

A VERSION OF κ -MILLER FORCING

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ABSTRACT. Let κ be an uncountable cardinal such that $2^{<\kappa} = \kappa$ or just $\text{cf}(\kappa) > \omega$, $2^{2^{<\kappa}} = 2^\kappa$, and $([\kappa]^\kappa, \supseteq)$ collapses 2^κ to ω . We show under these assumptions the κ -Miller forcing with club many splitting nodes collapses 2^κ to ω and adds a κ -Cohen real.

1. INTRODUCTION

Many of the tree forcings on the classical Baire space have various analogues for higher cardinals. Here we are concerned with Miller forcing [4]. For a κ -version of Miller forcing, in addition to superperfectness one usually requires (see, e.g., [2, Section 5.2]) limits of length $< \kappa$ of splitting nodes be splitting nodes as well and that splitting mean splitting into a club. In this paper we investigate a version of κ -Miller forcing where this latter requirement is waived. We show: If $\text{cf}(\kappa) > \omega$, $\text{cf}(\kappa) = \kappa$ or $\text{cf}(\kappa) < 2^{\text{cf}(\kappa)} \leq \kappa$, $2^{2^{<\kappa}} = 2^\kappa$, and there is a κ -mad family of size 2^κ , then this variant of Miller forcing is related to the forcing $([\kappa]^\kappa, \supseteq)$ and collapses 2^κ to ω . In particular, if $\omega < \kappa^{<\kappa} = \kappa$, then our four premises are fulfilled.

Throughout the paper we let κ be an uncountable cardinal. We write \triangleleft for end extension of functions whose domains are ordinals. If $\text{dom}(t), i$ are ordinals, we write $t \hat{\ } \langle i \rangle$ for the concatenation of t with the singleton function $\{(0, i)\}$, i.e., $t \hat{\ } \langle i \rangle = t \cup \{(\text{dom}(t), i)\}$. We denote forcing orders in the form $(\mathbb{P}, \leq_{\mathbb{P}})$ and let $p \leq_{\mathbb{P}} q$ mean that q is *stronger* than p . We write $\lambda >_{\kappa}$ for the set of functions $f: \alpha \rightarrow \kappa$ for some $\alpha < \lambda$. The domain α of f is also called the length of f . The set of subsets of κ of size κ is denoted by $[\kappa]^\kappa$.

Definition 1.1. (1) \mathbb{Q}_κ^1 is the forcing $([\kappa]^\kappa, \supseteq)$.
 (2) \mathbb{Q}_κ^2 is the following version of κ -Miller forcing: Conditions are trees $T \subseteq {}^{\kappa >} \kappa$ that are κ *superperfect*: for each $s \in T$ there is $s \triangleleft t$ such that t is a κ -splitting node of T (short $t \in \text{spl}(T)$). A node $t \in T$ is called a κ -*splitting node* if

$$\text{set}_p(t) = \{i < \kappa : t \hat{\ } \langle i \rangle \in T\}$$

Date: March 5, 2019.

2010 Mathematics Subject Classification. Primary 03E05; Secondary 03E04, 03E15.

Key words and phrases. Forcing with higher perfect trees.

This research, no. 1154 on the second author's list, was partially supported by European Research Council grant 338821.

has size κ . We furthermore require that the limit of an increasing in the tree order sequence of length less than κ of κ -splitting nodes is a κ -splitting node if it has length less than κ .

For $p, q \in \mathbb{Q}_\kappa^2$ we write $p \leq_{\mathbb{Q}_\kappa^2} q$ if $q \subseteq p$. So subtrees are stronger conditions.

- (3) For $p \in \mathbb{Q}_\kappa^2$ and $\eta \in p$ we let $\text{succ}_p(\eta) = \{\eta' \in {}^{\kappa >} \kappa : (\exists i \in \kappa)(\eta' = \eta \hat{\ } i \in p)\}$.
- (4) Let $\eta \in p \in \mathbb{Q}_\kappa^2$. We let $p^{(\eta)} = \{\nu \in p : \nu \trianglelefteq \eta \vee \eta \trianglelefteq \nu\}$.
- (5) For $a, b \subseteq \kappa$ we write $a \subseteq_\kappa^* b$ if $|a \setminus b| < \kappa$.

Each of the two forcing orders \mathbb{P} has a weakest element, denoted by $0_{\mathbb{P}}$. Namely, \mathbb{Q}_κ^1 has as a weakest element $0_{\mathbb{Q}_\kappa^1} = \kappa$, and \mathbb{Q}_κ^2 has as a weakest element the full tree ${}^{\kappa >} \kappa$. We write $\mathbb{P} \Vdash \varphi$ if the weakest condition $0_{\mathbb{P}}$ forces φ .

2. RESULTS ABOUT \mathbb{Q}_κ^1

We will apply the following result for $\chi = 2^\kappa$.

Theorem 2.1. ([5, Theorem 0.5])

- (1) Under the assumption of an antichain of size χ in \mathbb{Q}_κ^1 , \mathbb{Q}_κ^1 collapses χ to \aleph_0 if $\aleph_0 < \text{cf}(\kappa) = \kappa$ or if $\aleph_0 < \text{cf}(\kappa) < 2^{\text{cf}(\kappa)} \leq \kappa$.
- (2) Under the assumption of an antichain of size χ in \mathbb{Q}_κ^1 , \mathbb{Q}_κ^1 collapses χ to \aleph_1 in the case of $\aleph_0 = \text{cf}(\kappa)$.

Definition 2.2. A family $\mathcal{A} \subseteq [\kappa]^\kappa$ is called a κ -almost disjoint family if for $A \neq B \in \mathcal{A}$, $|A \cap B| < \kappa$. A κ -almost disjoint family of size at least κ that is maximal is called a κ -mad family.

Observation 2.3. If $2^{<\kappa} = \kappa$, there is a κ -mad family $\mathcal{A} \subseteq [\kappa]^\kappa$ of size 2^κ .

Proof. We let $f: {}^{\kappa >} 2 \rightarrow \kappa$ be an injection. We assign to each branch b of ${}^{\kappa >} 2$ a set $a_b = \{f(s) : s \in b\}$. Then we complete the resulting family $\{a_b : b \text{ branch of } {}^{\kappa >} 2\}$ to a maximal κ -almost disjoint family. \square

Observation 2.4. If \mathbb{Q}_κ^1 collapses 2^κ to ω , then there is a κ -mad family \mathcal{A} of size 2^κ .

Proof. \mathbb{Q}_κ^1 cannot have the 2^κ -c.c. Hence there is an antichain of size 2^κ . This is a κ -ad family, and we extend it to a κ -mad family. \square

For further use, we indicate the hypothesis for each technical step.

Lemma 2.5. Suppose that \mathbb{Q}_κ^1 collapses 2^κ to ω . Then there is a \mathbb{Q}_κ^1 -name $\tau: \aleph_0 \rightarrow 2^\kappa$ for a surjection, and there is a labelled tree $\mathcal{T} = \langle (a_\eta, n_\eta, \varrho_\eta) : \eta \in {}^{\omega >} (2^\kappa) \rangle$ with the following properties

- (a) $a_\emptyset = \kappa$ and for any $\eta \in {}^{\omega >} (2^\kappa)$, $a_\eta \in [\kappa]^\kappa$.
- (b) $\eta_1 \triangleleft \eta_2$ implies $a_{\eta_1} \supseteq a_{\eta_2}$.

- (c) $n_\eta \in [\text{lg}(\eta) + 1, \omega)$.
- (d) If $a \in [\kappa]^\kappa$ then there is some $\eta \in {}^{\omega>}(2^\kappa)$ such that $a \supseteq a_\eta$.
- (e) If $\eta \hat{\langle} \beta \rangle \in T$ then $a_{\eta \hat{\langle} \beta \rangle}$ forces $\tau \upharpoonright n_\eta = \varrho_{\eta \hat{\langle} \beta \rangle}$ for some $\varrho_{\eta \hat{\langle} \beta \rangle} \in {}^{n_\eta}(2^\kappa)$, such that the $\varrho_{\eta \hat{\langle} \beta \rangle}$, $\beta \in 2^\kappa$, are pairwise different. Hence for any $\eta \in {}^{\omega>}(2^\kappa)$, the family $\{a_{\eta \hat{\langle} \alpha \rangle} : \alpha < 2^\kappa\}$ is a κ -ad family in $[a_\eta]^\kappa$.

Proof. Let τ be a \mathbb{Q}_κ^1 -name such that $\mathbb{Q}_\kappa^1 \Vdash \tau : \aleph_0 \rightarrow 2^\kappa$ is onto. For $\alpha < 2^\kappa$ let AP_α be the set of objects \bar{m} satisfying

- (*)₁(1.1) $\bar{m} = (T, \bar{a}, \bar{n}, \bar{\varrho}) = (T_{\bar{m}}, \bar{a}_{\bar{m}}, \bar{n}_{\bar{m}}, \bar{\varrho}_{\bar{m}})$.
- (1.2) T is a subtree of $({}^{\omega>}(2^\kappa), \triangleleft)$ of cardinality $\leq |\alpha| + \kappa$ and $\langle \rangle \in T$.
- (1.3) $\bar{a} = \langle a_\eta : \eta \in T \rangle$ fulfils $\eta \triangleleft \nu \rightarrow a_\nu \subseteq a_\eta$ and $a_\langle \rangle = \kappa$ and $a_\eta \in [\kappa]^\kappa$.
- (1.4) $\bar{n} = \langle n_\eta : \eta \in T \rangle$ fulfils $\text{dom}(\varrho_{\eta \hat{\langle} \beta \rangle}) = n_\eta > \text{lg}(\eta)$ for any $\eta \hat{\langle} \beta \rangle \in T$.
- (1.5) If $\eta \hat{\langle} \beta \rangle \in T$, then $a_{\eta \hat{\langle} \beta \rangle}$ forces a value to $\tau \upharpoonright n_\eta$ called $\varrho_{\eta \hat{\langle} \beta \rangle}$ and for $\beta \neq \gamma$ we have $\varrho_{\eta \hat{\langle} \beta \rangle} \neq \varrho_{\eta \hat{\langle} \gamma \rangle}$. Hence for any $\eta \hat{\langle} \beta \rangle, \eta \hat{\langle} \gamma \rangle \in T_{\bar{m}}$, $\beta \neq \gamma$ implies $a_{\eta \hat{\langle} \beta \rangle} \cap a_{\eta \hat{\langle} \gamma \rangle} \in [\kappa]^{<\kappa}$.
- (1.6) For $\eta \in T_{\bar{m}}$, we let

$$\text{Pos}(a_\eta, n_\eta) = \{\varrho \in {}^{n_\eta}(2^\kappa) : a_\eta \Vdash_{\mathbb{Q}_\kappa^1} \tau \upharpoonright n_\eta \neq \varrho\},$$

and require that the latter has cardinality 2^κ .

In the next items we state some properties of AP_α that are derived from (*)₁.

- (*)₂ $AP = \bigcup \{AP_\alpha : \alpha < 2^\kappa\}$ is ordered naturally by \leq_{AP} , which means end extension.
- (*)₃ (a) AP_α is not empty and increasing in α .
- (b) For infinite α , AP_α is closed under unions of increasing sequences of length $< |\alpha|^+$.
- (*)₄ Let $\gamma < 2^\kappa$. If $\bar{m} \in AP_\gamma$ and $\eta \in T_{\bar{m}}$ and $\eta \hat{\langle} \alpha \rangle \notin T_{\bar{m}}$ then there is $\bar{m}' \in AP_\gamma$ such that $\bar{m} \leq_{AP} \bar{m}'$ and $T_{\bar{m}'} = T_{\bar{m}} \cup \{\eta \hat{\langle} \alpha \rangle\}$.

Proof: For $\eta \in T_{\bar{m}}$,

$$\mathcal{U} = \text{Pos}(a_\eta, n_\eta) = \{\varrho \in {}^{n_\eta}(2^\kappa) : a_\eta \Vdash_{\mathbb{Q}_\kappa^1} \tau \upharpoonright n_\eta \neq \varrho\} \text{ has size } 2^\kappa,$$

whereas

$$\Lambda_\eta = \{\varrho_{\eta \hat{\langle} \beta \rangle} \upharpoonright n_\eta : \beta \in 2^\kappa \wedge \eta \hat{\langle} \beta \rangle \in T_{\bar{m}}\}$$

is of size $\leq |T_{\bar{m}}| \leq |\gamma| + \kappa$. Hence we can choose $\varrho_* \in \mathcal{U} \setminus \Lambda_\eta$ and $b_* \in [a_\eta]^\kappa$ such that $b_* \Vdash_{\mathbb{Q}_\kappa^1} \varrho_* = \tau \upharpoonright n_\eta$. We let $\varrho_{\eta \hat{\langle} \alpha \rangle} = \varrho_*$. Since b_* forces a value of $\tau \upharpoonright n_\eta$ that is incompatible with the one forced by $a_{\eta \hat{\langle} \beta \rangle}$ for any $\eta \hat{\langle} \beta \rangle \in T_{\bar{m}}$, the set b_* is κ -almost disjoint from $a_{\eta \hat{\langle} \beta \rangle}$ for any $\eta \hat{\langle} \beta \rangle \in T_{\bar{m}}$. We take $b_* = a_{\bar{m}', \eta \hat{\langle} \alpha \rangle} \subseteq a_{\bar{m}, \eta}$.

Since $\text{cf}(2^\kappa) > \aleph_0$ and since

$$|\{\text{range}(\varrho) : \varrho \in {}^{\omega>}(2^\kappa) \wedge b_* \Vdash_{\mathbb{Q}_\kappa^1} \tau \upharpoonright n \neq \varrho\}| = 2^\kappa,$$

there is an n such that

$$\text{Pos}(b_*, n) = \{\varrho \in {}^n(2^\kappa) : b_* \not\Vdash_{\mathbb{Q}_\kappa^1} \mathcal{T} \upharpoonright n \neq \varrho\}$$

has cardinality 2^κ . We take the minimal one and let it be $n_{\eta \hat{\ } \langle \alpha \rangle}$.

(*)₅ If $\bar{m} \in AP_\alpha$ and $a \in [\kappa]^\kappa$ then there is some $\bar{m}' \geq \bar{m}$, such that there is $\eta \in T_{\bar{m}'}$ with $a_{\bar{m}', \eta} \subseteq a$.

Let

$$\mathcal{U}_a = \{\varrho \in {}^{\omega >}(2^\kappa) : a \not\Vdash_{\mathbb{Q}_\kappa^1} \varrho \not\triangleleft \mathcal{T}\},$$

i.e.

$$\mathcal{U}_a = \{\varrho \in {}^{\omega >}(2^\kappa) : (\exists b \geq_{\mathbb{Q}_\kappa^1} a)(b \Vdash_{\mathbb{Q}_\kappa^1} \varrho \triangleleft \mathcal{T})\}.$$

This set has cardinality 2^κ because $\mathbb{Q}_\kappa^1 \Vdash \mathcal{T} : \omega \rightarrow 2^\kappa$ is onto. We take n minimal such that

$$\mathcal{U}_{a, n} = \{\varrho \in {}^n(2^\kappa) : (\exists b \geq_{\mathbb{Q}_\kappa^1} a)(b \Vdash_{\mathbb{Q}_\kappa^1} \varrho \triangleleft \mathcal{T})\}$$

has size 2^κ . We let

$$\text{set}_n^+(\bar{m}) = \{\varrho_\eta : \eta \in T_{\bar{m}}, \text{lg}(\varrho_\eta) \geq n\}.$$

Clearly $|\text{set}_n^+(\bar{m})| \leq |T_{\bar{m}}| \leq |\gamma| + \kappa$. Thus we can take $\varrho_a \in \mathcal{U}_{a, n}$ that is incompatible with every element of $\text{set}_n^+(\bar{m})$. We take some $b_a \in [a]^\kappa$ such that $b_a \Vdash_{\mathbb{Q}_\kappa^1} \varrho_a \triangleleft \mathcal{T}$. The set

$$\Lambda_a = \{\eta \in T_{\bar{m}} : b_a \subseteq_{\leq \kappa}^* a_\eta\}$$

is \triangleleft -linearly ordered by (*)₁ clauses 1.3 and 1.5 and $\langle \rangle \in \Lambda_a$. Since b_a does not pin down \mathcal{T} , Λ_a has a \triangleleft -maximal member η_a . Now we take $\alpha_* = \min\{\beta : \eta_a \hat{\ } \langle \beta \rangle \notin T_{\bar{m}}\}$. For any $\eta_a \hat{\ } \langle \beta \rangle \in T_{\bar{m}}$ we have $\varrho_{\eta_a \hat{\ } \langle \beta \rangle}$ and ϱ_a are incompatible, and hence $a_{\eta_a \hat{\ } \langle \beta \rangle} \cap b_a \in [\kappa]^{< \kappa}$. Now we choose $b_a^1 \in [b_a]^\kappa$ and ϱ_a^* such that $b_a^1 \Vdash_{\mathbb{Q}_\kappa^1} \varrho_a^* \triangleleft \mathcal{T}$ and $\text{lg}(\varrho_a^*) \geq n_{\bar{m}, \eta_a} > \text{lg}(\eta_a)$.

We let

$$\begin{aligned} T_{\bar{m}'} &= T_{\bar{m}} \cup \{\eta_a \hat{\ } \langle \alpha_* \rangle\}, \\ a_{\eta_a \hat{\ } \langle \alpha_* \rangle} &= b_a^1, \end{aligned}$$

We let $n_{\eta_a \hat{\ } \langle \alpha_* \rangle}$ be the minimal n such that $|\text{Pos}(b_a^1, n)| \geq 2^\kappa$. So (*)₅ holds.

Now we are ready to construct \mathcal{T} as in the statement of the lemma. We do this by recursion on $\alpha \leq 2^\kappa$. First we enumerate $[\kappa]^\kappa$ as $\langle c_\alpha : \alpha < 2^\kappa \rangle$, and we enumerate ${}^{\omega >}(2^\kappa)$ as $\langle \eta_\alpha : \alpha < 2^\kappa \rangle$ such that $\eta_\alpha \triangleleft \eta_\beta$ implies $\alpha < \beta$. We choose an increasing sequence \bar{m}_α by induction on $\alpha < 2^\kappa$. We start with the tree $\{\langle \rangle\}$, $a_\langle \rangle = \kappa$, $\varrho_\langle \rangle = \emptyset$, $n_\langle \rangle$ be minimal such that $|\text{Pos}(\kappa, n)| = 2^\kappa$. In the odd successor steps we take $\bar{m}_{2\alpha+1} \geq_{AP} \bar{m}_\alpha$ so that $a_\eta \subseteq c_\alpha$ for some $\eta \in T_{2\alpha+1}$. This is done according to (*)₅. In the even successor steps we take $\bar{m}_{2\alpha+2} \geq_{AP} \bar{m}_{2\alpha+1}$ such that $\eta_\alpha \in T_{2\alpha+2}$. Since all initial segments of η_α appeared among the η_β , $\beta < \alpha$, $\bar{m}_{2\alpha+2}$ is found according to (*)₄. In the limit steps we take unions. Then \mathcal{T} that is given by the the last three

components of \bar{m}_{2^κ} has properties (a) to (e). \square

Since $\tau = \mathcal{T}[G]$ is not in \mathbf{V} , for any \mathcal{T} as in Lemma 2.5 no sequence of first components of a branch, i.e., no $\langle a_{f \upharpoonright n} : n \in \omega \rangle$, $f \in \omega(2^\kappa) \cap \mathbf{V}$, has a \subseteq_κ^* -lower bound.

3. TRANSFER TO \mathbb{Q}_κ^2

In this section we use the tree \mathcal{T} from Lemma 2.5 for finding \mathbb{Q}_κ^2 -names.

Definition 3.1. Let μ, λ be cardinals. For $\nu, \nu' \in {}^\lambda > \mu$ we write $\nu \perp \nu'$ if $\nu \not\leq \nu'$ and $\nu' \not\leq \nu$.

Typical pairs (λ, μ) are $(\omega, 2^\kappa)$ and (κ, κ) .

An important tool for the analysis of \mathbb{Q}_κ^2 is the following particular kind of fusion sequence $\langle p_\alpha : \alpha < \kappa^{<\kappa} \rangle$ in \mathbb{Q}_κ^2 . Since we do not suppose $\kappa^{<\kappa} = \kappa$, a fusion sequence can be longer than κ . An important property is that for each $\nu \in {}^\kappa > \kappa$ there is at most one $\alpha < \kappa^{<\kappa}$ such that $\text{set}_{p_\alpha}(\nu) \supseteq \text{set}_{p_{\alpha+1}}(\nu)$.

Lemma 3.2. Let $\langle \nu_\alpha : \alpha < \kappa^{<\kappa} \rangle$ be an injective enumeration of $\kappa^{<\kappa}$ such that

$$(3.1) \quad \nu_\alpha \triangleleft \nu_\beta \rightarrow \alpha < \beta.$$

Let $\langle p_\alpha, \nu_\alpha, c_\alpha : \alpha < \kappa^{<\kappa} \rangle$ be a sequence such that for any $\alpha \leq \lambda$ the following holds:

(a) $p_0 \in \mathbb{Q}_\kappa^2$.

(b1) If $\alpha = \beta + 1 < \kappa^{<\kappa}$ and $\nu_\beta \in \text{sp}(p_\beta)$, then

$$c_\beta \in [\text{suc}_{p_\beta}(\nu_\beta)]^\kappa \text{ and}$$

$$p_\alpha = p_\beta(\nu_\beta, c_\beta) := \bigcup \{ p_\beta^{\langle \nu_\beta \hat{\ }^i \rangle} : i \in c_\beta \} \cup \bigcup \{ p_\beta^{(\eta)} : \eta \not\leq \nu_\beta \wedge \nu_\beta \not\leq \eta \}$$

(b2) If $\alpha = \beta + 1 < \kappa^{<\kappa}$ and $\nu_\beta \notin \text{spl}(p_\beta)$ then $p_\alpha = p_\beta$.

(c) $p_\alpha = \bigcap \{ p_\beta : \beta < \alpha \}$ for limit $\alpha \leq \kappa^{<\kappa}$.

Then for any $\lambda \leq \kappa^{<\kappa}$, $p_\lambda \in \mathbb{Q}_\kappa^2$ and $\forall \beta < \lambda$, $p_\beta \leq_{\mathbb{Q}_\kappa^2} p_\lambda$.

Proof. We go by induction on λ . The case $\lambda = 0$ and the successor steps are obvious. So we assume that $\lambda \leq \kappa^{<\kappa}$ is a limit ordinal and $p_\alpha \in \mathbb{Q}_\kappa^2$ for $\alpha < \lambda$. Since $\emptyset \in p_\lambda$, p_λ is not empty, and p_λ clearly is a tree. Let $t \in p_\lambda$. We show that there is $t' \supseteq t$ that is a splitting node in p_λ . We fix the smallest α such that $\nu_\alpha \supseteq_{p_0} t$ is a splitting node in p_0 . Then in p_0 there are no splitting nodes in $\{s : t \trianglelefteq s \triangleleft \nu_\alpha\}$. Hence $\nu_\alpha \in \text{spl}(p_\beta)$ for any $\beta \in [0, \lambda]$.

Now we show that the limit of splitting nodes in p_λ is a splitting node. Let $\gamma < \lambda$ and let $\langle \nu^i : i < \gamma \rangle$ be an \triangleleft -increasing sequence of splitting nodes of p_λ with union $\nu \in \kappa^{<\kappa}$. Then ν is a splitting node of each p_α , $\alpha < \lambda$, and also in p_λ since $\langle \text{set}_{p_\alpha}(\nu) : \alpha < \lambda \rangle$ has at most two entries and their intersection has size κ . \square

We need yet another type of fusion sequence.

Definition 3.3. Let $p \in \mathbb{Q}_\kappa^2$ and let $\nu \in \text{spl}(p)$.

(1) Let $i \in \text{set}_p(\nu)$. We say η is the *shortest splitting node above $\nu \hat{\ } i$* in p and write $\eta = \text{next}_p(\nu \hat{\ } i)$ if η is the shortest splitting point in p such that $\eta \supseteq \nu \hat{\ } i$. Equality is allowed.

(2) We say $F \subseteq p$ is the *front of next splitting nodes above ν* in p , if

$$F = \{\eta' \in \text{spl}(p) : \exists(\eta \in \text{suc}_p(\nu))(\eta' = \text{next}_p(\eta))\}.$$

Lemma 3.4. Let $\langle \nu_\alpha : \alpha < \kappa^{<\kappa} \rangle$ be an injective enumeration of $\kappa^{<\kappa}$ such that

$$(3.2) \quad \nu_\alpha \triangleleft \nu_\beta \rightarrow \alpha < \beta.$$

Let $\langle p_\alpha, \nu_\alpha, c_\alpha, F_\alpha : \alpha < \kappa^{<\kappa} \rangle$ be a sequence such that for any $\alpha \leq \lambda$ the following holds:

(a) $p_0 \in \mathbb{Q}_\kappa^2$.

(b1) If $\alpha = \beta + 1 < \kappa^{<\kappa}$ and $\nu_\beta \in \text{sp}(p_\beta)$, then $c_\beta \in [\text{suc}_{p_\beta}(\nu_\beta)]^\kappa$, F_β contains for each $i \in c_\beta$ exactly one $\eta \in \text{spl}(p_\beta^{\langle \nu_\beta \hat{\ } i \rangle})$, and

$$p_\alpha = p_\beta(\nu_\beta, c_\beta, F_\beta) := \bigcup \{p_\beta^{\langle \eta \rangle} : i \in c_\beta, \eta \in F_\beta\} \\ \cup \bigcup \{p_\beta^{\langle \eta \rangle} : \eta \not\triangleleft \nu_\beta \wedge \nu_\beta \not\triangleleft \eta\}.$$

Note that this implies that F_β is the front of next splitting nodes of p_α above ν_β .

(b2) If $\alpha = \beta + 1 < \kappa^{<\kappa}$ and $\nu_\beta \notin \text{spl}(p_\beta)$ then $p_\alpha = p_\beta$.

(c) $p_\alpha = \bigcap \{p_\beta : \beta < \alpha\}$ for limit $\alpha \leq \kappa^{<\kappa}$.

Then for any $\lambda \leq \kappa^{<\kappa}$, $p_\lambda \in \mathbb{Q}_\kappa^2$ and $\forall \beta < \lambda$, $p_\beta \leq_{\mathbb{Q}_\kappa^2} p_\lambda$.

Proof. We go by induction on λ . The case $\lambda = 0$ and the successor steps are obvious. So we assume that $\lambda \leq \kappa^{<\kappa}$ is a limit ordinal and $p_\alpha \in \mathbb{Q}_\kappa^2$ for $\alpha < \lambda$. Since $\emptyset \in p_\lambda$, p_λ is not empty, and p_λ clearly is a tree. Let $t \in p_\lambda$. We show that there is $t' \supseteq t$ that is a splitting node in p_λ . We fix the smallest α such that $\nu_\alpha \supseteq_{p_0} t$ is a splitting node in p_0 . Then in p_0 there are no splitting nodes in $\{s : t \trianglelefteq s \triangleleft \nu_\alpha\}$. Hence $\nu_\alpha \in \text{spl}(p_\beta)$ for any $\beta \in [0, \lambda]$.

Now we show that the limit of splitting nodes in p_λ is a splitting node. Let $\gamma < \lambda$ and let $\langle \nu^i : i < \gamma \rangle$ be an \triangleleft -increasing sequence of splitting nodes of p_λ with union $\nu \in \kappa^{<\kappa}$. Then ν is a splitting node of each p_α , $\alpha < \lambda$, and also in p_λ since $\langle \text{set}_{p_\alpha}(\nu) : \alpha < \lambda \rangle$ has at most two entries and their intersection has size κ . \square

In the special case $F_\beta = \{\nu_\beta \hat{\ } j : j \in c_\beta\}$, the construction of Lemma 3.4 coincides with the simpler construction from Lemma 3.2.

Definition 3.5. We assume \mathbb{Q}_κ^1 collapses 2^κ to ω . Let \mathcal{T} and $\mathcal{T} = \langle (a_\eta, n_\eta, \varrho) : \eta \in {}^{\omega>}(2^\kappa) \rangle$ be as in Lemma 2.5. Now let $Q_{\mathcal{T}}$ be the set of κ -Miller trees p such that for every $\nu \in \text{spl}(p)$ there is $\eta_{p,\nu} = \eta_\nu \in {}^{\omega>}(2^\kappa)$ such that

$$(3.3) \quad \text{set}_p(\nu) = \{\varepsilon \in \kappa : \nu \hat{\ } \varepsilon \in p\} = a_{\eta_\nu}.$$

By the properties of \mathcal{T} , the node $\eta_{p,\nu}$ is unique.

Lemma 3.6. *Assume that \mathbb{Q}_κ^1 collapses 2^κ to ω , let \mathcal{T} be chosen as in Lemma 2.5, and let $Q_{\mathcal{T}}$ be defined from \mathcal{T} as above. Then $Q_{\mathcal{T}}$ is dense in \mathbb{Q}_κ^2 .*

Proof. Let $p_0 = T \in \mathbb{Q}_\kappa^2$. Let $\langle \nu_\alpha : \alpha < \kappa^{<\kappa} \rangle$ be an injective enumeration of $\kappa^{<\kappa}$ with property (3.2). We now define fusion sequence $\langle p_\alpha, \nu_\alpha, c_\alpha : \alpha \leq \kappa^\kappa \rangle$ according to the pattern in Lemma 3.2 in order to find $p_{\kappa^{<\kappa}} \geq T$ such that $p_{\kappa^{<\kappa}} \in Q_{\mathcal{T}}$.

Suppose that p_α and ν_α are given. If ν_α is not in p_α or is not a splitting node in p_α , then we let $p_{\alpha+1} = p_\alpha$. If $\nu_\alpha \in \text{spl}(p_\alpha)$, then according to Lemma 2.5 clause (d) there is $\eta \in {}^{\omega>} (2^\kappa)$ such that $\text{suc}_{p_\alpha}(\nu_\alpha) \supseteq a_\eta$. We choose such an η of minimal length and call it $\eta(\alpha)$.

Then we strengthen p_α to

$$(3.4) \quad p_{\alpha+1} = \bigcup \{ p_\alpha^{\langle \nu' \rangle} : \nu' = \nu_\alpha \hat{\ } \langle i \rangle \wedge i \in a_{\eta(\alpha)} \} \cup \bigcup \{ p_\alpha^{\langle \eta \rangle} : \eta \not\triangleleft \nu_\alpha \wedge \nu_\alpha \not\triangleleft \eta \}.$$

Now we have that

$$\eta_{p_{\alpha+1}, \nu_\alpha} = \eta(\alpha), c_\alpha = a_{\eta(\alpha)}.$$

For limit ordinals $\lambda \leq \kappa^{<\kappa}$, we let $p_\lambda = \bigcap \{ p_\beta : \beta < \lambda \}$. Since the sequence $\langle p_\alpha, \nu_\alpha, c_\alpha : \alpha \leq \kappa^{<\kappa} \rangle$ matches the pattern in Lemma 3.2, we have $p_{\kappa^{<\kappa}} \in \mathbb{Q}_\kappa^2$. By construction, for any $\alpha < \kappa^{<\kappa}$ for any $\delta \in [\alpha + 1, \kappa^{<\kappa})$, $\nu_\alpha \in \text{spl}(p_\delta)$ implies

$$\text{set}_{p_{\alpha+1}}(\nu_\alpha) = \text{set}_{p_\delta}(\nu_\alpha) = a_{\eta(\alpha)}.$$

Hence the condition $p = p_{\kappa^{<\kappa}}$ fulfils Equation (3.3) in its splitting node ν_α with witness $\eta_{p, \nu_\alpha} = \eta(\alpha)$. Since all nodes are enumerated, we have $p_{\kappa^{<\kappa}} \in Q_{\mathcal{T}}$. \square

We use only the inclusion $\text{set}_p(\nu) \subseteq a_{\eta_\nu}$ from Definition 3.5.

Definition 3.7. We assume that \mathbb{Q}_κ^1 collapses 2^κ to ω and the \mathcal{T} is as in Lemma 2.5. For $T \in Q_{\mathcal{T}}$ and a splitting node ν of T we set $\varrho_{T,\nu} := \varrho_{\eta_{T,\nu}} \in {}^{\omega>} (2^\kappa)$. Recall $\eta_{T,\nu}$ is defined in Def. 3.5, and ϱ is a component of \mathcal{T} .

For $p \in Q_{\mathcal{T}}$, the relation $\nu \trianglelefteq \nu' \in p$ does neither imply $\eta_\nu \trianglelefteq \eta_{\nu'}$ nor $\varrho_\nu \trianglelefteq \varrho_{\nu'}$. However, $\eta_\nu \triangleleft \eta_{\nu'}$ implies $a_{\eta_\nu} \supset a_{\eta_{\nu'}}$ and $\varrho_\nu \triangleleft \varrho_{\nu'}$.

Observation 3.8. *We assume that \mathbb{Q}_κ^1 collapses 2^κ to ω . Let $p_1, p_2 \in Q_{\mathcal{T}}$. If $p_1 \leq_{\mathbb{Q}_\kappa^2} p_2$ then for $\nu \in \text{spl}(p_2)$ we have $\nu \in \text{spl}(p_1)$ and $\varrho_{p_1,\nu} \trianglelefteq \varrho_{p_2,\nu}$.*

We introduce dense sets:

Definition 3.9. We assume that \mathbb{Q}_κ^1 collapses 2^κ to ω . Let $n \in \omega$.

$$D_n = \{ p \in Q_{\mathcal{T}} : (\forall \nu \in \text{spl}(p)) (\text{lg}(\varrho_{p,\nu}) > n) \}.$$

D_n is open dense in $Q_{\mathcal{T}}$ and the intersection of the D_n is empty. The following technical lemma is the first step of a transformation of a \mathbb{Q}_{κ}^1 -name of a surjection from ω onto 2^κ into a \mathbb{Q}_{κ}^2 -name of such a surjection.

Lemma 3.10. *We assume that \mathbb{Q}_{κ}^1 collapses 2^κ to ω , $\text{cf}(\kappa) > \omega$ and $2^{(\kappa < \kappa)} = 2^\kappa$. Let $\langle T_\alpha : \alpha < 2^\kappa \rangle$ enumerate \mathbb{Q}_{κ}^2 such that each Miller tree appears 2^κ times. There is $\langle (p_\alpha, n_\alpha, \bar{\gamma}_\alpha) : \alpha < 2^\kappa \rangle$ such that*

- (a) $n_\alpha < \omega$,
- (b) $p_\alpha \in D_{n_\alpha}$ and $p_\alpha \geq T_\alpha$.
- (c) If $\beta < \alpha$ and $n_\beta \geq n_\alpha$ then $p_\beta \perp p_\alpha$.
- (d) $\bar{\gamma}_\alpha = \langle \gamma_{\alpha, \nu} : \nu \in \text{spl}(p_\alpha) \rangle$.
- (e) $(\forall \nu \in \text{spl}(p_\alpha))(a_{\eta_{p_\alpha, \nu}} \Vdash_{\mathbb{Q}_{\kappa}^1} \gamma_{\alpha, \nu} \in \text{range}(\varrho_{p_\alpha, \nu}))$.
- (f) $\gamma_{\alpha, \nu} \in 2^\kappa \setminus W_{< \alpha, \nu}$ with

$$W_{< \alpha, \nu} = \bigcup \{ \text{range}(\varrho_{p_\beta, \nu}) : \beta < \alpha, \nu \in \text{spl}(p_\beta) \}.$$

Proof. Assume that $\langle (p_\beta, n_\beta, \bar{\gamma}_\beta) : \beta < \alpha \rangle$ has been defined and we are to define $(p_\alpha, n_\alpha, \bar{\gamma}_\alpha)$. Note that the p_β need not be increasing in strength.

- (\oplus)₁ The choice of the a_η in Lemma 2.5 and the choice $Q_{\mathcal{T}}$ and of $\eta_{p_\beta, \nu}$ for $\nu \in \text{spl}(p_\beta)$, $\beta < \alpha$, imply that the set $W_{< \alpha, \nu}$ is well defined and of cardinality $\leq |\alpha| + \aleph_0 < 2^\kappa$. Hence we can choose $\gamma_{\alpha, \nu} \in 2^\kappa \setminus W_{< \alpha, \nu}$.
- (\oplus)₂ With the fusion Lemma 3.2 we choose $q_\alpha \geq T_\alpha$, $q_\alpha \in Q_{\mathcal{T}}$, such that

$$(\forall \nu \in \text{spl}(q_\alpha))(a_{\eta_{q_\alpha, \nu}} \Vdash_{\mathbb{Q}_{\kappa}^1} \gamma_{\alpha, \nu} \in \text{range}(\varrho_{q_\alpha, \nu})).$$

- (\oplus)₃ Let $q \in \mathbb{Q}_{\kappa}^2$. For $n \in \omega$ and $\nu \in \text{spl}(q)$ we let

$$\mathcal{U}_{\alpha, \nu, n}(q) = \{ \beta < \alpha : n_\beta = n, \nu \in \text{spl}(p_\beta) \wedge |\text{set}_q(\nu) \cap \text{set}_{p_\beta}(\nu)| = \kappa \}.$$

$$\mathcal{U}_{\alpha, \nu}(q) = \bigcup \{ \mathcal{U}_{\alpha, \nu, n}(q) : n \in \omega \}.$$

- (\oplus)₄ (a) If $n \in \omega$ and $\nu \in \text{spl}(q_\alpha)$ then

$$\beta \in \mathcal{U}_{\alpha, \nu}(q_\alpha) \rightarrow \varrho_{p_\beta, \nu} \trianglelefteq \varrho_{q_\alpha, \nu}.$$

This is seen as follows. We let $a = \text{set}_{p_\beta}(\nu) \cap \text{set}_{q_\alpha}(\nu)$. Since $\beta \in \mathcal{U}_{\alpha, \nu}(q_\alpha)$, $a \in [\kappa]^\kappa$. Clearly $a \Vdash_{\mathbb{Q}_{\kappa}^1} \tau \triangleright \varrho_{p_\beta, \nu}, \varrho_{q_\alpha, \nu}$. So either $\varrho_{p_\beta, \nu} \triangleleft \varrho_{q_\alpha, \nu}$ or $\varrho_{p_\beta, \nu} \triangleright \varrho_{q_\alpha, \nu}$. However, since $\gamma_{\alpha, \nu} \in \text{range}(\varrho_{q_\alpha, \nu}) \setminus W_{< \alpha, \nu}$, only $\varrho_{q_\alpha, \nu} \triangleright \varrho_{p_\beta, \nu}$ is possible.

- (b) So for $\nu \in \text{spl}(q_\alpha)$, the set $\{ \varrho_{p_\beta, \nu} : \beta \in \mathcal{U}_{\alpha, \nu}(q_\alpha) \}$ has at most $\text{lg}(\varrho_{q_\alpha, \nu})$ elements.
- (c) The assignment $\beta \mapsto \varrho_{p_\beta, \nu}$ is defined between $\mathcal{U}_{\alpha, \nu}(q_\alpha)$ and $\{ \varrho_{p_\beta, \nu} : \beta \in \mathcal{U}_{\alpha, \nu}(q_\alpha) \}$. According to properties (e) and (f) in the induction hypothesis, the assignment is injective, and hence $|\mathcal{U}_{\alpha, \nu}(q_\alpha)| \leq \text{lg}(\varrho_{q_\alpha, \nu})$.
- (d) We state for further use that $\mathcal{U}_{\alpha, \nu}(q_\alpha)$ is finite and for any $q \geq q_\alpha$, $\mathcal{U}_{\alpha, \nu}(q) \subseteq \mathcal{U}_{\alpha, \nu}(q_\alpha)$.

(\oplus)₅ We look at the cone above q_α and show:

$$(3.5) \quad \begin{aligned} & (\forall q \geq q_\alpha)(\forall \nu \in \text{spl}(q))(\exists r_{\alpha,\nu} \geq_{\mathbb{Q}_\kappa^2} q) \\ & (\exists c \in [\text{set}_q(\nu)]^\kappa)(\exists F \subseteq \{\eta \in \text{spl}(q) : \eta \triangleright \nu\}) \\ & (r_{\alpha,\nu} = q(\nu, c, F) \wedge (\forall \beta \in \mathcal{U}_{\alpha,\nu}(q_\alpha))(r_{\alpha,\nu}^{(\nu)} \perp p_\beta^{(\nu)} \vee p_\beta^{(\nu)} \leq r_{\alpha,\nu}^{(\nu)})). \end{aligned}$$

How do we find $r_{\alpha,\nu} = r_{\alpha,\nu}(q)$? Given $q \geq_{\mathbb{Q}_\kappa^2} q_\alpha$, $\nu \in \text{spl}(q)$ we enumerate $\mathcal{U}_{\alpha,\nu}(q_\alpha)$ as $\beta_0, \dots, \beta_{k-1}$. We let $r_0 = q$ and by induction on $i \leq k$ we define r_i , increasing in strength, with $\nu \in \text{spl}(r_i)$ and $c_i = \text{set}_{r_i}(\nu)$. Thus the c_i are \subseteq -decreasing sets of size κ . Given r_i , we distinguish cases:

First case: $\beta_i \notin \mathcal{U}_{\alpha,\nu}(r_i)$. Then there is $c_{i+1} \in [\text{set}_{r_i}(\nu)]^\kappa$, $c_{i+1} \cap \text{set}_{p_{\beta_i}}(\nu) = \emptyset$. We let $r_{i+1} = r_i(\nu, c_{i+1})$ and thus have $r_{i+1}^{(\nu)} \perp p_{\beta_i}$.

Second case: $\beta_i \in \mathcal{U}_{\alpha,\nu}(r_i)$. We let

$$c_i = \{j \in \text{set}_{r_i}(\nu) : r_i^{\langle \nu \hat{\ } j \rangle} \geq p_{\beta_i}^{\langle \nu \hat{\ } j \rangle}\} \cup \{j \in \text{set}_{r_i}(\nu) : r_i^{\langle \nu \hat{\ } j \rangle} \not\geq p_{\beta_i}^{\langle \nu \hat{\ } j \rangle}\}.$$

If $c_{i,1} = \{j \in \text{set}_{r_i}(\nu) : r_i^{\langle \nu \hat{\ } j \rangle} \geq p_{\beta_i}^{\langle \nu \hat{\ } j \rangle}\}$ has size κ , then we let $c_{i+1} = c_{i,1}$ and $r_{i+1} = r_i(\nu, c_{i+1})$ and thus get $r_{i+1}^{(\nu)} \geq p_{\beta_i}$.

If $|c_{i,1}| < \kappa$, then $c_{i,2} = \{j \in \text{set}_{r_i}(\nu) : r_i^{\langle \nu \hat{\ } j \rangle} \not\geq p_{\beta_i}^{\langle \nu \hat{\ } j \rangle}\}$ has size κ , and we let $c_{i+1} = c_{i,2}$. For $j \in c_{i+1}$, $r_i^{\langle \nu \hat{\ } j \rangle} \not\geq p_{\beta_i}^{\langle \nu \hat{\ } j \rangle}$. Thus we can find a node in the $r_i^{\langle \nu \hat{\ } j \rangle} \setminus p_{\beta_i}^{\langle \nu \hat{\ } j \rangle}$ and above this node we find a splitting node of r_i . We take this latter splitting node into r_{i+1} as the direct successor splitting node to $\nu \hat{\ } j$. Doing so for every $j \in c_{i+1}$ we get $F_{\nu,i}$, a front strictly above ν in $r_{i+1} = r_i(\nu, c_{i+1}, F_{\nu,i})$. Again we get $r_{i+1}^{(\nu)} \perp p_{\beta_i}$.

In the end we let $r_{\alpha,\nu} = r_k$. There is a front F that contains for each $j \in c_k$ the shortest splitting node of r_k above $\nu \hat{\ } j$. Thus we have $r_k = r_{\alpha,\nu} = q(\nu, c_k, F)$ and $r_{\alpha,\nu}$ fulfils (3.5).

(\oplus)₆ Now we use (\oplus)₅ iteratively along all $\nu \in \kappa^{<\kappa}$ to find a fusion sequence $\langle r_{\alpha,\nu}, \nu, c_\nu, F_\nu : \nu < \kappa^{<\kappa} \rangle$ with starting point $q_\alpha = r_{0,\nu_0}$. In this sequence, $r_{\alpha,\nu}$ is chosen as $r_{\alpha,\nu}(q)$ in (\oplus)₅ for $q = \bigcap_{\beta < \alpha} r_\beta$, if $\nu \in \text{spl}(q)$. If $\nu \notin \text{spl}(q)$, then $r_{\alpha,\nu} = q$. Then we apply the fusion Lemma 3.4 and get an upper bound r_α of $r_{\alpha,\nu}$, $\nu \in \kappa^{>\kappa}$. Note $r_\alpha^{(\nu)} \perp p_\beta$ iff $r_\alpha^{(\nu)} \perp p_\beta^{(\nu)}$ and $r_\alpha^{(\nu)} \geq p_\beta$ iff $r_\alpha^{(\nu)} \geq p_\beta^{(\nu)}$. Hence $r_\alpha \geq q_\alpha$ and

$$(\forall \nu \in \text{spl}(r_\alpha))(\forall \beta \in \mathcal{U}_{\alpha,\nu}(q_\alpha))(r_\alpha^{(\nu)} \perp p_\beta \vee p_\beta \leq r_\alpha^{(\nu)}).$$

(\oplus)₇ Finally we choose n_α and p_α . There are k and ν such that $n < \omega$ and $\nu \in \text{spl}(r_\alpha)$ such that $p_\alpha = r_\alpha^{(\nu)}$ fulfils

$$(\forall \beta < \alpha)(n_\beta \geq k \rightarrow p_\alpha \perp p_\beta).$$

Proof of existence. By induction on $k \in \omega$ we try to find $\langle \nu_k, \beta_k : k \in \omega \rangle$ such that

- (a) $\nu_k \in \text{spl}(r_\alpha)$,
- (b) $\nu_k \triangleleft \nu_m$ for $k < m$,
- (c) $\beta_k < \alpha$ and $n_{\beta_k} \geq k$ and $r_\alpha^{(\nu_k)} \geq p_{\beta_k}$.

If we succeed, then $\nu_* = \bigcup \{\nu_k : k \in \omega\} = \nu^* \in \text{spl}(r_\alpha)$ by Definition 1.1 (2). Here we use that $\text{cf}(\kappa) > \omega$. Hence

$$r_\alpha^{(\nu^*)} \in Q_{\mathcal{T}} \cap \bigcap \{D_k : k < \omega\} \text{ and}$$

$a_{\eta_{r_\alpha, \nu^*}}$ determines in $\Vdash_{\mathbb{Q}_\kappa^1}$ for any $k < \omega$ the value of $\mathcal{T} \upharpoonright k$.

This is a contradiction.

So there is a smallest k such that ν_k cannot be defined. We let $n_\alpha = k$. We let p_α be a strengthening of $r_\alpha^{(\nu_{k-1})}$ such that $p_\alpha \in D_{n_\alpha}$. For finding such a strengthening we again invoke the fusion Lemma 3.2.

We show that $p_\alpha \perp p_\beta$ for $\beta < \alpha$ with $n_\beta \geq k$. Otherwise, having arrived at $r_\alpha^{(\nu_{k-1})}$ we find some β_k, α such that $n_{\beta_k} \geq k$ and $r_\alpha^{(\nu_{k-1})}$ is compatible with p_{β_k} . Then we can prolong ν_{k-1} to a splitting node $\nu_k \in \text{spl}(p_{\beta_k}) \cap \text{spl}(r_\alpha)$. By the choice of r_α the latter implies that $r_\alpha^{(\nu_k)} \geq p_{\beta_k}$. However, now we would have found ν_k, β_k as required in contradiction to the choice of k . □

Remark 3.11. Conditions (a) to (c) of Lemma 3.10 yield: For any $k < \omega$,

$$\{p_\alpha : n_\alpha \geq k\} \text{ is dense in } \mathbb{Q}_\kappa^2.$$

Proof. Let k and p be given. There is α_0 such that $T_{\alpha_0} \in D_0$ and $T_{\alpha_0} \geq_{\mathbb{Q}_\kappa^2} p$. Then $p_{\alpha_0} \geq T_{\alpha_0}$ and $n_{\alpha_0} \geq k$. Then there is $\alpha_1 > \alpha_0$ such that $T_{\alpha_1} \geq_{\mathbb{Q}_\kappa^2} p_{\alpha_0}$. Then $p_{\alpha_1} \geq T_{\alpha_1}$ and hence by condition (c), $n_{\alpha_1} > n_{\alpha_0} \geq 0$. We can repeat the argument $k - 1$ times. □

Now we drop the component $\bar{\gamma}_\alpha$ from a sequence $\langle p_\alpha, n_\alpha, \bar{\gamma}_\alpha : \alpha < 2^\kappa \rangle$ given by Lemma 3.10. Then we get a sequence with properties (a), (b), and a weakening (c) with the property stated in the remark.

Lemma 3.12. *We assume that \mathbb{Q}_κ^1 collapses 2^κ to ω , $\text{cf}(\kappa) > \omega$ and $2^{(2^{<\kappa})} = 2^\kappa$. Let $\langle T_\alpha : \alpha < 2^\kappa \rangle$ enumerate all Miller trees that such each tree appears 2^κ times. If $\langle (p_\alpha, n_\alpha) : \alpha < 2^\kappa \rangle$ are such that*

- (a) $n_\alpha < \omega$,
- (b) $p_\alpha \in D_{n_\alpha}$ and $p_\alpha \geq T_\alpha$,
- (c) if $\beta < \alpha$ and $n_\beta = n_\alpha$ then $p_\beta \perp p_\alpha$,
- (d) for any $k \in \omega$, $\{p_\alpha : n_\alpha \geq k\}$ is dense in \mathbb{Q}_κ^2 .

Then there is a \mathbb{Q}_κ^2 -name \mathcal{T}' for a surjection of ω onto 2^κ .

Proof. Let G be a \mathbb{Q}_κ^2 -generic filter over \mathbf{V} . We define $\tau(n)$, a \mathbb{Q}_κ^2 -name by $\tau(n)[G] = \alpha$ if $p_\alpha \in G$ and $n_\alpha = n$. The name τ is a name of a function by (c). By (d), the domain of τ is forced to be infinite. For any $p \in \mathbb{Q}_\kappa^2$ we let $U_p = \{\alpha : T_\alpha = p\}$. U_p is of size 2^κ , in particular for $\alpha \in 2^\kappa$ we have $|U_{p_\alpha}| = 2^\kappa$. Hence there is $f: 2^\kappa \rightarrow 2^\kappa$ such that for any $\alpha, \gamma \in 2^\kappa$ and $\exists \beta \in U_{p_\gamma}$ with $f(\beta) = \alpha$. We let $\tau'(n) = f(\tau(n))$. Next we show

$$\mathbb{Q}_\kappa^2 \Vdash \text{range}(\tau') = 2^\kappa.$$

Suppose $p \in Q_\tau$ and $\alpha < 2^\kappa$ are given. By construction the sequence $\{p_\beta : \beta < 2^\kappa\}$ is dense. Let $p \leq p_\gamma$. Then there is $\beta \in U_{p_\gamma}$, with $f(\beta) = \alpha$. However, $\beta \in U_{p_\gamma}$ means $T_\beta = p_\gamma \leq p_\beta$ by construction. By the definition of τ , $p_\beta \Vdash \tau(n_\beta) = \beta$, so $p_\beta \Vdash f(\tau(n_\beta)) = \alpha$. \square

So we can sum up:

Theorem 3.13. *We assume that \mathbb{Q}_κ^1 collapses 2^κ to ω and $\text{cf}(\kappa) > \omega$ and $2^{(\kappa^{<\kappa})} = 2^\kappa$. Then the forcing with \mathbb{Q}_κ^2 collapses 2^κ to \aleph_0 .*

4. κ -COHEN REALS AND THE LEVY COLLAPSE

Another vice of a κ -tree forcing is to add κ -Cohen reals. In this section we show that under the above conditions, \mathbb{Q}_2^κ adds Cohen reals and is equivalent to the Levy collapse of 2^κ to \aleph_0 .

Lemma 4.1. *If \mathbb{P} collapses 2^κ to \aleph_0 , $\text{cf}(\kappa) > \aleph_0$, and $2^{2^{<\kappa}} = 2^\kappa$, then \mathbb{Q}_2^κ adds a κ -Cohen real.*

Proof. Let G be \mathbb{Q}_κ^2 -generic over \mathbf{V} . Let $f: \omega \rightarrow 2^{<\kappa}$ be a function in $\mathbf{V}[G]$, such that $(\forall \eta \in 2^{<\kappa})(\exists^\infty k f(k) = \eta)$. Such a function exists since $2^{<\kappa} \leq 2^\kappa$.

Since $2^{2^{<\kappa}} = 2^\kappa$, we can enumerate all antichains in $\mathbb{C}(\kappa)$ in $\alpha_* \leq 2^\kappa$ many steps. In $\mathbf{V}[G]$, α_* is countable. We list it as $\langle \alpha_n : n < \omega \rangle$. Now we choose $\eta_n \in \mathbb{C}(\kappa)^\mathbf{V}$ by induction on n in $\mathbf{V}[G]$: $\eta_0 = \emptyset$. Given η_n we choose k_n such that $f(k_n) = \eta_n$ and then we choose $\eta_{n+1} \supseteq \eta_n$, such that $\eta_{n+1} \in I_{\alpha_n}$. Then $\{\eta : (\exists n < \omega)(\eta \trianglelefteq f(k_n))\}$ is a $\mathbb{C}(\kappa)$ -generic filter over \mathbf{V} and it exists in $\mathbf{V}[G]$, since it is definable from $\{f(k_n) : n < \omega\}$. \square

Two forcings $\mathbb{P}_1, \mathbb{P}_2$ are said to be equivalent if their regular open algebras $\text{RO}(\mathbb{P}_i)$ coincide (for a definition of the regular open algebra of a poset, see, e.g., [3, Corollary 14.12]). Some forcings are characterised up to equivalence just by their size and their collapsing behaviour.

Definition 4.2. Let B be a Boolean algebra. We write $B^+ = B \setminus \{0\}$. A subset $D \subseteq B^+$ is called *dense* if $(\forall b \in B^+)(\exists d \in D)(d \leq b)$. The *density* of a Boolean algebra B is the least size of a dense subset of B . A Boolean algebra B has uniform density if for every $a \in B^+$, $B \upharpoonright a$ has the same density. The *density* of a forcing order $(\mathbb{P}, <)$ is the density of the regular open algebra $\text{RO}(\mathbb{P})$.

Lemma 4.3. [3, Lemma 26.7]. *Let $(Q, <)$ be a notion of forcing such that $|Q| = \lambda > \aleph_0$ and such that Q collapses λ onto \aleph_0 , i.e.,*

$$0_Q \Vdash_Q |\check{\lambda}| = \aleph_0.$$

Then $\text{RO}(Q) = \text{Levy}(\aleph_0, \lambda)$.

Lemma 4.4. *If \mathbb{Q}_κ^1 collapses 2^κ to \aleph_0 , then \mathbb{Q}_κ^1 is equivalent of $\text{Levy}(\aleph_0, 2^\kappa)$.*

Proof. \mathbb{Q}_κ^1 has size 2^κ . Hence Lemma 4.3 yields $\text{RO}(\mathbb{Q}_\kappa^1) = \text{Levy}(\aleph_0, 2^\kappa)$. \square

Definition 4.5. A Boolean algebra is (θ, λ) -nowhere distributive if there are antichains $\bar{p}^\varepsilon = \langle p_\alpha^\varepsilon : \alpha < \alpha_\varepsilon \rangle$ of \mathbb{P} for $\varepsilon < \theta$ such that for every $p \in \mathbb{P}$ for some $\varepsilon < \theta$

$$|\{\alpha < \alpha_\varepsilon : p \not\leq p_\alpha^\varepsilon\}| \geq \lambda.$$

Lemma 4.6. [1, Theorem 1.15] *Let $\theta < \lambda$ be regular cardinals.*

(1) *Suppose that \mathbb{P} has the following properties (a) to (c).*

(a) *\mathbb{P} is a (θ, λ) -nowhere distributive forcing notion,*

(b) *\mathbb{P} has density λ ,*

(c) *in case $\theta > \aleph_0$, \mathbb{P} has a θ -complete subset S . The latter means:*
 $(\forall B \in [S]^{<\theta})(\exists s \in S)(\forall b \in B)(b \leq_{\mathbb{P}} s)$.

Then \mathbb{P} is equivalent to $\text{Levy}(\theta, \lambda)$.

(2) *Under (a) and (b) \mathbb{P} collapses λ to θ (and may or may not collapse \aleph_0).*

Proposition 4.7. *If there is a κ -mad family of size 2^κ the forcing \mathbb{Q}_κ^1 is $(\aleph_0, 2^\kappa)$ -nowhere distributive.*

Proof. Lemma 2.5 gives \mathcal{T} such that $\bar{p}^n = \{a_\eta : \eta \in {}^n(2^\kappa)\}$, $n \in \omega$, witnesses $(\aleph_0, 2^\kappa)$ -nowhere distributivity. \square

By Lemma 4.3 and Theorem 3.13 we get:

Proposition 4.8. *If \mathbb{Q}_κ^1 collapses 2^κ to \aleph_0 , $\text{cf}(\kappa) > \aleph$ and $2^{(\kappa^{<\kappa})} = 2^\kappa$ then \mathbb{Q}_κ^2 is equivalent to $\text{Levy}(\aleph_0, 2^\kappa)$.*

Acknowledgement: We thank Marlene Koelbing for pointing out a gap in an earlier version.

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