*)^{(2)}$ and $i < \iota$ we have $r_{h_i(\alpha, \beta)} \leq \ell_*$,

(*)₈^b $u_{\mathbf{m}} = \{\eta_\alpha \upharpoonright \ell_* : \alpha \in w_*\}$ and $\eta_\alpha \upharpoonright \ell_* \neq \eta_\beta \upharpoonright \ell_*$ for distinct $\alpha, \beta \in w_*$,

(*)₈^c $\bar{h}_{\mathbf{m}} = \langle h_i^{\mathbf{m}} : i < \iota \rangle$, where

$$h_i^{\mathbf{m}} : (u_{\mathbf{m}})^{(2)} \rightarrow M : (\eta_\alpha \upharpoonright \ell_*, \eta_\beta \upharpoonright \ell_*) \mapsto h_i(\alpha, \beta),$$

(*)₈^d $\bar{g}_{\mathbf{m}} = \langle g_i^{\mathbf{m}} : i < \iota \rangle$, where

$$g_i^{\mathbf{m}} : (u_{\mathbf{m}})^{(2)} \rightarrow \bigcup_{m < M} (t_m \cap {}^{\ell_* 2}) : (\eta_\alpha \upharpoonright \ell_*, \eta_\beta \upharpoonright \ell_*) \mapsto g_i(\alpha, \beta) \upharpoonright \ell_*$$

In the above situation we will write $\mathbf{m} = \mathbf{m}(\ell_*, w_*) = \mathbf{m}^p(\ell_*, w_*)$. (Note that w_* is not determined uniquely by \mathbf{m} and we may have $\mathbf{m}(\ell, w_0) = \mathbf{m}(\ell, w_1)$ for distinct $w_0, w_1 \subseteq w$. Also, the conditions (*)₈^a–(*)₈^d alone do

not necessarily determine an element of $\mathbf{M}_{\ell,k}^p$, but clearly for each $w_* \subseteq w$ of size ≥ 5 we have $\mathbf{m}^p(n^p, w_*) \in \mathcal{M}^p$.

- (*)₉ If $\mathbf{m}(\ell, w_0), \mathbf{m}(\ell, w_1) \in \mathcal{M}$, $\rho \in {}^\ell 2$ and $\mathbf{m}(\ell, w_0) \dot{=} \mathbf{m}(\ell, w_1) + \rho$, then $\text{rk}(w_0) = \text{rk}(w_1)$, $\zeta(w_0) = \zeta(w_1)$, $k(w_0) = k(w_1)$ and if $\alpha \in w_0$, $\beta \in w_1$ are such that $|\alpha \cap w_0| = k(w_0) = k(w_1) = |\beta \cap w_1|$, then $(\eta_\alpha \upharpoonright \ell) + \rho = \eta_\beta \upharpoonright \ell$.
- (*)₁₀ If $\mathbf{m}(\ell_*, w_*) \in \mathcal{M}$, $\alpha \in w_*$, $|\alpha \cap w_*| = k(w_*)$, $\text{rk}(w_*) = -1$, and $\mathbf{m}(\ell_*, w_*) \sqsubseteq^* \mathbf{n} \in \mathcal{M}$, then $|\{\nu \in u_{\mathbf{n}} : (\eta_\alpha \upharpoonright \ell_*) \leq \nu\}| = 1$.
- (*)₁₁ If $\rho_i^0, \rho_i^1 \in \bigcup_{m < M} (t_m \cap {}^{n^2})$ (for $i < \iota$) are such that
 - (a) there are no repetitions in $\langle \rho_i^0, \rho_i^1 : i < \iota \rangle$, and
 - (b) $\rho_i^0 + \rho_i^1 = \rho_j^0 + \rho_j^1$ for $i < j < \iota$,
 then for some $\alpha, \beta \in w$ we have

$$\{\{\rho_i^0, \rho_i^1\} : i < \iota\} = \{\{g_i(\alpha, \beta), g_i(\beta, \alpha)\} : i < \iota\}.$$

To define the order \leq of \mathbb{P} we declare for $p, q \in \mathbb{P}$ that $p \leq q$ if and only if

- $w^p \subseteq w^q$, $n^p \leq n^q$, $M^p \leq M^q$, and
- $t_m^p = t_m^q \cap n^p \geq 2$ and $r_m^p = r_m^q$ for all $m < M^p$, and
- $\eta_\alpha^p \leq \eta_\alpha^q$ for all $\alpha \in w^p$, and
- $h_i^q \upharpoonright (w^p)^{(2)} = h_i^p$ and $g_i^p(\alpha, \beta) \leq g_i^q(\alpha, \beta)$ for $i < \iota$ and $(\alpha, \beta) \in (w^p)^{(2)}$.

Claim 4.4.1. Assume $p = (w, n, M, \bar{\eta}, \bar{t}, \bar{h}, \bar{g}, \mathcal{M}) \in \mathbb{P}$. If $\mathbf{m} \in \mathbf{M}_{\ell,k}^p$ is such that $\ell_{\mathbf{m}} = n$ and $|u_{\mathbf{m}}| \geq 5$, then for some $\rho \in {}^{n^2}$ and $\mathbf{n} \in \mathcal{M}$ we have $(\mathbf{m} + \rho) \dot{=} \mathbf{n}$.

Proof of the Claim. Let $\mathbf{m} \in \mathbf{M}_{\ell,k}^p$ be such that $\ell_{\mathbf{m}} = n$. It follows from Definition 3.5(d) and clauses (*)₆ + (*)₁₁ that

$$(\square) \text{ for every } (\nu, \eta) \in (u_{\mathbf{m}})^{(2)} \text{ there is } (\alpha, \beta) \in w^{(2)} \text{ such that } \nu + \eta = \eta_\alpha + \eta_\beta.$$

By Lemma 4.3 for some ρ we have $u_{\mathbf{m}} + \rho \subseteq \{\eta_\alpha : \alpha \in w\}$. Let $w_0 = \{\alpha \in w : \eta_\alpha + \rho \in u_{\mathbf{m}}\}$ and $\mathbf{n} = \mathbf{m}^p(n, w_0) \in \mathcal{M}$. Using clause (*)₁₁ again we easily conclude $(\mathbf{m} + \rho) \dot{=} \mathbf{n}$. (Note that since $t_m \cap t_{m'} \cap {}^{n^2} = \emptyset$ for $m < m' < M$, $h_i^{\mathbf{m}}(\eta, \nu)$ is determined by $g_i^{\mathbf{m}}(\eta, \nu)$.) \square

Claim 4.4.2. (1) $\mathbb{P} \neq \emptyset$ and (\mathbb{P}, \leq) is a partial order.

(2) For each $\beta < \lambda$ and $n_0, M_0 < \omega$ the set

$$D_\beta^{n_0, M_0} = \{p \in \mathbb{P} : n^p > n_0 \wedge M^p > M_0 \wedge \beta \in w^p\}$$

is open dense in \mathbb{P} .

Proof of the Claim. (1) Should be clear.

(2) Let $p \in \mathbb{P}$, $\beta \in \lambda \setminus w^p$. Put $N = |w^p| \cdot \iota + 2$.

We will define a condition $q \in \mathbb{P}$ such that $q \geq p$ and

$$w^q = w^p \cup \{\beta\}, \quad n^q = n^p + N > n^p + 1, \quad M^q = M^p + N - 2 > M^p + 1.$$

For $\alpha \in w^p$ we set $\eta_\alpha^q = \eta_\alpha^p \frown \underbrace{\langle 0, \dots, 0 \rangle}_N$ and we also let $\eta_\beta^q = \underbrace{\langle 0, \dots, 0 \rangle}_{n^p+1} \frown \underbrace{\langle 1, \dots, 1 \rangle}_{N-1}$.

Next, if $(\alpha_0, \alpha_1) \in (w^p)^{(2)}$, then for all $i < \iota$

$$h_i^q(\alpha_0, \alpha_1) = h_i^p(\alpha_0, \alpha_1) \quad \text{and} \quad g_i^q(\alpha_0, \alpha_1) = g_i^p(\alpha_0, \alpha_1) \frown \underbrace{\langle 0, \dots, 0 \rangle}_N.$$

If $\alpha \in w^p$ and $j = |w^p \cap \alpha|$, then for $i < \iota$:

- $g_i^q(\alpha, \beta) = \underbrace{\langle 0, \dots, 0 \rangle}_{n^p} \frown \langle 1 \rangle \frown \underbrace{\langle 0, \dots, 0 \rangle}_{j\iota+i+1} \frown \underbrace{\langle 1, \dots, 1 \rangle}_{N-j\iota-i-2}$,
- $g_i^q(\beta, \alpha) = \eta_\alpha^p \frown \underbrace{\langle 1, \dots, 1 \rangle}_{j\iota+i+2} \frown \underbrace{\langle 0, \dots, 0 \rangle}_{N-j\iota-i-2}$,
- $h_i^q(\beta, \alpha) = h_i^q(\alpha, \beta) = M^p + j\iota + i$.

We also set:

- if $m < M^p$, then $r_m^q = r_m^p$ and
 $t_m^q = \{\eta \in n^q \geq 2 : \eta \upharpoonright n^p \in t_m^p \wedge (\forall j < n^q)(n^p \leq j < |\eta| \Rightarrow \eta(j) = 0)\}$
and
- if $M^p \leq m < M^q$, $m = M^p + j\iota + i$, $i < \iota$ and $j < |w^p|$, then $r_m^q = n^q$ and
 $t_m^q = \{g_i^q(\alpha, \beta) \upharpoonright \ell, g_i^q(\beta, \alpha) \upharpoonright \ell : \ell \leq n^q\}$,

where $\alpha \in w^p$ is such that $|\alpha \cap w^p| = j$.

Now letting \mathcal{M}^q be defined as in $(*)_8$ we check that

$$q = (w^q, n^q, M^q, \bar{\eta}^q, \bar{t}^q, \bar{r}^q, \bar{h}^q, \bar{g}^q, \mathcal{M}^p) \in \mathbb{P}.$$

Demands $(*)_1$ – $(*)_8$ are pretty straightforward.

RE $(*)_9$: To justify clause $(*)_9$, suppose that $\mathbf{m}^q(\ell, w_0), \mathbf{m}^q(\ell, w_1) \in \mathcal{M}^q$, $\rho \in \ell_2$ and $\mathbf{m}^q(\ell, w_0) \dot{=} \mathbf{m}^q(\ell, w_1) + \rho$, and consider the following two cases.

CASE 1: $\beta \notin w_0 \cup w_1$

Then letting $\ell^* = \min(\ell, n^p)$ and $\rho^* = \rho \upharpoonright \ell^*$ we see that $\mathbf{m}^p(\ell^*, w_0) \dot{=} \mathbf{m}^p(\ell^*, w_1) + \rho^*$ (and both belong to \mathcal{M}^p). Hence clause $(*)_9$ for p applies.

CASE 2: $\beta \in w_0 \cup w_1$

Say, $\beta \in w_0$. If $\alpha \in w_0 \setminus \{\beta\}$, then $h_i^q(\alpha, \beta) = h_i^q(\beta, \alpha) \geq M^p$ and $r_{h_i^q(\alpha, \beta)}^q = n^q$.

Consequently, $\ell = n^q$. Moreover,

$$(\gamma, \delta) \in (w^q)^{(2)} \wedge h_j^q(\gamma, \delta) = h_i^q(\alpha, \beta) \Rightarrow \{\gamma, \delta\} = \{\alpha, \beta\}.$$

Therefore, $\beta \in w_1$ and $w_1 = w_0$ and since $|w_1| \geq 5$, the linear independence of $\bar{\eta}$ implies $\rho = \mathbf{0}$.

RE $(*)_{10}$: Concerning clause $(*)_{10}$, suppose that $\mathbf{m}^q(\ell_0, w_0), \mathbf{m}^q(\ell_1, w_1) \in \mathcal{M}^q$, $\alpha \in w_0$, $|\alpha \cap w_0| = k(w_0)$, $\text{rk}(w_0) = -1$, and $\mathbf{m}^q(\ell_0, w_0) \sqsubseteq^* \mathbf{m}^q(\ell_1, w_1)$. Assume towards contradiction that there are $\alpha_0, \alpha_1 \in w_1$ such that

$$\eta_{\alpha_0}^q \upharpoonright \ell_1 \neq \eta_{\alpha_1}^q \upharpoonright \ell_1 \wedge \eta_{\alpha_0}^q \upharpoonright \ell_0 \triangleleft \eta_{\alpha_0}^q \wedge \eta_{\alpha_1}^q \upharpoonright \ell_0 \triangleleft \eta_{\alpha_1}^q.$$

Suppose $\beta \in w_0 \cup w_1$. Then looking at the function h_i^q in a manner similar to considerations for clause $(*)_9$ we get $\beta \in w_0 \cap w_1$. Let $\beta' \in w_0 \setminus \{\beta\}$. Then $h_0^q(\beta, \beta') \geq M^p$ and hence $r_{h_0^q(\beta, \beta')}^q = n^q = \ell_0 = \ell_1$, contradicting our assumptions. Therefore $\beta \notin w_0 \cup w_1$. But then we immediately get contradiction with clause $(*)_{10}$ for p .

RE $(*)_{11}$: Let us argue that $(*)_{11}$ is satisfied as well and for this suppose that $\rho_i^0, \rho_i^1 \in \bigcup_{m < M^q} (t_m \cap n^q 2)$ (for $i < \iota$) are such that

- there are no repetitions in $\langle \rho_i^0, \rho_i^1 : i < \iota \rangle$, and
- $\rho_i^0 + \rho_i^1 = \rho_j^0 + \rho_j^1$ for $i < j < \iota$.

Clearly, if

(\odot)₁ all ρ_i^0, ρ_i^1 are from $\bigcup_{m < M^p} t_m$,

then we may use the condition ($*$)₁₁ for p and conclude that for some $\alpha_0, \alpha_1 \in w^p$ we have

$$\{\{\rho_i^0, \rho_i^1\} : i < \iota\} = \{\{g_i(\alpha_0, \alpha_1), g_i(\alpha_1, \alpha_0)\} : i < \iota\}.$$

Now note that if $\rho_0, \rho_1, \rho_2, \rho_3 \in \bigcup_{m < M^q} (t_m \cap n^q 2)$, $\rho_0 + \rho_1 = \rho_2 + \rho_3$ and $\rho_0 \in \bigcup_{m < M^p} (t_m \cap n^q 2)$ but $\rho_1 \notin \bigcup_{m < M^p} (t_m \cap n^q 2)$, then $\{\rho_0, \rho_1\} = \{\rho_2, \rho_3\}$. Hence easily, if (\odot)₁ fails we must have

$$(\odot)_2 \rho_i^0, \rho_i^1 \in \bigcup_{m=M^p}^{M^q-1} (t_m \cap n^q 2) \text{ for } i < \iota.$$

But then necessarily

$$\{\{\rho_i^0 \upharpoonright [n^p, n^q], \rho_i^1 \upharpoonright [n^p, n^q]\} : i < \iota\} \subseteq \{\{g_i(\alpha, \beta) \upharpoonright [n^p, n^q], g_i(\beta, \alpha) \upharpoonright [n^p, n^q]\} : i < \iota, \alpha \in w^p\}.$$

(Use Lemma 4.3(2), remember $\iota \geq 3$.) Since $(g_i(\alpha, \beta) + g_i(\beta, \alpha)) \upharpoonright n^p = \eta_\alpha^p$ we easily conclude that for some $\alpha \in w^p$ we have

$$\{\{\rho_i^0, \rho_i^1\} : i < \iota\} = \{\{g_i(\alpha, \beta), g_i(\beta, \alpha)\} : i < \iota\}.$$

Finally, it should be clear that q is a condition stronger than p . \square

Claim 4.4.3. *The forcing notion \mathbb{P} has the Knaster property.*

Proof of the Claim. Suppose that $\langle p_\xi : \xi < \omega_1 \rangle$ is a sequence of pairwise distinct conditions from \mathbb{P} and let

$$p_\xi = (w_\xi, n_\xi, M_\xi, \bar{\eta}_\xi, \bar{t}_\xi, \bar{r}_\xi, \bar{h}_\xi, \bar{g}_\xi, \mathcal{M}_\xi)$$

where $\bar{\eta}_\xi = \langle \eta_\alpha^\xi : \alpha \in w_\xi \rangle$, $\bar{t}_\xi = \langle t_m^\xi : m < M_\xi \rangle$, $\bar{r}_\xi = \langle r_m^\xi : m < M_\xi \rangle$, and $\bar{h}_\xi = \langle h_i^\xi : i < \iota \rangle$, $\bar{g}_\xi = \langle g_i^\xi : i < \iota \rangle$. By a standard Δ -system cleaning procedure we may find an uncountable set $A \subseteq \omega_1$ such that the following demands ($*$)₁₂–($*$)₁₅ are satisfied.

- ($*$)₁₂ $\{w_\xi : \xi \in A\}$ forms a Δ -system.
- ($*$)₁₃ If $\xi, \varsigma \in A$, then $|w_\xi| = |w_\varsigma|$, $n_\xi = n_\varsigma$, $M_\xi = M_\varsigma$, and $t_m^\xi = t_m^\varsigma$ and $r_m^\xi = r_m^\varsigma$ (for $m < M_\xi$).
- ($*$)₁₄ If $\xi < \varsigma$ are from A and $\pi : w_\xi \rightarrow w_\varsigma$ is the order isomorphism, then
 - (a) $\pi(\alpha) = \alpha$ for $\alpha \in w_\xi \cap w_\varsigma$,
 - (b) if $\emptyset \neq v \subseteq w_\xi$, then $\text{rk}(v) = \text{rk}(\pi[v])$, $\zeta(v) = \zeta(\pi[v])$ and $k(v) = k(\pi[v])$,
 - (c) $\eta_\alpha^\xi = \eta_{\pi(\alpha)}^\varsigma$ (for $\alpha \in w_\xi$),
 - (d) $g_i(\alpha, \beta) = g_i(\pi(\alpha), \pi(\beta))$ and $h_i(\alpha, \beta) = h_i(\pi(\alpha), \pi(\beta))$ for $(\alpha, \beta) \in (w_\xi)^{(2)}$ and $i < \iota$,

and

- ($*$)₁₅ $\mathcal{M}_\xi = \mathcal{M}_\varsigma$ (this actually follows from the previous demands).

Following the pattern of Claim 4.4.2(2) we will argue that for distinct ξ, ς from A the conditions p_ξ, p_ς are compatible. So let $\xi, \varsigma \in A$, $\xi < \varsigma$ and let $\pi : w_\xi \rightarrow w_\varsigma$ be the order isomorphism. We will define $q = (w, n, M, \bar{\eta}, \bar{t}, \bar{r}, \bar{h}, \bar{g}, \mathcal{M})$ where $\bar{\eta} = \langle \eta_\alpha : \alpha \in w \rangle$, $\bar{t} = \langle t_m : m < M \rangle$, $\bar{r} = \langle r_m : m < M \rangle$, and $\bar{h} = \langle h_i : i < \iota \rangle$, $\bar{g} = \langle g_i : i < \iota \rangle$.

Let $w_\xi \cap w_\zeta = \{\alpha_0, \dots, \alpha_{k-1}\}$, $w_\xi \setminus w_\zeta = \{\beta_0, \dots, \beta_{\ell-1}\}$ and $w_\zeta \setminus w_\xi = \{\gamma_0, \dots, \gamma_{\ell-1}\}$ be the increasing enumerations.

We set $N_0 = \iota \cdot \ell(\ell + k) + \iota \cdot \frac{\ell(\ell-1)}{2} + 1$, $N = N_0 + \ell + 1$, and we define

- (*)₁₆ $w = w_\xi \cup w_\zeta$, $n = n_\xi + N$, and $M = M_\xi + 1$;
 (*)₁₇ $\eta_\alpha = \eta_\alpha \frown \underbrace{\langle 0, \dots, 0 \rangle}_N$ for $\alpha \in w_\xi$ and we also let for $c < \ell$

$$\eta_{\gamma_c} = \eta_{\gamma_c}^\xi \frown \langle 0 \rangle \frown \underbrace{\langle 1, \dots, 1 \rangle}_{N_0} \frown \underbrace{\langle 0, \dots, 0 \rangle}_c \frown \underbrace{\langle 1, \dots, 1 \rangle}_{\ell-c}.$$

Next we are going to define $h_i(\alpha, \beta)$ and $g_i(\alpha, \beta)$ for $(\alpha, \beta) \in w^{(2)}$. For $d < N_0$ let

$$\nu_d = \underbrace{\langle 0, \dots, 0 \rangle}_d \frown \langle 1 \rangle \frown \underbrace{\langle 0, \dots, 0 \rangle}_{N_0-d-1} \in {}^{N_0}2, \quad \text{and} \quad \nu_d^* = \mathbf{1} + \nu_d \in {}^{N_0}2$$

and note that $\{\nu_d : d < N_0 - 1\} \cup \{\mathbf{1}\}$ are linearly independent in ${}^{N_0}2$. Fix a bijection

$$\Theta : (k \times \ell \times \iota \times \{0\}) \cup (\{(a, b) \in \ell^2 : a < b\} \times \iota \times \{1\}) \cup (\ell \times \ell \times \iota \times \{2\}) \longrightarrow N_0 - 1$$

and define h_i, g_i as follows.

- (*)₁₈^a If $(\alpha, \beta) \in (w_\xi)^{(2)}$ and $i < \iota$, then
 $h_i(\alpha, \beta) = h_i^\xi(\alpha, \beta)$ and $g_i(\alpha, \beta) = g_i^\xi(\alpha, \beta) \frown \underbrace{\langle 0, \dots, 0 \rangle}_N$.
 (*)₁₈^b If $a < k$, $c < \ell$ and $i < \iota$, then $h_i(\alpha_a, \gamma_c) = h_i^\xi(\alpha_a, \gamma_c)$ and $h_i(\gamma_c, \alpha_a) = h_i^\xi(\gamma_c, \alpha_a)$, and

$$g_i(\alpha_a, \gamma_c) = g_i^\xi(\alpha_a, \gamma_c) \frown \langle 1 \rangle \frown \nu_{\Theta(a,c,i,0)} \frown \underbrace{\langle 0, \dots, 0 \rangle}_\ell \quad \text{and}$$

$$g_i(\gamma_c, \alpha_a) = g_i^\xi(\gamma_c, \alpha_a) \frown \langle 1 \rangle \frown \nu_{\Theta(a,c,i,0)}^* \frown \underbrace{\langle 0, \dots, 0 \rangle}_c \frown \underbrace{\langle 1, \dots, 1 \rangle}_{\ell-c}.$$

- (*)₁₈^c If $b < c < \ell$ and $i < \iota$, then $h_i(\gamma_b, \gamma_c) = h_i^\xi(\gamma_b, \gamma_c)$, $h_i(\gamma_c, \gamma_b) = h_i^\xi(\gamma_c, \gamma_b)$, and

$$g_i(\gamma_b, \gamma_c) = g_i^\xi(\gamma_b, \gamma_c) \frown \langle 1 \rangle \frown \nu_{\Theta(b,c,i,1)} \frown \underbrace{\langle 0, \dots, 0 \rangle}_b \frown \underbrace{\langle 1, \dots, 1 \rangle}_{\ell-b} \quad \text{and}$$

$$g_i(\gamma_c, \gamma_b) = g_i^\xi(\gamma_c, \gamma_b) \frown \langle 1 \rangle \frown \nu_{\Theta(b,c,i,1)} \frown \underbrace{\langle 0, \dots, 0 \rangle}_c \frown \underbrace{\langle 1, \dots, 1 \rangle}_{\ell-c}$$

(note: ν_Θ not ν_Θ^*).

- (*)₁₈^d If $b < \ell$, $c < \ell$ and $b \neq c$ and $i < \iota$, then $h_i(\beta_b, \gamma_c) = h_i(\gamma_c, \beta_b) = M_\xi = M_\zeta$, and

$$g_i(\beta_b, \gamma_c) = g_i^\xi(\beta_b, \beta_c) \frown \langle 1 \rangle \frown \nu_{\Theta(b,c,i,2)} \frown \underbrace{\langle 0, \dots, 0 \rangle}_c \frown \underbrace{\langle 1, \dots, 1 \rangle}_{\ell-c} \quad \text{and}$$

$$g_i(\gamma_c, \beta_b) = g_i^\xi(\gamma_c, \gamma_b) \frown \langle 1 \rangle \frown \nu_{\Theta(b,c,i,2)}^* \frown \underbrace{\langle 0, \dots, 0 \rangle}_\ell.$$

$$\begin{aligned}
(*)_{18}^e \text{ If } b < \ell \text{ and } i < \iota, \text{ then } h_i(\beta_b, \gamma_b) &= h_i(\gamma_b, \beta_b) = M_\xi = M_\zeta, \text{ and} \\
g_i(\beta_b, \gamma_b) &= \eta_{\beta_b}^\xi \frown \langle 1 \rangle \frown \nu_{\Theta(b,b,i,2)} \frown \underbrace{\langle 0, \dots, 0 \rangle}_b \frown \underbrace{\langle 1, \dots, 1 \rangle}_{\ell-b} \quad \text{and} \\
g_i(\gamma_b, \beta_b) &= \eta_{\gamma_b}^\zeta \frown \langle 1 \rangle \frown \nu_{\Theta(b,b,i,2)}^* \frown \underbrace{\langle 0, \dots, 0 \rangle}_\ell.
\end{aligned}$$

We also set:

$$\begin{aligned}
(*)_{19} \text{ } r_m &= r_m^\xi \text{ for } m < M_\xi, r_{M_\xi} = n \text{ and if } m < M_\xi, \text{ then} \\
t_m &= \left\{ \eta \in {}^{n \geq 2} : \eta \upharpoonright n_\xi \in t_m^\xi \wedge (\forall j < n)(n \leq j < |\eta| \Rightarrow \eta(j) = 0) \right\} \cup \\
&\quad \left\{ g_i(\delta, \varepsilon) \upharpoonright n' : (\delta, \varepsilon) \in w^{(2)}, i < \iota, \text{ and } n' \leq n \text{ and } h_i(\delta, \varepsilon) = m \right\} \\
&\text{and} \\
t_{M_\xi} &= \left\{ g_i(\delta, \varepsilon) \upharpoonright n' : (\delta, \varepsilon) \in w^{(2)}, i < \iota, \text{ and } n' \leq n \text{ and } h_i(\delta, \varepsilon) = M_\xi \right\}.
\end{aligned}$$

Now letting \mathcal{M} be defined by $(*)_8$ we claim that

$$q = (w, n, M, \bar{\eta}, \bar{t}, \bar{r}, \bar{h}, \bar{g}, \mathcal{M}) \in \mathbb{P}.$$

Demands $(*)_1 - (*)_8$ are pretty straightforward.

RE $(*)_9$: To justify clause $(*)_9$, suppose that $\mathbf{m}(\ell, w'), \mathbf{m}(\ell, w'') \in \mathcal{M}$, $\rho \in \ell_2$ and $\mathbf{m}(\ell, w') \dot{+} \mathbf{m}(\ell, w'') + \rho$, and consider the following three cases.

CASE 1: $w' \subseteq w_\xi$

Then for each $(\delta, \varepsilon) \in (w')^{(2)}$ we have $h_i(\delta, \varepsilon) < M_\xi$, so this also holds for $(\delta, \varepsilon) \in (w'')^{(2)}$. Consequently, either $w'' \subseteq w_\xi$ or $w'' \subseteq w_\zeta$.

If $w'' \subseteq w_\xi$, then let $\ell' = \min(\ell, n_\xi)$ and consider $\mathbf{m}^{p_\xi}(w', \ell'), \mathbf{m}^{p_\xi}(w'', \ell') \in \mathcal{M}_\xi$. Using clause $(*)_9$ for p_ξ we immediately obtain the desired conclusion.

If $w'' \subseteq w_\zeta$, then we let $\ell' = \min(\ell, n_\xi)$ and we consider $\mathbf{m}^{p_\xi}(w', \ell')$ and $\mathbf{m}^{p_\xi}(\pi^{-1}[w''], \ell')$ (both from \mathcal{M}_ξ). By $(*)_{14}$, clause $(*)_9$ for p_ξ applies to them and we get

- $\text{rk}(w') = \text{rk}(\pi^{-1}[w'']), \zeta(w') = \zeta(\pi^{-1}[w'']), k(w') = k(\pi^{-1}[w''])$ and
- if $\delta \in w', \varepsilon \in \pi^{-1}[w'']$ are such that $|\delta \cap w'| = k(w') = k(\pi^{-1}[w'']) = |\varepsilon \cap \pi^{-1}[w'']|$, then $(\eta_\delta^{p_\xi} \upharpoonright \ell') + \rho = \eta_\varepsilon^{p_\xi} \upharpoonright \ell'$.

By $(*)_{14}$ this immediately implies the desired conclusion.

CASE 2: $w' \subseteq w_\zeta$

Same as the previous case, just interchanging ξ and ζ .

CASE 3: $w' \setminus w_\xi \neq \emptyset \neq w' \setminus w_\zeta$

Then for some $(\delta, \varepsilon) \in (w')^{(2)}$ we have $h_i(\delta, \varepsilon) = M_\xi$, so necessarily $\ell = r_{M_\xi} = n$. Hence $\{\eta_\alpha : \alpha \in w'\} = \{\eta_\alpha + \rho : \alpha \in w''\}$ and since $|w'| \geq 5$, the linear independence of $\bar{\eta}$ implies $\rho = \mathbf{0}$ and $w' = w''$ and the desired conclusion follows.

RE $(*)_{10}$: Let us prove clause $(*)_{10}$ now. Suppose that $\mathbf{m}(\ell_0, w'), \mathbf{m}(\ell_1, w'') \in \mathcal{M}$, $\delta \in w', |\delta \cap w'| = k(w'), \text{rk}(w') = -1$, and $\mathbf{m}(\ell_0, w') \sqsubseteq^* \mathbf{m}(\ell_1, w'')$. Assume towards contradiction that there are $\varepsilon_0, \varepsilon_1 \in w''$ such that

$$(\otimes)_0 \eta_{\varepsilon_0} \upharpoonright \ell_1 \neq \eta_{\varepsilon_1} \upharpoonright \ell_1 \text{ and } \eta_\delta \upharpoonright \ell_0 \triangleleft \eta_{\varepsilon_0} \text{ and } \eta_\delta \upharpoonright \ell_0 \triangleleft \eta_{\varepsilon_1}.$$

Without loss of generality $|w''| = |w'| + 1 \geq 6$.

Since we must have $\ell_0 < n$, for no $\alpha, \beta \in w'$ we can have $h_i(\alpha, \beta) = M_\xi$. Therefore either $w' \subseteq w_\xi$ or $w' \subseteq w_\zeta$. Also,

(\otimes)₁ if $(\alpha, \beta) \in (w'')^{(2)} \setminus \{(\varepsilon_0, \varepsilon_1), (\varepsilon_1, \varepsilon_0)\}$ then $h_i(\alpha, \beta) < M_\xi$ for $i < \iota$.

Note that

(\otimes)₂ if $(\alpha, \beta) \in (w_\xi)^{(2)} \cup (w_\zeta)^{(2)}$ then $\min(\{\ell : \eta_\alpha(\ell) \neq \eta_\beta(\ell)\}) < n_\xi$ and there are no repetitions in the sequence $\langle g_i(\alpha, \beta) \upharpoonright n_\xi, g_i(\beta, \alpha) \upharpoonright n_\xi : i < \iota \rangle$.

Let $\ell^* = \min(\ell_1, n_\xi)$.

Now, if $w' \cup w'' \subseteq w_\xi$, then considering $\mathbf{m}(\ell_0, w')$ and $\mathbf{m}(\ell^*, w'')$ (and remembering (\otimes)₂) we see that $\ell_0 < n_\xi$, $\mathbf{m}^{p_\xi}(\ell_0, w') \sqsubseteq^* \mathbf{m}^{p_\xi}(\ell^*, w'')$ and they have the property contradicting ($*$)₁₀ for p_ξ .

If $w' \cup w'' \subseteq w_\zeta$, then in a similar manner we get contradiction with ($*$)₁₀ for p_ζ .

If $w' \subseteq w_\xi$ and $w'' \subseteq w_\zeta$ then one easily verifies that $\ell_0 < n_\xi$ and $\mathbf{m}^{p_\xi}(\ell_0, w') \sqsubseteq^* \mathbf{m}^{p_\xi}(\ell^*, \pi^{-1}[w''])$ provide a counterexample for ($*$)₁₀ for p_ξ . Similarly if $w' \subseteq w_\zeta$ and $w'' \subseteq w_\xi$.

Consequently, the only possibility left is that $w'' \setminus w_\xi \neq \emptyset \neq w'' \setminus w_\zeta$ and it follows from (\otimes)₁ that $|w'' \setminus w_\xi| = |w'' \setminus w_\zeta| = 1$. Let $\{\beta_b\} = w'' \setminus w_\zeta$ and $\{\gamma_c\} = w'' \setminus w_\xi$; then $\{\varepsilon_0, \varepsilon_1\} = \{\beta_b, \gamma_c\}$.

Assume $w' \subseteq w_\xi$ (the case when $w' \subseteq w_\zeta$ can be handled similarly). If we had $b \neq c$, then $\eta_{\beta_b} \upharpoonright n_\xi = \eta_{\beta_b}^{p_\xi} \upharpoonright n_\xi \neq \eta_{\gamma_c}^{p_\xi} \upharpoonright n_\xi = \eta_{\gamma_c} \upharpoonright n_\xi$. Since $w'' \subseteq (w_\xi \cap w_\zeta) \cup \{\beta_b, \gamma_c\}$ we could see that $\ell_0 < n_\xi$ and $\mathbf{m}^{p_\xi}(\ell_0, w') \sqsubseteq^* \mathbf{m}^{p_\xi}(\ell^*, \pi^{-1}[w''])$ would provide a counterexample for ($*$)₁₀ for p_ξ . Consequently, $b = c$ and $\ell_1 > n_\xi$. Now, remembering (\otimes)₀, $\eta_\delta^{p_\xi} \upharpoonright \ell_0 = \eta_{\beta_b}^{p_\xi} \upharpoonright \ell_0$ and $\mathbf{m}^{p_\xi}(\ell_0, w') \doteq \mathbf{m}^{p_\xi}(\ell_0, w'' \setminus \{\gamma_b\})$, so by ($*$)₉ for p_ξ we conclude

$$\text{rk}(w'' \setminus \{\gamma_b\}) = -1 \quad \text{and} \quad |\beta_b \cap (w'' \setminus \{\gamma_b\})| = k(w'' \setminus \{\gamma_b\}).$$

Let $\zeta^* = \zeta(w'' \setminus \{\gamma_b\})$ and $k^* = k(w'' \setminus \{\gamma_b\})$. For $\varepsilon \in A \setminus \{\xi\}$ let $\pi^\varepsilon : w_\xi \rightarrow w_\varepsilon$ be the order isomorphism and let $\gamma(\varepsilon) \in \pi^\varepsilon[w'' \setminus \{\gamma_b\}]$ be such that $|\pi^\varepsilon[w'' \setminus \{\gamma_b\}] \cap \gamma(\varepsilon)| = k^*$ (necessarily $\gamma(\varepsilon) = \pi^\varepsilon(\beta_b) \in w_\varepsilon \setminus w_\xi$). Then

- $\pi^\varepsilon[w'' \setminus \{\gamma_b\}] = (w'' \cap (w_\xi \cap w_\varepsilon)) \cup \{\gamma(\varepsilon)\} = w'' \setminus \{\beta_b, \gamma_b\} \cup \{\gamma(\varepsilon)\}$,
- $\text{rk}(\pi^\varepsilon[w'' \setminus \{\gamma_b\}]) = -1$, and $\zeta(\pi^\varepsilon[w'' \setminus \{\gamma_b\}]) = \zeta^*$, and
- $k(\pi^\varepsilon[w'' \setminus \{\gamma_b\}]) = k^* = |\pi^\varepsilon[w'' \setminus \{\gamma_b\}] \cap \gamma(\varepsilon)|$.

Hence $\mathbb{M} \models R_{|w'|, \zeta^*}[w'' \setminus \{\beta_b, \gamma_b\} \cup \{\gamma(\varepsilon)\}]$ for each $\varepsilon \in A \setminus \{\xi\}$. Consequently, the set

$$\left\{ \alpha < \lambda : \mathbb{M} \models R_{|w'|, \zeta^*}[w'' \setminus \{\beta_b, \gamma_b\} \cup \{\alpha\}] \right\}$$

is uncountable, contradicting (\otimes)_e.

RE ($*$)₁₁ : Let us argue that ($*$)₁₁ is satisfied as well and for this suppose that $\rho_i^0, \rho_i^1 \in \bigcup_{m < M} (t_m \cap {}^n 2)$ (for $i < \iota$) are such that

- (a) there are no repetitions in $\langle \rho_i^0, \rho_i^1 : i < \iota \rangle$, and
- (b) $\rho_i^0 + \rho_i^1 = \rho_j^0 + \rho_j^1$ for $i < j < \iota$.

Clearly, if all ρ_i^0, ρ_i^1 are form $\rho \frown \underbrace{(0, \dots, 0)}_N$, then we may use condition ($*$)₁₁ for p_ξ

and conclude that for some $\alpha_0, \alpha_1 \in w_\xi$ we have

$$\{\{\rho_i^0, \rho_i^1\} : i < \iota\} = \{\{g_i(\alpha_0, \alpha_1), g_i(\alpha_1, \alpha_0)\} : i < \iota\}.$$

So assume that we are not in the situation when all ρ_i^0, ρ_i^1 are form $\rho \frown \underbrace{(0, \dots, 0)}_N$.

Note that if $\rho \in \bigcup_{m < M} (t_m \cap {}^{n_2})$ and $\rho(n_\xi) = 0$, then $\rho \upharpoonright [n_\xi, n) = \mathbf{0}$. Hence, remembering definitions in $(*)_{18}$, if $\rho_0, \rho_1, \rho_2, \rho_3 \in \bigcup_{m < M} (t_m \cap {}^{n_2})$, $\rho_0 + \rho_1 = \rho_2 + \rho_3$ and $\rho_0(n_\xi) = 0$ but $\rho_1(n_\xi) = 1$, then $\{\rho_0, \rho_1\} = \{\rho_2, \rho_3\}$. Therefore, under current assumption, we must have $\rho_i^0(n_\xi) = \rho_i^1(n_\xi) = 1$ for all $i < \iota$. Define

$$\begin{aligned} B &= \{(\alpha_a, \gamma_c) : a < k \ \& \ c < \ell\}, \\ C &= \{(\gamma_b, \gamma_c) : b < c < \ell\}, \\ D &= \{(\beta_b, \gamma_c) : b < \ell \ \& \ c < \ell \ \& \ b \neq c\}, \\ E &= \{(\beta_b, \gamma_b) : b < \ell\}. \end{aligned}$$

(These four sets correspond to the conditions $(*)_{18}^b - (*)_{18}^e$ in the definition of g_i .) Clearly, $\rho_i^0(n_\xi) = \rho_i^1(n_\xi) = 1$ implies that

$$\rho_i^0, \rho_i^1 \in \{g_j(\varepsilon_0, \varepsilon_1), g_j(\varepsilon_1, \varepsilon_0) : (\varepsilon_0, \varepsilon_1) \in B \cup C \cup D \cup E, j < \iota\}.$$

Note also that for each $d < N_0 - 1$,

- $(\boxtimes)_a$ the set $\{\rho \in \bigcup_{m < M} (t_m \cap {}^{n_2}) : \rho \upharpoonright (n_\xi, n_\xi + N_0] = \nu_d\}$ is not empty but it has at most two elements, and
- $(\boxtimes)_b$ $|\{\rho \in \bigcup_{m < M} (t_m \cap {}^{n_2}) : \rho \upharpoonright (n_\xi, n_\xi + N_0] = \nu_d\}| = 2$ if and only if $d = \Theta(b, c, i, 1)$ for some $b < c < \ell$ and $i < \iota$, and
- $(\boxtimes)_c$ the set $\{\rho \in \bigcup_{m < M} (t_m \cap {}^{n_2}) : \rho \upharpoonright (n_\xi, n_\xi + N_0] = \nu_d^*\}$ has at most one element, and
- $(\boxtimes)_d$ $\{\rho \in \bigcup_{m < M} (t_m \cap {}^{n_2}) : \rho \upharpoonright (n_\xi, n_\xi + N_0] = \nu_d^*\} = \emptyset$ if and only if $d = \Theta(b, c, i, 1)$ for some $b < c < \ell$ and $i < \iota$.

Now consider $\rho_i^0 \upharpoonright (n_\xi, n_\xi + N_0]$, $\rho_i^1 \upharpoonright (n_\xi + 1, n_\xi + N_0]$ for $i < \iota$.

If for some $(i, x) \neq (j, y)$ we have $\rho_i^x \upharpoonright (n_\xi, n_\xi + N_0] = \rho_j^y \upharpoonright (n_\xi, n_\xi + N_0]$, then (using $(\boxtimes)_a - (\boxtimes)_d$ and the linear independence of ν_d 's) we must also have that

$$\rho_i^0 \upharpoonright (n_\xi, n_\xi + N_0] = \rho_i^1 \upharpoonright (n_\xi, n_\xi + N_0] \quad \text{for all } i < \iota.$$

Thus, for every $i < \iota$ there are $b < c < \ell$ and $j < \iota$ such that

$$\{\rho_i^0, \rho_i^1\} = \{g_j(\gamma_b, \gamma_c), g_j(\gamma_c, \gamma_b)\}.$$

Since for $b < c < \ell$ we have

$$(g_j(\gamma_b, \gamma_c) + g_j(\gamma_c, \gamma_b)) \upharpoonright (N_0, N_0 + \ell] = \underbrace{\langle 0, \dots, 0 \rangle}_b \wedge \underbrace{\langle 1, \dots, 1 \rangle}_{c-b} \wedge \underbrace{\langle 0, \dots, 0 \rangle}_{\ell-c}$$

we immediately get that (in the current situation) for some $b < c < \ell$ we have

$$\{\{\rho_i^0, \rho_i^1\} : i < \iota\} = \{\{g_i(\gamma_b, \gamma_c), g_i(\gamma_c, \gamma_b)\} : i < \iota\}.$$

So let us assume that $\rho_i^x \upharpoonright (n_\xi, n_\xi + N_0] \neq \rho_j^y \upharpoonright (n_\xi, n_\xi + N_0]$ for all distinct $(i, x), (j, y) \in \iota \times 2$. Since $\{\mathbf{1}, \nu_0, \dots, \nu_{N_0-2}\}$ are linearly independent we may use Lemma 4.3(2) to conclude that

$$\left\{ \{\rho_i^0 \upharpoonright (n_\xi, n_\xi + N_0], \rho_i^1 \upharpoonright (n_\xi, n_\xi + N_0]\} : i < \iota \right\} \subseteq \left\{ \{\nu_d, \nu_d^*\} : d < N_0 - 1 \right\}.$$

Consequently, we easily deduce that

$$\left\{ \{\rho_i^0, \rho_i^1\} : i < \iota \right\} \subseteq \left\{ \{g_i(\varepsilon_0, \varepsilon_1), g_i(\varepsilon_1, \varepsilon_0)\} : i < \iota \ \& \ (\varepsilon_0, \varepsilon_1) \in B \cup D \cup E \right\}.$$

Using the linear independence of η_ε^s 's and the definitions of g_i 's in $(*)_{18}$ one checks that the three sets

$$\begin{aligned} & \{g_i(\varepsilon_0, \varepsilon_1) + g_i(\varepsilon_1, \varepsilon_0) : (\varepsilon_0, \varepsilon_1) \in B, i < \iota\}, \\ & \{g_i(\varepsilon_0, \varepsilon_1) + g_i(\varepsilon_1, \varepsilon_0) : (\varepsilon_0, \varepsilon_1) \in D, i < \iota\}, \\ & \{g_i(\varepsilon_0, \varepsilon_1) + g_i(\varepsilon_1, \varepsilon_0) : (\varepsilon_0, \varepsilon_1) \in E, i < \iota\} \end{aligned}$$

are pairwise disjoint. Therefore, $\{\{\rho_i^0, \rho_i^1\} : i < \iota\}$ must be included in (exactly) one of the sets

$$\begin{aligned} & \{g_i(\varepsilon_0, \varepsilon_1), g_i(\varepsilon_1, \varepsilon_0)\} : i < \iota \text{ \& } (\varepsilon_0, \varepsilon_1) \in B\}, \\ & \{g_i(\varepsilon_0, \varepsilon_1), g_i(\varepsilon_1, \varepsilon_0)\} : i < \iota \text{ \& } (\varepsilon_0, \varepsilon_1) \in D\}, \text{ or} \\ & \{g_i(\varepsilon_0, \varepsilon_1), g_i(\varepsilon_1, \varepsilon_0)\} : i < \iota \text{ \& } (\varepsilon_0, \varepsilon_1) \in E\}. \end{aligned}$$

But now we easily check that for some $(\varepsilon_0, \varepsilon_1) \in B \cup D \cup E$ we must have

$$\{\{\rho_i^0, \rho_i^1\} : i < \iota\} = \{g_i(\varepsilon_0, \varepsilon_1), g_i(\varepsilon_1, \varepsilon_0)\} : i < \iota\}.$$

This completes the verification that $q = (w, n, M, \bar{\eta}, \bar{t}, \bar{r}, \bar{h}, \bar{g}, \mathcal{M}) \in \mathbb{P}$. It should be clear that q is stronger than both p_ξ and p_ζ . \square

Define \mathbb{P} -names \mathcal{T}_m and η_α (for $m < \omega$ and $\alpha < \lambda$) by

$$\begin{aligned} \Vdash_{\mathbb{P}} \text{“ } \mathcal{T}_m = \bigcup \{t_m^p : p \in \mathcal{G}_{\mathbb{P}} \wedge m < M^p\} \text{”}, \text{ and} \\ \Vdash_{\mathbb{P}} \text{“ } \eta_\alpha = \bigcup \{\eta_\alpha^p : p \in \mathcal{G}_{\mathbb{P}} \wedge \alpha \in w^p\} \text{”}. \end{aligned}$$

Claim 4.4.4. (1) For each $m < \omega$ and $\alpha < \lambda$,
 $\Vdash_{\mathbb{P}} \text{“ } \eta_\alpha \in {}^\omega 2 \text{ and } \mathcal{T}_m \subseteq {}^{\omega > 2}$ is a tree without terminal nodes ”.

(2) $\Vdash_{\mathbb{P}} \text{“ } \bigcup_{m < \omega} \lim(\mathcal{T}_m) \text{ is a } 2\iota\text{-npots set } \text{”}.$

Proof of the Claim. (1) By Claim 4.4.2 (and the definition of the order in \mathbb{P}).

(2) Let $G \subseteq \mathbb{P}$ be a generic filter over \mathbf{V} and let us work in $\mathbf{V}[G]$.

Let $k = 2\iota$ and $\bar{T} = \langle (\mathcal{T}_m)^G : m < \omega \rangle$.

Suppose towards contradiction that $B = \bigcup_{m < \omega} \lim((\mathcal{T}_m)^G)$ is a k -**npots** set. Then,

by Proposition 3.11, $\text{NDRK}(\bar{T}) = \infty$. Using Lemma 3.10(5), by induction on $j < \omega$ we choose $\mathbf{m}_j, \mathbf{m}_j^* \in \mathbf{M}_{\bar{T}, k}$ and $p_j \in G$ such that

- (i) $\text{ndrk}(\mathbf{m}_j) \geq \omega_1$, $|u_{\mathbf{m}_j}| > 5$ and $\mathbf{m}_j \sqsubseteq \mathbf{m}_j^* \sqsubseteq \mathbf{m}_{j+1}$,
- (ii) for each $\nu \in u_{\mathbf{m}_j^*}$ the set $\{\eta \in u_{\mathbf{m}_{j+1}} : \nu \triangleleft \eta\}$ has at least two elements,
- (iii) $p_j \leq p_{j+1}$, $\ell_{\mathbf{m}_j} \leq \ell_{\mathbf{m}_j^*} = n^{p_j} < \ell_{\mathbf{m}_{j+1}}$ and $\text{rng}(h_i^{\mathbf{m}_j}) \subseteq M^{p_j}$ for all $i < \iota$,
and
- (iv) $|\{\eta \upharpoonright n^{p_j} : \eta \in u_{\mathbf{m}_{j+1}}\}| = |u_{\mathbf{m}_j}| = |u_{\mathbf{m}_j^*}|$.

Then, by (iii)+(iv), $\mathbf{m}_j, \mathbf{m}_j^* \in \mathbf{M}_{\bar{T}, k}^{n^{p_j}}$. It follows from Claim 4.4.1 that for some $w_j \subseteq w^{p_j}$ and $\rho_j \in n^{p_j} 2$ we have $(\mathbf{m}_j^* + \rho_j) \dot{\sqsubseteq} \mathbf{m}^{p_j}(n^{p_j}, w_j) \in \mathcal{M}^{p_j}$.

Fix j for a moment and consider $\mathbf{m}^{p_j}(n^{p_j}, w_j) \in \mathcal{M}^{p_j}$ and $\mathbf{m}^{p_{j+1}}(n^{p_{j+1}}, w_{j+1}) \in \mathcal{M}^{p_{j+1}}$. Since $(\mathbf{m}_j^* + (\rho_{j+1} \upharpoonright n^{p_j})) \sqsubseteq (\mathbf{m}_{j+1}^* + \rho_{j+1}) \dot{\sqsubseteq} \mathbf{m}^{p_{j+1}}(n^{p_{j+1}}, w_{j+1})$, we may choose $w_j^* \subseteq w_{j+1}$ such that

$$(\mathbf{m}_j^* + (\rho_{j+1} \upharpoonright n^{p_j})) \dot{\sqsubseteq} \mathbf{m}^{p_{j+1}}(n^{p_j}, w_j^*) \sqsubseteq^* \mathbf{m}^{p_{j+1}}(n^{p_{j+1}}, w_{j+1})$$

(and the latter two belong to $\mathcal{M}^{p_{j+1}}$). Then also

$$\mathbf{m}^{p_{j+1}}(n^{p_j}, w_j^*) \dot{\sqsubseteq} \mathbf{m}^{p_j}(n^{p_j}, w_j) + (\rho_j + \rho_{j+1} \upharpoonright n^{p_j}) = \mathbf{m}^{p_{j+1}}(n^{p_j}, w_j) + (\rho_j + \rho_{j+1} \upharpoonright n^{p_j}),$$

so by clause $(*)_9$ for p_{j+1} we have

$$\text{rk}(w_j^*) = \text{rk}(w_j).$$

Clause (ii) of the choice of \mathbf{m}_{j+1} implies that

$$(\forall \gamma \in w_j^*)(\exists \delta \in w_{j+1} \setminus w_j^*)(\eta_\gamma^{p_{j+1}} \upharpoonright n^{p_j} = \eta_\delta^{p_{j+1}} \upharpoonright n^{p_j}).$$

Let $\delta(\gamma)$ be the smallest $\delta \in w_{j+1} \setminus w_j^*$ with the above property and let $w_j^*(\gamma) = (w_j^* \setminus \{\gamma\}) \cup \{\delta(\gamma)\}$. Then, for $\gamma \in w_j^*$, $\mathbf{m}^{p_{j+1}}(n^{p_j}, w_j^*(\gamma)) \in \mathcal{M}^{p_{j+1}}$ and

$$\mathbf{m}^{p_{j+1}}(n^{p_j}, w_j^*) \dot{=} \mathbf{m}^{p_{j+1}}(n^{p_j}, w_j^*(\gamma)) \sqsubseteq^* \mathbf{m}^{p_{j+1}}(n^{p_{j+1}}, w_{j+1}).$$

So by clause $(*)_9$ we know that for each $\gamma \in w_j$:

$$\text{rk}(w_j^*(\gamma)) = \text{rk}(w_j^*), \quad \zeta(w_j^*(\gamma)) = \zeta(w_j^*), \quad \text{and} \quad k(w_j^*(\gamma)) = k(w_j^*).$$

Let $n = |w_j^*|$, $\zeta = \zeta(w_j^*)$, $k = k(w_j^*)$, and let $w_j^* = \{\alpha_0, \dots, \alpha_k, \dots, \alpha_{n-1}\}$ be the increasing enumeration. Let $\alpha_k^* = \delta(\alpha_k)$. Then clause $(*)_9$ also gives that $w_j^*(\alpha_k) = \{\alpha_0, \dots, \alpha_{k-1}, \alpha_k^*, \alpha_{k+1}, \dots, \alpha_{n-1}\}$ is the increasing enumeration. Now,

$$\begin{aligned} \mathbb{M} &\models R_{n,\zeta}[\alpha_0, \dots, \alpha_{k-1}, \alpha_k, \alpha_{k+1}, \dots, \alpha_{n-1}] && \text{and} \\ \mathbb{M} &\models R_{n,\zeta}[\alpha_0, \dots, \alpha_{k-1}, \alpha_k^*, \alpha_{k+1}, \dots, \alpha_{n-1}], \end{aligned}$$

and consequently if $\text{rk}(w_j^*) \geq 0$, then

$$\text{rk}(w_{j+1}) \leq \text{rk}(w_j^* \cup \{\alpha_k^*\}) < \text{rk}(w_j^*) = \text{rk}(w_j)$$

(remember $(\otimes)_d$).

Now, unfixing j , suppose that we constructed w_{j+1}, w_j^* for all $j < \omega$. It follows from our considerations above that for some $j_0 < \omega$ we must have:

- (a) $\text{rk}(w_{j_0}^*) = -1$, and
- (b) $\mathbf{m}^{p_{j_0+1}}(n^{p_{j_0}}, w_{j_0}^*) \sqsubseteq^* \mathbf{m}^{p_{j_0+1}}(n^{p_{j_0+1}}, w_{j_0+1})$ (and both belong to $\mathcal{M}^{p_{j_0+1}}$),
- (c) for every $\alpha \in w_{j_0}^*$ we have

$$|\{\beta \in w_{j_0+1} : \eta_\alpha^{p_{j_0+1}} \upharpoonright n^{p_{j_0}} \triangleleft \eta_\beta^{p_{j_0+1}}\}| > 1.$$

However, this contradicts clause $(*)_{10}$ (for p_{j_0+1}). □

Corollary 4.5. *Assume $\mathbf{MA} + \neg\mathbf{CH}$, $\lambda < \mathfrak{c}$ and $\text{NPr}_{\omega_1}(\lambda)$, $3 \leq \iota < \omega$. Then there exists a Σ_2^0 2ι -**npots**-set $B \subseteq {}^\omega 2$ which has λ many pairwise 2ι -nondisjoint translations.*

Proof. Standard modification of the proof of Theorem 4.4. □

Corollary 4.6. *Assume $\text{NPr}_{\omega_1}(\lambda)$ and $\lambda = \lambda^{\aleph_0} < \mu = \mu^{\aleph_0}$, $3 \leq \iota < \omega$. Then there is a ccc forcing notion \mathbb{Q} of size μ forcing that*

- (a) $2^{\aleph_0} = \mu$ and
- (b) *there is a Σ_2^0 2ι -**npots**-set $B \subseteq {}^\omega 2$ which has λ many pairwise 2ι -nondisjoint translates but not λ^+ such translates.*

Proof. Let \mathbb{P} be the forcing notion given by Theorem 4.4 and let $\mathbb{Q} = \mathbb{P} * \mathbb{C}_\mu$. Use Proposition 3.3(4) to argue that the set B added by \mathbb{P} is a **npots**-set in $\mathbf{V}^{\mathbb{Q}}$. By 3.3(3) this set cannot have λ^+ pairwise 2ι -nondisjoint translates, but it does have λ many pairwise 2ι -nondisjoint translates (by absoluteness). □

Remark 4.7. It follows from Proposition 3.3(1,2), that if there exists a Σ_2^0 **pots**-set $B \subseteq {}^\omega 2$ such that for some set $A \subseteq {}^\omega 2$ we have $(B+a) \cap (B+b) \neq \emptyset$ for all $a, b \in A$, then $\text{std}(B) \subseteq {}^\omega 2 \times {}^\omega 2$ is a Σ_2^0 set which contains a $|A|$ -square but no perfect square. Thus Corollary 4.6 is a slight generalization of Shelah [6, Theorem 1.13].

5. OPEN QUESTIONS

Problem 5.1. Is it consistent that for every Borel set $B \subseteq {}^\omega 2$,

if there is $H \subseteq {}^\omega 2$ of size \aleph_1 such that $|(B+x) \cap (B+y)| \geq 6$ for all $x, y \in H$, then there is a perfect set P such that $|(B+x) \cap (B+y)| \geq 6$ for all $x, y \in P$? (Compare this with Proposition 3.3(3).)

Problem 5.2. Is it consistent to have a Borel set $B \subseteq {}^\omega 2$ such that

- for some uncountable set H , $(B+x) \cap (B+y)$ is uncountable for every $x, y \in H$, but
- for every perfect set P there are $x, y \in P$ with $(B+x) \cap (B+y)$ countable?

Problem 5.3. Is it consistent to have a Borel set $B \subseteq {}^\omega 2$ such that

- B has uncountably many pairwise disjoint translations, but
- there is no perfect set of pairwise disjoint translations of B . ?

REFERENCES

- [1] Marek Balcerzak, Andrzej Roslanowski, and Saharon Shelah. Ideals without ccc. *Journal of Symbolic Logic*, 63:128–147, 1998. arxiv:math.LO/9610219.
- [2] Tomek Bartoszyński and Haim Judah. *Set Theory: On the Structure of the Real Line*. A K Peters, Wellesley, Massachusetts, 1995.
- [3] Thomas Jech. *Set theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.
- [4] Andrzej Roslanowski and Vyacheslav V. Rykov. Not so many non-disjoint translations, submitted. arxiv:1711.04058.
- [5] Andrzej Roslanowski and Saharon Shelah. Borel sets without perfectly many overlapping translations, II. In progress.
- [6] Saharon Shelah. Borel sets with large squares. *Fundamenta Mathematicae*, 159:1–50, 1999. arxiv:math.LO/9802134.
- [7] Piotr Zakrzewski. On Borel sets belonging to every invariant ccc σ -ideal on $2^{\mathbb{N}}$. *Proc. Amer. Math. Soc.*, 141:1055–1065, 2013.

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