Maximal models up to the first measurable in ZFC

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In this paper we prove in ZFC the existence of a *complete sentence* ϕ of $L_{\omega_1,\omega}$ such that ϕ has maximal models in a set of cardinals λ that is cofinal in the first measurable μ while ϕ has no maximal models in any $\chi \geq \mu$. This paper contributes to the study of Hanf numbers for infintary logics. Works such as [BKS09, BKS16, BS17b, KLH16] study the behavior of maximal in context where the class has a bounded number of models. We list now some properties that are true in every cardinality for first order logic but are true only eventually for complete sentences of $L_{\omega_1,\omega}$ or, more generally, for abstract elementary classes, and compare the cardinalities (the Hanf number) at which the eventual behavior must begin. Every model of a first order theory has a proper elementary extension and so each theory has arbitrarily large models. Moreover, the amalgamation property holds for every complete first order theory. Morley [Mor65] showed that every sentence of $L_{\omega_1,\omega}$ that has models up to $\beth_{\omega_1}$ has arbitrarily large models and provided counterexamples showing that cardinal was minimal. Thus he showed the Hanf number for existence of $L_{\omega_1,\omega}$ is $\beth_{\omega_1}$. Hjorth [Hjo02], by a much more complicated argument, showed there are *complete* sentences ϕ_α for $\alpha < \omega_1$ such that ϕ_α has a model in $\aleph_\alpha$ and no larger so the Hanf number for complete sentences is $\aleph_{\omega_1}$. Boney and Unger [BU17], building on [She13] show that the Hanf number 'for all AEC's are tame' is the first strongly compact cardinal. They also show the analogous property for various variants on tameness is equivalent to the existence of almost (weakly compact, measurable, strongly compact). The result here shows the Hanf number for extendability (every $L_{\omega_1,\omega}$-model has an elementary extension) is the first measurable cardinal. However, [BB17] show that an upper bound on the Hanf number for amalgamation is the first strongly compact. The actual value remains open.

In [BS17a], we proved a theorem with the same conclusion as the main result here; the earlier proof required that $\lambda = \lambda^{<\lambda}$, and that there is an $S \subseteq S^{\lambda}_{\aleph_0}$, that is stationary non-reflecting, and $\diamond_S$ holds. Here, we show *in ZFC* that the sentence ϕ defined in

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[BS17a] has maximal models cofinally in μ. The additional hypotheses in [BS17a] allow one to demand that if N is a submodel with cardinality $< \lambda$ of the P_0-maximal model, N is $\boldsymbol{K}_1$-free (Definition 2.2); that property fails for the example here. The existence of such a ϕ which is *not complete* is well-known (e.g. [Mag16]).

The first section of the paper defines the class of models $\boldsymbol{K}_{-1}$ and explains the connections with [BS17a]. In Section 2 we construct in ZFC, for cofinally many λ less than the first measurable, a P_0-maximal model $M_* \in \boldsymbol{K}_{-1}$. Subsection 2.1 is a completely independent set theoretic argument for the existence of a Boolean algebra with certain specified properties in any cardinal λ of the form $\lambda = 2^\mu$ that is less than the first measurable. Subsection 2.2 builds on this result to find a P_0-maximal model in $\boldsymbol{K}_{-1}$ with cardinality λ satisfying certain further restrictions. Finally in Section 3, this model is converted to the maximal model of $\boldsymbol{K}_2$, the class of model of the comoplete sentence (ϕ) from [BS17a].

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1 Preliminaries

{prelim}

We include as needed definitions of the classes of model $\boldsymbol{K}_{-1}, \boldsymbol{K}_1, \boldsymbol{K}_2$ introduced in [BS17a]. For each i, $\boldsymbol{K}^i_{<\aleph_0}$ denotes the class of finitely generated members of $\boldsymbol{K}_i$. Occasionally we fall into the notation $\hat{\boldsymbol{K}}$ for the direct limits of a class $\boldsymbol{K}$ of finitely generated models.

{deftau}

Definition 1.1 *τ is a vocabulary with unary predicates P_0, P_1, P_2, P_4, binary R, E, $\wedge, \vee, \leq$ unary functions $^-$, G_1, H_1, n unary functions $g_{n,i}$ for each n, constants 0,1 and unary functions F_n, for $n < \omega$. $\leq$ is a partial order on P^M_1 and the Boolean algebra can be defined from it.*

We occasionally use the notations $(\forall^\infty n)$ and $(\exists^\infty n)$ to mean 'for all but finitely many' and 'for infinitely many' respectively. It is easy to see that $\boldsymbol{K}_{-1}$ is $L_{\omega_1,\omega}$-axiomatizable but far from complete.

{f1}

Definition 1.2 ($\boldsymbol{K}_{-1}$) *$M \in \boldsymbol{K}^{-1}_{<\aleph_0}$ is the class of* finitely generated *structures M satisfying the following conditions.*

Note that b_* is not a function symbol in τ.

1. *P^M_0, P^M_1, P^M_2 partition M.*
2. *$(P^M_1, 0, 1, \wedge, \vee, \leq, ^-)$ is a Boolean algebra ($^-$ is complement). We may also write $\boldsymbol{B}_M$ or $\boldsymbol{B}[M]$ for P^M_1. We also consider ideals and restrictions to them of the relations/operations except for complement.*
3. *$R \subset P^M_0 \times P^M_1$ with $R(M, b) = \{a : R^M(a, b)\}$ and the set of $\{R(M, b) : b \in P^M_1\}$ is a Boolean algebra. $f^M : P^M_1 \mapsto \mathcal{P}(P^M_0)$ by $f^M(b) = R(M, b)$ is a Boolean algebra homomorphism into $\mathbb{P}(P^M_0)$.*

Note that f is not[1] in τ; it is simply a convenient abbreviation for the relation between the Boolean algebra P_1^M and the set algebra on P_0 by the map $b \mapsto R(M,b)$.

4. *$P_{4,n}^M$ is the set containing each join of n distinct atoms from M; P_4^M is the union of the $P_{4,n}^M$ and so is an ideal. That is, P_4^M is the set of all finite joins of atoms.*

   There is an element $b^* \in P_1^M$ and for each n, $P_{4,n}^M = \{c : c \leq^M b_*\}$.

5. *G_1^M is a bijection from P_0^M onto $P_{4,1}^M$ such that $R(M, G_1^M(a)) = \{a\}$. (Note that $P_0^M = \emptyset$ is allowed.*

6. *P_2^M is finite (and may be empty). Further, for each $c \in P_2^M$ the $F_n^M(c)$ are functions from P_2^M into P_1^M. Note that it is allowed that for all but finitely many n, $F_n^M(c) = 0_{P_1^M}$.*

7. *If $a \in P_{4,1}^M$ and $c \in P_2^M$ then[2] $(\forall^\infty n)\ a \not\leq_M F_n^M(c)$. This implies $\bigcap_n \{x\colon (G_1(x) \in F_n^M(c)\} = \emptyset$.*

8. *P_1^M is generated as a Boolean algebra by $P_4^M \cup \{F_n^M(c)\colon c \in P_2^M, n \in \omega\} \cup X$ where X is a finite subset of P_1^M.*

{f2}

Definition 1.3 *1. K_{-1} is the class of τ structures M such that every finitely generated substructure of M is in $\boldsymbol{K}^{-1}_{<\aleph_0}$. K^{-1}_μ is the members of K_{-1} with cardinality μ.*

2. We say $M \in K_{-1}$ is atomic *if P_1^M is atomic as a Boolean algebra. That is, P_4^M is dense in $\boldsymbol{B}_M$.*

2 The first approximation

{1approx}

In this section we construct in ZFC, for cofinally many λ less than the first measurable, a P_0-maximal model $M_* \in \boldsymbol{K}_{-1}$. In Section 3 we 'correct' that model to a model of the complete sentence ϕ of $L_{\omega_1,\omega}$ defined in [BS17a]. Subsection 2.1 can be read completely independently; it has no reliance on Section 1. In particular, there is no mention here of independence of sets in the Boolean algebra.

2.1 Set theoretic construction of a Boolean algebra

{stba}

The goal of this subsection is to prove the property $\boxplus$ in ZFC. The class $\boldsymbol{K}_{-1}$ plays no role in section. The arguments here are similar to those around page 7 of [GS05]. In Subsection 2.2 we deduce Theorem 2.2.4 from $\boxplus$, showing there is a nicely free P_0-maximal (Definition 2.1) model in $\boldsymbol{K}_{-1}$.

Definition 2.1.1 **($\boxplus(\lambda)$)** *denotes: There are a Boolean algebra $\mathbb{B} \subset \mathcal{P}(\lambda)$ with $|\mathbb{B}| = \lambda$ and a set $\mathcal{A} \subseteq {}^\omega\mathbb{B}$ such that:* {boxplus}

[1]The subsets of P_0^M are *not* elements of M.

[2]Read $\forall^\infty$ as 'for all but finitely many'.

i) $\mathcal{A}$ *has cardinality* λ *and if* $\overline{A} = \{A_n : n \in \omega\} \in \mathcal{A}$ *then for* $\alpha < \lambda$ *for all but finitely many* n, $\alpha \notin A_n$.

ii) $\mathbb{B}$ *includes the finite subsets of* λ*; but is such that for every non-principal ultrafilter* D *of* λ *(equivalently of* $\mathbb{B}$ *and disjoint from* $\lambda^{<\omega}$*) for some sequence* $\langle A_n : n \in \omega\rangle \in \mathcal{A}$*, there are infinitely many* n *with* $A_n \in D$.

We may say that $(\mathbb{B}, \mathcal{A})$ witness uniform $\aleph_1$-incompleteness.

{boxthm}

Lemma 2.1.2 (ZFC) *Assume for some* μ, $\lambda = 2^\mu$ *and* $\lambda <$ *first measurable, then* $\boxplus$ *of 2.1.1 holds.*

We need the following structure.

{f12.5}

Definition 2.1.3 *1. Fix the vocabulary* σ *with unary predicates* P, U*, a binary predicate* C*, and a binary function* F_2.

2. *Let* $\langle C_\alpha : \alpha < \lambda\rangle$ *list* without repetitions $\mathcal{P}(\mu)$ *such that* $C_0 = \emptyset$ *and also let* $\langle f_\alpha : \mu \leq \alpha < \lambda\rangle$ *list* ${}^\mu\omega$.

3. *Define the* σ*-structure* M *by:*

   (a) The universe of M *is* λ*;*

   (b) $P^M = \omega$*;* $U^M = \mu$,

   (c) $C(x, y)$ *is binary relation on* $U \times M$ *defined by* $C(x, \alpha)$ *if and only* $x \in C_\alpha$. *Note that* C *is extensional. I.e., elements of* M *uniquely code subsets of* U^M.

   (d) Let $F_2^M(\alpha, \beta)$ *map* $M \times U^M \to P^M$ *by* $F_2^M(\alpha, \beta) = f_\alpha(\beta)$ *for* $\alpha < \lambda$, $\beta < \mu$*;*

   (e) $F_2^M(\alpha, \beta) = 0$ *for* $\alpha < \lambda$ *and* $\beta \in [\mu, \lambda)$.

We use the following, likely well-known, fact pointed out to us by Sherwood Hachtman.

{hacht}

Fact 2.1.4 *Let* $D \subseteq \mathcal{P}(X)$ *and suppose that for each partition* $Y \subseteq \mathcal{P}(X)$ *of* X *into at most countably many sets,* $|D \cap Y| = 1$*. Then,* D *is a countably complete ultrafilter.*

We need the following lemma about M before finding in M a representative of $\boxplus$.

{f12.7}

Lemma 2.1.5 *If* λ *is less than the first measurable cardinal and* $\lambda = 2^\mu$ *for some* μ *there is a model* M*, with* $|M| = \lambda$*, and a countable vocabulary with* P^M *denoting the natural numbers such that every first order proper elementary extension of* M *properly extends* P^M.

Proof. To show the M in Definition 2.1.3 satisfies Lemma 2.1.2, we first show that any proper elementary extension of M extends U^M. Suppose for contradiction there exists $\alpha' \in N - M$ but $U^N = U^N$. By the full listing of the C_α, there is a $\beta \in V^M$

with $\{x : N \models C(x,\beta)\} = \{x : N \models C(x,\alpha')\}$. This contradicts extensionality of the relation C in N; but C is extensional in the elementary submodel M.

Now we show that if $U^M \subsetneq U^N$ and $P^M = P^N$, then there is a countably complete non-principal ultrafilter on μ, contradicting that μ is not measurable. Note that the sequence $\langle f_\alpha \colon \mu \leq \alpha < \lambda\rangle$ can be viewed as a list of all non-trivial partitions of μ into at most countably many pieces. Let $\nu^* \in U^N - U^M$. For $\alpha \in N$, denote $F_2^N(\alpha,\nu^*)$ by n_α. Since $P^M = P^N$, $n_\alpha \in M$. By elementarity, for $\alpha \in M, \eta \in U^M$, $F_2^N(\alpha,\eta) = F_2^M(\alpha,\eta) = f_\alpha(\eta)$. Now, let

$$D = \{x \subseteq U^M : x \neq \emptyset \wedge (\exists \alpha \in M)\; x \supseteq f_\alpha^{-1}(n_\alpha)\}.$$

We show D satisfies the conditions from Fact 2.1.4. Let W be a partition, indexed by f_α. Then $f_\alpha^{-1}(n_\alpha) \neq \emptyset$ and is in D. Suppose for contradiction there are $x_0 \neq x_1$ in W that are both in D. Then, there are $\alpha_i \in M$ such that $x_i \in W \cap D$ contains $f_{\alpha_i}^{-1}(n_{\alpha_i})$ for $i = 0, 1$. So, $N \models F_2(\alpha_i,\nu^*) = n_{\alpha_i}$ for $i = 1, 2$. Since $\alpha_i \in M$ and $M \prec N$, $M \models \exists x(F_2(\alpha_0,x) = n_{\alpha_0} \wedge F_2(\alpha_1,x) = n_{\alpha_1}$. So, by Definition 2.1.3 (d), for any witness a in M for this formula, $a \in x_0 \cap x_1$; but $x_0 \cap x_1 = \emptyset$ since W is a partition.

Finally, D is non-principal on U^M since if it were generated by an $a \in U^M$,

$$D = \{x \subseteq U : (\exists \alpha)\; x \supseteq f_\alpha^{-1}(n_\alpha)\} = \{x \subseteq U : a \in x\}.$$

Since $\{a\} \in D$, for some $\alpha_0 \in M$, $\{a\} = f_{\alpha_0}^{-1}(n_{\alpha_0})$. Note that $\alpha_0 \in M$, because the definition of D is about the model M. That is, $M \models \exists! y F_2(\alpha_0,y) = n_{\alpha_0}$. But $N \models F_2(\alpha_0,a) = n_{\alpha_0} \wedge F_2(\alpha_0,\nu^*) = n_{\alpha_0}$. This contradicts the assumption $M \prec N$ and completes the proof. $\square_{2.1.5}$

The following claim completes the proof of Lemma 2.1.2

{f12.8}

Claim 2.1.6 *If $\mathbb{B}$ is the boolean algebra of definable formulas in the M defined in Definition 2.1.3, there is an $\mathcal{A}$ such that $(\mathbb{B},\mathcal{A})$ is uniformly $\aleph_1$-incomplete so $\boxplus(\lambda$.*

Proof. We may assume τ has Skolem functions for M and then define $\mathbb{B}$ and $\mathcal{A}$ as follows to satisfy $\boxplus$.b. Let $\mathbb{B}$ be the Boolean algebra of definable subsets of M.

$$\mathbb{B} = \{X \subseteq M \colon \text{ for some } \tau\text{-formula } \phi(\mathbf{x},\mathbf{y}) \text{ and } \mathbf{b} \in {}^{\lg(\mathbf{y})}M,\; \phi(M,\mathbf{b}) = X.\}$$

Note $\mathbb{B}$ is a Boolean algebra of cardinality λ with the normal operations. We define the Skolem functions a little differently than usual; as maps σ_ϕ from M^{n+1} to M for formulas $\phi(x,w,\mathbf{y})$ such that $\phi(\sigma_{\phi(x,w,\mathbf{y})}(b,\boldsymbol{a}),b,\boldsymbol{a})$. However, we specialize the Skolem functions by considering the unary function arising from fixing the $\mathbf{y}$ entry of $\sigma_\phi(w,\mathbf{y})$ to obtain $\sigma_\phi(w,\boldsymbol{a})$.

$$\begin{aligned} A_n^{\sigma_\phi(x,w,\boldsymbol{a})} &= \{\alpha < \lambda \colon \phi(\sigma_\phi^M(\alpha,\boldsymbol{a}),\alpha,\boldsymbol{a}) \wedge P(\sigma_\phi^M(\alpha,\boldsymbol{a})) \wedge \sigma_\phi^M(\alpha,\boldsymbol{a}) \not\leq n\} \\ &\cup \{\alpha < \lambda \colon n = 0 \wedge \neg P(\sigma_\phi^M(\alpha,\boldsymbol{a})). \end{aligned}$$

and then letting $\overline{A}_{\sigma_\phi(x,w,\boldsymbol{a})} = \langle A_n^{\sigma_\phi(w,\boldsymbol{a})} \colon n < \omega\rangle$

$$(*)\quad \mathcal{A} = \{\overline{A}_{\sigma_\phi(x,\boldsymbol{a})} : \text{ for some } \tau_M - \text{term } \sigma_\phi(x,w,\mathbf{y}) \text{ and } \boldsymbol{a} \in {}^{\lg(\mathbf{y})}M.\}$$

For each α, for each $0 < m < \omega$ and $\overline{A} = \overline{A}_{\sigma_\phi(x,\alpha,\mathbf{b})}$, the set $\{m \colon \alpha \in A_m\}$ is finite, bounded by $\sigma_\phi(\alpha, \boldsymbol{a})$. Thus, clause i) of $\boxplus$ is satisfied.

We now show Clause ii) of $\boxplus$. Let D be an arbitrary non-principal ultrafilter on λ and where $\psi(v, \mathbf{y})$ is a first order τ-formula and $\mathbf{y}, \boldsymbol{a}$ have the same length, define the type $p_D(x) = p(x)$ as:

$$p(x) = \{\psi(v, \boldsymbol{a}) \colon \{\alpha \in M \colon M \models \phi(\alpha, \boldsymbol{a})\} \in D\}.$$

Since D is an ultrafilter, p is a complete type over M. So there is an elementary extension N of M where an element d realizes p. Let N be the Skolem hull of $M \cup \{d\}$. Since D is non-principal, so is p; thus, $N \neq M$. By Lemma 2.1.5, we can choose $c \in P^N - P^M$. Since, N is the Skolem hull of $M \cup \{d\}$ there is a Skolem term $\sigma(u, \mathbf{y})$ and $\boldsymbol{a} \in M$ such that $c = \sigma^N(d, \boldsymbol{a})$. Since $c \notin M$, for each $n \in P^M$, $N \models \bigwedge_{k<n} c \neq k$ so $N \models \bigwedge_{k<n} \sigma(d, \boldsymbol{a}) \neq k$ so $\bigwedge_{k<n} \sigma(x, \boldsymbol{a}) \neq k$ is in p. That is, for each n, $A_n^{\sigma_\phi(x,w,\boldsymbol{a})}$ is in D.

2.2 A P_0-maximal model in $\boldsymbol{K}_{-1}$

{mtconst}

In this section we prove Theorem 2.2.4, invoking Theorem 2.1.2. To even state the new result, we need some new definitions as well as recalling Definition 1.2.7.

{f2*}

Definition 2.1 *We say $M \in K_{-1}$ is P_0-maximal (in K_{-1}) if $M \subseteq N$ and $N \in K_{-1}$ implies $P_0^M = P_0^N$.*

We now introduce the requirement that the Boolean algebras constructed will, when the atoms are factored out, be free. Moreover, different $c \in P_2^N$ generate disjoint sets $F_n^N(c)$ as c varies. This strong requirement is used inductively in Section 2.2 to construct the first approximation. The correction in Section 3 loses this disjointness (and thus freeness).

{b9}

Definition 2.2 (Nicely Free) *We say $M \in K_{-1}$ is nicely free when $|P_1^M| = \lambda$ and there is a $\mathbf{b} = \langle b_\alpha : \alpha < \lambda \rangle$ such that*

(a) $b_\alpha \in P_1^M - P_4^M$;

(b) $\langle b_\alpha / P_4^M : \alpha < \lambda \rangle$ generate P_1^M / P_4^M freely;

(c) there is a set $Y \subset P_2^M$ of cardinality λ and a sequence $\langle u_c \colon c \in Y \rangle$ of pairwise disjoint sets of distinct ordinals such that, with $u_c = \{F_n^M(c) \colon n < \omega\}$ the collection u_c partition a subset of the basis $\langle b_\alpha : \alpha < \lambda \rangle$.

{f5}

Definition 2.2.1 ($\mathrm{uf}(M)$) *For $M \in K_{-1}$, let $\mathrm{uf}_1(M) = \mathrm{uf}(M)$ be the set of ultrafilters D of the Boolean Algebra P_1^M such that $D \cap P_{4,1}^M = \emptyset$ and for each $c \in P_2^M$ only finitely many of the $F_n^M(c)$ are in D.*

For applications we rephrase this notion with the following terminology. For any $M \in \boldsymbol{K}_{-1}$ and $d \in P_2^M$, let $S_d^M(D) = \{n : F_n^M(d) \in D\}$. So $\mathrm{uf}(M) = \emptyset$ if and only if for every ultrafilter D on P_1^M, there exists a $d \in P_2^M$ such that $S_d^M(D)$ is infinite.

We use the following standard properties of a Boolean algebra B and ideal I in proving Lemma 2.2.3 and Claim 2.2.7 from Definition 2.2.6.

{quotprop}

Fact 2.2.2 *1. $b \wedge c \in I$ implies b/I and c/I are disjoint.*

2. $b \triangle c \in I$ implies $b/I = c/I$.

3. $b - c \in I$ implies $b/I \leq c/I$.

For our collection of structures $\boldsymbol{K}_{-1}$, we can characterize P_0-maximality in terms of ultrafilters.

{f8}

Lemma 2.2.3 *An $M \in \boldsymbol{K}_{-1}$ is P_0-maximal if and only if* $\mathrm{uf}(M) = \emptyset$.

Proof. Suppose M is extended by an element $d \in P_0^N$. Then $\{b \in M : R^N(d,b)\}$ is an ultrafilter D_0 of the Boolean algebra P_1^M. To see D_0 is non-principal suppose there is a $b_0 \in P_1^M$ such that $D_0 = \{b \in M : b_0 \leq b\}$. Note $b_0 = G_1^M(a)$ for some $a \in P_0^M$. But $N \models G_1^N(d) \not\geq b_0$.

For each $c \in P_2^M$, since $N \in \boldsymbol{K}_{-1}$, by clause 7 of Definition 1.2, for all $a \in P_0^N$ and all but finitely many n, $G_1^N(a) \not\leq F_n^N(c)$. Since $F_n^N(c) = F_n^M(c)$ only finitely many of the $F_n^M(c)$ can be in D_0, which implies $D_0 \in \mathrm{uf}(M)$. By contraposition we have the right to left.

Conversely, if $D \in \mathrm{uf}(M)$, we can construct an extension by adding an element $d \in P_0^N$ satisfying $R^N(d,b)$ iff $b \in D$. Let P_1^N be the Boolean algebra generated by $P_1^N \cup \{G_1(d)\}$ modulo the ideal generated by $\{G_1^N(d) - b : b \in D\}$; this implies that in the quotient $G_1(d) \leq b$. (Compare Fact 2.2.2). It is easy to check $N \in \boldsymbol{K}_{-1}$. $\square_{2.2.3}$

Here is the main theorem of Section 2. In the following proof the hypotheses $\lambda = 2^\mu$ and λ is less than the first measurable cardinal are used essentially as the hypotheses for proving $\boxplus$, the existence of a uniformly $\aleph_1$-incomplete boolean algebra. The rest of the argument depends on $\lambda = \lambda^{\aleph_0}$, which follows from $\lambda = 2^\mu$. Recall Definition 2.1 of P_0-maximal. By constructing a nicely free model, we introduce the independence requirements on the $F_n(c)$ at this stage.

{f11a}

Theorem 2.2.4 *If for some μ, $\lambda = 2^\mu$ and λ is less than the first measurable cardinal then there is a P_0-maximal model M in $\boldsymbol{K}_{-1}$ such that $|P_i^M| = \lambda$ (for $i = 0,1,2$), P_1^M is an atomic Boolean algebra,* $\mathrm{uf}(M) = \emptyset$*, and M is nicely free.*

Proof. We first construct by induction a model in $\boldsymbol{K}^{-1}$. The hypothesis $\boxplus$ appears in the construction in Specification f) and in the proof that the construction works in considering possibility 2. We choose M_ϵ, D_ϵ and other auxiliaries by induction for $\epsilon \leq \omega + 1$ to satisfy the following *specifications* of the construction.

{oplus}

Construction 2.2.5 (Specifications) *(a) For $\epsilon \leq \omega+1$, M_ϵ is a continuous increasing chain of members of K_λ^{-1} with each P_1^M atomic and $P_1^{M_{\omega+1}} = P_1^{M_\omega}$;*

{clb}

(b) For all $\epsilon \leq \omega$, $|P_i^{M_\epsilon}| = \lambda$ and $P_i^{M_\omega} = P_i^{M_{\omega+1}}$ for $i = 0,1$;

{clc}

(c) For all $\epsilon \leq \omega$, $P_1^{M_\epsilon}/P_4^{M_\epsilon}$ is a free Boolean algebra;

{cld}

(d) *(i) If* $\epsilon < \omega$, $D_\epsilon \in \mathrm{uf}(M_\epsilon)$.

(ii) If $\epsilon = 0$, *then* $\langle b_{-1,\alpha} \colon \alpha < \lambda \rangle$ *is a free basis of* $P_1^{M_0}/P_4^{M_0}$, *listed without repetition, and* $\langle F_n^{M_0}(c) \colon n < \omega, c \in P_2^{M_0} \rangle$ *lists* $\langle b_{-1,\alpha} \colon \alpha < \lambda \rangle$ *without repetition.*

(iii) if $\epsilon = \zeta + 1 < \omega$ *then there is a free basis* $\langle b_{\zeta,\alpha}/P_4^{M_\zeta} \colon \alpha < \lambda \rangle$ *of* $P_1^{M_\epsilon}/P_4^{M_\epsilon}$ *over* $P_1^{M_\zeta}/P_4^{M_\zeta}$. *Note* $b_{\zeta,\alpha} \in P_1^{M_\epsilon} - P_1^{M_\zeta}$.

{cle}

(e) if $\epsilon = \omega + 1$, *for each* $\overline{d} \in {}^\omega(P_1^{M_{\omega+1}} - P_4^{M_{\omega+1}})$ *such that for each* $a \in P_0^{M_\omega}$ *for all but finitely many* n, $a \notin R(M_\omega, d_n)$, *then for some* $c \in P_2^{M_{\omega+1}}$, $F_n^{M_{\omega+1}}(c) = d_n$; *(We will in fact have that* $P_1^{M_{\omega+1}} = P_1^{M_\omega}$ *and* $P_4^{M_{\omega+1}} = P_4^{M_\omega}$.*)*

{clf}

(f) $\epsilon = \zeta + 1 < \omega$:

Let $\mathbb{B}$ *and* $\mathcal{A}$ *be as in Definition 2.1.1. There is a* 1-1 *function* f_ϵ *from* λ *onto* $P_{4,1}^{M_\epsilon}$ *such that:*

i) for every $X \in \mathbb{B}$ *(from* $\boxplus$*) there is a* $b = b_X \in P_1^{M_\epsilon}$ *such that*

$$\{\alpha < \lambda \colon f_\epsilon(\alpha) \leq_{M_\epsilon} b_X\} = \{\alpha < \lambda \colon \alpha \in X\};$$

ii) for each $\overline{A} = \langle A_n : n < \omega \rangle \in \mathcal{A}$ *there is a* $c \in P_2^{M_\epsilon}$ *such that for each* n:

$$A_n = \{\alpha < \lambda \colon f_\epsilon(\alpha) \leq_{P_1^{M_\epsilon}} F_n^{M_\epsilon}(c)\}.$$

Carrying out the construction.

Below, the element b_{ζ,a_α} is the b_{A_α} from Specification 2.2.5.f.(i).

case 1: When $\epsilon = 0$, take $P_1^{M_0}$ as the Boolean algebra generated by a set $P_{4,1}^{M_0}$ of cardinality λ along with a set $\{b_{-1,\alpha} \colon \alpha < \lambda\}$ of independent subsets of $\mathcal{P}(\lambda)$. Let G_1 be a bijection between a set $P_0^{M_0}$ and $P_{4,1}^{M_0}$. Set $P_4^{M_0}$ as the ideal generated by the image of G_1. For $a \in P_0^{M_0}$ and $b \in P_1^M$, define $R^{M_0}(a,b)$ to hold if $G_1(a) \leq b$. Set $P_2^{M_0} = \emptyset$ and so there are no F_n^M to define. Thus, any non-principal ultrafilter on $P_1^{M_0}$ is in $\mathrm{uf}(M_0)$.

case 2: For $\epsilon = \omega$, $M_\omega = \bigcup_{n<\omega} M_n$.

case 3: If $\epsilon = \zeta + 1 < \omega$, the main effort is to verify clauses (c), (d), and (f) of Specification 2.2.5.

Now, to construct M_ϵ:

i Recall that $D_\zeta \in \mathrm{uf}(M_\zeta)$.

ii choose a set $B_\epsilon \subseteq \mathcal{P}(\lambda)$; with $B_\epsilon \cap M_\zeta = \emptyset$ and $|B_\epsilon| = \lambda$ as the new atoms introduced at this stage.

iii Let f_ϵ be a one-to-one function from λ onto $B_\epsilon \cup P_{4,1}^{M_\zeta}$.

iv Let $\langle X_\gamma \colon \gamma < \lambda \rangle$ list the elements of $\mathbb{B}$ from $\boxplus$.(ii) with $X_0 = \emptyset$.

v Fix a sequence $\{b_{\zeta,\alpha} : \alpha < \lambda\}$, which are distinct and not in $M_\zeta \cup B_\epsilon$, and let $\mathbb{B}'_\zeta$ be the Boolean Algebra generated freely by $P_1^{M_\zeta} \cup \{b_{\zeta,\alpha} : \alpha < \lambda\} \cup \{f_\epsilon(\alpha) : \alpha < \lambda\}$.

Using Lemma 2.2.2, we apply the following definition at the successor stage.

{defI}

Definition 2.2.6 (Ideal) *Let I_ζ be the ideal of $\mathbb{B}'_\zeta$ generated by:*

(i) $\sigma(a_0, \dots a_m)$ *when* $\sigma(x_0, \dots x_m)$ *is a Boolean term,* $a_0, \dots a_m \in P_1^{M_\zeta}$ *and* $P_1^{M_\zeta} \models \sigma(a_0, \dots a_m) = 0$.

The next two clauses aim to show that in M_ζ/I_ζ, the element $b_{\zeta,\gamma}$ is the b_{X_γ} from Specification 2.2.5 f.i). That is, $\{\alpha < \lambda : f_\epsilon(\alpha) \leq_{M_\epsilon} X_\gamma\} = \{\alpha < \lambda : \alpha \in X_\gamma\}$. Recall (Definition 2.1.1) that the X_γ enumerate $\mathbb{B}$ and are subsets of λ.

(ii) $f_\epsilon(\alpha) - b_{\zeta,\gamma}$ *when* $\alpha \in X_\gamma$ *and* $\alpha, \gamma < \lambda$.

(iii) $b_{\zeta,\gamma} \wedge f_\epsilon(\alpha)$ *when* $\alpha \in \lambda - X_\gamma$ *and* $\alpha, \gamma < \lambda$.

To show the $f_\epsilon(\gamma)$ are disjoint atoms we add:

(iv) For any $f_\epsilon(\gamma)$ and any $b \in \mathbb{B}'_\zeta$ either $(f_\epsilon(\gamma) \wedge b) \in I_\zeta$ or $(f_\epsilon(\gamma) - b) \in I_\zeta$.

(v) $f_\epsilon(\gamma_1) \wedge f_\epsilon(\gamma_2)$ *when* $\gamma_1 < \gamma_2 < \lambda$*;*

(vi) $f_\epsilon(\alpha) - b$ *when* $\alpha < \lambda$, $f_\epsilon(\alpha) \notin P_{4,1}^{M_\zeta}$ *and* $b \in D_\zeta$.

This asserts: Every new atom is below each $b \in D_\zeta$ and is used at the end of case 3 of the construction.

Let $\mathbb{B}''_\zeta = \mathbb{B}'_\zeta / I_\zeta$. Applying Fact 2.2.2, we see from Definition 2.2.6:

{succ}

Claim 2.2.7 *The structure $P_1^{M_\zeta}$ is embedded as a Boolean algebra into $\mathbb{B}''_\zeta$ by the map $b \mapsto b/I_\zeta$ and*

1. *For $\gamma < \lambda$, $f_\epsilon(\gamma)/I_\zeta$ is an atom of $\mathbb{B}''_\zeta$;*
2. *If $b \in P_1^{M_\zeta}$ is non-zero, then $bI_\zeta \geq_{\mathbb{B}''_\zeta} f_\epsilon(\gamma)$ for some $\gamma < \lambda$.*

We take a further quotient of $P_1^{M_\zeta}$. Let

$$J_\zeta = \{b \in P_1^{M_\zeta} : b/I_\zeta \wedge_{\mathbb{B}''_\zeta} f_\epsilon(\gamma) = 0 \text{ for every } \gamma < \lambda\}.$$

Then J_ζ is an ideal of $P_1^{M_\zeta}$ extending I_ζ so $b \mapsto b/J_\zeta$ is a homomorphism. Further, $f_\epsilon(\gamma)$ is an atom of $P_1^{M_\zeta}/J_\zeta$ for $\gamma < \lambda$. These atoms are distinct and dense in $P_1^{M_\zeta}/J_\zeta$.

Notation 2.2.8 *Let $\mathbb{B}_\epsilon$ be $P_1^{M_\zeta}/J_\zeta$.*

Now we define M_ϵ by setting $P_1^{M_\epsilon} = \mathbb{B}_\epsilon$ which contains $P_1^{M_\zeta}$. $P_{4,1}^{M_\epsilon}$ is the injective image in $P_1^{M_\epsilon}$ of $P_{4,1}^{M_\zeta} \cup B_\epsilon$. For $a \in P_{4,1}^{M_\epsilon}$ and $b \in P_1^{M_\epsilon}$ set $R^{M_\epsilon}(a,b)$ if some γ, $a = f_\epsilon(\gamma)/J_\epsilon$ and $f_\epsilon(\gamma)/J_\epsilon \leq_{\mathbb{B}_\epsilon} b/J_\zeta$. Finally, let D_ϵ be the ultrafilter on $P_1^{M_\epsilon}$ generated by

$$D_\zeta \cup \{j_\epsilon(-b_{\zeta,\gamma}) \colon \gamma < \lambda\} \cup \{j_\epsilon(-f_\epsilon(\gamma)) \colon \gamma < \lambda\}.$$

By Claim 2.2.7, we have the cardinality and atomicity conditions of Specification 2.2.5.(a) and (b); the definition of I guarantees, (c) and (d).(ii), (d).(iii). We verify $M_\epsilon \in \boldsymbol{K}_{-1}$ below. The first set of new elements in D_ϵ show along with (our later) definition of $F_n^{M_\epsilon}(c)$ show $D_\epsilon \in \mathrm{uf}(M_\epsilon)$ (as no new $F_n(c$ is in D_ϵ); the second set show D_ϵ is non-principal. Note that Specification 2.2.5.(e) does not apply except in the $\omega + 1$st stage of the construction.

For Specification 2.2.5 (f) (i), let X be a set of atoms of M_ϵ and note that we can choose b_X by conditions ii) and iii) in Definition 2.2.6 of I_ζ.

We can choose $P_2^{M_\epsilon}$ and $F_n^{M_\epsilon}$ to satisfy Specification 2.2.5 (f) (ii). Fix an $\overline{A} \in \mathcal{A}$ (as given by $\boxplus$). Fix a $c = c_{\overline{A}}$ and define, using the last paragraph, the $F_n^{M_\epsilon}(c)$ as b_{A_n}, so that for each n, $A_n = \{\alpha < \lambda : f_\epsilon(\alpha) \leq_{P_1^{M_\epsilon}} F_n^{M_\epsilon}(c)\}$. These are the only new $c \in P_2^{M_\epsilon}$.

Thus, it remains only to show that $M_\epsilon \in \boldsymbol{K}_{-1}$. I.e., that M_ϵ satisfies Definition 1.2.(7):

(⧫) If $a \in P_{4,1}^{M_\epsilon}$ and $c \in P_2^{M_\epsilon}$ then $(\forall^\infty n)\ a \not\leq_{M_\epsilon} F_n^{M_\epsilon}(c)$.

If $c \in P_2^{M_\zeta}$, $F_c^{M_\epsilon} = F_c^{M_\zeta} \in P_1^{M_\zeta}$ and we know by induction that ⧫ holds for $a \in P_{4,1}^{M_\zeta}$. For $a \in P_{4,1}^{M_\epsilon} - P_{4,1}^{M_\zeta}$, Definition 1.2.5. and condition (v) on I_ζ (from Definition 2.2.6) imply $a \leq_{M_\epsilon} b$ for every $b \in D_\zeta$. As $c \in P_2^{M_\zeta}$, since $D_\zeta \in \mathrm{uf}(M_\zeta)$ all but finitely many $F_n^\zeta(c)$, call them e_n, are *not* in D_ζ. So $e_n^- \in D_\zeta$. That is, $a \leq_{M_\epsilon} e_n^-$; so $a \wedge_{M_\epsilon} e_n = \emptyset$ as required.

If $c \in P_2^{M_\epsilon} - P_2^{M_\zeta}$ then by our choice of $P_2^{M_\epsilon}$ and the $F_2^{M_\epsilon}$, there is an $\overline{A}_c$ that is enumerated by the $F_2^{M_\epsilon}(c)$ and satisfies ⧫ by (i) of $\boxplus$ (Definition 2.1.1.(i)). This completes the verification of ⧫ at stage ϵ and the M_ϵ satisfies all the specifications of the induction.

<u>case 4:</u> $\epsilon = \omega + 1$:

Only clauses (b) and (e) of Specification 2.2.5 are relevant. Define $P_2^{M_\epsilon}$ and $F_n^{M_\epsilon}$ to satisfy clause (e). Since $P_i^{M_\epsilon} = P_i^{M_\omega}$ for $i = 0, 1$, specification c) is immediate. This completes the construction.

The construction suffices.

Having completed the induction, let $M = M_{\omega+1}$. Using specifications c) and d) of 2.2.5, it is straightforward to verify that $M \in \boldsymbol{K}_{-1}$ and the Boolean algebra is atomic. By (b), $M_i^{M_\omega}$ for $i = 0, 1$ have cardinality λ. And by (f), the same holds for $M_2^{M_{\omega+1}}$.

We now show M is nicely free. Let $\mathbf{b} = \langle b'_\beta \colon \beta < \lambda\rangle$ enumerate $\langle b_{n,\alpha} \colon n < \omega, \alpha < \lambda\rangle$ without repetition. We show $\mathbf{b}$ satisfies the requirements in Definition 2.2 of nicely free. By Specifications 2.2.5. (c), (d) and since P_1^M is constructed as the union of the $P_1^{M_n}$, P_1^M/P_4^M is generated freely by $\mathbf{b}$. Finally, clause c) of Definition 2.2 holds by clause (d).ii) of Specification 2.2.5.

The crux is to show $M = M_{\omega+1}$ is P_0-maximal. For this, assume for a contradiction:

(*) P_0^M is not maximal; thus, by Lemma 2.2.3, there is a $D \in \mathrm{uf}(M_{\omega+1}) = \mathrm{uf}(M_\omega)$.

For every $n < \omega$, we ask the following question: is there a $d \in D$ such that $R(M_\omega, d) \cap M_n = \emptyset$?

Possibility 1 : For every $n < \omega$, the answer is yes, exemplified by $d_n \in D$. Now for each $a \in P_0^{M_n}$, $a \notin R(M_\omega, d_m)$ for all $m \geq n$. So the sequence $\overline{d} = \langle d_n : n < \omega \rangle$ satisfies the hypothesis of Specification 2.2.5.(e) and so there is a $c \in P_2^M$ such that for each $n < \omega$, $F_n^M(c) = d_n$. Thus, recalling Definition 2.2.1, $D \notin \mathrm{uf}(M)$.

Possibility 2 : For some $n < \omega$, there is no such d_n; without loss of generality, assume $n > 0$. We apply specification f) with $\epsilon = n$. Recall that f_n is a 1-1 map from λ onto $P_{4,1}^{M_n}$. Let g_1 be the following homomorphism from the Boolean algebra $P_1^{M_{\omega+1}} = P_1^{M_\omega}$ into $\mathcal{P}(\lambda) : g_1(b) = \{\alpha < \lambda : f_n(\alpha) \leq_{\mathbb{B}_{M_\omega}} b\}$. By Specification f.i) of 2.2.5, the Boolean algebra $\mathbb{B}$ provided by $\boxplus$ is contained in the range of g_1.

Let $\mathcal{I}_n$ denote the ideal of P_1^M generated by $P_{4,1}^M - P_{4,1}^{M_n}$. Since D is non-principal, $\mathcal{I}_n \cap D = \emptyset$. Now, g_1 maps any $b \in P_1^{M_\omega} - P_4^{M_\omega}$ (and, thus, any $b \in P_1^{M_\omega} - \mathcal{I}_n$) to a nonempty subset of λ. Recalling $\mathcal{I}_n \cap D = \emptyset$, g_1 embeds the quotient algebra $P_1^{M_{\omega+1}}/\mathcal{I}_n$ into the Boolean Algebra $\mathcal{P}(\lambda)$. Hence, $D_1 = g_1''(D)$ is an ultrafilter of the Boolean Algebra $\mathrm{rg}(\mathrm{g}_1)$ and so $D_2 = D_1 \cap \mathbb{B}$ is an ultrafilter of the Boolean algebra $\mathbb{B}$. Now, for any $\alpha < \lambda$, $\{\alpha\} \notin D_2$ because $f_n(\alpha) \in P_{4,1}^{M_\omega}$ and so $\{f_n(\alpha)\}$ is not in D. So $\{\alpha\} \notin D_1$. Thus, $\lambda - \{\alpha\} \in D_1$ and so $\lambda - \{\alpha\} \in D_2$. So $\{\alpha\} \notin D_2$ as promised.

Now we apply the second clause of $\boxplus$ to the ultrafilter D_2. Since we satisfied specification f.ii) in the construction, we can conclude there is $\overline{A} = \langle A_n : n < \omega \rangle \in \mathcal{A}$ such that for infinitely many k, A_k is in D_2. Thus, $u = \{k \colon A_k \in D\}$ is infinite. We will finish the proof by showing there is a c such that $u = u_c$ (Definition 2.2) is the set of images of the $F_n^M(c)$.

Since each $A_k \in \mathbb{B}$, $A_k \in \mathrm{rg}(\mathrm{g}_1)$. So we can choose $d_k \in P_1^{M_\omega}$ with $g_1(d_k) = A_k$. As $A_k \in D$, by the choice of D_1, D_2 we have d_k is in the ultrafilter D from the hypothesis for contradiction: (*).

We show the sequence $\overline{d} = \langle d_k : k < \omega \rangle$ satisfies the hypothesis of clause e of Specification 2.2.5. First, $d_k \in P_1^{M_\omega} - P_4^{M_\omega}$ as D is a non-principal ultrafilter on $P_1^{M_\omega}$ so the first hypothesis is satisfied. Further, for every $a \in P_0^{M_\omega}$ all but finitely many k, $G_1^{M_\omega}(a) \nleq_{M_\omega} d_k$ because $\overline{A} \in \mathcal{A}$, which implies by $\boxplus$ ii) that for every $\alpha < \lambda$, for some k_α, we have $k \geq k_\alpha$ implies $\alpha \notin A_k$. Now by the definition of g_1, recalling $g_1(d_k) = A_k$, we have $k \geq k_\alpha$ implies $f_k(\alpha) \nsubseteq d_k$ (in $P_1^{M_\omega}$). So by Specification 2.2.5. f.ii), there is a $c \in P_2^{M_n}$ such that if for all $k < \omega$, $F_k^{M_n}(c) = d_k$. So, for each finite k, $d_k \in D$ and $F_k^{M_{\omega+1}}(c) = d_k$. This contradicts $D \in \mathrm{uf}(M_{\omega+1})$ and we finish. $\square_{2.2.4}$

3 Correcting M_* to a model of $\boldsymbol{K}_2$

{corr}

We now 'correct' the structure M_* constructed in Section 2 to obtain a P_0-maximal model (Definition 2.1) of the complete sentence constructed in [BS17a]. In Theorem 3.18 we modify M_*, to construct a model $M \in \boldsymbol{K}_2$ with $P_2^M \subseteq P^{M_*}$ and redefining the F_n^M, but retaining $M \restriction (P_0^M \cup P_1^M) = M_* \restriction (P_0^{M_*} \cup P_1^{M_*})$. The old values of $F_n^{M_*}$ will be used to divide the work of ensuring each ultrafilter D is not in $\mathrm{uf}(M)$ by for each D attending to only those c with infinitely many $F_n(c)$ in D.

For this we need to introduce some terminology from [BS17a]. We first describe the finitely generated models.

{k0}

Definition 3.1 ($\boldsymbol{K}^1_{<\aleph_0}$ Defined) *M is in the class of structures $\boldsymbol{K}^1_{<\aleph_0}$ if $M \in \boldsymbol{K}^{-1}_{\aleph_0}$ and there is a* witness *$\langle n_*, \boldsymbol{B}, b_*\rangle$ such that:*

1. *$b_* \in P_1^M$ is the supremum of the finite joins of atoms in P_1^M. Further, for some k, $\bigcup_{j\le k} P^M_{4,j} = \{c : c \le b_*\}$ and for all $n > k$, $P^M_{4,n} = \emptyset$.*
2. *$\boldsymbol{B} = \langle B_n : n \ge n_*\rangle$ is an increasing sequence of finite Boolean subalgebras of P_1^M.*
3. *$B_{n_*} \supsetneq \{c \in P_1^M : c \le b_*\} = P_4^M$; it is generated by the subset $P_4^M \cup \{F_n^M(c) \colon n < n_*, c \in P_2^M\}$.*

   Moreover, the Boolean algebra B_{n_*} is free over the ideal P_4^M (equivalently, B_{n_*}/P_4^M is a free Boolean algebra[3]).
4. *$\bigcup_{n\ge n_*} B_n = P_1^M$.*
5. *P_2^M is finite and not empty. Further, for each $c \in P_2^M$ the $F_n^M(c)$ for $n < \omega$ are independent over $\{0\}$.*
6. *The set $\{F_m(c) : m \ge n_*, c \in P_2^M\}$ (the enumeration is without repetition) is free from B_{n_*} over $\{0\}$. $B_{n_*} \supsetneq P_4^M$ and $F_m(c) \wedge b_* = 0$ for $m \ge n_*$. (In this definition, $0 = 0^{P_1^M}$.)*

   In detail, let $\sigma(\ldots x_{c_i}\ldots)$ be a Boolean algebra term in the variables x_{c_i} (where the c_i are in P_2^M which is not identically 0. Then, for finitely many $n_i \ge n_*$ and a finite sequence of $c_i \in P_2^M$:

   $$\sigma(\ldots F_n(c_i)\ldots) > 0$$

   and some $n < \omega$. Further, for any non-zero $d \in B_{n_*}$ with $d \wedge b_* = 0$, (i.e. $d \in B_n - P_M^4$),

   $$\sigma(\ldots F_{n_i}(c)\ldots) \wedge d > 0.$$
7. *For every $n \ge n_*$, B_n is generated by $B_{n_*} \cup \{F_m(c) : n > m \ge n_*, c \in P_2^M\}$. Thus P_1^M and so M is generated by $B_{n_*} \cup P_2^M$.*

[3] A further equivalence: $|Atom(B_{n_*})| - |P^M_{4,1}|$ is a power of two.

Note that the free generation in item 6 of Definition 3.1 is not preserved by arbitrary direct limits and so is not a property of each model in $\boldsymbol{K}_1$. In particular, as M_* is corrected to a model of $\boldsymbol{K}_1$, we check this property only for finitely generated submodels as it will be false in general.

Recall some terminology from [BS17a].

{richname}

Definition 3.2 ($\boldsymbol{K}_2$ defined) *1. $\boldsymbol{K}_1$ denotes the collection of all direct limits of models in $\boldsymbol{K}^1_{<\aleph_0}$.*

2. We say a model M in $\boldsymbol{K}_1$ is rich if for any $N_1, N_2 \in \boldsymbol{K}^1_{<\aleph_0}$ with $N_1 \subseteq N_2$ and $N_1 \subseteq M$, there is an embedding of N_2 into M over N_1.

Since $\boldsymbol{K}^1_{<\aleph_0}$ has joint embedding, amalgamation and only countably many finitely generated models, we construct in the usual way a generic model.

{getgen}

Fact 3.3 ($\boldsymbol{K}_2$ defined) *There is a countable generic model M for $\boldsymbol{K}_0$ (Corollary 3.2.19 of [BS17a]. We denote its Scott sentence by ϕ. $\boldsymbol{K}_2$ is the class of models of this ϕ. $\boldsymbol{K}_2$ is also the class of rich models in $\boldsymbol{K}_1$.*

We now describe the transformation the current section makes on the result of Section 2.

{correnum2}

Remark 3.4 (The Corrections) *1. The structures constructed in this Section are subsets of M_*; the F_n are redefined so the new structures are substructures only the of reduct of M_* to $\tau - \{F_n : n < \omega\}$.*

2. In particular, for all the M considered here $P_1^M = P_1^{M_*}$ and these Boolean algebras have the same set of ultrafilters. However, $\mathrm{uf}(M) \neq \mathrm{uf}(M^*)$ *as the definition of* uf *depends on properties of the F_n.*

3. The set $\{F_n^M(c) : c \in P_2^M\}$ is not required to be an independent subset in $\boldsymbol{K}_{-1}$; the final constructed model is not *nicely free.*

4. Claim 3.15 demands a sequence of finite Boolean algebras B_n to witness membership in $\boldsymbol{K}_1$ (not required for $\boldsymbol{K}_{-1}$) in Section 2 and [BS17a].

5. In [BS17a], the proof that a non-maximal model in λ makes λ measurable depends on $\diamond$. In contrast, we work here in ZFC; the combinatorial principle is the small black box of Lemma 3.20.

The main goal of this section is to prove:

{realthm}

Theorem 3.5 *If λ is less that the first measurable cardinal and for some μ, $2^\mu = \lambda$ and $2^{\aleph_0} < \lambda$, then there is a P_0-maximal model in $\boldsymbol{K}_2$ of cardinality λ.*

The requirement that for some μ, $2^\mu = \lambda$ is needed only to guarantee (by Theorem 2.2.4) there is a model M_* in λ satisfying Context 3.6, which summarises the results of the construction in Theorem 2.2.4, specifically to fix our assumptions for this section.

{hyp}

Context 3.6 *1. $P_1^{M_*}$ is an atomic Boolean algebra and M_* is P_0-maximal. Further, $|P_i^{M_*}| = \lambda$ for $i = 0, 1$.*

2. *$P_{4,1}^{M_*}$ is the set of atoms of M_*.*

3. *M_* is nicely free (Definition 2.2); in particular, $P_1^{M_*}/P_4^{M_*}$ is a free Boolean algebra of cardinality λ.*

We now 'correct' M_* to a model in $\boldsymbol{K}_2$. In order to make these corrections, we lay out some notation for the generating set of $P_1^{M_*}$, the free basis of the boolean algebra $P_1^{M_*}/P_4^{M_*}$, and the indexing of the tasks performed in the construction.

{f33}

Notation 3.7 *We define a family of trees of sequences:*

1. *Let $\mathcal{T}_\alpha = \{\langle\rangle\} \cup \{\alpha\widehat{\ }\eta; \eta \in {}^{<\omega}3\}$ and $\mathcal{T} = \bigcup_{\alpha<\lambda} \mathcal{T}_\alpha$.*
2. *$\lim(\mathcal{T}_\alpha)$ is the collection of paths through $\mathcal{T}_\alpha$.*

Combining the specifications for constructing M_* (Specification 2.2.5) and the Definition 2.2 of nicely free,

{f34}

Claim 3.8 (Fixing Notation) *Without loss of generality, we may assume:*

1. *The universe of M_* is λ and the 0 of $P_1^{M_*}$ is the ordinal 0.*
2. *We can choose sequences of elements of $P_1^{M_*}$, $\mathbf{b} = \langle b_\eta \colon \eta \in \mathcal{T}\rangle$ so that their images in the natural projection of $P_1^{M_*}$ on $P_1^{M_*}/P_4^{M_*}$ freely generate $P_1^{M_*}/P_4^{M_*}$.*
3. *For every $a \in P_{4,1}^{M_*}$ and the even ordinals $\alpha < \lambda$, for some n, for any $\nu, \rho \in \mathcal{T}_\alpha$ with $\lg(\eta) \geq n$ and $\lg(\rho) \geq n$, $a \leq_{P_1^{M_*}} b_\nu$ if and only if $a \leq_{P_1^{M_*}} b_\rho$.*

Proof. The only difficulty is deducing from c) of Definition 2.2 that 3) holds. For that, for even α, let $\{b'_{\omega\alpha+n} \colon n < \omega\}$ enumerate $u_c = \{F_n^{M_*}(c) : n < \omega\}$ (from Definition 2.2.c) for the αth c in some enumeration of $P_2^{M_*}$.

Now for $\alpha > 0$, let $\langle b_\eta \colon \eta \in \mathcal{T}_\alpha \setminus \{\langle\rangle\}\rangle$ list $\{b'_{\omega\alpha+n} \colon n < \omega\}$ without repetition and $\langle b_\eta \colon \eta \in \mathcal{T}_0\rangle$ list $\{b'_n \colon n < \omega\}$.

Now by Definition 1.2.7 we have. For every $a \in P_{4,1}^{M_*}$ for all but finitely many n, $a \cap b'_{\omega\alpha+n} = 0_{P_{4,1}^{M_*}}$; whence all but finitely many of the $b_\nu \in \mathcal{T}_\alpha$ with $\nu(\alpha) \neq 0$, $a \cap b_\nu = 0_{P_{4,1}^{M_*}}$. Then re-enumerate this sequence as $\langle b_\eta \colon \eta \in \{\alpha\widehat{\ }\eta; \eta \in {}^{<\omega}3\}\rangle$. Since for each n, the intersection of the $F_n(c)$ is empty, the clause (3) follows since for all sufficiently large n, $a \not\leq F_n(c)$. $\square_{3.8}$

Note that Claim 3.8 provides a 1-1 map from $P_2^{M_*}$ to ordinals less than λ.

We introduce the collection of models which is the starting point for the following construction.

{f37}

Definition 3.9 ($\mathbb{M}_1$ **Defined**) *Let $\mathbb{M}^1 = \mathbb{M}^1_\lambda$ be the set of $M \in \boldsymbol{K}_{-1}$ such that the universe of M is contained in λ, the universe of M_*, and for $i < 2$, $P_i^M = P_i^{M_*}$, $M \restriction (P_0^M \cup P_1^M) = M_* \restriction (P_0^{M_*} \cup P_1^{M_*})$ while $P_2^M \subseteq P_2^{M_*}$.*

The posited M_* differs from any $M \in \mathbb{M}_1$ only in that P_2^M may be a proper subset of P_2^{M*} and the $F_n^M(c)$ need not equal the $F_n^{M_*}(c)$.

We now spell out the tasks which must be completed to correct M_* to the required member of $\boldsymbol{K}_2$. The values of $F_n^{M_*}(c)$ are used as oracles.

{f39}

Definition 3.10 *1. Let $\boldsymbol{T}_1$, the set of 1-tasks, be the set of pairs (N_1, N_2) such that:*

(a) $N_1 \subseteq N_2 \subseteq \lambda$

(b) $N_1, N_2 \in \boldsymbol{K}^1_{<\aleph_0}$

(c) $N_1 \subset M$ *for some* $M \in \mathbb{M}_1$.

2. *Let $\boldsymbol{T}_2$, the set of 2-tasks, be the set of $c \in P_2^{M_*}$.*

3. $\boldsymbol{T} = \boldsymbol{T}_1 \cup \boldsymbol{T}_2$.

4. *Let $\langle \mathbf{t}_\alpha : \alpha < \lambda \rangle$ enumerate $\boldsymbol{T}$.*

Note $|\boldsymbol{T}_1| = |\boldsymbol{T}_2| = |\boldsymbol{T}|$.

{f41}

Definition 3.11 *The task $\mathbf{t}$ is* relevant *to the structure M if $M \in \mathbb{M}_1$ and i) if $\mathbf{t}$ is 1-task (N_1, N_2) then $N_1 \subseteq M$ or ii) if $\mathbf{t}$ is a 2-task $\{c\}$ and $c \in P_2^{M_*}$.*

We say $M \in \mathbb{M}_1$ satisfies the task $\mathbf{t}$ *if either:*

A) $\mathbf{t} = (N_1, N_2) \in \boldsymbol{T}_1$ *(so $N_1 \subset M$) and there exists an embedding of N_2 into M over N_1.*

B) $\mathbf{t} = \{c\}$*, where $c \in P_2^{M_*}$, is in $\boldsymbol{T}_2$ and for every ultrafilter D on P_1^M, such that for infinitely many n, $F_n^{M_*}(c) \in D$, there is a $d \in P_2^M$ such that for infinitely many n, $F_n^M(d) \in D$.*

Recall Definition 2.2.1 of $\mathrm{uf}(M)$ and Lemma 2.2.3 connecting $\mathrm{uf}(M)$ with P_0-maximality of M.

{f44}

Claim 3.12 *If $M \in \mathbb{M}_1$ satisfies all tasks in $\boldsymbol{T}$ and is in $\boldsymbol{K}_1$ then M is P_0-maximal and, in particular, satisfying the tasks in $\boldsymbol{T}_1$ guarantees it is in $\boldsymbol{K}_2$.*

Proof. For P_0-maximality of M, it suffices, by Lemma 2.2.3, to show $\mathrm{uf}(M) = \emptyset$. But, since $\mathrm{uf}(M_*) = \emptyset$, for every ultrafilter D on $P_1^{M_*}$ there is $c \in P_2^{M_*}$ with $S_c^{M*}(D)$ infinite and satisfying task d means there is $d \in P_2^M$ such that $S_d^M(D)$ is infinite and so not in $\mathrm{uf}(M)$. Since M and M^* have the same ultrafilters, this implies $\mathrm{uf}(M) = \emptyset$, as required. The second assertion follows by realizing that satisfying all the tasks in $\boldsymbol{T}_1$ establishes the model is rich, which suffices by Fact 3.3. $\square_{3.12}$.

Thus our job is reduced to showing each ultrafilter D is countably incomplete. Definition 3.13 lays out the use of the generating elements b_η in correcting the F_n^{M*} to require independence while maintaining that infinite intersections of members of the ultrafilter under consideration are empty. The infinite sequence η_d will guide the choice of possibilities for $F_n(d)$; the black box in Fact 3.19 will ensure the $F_n(d)$ are in the D under consideration.

We define a class $\mathbb{M}_2 \subseteq \mathbb{M}_1$ such that for each $d \in P_2^M \in \mathbb{M}_2$ there is an ordinal α_d, a tree of elements of P_1^M, indexed by sequences in $\mathcal{T}_{\alpha_d} \subseteq {}^{<\omega}3$, a target path η_d through that tree and a sequence $a_{d,n}$, whose indices are not in $\mathcal{T}_{\alpha_d}$, but which satisfy that each $a \in P_{4,1}^{M_*} = P_{4,1}^M$ is in at most finitely many $a_{d,n}$. Further, elements indexed by $\mathcal{T}_\alpha$ are combined with the $a_{d,n}$ to get values of the $F_n^M(d)$ which are both independent and satisfy $\bigcap_{n<\omega} F_n^M(d) = \emptyset$. Different d's may have the same α but not the same η.

{f50}

Definition 3.13 **($\mathbb{M}_2$ Defined)** *Let $\mathbb{M}_2$ be the set of $M \in \mathbb{M}_1$ such that there is a sequence $w = \langle(\alpha_d, \eta_d, a_{d,n}) \colon d \in P_2^M, n < \omega\rangle$ witnessing the membership, which means:*

A (a) For each $d \in P_2^M$, $\alpha_d < \lambda$ is even and $d_1 \neq d_2$ implies $\eta_{d_1} \neq \eta_{d_2}$. (Note that it is possible that $d_1 \neq d_2$ implies $\alpha_{d_1} = \alpha_{d_2}$

(b) $\langle\alpha_d\rangle \lhd \eta_d \in \lim(\mathcal{T}_{\alpha_d})$.

B The $a_{d,n}$ are in $P_1^{M_*}$ and for each $d \in P_2^M$ and $n < \omega$, there are $\nu_1[d,n] \neq \nu_2[d,n]$ in ${}^{n+1}3$ such that:

(a) For a fixed function $\mathbf{n}_M \colon P_2^M \to \omega$, we have, for every $n \geq \mathbf{n}_M(d)$:

$$F_n^M(d) = (b_{\nu_1[d,n]} \triangle b_{\nu_2[d,n]}) \triangle a_{d,n};$$

For $n < \mathbf{n}_M(d)$, $F_n^M(d) = F_n^{M_*}(d)$.

(b) $\eta_d \restriction n \lhd \nu_1[d,n]$ and $\eta_d \restriction n \lhd \nu_2[d,n]$;

(c) for each $a \in P_{4,1}^{M_*}$ and each $d \in P_2^M$, there are only finitely many n with $a \leq_{P_1^{M_*}} a_{d,n}$.

C For each $Y \subseteq P_2^M$ there is a list $\langle d_\ell : \ell < |Y|\rangle$ of Y such that: $(*)$ for every $\ell < |Y|$, letting α_ℓ abbreviate α_{d_ℓ}, we have

$$W_\ell = \{a_{d_k,n} : k \leq \ell \wedge n < \omega\} \cup \{b_\nu \colon \nu(0) \neq \alpha_\ell, \alpha_k \neq \alpha_\ell\} \cup \{F_i^M(d_k) : i < \mathbf{n}_M(d_k), k \leq \ell, d_k \neq d_\ell\}$$

is included in the subalgebra $\mathbb{B}_\ell$ of $P_1^{M_*}$ generated by

$$\{b_\nu \colon \nu(0) \neq \alpha_\ell \wedge \nu \in \mathcal{T}\} \cup \{b_{\langle\rangle}\} \cup P_{4,1}^{M_*}.$$

(a) The d_ℓ list Y without repetition.

(b) If $i_1 < i_2 < i_3 < |Y|$ and $\alpha_{i_1} = \alpha_{i_3}$ then $\alpha_{i_2} = \alpha_{i_1}$.

The following facts about the relation of symmetric difference and ultrafilters are central for calculations below.

{backgrbauf}

Remark 3.14 *Recall that the operation of symmetric difference is associative.*

1. *(for 3.15) Suppose $\mathbb{B}_1 \subseteq \mathbb{B}_2$ are Boolean algebras with $a \in \mathbb{B}_1$, and $b \neq c$ are in $\mathbb{B}_2$, and $\{b, c\}$ is independent over $\mathbb{B}_1$ in $\mathbb{B}_2$. Then*

   The element $(b \vartriangle c) \vartriangle a \in \mathbb{B}_2$ is independent from $\mathbb{B}_1$.

   Starting from infinite independent sequences $\mathbf{b}_1, \mathbf{b}_2 \in P_1^{M_*}$ and an infinite independent sequence of $a_{d,n}$ we can prove by induction that the $F_n^M(d)$ are independent.

2. *(for 3.18) Let D be an ultrafilter on a Boolean algebra $\mathbb{B}$. Note: ($a \in D$ iff $b \in D$) if and only if $a \vartriangle b \notin D$.*

   If $a_0, a_1, a_2 \in \mathbb{B}$ are distinct and ($a_0 \in D$ iff $a_1 \in D$) then at least one of $a_i \vartriangle a_j \notin D$ (since the intersection over all pairs i, j of the $a_i \vartriangle a_j$ is empty).

   More importantly for our use later, ($a_0 \in D$ iff $a_1 \in D$) iff

$$(a_0 \vartriangle a_1 \vartriangle a_2) \in D \leftrightarrow a_2 \in D.$$

3. *(for 3.18) 2) implies that if D is an ultrafilter of $\mathbb{B}_2$ and ($b \in D \leftrightarrow c \in D$) and $a \notin D$ then*

   - $b \triangle c \notin D$
   - $(b \triangle c) \triangle a \notin D$.

We will show in Theorem 3.15 that members of $\mathbb{M}_2$ are in $\boldsymbol{K}_1$ and then in Theorem 3.18 that there are structures in $\mathbb{M}_2$ that are in $\boldsymbol{K}_2$. Two main features distinguish $\boldsymbol{K}_1$ from $\boldsymbol{K}_{-1}$. The $F_n(d)$ retain the intersection properties from $\boldsymbol{K}_{-1}$ but also must be independent; membership of an M in $\boldsymbol{K}_1$ from [BS17a] must be witnessed by the construction for a countable substructure $M' \subset M$ of a family of finite Boolean algebras satisfying Definition 3.1.2 and .3.

{f53}

Theorem 3.15 *If $M \in \mathbb{M}_2$, then $M \in \boldsymbol{K}_1$.*

Proof. Let $Y \subset P_2^M$ and $X \subset P_1^M$ be finite ; we shall find $N \in \boldsymbol{K}^1_{<\aleph_0}$ such that $Y \cup X \subseteq N \subseteq M$; this suffices. As, by the definition of $\boldsymbol{K}_1$, M is a direct limit of finitely generated structures in $\boldsymbol{K}^1_{<\aleph_0}$.

Our two main tasks are to find such an N in which i) the F_n^N satisfy property 6 of Definition 3.1 and ii) there is a sequence of finite Boolean algebras $\mathbb{B}_{k_n}$ witnessing 2 and 3 of Definition 3.1. First we attack i).

Let the sequence $\langle(\alpha_d, \eta_d, a_{d,k})\colon d \in P_2^M, k < \omega\rangle$ *witness* $M \in \mathbb{M}_2$ as in Definition 3.13. Let $\langle d_i : i < n\rangle$ enumerate Y without repetition and denote, for $i < n$, η_{d_i} by η_i and α_{d_i} by α_i. Without loss, the $\langle \eta_i(0)\colon i < n\rangle$ are non-decreasing. Fix k_1 such that

- $k_1 \geq n_M(d_i)$ (see Definition 3.13.B) for all $i < n$,
- $\langle \eta_i \restriction k_1 \colon i < n\rangle$ are distinct for $i < n$, and
- $X_0 = X \cup \{F_k^M(d_i)\colon i < |Y| = n, k \leq \mathbf{n}_M(d_i)\}$ is contained in the subalgebra generated by $\{b_\nu \colon \eta_i \restriction k_1 \ntrianglelefteq \nu$ for $i < |Y|\} \cup \{b_{\langle\rangle}\} \cup P_{4,1}^M$.

(None of the $F^M_k(d_i) = b_\sigma$ with $\sigma(0)$ equal to an α_i for $i < n$, by clause C of Definition 3.13; we avoid X by choosing k_1 big enough.)

Claim 3.16 $A = \langle F^M_k(d_i)\colon k \geq k_1$ and $i < n\rangle$ *is independent in* P^M_1 *over* X_0 *modulo the atoms and the elements are distinct.* {f53.5}

Proof. We prove this claim by showing by induction on $\ell \leq |Y| = n$:

$$(\oplus_\ell)\ \ A_\ell = \langle F^M_k(d_i)\colon k \geq k_1 \text{ and } i < \ell\rangle$$

is independent in P^M_1 over X_ℓ modulo the atoms, where we have defined X_0 and for $0 < \ell \leq |Y|$,

$$X_{\ell+1} = X_\ell \cup \{F^M_k(d_i)\colon k < k_1, i < \ell\}.$$

Note that $A = A_n$. The independence of A_ℓ over all the $F^M_k(d_i)$ with $d_i \in Y$ for $k < n_M(d_i)$ is clear because since they are in X_0. The induction on ℓ adds incrementally the independence of the tail of the $F^M_k(d_\ell)$ over the $F^M_k(d_i)$ for $n_M(d_i) \leq k < k_1$ and $i < \ell \leq |Y|$.

Now by Definition 3.13.C, $X_\ell \subseteq W_\ell$ is contained in $\check{\mathbb{B}}_\ell$, the subalgebra generated by

$$Z_\ell = \{b_\nu : \nu_{d_\ell} \upharpoonright k_{\ell+1} \not\trianglelefteq \nu\} \cup \{b_{\langle\rangle}\} \cup P^M_{4,1}.$$

Think of the $\check{\mathbb{B}}_\ell \subseteq \mathbb{B}_\ell$ as omitting certain cones. The crucial point is that $b_{\nu_1[d_\ell,n]}$ and $b_{\nu_2[d_\ell,n]}$ are *not* in $\check{\mathbb{B}}_\ell$. Thus, Claim 3.8.2 and Definition 3.13.b imply the infinite set $\{b_{\nu_i[\eta_\ell,k]}\colon i \in \{0,1\}, k \geq k_1, \rangle$ is independent over $\check{\mathbb{B}}_\ell$. For convenience we write $a_{k,i}$ for $a_{d_k,i}$.

By Definition 3.13.C, looking at the first term in the union and comparing X_ℓ with the generators of W_ℓ in that definition, each $a_{k,i}$, for $k < \ell, i < \omega$ is in $\check{\mathbb{B}}_\ell$, and for each $k < \ell$ and $i < \mathbf{n}(d_\ell)$, $F^M_i(d_k) \in \check{\mathbb{B}}_\ell$ (3rd term of the definition of Z_ℓ). Compare also the definition of η_{d_i} in A(b) of that definition.

But, using the crucial point, we also claim $F^M_i(d_\ell) \in \check{\mathbb{B}}_\ell$ when $\ell < k_1$ for $i \geq \mathbf{n}(d_\ell)$, since, setting $d = d_k$ for notational simplicity,

$$F^M_n(d_k) = F^M_n(d) = (b_{\nu_1[d,n]} \bigtriangleup b_{\nu_2[d,n]}) \bigtriangleup a_{d,n},$$

by B.(b). For each i and k, $\nu_i[d_i,k]$ is $\trianglelefteq$ above $\langle\alpha_{d_i}\rangle$ and so the set $V = \{\nu_i[d_i,k]; i < \ell, k < \omega\}$ independent from $\mathbb{B}_\ell$ and $X_\ell \subset \mathbb{B}_\ell$ so V is independent from X_ℓ. Since we noticed the $a_{k,i} \in \mathbb{B}_\ell$, Lemma 3.14 implies $\{F^M_k(d_\ell)\colon k \geq k_2\}$ is independent over $\mathbb{B}_\ell$. By the induction hypothesis we complete the proof. $\square_{3.16}$

Applying Claim 3.8.3, for sufficiently large n, e.g. $n \geq \mathbf{n}_d$, $a \lneq (b_{\nu_1[d,n]} \bigtriangleup b_{\nu_2[d,n]})$ and by hypothesis, the $a_{d,n}$ satisfy the same condition. Thus, for sufficiently large n, $a \lneq F^M_n(d_\ell)$.

This completes task i). To finish the proof of Theorem 3.15 by satisfying conditions 2-2 of Definition 3.1, we must find a sequence of finite Boolean algebras B_n witnessing that $X \cup Y$ is contained in a member of $\boldsymbol{K}^1_{<\aleph_0}$.

Recall that M_* is generated by $\{b_\nu : \nu \in \mathcal{T}\} \cup P^{M_*}_{4,1} \cup \{b_{<>}\}$. As X is finite, there is a k_2 such that X is contained in the finite subalgebra of $P^{M_*}_1$ generated by $\{b_\nu\colon \nu \in \mathcal{T}, \lg(\nu) < k_2\} \cup P^{M_*}_{4,1}$.

We now choose $N \subseteq M$ with $P_1^N = \bigcup_{m \geq k_2} B_m$, $P_2^N = Y$, $P_0^N = \{G_1^{-1}(a) \colon a \in P_{4,1}^M \cap P_1^N\}$.

Define the B_m for $m \geq k_2$, $N \in \boldsymbol{K}_1$ as follows. Let B_{k_2} be the subalgebra of M_* generated by $X \cup \{F_k^M(d_\ell) \colon k \leq k_2, \ell < |Y|\}$. For $m \geq k_2$, let B_m be generated by $X \cup \{F_k^M(d_\ell) \colon k < m, \ell < |Y|\}$. Without loss of generality (using the choice of $\mathbf{b}$ from Claim 3.8), we can demand each $\mathbb{B}_{k_i}$ is a finite free Boolean algebra. This sequence witnesses that $M \in \boldsymbol{K}_1$. $\square_{3.15}$

Now we show $\mathbb{M}_2$ is non-empty and at least one member satisfies all the tasks.

Notation 3.17 *We can enumerate $\boldsymbol{T}$ as $\langle t_\alpha \colon \alpha < \lambda \rangle$ such that each task appears λ times, as we assumed in Hypothesis 3.6 that $\lambda = \lambda^{\aleph_0}$.*

For Claim 3.18, realizing all the tasks, $\lambda > 2^{\aleph_0}$ suffice; the requirement in Lemma 2.1.5 that $\lambda = 2^\mu$ is used to get maximal models.

{f56}

Theorem 3.18 *There is an $M \in \mathbb{M}_2$ that satisfies all the tasks. Thus $M \in \boldsymbol{K}_2$.*

Proof. We choose M_α by induction on $\alpha \leq \lambda$ such that:

1. $\mathbf{w}_\alpha$ witnesses $M_\alpha \in \mathbb{M}_2$ (Definition 3.13). And for $\beta < \alpha$, w_α extends w_β. That is, for $d \in P_2^{M_\beta}$, $\alpha_d[\mathbf{w}_\alpha] = \alpha_d[\mathbf{w}_\beta]$, $\eta_d[\mathbf{w}_\alpha] = \eta_d[\mathbf{w}_\beta]$, etc..
2. $P_2^{M_\alpha} \subseteq P_2^{M_*}$ has cardinality at most $|\alpha| + 2^{\aleph_0}$.
3. if $\alpha = \beta + 1$ and t_β is relevant to M_β, M_α satisfies task $\mathbf{t}_\beta$.

case 1 If $\alpha = 0$, set $M_0 = M_* \restriction (P_0^{M_*} \cup P_1^{M_*})$.

case 2 Take unions at limits.

case 3 $\alpha = \beta + 1$ and $\mathbf{t}_\beta \in \boldsymbol{T}_1$; say, $\mathbf{t}_\beta = (N_1, N_2)$.

If N_1 is not embedded in M_β then the task is irrelevant and let $M_\alpha = M_\beta$. Let $\langle c_i \colon i < m \rangle$ enumerate $P_2^{N_2} - P_2^{N_1}$ and $\langle d_i \colon i < m \rangle$ enumerate the first m elements of $P_2^{M_*} - P_2^{M_\beta}$.

By induction, since $M_\beta \in \mathbb{M}_2$ there are witnesses $w_\beta = \langle a_{d,k}, \eta_d, \alpha_d \rangle$ (formally $\langle a_{d,k}^\beta, \eta_d, \alpha_d^\beta \rangle$ for $d \in P_2^{M_\beta}$. By Definition 3.13.C, we can fix $U_\alpha \subseteq \lambda$ of cardinality $\leq |\alpha| + 2^{\aleph_0}$ such that: {Ualph}

$$(*)\{a_{d,k} \colon k < \omega, d \in P_2^{M_\beta}\} \cup \{b_\nu \colon (\exists d \in P_2^{M_\beta}) \langle \alpha_d \rangle \trianglelefteq \nu \in \mathcal{T}_{\alpha_d}\} \cup P_{4,1}^{M_*}$$

is included in the subalgebra of M_* generated by the

$$\{b_\rho \colon \exists \beta \in U_\alpha, \langle \beta \rangle \trianglelefteq \rho \in \mathcal{T}_\beta\} \cup \{b_{\langle\rangle}\} \cup P_{4,1}^{M_*}.$$

Let M_α extend the universe of M_β by adding $\langle d_i : i < m \rangle \subset P_2^{M_*}$. Note that the domain of M_α is a subset of M_*, but M_α is not a substructure of M_*; we are about to define the $F_k^{M_\alpha}$ at the d_i. Let k_* be large enough and let B be a finite Boolean sub-algebra of $P_1^{N_2}$ and $b_* \in B$ be as in Definition 3.1.3 of $\boldsymbol{K}_1$. In particular b_* is a finite union of atoms of P_1^{M*}, which are in $P_1^{N_2}$, and $P_1^{N_2}$ is generated freely over $P_1^{N_1} \cup B$ by $\{F_k^{N_2}(c_i)\colon k_* \leq k < \omega, i < m\}$.

To extend the witnesses to M_α, let $\langle(\beta_i, \eta_i)\colon i < m\rangle$ be such that the β_i are a strictly increasing list of the first m even members of $\lambda - U_\alpha$ with $\eta_i \in \mathcal{T}_{\beta_i}$. Let $a_{d_i,k}$ be the 0 of $P_1^{M_*}$ for $i < m, k < \omega$.

We first map B into M_β; map atoms $a \in P_{4,1}^{N_2} - P_{4,1}^{N_1}$ into atoms a' in $P_{4,1}^{M_\beta} - P_{4,1}^{N_1}$. Then map the finitely many $F_k^{N_2}(c_i)$ for $k < n_*, i < m$ to $b'_{k,i}$ which are in $P_1^{N_1}$ and independent over B_{n_*} for N_1. Now let $F_k^{M_\alpha}(d_i)$ be the join of $b'_{k,i}$ with all the $a \in P_{4,1}^{N_1}$ that lie below $F_k^{N_2}(c_i)$ and the a' such that $a \in P_{4,1}^{N_2} - P_{4,1}^{N_1}$ and $a \leq F_k^{N_2}(c_i)$.

Now, the $\{b_{(\eta_i \restriction k)\frown<0>} \triangle b_{(\eta_i \restriction k)\frown<1>} : i < m\}$ are independent[4] over $P_1^{N_1}$.

So we can define h_β to embed the Boolean algebra $P_1^{N_2}$ into $P_1^{M_*}$ over $P_1^{N_1}$ such that $k \geq k_*$ implies

$$h_\beta(F_k^{N_2}(c_j)) = b_{\eta_j \restriction k\frown 0} \triangle b_{\eta_j \restriction k\frown 1}.$$

By Claim 3.8.3, since the β_i are even, for each $a \in P_{4,1}^{M_*}$, for some n, for any $\nu, \rho \in \mathcal{T}_\alpha$ with $\lg(\eta) \geq n$ and $\lg(\rho) \geq n$ then $a \leq_{P_1^{M_*}} b_\nu$ if and only if $a \leq_{P_1^{M_*}} b_\rho$. For each j and k, $b_{\eta_j \restriction k\frown 0} \triangle b_{\eta_j \restriction k\frown 1} = (b_{\eta_j \restriction k\frown 0} \triangle b_{\eta_j \restriction k\frown 1}) \triangle 0_{P_1^{M_*}}$. So setting $a_{d_i,k} = 0$ for $i < m$, we have:

$$F_k^{M_\alpha}(d_j) = (b_{\eta_j \restriction k\frown 0} \triangle b_{\eta_j \restriction k\frown 1}) \triangle 0_{P_1^{M_*}},$$

and for each $a \in P_{4,1}^{M_\alpha}$, for some n, $a \not\leq_{P_1^{M_*}} F_k^{M_\alpha}(d_j)$. Thus, $M_\alpha \in \mathbb{M}_1$ and so in $M_\alpha \in \mathbb{M}_2$ as required.

Before taking up Case 4, we introduce a further tool, a simple black box, which is a variant on Fact 1.5 of [She].

{309fact}

Lemma 3.19 *There is an assignment to each $\eta \in {}^\chi\lambda$ of a function f_η, whose domain is the set of initial segments of η and with range is contained in λ such that for any $f \in {}^{<\chi}\lambda$, there is and $\eta_* \in {}^\chi\lambda$, $f_{\eta_*} \subset f$.*

Proof. Define the sequence f_η by for $\alpha < \chi$,

$$f_\eta(\eta \restriction \alpha) = \eta(\alpha).$$

Now define $\eta_* \in {}^\chi\lambda$ by $\eta_*(0) = f(0)$, taking unions at limits and for successor $\alpha = \beta + 1$, $\eta_*(\alpha) = f(\eta_* \restriction \beta)$.

[4] Suppose one takes any partition of an independent set and chooses for each block one element which is a finite Boolean combination of elements from that block. Then, that set of elements is independent.

$\square_{3.19}$

We derive the relevant property for our purpose, using $\chi = 4$ and $\lambda = 3$.

{309con}

Lemma 3.20 *For any $\beta < \lambda$, and each $\eta \in \lim(\mathcal{T}_\beta)$ there are $\rho_\eta \in {}^{<\omega}3$ indexed as $\overline{\rho} = \langle \rho_\eta \colon \eta \in \lim(\mathcal{T}_\beta) \rangle)$ such that*

for any g that maps $\mathcal{T}_\beta$ to 3, for some $\eta \in \lim(\mathcal{T}_\beta)$, for $1 \leq n < \omega$:

$$\rho_\eta(n) = g(\eta \restriction n).$$

Proof. Define $\rho_\eta(n)$ by induction: $\rho_\eta(0) = 0$; $\rho_\eta(n+1) = f_\eta(\eta \restriction n)$ where f_η is chosen by Lemma 3.19. $\square_{3.20}$

case 4 $\alpha = \beta + 1$ and $\mathbf{t}_\beta \in \boldsymbol{T}_2$; say, $\mathbf{t}_\beta = c$.

Let γ be an even ordinal such that $\gamma \neq \alpha_d$ for any $d \in P_2^{M_\beta}$ and $\gamma \ntrianglelefteq \nu$ if $b_\nu = a_{d,k}$ for some $k < \omega$ and $d \in P_2^{M_\beta}$. Let $\langle d_\eta \colon \eta \in \lim \mathcal{T}_\gamma \rangle$ be a set of pairwise distinct elements of $P_2^{M_*} - P_2^{M_\beta}$.

We define M_α. We have a witness $\langle a^\beta_{d,k}, \eta_d, \alpha^\beta_d \rangle$' that $M_\beta \in \mathbb{M}_2$; we extend it to a witness for M_α.

1. Let M_α be generated by $M_\beta \cup \{d_\eta \colon \eta \in \lim(\mathcal{T}_\gamma)\}$. Choose ρ_η depending on η by Lemma 3.20. To define $F_k^{M_\alpha}(d_\eta)$, for each $\eta \in \lim \mathcal{T}_\gamma$ and $k < \omega$, choose $i_0 < i_1 \leq 2$ that are different from $\rho_\eta(k)$; Recalling $c = \mathbf{t}_\beta$, let

$$F_k^{M_\alpha}(d_\eta) = (b_{\eta \restriction k \frown i_0} \triangle b_{\eta \restriction k \frown i_1}) \triangle (F_k^{M_*}(c)).$$

2. (a) Let $\langle d_\nu \colon \nu \in \lim \mathcal{T}_\gamma \rangle$ be a set of pairwise distinct elements of $P_2^{M_*} - P_2^{M_\beta}$.
   (b) Define $\{\langle \alpha_d, \eta_d \rangle \colon d \in P_2^{M_\alpha} - P_2^{M_\beta}$ by setting $\alpha_d = \gamma$ and if $d = d_\nu$ then $\eta_d = \nu\}$.

We must show $M_\alpha \in \mathbb{M}_2$ and satisfies task $\mathbf{t}_\beta$. To verify $\mathbf{t}_\beta$, suppose D is an ultrafilter on $P_1^{M_*}$ such that the set $S_c^{M_*}(D) = \{n \colon F_n^{M_*}(c) \in D\}$ is infinite (Definition 2.2.1). Define g mapping $\mathcal{T}_\gamma$ to 3 so that if $\nu \in \mathcal{T}_\gamma$ has length k, and neither of $i_o < i_1 \leq 2$ are equal to $g(\nu)$, then $b_{\nu \frown i_0} \in D$ if and only if $b_{\nu \frown i_1} \in D$. Now by the black box (i.e. the choice of the ρ_η) there is an $\eta \in \lim(\mathcal{T}_\gamma)$ such that for every $n \geq 1$, $\rho_\eta(n) = g(\eta \restriction n)$. By Remark 3.14.2, $F_n^{M_\alpha}(d_\eta) \in D$ for infinitely many n, so $\mathbf{t}_\beta$ is satisfied.

The main difficulty in showing $M_\alpha \in \mathbb{M}_2$ is to show $M_\alpha \in \boldsymbol{K}_{-1}$. We must verify Definition 1.2.7: for every $a \in P_{4,1}^{M_\alpha}$, and every $d \in P_2^{M_\alpha} - \in P_2^{M_\beta}$, and all but finitely many n, $F_n^{M_\alpha}(d) \wedge a = 0_{P_1^{M_\alpha}}$. As $M_* \in \boldsymbol{K}_{-1}$, by Definition 1.2.7 we have for every large enough n, $P_1^{M_\alpha} = P_1^{M_*} \models F_n^{M_*}(c) \wedge a = 0_{P_1^{M_\alpha}}$. Now, recall from 3.8.3, that for every $a \in P_{4,1}^{M_*}$ and the even ordinals $\alpha < \lambda$, if for some n, for any $\nu, \rho \in \mathcal{T}_\alpha$ with $\lg(\eta) \geq n$ and $\lg(\rho) \geq n$ then $a \leq_{P_1^{M_*}} b_\nu$ if and only if $a \leq_{P_1^{M_*}} b_\rho$. As γ is even, it follows that for every $a \in P_{4,1}^{M_*}$, and large enough n, $a \wedge b_{\eta \restriction n} = 0_{P_1^{M_*}}$. The other conditions to show $M_\alpha \in \mathbb{M}_2$ are routine.

❚ Is the next paragraph needed?

Since $\mathrm{uf}(M_*) = \emptyset$, for any non-principal ultrafilter D on $P_1^{M_*}$, there is a c such that the set $S_c^{M_*}(D) = \{n \colon F_n^{M_*}(c) \in D\}$ is infinite (Definition 2.2.1). This c will be addressed in the construction of M so $D \notin \mathrm{uf}(M)$. $\square_{3.18}$

Conclusion 3.21 *We have found a P_0-maximal $M \in \boldsymbol{K}_2$ with all $|P_i^M| = \lambda$. As in [BS17a], for every λ less than the first measurable, since $M \in \boldsymbol{K}_2$ implies $|M| \leq 2^{P_0^M}$, there is a maximal model $M \in \boldsymbol{K}_2$ with $2^\lambda \leq |M| < 2^{2^\lambda}$.*

Note that the model M contains uncountably many elements $d_\eta \in P_2^M$, which were constructed in case 4, such that for some α_d, each of the $\eta(0) = \alpha_d$, but η and η' first differ at $k \geq k_*$ and $F_k^{M_\alpha}(d_\eta) = F_k^{M_\alpha}(d'_\eta)$ as, $F_k^{M_\alpha}(d_\eta) = (b_{\eta \restriction k \frown i_0} \triangle b_{\eta \restriction \frown i_1}) \triangle (F_k^{M_*}(c))$. This contradicts nice freeness. In contrast, in the P_0-maximal model constructed in [BS17a] using diamond, was $\boldsymbol{K}_1$-free for subalgebras of cardinality $< \lambda$.

Question 3.22

1. *Is there a κ such that if a complete sentence has a maximal model in cardinality κ, it has maximal models in cardinalities cofinal in the first measurable?*
2. *Is there a complete sentence that has maximal models cofinally in some κ with $\beth_{\omega_1} < \kappa < \mu$ where μ is the first measurable, but no larger models are maximals. Could the first inaccessible be such a κ?*

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