ON FINITARY HINDMAN NUMBERS

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Abstract. Spencer asked whether the Paris-Harrington version of the Folkman-Sanders theorem has primitive recursive upper bounds. We give a positive answer to this question.

1. Introduction

Inspired by Paris-Harrington's strengthening of the finite Ramsey theorem [5], Spencer defined in a similar way the following numbers (which we denote by $\mathrm{Sp}(m,c)$), strengthening the Folkman-Sanders theorem [6] [1]. Let $\mathrm{Sp}(m,c)$ be the least integer k such that whenever $[k]=\{1,\dots,k\}$ is c-colored then there is $H=\{a_0,\dots,a_{l-1}\}\subset[k]$ such that $\sum H$ (sums of elements of H with no repetition) is monochromatic and $m\leq \min H\leq l$. As in the case of Paris-Harrington's theorem which is deduced from the infinite Ramsey theorem, the existence of the Spencer numbers $\mathrm{Sp}(m,c)$ is also easily deduced from the infinite version of the Folkman-Sanders theorem, namely Hindman's theorem [4]. Spencer asked whether $\mathrm{Sp}(m,c)$ is primitive recursive[2]. In this paper we give a positive answer to this question. In fact we define the more general numbers $\mathrm{Sp}(m,p,c)$ and show that it is in $\mathcal{E}_5$ of the Grzegorczyk hierarchy of primitive recursive functions. This means that the rate of the growth of the Spencer function is much slower than the Paris-Harrington function which grows faster than every primitive recursive function. We refer the reader to Section 2.7. of [3] for getting information about the growth rate of the functions in class $\mathcal{E}_5$ which are called WOW functions there. It contains sufficient information to be convinced why our proof implies that the function $\mathrm{Sp}(m,p,c)$ is in class $\mathcal{E}_5$. We also refer the reader to [2] for some Ackermannian bounds in both directions for the Paris-Harrington numbers.

Definition 1.1. *For positive integers m,p,c, let $Sp(m,p,c)$ be the least integer k such that whenever $[k]=\{1,\dots,k\}$ is c-colored then there is $H=\{a_0,\dots,a_{l-1}\}\subset[k]$ (with $a_0<\cdots<a_{l-1}$) such that*

(i) *$\sum H$ is monochromatic,*
(ii) *$m\leq a_0$, $p\leq l$ and $a_{p-1}\leq l$.*

To prove our theorem we use the bounds given in [7] for the numbers $\mathrm{U}(n,c)$ for the disjoint unions theorem. We also need to consider the finitary Hindman numbers $\mathrm{Hind}(n,c)$ defined below. Let's first fix some notations. Let A,B be finite

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[1] According to Soifer, this should be called the Arnautov-Folkman-Sanders theorem. See [6], pp. 305.

[2] Spencer asked Shelah the question during the workshop: **Combinatorics: Challenges and Applications**, celebrating Noga Alon's 60th birthday, Tel Aviv University, January 17-21, 2016.

subsets of $\mathbb{N}$, by $A < B$ we mean $\max A < \min B$. If T is a collection of pairwise disjoint sets, then $NU(T)$ will denote the set of non-empty unions of elements T. Also by $T = \{A_0, \ldots, A_{l-1}\}_<$ we mean that the elements of T are finite non-empty subsets of $\mathbb{N}$ and $A_0 < \cdots < A_{l-1}$. We also need the following notation. Let $A = \{a_0, \ldots, a_n\}$ be a finite subset of $\mathbb{N}$. Let $\exp_2(A)$ denote $2^{a_0} + \cdots + 2^{a_n}$. We will use the simple fact that if A, B are two nonempty disjoint finite subsets of $\mathbb{N}$, then $\exp_2(A \cup B) = \exp_2(A) + \exp_2(B)$. Also we have $A \neq B$ iff $\exp_2(A) \neq \exp_2(B)$. We denote the collection of nonempty subsets of S by $\mathcal{P}^+(S)$.

Definition 1.2. *For positive integers n, c, let $U(n,c)$ be the least integer k with the following property. For any pairwise disjoint sets $A_0, \ldots, A_{k-1}$, if $NU\{A_0, \ldots, A_{k-1}\}$ is c-colored, then there are pairwise disjoint sets $d_0, \ldots, d_{n-1}$ such that*

(i) *$d_i \in NU\{A_0, \ldots, A_{k-1}\}$ for $i = 0, \ldots, n-1$,*
(ii) *$NU\{d_0, \ldots, d_{n-1}\}$ is monochromatic.*

Theorem 1.3 (Taylor, [7])**.** *$U(n,c)$ is a tower function.*

Definition 1.4. *For positive integers n, c, let Hind(n,c) be the least integer k such that whenever $NU\{A_0, \ldots, A_{k-1}\}_<$ is c-colored, then there is $\{d_0, \ldots, d_{n-1}\}_<$ such that*

(i) *$d_i \in NU\{A_0, \ldots, A_{k-1}\}_<$ for $i = 0, \ldots, n-1$,*
(ii) *$NU\{d_0, \ldots, d_{n-1}\}_<$ is monochromatic.*

It is also known that

Theorem 1.5 ([1], Proposition 2.19.)**.** *Hind(n,c) lies in $\mathcal{E}_4$ of the Grzegorczyk hierarchy.*

2. Spencer Numbers

Let m, p, c be positive integers and let $k_* = \text{Hind}(p+1, c)$. We inductively define a sequence of positive integers $\langle n_i; i < k_* + 1\rangle$ as follows.

(i) n_0 is the least integer with $m \leq 2^{n_0}$,
(ii) $m_i = 2^{\sum_{j=0}^{i} n_j}$,
(iii) $\alpha_i = 2^{k_* - i - 1 + \sum_{j=1}^{i} n_j}$,
(iv) $n_{i+1} = \text{U}(m_i, c^{\alpha_i})$.

Theorem 2.1. *For all positive integers m, p, c we have $Sp(m,p,c) \leq 2^{n_{k_*}}$.*

Proof. Let $\mathbf{c}$ be a c-coloring of $\{1, \ldots, 2^{n_{k_*}}\}$. We will find $H = \{a_0, \ldots, a_{l-1}\} \subseteq [2^{n_{k_*}}]$ satisfying the requirements of Definition 1.1. For $0 \leq i \leq k_* - 1$ we first define the following intervals of positive integers

$$S_i = [n_0 + \cdots + n_i, n_0 + \cdots + n_{i+1} - 1].$$

So $|S_i| = n_{i+1}$ and $S_i < S_{i+1}$. Set $S^* = \bigcup_{i=0}^{k_*-1} S_i$. Let $\mathbf{c}^*$ be a c-coloring of $\mathcal{P}^+(S^*)$ defined by $\mathbf{c}^*(A) = \mathbf{c}(\exp_2(A))$. For the next step, we shall find specific pairwise disjoint subsets $w_{i,s} \subseteq S_i$ for $0 \leq i \leq k_* - 1$, $0 \leq s < m_i$ by reverse induction on $0 \leq i \leq k_* - 1$. Let $\mathbf{c}_i$ be a coloring of $\mathcal{P}^+(S_i)$ defined as follows. For every $u, v \in \mathcal{P}^+(S_i)$, we put $\mathbf{c}_i(u) = \mathbf{c}_i(v)$ if for all $A \in \mathcal{P}(\bigcup_{j<i} S_j)$ and all $B \subseteq \{i+1, \ldots, k_* - 1\}$, we have

$$\mathbf{c}^*\big(A \cup u \cup \bigcup_{j \in B} w_{j,0}\big) = \mathbf{c}^*\big(A \cup v \cup \bigcup_{j \in B} w_{j,0}\big). \tag{1}$$

As $|\mathcal{P}(\bigcup_{j<i} S_j)| = 2^{\sum_{j=1}^{i} n_j}$ and $|\mathcal{P}(\{i+1,\dots,k_*-1\})| = 2^{k_*-i-1}$, we observe that the number of colors of $\mathbf{c}_i$ is at most c^{α_i} where $\alpha_i = 2^{k_*-i-1+\sum_{j=1}^{i} n_j}$. So from $n_{i+1} = \mathrm{U}(m_i, c^{\alpha_i})$ it follows that there are pairwise disjoint subsets $w_{i,s} \subseteq S_i$ for $0 \le s < m_i$ such that $NU\{w_{i,0},\dots,w_{i,m_i-1}\}$ is $\mathbf{c}_i$-monochromatic. It is clear by construction that for $i_1 < i_2$ we have $w_{i_1,j_1} < w_{i_2,j_2}$. Now consider

$$NU\{w_{0,0}, w_{1,0}, \dots, w_{k_*-1,0}\}_<$$

with the coloring $\mathbf{c}^*$. Recall that $k_* = \mathrm{Hind}(p+1,c)$, then there is $\{v_0,\dots,v_p\}_<$ such that

(i) $v_i \in NU\{w_{0,0}, w_{1,0}, \dots, w_{k_*-1,0}\}_<$ for $0 \le i \le p$,
(ii) $NU\{v_0,\dots,v_p\}_<$ is $\mathbf{c}^*$-monochromatic.

Assume that $v_p = w_{e_1,0} \cup \cdots \cup w_{e_r,0}$ and $l^* = m_{e_1}$. Now set

$$v_{p+1} = w_{e_1,1} \cup \cdots \cup w_{e_r,1},$$

$$v_{p+2} = w_{e_1,2} \cup \cdots \cup w_{e_r,2},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$v_{p+l^*-1} = w_{e_1,l^*-1} \cup \cdots \cup w_{e_r,l^*-1}.$$

Note that $v_0,\dots,v_{p+l^*-1}$ are pairwise disjoint. We claim the desired $H = \{a_0,\dots,a_{l-1}\}$ is obtained by putting $l = p + l^*$ and $a_i = \exp_2(v_i)$. First observe that

$$a_0 = \exp_2(v_0) \ge 2^{n_0} \ge m.$$

Let $v_{p-1} = w_{d_1,0} \cup \cdots \cup w_{d_q,0}$. Also $v_{p-1} < v_p$ implies $d_q < e_1$, so we have

$$\begin{aligned} a_{p-1} = \exp_2(v_{p-1}) &= \exp_2(w_{d_1,0}) + \cdots + \exp_2(w_{d_q,0}) \\ &\le \exp_2(S_{d_1}) + \cdots + \exp_2(S_{d_q}) \\ &\le 2^{n_0} + 2^{n_0+1} + \cdots + 2^{n_0+n_1+\cdots+n_{d_q+1}-1} \\ &\le 2^{n_0+n_1+\cdots+n_{d_q+1}} = m_{d_q+1} \le m_{e_1} = l^* \le l. \end{aligned}$$

Note that $a_0 < a_1 < \cdots < a_{p-1}$, and also $a_{p-1} < a_i$ for $i \ge p$. This is enough for our purpose and there is no need to know the order of $\{a_p, a_{p+1},\dots,a_{l-1}\}$. It remains to show that $\sum H$ is $\mathbf{c}$-monochromatic. This is equivalent to saying that $NU\{v_0,\dots,v_{l-1}\}$ is $\mathbf{c}^*$-monochromatic. Recall that $NU\{v_0,\dots,v_p\}$ is $\mathbf{c}^*$-monochromatic. Let

$$A_1 \in NU\{v_0,\dots,v_{p-1}\}, \;\; B_1 \in \{A_1,\emptyset\}, \;\; A_2 \in NU\{v_p,\dots,v_{l-1}\}.$$

Obviously $\mathbf{c}^*(A_1) = \mathbf{c}^*(v_p)$. So we will finish if we show $\mathbf{c}^*(B_1 \cup A_2) = \mathbf{c}^*(v_p)$. This will be done by iterated application of the relation (1) when $u, v \in NU\{w_{i,0},\dots,w_{i,m_i-1}\}$. First note that we can write A_2 as

$$\bigcup_{i\in I} w_{e_1,i} \cup \bigcup_{i\in I} w_{e_2,i} \cup \ldots \cup \bigcup_{i\in I} w_{e_r,i}$$

for some $I \subseteq \{0, 1, \ldots, l^* - 1\}$. Finally

$$\begin{aligned} \mathbf{c}^*(v_p) = \mathbf{c}^*(B_1 \cup v_p) &= \mathbf{c}^*\big(B_1 \cup w_{e_1,0} \cup w_{e_2,0} \cup \cdots \cup w_{e_r,0}\big) \\ &= \mathbf{c}^*\big(B_1 \cup \bigcup_{i \in I} w_{e_1,i} \cup w_{e_2,0} \cup \cdots \cup w_{e_r,0}\big) \\ &= \mathbf{c}^*\big(B_1 \cup \bigcup_{i \in I} w_{e_1,i} \cup \bigcup_{i \in I} w_{e_2,i} \cup \cdots \cup w_{e_r,0}\big) = \cdots \\ &= \mathbf{c}^*\big(B_1 \cup \bigcup_{i \in I} w_{e_1,i} \cup \bigcup_{i \in I} w_{e_2,i} \cup \ldots \cup \bigcup_{i \in I} w_{e_r,i}\big) \\ &= \mathbf{c}^*(B_1 \cup A_2). \end{aligned}$$

□

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