HALFWAY NEW CARDINAL CHARACTERISTICS

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Abstract. We introduce several cardinal characteristics related to the splitting number $\mathfrak{s}$, the reaping number $\mathfrak{r}$ and the independence number $\mathfrak{i}$ and prove bounds and consistency results.

1. Introduction

This research forms part of the study of cardinal characteristics of the continuum. For a general overview of cardinal characteristics, see [Bla10], [Hal17, chapter 9] and [Vau90] as well as [BJ95]. Based on the well-known cardinal characteristics

- $\mathfrak{s} := \min\{|\mathcal{S}| \mid \mathcal{S} \subseteq [\omega]^\omega \text{ and } \forall\, X \in [\omega]^\omega\ \exists\, S \in \mathcal{S} \colon |X \cap S| = |X \setminus S| = \aleph_0\}$ (the splitting number),
- $\mathfrak{r} := \min\{|\mathcal{R}| \mid \mathcal{R} \subseteq [\omega]^\omega \text{ and } \nexists\, X \in [\omega]^\omega\ \forall\, R \in \mathcal{R} \colon |R \cap X| = |R \setminus X| = \aleph_0\}$ (the reaping number), and
- $\mathfrak{i} := \min\{|\mathcal{I}| \mid \mathcal{I} \subseteq [\omega]^\omega \text{ and } \forall\, \mathcal{A} \mathbin{\dot\cup} \mathcal{B} \subseteq \mathcal{I} \colon \left|\bigcap_{A \in \mathcal{A}} A \cap \bigcap_{B \in \mathcal{B}} (\omega \setminus B)\right| = \aleph_0\}$ (the independence number),

we were inspired to define specialised variants of these (all of them related in some way to asymptotic density, in particular asymptotic density $1/2$) and successfully proved a number of bounds and consistency results for them.

We use the standard notation; in addition to $\mathfrak{s}$, $\mathfrak{r}$ and $\mathfrak{i}$ mentioned above, we will refer to a few other well-known cardinal characteristics.

Given an ideal $\mathcal{I}$ on some base set X, we can define four cardinal characteristics:

- the additivity number $\operatorname{add}(\mathcal{I}) := \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{A} \notin \mathcal{I}\}$,
- the covering number $\operatorname{cov}(\mathcal{I}) := \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{A} = X\}$,
- the uniformity number $\operatorname{non}(\mathcal{I}) := \min\{|Y| \mid Y \subseteq X \text{ and } Y \notin \mathcal{I}\}$, and
- the cofinality $\operatorname{cof}(\mathcal{I}) := \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq \mathcal{I} \text{ and } \forall\, B \in \mathcal{I}\ \exists\, A \in \mathcal{A} \colon B \subseteq A\}$.

In particular, we will refer to these cardinal characteristics for

- the ideal $\mathcal{N} := \{A \subseteq 2^\omega \mid \lambda(A) = 0\}$ of *Lebesgue null sets* and

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- the ideal $\mathcal{M} := \{A \subseteq \omega^\omega \mid A = \bigcup_{n<\omega} A_n$ and $\forall\, n < \omega\colon A_n$ nowhere dense$\}$ of *meagre sets*.

Finally, we will refer to two more cardinal characteristics:

- $\mathfrak{b} := \min\{|B| \mid B \subseteq \omega^\omega$ and $\forall\, g \in \omega^\omega\ \exists\, f \in B\colon f \not\leq^* g\}$ (the unbounding number) and
- $\mathfrak{d} := \min\{|D| \mid D \subseteq \omega^\omega$ and $\forall\, g \in \omega^\omega\ \exists\, f \in D\colon g \leq^* f\}$ (the dominating number).

We will use the following concept in a few of the proofs:

Definition 1.1. A *chopped real* is a pair (x, Π) where $x \in 2^\omega$ and Π is an interval partition of ω. We say a real $y \in 2^\omega$ *matches* (x, Π) if $y{\restriction}_I = x{\restriction}_I$ for infinitely many $I \in \Pi$.

We note that the set $\mathrm{Match}(x, \Pi)$ of all reals matching (x, Π) is a comeagre set (see [Bla10, Theorem 5.2]).

We remark that we will not rigidly distinguish between a real r in 2^ω and the set $R := r^{-1}(1)$, or conversely, between a subset of ω and its characteristic function.

The paper is structured as follows. In section 2, we introduce and work on several cardinal characteristics related to $\mathfrak{s}$. In section 3, we conduct a particularly sophisticated proof for a consistency claim from the preceding section. In section 4, we introduce and work on cardinal characteristics mostly related to $\mathfrak{r}$ and $\mathfrak{i}$. The final section 5 summarises the open questions.

2. Characteristics Related to $\mathfrak{s}$

Recall the following concepts from number theory.

Definition 2.1. For $X \in [\omega]^\omega$ and $0 < n < \omega$, define the *initial density* (of X up to n) as

$$d_n(X) := \frac{|X \cap n|}{n}$$

and the *lower* and *upper density* of X as

$$\underline{d}(X) := \liminf_{n\to\infty}(d_n(X)) \qquad \text{and} \qquad \bar{d}(X) := \limsup_{n\to\infty}(d_n(X)),$$

respectively. In case of convergence of $d_n(X)$, call

$$d(X) := \lim_{n\to\infty}(d_n(X))$$

the *asymptotic density* or just the *density* of X.

We define four relations on $[\omega]^\omega \times [\omega]^\omega$ and their associated cardinal characteristics.

Definition 2.2. Let $S, X \in [\omega]^\omega$. We define the following relations:

- *S bisects X in the limit* (or just *S bisects X*), written as $S \mid_{1/2} X$, if

$$\lim_{n\to\infty} \frac{|S \cap X \cap n|}{|X \cap n|} = \lim_{n\to\infty} \frac{d_n(S \cap X)}{d_n(X)} = \frac{1}{2}.$$

- For $0 < \varepsilon < 1/2$, S *ε-almost bisects* X, written as $S \mid_{1/2\pm\varepsilon} X$, if for all but finitely many $n < \omega$ we have
$$\frac{|S \cap X \cap n|}{|X \cap n|} = \frac{d_n(S \cap X)}{d_n(X)} \in \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right).$$
- S *weakly bisects* X, written as $S \mid^w_{1/2} X$, if for any $\varepsilon > 0$, for infinitely many $n < \omega$ we have
$$\frac{|S \cap X \cap n|}{|X \cap n|} = \frac{d_n(S \cap X)}{d_n(X)} \in \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right).$$
- S *bisects X infinitely often*, written as $S \mid^\infty_{1/2} X$, if for infinitely many $n < \omega$ we have
$$\frac{|S \cap X \cap n|}{|X \cap n|} = \frac{d_n(S \cap X)}{d_n(X)} = \frac{1}{2}.$$

Definition 2.3. We say a family $\mathcal{S}$ of infinite sets is

$$\begin{cases} \text{bisecting (in the limit)} \\ \varepsilon\text{-almost bisecting} \\ \text{weakly bisecting} \\ \text{infinitely often bisecting} \end{cases}$$

if for each $X \in [\omega]^\omega$ there is some $S \in \mathcal{S}$ such that

$$\begin{cases} S \text{ bisects } X \text{ (in the limit)} \\ S\ \varepsilon\text{-almost bisects } X \\ S \text{ weakly bisects } X \\ S \text{ bisects } X \text{ infinitely often} \end{cases}$$

and denote the least cardinality of such a family by $\mathfrak{s}_{1/2}$, $\mathfrak{s}_{1/2\pm\varepsilon}$, $\mathfrak{s}^w_{1/2}$, $\mathfrak{s}^\infty_{1/2}$, respectively.

We remark that equivalently, we could define $X \mid^\infty_{1/2} Y$ by
$$\liminf_{n\to\infty} \frac{|X \cap Y \cap n|}{|Y \cap n|} \leq \frac{1}{2} \leq \limsup_{n\to\infty} \frac{|X \cap Y \cap n|}{|Y \cap n|}.$$

Theorem 2.4. *The relations shown in Figure 1 hold.*

Proof. Recall that it is known that $\mathfrak{s} \leq \operatorname{non}(\mathcal{M})$ and $\mathfrak{s} \leq \operatorname{non}(\mathcal{N})$ (see e. g. [Bla10, Theorem 5.19]) as well as $\mathfrak{s} \leq \mathfrak{d}$ (see e. g. [Hal17, Theorem 9.4] or [Bla10, Theorem 8.13]).

$\mathfrak{s} \leq \mathfrak{s}^w_{1/2} \leq \mathfrak{s}^\infty_{1/2}$: An infinitely often bisecting real is a weakly bisecting real (being equal to $1/2$ infinitely often implies entering an arbitrary ε-neighbourhood of $1/2$ infinitely often), and a weakly bisecting real is a splitting real (if a real X does not split another real Y, the relative initial density of X in Y, that is
$$\frac{d_n(X \cap Y)}{d_n(Y)},$$

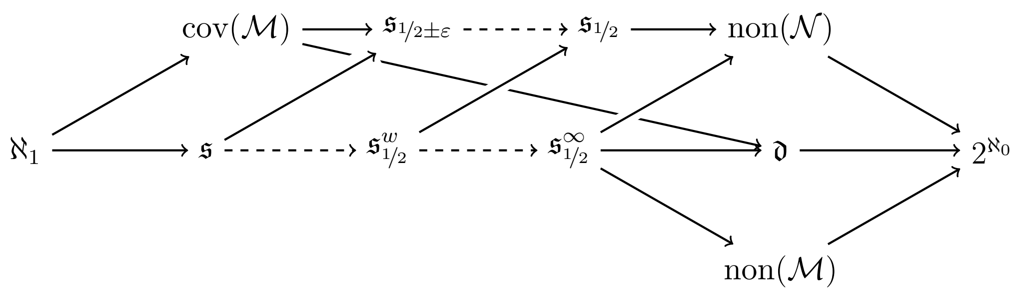

FIGURE 1. The ZFC-provable and/or consistent inequalities between $\mathfrak{s}_{1/2}$, $\mathfrak{s}_{1/2\pm\varepsilon}$, $\mathfrak{s}^w_{1/2}$, $\mathfrak{s}^\infty_{1/2}$ and other well-known cardinal characteristics, where $\longrightarrow$ means "$\leq$, consistently $<$" and $\dashrightarrow$ means "$\leq$, possibly $=$".

cannot be close to $1/2$ infinitely often). Hence a family witnessing the value of $\mathfrak{s}^\infty_{1/2}$ gives an upper bound for the value of $\mathfrak{s}^w_{1/2}$ (and analogously for $\mathfrak{s} \leq \mathfrak{s}^w_{1/2}$).

$\boldsymbol{\mathfrak{s} \leq \mathfrak{s}_{1/2\pm\varepsilon} \leq \mathfrak{s}_{1/2}}$: The first claim follows since an ε-almost bisecting real is a splitting real by the fact that finite sets have density 0 and cofinite sets have density 1, and hence if X does not split Y, the relative initial densities of X and $\omega \setminus X$ in Y tend to 0 and 1, respectively (or vice versa). The second claim follows since a bisecting real is an ε-almost bisecting real by definition.

$\mathbf{cov}(\boldsymbol{\mathcal{M}}) \leq \boldsymbol{\mathfrak{s}_{1/2\pm\varepsilon}}$: Given a family $\mathcal{S}$ witnessing the value of $\mathfrak{s}_{1/2\pm\varepsilon}$, take $S \in \mathcal{S}$. Define a chopped real based on S with the interval partition having the partition boundaries at the $n!$-th elements of S; the sets matching this chopped real form a comeagre set which consists of reals not halved by S (as the matching intervals grow longer and longer, "pulling" the relative initial density above $1 - 1/n$). Hence the family $E(S)$ of those reals that *are* ε-almost bisected by S is a meagre set (as its complement is a superset of a comeagre set), and $\{E(S) \mid S \in \mathcal{S}\}$ is a 2^ω-covering consisting of meagre sets.

$\boldsymbol{\mathfrak{s}^w_{1/2} \leq \mathfrak{s}_{1/2}}$: A bisecting real is a weakly splitting real – for the relative density to converge to $1/2$, it has to eventually be arbitrarily close to $1/2$, and hence also within an arbitrary ε-neighbourhood of $1/2$ infinitely often. The same argument using the families witnessing the cardinal characteristics holds.

$\boldsymbol{\mathfrak{s}^\infty_{1/2}} \leq \mathbf{non}(\boldsymbol{\mathcal{M}})$: For a given $X \in [\omega]^\omega$, we show that the set $B(X)$ of reals bisecting X infinitely often (contains and hence) is a comeagre set. For any $F \notin \mathcal{M}$, $F \cap B(X)$ is non-empty, hence it contains a real bisecting X infinitely often.

Given X as above, let $f(n) := \sum_{k=0}^{n} k!$ and define an interval partition Π with partition boundaries precisely after the $f(2n)$-th elements of X. Define a chopped real (S, Π) as follows: Let $S \cap (\omega \setminus X) = \varnothing$ (i. e. S contains no elements not in X). For each $0 < n < \omega$, the n-th interval $I_n \in \Pi$ contains at least $(2n-1)! + (2n)!$ elements of X. Let S skip the first $(2n-1)!$ of these elements and contain the rest. Any real that matches (S, Π) indeed has a lower relative density of 0 in X and an upper relative density of 1 in X and hence bisects X infinitely often. The set of all reals matching (S, Π) is comeagre, as required to finish the proof above.

$\boldsymbol{\mathfrak{s}^\infty_{1/2} \leq \mathfrak{d}}$: Let $\mathcal{D}$ be a dominating family. Without loss of generality assume that every member g of $\mathcal{D}$ is strictly increasing and satisfies $g(0) > 0$. Let $X \in [\omega]^\omega$ and let f_X be its enumeration. Pick a $g_X =: g$ from $\mathcal{D}$ that dominates f_X and define $G\colon \omega \to \omega$ by $G(n) := g^{(n+1)}(0)$ for every $n < \omega$. Then, for sufficiently large n,

$$G(n) \leq f_X(G(n)) < g(G(n)) = G(n+1).$$

Hence (for sufficiently large n) every interval $[G(n), G(n+1))$ contains at least one element of X and at most $G(n+1) - G(n)$ many. Now iteratively define a function $\Gamma\colon \omega \to \omega$ by $\Gamma(0) := 0$, $\Gamma(1) := G(0) = g(0)$ and $\Gamma(n+1) := G\big(\sum_{k=0}^{n} \Gamma(k)\big) = G(\Sigma_n)$ and consider the interval partition with partition boundaries $\langle \Gamma(n) \mid n < \omega \rangle$; for sufficiently large n, every interval

$$\begin{aligned} I_n := &\Big[\Gamma(n), \Gamma(n+1)\Big) = \Big[G\big(\sum_{k=0}^{n-1}(\Gamma(k))\big), G\big(\sum_{k=0}^{n}(\Gamma(k))\big)\Big) \\ = &\Big[G(\Sigma_{n-1}), G(\Sigma_{n-1}+1)\Big) \cup \ldots \cup \Big[G(\Sigma_{n-1} + \Gamma(n) - 1), G(\Sigma_{n-1} + \Gamma(n))\Big) \end{aligned}$$

contains at least $\Gamma(n)$ many elements of X and at most $\Gamma(n+1) - \Gamma(n)$ many of them.

The real defined as the union of every other interval, i. e. the intervals $I_{2k} = [\Gamma(2k), \Gamma(2k+1))$, will yield a real Y_X bisecting X infinitely often: Since the number of elements of X which are in any interval I_n is at least as large as the lower boundary of I_n, and since Y_X is defined to alternate between consecutive intervals, this means the relative initial density infinitely often reaches $1/2$, as each I_{2k} "pushes" the relative initial density above $1/2$ (and each I_{2k+1}, which is disjoint from Y_X, "pulls" it below $1/2$).

$\mathfrak{s}^{\infty}_{1/2} \leq$ **non**$(\mathcal{N})$: Given some $X \in [\omega]^\omega$ with enumerating function f_X and a Lebesgue-random set S (i. e. such that $\forall n < \omega\colon \Pr[n \in S] = 1/2$), the function $g(n) := |X \cap S \cap f_X(n)| - n/2$ defines a balanced random walk with step size $1/2$, since

$$g(n+1) - g(n) = \begin{cases} +1/2 & f_X(n) \in S, \\ -1/2 & f_X(n) \notin S. \end{cases}$$

From probability theory we know that for almost all S, $g(n)$ will be 0 infinitely often. Equivalently, almost surely,

$$\frac{g(n)}{n} + \frac{1}{2} = \frac{|X \cap S \cap f_X(n)|}{n}$$

will be $1/2$ infinitely often.

In other words, for any $X \in [\omega]^\omega$, the set of all S not bisecting X infinitely often is a null set. By contraposition, for any $X \in [\omega]^\omega$, any non-null set contains a set S that bisects X infinitely often.

$\mathfrak{s}_{1/2} \leq \mathrm{non}(\mathcal{N})$: Let $X \in [\omega]^\omega$ and $F \notin \mathcal{N}$. Enumerating $X =: \{x_0, x_1, x_2, \dots\}$, we define functions $f_{X,n}$ and f_X as follows:

$$f_{X,n} \colon [\omega]^\omega \to \{0,1\} \colon Y \mapsto \begin{cases} 0 & x_n \notin Y \\ 1 & x_n \in Y \end{cases}$$

$$f_X \colon [\omega]^\omega \to [0,1] \colon Y \mapsto \begin{cases} \lim_{k\to\infty} \dfrac{\sum_{n=1}^{k} f_{X,n}(Y)}{k} & \text{if the limit exists} \\ 0 & \text{otherwise} \end{cases}$$

It is clear that $\lambda(f_{X,n}^{-1}(\{1\})) = 1/2$. Hence, the $f_{X,n}$ are identically distributed random variables on the probability space $[\omega]^\omega$ with probability measure the Lebesgue measure λ. Moreover, they are independent and have finite variance. By the law of large numbers it follows that f_X is almost surely equal to $1/2$, in other words $\lambda(f_X^{-1}(\{1/2\})) = 1$. This means that with

$$S_X := \{Y \in [\omega]^\omega \mid f_X(Y) = 1/2\} = \{Y \in [\omega]^\omega \mid Y \mid_{1/2} X\},$$

we have that $\lambda(S_X) = 1$ and hence $S_X \notin \mathcal{N}$. Hence $F \cap S_X \neq \varnothing$ and there is some $S \in F$ such that $S \mid_{1/2} X$. Since all this holds for any $X \in [\omega]^\omega$, we have $\mathfrak{s}_{1/2} \leq \mathrm{non}(\mathcal{N})$.

$\mathrm{Con}(\mathrm{non}(\mathcal{M}) < \mathfrak{s}_{1/2\pm\varepsilon})$: This is implied by $\mathrm{Con}(\mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}))$ as witnessed by the Cohen model.

$\mathrm{Con}(\mathfrak{s}_{1/2}^\infty < \mathfrak{s}_{1/2\pm\varepsilon})$: This also follows from $\mathrm{Con}(\mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}))$.

$\mathrm{Con}(\mathfrak{s}_{1/2}^\infty < \mathrm{non}(\mathcal{M}))$, $\mathrm{Con}(\mathfrak{s}_{1/2}^\infty < \mathfrak{d})$ and $\mathrm{Con}(\mathfrak{s}_{1/2}^\infty < \mathrm{non}(\mathcal{N}))$: In the Cohen model, we have $\aleph_1 = \mathfrak{s} = \mathfrak{s}_{1/2}^\infty = \mathrm{non}(\mathcal{M}) < \mathrm{non}(\mathcal{N}) = \mathfrak{d}$; and in the random model, we have $\aleph_1 = \mathfrak{s}_{1/2}^\infty = \mathfrak{d} < \mathrm{non}(\mathcal{M})$.

$\mathrm{Con}(\mathrm{cov}(\mathcal{M}) < \mathfrak{s} \leq \mathfrak{s}_{1/2})$: In the Mathias model, we have $\mathrm{cov}(\mathcal{M}) < \mathfrak{s} = 2^{\aleph_0}$, see [Hal17, Theorem 26.14].

$\mathrm{Con}(\mathfrak{s}_{1/2} < \mathrm{non}(\mathcal{N}))$: See Theorem 3.5 in the subsequent section. □

Finally, we remark that $\mathfrak{b}$ is incomparable with all of our newly defined cardinal characteristics. This is because in the Cohen model, $\mathfrak{s}$ is strictly above $\mathfrak{b}$ and so are all of our characteristics; and in the Laver model, $\mathrm{non}(\mathcal{N})$ is strictly below $\mathfrak{b}$ and so are all of our characteristics.

3. Separating $\mathfrak{s}_{1/2}$ and $\mathrm{non}(\mathcal{N})$

To prove $\mathrm{Con}(\mathfrak{s}_{1/2} < \mathrm{non}(\mathcal{N}))$, we will use a typical creature forcing construction to increase $\mathrm{non}(\mathcal{N})$ and show that the forcing poset does not increase $\mathfrak{s}_{1/2}$.

We will not go into too much detail regarding creature forcing; see [RS99] for the most general and most detailed explanation. The specific forcing poset we use here also appears in [FGKS17] and [GK18].

Definition 3.1. We define a forcing poset $\mathbb{P}$ as follows: A condition $p \in \mathbb{P}$ is a sequence of *creatures* $p(k)$ such that each $p(k)$ is a non-empty subset of

$$\mathrm{POSS}_k := \left\{ F \subseteq 2^{I_k} \;\middle|\; \frac{|F|}{|2^{I_k}|} \geq 1 - \frac{1}{2^{a_k}} \right\}$$

for some sufficiently large consecutive intervals $I_k \subseteq \omega$ and strictly increasing $a_k < \omega$ (for our construction, let I_k be an interval of length 2^{2^k} and let $a_k := k$) and such that, letting the *norm* $\|\cdot\|$ of a creature C be defined by $\|C\| := \log_2 |C|$, p fulfils $\limsup_{k\to\infty} \|p(k)\| = \infty$. The order is $q \leq p$ iff $q(k) \subseteq p(k)$ for all $k < \omega$ (i. e. stronger conditions consist of smaller subsets of POSS_k). Note that $\mathbb{P} \neq \varnothing$ since $\limsup_{k\to\infty} \|\mathrm{POSS}_k\| = \infty$.

Given a condition p such as above, the finite initial segments in $p{\restriction}_{k+1}$ (for $k < \omega$) are sometimes referred to as *possibilities* and denoted by $\mathrm{poss}(p, \leq k) := \prod_{\ell \leq k} [p(\ell)]^1 = \{\langle \{z(\ell)\} \mid \ell \leq k \rangle \mid \forall \ell \leq k \colon z(\ell) \in p(\ell)\}$. We may also use the notation $\mathrm{poss}(p, <k) := \mathrm{poss}(p, \leq k - 1)$. When $\eta \in \mathrm{poss}(p, \leq k)$, we write $p \wedge \eta$ to denote $\eta^\frown p{\restriction}_{[k+1,\omega)}$.[1]

Define the forcing poset $\mathbb{Q}$ as the countable support product $\mathbb{Q} := \prod_{\alpha<\omega_2} \mathbb{Q}_\alpha$, where each $\mathbb{Q}_\alpha = \mathbb{P}$. We will work with the dense subset of *modest* conditions of $\mathbb{Q}$, i. e. conditions $p \in \mathbb{Q}$ such that for each $k < \omega$, there is at most one index α_k such that $|p(\alpha_k, k)| > 1$. We call such creatures $p(\alpha_k, k)$ *non-trivial.* (An easy bookkeeping argument shows that the modest conditions do indeed form a dense subset of $\mathbb{Q}$.) Modest conditions p have the advantage that for each $k < \omega$, $\mathrm{poss}(p, <k)$ is finite and even bounded by $\mathrm{maxposs}(<k) := \prod_{j<k} |\mathrm{POSS}_k|$, which makes iterating over all possibilities below a certain level possible.

By the usual Δ-system argument, CH implies that $\mathbb{Q}$ is $\aleph_2$-cc. (For details, see [FGKS17, Lemma 3.3.1] or [GK18, Lemma 4.18].) By the usual creature forcing arguments, it is clear that $\mathbb{Q}$ satisfies the finite version of Baumgartner's axiom A and hence is proper and ω^ω-bounding, that $\mathbb{Q}$ continuously reads all reals and that $\mathbb{Q}$ preserves all cardinals and cofinalities. (For details, see [FGKS17, section 5] or [GK18, sections 6–7].) In particular, given any condition $p \in \mathbb{Q}$ and any name $\dot{r}$ for a real, we can find $q \leq p$ such that each $\eta \in \mathrm{poss}(q, <k)$ already decides $\dot{r}{\restriction}_{\min(I_k)}$ (which we refer to as "*q reads $\dot{r}$ rapidly*"). We will reproduce an abbreviated version of the proof of $V^{\mathbb{Q}} \vDash \mathrm{non}(\mathcal{N}) \geq \aleph_2$ here:

Lemma 3.2. *Assuming* CH *in the ground model,* $\mathbb{Q}$ *forces that* $\mathrm{non}(\mathcal{N}) \geq \aleph_2$.

Proof. First, note that for $\alpha < \omega_2$, the generic object $\dot{R}_\alpha$ is a sequence of $\dot{R}_\alpha(k) \subseteq 2^{I_k}$ of relative size at least $1 - 1/2^{a_k}$. Since $\langle a_k \mid k < \omega \rangle$ is strictly increasing, it is clear that

$$\prod_{k<\omega} \left(1 - \frac{1}{2^{a_k}}\right) > 0$$

[1] The usual creature forcing notation defines the set of possibilities more abstractly as $\mathrm{poss}(p, \leq k) := \prod_{\ell \leq k} p(\ell)$ and defines $p \wedge \eta$ as a condition with an extended *trunk* (a concept which we did not deem necessary to introduce in our paper). Since working with possibilities η as sequences of singletons suffices for our proofs and is conceptually easier, we instead opted for this simpler definition.

and hence the set

$$\{r \in 2^\omega \mid \forall\, k < \omega \colon r{\restriction}_{I_k} \in \dot{R}_\alpha(k)\}$$

is positive and

$$\dot{N}_\alpha := \{r \in 2^\omega \mid \exists^\infty\, k < \omega \colon r{\restriction}_{I_k} \notin \dot{R}_\alpha(k)\}$$

is a name for a null set.

Now, given a name $\dot{r} \in 2^\omega$ for a real and a $p \in \mathbb{Q}$ which reads $\dot{r}$ rapidly, we can pick an $\alpha < \omega_2$ not in the support of p and add it to the support to get a (without loss of generality) modest condition p'; then p' still reads $\dot{r}$ rapidly not using the index α. Since we only require the lim sup of the norms to go to infinity, one can then show that $p' \Vdash \dot{r} \in \dot{N}_\alpha$. From this fact and $\aleph_2$-cc, it follows that for any $\kappa < \omega_2$, any sequence of names of reals $\langle \dot{r}_i \mid i < \kappa \rangle$ is contained in a null set of $V^{\mathbb{Q}}$.[2] ☐

We will now prove that the ground model reals are a bisecting family in $V^{\mathbb{Q}}$. To show this, we will use the following combinatorial lemma.

Lemma 3.3. *If $R, S \subseteq \omega$ are disjoint finite sets of sizes r and s, respectively, $s = c \cdot r$ for some $c > 1$, and $A \subseteq R$, $B \subseteq S$ such that*

$$\frac{|B|}{|S|} \in \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right)$$

for some $\varepsilon > 0$, then

$$\frac{|A \cup B|}{|R \cup S|} \in \left(\frac{1}{2} - \varepsilon - \frac{1}{c}, \frac{1}{2} + \varepsilon + \frac{1}{c}\right).$$

Proof. Since

$$\frac{1}{1 + 1/c} \geq 1 - \frac{1}{c},$$

we have the lower bound

$$\begin{aligned}\frac{|A \cup B|}{|R \cup S|} &> \frac{s \cdot (1/2 - \varepsilon)}{r + s} = \frac{s \cdot (1/2 - \varepsilon)}{s \cdot 1/c + s} = \frac{1/2 - \varepsilon}{1 + 1/c} \\ &\geq \left(\frac{1}{2} - \varepsilon\right)\left(1 - \frac{1}{c}\right) \geq \frac{1}{2} - \varepsilon - \frac{1}{c}.\end{aligned}$$

For the upper bound, we get

$$\begin{aligned}\frac{|A \cup B|}{|R \cup S|} &< \frac{r + s \cdot (1/2 + \varepsilon)}{r + s} = \frac{s \cdot 1/c + s \cdot (1/2 + \varepsilon)}{s \cdot 1/c + s} \\ &= \frac{1/2 + \varepsilon + 1/c}{1 + 1/c} \leq \frac{1}{2} + \varepsilon + \frac{1}{c}.\end{aligned}$$

☐

[2] The actual argument for $p \Vdash \dot{r} \in \dot{N}_\alpha$ involves a slightly more complicated norm than we defined above; however, since the parameters of the creature forcing poset $\mathbb{P}$ are immaterial for the more complicated proof in Lemma 3.4 below, we opted to omit the details for this paper. Details can be found in [GK18, section 11].

Lemma 3.4. *$2^\omega \cap V$ is a bisecting family in $V^{\mathbb{Q}}$.*

Proof. We will show the following: Given a modest condition $p \in \mathbb{Q}$ and a name $\dot{Y}$ for a real, we can find $q \le p$ and a ground model real X such that $q \Vdash X \mid_{1/2} \dot{Y}$.

In order to do this, we will construct $p^* \le p$ as well as $m_0 := 0 < m_1 < m_2 < \dots$ and choose $\langle P_i \mid i < \omega \rangle$ with $P_0 := 1/2$, $P_i > 0$ for all $i < \omega$ and $\lim_{i\to\infty} P_i = 0$ such that the following statements hold:

(i) The condition p^* is not only modest, but even fulfils that for each interval $J_i := [m_i, m_{i+1})$, there is exactly one $k_i \in J_i$ such that $|p^*(\alpha_{k_i}, k_i)| > 1$, i. e. such that the creature $C_i := p^*(\alpha_{k_i}, k_i)$ is non-trivial.

(ii) Due to continuous reading, we can find for each $\eta \in \operatorname{poss}(p^*, {<}k_i)$ and each $S \in C_i$ finite sets $Y_{\eta,S} \subseteq m_{i+1}$ and $Z_{\eta,S} \subseteq J_i$ such that

$$p^* \wedge (\eta^\frown\{S\}) \Vdash \dot{Y}\restriction_{m_{i+1}} = Y_{\eta,S} \text{ and } \dot{Y}\restriction_{J_i} = Z_{\eta,S}.$$

(iii) Note that due to property (i), $N_{i+1} := |\operatorname{poss}(p^*, {<}m_{i+1})| = |\operatorname{poss}(p^*, {\le}k_i)|$ only depends on the i-th creature $C_i = p^*(\alpha_{k_i}, k_i)$, since from $k_i + 1$ to m_{i+1}, there are only singletons in p^*. Hence we can choose m_{i+1} such that $m_{i+1} \gg N_{i+1}$.

(iv) For all $0 < i < \omega$, we have $N_i \ge i^6$. Additionally, let $N_1 = |C_0| \ge 100$. (This is possible without loss of generality since we can just "skip" creatures which do not have sufficiently many elements to fulfil these bounds.)

(v) Letting the name $\dot{M}_i$ denote the number of elements in $\dot{Y}\restriction_{[m_i, m_{i+1})}$, we can ensure that p^* forces for all $i < \omega$ that $\dot{M}_i \ge \max\{2i \cdot m_i, N_{i+1}\}$.

(vi) Letting $E_i := \lceil N_i \cdot P_i \rceil$, letting $e_i(\eta, S)$ be the E_i-th element of $Z_{\eta,S}$ and letting $e_i := \max_{\eta,S} e_i(\eta, S)$, we can finally choose m_{i+1} large enough such that $m_i + e_i < m_{i+1}$.

We now make a probabilistic argument using the following formulation of Chernoff's bound (see [AS16, Theorem A.1.1]): Given mutually independent random variables $\langle x_i \mid 1 \le i \le k \rangle$ with $\Pr[x_i = 0] = \Pr[x_i = 1] = 1/2$ for all $1 \le i \le k$ and letting $S_k := \sum_{1 \le i \le k} x_i$, it follows that for any $a > 0$,

$$\Pr\left[S_k - \frac{k}{2} > a\right] < \exp\left(-\frac{a^2}{2k}\right).$$

We use this bound as follows: Fix $n < \omega$. Let X be some randomly chosen subset of J_n and denote the probability space by Ω. Fix $\eta \in \operatorname{poss}(p^*, {<}k_n)$, $S \in C_n$ and $m \in J_n$ with $m \ge m_n + e_n(\eta, S)$. We consider the probability that this randomly chosen X does *not* bisect $Z_{\eta,S} \cap m$ with error at most $\frac{1}{2n}$; denote this event by $\mathrm{FAIL}(X, \eta, S, m)$.

Let $k \ge E_n$ denote the number of elements in $Z_{\eta,S} \cap m$. Then the choice of X (or, more precisely, the choice of the initial part of X relevant for this argument) amounts to tossing k fair coins x_j with values in $\{0, 1\}$, summing up the results and dividing by k, and comparing the gap between the result and $1/2$. By Chernoff's bound above we have

$$\Pr[\mathrm{FAIL}(X,\eta,S,m)] = \Pr\left[\sum_{1\le i\le k}\frac{x_i}{k}-\frac{1}{2} > \frac{1}{2n}\right] = \Pr\left[\sum_{1\le i\le k} x_i - \frac{k}{2} > \frac{k}{2n}\right]$$
$$< \exp\left(-\frac{(k/2n)^2}{2k}\right) = \exp\left(-\frac{k}{8n^2}\right).$$

Hence the probability of failing for at least one $m \in J_n$ (with $Z_{\eta,S}\cap m \ge E_n$) is bounded as follows (note that we only have to sum over the elements of $Z_{\eta,S}\cap m$):

$$\Pr[\mathrm{FAIL}(X,\eta,S)] := \Pr[\exists\, m \ge m_n + e_n(\eta,S)\colon\ \mathrm{FAIL}(X,\eta,S,m)]$$
$$< \sum_{k\ge E_n}\exp\left(-\frac{k}{8n^2}\right) = \frac{\exp(-E_n/8n^2)}{1-\exp(-1/8n^2)}$$

Using the fact that $\frac{1}{1-\exp(-x)} \le \frac{2}{x}$ for $x\in(0,1)$, we get

$$\Pr[\mathrm{FAIL}(X,\eta,S)] < 16n^2\cdot\exp\left(-\frac{E_n}{2n^2}\right) = 16n^2\cdot\exp\left(-\frac{\lceil N_n\cdot P_n\rceil}{2n^2}\right).$$

For the final step of our probabilistic estimate, we want to bound the probability of failing for at least one η, and we get

$$\Pr[\mathrm{FAIL}(X,S)] := \Pr[\exists\,\eta\colon\ \mathrm{FAIL}(X,\eta,S)] \le N_n\cdot 16n^2\cdot\exp(-\lceil N_n\cdot P_n\rceil/2n^2) =: \delta_n.$$

It is easy to see that $\delta_n < 1/2$ holds for e. g. $P_n := \max\{1/2, 1/n\}$ and $N_n \ge \min\{n^6, 100\}$, which holds by property (iv).

Now we make the following observation: If we count the number of pairs $\{\langle X,S\rangle \mid X\in\Omega, S\in C_n\}$ with $\mathrm{FAIL}(X,S)$, this total number of failures is bounded from above by $\delta_n\cdot|C_n|\cdot|\Omega|$. If we now assume that for each $X\in\Omega$, the number of $S\in C_n$ with $\mathrm{FAIL}(X,S)$ is at least F, then the total number of failures is bounded from below by $F\cdot|\Omega|$ – but this shows that $F\le\delta_n\cdot|C_n| < |C_n|/2$.

Summing up the entire probabilistic argument, this means that we can find some $X =: X_n \subseteq J_n$ and some $D_n\subseteq C_n$ with $|D_n| > |C_n|/2$ (and hence $\|D_n\| > \|C_n\| - 1$) such that for each $\eta\in\mathrm{poss}(p^*, {<}k_n)$, each $S\in D_n$ and each $m\ge m_n + e_n(\eta,S)$, we have that

$$\frac{|X_n\cap Z_{\eta,S}\cap m|}{|Z_{\eta,S}\cap m|} \in \left(\frac{1}{2}-\frac{1}{2n}, \frac{1}{2}+\frac{1}{2n}\right).$$

Now we perform the usual fusion construction, starting with $q_0 := p^*$, shrinking the creature C_n to D_n in the n-th step (and keeping everything below that from q_{n-1}), and constructing a fusion condition $q := \bigcap_{n<\omega} q_n$ as well as sets $X_n\subseteq J_n$. It is clear that the q constructed this way is a valid condition. We now claim that the set $X := \bigcup_{n<\omega} X_n$ is as required; in particular, we claim that for each $\varepsilon > 0$, there is an m_ε such that for all $m\ge m_\varepsilon$, we have

$$q \Vdash \frac{|X\cap\dot Y\cap m|}{|\dot Y\cap m|} \in \left(\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon\right).$$

We prove this inductively and will show that the error at any point $m<\omega$ is bounded by an expression that goes to 0 as n goes to infinity. Let $X_{<n} := \bigcup_{i<n} X_i$ for each $n<\omega$. For our induction hypothesis, assume that we already know that at m_n, the bisection error of $X_{<n}$ with each possible $Y_{\eta,S}\restriction_{m_n}$ is at most $^1/_{n-1}$. For each $m \in [m_n+1, m_{n+1}]$, we now have to consider the bisection error of $X_{<n+1}$ at m with each such $Y_{\eta,S}$.

- For $m \in [m_n+1, m_n+e_n(\eta,S))$, note that $Y_{\eta,S}\restriction_{m_n}$ has at least N_n elements by property (v), while $Y_{\eta,S}\restriction_{[m_n,m]}$ has at most $E_n = N_n \cdot P_n$ elements by property (vi). Thus we can apply Lemma 3.3 with $R := Y_{\eta,S}\restriction_{[m_n,m]}$, $S := Y_{\eta,S}\restriction_{m_n}$, $\varepsilon := {}^1/_{n-1}$ and some $c > {}^1/_{P_n}$ to get
$$\begin{aligned}\frac{|X_{<n+1} \cap Y_{\eta,S} \cap m|}{|Y_{\eta,S} \cap m|} &\in \left(\frac{1}{2} - \frac{1}{n-1} - \frac{1}{c}, \frac{1}{2} + \frac{1}{n-1} + \frac{1}{c}\right)\\ &\subseteq \left(\frac{1}{2} - \frac{1}{n-1} - P_n, \frac{1}{2} + \frac{1}{n-1} + P_n\right)\\ &\subseteq \left(\frac{1}{2} - \frac{2}{n-1}, \frac{1}{2} + \frac{2}{n-1}\right).\end{aligned}$$
- For $m \in [m_n + e_n(\eta,S), m_{n+1}]$, it is clear that
$$\frac{|X_{<n+1} \cap Y_{\eta,S} \cap m|}{|Y_{\eta,S} \cap m|} \in \left(\frac{1}{2} - \frac{1}{n-1}, \frac{1}{2} + \frac{1}{n-1}\right),$$
since the error on $Y_{\eta,S}\restriction_{m_n}$ is at most $^1/_{n-1}$ and the error on $Y_{\eta,S}\restriction_{[m_n,m]}$ is at most $^1/_n$.
- For $m = m_{n+1}$, however, we have to show even more to ensure that our induction hypothesis remains true for the next step. So note that $Y_{\eta,S}\restriction_{m_n}$ has at most m_n elements, while $Y_{\eta,S}\restriction_{[m_n,m_{n+1}]}$ has at least $2n \cdot m_n$ elements by property (v). Thus we can apply Lemma 3.3 once more with $R := Y_{\eta,S}\restriction_{m_n}$, $S := Y_{\eta,S}\restriction_{[m_n,m_{n+1}]}$, $\varepsilon := {}^1/_{2n}$ and some $c \geq 2n$ to get
$$\begin{aligned}\frac{|X_{<n+1} \cap Y_{\eta,S} \cap m_{n+1}|}{|Y_{\eta,S} \cap m_{n+1}|} &\in \left(\frac{1}{2} - \frac{1}{2n} - \frac{1}{c}, \frac{1}{2} + \frac{1}{2n} + \frac{1}{c}\right)\\ &\subseteq \left(\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}\right),\end{aligned}$$
which is precisely the induction hypothesis for $n+1$.

Given any $\varepsilon > 0$, pick some n_ε such that $\frac{2}{n_\varepsilon - 1} < \varepsilon$ and let $m_\varepsilon := m_{n_\varepsilon}$. Then for all $m \geq m_\varepsilon$, by the bounds above
$$q \Vdash \frac{|X \cap \dot{Y} \cap m|}{|\dot{Y} \cap m|} \in \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right),$$
finishing the proof. ☐

Theorem 3.5. $\mathrm{Con}(\mathfrak{s}_{1/2} < \mathrm{non}(\mathcal{N}))$.

Proof. Assume CH in the ground model; then the statement follows by combining Lemma 3.2 and Lemma 3.4. ☐

4. Characteristics Related to $\mathfrak{r}$ and $\mathfrak{i}$

We define a second set of properties more closely related to $\mathfrak{i}$, although $\mathfrak{s}$ does reappear in this section.

Definition 4.1. A set $X \in [\omega]^\omega$ is *moderate* if $\underline{d}(X) > 0$ as well as $\bar{d}(X) < 1$.[3]

Definition 4.2. A family $\mathcal{I}_* \subseteq [\omega]^\omega$ is *statistically independent* or *$*$-independent* if for any set $X \in \mathcal{I}_*$ we have that X is moderate and for any finite subfamily $\mathcal{E} \subseteq \mathcal{I}_*$, the following holds:

$$\lim_{n\to\infty} \left(\frac{d_n\left(\bigcap_{E\in\mathcal{E}} E\right)}{\prod_{E\in\mathcal{E}} d_n(E)} \right) = 1$$

In the case of convergence of $d_n(X)$, this simplifies to asking for $0 < d(X) < 1$ to hold for all $X \in \mathcal{I}_*$ and

$$\prod_{E\in\mathcal{E}} d(E) = d\big(\bigcap_{E\in\mathcal{E}} E\big)$$

to hold for any finite subfamily $\mathcal{E} \subseteq \mathcal{I}_*$.

We denote the least cardinality of a maximal $*$-independent family by $\mathfrak{i}_*$.

Recall that a family $\mathcal{I}$ of subsets of ω is called *independent* if for any disjoint finite subfamilies $\mathcal{A}, \mathcal{B} \subseteq \mathcal{I}$, the set

$$\bigcap_{A\in\mathcal{A}} A \cap \bigcap_{B\in\mathcal{B}} (\omega \setminus B)$$

is infinite. Generalising this notion leads to the following definitions (which are more obviously related to the classical $\mathfrak{i}$):

Definition 4.3. Let $\rho \in (0,1)$. A family $\mathcal{I}_\rho \subseteq [\omega]^\omega$ is *ρ-independent* if for any disjoint finite subfamilies $\mathcal{A}, \mathcal{B} \subseteq \mathcal{I}_\rho$, the following holds:

$$d\left(\bigcap_{A\in\mathcal{A}} A \cap \bigcap_{B\in\mathcal{B}} (\omega \setminus B) \right) = \rho^{|\mathcal{A}|} \cdot (1-\rho)^{|\mathcal{B}|},$$

which simplifies to $= {}^1\!/_2{}^{|\mathcal{A}|+|\mathcal{B}|}$ in the case of $\rho = {}^1\!/_2$. This definition is equivalent to demanding that for any finite $\mathcal{A} \subseteq \mathcal{I}_\rho$, the following holds:

$$d\left(\bigcap_{A\in\mathcal{A}} A \right) = \rho^{|\mathcal{A}|}$$

We denote the least cardinality of a maximal ρ-independent family by $\mathfrak{i}_\rho$.

Recalling the definition of $\mathfrak{r}$ as the least cardinality of a family $\mathcal{R} \subseteq [\omega]^\omega$ such that no $S \in [\omega]^\omega$ splits every $R \in \mathcal{R}$, we naturally arrive at the following definition:

Definition 4.4. A family $\mathcal{R}_{1/2} \subseteq [\omega]^\omega$ is *${}^1\!/_2$-reaping* if there is no $S \in [\omega]^\omega$ bisecting all $R \in \mathcal{R}_{1/2}$. We denote the least cardinality of a ${}^1\!/_2$-reaping family by $\mathfrak{r}_{1/2}$.

Given the above, the natural question is: Can we define $\mathfrak{r}_*$ analogously? Consider the following definition:

[3] Actually, it would suffice to demand $\bar{d}(X) > 0$ as well as $\underline{d}(X) < 1$, though one would have to modify a few of the subsequent proofs.

Definition 4.5. A family $\mathcal{R}_* \subseteq [\omega]^\omega$ is *statistically reaping* or $*$-*reaping* if

$$\nexists\, S \in [\omega]^\omega \text{ moderate such that } \forall\, X \in \mathcal{R}_*\colon \lim_{n\to\infty}\left(\frac{d_n\,(S\cap X)}{d_n(S)\cdot d_n(X)}\right) = 1.$$

We denote the least cardinality of a $*$-reaping family by $\mathfrak{r}_*$.

The motivation for this is as follows: Considering the analogous definitions for $\mathfrak{r}$, we might call $\mathcal{I}$ *maximal quasi-independent* if there is no X such that for all $Y \in \mathcal{I}$ we have that X splits Y and X splits $\omega \smallsetminus Y$ (i. e. X and Y are independent for all $Y \in \mathcal{I}$). It is obvious that a reaping family is also maximal quasi-independent; the converse can easily be derived by taking a maximal quasi-independent family and saturating it (without increasing its size) by adding the complements of all its sets, resulting in a reaping family. By this train of thought, it makes sense to take Definition 4.5 as the defining property of a $*$-reaping family.

Dualising the definition of $*$-reaping leads to the following, final definition:

Definition 4.6. A family $\mathcal{S}_* \subseteq [\omega]^\omega$ is *statistically splitting* or $*$-*splitting* if

$$\forall\, X \in [\omega]^\omega\ \exists\, S \in \mathcal{S}_* \text{ moderate}\colon \lim_{n\to\infty}\left(\frac{d_n\,(S\cap X)}{d_n(S)\cdot d_n(X)}\right) = 1.$$

We denote the least cardinality of a $*$-splitting family by $\mathfrak{s}_*$.

Theorem 4.7. *The relations shown in Figure 2 hold.*

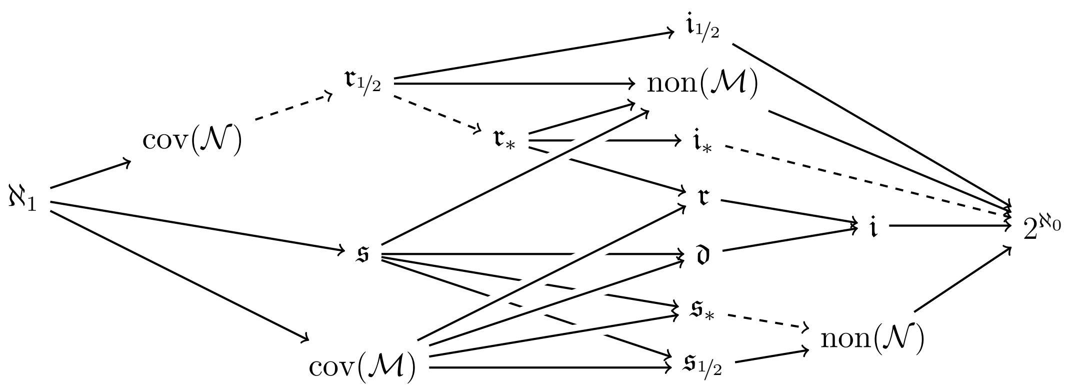

FIGURE 2. The ZFC-provable and/or consistent inequalities between $\mathfrak{i}_{1/2}$, $\mathfrak{i}_*$, $\mathfrak{r}_{1/2}$, $\mathfrak{r}_*$, $\mathfrak{s}_{1/2}$, $\mathfrak{s}_*$ and other well-known cardinal characteristics, where $\longrightarrow$ means "$\leq$, consistently $<$" and $\dashrightarrow$ means "$\leq$, possibly $=$".

Proof. $\mathbf{cov(\mathcal{N})} \leq \mathfrak{r}_{1/2}$ and $\mathfrak{s}_* \leq \mathbf{non(\mathcal{N})}$: Both proofs are analogous to the proof of $\mathfrak{s}_{1/2} \leq \mathrm{non}(\mathcal{N})$.

For the first claim, let $\mathcal{R}_{1/2}$ be a family witnessing the value of $\mathfrak{r}_{1/2}$. By the argument for $\mathfrak{s}_{1/2} \leq \mathrm{non}(\mathcal{N})$ in the proof of Theorem 2.4, the family

$$\{[\omega]^\omega \smallsetminus \mathcal{S}_R \mid R \in \mathcal{R}_{1/2}\}$$

is a covering of $\mathcal{N}$. (Recall that $[\omega]^\omega \smallsetminus \mathcal{S}_R \in \mathcal{N}$ for $R \in \mathcal{R}_{1/2}$.)

For the second claim, let $X \in [\omega]^\omega$ and $F \notin \mathcal{N}$. As seen above, letting

$$S_X = \{Y \in [\omega]^\omega \mid Y \,|_{1/2}\, X\},$$

we have that $\lambda(S_X) = 1$ and hence $S_X \notin \mathcal{N}$. Moreover, this is true in particular for $X = \omega$ and

$$S_\omega = \{Y \in [\omega]^\omega \mid Y \,|_{1/2}\, \omega\} = \{Y \in [\omega]^\omega \mid d(Y) = 1/2\}.$$

Since then $F \cap S_X \cap S_\omega \neq \varnothing$, there is some $S \in F$ such that $S \,|_{1/2}\, X$ and $d(S) = 1/2$, which implies $S \,|_*\, X$.

Since all this is true for any $X \in [\omega]^\omega$, we have $\mathfrak{s}_* \leq \mathrm{non}(\mathcal{N})$.

$\mathfrak{r}_{1/2} \leq \mathfrak{r}_*$: Let $\mathcal{R}_*$ be a $*$-reaping family and let $\mathcal{R}_{1/2} := \mathcal{R}_* \cup \{\omega\}$; clearly, $|\mathcal{R}_{1/2}| = |\mathcal{R}_*|$. Now, any S which bisects all $R \in \mathcal{R}_{1/2}$ also $*$-splits all $R \in \mathcal{R}_*$ – this follows from the fact that $S \,|_{1/2}\, \omega$ implies $d(S) = 1/2$, and hence for any $R \in \mathcal{R}_*$, we now have

$$\frac{d_n(S \cap R)}{d_n(S) \cdot d_n(R)} = \frac{d_n(S \cap R)}{d_n(R)} \cdot \frac{1}{d_n(S)} \to 1,$$

since $S \,|_{1/2}\, R$ implies that the first factor converges to $1/2$, while $d(S) = 1/2$ implies that the second factor converges to 2.

$\mathfrak{r}_{1/2} \leq \mathbf{non}(\mathcal{M})$: Since the set of all reals bisected by a fixed real S is a meagre set (by the argument for $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{s}_{1/2\pm\varepsilon}$), a non-meagre set contains some real not bisected by S and hence is $1/2$-reaping.

$\mathfrak{r}_* \leq \mathbf{non}(\mathcal{M})$: This is analogous to the proof of $\mathfrak{r}_{1/2} \leq \mathrm{non}(\mathcal{M})$, since the set of all reals $*$-split by a fixed moderate real S is a meagre set, as well. To see this, define a chopped real based on S with the interval partition having the partition boundaries at the $n!$-th elements of S; the sets matching this chopped real form a comeagre set which consists of reals X not $*$-split by S: As the matching intervals grow longer and longer, they "pull" $\frac{d_n(S\cap X)}{d_n(X)}$ above $1 - 1/n$, which implies that $\frac{d_n(S\cap X)}{d_n(S)\cdot d_n(X)}$ cannot converge to 1 as $d_n(S)$ does not converge to 1 by the moderacy of S.

$\mathbf{cov}(\mathcal{M}) \leq \mathfrak{s}_*$: This is analogous to the proof of $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{s}_{1/2}$ by the same argument as in the proof of $\mathfrak{r}_* \leq \mathrm{non}(\mathcal{M})$.

$\mathfrak{s} \leq \mathfrak{s}_*$: Let $\mathcal{S}_*$ be a family witnessing the value of $\mathfrak{s}_*$ and let $X \in [\omega]^\omega$ be arbitrary. We will prove by contradiction that there must be some $S \in \mathcal{S}_*$ splitting X. Suppose not, that is, suppose that for any $S \in \mathcal{S}_*$, either (a) $S \cap X$ is finite or (b) $S \cap X$ is cofinite. In case (a), we use the fact that S is moderate to see that $d_n(S)$ must eventually be bounded from below by some ε, and the fact that $S \cap X$ is finite to see that $|S \cap X \cap n|$ is bounded by some k^*. Letting $k_n := |X \cap n|$, this eventually yields

$$\frac{d_n\,(S \cap X)}{d_n(S) \cdot d_n(X)} \leq \frac{k^*/n}{\varepsilon \cdot k_n/n} = \frac{k^*}{\varepsilon \cdot k_n} \to 0.$$

Similarly, in case (b) we use the moderacy of S to see that $d_n(S)$ is eventually bounded from above by some $1 - \delta$, and the fact that $S \cap X$ is cofinite to see that $|S \cap X \cap n|$ is bounded from below by $k_n - k^*$ for some k^*. (This bound simply states that after some finite aberrations, S contains all elements of X.) Taken together, we eventually have

$$\frac{d_n\,(S\cap X)}{d_n(S)\cdot d_n(X)} \geq \frac{{(k_n-k^*)}/{n}}{(1-\delta)\cdot {k_n}/{n}} = \frac{1}{1-\delta} - \frac{k^*}{(1-\delta)\cdot k_n} \to \frac{1}{1-\delta} = 1+\varepsilon$$

for some $\varepsilon > 0$. In summary, for all $S \in \mathcal{S}_*$ we have that S does not $*$-split X, and hence $\mathcal{S}_*$ could not have been a witness for the value of $\mathfrak{s}_*$.

$\mathfrak{r}_{1/2} \leq \mathfrak{i}_{1/2}$ and $\mathfrak{r}_* \leq \mathfrak{i}_*$: For the first claim, let $\mathcal{I}_{1/2}$ be a maximal $1/2$-independent family. Define

$$\mathcal{R}_{1/2} := \left\{ \bigcap_{A\in\mathcal{A}} A \cap \bigcap_{B\in\mathcal{B}} (\omega \setminus B) \;\middle|\; \mathcal{A},\mathcal{B} \subseteq \mathcal{I}_{1/2}, \mathcal{A}\cap\mathcal{B} = \varnothing \right\}.$$

Then $\mathcal{R}_{1/2}$ is a $1/2$-reaping family, since the existence of an $S \in [\omega]^\omega$ bisecting each $R \in \mathcal{R}_{1/2}$ (in the limit) would contradict the maximality of $\mathcal{I}_{1/2}$.

The proof of the second claim is analogous: Take all finite tuples of sets in the witness $\mathcal{I}_*$ of the value of $\mathfrak{i}_*$ and collect their Boolean combinations in a family $\mathcal{R}_*$; this family must then be $*$-reaping, because a set S $*$-splitting each $R \in \mathcal{R}_*$ would violate the maximality of $\mathcal{I}_*$, and thus $\mathcal{R}_*$ witnesses $\mathfrak{r}_* \leq \mathfrak{i}_*$.

$\mathfrak{i}_\rho \leq 2^{\aleph_0}$ and $\mathfrak{i}_* \leq 2^{\aleph_0}$: For $\mathfrak{i}_\rho$, consider the collection $\mathcal{I}_\rho$ of all ρ-independent families. Now, $\mathcal{I}_\rho$ has finite character, i.e. for each $I \subseteq 2^{\aleph_0}$, I belongs to $\mathcal{I}_\rho$ iff every finite subset of I belongs to $\mathcal{I}_\rho$. Hence we can apply Tukey's lemma and see that $\mathcal{I}_\rho$ has a maximal element with respect to inclusion. Therefore, $\mathfrak{i}_\rho$ is well defined and hence $\mathfrak{i}_\rho \leq 2^{\aleph_0}$. The proof for $\mathfrak{i}_*$ is analogous.

Con($\mathfrak{r}_* < \mathfrak{r}$): This follows from Con(non($\mathcal{M}$) $<$ cov($\mathcal{M}$)), but we also have an explicit proof of this.

We will show that Cohen forcing does not increase $\mathfrak{r}_*$ due to the ground model reals remaining $*$-reaping; we already know that Cohen forcing increases $\mathfrak{r}$, proving our consistency statement.

Let $\dot{X}$ be a $\mathbb{C}$-name for a real. We will construct a ground model real Y such that for any $q \in \mathbb{C}$, we can find $r \leq q$ such that $r \Vdash \dot{X} \not\mid_* Y$.

Let $\varphi(\dot{X})$ be the statement $\forall\, k\ \exists\, \ell_0, \ell_1 > k\colon \dot{X}(\ell_0) = 0 \wedge \dot{X}(\ell_1) = 1$. Let $D_{\mathrm{good}} := \{p \in \mathbb{C} \mid p \Vdash \varphi(\dot{X})\}$ and $D_{\mathrm{bad}} := \{p \in \mathbb{C} \mid p \Vdash \neg\varphi(\dot{X})\}$ and note that $D := D_{\mathrm{good}} \cup D_{\mathrm{bad}}$ is open dense in $\mathbb{C}$. Since it is clear that any $q \in D_{\mathrm{bad}}$ already forces that $\dot{X}$ is not moderate, we only need to consider $q \in D_{\mathrm{good}}$.

Now pick an enumeration $\langle p_k \mid k < \omega\rangle$ of D_{good} which enumerates each element infinitely often. In the following argument, for each $k < \omega$, let $L_k := \sum_{\ell \leq k} \ell_k$.

- For $k = 0$, we find $q_0 \leq p_0$, $\ell_0 \geq 2$ and $A_0 \subseteq [0, \ell_0)$ such that q_0 decides $\dot{X}{\restriction}_{\ell_0}$, $q_0 \Vdash \dot{X}{\restriction}_{\ell_0} = A_0$ and such that $|A_0| \geq 1$, $|[0,\ell_0) \setminus A_0| \geq 1$, and at least one of these two inequalities is an equality.
- For $0 < k < \omega$, we find $q_k \leq p_k$, $\ell_k < \omega$ and $A_k \subseteq [L_{k-1}, L_k)$ such that q_k decides $\dot{X}{\restriction}_{L_k}$, $q_k \Vdash \dot{X}{\restriction}_{[L_{k-1},L_k)} = A_k$ and such that $|A_k| \geq 3L_{k-1}$,

$|[L_{k-1}, L_k) \setminus A_k| \geq 3L_{k-1}$, and at least one of these inequalities is an equality.

Define Y piecewise by $Y\restriction_{[L_{k-1},L_k)} := A_k$. Assume $\dot{X}$ $*$-splits Y; then there must be some $q \in \mathbb{C}$ forcing this. It is clear that $q \perp D_{\text{bad}}$. In particular, this means that q forces that for any $\varepsilon > 0$, there is some $m_\varepsilon < \omega$ such that for any $j > m_\varepsilon$,

$$\frac{d_j(\dot{X} \cap Y)}{d_j(\dot{X}) \cdot d_j(\dot{Y})} > 1 - \varepsilon.$$

Pick some sufficiently small ε, say $\varepsilon := 2/9$, and find $n < \omega$ such that $p_n = q$ and $L_n > m_{1/4}$. Letting O_n and I_n be the number of 0s and 1s in A_n, respectively, $q_n \leq q$ forces

$$d_{L_n}(\dot{X} \cap Y) \leq \frac{L_{n-1}}{L_n},$$
$$d_{L_n}(\dot{X}) \geq \frac{I_n}{L_n},$$
$$d_{L_n}(Y) \geq \frac{O_n}{L_n}.$$

Without loss of generality, $O_n = 3L_{n-1}$ and $I_n = 3L_{n-1} + \Delta$ for some $\Delta < \omega$. Then q_n forces

$$\frac{d_{L_n}(\dot{X} \cap Y)}{d_{L_n}(\dot{X}) \cdot d_{L_n}(\dot{Y})} \leq \frac{\frac{L_{n-1}}{L_n}}{\frac{O_n I_n}{L_n^2}} = \frac{L_{n-1}L_n}{O_n I_n} = \frac{L_{n-1}(L_{n-1} + O_n + I_n)}{O_n I_n}$$
$$= \frac{L_{n-1}(7L_{n-1} + \Delta)}{3L_{n-1}(3L_{n-1} + \Delta)} = \frac{7L_{n-1} + \Delta}{3 \cdot (3L_{n-1} + \Delta)},$$

which is strictly decreasing in Δ and is $7/9$ for $\Delta = 0$. This contradicts the assumption on q, proving that $\dot{X}$ does not $*$-split Y in $V^{\mathbb{C}}$.

Hence assuming CH in the ground model and forcing with $\mathbb{C}_\lambda$ for some $\lambda \geq \aleph_2$ with $\lambda = \lambda^{\aleph_0}$ gives us $V^{\mathbb{C}_\lambda} \vDash \mathfrak{r}_* = \aleph_1 < \lambda = \mathfrak{r} = \mathfrak{c}$.

Con($\mathfrak{r}_{1/2}$ < non($\mathcal{M}$)) and Con($\mathfrak{r}_*$ < non($\mathcal{M}$)): This follows from Con($\mathfrak{r} <$ non($\mathcal{M}$)), see [BJ95, Model 7.5.9].

Con($\mathfrak{s} < \mathfrak{s}_*$): Just like Con($\mathfrak{r}_* < \mathfrak{r}$), this follows from Con(non($\mathcal{M}$) < cov($\mathcal{M}$)), but once more, we also have an explicit proof of this.

We will show that Cohen forcing increases $\mathfrak{s}_*$ due to the Cohen real not being $*$-split by any real from the ground model; we already know that Cohen forcing keeps $\mathfrak{s}$ small, proving our consistency statement.

The proof uses the same technique as the one for $\mathfrak{s} \leq \mathfrak{s}_*$: Given some moderate $X \in [\omega]^\omega \cap V$, with moderacy in the sense of $\bar{d}(X) = 1 - 2\varepsilon$ and $d_n(X) < 1 - \varepsilon$ for all $n \geq n_0$ for some n_0, we will show that the assumption that there is a condition forcing $X \mid_* \dot{C}$, i. e. that X $*$-splits the Cohen real, leads to a contradiction.

So suppose that there were some $p \in \mathbb{C}$ such that $p \Vdash X \mid_* \dot{C}$; more specifically, suppose that for some n_1, even $p \Vdash \frac{d_n(X \cap \dot{C})}{d_n(X) \cdot d_n(\dot{C})} < 1 - \delta$ for all $n \geq n_1$, where $\delta := \frac{\varepsilon/2}{1-\varepsilon}$.

We now define $q \leq p$ as follows: Let n_2 be large enough such that

$$\frac{|p|}{|X \cap n_2|} < \frac{\varepsilon}{2} \quad \Longleftrightarrow \quad \frac{2 \cdot |p|}{\varepsilon} < |X \cap n_2|;$$

this is possible due to the moderacy of X (which implies X is infinite). Let $k := \max\{n_0, n_1, n_2\}$ and $q := p^\frown \chi_X \restriction_{[|p|+1,k]}$, that is, extend p by the next $k - |p|$ values of the characteristic function of X. Then we have

$$\frac{d_k(X \cap \dot{C})}{d_k(X) \cdot d_k(\dot{C})} > \frac{1}{1-\varepsilon} \cdot \frac{d_k(X \cap \dot{C})}{d_k(\dot{C})}$$

by the moderacy of X. By our choice of q, we have

$$q \Vdash \frac{d_k(X \cap \dot{C})}{d_k(\dot{C})} = \frac{|X \cap \dot{C} \cap k|}{|\dot{C} \cap k|} \geq \frac{|X \cap k| - |p|}{|X \cap k|} = 1 - \frac{|p|}{|X \cap k|} > 1 - \frac{\varepsilon}{2},$$

with the first inequality being an equality in the "worst case" of $X\restriction_{|p|+1} \equiv 1$ and $(p = q\restriction_{|p|+1} =)\, \dot{C}\restriction_{|p|+1} \equiv 0$. This implies that

$$q \Vdash \frac{d_k(X \cap \dot{C})}{d_k(X) \cdot d_k(\dot{C})} > \frac{1 - \varepsilon/2}{1-\varepsilon} = 1 + \delta,$$

contradictory to the original assumption on p.

Con(cov($\mathcal{M}$) $< \mathfrak{s} \leq \mathfrak{s}_*$): Follows as in the proof of Con(cov($\mathcal{M}$) $< \mathfrak{s} \leq \mathfrak{s}_{1/2}$).

Con($\mathfrak{r}_{1/2} < \mathfrak{i}_{1/2}$) and Con($\mathfrak{r}_* < \mathfrak{i}_*$): See Lemma 4.8 and Corollary 4.9 below.

Con($\mathfrak{i}_{1/2} < 2^{\aleph_0}$): This follows from Lemma 4.11 below. □

Lemma 4.8. Con($\mathfrak{r}_{1/2} < \mathfrak{i}_{1/2}$).

Proof. We will prove the following: Assume CH in the ground model and let $\lambda > \mu > \aleph_1$ be regular cardinals with $\lambda = \lambda^{\aleph_0}$. Then there is a forcing extension satisfying $\mathrm{add}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \mathfrak{r}_{1/2} = \mu$ and $\mathfrak{c} = \mathfrak{i}_{1/2} = \lambda$.

We prove this by using the forcing $\mathbb{P}\restriction_{(L,\mathcal{I})}$ and the model from [Bre02, Proposition 4.7]; this is essentially the fifth author's original template model (see [Bre02, Theorem 3.3]) with localisation forcing instead of Hechler forcing. It is shown in [Bre02] that this model satisfies $\mathrm{add}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \mu$; since we know that $\mathrm{add}(\mathcal{N}) \leq \mathrm{cov}(\mathcal{N}) \leq \mathfrak{r}_{1/2} \leq \mathrm{non}(\mathcal{M}) \leq \mathrm{cof}(\mathcal{N})$, we also have $\mathfrak{r}_{1/2} = \mu$.

To show that $\mathfrak{i}_{1/2} = \lambda$ holds in this model, we use the isomorphism-of-names argument from [Bre02, Theorem 3.3]. Although the original proof of Theorem 3.3 uses Hechler forcing, it was already remarked in [Bre02] that this is irrelevant to the isomorphism-of-names argument as long as we use the same template. We will not reproduce the full extent of the argument here, but instead only point out the few differences.[4]

Let $\dot{\mathcal{A}} = \{\dot{A}^\alpha \mid \alpha < \kappa\}$ be a name for a $1/2$-independent family of size $\kappa < \lambda$; we have to show that $\dot{\mathcal{A}}$ is not maximal in $V^{\mathbb{P}\restriction_{(L,\mathcal{I})}}$. By $\mathfrak{r}_{1/2} \leq \mathfrak{i}_{1/2}$, we may assume $\mu \leq \kappa$; for technical reasons, we actually want to assume that $\omega_2 \cdot 2 \leq \kappa$. We now obtain the B^α as in the proof of Theorem 3.3 and use them to construct B^κ and the name $\dot{A}^\kappa$ in the same way. The pruning arguments and other details of

[4] For a general approach to and explanation of template forcing, see [Bre05].

the construction depend neither on the specific forcing poset nor on the particular properties of the names $\dot{A}^\alpha$, but only on the structure of the template, so every step of the proof works exactly as in [Bre02].

The only part we need to replace is the final paragraph ([Bre02, p. 23]). We instead observe that for any finite $F \subseteq \kappa$, we can find $\alpha < \omega_1$

- such that $B^\kappa \cup \bigcup_{\beta \in F} B^\beta$ and $B^\alpha \cup \bigcup_{\beta \in F} B^\beta$ are order isomorphic via the mapping fixing nodes of $\bigcup_{\beta \in F} B^\beta$ and moving B^κ to B^α, and
- such that the template restricted to $B^\kappa \cup \bigcup_{\beta \in F} B^\beta$ is basically the same as the template restricted to $B^\alpha \cup \bigcup_{\beta \in F} B^\beta$.[5]

Hence the posets $\mathbb{P}\restriction_{B^\kappa \cup \bigcup_{\beta \in F} B^\beta}$ and $\mathbb{P}\restriction_{B^\alpha \cup \bigcup_{\beta \in F} B^\beta}$ are isomorphic (and both are subforcings of the forcing poset $\mathbb{P}\restriction_{(L,\mathcal{I})}$). Since we know that $\mathbb{P}\restriction_{B^\alpha \cup \bigcup_{\beta \in F} B^\beta}$ forces that $\{\dot{A}^\alpha\} \cup \{\dot{A}^\beta \mid \beta \in F\}$ is a $1/2$-independent family, $\mathbb{P}\restriction_{B^\kappa \cup \bigcup_{\beta \in F} B^\beta}$ forces that $\{\dot{A}^\kappa\} \cup \{\dot{A}^\beta \mid \beta \in F\}$ is a $1/2$-independent family. Since $F \subseteq \kappa$ was arbitrary, this shows that $\{\dot{A}^\alpha \mid \alpha \leq \kappa\}$ is forced to be a $1/2$-independent family in $V^{\mathbb{P}\restriction_{(L,\mathcal{I})}}$, which shows that $\dot{\mathcal{A}}$ is not maximal in $V^{\mathbb{P}\restriction_{(L,\mathcal{I})}}$. $\square$

We remark that the construction in [Bre03] can be modified analogously to show that $\mathfrak{i}_{1/2}$ can have countable cofinality.

Corollary 4.9. Con($\mathfrak{r}_* < \mathfrak{i}_*$).

Proof. Replacing the names for $1/2$-independent families $\dot{\mathcal{A}}$ with names for $*$-independent families, the same proof as in Lemma 4.8 shows the analogous result. $\square$

For the final proof of this section, we will require another combinatorial lemma.

Lemma 4.10. *If $R, S \subseteq \omega$, $0 < r < 1$, $\varepsilon > 0$ and $m < n$ are such that*

$$\frac{|R \cap m|}{m} \in (r - \varepsilon, r + \varepsilon)$$

and for all ℓ with $m \leq \ell \leq n$, we have

$$\frac{|S \cap \ell|}{\ell} \in (r - \varepsilon, r + \varepsilon),$$

then for all ℓ with $m \leq \ell \leq n$, we have

$$\frac{|(R \cap m) \cup (S \cap [m, \ell))|}{\ell} \in (r - 3\varepsilon, r + 3\varepsilon).$$

Proof. Suppose this were false for some $\ell^* \geq m$; then without loss of generality,

$$\frac{|(R \cap m) \cup (S \cap [m, \ell^*))|}{\ell^*} \geq r + 3\varepsilon.$$

Since

$$\frac{|R \cap m|}{m} < r + \varepsilon,$$

[5] Using the terms of [Bre02], this means α is such that $\mathcal{I}\restriction_{B^\kappa \cup \bigcup_{\beta \in F} B^\beta}$ is an innocuous extension of the image of $\mathcal{I}\restriction_{B^\alpha \cup \bigcup_{\beta \in F} B^\beta}$.

we get

$$\frac{|S \cap [m, \ell^*)|}{\ell^*} \geq r + 3\varepsilon - \frac{m}{\ell^*}(r + \varepsilon).$$

But then

$$\frac{|S \cap m|}{m} > r - \varepsilon$$

implies

$$\begin{aligned}\frac{|S \cap \ell^*|}{\ell^*} &= \frac{|(S \cap m) \cup (S \cap [m, \ell^*))|}{\ell^*} > \frac{m}{\ell^*}(r - \varepsilon) + r + 3\varepsilon - \frac{m}{\ell^*}(r + \varepsilon) \\ &= r + 3\varepsilon - \frac{2m}{\ell^*} \cdot \varepsilon \geq r + \varepsilon,\end{aligned}$$

which is a contradiction. □

Lemma 4.11. $\mathrm{Con}(\mathfrak{i}_{1/2} < \mathfrak{i})$.

Proof. The proof is analogous to the classical proof of $\mathrm{Con}(\aleph_1 = \mathfrak{a} < 2^{\aleph_0})$ (see e. g. [Hal17, Proposition 18.5]).

Assume CH in the ground model and let $\lambda \geq \aleph_2$. We force with the λ-Cohen forcing poset $\mathbb{C}_\lambda$; letting G be a $\mathbb{C}_\lambda$-generic filter, it is clear that $V[G] \vDash \mathfrak{i} = 2^{\aleph_0} = \lambda$. We will now show $V[G] \vDash \mathfrak{i}_{1/2} = \aleph_1$ by constructing a maximal $1/2$-independent family $\mathcal{A}$ in the ground model such that $\mathcal{A}$ remains maximal $1/2$-independent in $V[G]$. By the usual arguments, it suffices to consider what happens to a countably infinite $1/2$-independent family when forcing with just $\mathbb{C} := \langle 2^{<\omega}, \subseteq \rangle$.

Let $\mathcal{A}_0 := \{A_n \subseteq [\omega]^{\aleph_0} \mid n < \omega\}$ be such a family. Fix (in the ground model) an enumeration $\{(p_\alpha, \dot{X}_\alpha) \mid \omega \leq \alpha < \omega_1\}$ of all pairs $(p, \dot{X})$ such that $p \in \mathbb{C}$ and $\dot{X}$ is a nice name for a subset of ω.[6] In particular, this means that for any $\langle \check{n}, p_1 \rangle, \langle \check{n}, p_2 \rangle \in \dot{X}$, either $p_1 = p_2$ or $p_1 \perp p_2$. Note that since $V \vDash$ CH, there are just $\aleph_1$ many nice names for subsets of ω in V.

We now construct $\mathcal{A}$ from $\mathcal{A}_0$ iteratively as follows: Let $\omega \leq \alpha < \omega_1$ and assume we have already defined sets $A_\beta \subseteq \omega$ for all $\beta < \alpha$. Below, we will construct $A_\alpha \subseteq \omega$ such that the following two properties hold:

(i) The family $\{A_\beta \mid \beta \leq \alpha\}$ is $1/2$-independent.
(ii) If $p_\alpha \Vdash |\dot{X}_\alpha| = \aleph_0 \wedge$ "$\{A_\beta \mid \beta < \alpha\} \cup \{\dot{X}_\alpha\}$ is $1/2$-independent", then for all $m < \omega$, the set $D^\alpha_m := \{q \in \mathbb{C} \mid \exists\, n \geq m \colon q \Vdash A_\alpha \cap [2^n, 2^{n+1}) = \dot{X}_\alpha \cap [2^n, 2^{n+1})\}$ is dense below p_α.

We first show that the $\mathcal{A} := \{A_\beta \mid \beta \leq \omega_1\}$ constructed this way is a maximal $1/2$-independent family in $V^{\mathbb{C}}$. Clearly, $\mathcal{A}$ is $1/2$-independent, so only maximality could fail. Suppose it were not maximal; then there is a condition p and a nice name $\dot{X}$ for a subset of ω such that $p \Vdash$ "$\mathcal{A} \cup \{\dot{X}\}$ is $1/2$-independent". Let α be such that $(p, \dot{X}) = (p_\alpha, \dot{X}_\alpha)$ and let $\varepsilon > 0$ be sufficiently small (e. g. $\varepsilon < 1/16$). We

[6] The reason the index set of the enumeration is $[\omega, \omega_1)$ instead of $[0, \omega_1)$ is just to make the notation more convenient.

can then find $q \leq p_\alpha$ and $m < \omega$ such that

$$(*_1) \qquad q \Vdash \frac{|A_\alpha \cap \dot{X}_\alpha \cap \ell|}{\ell} \in \left(\frac{1}{4} - \varepsilon, \frac{1}{4} + \varepsilon\right) \text{ for all } \ell \geq 2^m$$

(because p_α forces that $\{A_\alpha, \dot{X}_\alpha\}$ is $1/2$-independent) and

$$\frac{|A_\alpha \cap [2^n, 2^{n+1})|}{2^n} > \frac{1}{2} - \varepsilon \text{ for all } n \geq m.$$

Now by the density of D^α_m below p_α, we can find $r \leq q$ and some $n \geq m$ such that $r \Vdash A_\alpha \cap [2^n, 2^{n+1}) = \dot{X}_\alpha \cap [2^n, 2^{n+1})$. But this implies that

$$\begin{aligned} r \Vdash \frac{|A_\alpha \cap \dot{X}_\alpha \cap 2^{n+1}|}{2^{n+1}} &= \frac{1}{2} \cdot \frac{|A_\alpha \cap \dot{X}_\alpha \cap 2^n|}{2^n} + \frac{1}{2} \cdot \frac{|A_\alpha \cap \dot{X}_\alpha \cap [2^n, 2^{n+1})|}{2^n} \\ &> \frac{1/4 - \varepsilon}{2} + \frac{1/2 - \varepsilon}{2} = \frac{3}{8} - \varepsilon > \frac{1}{4} + \varepsilon, \end{aligned}$$

which contradicts Eq. ($*_1$).

We finally have to show that we can find such an A_α satisfying (i) and (ii) for any $\omega \leq \alpha < \omega_1$. We only have to consider those α such that $\dot{X}_\alpha$ satisfies the assumption in property (ii), since finding an A_α with property (i) is straightforward. Enumerate $\{A_\beta \mid \beta < \alpha\}$ as $\{B_n \mid n < \omega\}$. For $n < \omega$ and any partial function $f \colon n \to \{-1, 1\}$, we let

$$B^f := \bigcap_{i \in \operatorname{dom}(f)} B_i^{f(i)},$$

where $B_i^1 := B$ and $B_i^{-1} := \omega \setminus B$. We further pick some strictly decreasing sequence of real numbers $\langle \delta_n \mid n < \omega \rangle$ with $\delta_0 := 3$ and $\lim_{n\to\infty} \delta_n = 0$ and let $\langle q_n \mid n < \omega \rangle$ be some sequence enumerating all conditions below p_α infinitely often. We will now construct, by induction on $n < \omega$, conditions $r_n \leq q'_n \leq q_n$, a strictly increasing sequence of natural numbers $\langle k_n \mid n < \omega \rangle$ and initial segments $Z_n = A_\alpha \cap 2^{k_n}$ of A_α such that for all $n < \omega$ and all partial functions $f \colon n \to \{-1, 1\}$, the following four statements will hold (with $F := |\operatorname{dom}(f)| + 1$)

(R1) $\dfrac{|B^f \cap Z_n \cap 2^{k_n}|}{2^{k_n}}, \dfrac{|(B^f \setminus Z_n) \cap 2^{k_n}|}{2^{k_n}} \in \left(\dfrac{1}{2^F} - \dfrac{\delta_n}{3}, \dfrac{1}{2^F} + \dfrac{\delta_n}{3}\right)$,

(R2) $q'_n \Vdash \dfrac{|B^f \cap \dot{X}_\alpha \cap \ell|}{\ell}, \dfrac{|(B^f \setminus \dot{X}_\alpha) \cap \ell|}{\ell} \in \left(\dfrac{1}{2^F} - \dfrac{\delta_n}{3}, \dfrac{1}{2^F} + \dfrac{\delta_n}{3}\right)$
for all ℓ with $2^{k_n} \leq \ell \leq 2^{k_{n+1}}$,

(R3) $\dfrac{|B^f \cap Z_{n+1} \cap \ell|}{\ell}, \dfrac{|(B^f \setminus Z_{n+1}) \cap \ell|}{\ell} \in \left(\dfrac{1}{2^F} - \delta_n, \dfrac{1}{2^F} + \delta_n\right)$
for all ℓ with $2^{k_n} \leq \ell \leq 2^{k_{n+1}}$, and

(R4) $r_n \Vdash Z_{n+1} \cap [2^{k_n}, 2^{k_{n+1}}) = \dot{X}_\alpha \cap [2^{k_n}, 2^{k_{n+1}})$.

It is clear that (R1)–(R4) taken together for all $n < \omega$ imply that $A_\alpha := \bigcup_{n<\omega} Z_n$ is as required by (i) and (ii).

For $n = 0$, let $k_0 := 0$, $q'_0 := q_0$ and $Z_0 := \varnothing$; then (R1) and (R2) hold vacuously by our choice of δ_0, and there is nothing to show yet for (R3) and (R4).

Now assume that we have obtained k_n, $q'_n \leq q_n$ and Z_n such that (R1) and (R2) hold for n; we will construct $r_n \leq q'_n$, k_{n+1}, $q'_{n+1} \leq q_{n+1}$ and Z_{n+1} such that (R3) and (R4) hold for n and such that (R1) and (R2) hold for $n+1$. We first find $q'_{n+1} \leq q_{n+1}$ and $k'_n \geq k_n$ such that for all partial functions $f\colon n+1 \to \{-1,1\}$, we have that (with $F := |\operatorname{dom}(f)| + 1$)

$$q'_{n+1} \Vdash \frac{|B^f \cap \dot{X}_\alpha \cap \ell|}{\ell}, \frac{|(B^f \setminus \dot{X}_\alpha) \cap \ell|}{\ell} \in \left(\frac{1}{2^F} - \frac{\delta_{n+1}}{3}, \frac{1}{2^F} + \frac{\delta_{n+1}}{3}\right)$$

for all $\ell \geq 2^{k_n}$ (hence satisfying (R2) for $n+1$); this is possible since the assumption in property (ii) is true. Next we find $r_n \leq q'_n$ and a sufficiently large $k_{n+1} \geq k'_n$ such that for all partial functions $f\colon n+1 \to \{-1,1\}$, we have that (still with $F := |\operatorname{dom}(f)| + 1$)

$$(*_2) \qquad r_n \Vdash \frac{|B^f \cap \dot{X}_\alpha \cap 2^{k_{n+1}}|}{2^{k_{n+1}}}, \frac{|(B^f \setminus \dot{X}_\alpha) \cap 2^{k_{n+1}}|}{2^{k_{n+1}}} \in \left(\frac{1}{2^F} - \frac{\delta_{n+1}}{6}, \frac{1}{2^F} + \frac{\delta_{n+1}}{6}\right)$$

and that r_n decides $\dot{X}_\alpha \cap 2^{k_{n+1}}$; in particular, let $X_n \subseteq [2^{k_n}, 2^{k_{n+1}})$ be such that $r_n \Vdash \dot{X}_\alpha \cap [2^{k_n}, 2^{k_{n+1}}) = X_n$. All this is also possible since the assumption in property (ii) is true. Let $Z_{n+1} := Z_n \cup X_n$.

Now, (R4) holds for n by definition of Z_{n+1}. Apply Lemma 4.10 to $R := Z_n$, $S := \dot{X}_\alpha[r_n]$, $r := 1/2^F$, $\varepsilon := \delta_n$, $m := 2^{k_n}$ and $n := 2^{k_{n+1}}$ to see that (R3) for n follows from (R1) and (R2) for n and our choice of Z_{n+1}. Finally, (R1) for $n+1$ follows from Eq. ($*_2$), (R4) for n and the choice of a sufficiently large k_{n+1} (e.g. using the argument from Lemma 3.3).

By the usual arguments, our construction implies that $\mathcal{A}$ remains maximal $1/2$-independent in $V^{\mathbb{C}_\lambda}$. □

5. Open Questions

While we have shown that several of our newly defined cardinal characteristics are, in fact, new, there are still a number of open questions.

Question A. *We summarise the open questions related to Figure 1:*

(Q1) Does $\operatorname{Con}(\mathfrak{d} < \mathfrak{s}_{1/2\pm\varepsilon} \leq \mathfrak{s}_{1/2})$ *hold or is* $\mathfrak{s}_{1/2} \leq \mathfrak{d}$*? (If it is the latter, we already know* $\operatorname{Con}(\mathfrak{s}_{1/2} < \mathfrak{d})$ *by* $\operatorname{Con}(\operatorname{non}(\mathcal{N}) < \mathfrak{d})$*.)*

(Q2) Which of the following statements are true?

$$\operatorname{Con}(\mathfrak{s} < \mathfrak{s}^w_{1/2}) \quad \text{or} \quad \mathfrak{s} = \mathfrak{s}^w_{1/2}$$
$$\operatorname{Con}(\mathfrak{s}^w_{1/2} < \mathfrak{s}^\infty_{1/2}) \quad \text{or} \quad \mathfrak{s}^w_{1/2} = \mathfrak{s}^\infty_{1/2}$$
$$\operatorname{Con}(\mathfrak{s}_{1/2\pm\varepsilon} < \mathfrak{s}_{1/2}) \quad \text{or} \quad \mathfrak{s}_{1/2\pm\varepsilon} = \mathfrak{s}_{1/2}$$

(Q3) Given $\varepsilon > \varepsilon'$ *and an* ε*-almost bisecting family, can one (finitarily) modify it to get an* ε'*-almost bisecting family of equal size? (If yes, then* $\mathfrak{s}_{1/2\pm\varepsilon}$ *is independent of* ε*. If not, then* $\inf_{\varepsilon\in(0,1/2)} \mathfrak{s}_{1/2\pm\varepsilon}$ *and* $\sup_{\varepsilon\in(0,1/2)} \mathfrak{s}_{1/2\pm\varepsilon}$ *might be interesting characteristics, as well.)*

(Q4) Can characteristics in the upper row of the diagram consistently be smaller than ones in the lower row? Specifically, which of the following statements are true?

$$\begin{aligned} \mathrm{Con}(\mathfrak{s}_{1/2\pm\varepsilon} < \mathfrak{s}^w_{1/2}) &\quad \textit{or} \quad \mathfrak{s}_{1/2\pm\varepsilon} \geq \mathfrak{s}^w_{1/2} \\ \mathrm{Con}(\mathfrak{s}_{1/2\pm\varepsilon} < \mathfrak{s}^\infty_{1/2}) &\quad \textit{or} \quad \mathfrak{s}_{1/2\pm\varepsilon} \geq \mathfrak{s}^\infty_{1/2} \\ \mathrm{Con}(\mathfrak{s}_{1/2} < \mathfrak{s}^\infty_{1/2}) &\quad \textit{or} \quad \mathfrak{s}_{1/2} \geq \mathfrak{s}^\infty_{1/2} \end{aligned}$$

Question B. *We summarise the open questions related to Figure 2:*

(Q5) Is it consistent that $\mathfrak{i}_* < 2^{\aleph_0}$?
(Q6) Which relations between $\mathfrak{i}_{1/2}$, $\mathfrak{i}_*$ and $\mathfrak{i}$ are true or consistent?
(Q7) Are there any smaller upper bounds for $\mathfrak{i}_{1/2}$ and $\mathfrak{i}_*$?
(Q8) Which relations between $\mathfrak{s}_{1/2}$ and $\mathfrak{s}_*$ are true or consistent?
(Q9) Which of the following statements are true?

$$\begin{aligned} \mathrm{Con}(\mathrm{cov}(\mathcal{N}) < \mathfrak{r}_{1/2}) &\quad \textit{or} \quad \mathrm{cov}(\mathcal{N}) = \mathfrak{r}_{1/2} \\ \mathrm{Con}(\mathfrak{r}_{1/2} < \mathfrak{r}_*) &\quad \textit{or} \quad \mathfrak{r}_{1/2} = \mathfrak{r}_* \\ \mathrm{Con}(\mathfrak{s}_* < \mathrm{non}(\mathcal{N})) &\quad \textit{or} \quad \mathfrak{s}_* = \mathrm{non}(\mathcal{N}) \end{aligned}$$

We suspect that (Q5) might be provable (via $\mathrm{Con}(\mathfrak{i}_* < \mathfrak{i})$) using the same idea as in Lemma 4.11. If the probabilistic argument from Lemma 3.4 can be reproduced for $\mathfrak{s}_*$, a similar approach as in section 3 might also work to answer the third part of (Q9) and prove $\mathrm{Con}(\mathfrak{s}_* < \mathrm{non}(\mathcal{N}))$. Finally, since it is not too difficult to ensure that a creature forcing poset keeps $\mathrm{cov}(\mathcal{N})$ small (compare [FGKS17, Lemma 5.4.2] or [GK18, Lemma 7.7]), a clever creature forcing construction might be able to answer the first part of (Q9) and prove $\mathrm{Con}(\mathrm{cov}(\mathcal{N}) < \mathfrak{r}_{1/2})$.

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