CONTROLLING CARDINAL CHARACTERISTICS WITHOUT ADDING REALS

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ABSTRACT. We investigate the behavior of cardinal characteristics of the reals under extensions that do not add new $<\kappa$-sequences (for some regular κ).

As an application, we present models (assuming the consistency of three strongly compact cardinals) with 13 simultaneously different cardinal characteristics (including $\aleph_1$ and continuum): the characteristics in Cichoń's diagram, plus $\mathfrak{m}$, $\mathfrak{p}$ and $\mathfrak{h}$.

Additionally, we show how to get rid of the cardinal gaps of previous constructions with many different characteristics. For example, the set of values for the characteristics mentioned above, except for $\mathfrak{h}$, can be made $\{\aleph_n : 1 \leq n \leq 12\}$; alternatively, we can reduce any of the gaps to zero and thus replace any number of the $<$ in the chain of inequalities of the characteristics in Cichoń's diagram with $=$.

Without large cardinals, we get 9 different values (7 in Cichoń's diagram, plus $\mathfrak{m}$ and $\mathfrak{p}$).

1. Introduction

In this work we investigate how to preserve and how to change certain cardinal characteristics of the continuum in NNR extensions, i.e., extensions that do not add reals; or more generally that do not add $<\kappa$-sequences of ordinals for some regular κ. It is known that the "Blass-uniform" characteristics (see Definition 2.7) tend to keep their values in such extensions (cf. Section 3.1), and we give some explicit results in that direction. Other cardinal characteristics tend to keep a value θ only if $\theta < \kappa$. We will use this effect to combine various forcing notions (most of them already known) to get models with many simultaneously different "classical" characteristics.

In particular, we look at the entries of Cichoń's diagram, which we call *Cichoń-characteristics* (see Figure 1, we assume that the reader is familiar with this diagram), and the following characteristics:

Definition 1.1. Let $\mathcal{P}$ be a class of posets.

(1) $\mathfrak{m}(\mathcal{P})$ denotes the minimal cardinal where Martin's axiom for the posets in $\mathcal{P}$ fails. More explicitly, it is the minimal κ such that, for some poset $Q \in \mathcal{P}$, there is a collection $\mathcal{D}$ of size κ of dense subsets of Q such that there is no filter in Q intersecting all the members of $\mathcal{D}$.

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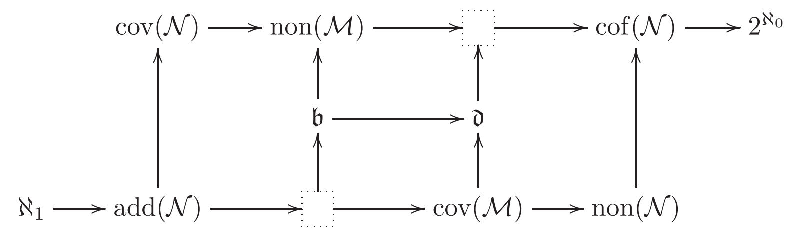

FIGURE 1. Cichoń's diagram with the two "dependent" values removed, which are $\mathrm{add}(\mathcal{M}) = \min(\mathfrak{b}, \mathrm{cov}(\mathcal{M}))$ and $\mathrm{cof}(\mathcal{M}) = \max(\mathrm{non}(\mathcal{M}), \mathfrak{d})$. An arrow $\mathfrak{x} \to \mathfrak{y}$ means that ZFC proves $\mathfrak{x} \leq \mathfrak{y}$.

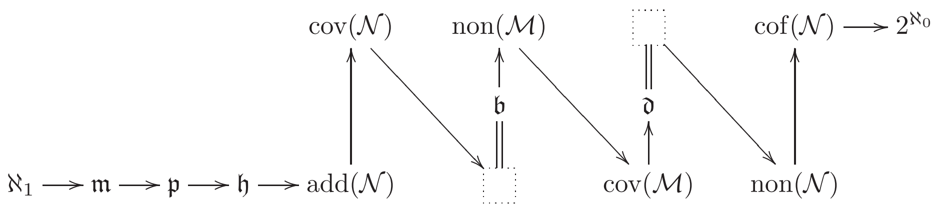

FIGURE 2. The model we construct in this paper; here $\mathfrak{x} \to \mathfrak{y}$ means that $\mathfrak{x} < \mathfrak{y}$ (when $\mathfrak{h}$ is omitted, any number of the $<$ signs can be replaced by $=$ as desired).
This model corresponds to "Version A" (vA*, Fig. 3). We also realise another ordering of the Cichoń values, called "Version B" (vB*, Fig. 4).

(2) $\mathfrak{m} := \mathfrak{m}(\text{ccc})$.
(3) Write $a \subseteq^* b$ iff $a \setminus b$ is finite. Say that $a \in [\omega]^{\aleph_0}$ is a *pseudo-intersection* of $F \subseteq [\omega]^\omega$ if $a \subseteq^* b$ for all $b \in F$.
(4) The *pseudo-intersection number* $\mathfrak{p}$ is the smallest size of a filter base of a free filter on ω that has no pseudo-intersection in $[\omega]^{\aleph_0}$.
(5) The *tower number* $\mathfrak{t}$ is the smallest order type of a $\subseteq^*$-decreasing sequence in $[\omega]^{\aleph_0}$ without pseudo-intersection.
(6) The *distributivity number* $\mathfrak{h}$ is the smallest size of a collection of dense subsets of $([\omega]^{\aleph_0}, \subseteq^*)$ whose intersection is empty.
(7) A family $D \subseteq [\omega]^{\aleph_0}$ is *groupwise dense* if
   (i) $a \subseteq^* b$ and $b \in D$ implies $a \in D$, and
   (ii) whenever $(I_n : n < \omega)$ is an interval partition of ω, there is some $a \in [\omega]^{\aleph_0}$ such that $\bigcup_{n \in a} I_n \in D$.

   The *groupwise density number* $\mathfrak{g}$ is the smallest size of a collection of groupwise dense sets whose intersection is empty.

The known ZFC provable relations between these cardinals are

$$\mathfrak{m} \leq \mathfrak{p} = \mathfrak{t} \leq \mathfrak{h} \leq \mathfrak{g}, \quad \mathfrak{m} \leq \mathrm{add}(\mathcal{N}), \quad \mathfrak{t} \leq \mathrm{add}(\mathcal{M}), \quad \mathfrak{h} \leq \mathfrak{b}, \quad \mathfrak{g} \leq \mathfrak{d}. \tag{1.2}$$

Also, with the exception of $\mathfrak{m}$ and $\mathfrak{d}$, all the cardinals in (1.2) are known to be regular (and uncountable), $2^{<\mathfrak{t}} = \mathfrak{c}$ and $\mathrm{cof}(\mathfrak{c}) \geq \mathfrak{g}$. For details see e.g. Blass [Bla10], but for $\mathfrak{p} = \mathfrak{t}$ see the result [MS16] with Malliaris.[1]

Very recently [GKS] constructed, modulo four strongly compact cardinals, a ZFC model where the ten (non-dependent) Cichoń-characteristics are pairwise different. This gives the order of these characteristics as displayed in Figures 2 and 3. This construction was later improved with Brendle and Cardona [BCM18], now only using three strongly compact cardinals. We will refer to this order of the characteristics, and the construction to get them, as "version A" from now on.

[KST19], with Tănasie, forced an alternative order of the same ten characteristics; again using four strongly compact cardinals. This alternative order is displayed in Figure 4 and is from now on called "Version B".

To continue with this line of work, we ask whether other classical cardinal characteristics of the continuum can be included and forced to be pairwise different. Our main result is that, in the consistency results corresponding to [BCM18] and [KST19], we can additionally force that $\aleph_1 < \mathfrak{m} < \mathfrak{p} < \mathfrak{h} = \mathfrak{g} < \mathrm{add}(\mathcal{N})$, thus yielding a model where 13 classical cardinal characteristics are pairwise different (modulo three strongly compact cardinals for version A, see Figure 2, and modulo four strongly compact cardinals for version B).

We now give an outline of this paper:

Preliminaries (Section 2). We review some parts of the "old" constructions that give 10 different values in Cichoń's diagram. In particular, we define Blass-uniform characteristics and the LCU and COB properties. We briefly remark on the history of the result.

Cardinal characteristics in extensions without new $<\kappa$-sequences (Section 3). We define some classes of cardinal characteristics and show how they are affected (or unaffected) by extensions that do not add new $<\kappa$-sequences for some regular κ; in particular: under $<\kappa$-distributive forcing extensions; under Boolean ultrapowers; and when intersecting the poset with some $<\kappa$-closed elementary submodel.

Dealing with $\mathfrak{m}$ (Section 4). Using classical methods originating from Barnett and Todorčević [Tod86, Tod89, Bar92], we modify the "old" iterations from [BCM18] and [KST19] (for the left hand side, without Boolean ultrapowers or large cardinals) to additionally force $\mathfrak{m} = \lambda_\mathfrak{m}$ and $\mathfrak{p} = \mathfrak{b}$ for any given regular value $\lambda_\mathfrak{m}$ between $\aleph_1$ and $\mathrm{add}(\mathcal{N})$.

Alternatively, we can modify the "old" constructions with Boolean ultrapowers to get, in addition to the Cichoń values, $\mathfrak{m} = \lambda_\mathfrak{m}$, under the condition that $\lambda_\mathfrak{m}$ is less than the smallest strongly compact cardinal used for the ultrapowers.

In addition to $\mathfrak{m}$, we can control the Knaster-numbers $\mathfrak{m}(k\text{-Knaster})$ as well. But this does not give a larger number of simultaneously different characteristics: All Knaster numbers bigger than $\aleph_1$ have the same value (which is also the value of $\mathfrak{m}(\text{precaliber})$). We give models for all possible constellations (at least for regular λ): All Knaster numbers can be $\aleph_1$ (in which case we can also force $\mathfrak{m}(\text{precaliber}) = \aleph_1$), and there can be a $k \geq 1$ such that $\mathfrak{m}(\ell\text{-Knaster}) = \aleph_1$ for all $1 \leq \ell < k$ and $\mathfrak{m}(\ell\text{-Knaster}) = \lambda$ for $\ell \geq k$.

[1] However, only the trivial inequality $\mathfrak{p} \leq \mathfrak{t}$ is used thorough this text.

Dealing with $\mathfrak{m}$(precaliber) (Section 5). We deal with a case that was left open in the previous section: We construct a model where all Knaster numbers are $\aleph_1$, and the precaliber number is some regular $\lambda > \aleph_1$. For this purpose, we construct a ccc poset P_{cal} forcing $\mathfrak{m}(\text{precaliber}) \leq \lambda$ and such that this inequality is preserved after λ-Knaster posets. Again, these constructions are compatible with the old constructions, assuming λ is small enough.

Dealing with $\mathfrak{p}$ (Section 6). Based on a result with Dow [DS], we show that the product of a κ-cc poset P with $\kappa^{<\kappa}$ may add a tower of length κ, while preserving the cardinal $\mathfrak{h}$ above κ and the values for the Cichoń-characteristics that were already forced by P.

Dealing with $\mathfrak{h}$ (Section 7). Given a poset P, we show how to obtain a complete subposet P' of P forcing smaller values to $\mathfrak{g}$ and $\mathfrak{c}$, while preserving certain other values for cardinal characteristics already forced by P. This method not only allows us to control $\mathfrak{g}$, but also lets us modify the value of $\mathfrak{c}$ to anything with cofinality above $\mathfrak{g}$.

Collapsing the gaps (Section 8). In the "old" models for 10 different Cichoń characteristics, we require compact cardinals between the left hand side characteristics. So we do not know, for example, that $\text{cov}(\mathcal{N})$ could be the successor of $\text{add}(\mathcal{N})$.

In this paper we show how to avoid this limitation. (However, strongly compact cardinals are still required for the consistency results.) After forcing 10 (or more) different values with a forcing P as before, we take products with collapse-posets to shorten the gaps between any pair of successive cardinals characteristics (including those on the left side). We can also make a gap vanish altogether, which gives the new result that in Cichoń's diagram, not only are the inequalities between all entries consistent, but we can even replace arbitrarily many of the $<$ with $=$.

We remark that these techniques are not compatible with the methods of Section 7, so we do not know how to control $\mathfrak{h}$ here.

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2. Preliminaries

2.1. **The old constructions.** In this paper, we will build on two constructions from [GKS, BCM18] and [KST19], which we call the "old constructions" and refer to as $\texttt{vA}^*$ and $\texttt{vB}^*$, respectively. They force different values to several (or all) entries of Cichoń's diagram. We will not describe these constructions in detail, but refer to the respective papers instead.

The "basic versions" of the constructions do not require large cardinals and give us different values for the "left hand side":

Theorem 2.1. *Assume that $\aleph_1 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$ are regular cardinals and $\lambda_4 \leq \lambda_5 \leq \lambda_6$.*

[BCM18] *If λ_5 is a regular cardinal and $\lambda_6^{<\lambda_3} = \lambda_6$, then there is a f.s. iteration $\bar{P}^{\texttt{vA}}$ of length of size λ_6 with cofinality λ_3, using iterands that are (σ, k)-linked for*

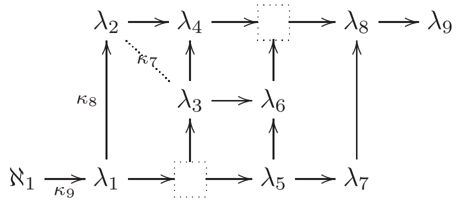

FIGURE 3. The vA* order.

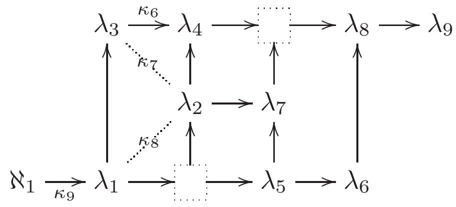

FIGURE 4. The vB* order.

every $k \in \omega$ (see Definition 4.1(3)), which forces

$$(\texttt{vA})\quad \mathrm{add}(\mathcal{N}) = \lambda_1,\ \mathrm{cov}(\mathcal{N}) = \lambda_2,\ \mathfrak{b} = \lambda_3,\ \mathrm{non}(\mathcal{M}) = \lambda_4,\ \mathrm{cov}(\mathcal{M}) = \lambda_5,\ \textit{and}\ \mathfrak{d} = \mathfrak{c} = \lambda_6.$$

[KST19] *If $\lambda_5 = \lambda_5^{<\lambda_4}$ and either $\lambda_2 = \lambda_3$,*[2] *or λ_3 is $\aleph_1$-inaccessible,*[3] *$\lambda_2 = \lambda_2^{<\lambda_2}$ and $\lambda_4^{\aleph_0} = \lambda_4$, then there is a f.s. iteration $\bar{P}^{\texttt{vB}}$ of length of size λ_5 with cofinality λ_4, using iterands that are (σ, k)-linked for every $k \in \omega$, that forces*

$$(\texttt{vB})\quad \mathrm{add}(\mathcal{N}) = \lambda_1,\ \mathfrak{b} = \lambda_2,\ \mathrm{cov}(\mathcal{N}) = \lambda_3,\ \mathrm{non}(\mathcal{M}) = \lambda_4,\ \textit{and}\ \mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \lambda_5.$$

These consistency results correspond to λ_1–λ_6 of Figure 3, and to λ_1–λ_5 of Figure 4, respectively.

Remark 2.2. Note that the hypothesis for vB is weaker than the hypothesis in the original reference [KST19], even more, GCH is not assumed at all. This strengthening is a result of simple modifications, which are presented in [Mej19b].

Both constructions can then be extended with Boolean ultrapowers (more precisely: compositions of finitely many successive Boolean ultrapowers), to make all values simultaneously different:

Theorem 2.3. *Assume $\aleph_1 < \lambda_1 < \lambda_2 < \lambda_3 \le \lambda_4 \le \lambda_5 \le \lambda_6 \le \lambda_7 \le \lambda_8 \le \lambda_9$.*
[BCM18] *If $\aleph_1 < \kappa_9 < \lambda_1 < \kappa_8 < \lambda_2 < \kappa_7 < \lambda_3$ such that*

(i) for $j = 7, 8, 9$, κ_j is strongly compact and $\lambda_j^{\kappa_j} = \lambda_j$,
(ii) λ_i is regular for $i \ne 6$ and
(iii) $\lambda_6^{<\lambda_3} = \lambda_6$,

then there is a f.s. ccc iteration $P^{\texttt{vA}*}$ (a Boolean ultrapower of $P^{\texttt{vA}}$) that forces the constellation of Figure 3:

$$(\texttt{vA}^*)\quad \mathrm{add}(\mathcal{N}) = \lambda_1,\ \mathrm{cov}(\mathcal{N}) = \lambda_2,\ \mathfrak{b} = \lambda_3,\ \mathrm{non}(\mathcal{M}) = \lambda_4,\ \mathrm{cov}(\mathcal{M}) = \lambda_5,\ \mathfrak{d} = \lambda_6,\ \mathrm{non}(\mathcal{N}) = \lambda_7,\ \mathrm{cof}(\mathcal{N}) = \lambda_8,\ \textit{and}\ \mathfrak{c} = \lambda_9.$$

[KST19] *If $\aleph_1 < \kappa_9 < \lambda_1 < \kappa_8 < \lambda_2 < \kappa_7 < \lambda_3 < \kappa_6 < \lambda_4$ such that*

(i) for $j = 6, 7, 8, 9$, κ_j is strongly compact and $\lambda_j^{\kappa_j} = \lambda_j$,
(ii) λ_i is regular for $i \ne 5$,
(iii) $\lambda_2^{<\lambda_2} = \lambda_2$, $\lambda_4^{\aleph_0} = \lambda_4$, $\lambda_5^{<\lambda_4} = \lambda_5$, and

[2] The result for the case $\lambda_2 = \lambda_3$ is easily obtained with techniques from Brendle [Bre91].
[3] A cardinal λ is *κ-inaccessible* if $\mu^\nu < \lambda$ for any $\mu < \lambda$ and $\nu < \kappa$.

(iv) λ_3 *is* $\aleph_1$*-inaccessible,*

then there is a f.s. ccc iteration $P^{\texttt{vB}*}$ *(a Boolean ultrapower of* $P^{\texttt{vB}}$*) that forces the constellation of Figure 4:*

$$(\texttt{vB}^*)\quad \mathrm{add}(\mathcal{N})=\lambda_1,\ \mathfrak{b}=\lambda_2,\ \mathrm{cov}(\mathcal{N})=\lambda_3,\ \mathrm{non}(\mathcal{M})=\lambda_4,$$
$$\mathrm{cov}(\mathcal{M})=\lambda_5,\ \mathrm{non}(\mathcal{N})=\lambda_6,\ \mathfrak{d}=\lambda_7,\ \mathrm{cof}(\mathcal{N})=\lambda_8,\ \textit{and}\ \mathfrak{c}=\lambda_9.$$

More specifically: For $i=6,7,8,9$ let j_i be a complete embedding associated with some suitable Boolean ultrapower from the completion of $\mathrm{Coll}(\kappa_i,\lambda_i)$, which yields $\mathrm{cr}(j_i)=\kappa_i$ and $\mathrm{cof}(j_i(\kappa_i))=|j_i(\kappa_i)|=\lambda_i$ (see a bit more details at the end of Subsection 2.2). Then $P^{\texttt{vA}*}=j_9(j_8(j_7(P^{\texttt{vA}})))$ forces the constellation of Figure 3. Analogously, $P^{\texttt{vB}*}=j_9(j_8(j_7(j_6(P^{\texttt{vB}}))))$ forces the constellation of Figure 4.

Remark 2.4. In the original results of Theorem 2.3, all the inequalities are assumed to be strict (though in vA* this is just from λ_6), but they can be equalities alternatively. Even more, equalities on the right side may disregard some strongly compact cardinals (and their corresponding hypothesis), for example, if $\lambda_j=\lambda_{j+1}$ (for some $j=6,7,8$) then the compact cardinal κ_j is not required, furthermore, the weaker assumption $\lambda_{9-j}\leq\lambda_{(9-j)+1}$ (for the dual cardinal characteristics) is allowed in this case.

Notation 2.5.
(1) Whenever we are investigating a characteristic $\mathfrak{x}$ and plan to force a specific value to it, we call this value $\lambda_{\mathfrak{x}}$. For example, for vA* $\lambda_2=\lambda_{\mathrm{cov}(\mathcal{N})}$, whereas for vB* $\lambda_2=\lambda_{\mathfrak{b}}$. Let us stress that calling a cardinal $\lambda_{\mathfrak{x}}$ is *not* an implicit *assumption* that $P\Vdash\mathfrak{x}=\lambda_{\mathfrak{x}}$ for the P under investigation; it is just an (implicit) declaration of intent.
(2) Whenever we base an argument on one of the old constructions above, and say "we can modify the construction to additionally force…", we implicitly assume that the desired values $\lambda_{\mathfrak{x}}$ for the "old" characteristics satisfy the assumptions we made in the "old" constructions (such as "$\lambda_{\mathfrak{x}}$ is regular").

Remark 2.6. We briefly remark on the history of the results of this section.

1. A (by now) classical series of results by various authors [Bar84, BJS93, CKP85, JS90, Kam89, Kra83, Mil81, Mil84, RS83] (summarized by Bartoszyński and Judah [BJ95]) shows that any assignments of $\{\aleph_1,\aleph_2\}$ to the Cichoń-characteristics that satisfy the well known ZFC restrictions is consistent. This leaves the questions how to show that many values can be simultaneously different.
2. The left hand side part of vA* was done in [GMS16] and uses eventually different forcing $\mathbb{E}$ to ensure $\mathrm{non}(\mathcal{M})\geq\lambda_{\mathrm{non}(\mathcal{M})}$ and ultrafilter-limits of $\mathbb{E}$ to show that $\mathfrak{b}$ remains small. It relies heavily on the notion of goodness, introduced in [JS90] (with Judah) and by Brendle [Bre91], and summarized in e.g. [GMS16] or [CM19] (with Cardona).
3. Based on this construction, [GKS] uses Boolean ultrapowers to get simultaneously different values for all (independent) Cichoń-characteristics, modulo four strongly compact cardinals.

   For this, the construction for the left hand side first has to be modified to get a ccc forcing starting with a ground model satisfying GCH.

   Then Boolean ultrapowers are applied to separate the cardinals on the right side. [KTT18] (with Tănasie and Tonti) gives an introduction to the

Boolean ultrapower construction. Such Boolean ultrapowers are applied four times, once for each pair of cardinals on the right side that are separated.

For this it is required that there is a strongly compact cardinal between two values corresponding to adjacent cardinals characteristics on the left side, so the cardinals on this side are necessarily very far apart.

4. [BCM18] improves the left hand side construction of [GMS16] to include $\mathrm{cov}(\mathcal{M}) < \mathfrak{d} = \mathrm{non}(\mathcal{N}) = \mathfrak{c}$ (see vA). This is achieved by using matrix iterations of partial Frechet-linked posets (the latter concept is originally from [Mej19a]). This construction can be performed directly under GCH (so the step to make the forcing compatible with GCH is not necessary here). Then the same method of Boolean ultrapowers as before can be applied, in the same way, to force different values for all Cichoń-characteristics, modulo three strongly compact cardinals (see vA*).
5. Both vB and vB* were done in [KST19]: The left hand side is based on ideas from [She00], and in particular the notion of finite additive measure (FAM) limit introduced there for random forcing. In addition, a creature forcing Q^2 similar to the one defined in [HS] (with Horowitz) is introduced, which forces $\mathrm{non}(\mathcal{M}) \geq \lambda_{\mathrm{non}(\mathcal{M})}$ and which has FAM-limits similar to random forcing (which is required to keep $\mathfrak{b}$ small). The rest of the construction is identical to the one in [GKS].

2.2. **Blass-uniform cardinal characteristics, LCU and COB.**

Definition 2.7. A *Blass-uniform cardinal characteristic* is a characteristic of the form

$$\mathfrak{d}_R := \min\{|D| : D \subseteq \omega^\omega \text{ and } (\forall x \in \omega^\omega)\,(\exists y \in D)\ xRy\}$$

for some Borel[4] R.

Such characteristics have been studied systematically since at least the 1980s by many authors, including Fremlin [Fre84], Blass [Bla93, Bla10] and Vojtáš [Voj93].

Note that its dual cardinal

$$\mathfrak{b}_R := \min\{|F| : F \subseteq \omega^\omega \text{ and } (\forall y \in \omega^\omega)\,(\exists x \in F)\ \neg xRy\}$$

is also Blass-uniform because $\mathfrak{b}_R = \mathfrak{d}_{R^\perp}$ where $xR^\perp y$ iff $\neg(yRx)$.

Remark 2.8. All Blass-uniform characteristics in this paper, and many others, such as those in Blass' survey [Bla10] or those in [GS93], are in fact of the form $\mathfrak{b}_R$ or $\mathfrak{d}_R$ for some Σ^0_2 relation R which is invariant under finite modifications of its arguments. When we restrict to such relations, there is no ambiguity as to which Blass-uniform cardinal characteristics are of the form $\mathfrak{b}_R$ and which are of the form $\mathfrak{d}_R$. It was shown by Blass [Bla93] that for such relations R we must have $\mathfrak{b}_R \leq \mathrm{non}(\mathcal{M})$ and $\mathfrak{d}_R \geq \mathrm{cov}(\mathcal{M})$, thus $\mathfrak{b}_R$ is always on the left side of Cichoń's diagram, and $\mathfrak{d}_R$ is on the right side.

[4]We could just as well assume that R is analytic or co-analytic. More specifically, for all results in this paper, it is enough to assume that R is absolute between the extensions we consider; in our case between extensions that do not add new reals. So even projective relations would be OK. However, all concrete relations that we will actually use are Borel, even of very low rank. Regarding "on ω^ω", see Remark 2.9.

Remark 2.9. It can be more practical to consider more generally relations on $X \times Y$ for some Polish spaces X, Y other than ω^ω, in particular as many examples of Blass-uniform cardinals are naturally defined in such spaces.

To cover such cases, one can either modify the definition, or use a Borel isomorphisms to translate the relation to ω^ω.

Example 2.10. The following are pairs of dual Blass-uniform cardinals $(\mathfrak{b}_R, \mathfrak{d}_R)$ for natural[5] Borel relations R:

(1) A cardinal on the left hand side of Cichoń's diagram and its dual on the right hand side: $(\text{add}(\mathcal{N}), \text{cof}(\mathcal{N}))$, $(\text{cov}(\mathcal{N}), \text{non}(\mathcal{N}))$, $(\text{add}(\mathcal{M}), \text{cof}(\mathcal{M}))$, $(\text{non}(\mathcal{M}), \text{cov}(\mathcal{M}))$, and $(\mathfrak{b}, \mathfrak{d})$.
(2) $(\mathfrak{s}, \mathfrak{r}) = (\mathfrak{b}_R, \mathfrak{d}_R)$ where $\mathfrak{s}$ is splitting number and $\mathfrak{r}$ the reaping number and R is the relation on $[\omega]^{\aleph_0}$ that states xRy iff "y does not split x".

We will often have a situation where $(\mathfrak{b}_R, \mathfrak{d}_R) = (\lambda_b, \lambda_d)$ is "strongly witnessed", as follows:

Definition 2.11. Fix a Borel relation R, λ a regular cardinal and μ an arbitrary cardinal. We define two properties:[6]

Linearly cofinally unbounded: $\text{LCU}_R(\lambda)$ means: There is a family $\bar{f} = (f_\alpha : \alpha < \lambda)$ of reals such that:

$$(\forall g \in \omega^\omega)\, (\exists \alpha \in \lambda)\, (\forall \beta \in \lambda \setminus \alpha)\, \neg f_\beta R g. \tag{2.12}$$

Cone of bounds: $\text{COB}_R(\lambda, \mu)$ means: There is a $<\lambda$-directed order $\trianglelefteq$ on μ,[7] and a family $\bar{g} = (g_s : s \in \mu)$ of reals such that

$$(\forall f \in \omega^\omega)\, (\exists s \in \mu)\, (\forall t \trianglerighteq s)\; f R g_t. \tag{2.13}$$

Fact 2.14. $\text{LCU}_R(\lambda)$ *implies* $\mathfrak{b}_R \leq \lambda \leq \mathfrak{d}_R$.
$\text{COB}_R(\lambda, \mu)$ *implies* $\mathfrak{b}_R \geq \lambda$ *and* $\mathfrak{d}_R \leq \mu$.

Remark 2.15. $\text{COB}_R(\lambda, \mu)$ clearly implies $\text{COB}_R(\lambda', \mu)$ whenever $\lambda' \leq \lambda$. The property $\text{COB}_R(2, \mu)$, the weakest of these notions, just says that there is a witness for $\mathfrak{d}_R \leq \mu$, or in other words: there is an R-dominating[8] family of size μ.

Also, $\text{COB}_R(\lambda, \mu)$ implies $\text{COB}_R(\lambda, \mu')$ whenever $\mu' \geq \mu$. And note that, for nontrivial R, $\text{COB}_R(\lambda, \mu)$ implies $\lambda \leq \mu$.

Informally, we call the objects $\bar{f}$ in the definition of LCU and $(\trianglelefteq, \bar{g})$ for COB "strong witnesses", and say that the corresponding cardinal inequalities (or equalities) are "strongly witnessed". For example, "$(\mathfrak{b}, \mathfrak{d}) = (\lambda_\mathfrak{b}, \lambda_\mathfrak{d})$ is strongly witnessed" means: for the natural relation R (namely, the relation $\leq^*$ of eventual dominance), we have $\text{COB}_R(\lambda_\mathfrak{b}, \lambda_\mathfrak{d})$, $\text{LCU}_R(\lambda_\mathfrak{b})$ and there is some regular $\lambda_0 \leq \lambda_\mathfrak{d}$ such that $\text{LCU}_R(\lambda)$ for all regular $\lambda \in [\lambda_0, \lambda_\mathfrak{d}]$ (this is to allow $\lambda_\mathfrak{d}$ to be singular as in `vA` and `vA*` of Theorems 2.1 and 2.3).

[5]The relations R used to define the following characteristics are "natural", but not entirely "canonical". For example, a different choice of a natural relation R such that $\mathfrak{b}_R = \mathfrak{s}$ leads to a different dual $\mathfrak{d}_R = \mathfrak{r}_\sigma$. See [Bla10, Example 4.6].

[6]In [BCM18] (and in other related work), a family with $\text{LCU}_R(\lambda)$ is said to be *strongly λ-R-unbounded of size λ*, while a family with $\text{COB}_R(\lambda, \mu)$ is said to be *strongly λ-R-dominating of size μ*.

[7]I.e., every subset of μ of cardinality $<\lambda$ has a $\trianglelefteq$-upper bound

[8]Formally: $D \subseteq \omega^\omega$ is R-dominating iff $(\forall x \in \omega^\omega)\, (\exists y \in D)\; xRy$.

Remark 2.16. The old constructions ((vA), (vB) in Theorem 2.1) use that we can first force strong witnesses to the left hand side, and then preserve strong witnesses in Boolean ultrapowers, so that in the final model all Cichoń-characteristics are strongly witnessed. In more detail, for each dual pair $(\mathfrak{x}, \mathfrak{y})$ in Cichoń's diagram, there is a natural relation $R_\mathfrak{x}$ such that $(\mathfrak{x}, \mathfrak{y}) = (\mathfrak{b}_{R_\mathfrak{x}}, \mathfrak{d}_{R_\mathfrak{x}})$. We use these natural relations (with one exception[9]) as follows: The initial forcing (without Boolean ultrapowers) is a f.s. iteration P of length δ and forces $\mathrm{LCU}_{R_\mathfrak{x}}(\mu)$ for all regular $\lambda_\mathfrak{x} \leq \mu \leq |\delta|$, and $\mathrm{COB}_{R_\mathfrak{x}}(\lambda_\mathfrak{x}, |\delta|)$.

Once we know that the initial forcing P gives strong witnesses for the desired values $\lambda_\mathfrak{x}$ for all "left-hand" values $\mathfrak{x}$ in Cichoń's diagram (and continuum for the cardinals $\geq \mathfrak{d}, \mathrm{non}(\mathcal{N})$ in vA* or $\geq \mathrm{cov}(\mathcal{M})$ in vB*), we use the following theorem to separate all the entries.

Theorem 2.17 ([KTT18, GKS])**.** *Let $\nu < \kappa$ and $\lambda \neq \kappa$ be uncountable regular cardinals, R a Borel relation, and let P be a ν-cc poset forcing that λ is regular. Assume that $j : V \to M$ is an elementary embedding into a transitive class M satisfying:*

(i) The critical point of j is κ.
(ii) M is $<\kappa$-closed.[10]
(iii) Whenever $\theta > \kappa$ and I is a $<\theta$-directed partial order, $j''I$ is cofinal in $j(I)$.

Then:

(a) $j(P)$ is a ν-cc forcing.
(b) If $P \Vdash \mathrm{LCU}_R(\lambda)$, then $j(P) \Vdash \mathrm{LCU}_R(\lambda)$.
(c) If $\lambda < \kappa$ and $P \Vdash \mathrm{COB}_R(\lambda, \mu)$, then $j(P) \Vdash \mathrm{COB}_R(\lambda, |j(\mu)|)$.
(d) If $\lambda > \kappa$ and $P \Vdash \mathrm{COB}_R(\lambda, \mu)$, then $j(P) \Vdash \mathrm{COB}_R(\lambda, \mu)$.

Proof. We include the proof for completeness. Property (a) is immediate by (ii). First note that j satisfies the following additional properties.

(iv) Whenever a is a set of size $<\kappa$, $j(a) = j''a$.
(v) If $\mathrm{cof}(\alpha) \neq \kappa$ then $\mathrm{cof}(j(\alpha)) = \mathrm{cof}(\alpha)$.
(vi) If $\theta > \kappa$, L is a set and $P \Vdash$"$(L, \dot{\trianglelefteq})$ is $<\theta$-directed" then $j(P) \Vdash$"$j''L$ is cofinal in $(j(L), j(\dot{\trianglelefteq}))$, and it is $<\theta$-directed".
(vii) $j(P) \Vdash$"$\mathrm{cof}(j(\lambda)) = \lambda$".

Item (iv) follows from (i), and (v) follows from (iii). We show (vi). Let L^* be the set of nice P-names of members of L, and order it by $\dot{x} \leq \dot{y}$ iff $P \Vdash \dot{x} \dot{\trianglelefteq} \dot{y}$. It is clear that $\leq$ is $<\theta$-directed on L^*. On the other hand, since any nice $j(P)$-name of a member of $j(L)$ is already in M by (ii) and (a), $j(L^*)$ is equal to the set of nice $j(P)$-names of members of $j(L)$. Therefore, by (iii), $j''L^*$ is cofinal in $j(L^*)$.

[9]The exception is the following: In vA, for the pair $(\mathfrak{x}, \mathfrak{y}) = (\mathrm{non}(\mathcal{M}), \mathrm{cov}(\mathcal{M}))$ it is forced $\mathrm{LCU}_{\neq^*}(\lambda_4)$, $\mathrm{LCU}_{\neq^*}(\lambda_5)$ and $\mathrm{COB}_{\neq^*}(\lambda_4, \lambda_5)$ (here $x \neq^* y$ iff $x(i) \neq y(i)$ for all but finitely many i); in vB, for $\mathfrak{x} = \mathrm{cov}(\mathcal{N})$, we use the natural relation $R_{\mathrm{cov}(\mathcal{N})}$ (defined as the set of all pairs (x, y) where the real y is in the F_σ set of full measure coded by x) only for COB. In this version, we do not know whether P forces $\mathrm{LCU}_{R_{\mathrm{cov}(\mathcal{N})}}(\lambda_{\mathrm{cov}(\mathcal{N})})$ (as we do not have sufficient preservation results for $R_{\mathrm{cov}(\mathcal{N})}$, more specifically, we do know whether (ρ, π)-linked posets are $R_{\mathrm{cov}(\mathcal{N})}$-good.) Instead, we use another relation R' (which defines different, anti-localization characteristics $(\mathfrak{b}_{R'}, \mathfrak{d}_{R'})$), for which ZFC proves $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{b}_{R'}$ and $\mathrm{non}(\mathcal{N}) \geq \mathfrak{b}_{R'}$. We can then show that P forces $\mathrm{LCU}_{R'}(\mu)$ for all regular $\lambda_{\mathrm{cov}(\mathcal{N})} \leq \mu \leq |\delta|$.

[10]I.e., $M^{<\kappa} \subseteq M$.

Note that $j''L^*$ is equal to the set of nice $j(P)$-names of members of $j''L$. Thus, (vi) follows.

For (vii), the case $\lambda < \kappa$ is immediate by (i) and (ii); when $\lambda > \kappa$, apply (vi) to $(L, \dot{\trianglelefteq}) = (\lambda, \leq)$ (the usual order) and $\theta = \lambda$.

To see (b), note that $M \vDash$"$j(P) \Vdash \mathrm{LCU}_R(j(\lambda))$" and, by (a) and (ii), the same holds inside V (because any nice name of an ordinal, represented by a maximal antichain on P, belongs to M, hence any nice name of a real, which in fact means that $j(P) \Vdash \mathrm{LCU}_R(\mathrm{cof}(j(\lambda)))$. By (vii) we are done.

Now assume $P \Vdash \mathrm{COB}_R(\lambda, \mu)$ witnessed by $(\dot{\trianglelefteq}, \dot{\bar{g}})$. This implies $M \models$"$j(P) \Vdash (j(\dot{\trianglelefteq}), j(\dot{\bar{g}}))$ witnesses $\mathrm{COB}_R(j(\lambda), j(\mu))$". If $\lambda < \kappa$ then $j(\lambda) = \lambda$ and it follows that $V \models$"$j(P) \Vdash \mathrm{COB}_R(\lambda, |j(\mu)|)$". In the case $\lambda > \kappa$ apply (vi) to conclude that $j(P)$ forces that $(j(\dot{\bar{g}}(\beta)) : \beta < \mu)$, with $j(\dot{\trianglelefteq})$ restricted to $j''\mu$, witnesses $\mathrm{COB}_R(\lambda, \mu)$. □

If κ is a strongly compact cardinal and $\theta^\kappa = \theta$, then there is an elementary embedding j associated with a Boolean ultrapower of the completion of $\mathrm{Coll}(\kappa, \theta)$ such that j satisfies (i)–(iii) of the preceding lemma and, in addition, for any cardinal $\lambda \geq \kappa$ such that either $\lambda \leq \theta$ or $\lambda^\kappa = \lambda$ holds, we have $\max\{\lambda, \theta\} \leq j(\lambda) < \max\{\lambda, \theta\}^+$ (see details in [KTT18, GKS]). Therefore, using only this lemma, it is easy to see how to get from the old constructions (Theorem 2.1) to the Boolean ultrapowers (Theorem 2.3), as described in Remark 2.16 (see details in [BCM18, Thm. 5.7] for `vA`* and [KST19, Thm. 3.1] for `vB`*).

3. Cardinal characteristics in extensions without new $<\kappa$-sequences

Let us consider $<\kappa$-distributive forcing extensions for some regular κ. (In particular these extensions are NNR, i.e., do not add new reals.) For such extensions, we can also preserve strong witnesses in some cases:

Lemma 3.1. *Assume that Q is θ-cc and $<\kappa$-distributive for κ regular uncountable, and let λ be a regular cardinal and R a Borel relation.*

(1) If $\mathrm{LCU}_R(\lambda)$, then $Q \Vdash \mathrm{LCU}_R(\mathrm{cof}(\lambda))$.
So if additionally $\lambda \leq \kappa$ or $\theta \leq \lambda$, then $Q \Vdash \mathrm{LCU}_R(\lambda)$.

(2) If $\mathrm{COB}_R(\lambda, \mu)$ and either $\lambda \leq \kappa$ or $\theta \leq \lambda$, then $Q \Vdash \mathrm{COB}_R(\lambda, |\mu|)$.
So for any λ, $\mathrm{COB}_R(\lambda, \mu)$ implies $Q \Vdash \mathrm{COB}_R(\min(|\lambda|, \kappa), |\mu|)$.

Proof. For (1) it is enough to assume that Q does not add reals: Take a strong witness for $\mathrm{LCU}_R(\lambda)$. This object still satisfies (2.12) in the Q-extension (as there are no new reals), but the index set will generally not be regular any more; we can just take a cofinal subset of order type $\mathrm{cof}(\lambda)$ which will still satisfy (2.12).

Similarly, a strong witness for $\mathrm{COB}_R(\lambda, \mu)$ still satisfies (2.13) in the Q extension. However, the index set is generally not $<\lambda$-directed any more, unless we either assume $\lambda \leq \kappa$ (as in that case there are no new small subsets of the partial order) or Q is λ-cc (as then every small set in the extension is covered by a small set from the ground model). □

If P forces strong witnesses, then any complete subforcing that includes names for all witnesses also forces strong witnesses:

Lemma 3.2. *Assume that R is a Borel relation, P' is a complete subforcing of P, λ regular and μ is a cardinal, both preserved in the P-extension.*

(a) *If $P \Vdash \mathrm{LCU}_R(\lambda)$ witnessed by some $\dot{\bar{f}}$, and $\dot{\bar{f}}$ is actually a P'-name, then $P' \Vdash \mathrm{LCU}_R(\lambda)$.*
(b) *If $P \Vdash \mathrm{COB}_R(\lambda,\mu)$ witnessed by some $(\dot{\trianglelefteq},\dot{\bar{g}})$, and $(\dot{\trianglelefteq},\dot{\bar{g}})$ is actually a P'-name, then $P' \Vdash \mathrm{COB}_R(\lambda,|\mu|)$.*

Proof. Let V_2 be the P-extension and V_1 the intermediate P'-extension. For LCU: (2.12) holds in V_2, $V_1 \subseteq V_2$ and $(f_i)_{i<\lambda} \in V_1$, and R is absolute between V_1 and V_2, so (2.12) holds in V_1. The argument for COB is similar. □

We now define three properties of cardinal characteristics (more general than Blass-uniform) that have implications for their behaviour in extensions without new $<\kappa$-sequences. We call these properties e.g. $\mathfrak{t}$-like to refer to the "typical" representative $\mathfrak{t}$. But note that this is very superficial: There is no deep connection or similarity to $\mathfrak{t}$ for all $\mathfrak{t}$-like characteristics, it is just that $\mathfrak{t}$ is a well known example for this property, and "$\mathfrak{t}$-like" seems easier to memorize than other names we came up with.

Definition 3.3. Let $\mathfrak{x}$ be a cardinal characteristic.

(1) $\mathfrak{x}$ is $\mathfrak{t}$-*like*, if it has the following form: There is a formula $\psi(x)$ (possibly with, e.g., real parameters) absolute between universe extensions that do not add reals,[11] such that $\mathfrak{x}$ is the smallest cardinality λ of a set A of reals such that $\psi(A)$.

All Blass-uniform characteristics are $\mathfrak{t}$-like; other examples are $\mathfrak{t}$, $\mathfrak{u}$, $\mathfrak{a}$ and $\mathfrak{i}$.

(2) $\mathfrak{x}$ is called $\mathfrak{h}$-*like*, if it satisfies the same, but with A being a family of sets of reals (instead of just a set of reals).

Note that $\mathfrak{t}$-like implies $\mathfrak{h}$-like, as we can include "the family of sets of reals is a family of singletons" in ψ. Examples are $\mathfrak{h}$ and $\mathfrak{g}$.

(3) $\mathfrak{x}$ is called $\mathfrak{m}$-*like*, if it has the following form: There is a formula φ (possibly with, e.g., real parameters) such that $\mathfrak{x}$ is the smallest cardinality λ such that $H(\leq\lambda) \vDash \varphi$.

Any infinite $\mathfrak{t}$-like characteristic is $\mathfrak{m}$-like: If ψ witnesses $\mathfrak{t}$-like, then we can use $\varphi = (\exists A)\,[\psi(A)\&(\forall a \in A)\ a \text{ is a real}]$ to get $\mathfrak{m}$-like (since $H(\leq\lambda)$ contains all reals). Examples are[12] $\mathfrak{m}$, $\mathfrak{m}$(Knaster), etc.

Actually, we will really only use $\mathfrak{m}$-like in this paper: We do not know anything about $\mathfrak{t}$-like characteristics in general, apart from the fact that they are both $\mathfrak{m}$-like and $\mathfrak{h}$-like; and will not apply the only observation we mention about $\mathfrak{h}$-like characteristics (which is that NNR extensions do not increase such characteristics).

Lemma 3.4. *Let $V_1 \subseteq V_2$ be models (possibly classes) of set theory (or a sufficient fragment), V_2 transitive and V_1 is either transitive or an elementary submodel of $H^{V_2}(\chi)$ for some large enough regular χ, such that $V_1 \cap \omega^\omega = V_2 \cap \omega^\omega$.*

(a) *If $\mathfrak{x}$ is $\mathfrak{h}$-like, then $V_1 \vDash \mathfrak{x} = \lambda$ implies $V_2 \vDash \mathfrak{x} \leq |\lambda|$.*

In addition, whenever κ is uncountable regular in V_1 and $V_1^{<\kappa} \cap V_2 \subseteq V_1$:

(b) *If $\mathfrak{x}$ is $\mathfrak{m}$-like, then $V_1 \vDash \mathfrak{x} \geq \kappa$ iff $V_2 \vDash \mathfrak{x} \geq \kappa$.*

[11] Concretely, if $M_1 \subseteq M_2$ are transitive (possibly class) models of a fixed, large fragment of ZFC, with the same reals, then ψ is absolute between M_1 and M_2.

[12] $\mathfrak{m}$ can be characterized as the smallest λ such that there is in $H(\leq\lambda)$ a ccc forcing Q and a family $\bar{D}$ of dense subsets of Q such that "there is no filter $F \subseteq Q$ meeting all D_i" holds.

(c) *If* $\mathfrak{x}$ *is* $\mathfrak{m}$*-like and* $\lambda < \kappa$*, then* $V_1 \vDash \mathfrak{x} = \lambda$ *iff* $V_2 \vDash \mathfrak{x} = \lambda$*.*

Proof. Assume V_1 is transitive. For (a), if ψ witnesses that $\mathfrak{x}$ is $\mathfrak{h}$-like, $A \in V_1$ and V_1 satisfies $\psi(A)$, then the same holds in V_2. For (b) and (c), note that $H^{V_1}(\leq \mu) = H^{V_2}(\leq \mu)$ for all $\mu < \kappa$ (easily shown by $\in$-induction).

The case $V_1 = N \preceq H^{V_2}(\chi)$ is similar. Note that $H^{V_2}(\chi)$ is a transitive subset of V_2, so (a) follows by the previous case. For (b) and (c), work inside V_2. Note that $\kappa \subseteq N$ (by induction). Whenever $\mu \leq \kappa$, μ is regular iff $N \models$"μ regular", and $H(\mu) \subseteq N$. So $N \models$"$H(\mu) \models \phi$" iff $H(\mu) \cap N \models \phi$.

Alternatively, the case $V_1 \preceq H^{V_2}(\chi)$ is a consequence of the first case. Work in V_2. Let $\pi : V_1 \to \bar{V}_1$ be the transitive collapse of V_1. Note that $\pi(x) = x$ for any $x \in \omega^\omega \cap V_1$, so $\omega^\omega \cap \bar{V}_1 = \omega^\omega \cap V_1 = \omega^\omega$. To see (a), $V_1 \vDash \mathfrak{x} = \lambda$ implies $\bar{V}_1 \vDash \mathfrak{x} = \pi(\lambda)$, so $\mathfrak{x} \leq |\pi(\lambda)| \leq |\lambda|$ by the transitive case.

Now assume $V_1^{<\kappa} \subseteq V_1$ (still inside V_2), so we also have $\bar{V}_1^{<\kappa} \subseteq \bar{V}_1$. To see (b), $V_1 \models \mathfrak{x} \geq \kappa$ iff $\bar{V}_1 \models \mathfrak{x} \geq \pi(\kappa) = \kappa$, iff $V_2 \models \mathfrak{x} \geq \kappa$ by the transitive case. Property (c) follows similarly by using $\pi(\lambda) = \lambda$ (when $\lambda < \kappa$). □

We apply this to three situations: Boolean ultrapowers, extensions by distributive forcings, and complete subforcings:

Corollary 3.5. *Assume that* κ *is uncountable regular,* $P \Vdash \mathfrak{x} = \lambda$*, and* *either*

(i) *Q is a P-name for a* $<\kappa$*-distributive forcing, and we set* $P^+ := P * Q$ *and* $j(\lambda) := \lambda$*;*
(ii) *or* P *is* ν*-cc for some* $\nu < \kappa$*,* $j : V \to M$ *is a complete embedding into a transitive* $<\kappa$*-closed model* M*,* $\mathrm{cr}(j) \geq \kappa$*, and we set* $P^+ := j(P)$*,*
(iii) *or* P *is* κ*-cc,* $M \preceq H(\chi)$ *is* $<\kappa$*-closed, and we set* $P^+ := P \cap M$ *and* $j(\lambda) := |\lambda \cap M|$*. (So* P^+ *is a complete subposet of* P*; and if* $\lambda \leq \kappa$ *then* $j(\lambda) = \lambda$*.)*

Then we get:

(a) *If* $\mathfrak{x}$ *is* $\mathfrak{m}$*-like and* $\lambda \geq \kappa$*, then* $P^+ \Vdash \mathfrak{x} \geq \kappa$*.*
(b) *If* $\mathfrak{x}$ *is* $\mathfrak{m}$*-like and* $\lambda < \kappa$*, then* $P^+ \Vdash \mathfrak{x} = \lambda$*.*
(c) *If* $\mathfrak{x}$ *is* $\mathfrak{h}$*-like then* $P^+ \Vdash \mathfrak{x} \leq |j(\lambda)|$*. Concretely,*
    for (i): $P^+ \Vdash \mathfrak{x} \leq |\lambda|$*;*
    for (ii): $P^+ \Vdash \mathfrak{x} \leq |j(\lambda)|$*;*
    for (iii): $P^+ \Vdash \mathfrak{x} \leq |\lambda \cap M|$*.*

Proof. **Case (i).** Follows directly from Lemma 3.4.

Case (ii). Since M is $<\kappa$-closed and P is ν-cc, P (or rather: the isomorphic image $j''P$) is a complete subforcing of $j(P)$. Let G be a $j(P)$-generic filter over V. As $j(P)$ is in M (and M is transitive), G is generic over M as well. Then $V_1 := M[G]$ is $< \kappa$ closed in $V_2 := V[G]$.

First note that V_1 and V_2 have the same $< \kappa$-sequences of ordinals. Let $\dot{\bar{x}} = (\dot{x}_i)_{i \in \mu}$ be a sequence of $j(P)$-names for members of M with $\mu < \kappa$. Each $\dot{x}_i$ is determined by an antichain, which has size $<\nu$ and therefore is in M, so each $\dot{x}_i$ is in M. Hence $\dot{\bar{x}}$ is in M.

By elementaricity, $P \Vdash \mathfrak{x} = \lambda$ implies $M \models$"$j(P) \Vdash \mathfrak{x} = j(\lambda)$". So $V_1 \models \mathfrak{x} = j(\lambda)$, and we can apply Lemma 3.4: In the case that $\mathfrak{x}$ is $\mathfrak{m}$-like, if $\lambda \geq \kappa$, then $j(\lambda) \geq j(\kappa) \geq \kappa$, so $V_2 \models \mathfrak{x} \geq \kappa$; If $\lambda < \kappa$, then $j(\lambda) = \lambda$, so $V_2 \models \mathfrak{x} = \lambda$; if $\mathfrak{x}$ is $\mathfrak{h}$-like, then $V_2 \models \mathfrak{x} \leq |j(\lambda)|$.

Case (iii). Let $\pi^0 : M \to \bar{M}$ be the transitive collapse. Set $\bar{P} := \pi^0(P) \in \bar{M}$. Note that $\pi^0(\kappa) = \kappa$ and that $\bar{M}$ is $<\kappa$-closed. Also, any condition in P is M-generic since, for any antichain A in P, $A \in M$ iff $A \subseteq M$ (by $<\kappa$-closedness).

Let G^+ be P^+-generic over V. We can extend G^+ to a P-generic G over V (as P^+ is a complete subforcing of P), and we get $G^+ = G \cap P^+ = G \cap M$. Now work in $V[G]$. Note that $M[G]$ is an elementary submodel of $H^{V[G]}(\chi)$ (and obviously not transitive), and that the transitive collapse $\pi : M[G] \to V_1$ extends π^0 (as there are no new elements of V in $M[G]$). We claim that $V_1 = \bar{M}[\bar{G}^+]$ where $\bar{G}^+ := \pi^{0\prime\prime} G^+$ (which is $\bar{P}$-generic over $\bar{M}$, also $\bar{G}^+ = \pi(G)$), and that $\bar{\tau}[\bar{G}^+] = \pi(\tau[G])$ for any P-name $\tau \in M$, where $\bar{\tau} := \pi^0(\tau)$.[13] So in particular, V_1 is a subset of $V_2 := V[G^+]$ (the P^+-generic extension of V) because π^0 and M (and therefore $\bar{M}$) are elements of V, so G^+ (and therefore $\bar{G}^+$) are elements of $V[G^+]$. In fact, $\bar{G}^+$ is $\bar{P}$-generic over V because $\bar{M}$ is $<\kappa$-closed and $\bar{P}$ is κ-cc, moreover, $V_2 = V[\bar{G}^+]$ (this is reflected by the fact that, in V, $\pi^0 \restriction P^+$ is an isomorphism between P^+ and $\bar{P}$).

We claim:

$(*)$ $\qquad V_2$ is an NNR extension of V_1, moreover V_1 is $<\kappa$-closed in V_2.

To show this, work in V. We argue with $\bar{P}$. Let τ be a $\bar{P}$-name of an element of $V_1 = \bar{M}[\bar{G}^+]$. So we can find a maximal antichain A in $\bar{P}$ and, for each $a \in A$, a $\bar{P}$-name σ_a in $\bar{M}$ such that $a \Vdash_{\bar{P}} \tau = \sigma_a$. Since $|A| < \kappa$ and $\bar{P} \subseteq \bar{M}$ and $\bar{M}$ is $<\kappa$-closed, A, as well as the function $a \mapsto \sigma_a$, are in $\bar{M}$. Mixing the names σ_a along A to a name $\sigma \in \bar{M}$, we get $\bar{M} \vDash a \Vdash_{\bar{P}} \sigma_a = \sigma$ for all $a \in A$, which implies $V \vDash a \Vdash_{\bar{P}} \sigma_a = \sigma$ because the forcing relation of atomic formulas is absolute. So $\bar{P} \Vdash \tau = \sigma$.

Now fix a $\bar{P}$ name $\vec{\tau} = (\tau_\alpha)_{\alpha<\mu}$ of a sequence of elements of V_1, with $\mu < \kappa$. Again we use closure of $\bar{M}$ and get a sequence $(\sigma_\alpha)_{\alpha<\mu}$ in $\bar{M}$ such that $\bar{P}$ forces that $\tau_\alpha = \sigma_\alpha[\bar{G}^+]$, and so the evaluation of the sequence $\vec{\tau}$ is in $\bar{M}[\bar{G}^+] = V_1$. This proves $*$.

Now assume that $\mathfrak{x}$ is either $\mathfrak{h}$-like or $\mathfrak{m}$-like, and $P \Vdash \mathfrak{x} = \lambda$. By elementaricity, this holds in M, so $\bar{M} \vDash \bar{P} \Vdash \mathfrak{x} = \pi^0(\lambda)$. Now let $\bar{G}^+$ be $\bar{P}$-generic over V, $V_1 := \bar{M}[\bar{G}^+]$ and $V_2 := V[\bar{G}^+]$, so $V_1 \models \mathfrak{x} = \pi^0(\lambda)$. If $\mathfrak{x}$ is $\mathfrak{h}$-like then, by Lemma 3.4(a), $V_2 \models \mathfrak{x} \leq |\pi^0(\lambda)| = |\lambda \cap M|$; if $\mathfrak{x}$ is $\mathfrak{m}$-like and $\lambda < \kappa$, then $V_1 \models \mathfrak{x} = \lambda$ and so the same is satisfied in V_2 by Lemma 3.4(c); otherwise, if $\lambda \geq \kappa$ then $V_1 \models \mathfrak{x} = \pi^0(\lambda) \geq \pi^0(\kappa) = \kappa$, so $V_2 \models \mathfrak{x} \geq \kappa$ by Lemma 3.4(b). □

3.1. **On the role of large cardinals in our construction.** It is known that NNR extensions will preserve Blass-uniform characteristics in the absence of at least some large cardinals. More specifically:

Lemma 3.6. *Assume that $0^\#$ does not exist. Let $V_1 \subseteq V_2$ be transitive class models with the same reals, and assume $V_1 \models \mathfrak{x} = \lambda$ for some Blass-uniform $\mathfrak{x}$. Then $V_2 \models \mathfrak{x} = |\lambda|$.*

(This is inspired by the deeper observation of Mildenberger [Mil98, Prop. 2.1], who uses the covering lemma [DJ82] for the Dodd-Jensen core model to show that in *cardinality preserving* NNR extensions, a measurable in an inner model is required to change the value of a Blass-uniform characteristic.)

[13] This can be proved by induction on the rank of τ, and uses that $M[G] \preceq H^{V[G]}(\chi)$.

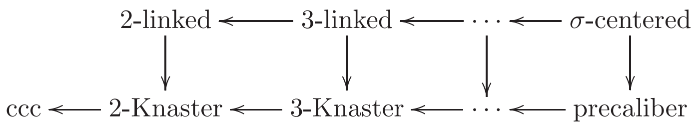

Figure 5. Some classes of ccc forcings

Proof. Fix a bijection in V_1 between the reals and some ordinal α. Assume that in V_2, $X \subseteq \omega^\omega$ witnesses that $\aleph_1 \leq \mathfrak{r} \leq \mu < |\lambda|$. Using the bijection, we can interpret X as a subset of α. According to Jensen's covering lemma in V_2, there is in L (and thus in V_1) some $X' \supseteq X$ such that $|X'| = |X|$ in V_2, in particular $|X'|^{V_2} < \lambda$. Therefore, $|X'|^{V_1} < \lambda$ as well; and, by absoluteness, V_1 thinks that X' witnesses $\mathfrak{r} < \lambda$, a contradiction. □

Recall the "old" Boolean ultrapower constructions `vA*`: Assume that we start with a forcing notion P forcing $\mathfrak{d} = 2^{\aleph_0} = \lambda_6$. We now use the elementary embedding $j = j_7 : V \to M$ with critical point κ_7, and set $P' := j(P)$. As we have seen, P' still forces $\mathfrak{d} = \lambda_6$, but $2^{\aleph_0} = \lambda_7 = |j(\kappa_7)|$.

So let G be a P'-generic filter over V (which is also M-generic). Set $V_1 := M[G]$ and $V_2 := V[G]$. Then V_1 is a $<\kappa$-complete submodel of V_2. By elementaricity, $M \models j(P) \Vdash \mathfrak{d} = j(\lambda_6)$. So $V_1 \models \mathfrak{d} = j(\lambda_6)$, whereas $V_2 \models \mathfrak{d} = \lambda_6 < |j(\lambda_6)|$.

Hence, for this specific constellation of models, some large cardinals (at least $0^\#$) are required (for our construction we actually use strongly compact cardinals).

4. Dealing with $\mathfrak{m}$

We show how to deal with $\mathfrak{m}$. It is easy to check that the old forcings (from Theorems 2.1 and 2.3) force $\mathfrak{m} = \aleph_1$, and can easily be modified to force $\mathfrak{m} = \text{add}(\mathcal{N})$ (by forcing with all small ccc forcings during the iteration). With a bit more work it is also possible to get $\aleph_1 < \mathfrak{m} < \text{add}(\mathcal{N})$.

Let us start by recalling the definitions of some well-known classes of ccc forcings:

Definition 4.1. Let λ be an infinite cardinal, $k \geq 2$ and let Q be a poset.

1. (1) Q is (λ, k)-*Knaster* if, for every $A \in [Q]^\lambda$, there is a $B \in [A]^\lambda$ which is k-linked (i.e., every $c \in [B]^k$ has a lower bound). We write k-*Knaster* for $(\aleph_1, k)$-Knaster; *Knaster* means 2-Knaster; $(\lambda, 1)$-*Knaster* denotes λ-cc and 1-*Knaster* denotes ccc.[14]
2. (2) Q *has precaliber* λ if, for every $A \in [Q]^\lambda$, there is a $B \in [A]^\lambda$ which is centered, i.e., every finite subset of B has a lower bound. We sometimes shorten "precaliber $\aleph_1$" to "precaliber".
3. (3) Q is (σ, k)-*linked* if there is a function $\pi : Q \to \omega$ such that $\pi^{-1}(\{n\})$ is k-linked for each n.
4. (4) Q is σ-*centered* if there is a function $\pi : Q \to \omega$ such that each $\pi^{-1}(\{n\})$ is centered.

The implications between these notions (for $\lambda = \aleph_1$) are listed in Figure 5. To each class C of forcing notions, we can define the Martin's Axiom number $\mathfrak{m}(C)$ in the usual way (recall Definition 1.1). An implication $C_1 \leftarrow C_2$ in the diagram

[14]This is just an abuse of notation that turns out to be convenient for stating our results.

corresponds to a ZFC inequality $\mathfrak{m}(C_1) \leq \mathfrak{m}(C_2)$. Recall that $\mathfrak{m}(\sigma\text{-centered}) = \mathfrak{p} = \mathfrak{t}$. Also recall that, in the old constructions, all iterands were (σ, k)-linked for all k.

Lemma 4.2. *(1) If there is a Suslin tree, then $\mathfrak{m} = \aleph_1$.*

(2) After adding a Cohen real c over V, in $V[c]$ there is a Suslin tree.

(3) Any Knaster poset preserves Suslin trees.

(4) The result of any finite support iteration of (λ, k)-Knaster posets (λ uncountable regular and $k \geq 1$) is again (λ, k)-Knaster.

(5) In particular, when $k \geq 1$, if P is a f.s. iteration of forcings such that all iterands are either (σ, k)-linked or smaller than λ, then P is (λ, k)-Knaster.

(6) Let C be any of the forcing classes of Figure 5, and assume $\mathfrak{m}(C) = \lambda > \aleph_1$. (Or just assume that C is a class of ccc forcings closed under $Q \mapsto Q^{<\omega}$, the finite support product of countably many copies of Q, and under $(Q, p) \mapsto \{q \leq p\}$ for $p \in Q$.)

If $Q \in C$, then every subset A of Q of size $<\lambda$ is "σ-centered in Q" (i.e., there is a function $\pi : A \to \omega$ such that every finite π-homogeneous subset of A has a common lower bound in Q).

So in particular, for all $\mu < \lambda$ of uncountable cofinality, Q has precaliber μ and is (μ, ℓ)-Knaster for all $\ell \geq 2$.

(7) $\mathfrak{m} > \aleph_1$ implies $\mathfrak{m} = \mathfrak{m}(\text{precaliber})$.
$\mathfrak{m}(k\text{-Knaster}) > \aleph_1$ implies $\mathfrak{m}(k\text{-Knaster}) = \mathfrak{m}(\text{precaliber})$.

Proof. (1): Clear. (2): See [She84, Tod89] or Velleman [Vel84]. (3): Recall that the product of a Knaster poset with a ccc poset is still ccc. Hence, if P is Knaster and T is a Suslin tree, then $P \times T = P * \check{T}$ is ccc, i.e., T remains Suslin in the P-extension.

(4): Well-known, see e.g. Kunen [Kun11, Lemma V.4.10] for $(\aleph_1, 2)$-Knaster. The proof for the general case is the same, see e.g. [Mej19a, Section 5].

(5): Clear, as (σ, k)-linked implies (μ, k)-Knaster (for all uncountable regular μ), and since every forcing of size $<\mu$ is (μ, k)-Knaster (for any k).

(6): First note that it is well known[15] that $\mathrm{MA}_{\aleph_1}(\text{ccc})$ implies that every ccc forcing is Knaster, and hence that the class C of ccc forcings is closed under $Q \mapsto Q^{<\omega}$. (For the other classes C in Figure 5, the closure is immediate.)

So let C be a closed class, $\mathfrak{m}(C) = \lambda > \aleph_1$, $Q \in C$ and $A \in [Q]^{<\lambda}$. Given a filter G in $Q^{<\omega}$ and $q \in Q$, set $c(q) = n$ iff n is minimal such that there is a $\bar{p} \in G$ with $p(n) = q$. Note that for all q, the set

$$D_q = \{p \in Q^{<\omega} : (\exists n \in \omega)\, q = p(n)\}$$

is dense, and that $c(q)$ is defined whenever G intersects D_q. Pick a filter G meeting all D_q for $q \in A$. This defines $c : A \to \omega$ such that $c(a_0) = c(a_1) = \cdots = c(a_{\ell-1}) = n$ implies that all a_i appear in $G(n)$ and thus they are compatible in Q. Hence, A is the union of countably many centered (in Q) subsets of Q.

(7): Follows as a corollary. □

This shows that it is not possible to simultaneously separate more than two Knaster numbers. More specifically: ZFC proves that there is a $1 \leq k^* \leq \omega$ and, if

[15]See, e.g., Jech [Jec03, 16.21] (and the historical remarks, where the result is attributed to (independently) Kunen, Rowbottom and Solovay), or [BJ95, 1.4.14] or Galvin [Gal80, Pg. 34].

$k^* < \omega$, a $\lambda > \aleph_1$, such that for all $1 \le \ell < \omega$

$$\mathfrak{m}(\ell\text{-Knaster}) = \begin{cases} \aleph_1 & \text{if } \ell < k^* \\ \lambda & \text{otherwise.} \end{cases} \tag{4.3}$$

(Recall that $\mathfrak{m}$(1-Knaster) $= \mathfrak{m}$(ccc) by definition.)

In this section, we will show how these constellations can be realized together with the previous values for the Cichoń-characteristics.

In the case $k^* < \omega$, we know that $\mathfrak{m}$(precaliber) $= \lambda$ as well. We briefly comment that $\mathfrak{m}$(precaliber) $= \aleph_1$ (in connection with the Cichoń-values) is possible too. In the next section, we will deal with the remaining case: $k^* = \omega$, i.e., all Knaster numbers are $\aleph_1$, while $\mathfrak{m}$(precaliber) $> \aleph_1$.

The central observation is the following, see [Tod86, Tod89] and [Bar92, Sect. 3].

Lemma 4.4. *Let $k \in \omega$, $k \ge 2$ and λ be uncountable regular. Let C be the finite support iteration of λ many copies of Cohen forcing. Assume that C forces that P is $(\lambda, k+1)$-Knaster. Then $C * P$ forces $\mathfrak{m}$(k-Knaster) $\le \lambda$.*

The same holds for $k = 1$ and $\lambda = \aleph_1$.

For $k = 1$ this trivially follows from Lemma 4.2: The first Cohen forcing adds a Suslin tree, which is preserved by the rest of the Cohen posets composed with P. So we get $\mathfrak{m} = \aleph_1$. The proof for $k > 1$ is done in the following two lemmas.

Remark 4.5. Adding the Cohen reals first is just for notational convenience. The same holds, e.g., in a f.s. iteration where we add Cohen reals on a subset of the index set of order type λ; and we assume that the (limit of the) whole iteration is $(\lambda, k+1)$-Knaster.

Lemma 4.6. *Under the assumption of Lemma 4.4, for $k \ge 1$: We interpret each Cohen real η_α $(\alpha \in \lambda)$ as an element of $(k+1)^\omega$. $C * P$ forces: For all $X \in [\lambda]^\lambda$,*

$$(\exists \nu \in (k+1)^{<\omega})\, (\exists \alpha_0, \dots, \alpha_k \in X)\, (\forall 0 \le i \le k) \nu^\frown i \lhd \eta_{\alpha_i} \tag{$\star$}$$

Proof. Let $p^* \in C * P$ force that $X \in [\lambda]^\lambda$. By our assumption, first note that $p^*{\restriction}\lambda$ forces that there is some $X' \in [\lambda]^\lambda$ and a $k+1$-linked set $\{r_\alpha : \alpha \in X'\}$ of conditions in P below $p^*(\lambda)$ such that $r_\alpha \Vdash_P \alpha \in X$ for any $\alpha \in X'$.

Since X' is a C-name, there is some $Y \in [\lambda]^\lambda$ and, for each $\alpha \in Y$, some $p_\alpha \le p^*{\restriction}\lambda$ in C forcing $\alpha \in X'$. We can assume that $\alpha \in \mathrm{dom}(p_\alpha)$ and, by thinning out Y, that $\mathrm{dom}(p_\alpha)$ forms a Δ-system with heart a below each $\alpha \in Y$, $\langle p_\alpha{\restriction}a : \alpha \in Y\rangle$ is constant, and that $p_\alpha(\alpha)$ is always the same Cohen condition $\nu \in (k+1)^{<\omega}$.

For each $\alpha \in Y$ let $q_\alpha \in C * P$ such that $q_\alpha{\restriction}\lambda = p_\alpha$ and $q_\alpha(\lambda) = r_\alpha$. It is clear that $\langle q_\alpha : \alpha \in Y\rangle$ is $k+1$-linked and that $q_\alpha \Vdash \alpha \in X$. Pick $\alpha_0, \dots, \alpha_k \in Y$ and $q \le q_{\alpha_0}, \dots, q_{\alpha_k}$. We can assume that $q{\restriction}\lambda$ is just the union of the $q_{\alpha_i}{\restriction}\lambda$. In particular, we can extend $q(\alpha_i) = \nu$ to $\nu^\frown i$, satisfying $(\star)$ after all. This proves the claim. □

Lemma 4.7. *Under the assumption of Lemma 4.4, for $k \ge 2$: In V^C define $R_{K,k}$ to be the set of finite partial functions $p : u \to \omega$, $u \subseteq \lambda$ finite, such that $(\star)$ fails for all p-homogeneous $X \subseteq u$. Then P forces the following:*

(a) *There is no filter on $R_{K,k}$ meeting all dense D_α $(\alpha \in \lambda)$, where we set $D_\alpha = \{p : \alpha \in \mathrm{dom}(u)\}$.*
(b) *$R_{K,k}$ is k-Knaster.*

Note that this proves Lemma 4.4, as $R_{K,k}$ is a witness.

Proof. Clearly each D_α is dense (as we can just use a hitherto unused color). If G is a filter meeting all D_α, then G defines a total function $p^* : \lambda \to \omega$, and there is some $n \in \omega$ such that $X := {p^*}^{-1}(\{n\})$ has size λ. So $(\star)$ holds for X, witnessed by some $\alpha_0, \dots, \alpha_k$. Now pick some $q \in G$ such that all α_i are in the domain of q. Then q contradicts the definition of $R_{K,k}$.

$R_{K,k}$ is k-Knaster: Given $(r_\alpha : u_\alpha \to \omega)_{\alpha\in\omega_1}$, we thin out so that u_α forms a Δ-system of sets of the same size and such that each r_α has the same "type", independent of α, where the type contains the following information: The color assigned to the n-the element of u_α; the (minimal, say) h such that all $\eta_\beta \restriction h$ are distinct for $\beta \in u_\alpha$, and $\eta_\beta \restriction h + 1$.

We claim that the union of k many such r_α is still in $R_{K,k}$: Assume towards a contradiction that there is a homogeneous set $\alpha_0, \dots, \alpha_k$ in $\bigcup_{i<k} u_i$ such that $(\star)$ holds for $\nu \in (k+1)^H$ for some $H \in \omega$. Assume $H \geq h$. Note that $\eta_\beta \restriction h$ are already distinct for the different β in the same u_i, so all $k+1$ many α_j have to be the n^*-th element of different u_i (n^* fixed), which is impossible as there are only k many u_i. So assume $H < h$. But then $\eta_\beta \restriction H + 1$ and the color of β both are determined by the position of β within u_i; so without loss of generality all the α_j are in the same u_i, which is impossible as $p_i : u_i \to \omega$ was a valid condition.

To summarize: P forces that there is a k-Knaster poset $R_{K,k}$ and λ many dense sets not met by any filter. Therefore P forces that $\mathfrak{m}(k\text{-Knaster}) \leq \lambda$. □

Lemma 4.8. *The forcing $P^{\mathtt{vA}}$ of Theorem 2.1 (without Boolean ultrapowers) can be modified to some forcing P' which still forces the same values to the Cichoń-characteristics, and additionally satisfies one of the following, where $\aleph_1 < \lambda \leq \lambda_{\mathrm{add}(\mathcal{N})}$ is regular:*

(1) Each iterand in P' is (σ, ℓ)-linked for all $\ell \geq 2$ and P' forces

$$\aleph_1 = \mathfrak{m} = \mathfrak{m}(\text{precaliber}) < \mathfrak{p} = \mathfrak{b}.$$

(2) Fix $k \geq 1$. Each iterand in P' is $k+1$-Knaster, and additionally either (σ, ℓ)-linked for all ℓ or of size less than λ. P' forces

$$\aleph_1 = \mathfrak{m} = \mathfrak{m}(k\text{-Knaster}) < \mathfrak{m}(k+1\text{-Knaster}) = \mathfrak{m}(\text{precaliber}) = \lambda \leq \mathfrak{p} = \mathfrak{b}.$$

(3) Each iterand in P' is either (σ, ℓ)-linked for all ℓ, or ccc of size less than λ. P' forces

$$\mathfrak{m} = \mathfrak{m}(\text{precaliber}) = \lambda \leq \mathfrak{p} = \mathfrak{b}.$$

The same can be done for $P^{\mathtt{vB}}$.

Proof. An argument like in [Bre91] works. We give the modification of $P^{\mathtt{vA}}$ (the one for $P^{\mathtt{vB}}$ is morally identical, we will indicate notational differences in footnotes). We first modify $P^{\mathtt{vA}}$ as follows:

We construct an iteration P with the same index set δ as $P^{\mathtt{vA}}$; we partition δ into two cofinal sets $\delta = S_{\mathrm{old}} \cup S_{\mathrm{new}}$ of the same size. For $\alpha \in S_{\mathrm{old}}$ we define Q_α as we defined Q^*_α for $P^{\mathtt{vA}}$. For $\alpha \in S_{\mathrm{new}}$, pick (by suitable book-keeping) a small (less than $\lambda_{\mathfrak{b}}$) σ-centered forcing Q_α.

As $\mathrm{cof}(\delta) \geq \lambda_{\mathfrak{b}}$, we get that P forces $\mathfrak{p} \geq \lambda_{\mathfrak{b}}$. Also, P still forces the same values to the "old" characteristics as $P^{\mathtt{vA}}$: From the point of $\lambda_{\mathfrak{b}}$, the new iterands are small, so they do not interfere with the ultrafilter argument[16] which shows that

[16] vB: the FAM argument

$\mathfrak{b} \leq \lambda_{\mathfrak{b}}$, nor do they interfere with goodness used for the characteristics intended to be larger than[17] $\mathfrak{b}$. Also, each new Q_α is σ-centered, and thus satisfies goodness for the characteristics intended to be smaller than or equal to[18] $\mathfrak{b}$.

Note that all iterands are still (σ, k)-linked for all k (as the new ones are even σ-centered).

To deal with ℓ-Knaster, note that the old constructions use Cohen forcings at "many" indices of the iteration; again for notational simplicity, we can assume that the first λ many indices use Cohen forcings; and we call these Cohen reals η_α ($\alpha \in \lambda$). Given ℓ, we can (and will) interpret the Cohen real η_α as an element of $(\ell+1)^\omega$.

(1) Recall from [Bar92, Sect. 2] that, after a Cohen real, there is a precaliber ω_1 poset Q^* such that no σ-linked poset adds a filter intersecting certain $\aleph_1$-many dense subsets of Q^*.[19] Therefore, the P we just constructed forces $\mathfrak{m}(\text{precaliber}) = \aleph_1$.
(2) Just as with the modification from $P^{\mathtt{vA}}$ to P, we now further modify P to force (by some bookkeeping) with all small (smaller than λ) $k+1$-Knaster forcings. So the resulting iteration obviously forces $\mathfrak{m}(k+1\text{-Knaster}) \geq \lambda$.

Note that now all iterands are either smaller than λ or σ-linked; and additionally all iterands are $k+1$-Knaster. So P is both $(\aleph_1, k+1)$-Knaster and (λ, ℓ)-Knaster for any ℓ. Again by Lemma 4.4, P forces both $\mathfrak{m}(k\text{-Knaster}) = \aleph_1$ and $\mathfrak{m}(\ell\text{-Knaster}) \leq \lambda$ for any ℓ, (which implies $\mathfrak{m}(k+1\text{-Knaster}) = \lambda$).

(3) This is very similar, but this time we use all small ccc forcings (not just the $k+1$-Knaster ones). This obviously results in $\mathfrak{m} \geq \lambda$; and the same argument as above shows that still $\mathfrak{m}(\ell\text{-Knaster}) \leq \lambda$ for all ℓ. □

This, together with Corollary 3.5, gives us 11 characteristics:

Corollary 4.9. *Given $\aleph_1 \leq \lambda_{\mathfrak{m}} < \kappa_9$ regular, we can modify $P^{\mathtt{vA}*}$ (and also $P^{\mathtt{vB}*}$) so that we additionally get $\mathfrak{p} \geq \kappa_9$ and one of the following:*

(1) $\mathfrak{m} = \mathfrak{m}(\text{precaliber}) = \aleph_1$.
(2) for a fixed $1 \leq k < \omega$, $\mathfrak{m}(k\text{-Knaster}) = \aleph_1$ and $\mathfrak{m}(k+1\text{-Knaster}) = \lambda_{\mathfrak{m}}$.
(3) $\mathfrak{m} = \mathfrak{m}(\text{precaliber}) = \lambda_{\mathfrak{m}}$.

Proof. We can assume $\lambda_{\mathfrak{m}} > \aleph_1$ (otherwise it is just case (1)). Fix $\sharp = 1, 2, 3$. Apply Boolean ultrapowers to the poset P' of Lemma 4.8($\sharp$), resulting in P''. We can apply Corollary 3.5(ii), more specifically the consequences (a) and (b): (b) implies that P'' forces ($\sharp$), while (a) implies that P'' forces $\mathfrak{p} \geq \kappa_9$. And just as in the "old" construction, we can use Theorem 2.17 to show that P'' forces the desired values to the Cichoń characteristics. □

5. Dealing with the precaliber number

Recall the possible constellations for the Knaster numbers given in (4.3). Note that if $k^* < \omega$, then $\mathfrak{m}(\text{precaliber}) = \lambda$ as well.

[17]Which are non($\mathcal{M}$), cov($\mathcal{M}$), $\mathfrak{d}$ and non($\mathcal{N}$) in vA, and cov($\mathcal{N}$), non($\mathcal{M}$) and cov($\mathcal{M}$) in vB.
[18]Which are add($\mathcal{N}$), cov($\mathcal{N}$) and $\mathfrak{b}$ in vA and add($\mathcal{N}$) and $\mathfrak{b}$ in vB.
[19]To be more precise, after one Cohen real there is a sequence $\bar{r} = \langle r_\alpha : \omega \to 2 : \alpha \in \omega_1 \text{ limit}\rangle$ such that, for any ladder system $\bar{c}$ from the ground model, the pair $(\bar{c}, \bar{r})$, as a ladder system coloring, cannot be uniformized in any stationary subset of ω_1. Furthermore, this property is preserved after any σ-linked poset. Also recall from [DS78] (with Devlin) that $\mathfrak{m}(\text{precaliber}) > \aleph_1$ implies that any ladder system coloring can be uniformized.

In this section, we construct models for all Knaster numbers being $\aleph_1$ and $\mathfrak{m}(\text{precaliber}) = \lambda$ for some given regular $\aleph_1 < \lambda \leq \text{add}(\mathcal{N})$ (and the "old" values for the Cichoń-characteristics, as in the previous section).

Definition 5.1. Let $\lambda > \aleph_1$ be regular. A condition $p \in P_{\text{cal}} = P_{\text{cal},\lambda}$ consists of

(i) finite sets $u_p, F_p \subseteq \lambda$,
(ii) a function $c_p : [u_p]^2 \to 2$,
(iii) for each $\alpha \in F_p$, a function $d_{p,\alpha} : \mathcal{P}(u_p \cap \alpha) \to \omega$ satisfying
 ($\star$) if $\alpha \in F_p$ and s_1, s_2 are 1-homogeneous (w.r.t. c_p) subsets of $u_p \cap \alpha$ with $d_{p,\alpha}(s_1) = d_{p,\alpha}(s_2)$, then $s_1 \cup s_2$ is 1-homogeneous.

The order is defined by $q \leq p$ iff $u_p \subseteq u_q$, $F_p \subseteq F_q$, $c_p \subseteq c_q$ and $d_{p,\alpha} \subseteq d_{q,\alpha}$ for any $\alpha \in F_p$.

Lemma 5.2. *P_{cal} has precaliber ω_1 (and in fact precaliber μ for any regular uncountable μ) and forces the following:*

(1) The generic functions $c : [\lambda]^2 \to \{0, 1\}$ and $d_\alpha : [\alpha]^{<\aleph_0} \to \omega$ for $\alpha < \lambda$ are totally defined.
(2) Whenever $(s_i)_{i \in I}$ is a family of finite, 1-homogeneous (w.r.t. c) subsets of α, and $d_\alpha(s_i) = d_\alpha(s_j)$ for $i, j \in I$, then $\bigcup_{i \in I} s_i$ is 1-homogeneous.
(3) If $A \subseteq [\lambda]^{<\aleph_0}$ is a family of size λ of pairwise disjoint sets, then there are two sets $u \neq v$ in A such that $c(\xi, \eta) = 0$ for any $\xi \in u$ and $\eta \in v$.
(4) Whenever $u \in [\lambda]^{<\aleph_0}$, the set $\{\eta < \lambda : \forall \xi \in u (c(\xi, \eta) = 1)\}$ is unbounded in λ.

Proof. For any $\alpha < \lambda$, the set of conditions $p \in P_{\text{cal}}$ such that $\alpha \in F_p$ is dense.

Starting with p such that $\alpha \notin F_p$, we set $u_q = u_p$, $F_q = F_p \cup \{\alpha\}$, and we pick new and unique values for all $d_{q,\alpha}(s)$ for $s \subseteq u_q \cap \alpha = u_p \cap \alpha$, as well as new and unique values for all $d_{q,\beta}(s)$ for $s \subseteq u_q \cap \beta$ with $\alpha \in s$. We have to show that $q \in P_{\text{cal}}$, i.e., that it satisfies ($\star$): Whenever s_1, s_2 satisfy the assumptions of ($\star$), then $\alpha \notin s_i$ (for $i = 1, 2$), as we would otherwise have chosen different values. So we can use that ($\star$) holds for p.

(1) and (4) *For any $\xi < \lambda$, the set of $q \in P_{\text{cal}}$ such that $\xi \in u_q$ is dense.*

Starting with p with $\xi \notin u_p$, we set $u_q = u_p \cup \{\xi\}$ and $F_q = F_p$. Again, pick new (and different) values for all $d_{q,\alpha}(s)$ with $\xi \in s$, and we can set $c(x, \xi)$ to whatever we want. The same argument as above shows that $q \in P_{\text{cal}}$. In particular we can set all $c(x, \xi) = 1$, which shows that P_{cal} forces (4).

(2) follows from ($\star$) for $I = \{1, 2\}$, and this trivially implies the case for arbitrary I. (For $x_1, x_2 \in \bigcup_{i \in I} s_i$, pick $i_1, i_2 \in I$ such that $x_1 \in s_{i_1}$ and $x_2 \in s_{i_2}$; then apply ($\star$) to $\{i_1, i_2\}$.)

Amalgamation. Let $p \in P$ and $u \subseteq u_p$ and $F \subseteq F_p$ (let us call u, F the "heart"). Then we define the *type* of p (with respect to the heart) as the following structure: Let i be the order-preserving bijection (Mostowski's collapse) of $u_p \cup F_p$ to some $N \in \omega$, which also translates the partial functions c_p and d_p and the subsets u_p and F_p. Then the type is the induced structure on N. Between any two conditions with same type there is a natural isomorphism.

Assume $p_0, p_1, \dots p_{n-1}$ are in P_{cal}. We set $u_i := u_{p_i}$, and we do the same for F, c, and d. Assume u_i and F_i form Δ-systems with hearts u and F, and that p_i have the same type for $i \in n$ with respect to u, F. Then all c_i and all d_i agree on the common domain.[20]

Then we define an "amalgamation" $q = q(p_1, p_2, \dots, p_{n-1})$ as follows: $u_q := \bigcup_{i\in n} u_i$, $F_q := \bigcup_{i\in n} F_i$, d_q extends all d_i and has a unique new value for each new element in its domain, c_q extends all c_i; and yet undefined $c_q(x, y)$ are set to 0 if $x, y > \max(F)$ (and 1 otherwise).

To see that $q \in P_{\text{cal}}$, assume that $\alpha \in F_q$ and s_1, s_2 are as in $(\star)$. This implies that $d_{q,\alpha}(s_k)$ for both $k = 1, 2$ were already defined[21] by one of the p_i (for $i \in n$), otherwise we would have picked a new value.

If they are both defined by the same p_i, we can use $(\star)$ for p_i. So assume otherwise, for notational simplicity assume that s_i is defined by p_i; and let $x_i \in s_i$. We have to show $c_q(x_1, x_2) = 1$. Note that $\alpha \in F_1 \cap F_2 = F$. If x_1 or x_2 are not in u, then we have set $c_q(x_1, x_2)$ to 1 (as $x_i < \alpha \in F$), so we are done. So assume $x_1, x_2 \in u$. The natural isomorphism between p_1 and p_2 maps s_1 onto some $s'_1 \subseteq u_2$, and we get that s'_1 is 1-homogeneous and that $d_{2,\alpha}(s_2) = d_{1,\alpha}(s_1) = d_{2,\alpha}(s'_1)$. So we use that p_2 satisfies $(\star)$ to get that $c_2(a, b) = 1$ for all $a \in s_2$ and $b \in s'_1$. As the isomorphism does not move x_1, we can use $a = x_2$ and $b = x_1$.

Precaliber. *P_{cal} has precaliber μ for any uncountable regular μ.*

Let $\{p_\xi : \xi < \mu\}$ be a set of conditions in P_{cal}. For $\xi < \mu$ denote $u_\xi := u_{p_\xi}$, $F_\xi := F_{p_\xi}$ and so on. We can assume that the u_ξ's and F_ξ's form Δ-systems with roots u and F respectively, and that the type of p_ξ does not depend on ξ. Then any finite subset of these conditions is compatible, witnessed by its amalgamation.

(3) Let $p \in P_{\text{cal}}$ and assume that p forces that $\dot{A} \subseteq [\lambda]^{<\aleph_0}$ is a family of size λ of pairwise disjoint sets. We can find, in the ground model, a family $A' \subseteq [\lambda]^{<\aleph_0}$ of size λ and conditions $p_v \leq p$ for $v \in A'$ such that $v \subseteq u_{p_v}$, and p_v forces $v \in \dot{A}$. We again thin out to a Δ-system as above; this time we can additionally assume that the heart of the F_v is below the non-heart parts of all u_v, i.e., that $\max(F)$ is below $u_v \setminus u$ for all v.

Pick any two p_v, p'_v in this Δ-system, and let q be the amalgamation defined above. Then q witnesses that p_v, p_w are compatible, which implies $v \cap w = 0$, i.e., v, w are outside the heart; which by construction of q implies that c_q is constantly zero on $v \times w$ (as their elements are above $\max(F)$). □

The poset $P_{\text{cal},\lambda}$ adds generic functions c and d_α. We now use them to define a precaliber ω_1 poset Q_{cal} witnessing $\mathfrak{m}(\text{precaliber}) \leq \lambda$:

Lemma 5.3. *In $V^{P_{\text{cal}}}$, define the poset $Q_{\text{cal}} := \{u \in [\lambda]^{<\aleph_0} : u \text{ is 1-homogeneous}\}$, ordered by $\supseteq$ (By 1-homogeneous, we mean 1-homogeneous with respect to c.) Then the following is satisfied (in $V^{P_{\text{cal}}}$):*

(1) Q_{cal} is an increasing union of length λ of centered sets (so in particular it has precaliber $\aleph_1$).
(2) For $\alpha < \lambda$, the set $D_\alpha := \{u \in Q_{\text{cal}} : u \not\subseteq \alpha\}$ is open dense. So Q_{cal} adds a cofinal generic 1-homogeneous subset of λ.
(3) There is no 1-homogeneous set of size λ (in $V^{P_{\text{cal}}}$). In other words, there is no filter meeting all D_α.

[20] I.e., for $\alpha < \beta$ in u, $c_i(\alpha, \beta) = c_j(\alpha, \beta)$, and for $\alpha \in F$ and $s \subseteq u$, $d_{i,\alpha}(s) = d_{j,\alpha}(s)$.
[21] By which we mean $\alpha \in F_i$ and $s_k \subseteq u_i$ for both $k = 1, 2$.

Proof. For (1) set $Q^\alpha_{\text{cal}} = Q_{\text{cal}} \cap [\alpha]^{<\aleph_0}$. Then $d_\alpha : Q^\alpha_{\text{cal}} \to \omega$ is a centering function, according to Lemma 5.2(2). Precaliber $\aleph_1$ is a consequence of $\lambda_{\text{cal}} > \aleph_1$.

Property (2) is a direct consequence of Lemma 5.2(4), and (3) follows from Lemma 5.2(3). □

This shows that $P_{\text{cal},\lambda} \Vdash \mathfrak{m}(\text{precaliber}) \leq \lambda$. We now show that this is preserved in further Knaster extensions.

Lemma 5.4. *In $V^{P_{\text{cal}}}$, assume that P' is a ccc λ-Knaster poset. Then, in $V^{P_{\text{cal}}*P'}$, $\mathfrak{m}$(precaliber) $\leq \lambda$.*

Proof. We claim that in $V^{P_{\text{cal}}*P'}$, Q_{cal} still has precaliber $\aleph_1$, and there is no filter meeting each open dense subset $D_\alpha \subseteq Q_{\text{cal}}$ for $\alpha < \lambda$.

Precaliber follows from Lemma 5.3(1). So we have to show that λ has no 1-homogeneous set (w.r.t. c) of size λ in $V^{P_{\text{cal}}*P'}$.

Work in $V^{P_{\text{cal}}}$ and assume that $\dot{A}$ is a P'-name and $p \in P'$ forces that $\dot{A}$ is in $[\lambda]^\lambda$. By recursion, find $A' \in [\lambda]^\lambda$ and $p_\zeta \leq p$ for each $\zeta \in A'$ such that $p_\zeta \Vdash \zeta \in \dot{A}$. Since P' is λ-Knaster, we may assume that $\{p_\zeta : \zeta \in A'\}$ is linked. By Lemma 5.2(3), there are $\zeta \neq \zeta'$ in A' such that $c(\zeta, \zeta') = 0$. So there is a condition q stronger that both p_ζ and $p_{\zeta'}$ forcing that $\zeta, \zeta' \in \dot{A}$ and $c(\zeta, \zeta') = 0$, i.e., that $\dot{A}$ is not 1-homogeneous. □

We can now add another case to Lemma 4.8:

Lemma 5.5. *Fix a regular uncountable $\lambda_{\text{cal}} \leq \lambda_{\text{add}(\mathcal{N})}$. The forcing $P^{\mathtt{vA}}$ of Theorem 2.1 (and analogously $P^{\mathtt{vB}}$) can be modified to some P' which still forces the same values to the Cichoń-characteristics, and additionally satisfies: Each iterand is either precaliber and of size $<\lambda_{\text{cal}}$, or (σ, k)-linked for all k. P' forces: for all $k \in \omega$, $\mathfrak{m}(k\text{-Knaster}) = \aleph_1$; $\mathfrak{m}(\text{precaliber}) = \lambda_{\text{cal}}$; and $\mathfrak{p} = \lambda_{\mathfrak{b}}$.*

Using Boolean ultrapowers and assuming $\lambda_{\text{cal}} < \kappa_9$, we get the old values for Cichoń's diagram, plus $\mathfrak{m}(k\text{-Knaster}) = \aleph_1$; $\mathfrak{m}(\text{precaliber}) = \lambda_{\text{cal}}$; and $\mathfrak{p} \geq \kappa_9$.

Proof. The case $\lambda_{\text{cal}} = \aleph_1$ was already dealt with in the previous section, so we assume $\lambda_{\text{cal}} > \aleph_1$. Just as in the proof of Lemma 4.8: We start with the forcing $P_{\text{cal},\lambda_{\text{cal}}}$. From then on, use (by bookkeeping) all precaliber forcings of size $< \lambda_{\text{cal}}$, all σ-centered ones of size $< \lambda_b$, and the forcings for the old construction. So each iterand either has precaliber $\aleph_1$ and is of size $< \lambda_{\text{cal}}$, or is (σ, k)-linked for any $k \geq 2$. Therefore, the limits are $k+1$-Knaster (for any k). Accordingly, the limit forces that each k-Knaster number is $\aleph_1$.

Also, each iterand is either of size $< \lambda_{\text{cal}}$ or σ-linked; so the limit is λ_{cal}-Knaster, and by Lemma 5.4 it forces that the precaliber number is $\leq \lambda_{\text{cal}}$; our bookkeeping gives $\geq \lambda_{\text{cal}}$. □

6. Products, dealing with $\mathfrak{p}$

We start reviewing a basic result in forcing theory.

Lemma 6.1 (Easton's lemma)**.** *Let κ be an uncountable cardinal, P a κ-cc poset and let Q be a $<\kappa$-closed poset. Then P forces that Q is $<\kappa$-distributive.*

Proof. See e.g. [Jec03, Lemma 15.19]. Note that there the lemma is proved for successor cardinals only, but literally the same proof works for any regular cardinal; for singular cardinals κ note that $<\kappa$-closed implies $<\kappa^+$-closed so we even get $<\kappa^+$-distributive. □

We will use the following assumption for Corollary 6.3 and Lemma 8.3:

Assumption 6.2. (1) κ is regular uncountable.

(2) $\theta \geq \kappa$, $\theta = \theta^{<\kappa}$.
(3) P is κ-cc and forces that $\mathfrak{x} = \lambda$ for some characteristic $\mathfrak{x}$ (so in particular λ is a cardinal in the P-extension).
(4) Q is $<\kappa$-closed.
(5) $P \Vdash Q$ is θ^+-cc.[22]
(6) We set $P^+ := P \times Q = P * Q$. We call the P^+-extension V'' and the intermediate P-extension V'.

(We will actually have $|Q| = \theta$, which implies (5)).

Let us list a few simple facts:

(P1) In V', all V-cardinals $\geq\kappa$ are still cardinals, and Q is a $<\kappa$-distributive forcing (due to Easton's lemma). So we can apply Lemma 3.1 and Corollary 3.5.
(P2) Let μ be the successor (in V or equivalently in V') of θ. So in V', Q is μ-cc and preserves all cardinals $\leq \kappa$ as well as all cardinals $\geq \mu$.
(P3) So if $V \models$"$\kappa \leq \nu \leq \theta$", then in V'', $\kappa \leq |\nu| < \mu$. The V'' successor of κ is $\leq \mu$.

We will use two instances of Q: Right now we use $Q = \kappa^{<\kappa}$, and in Section 8 Q is a collapse of θ to κ.

Corollary 6.3. *Assume $\kappa^{<\kappa} = \kappa$, P is κ-cc, and set $Q = \kappa^{<\kappa}$ (ordered by extension). Then P forces that Q^V preserves all cardinals and cofinalities. Assume $P \Vdash \mathfrak{x} = \lambda$ (in particular that λ is a cardinal), and let R be a Borel relation.*

(a) If $\mathfrak{x}$ is $\mathfrak{m}$-like: $\lambda < \kappa$ implies $P \times Q \Vdash \mathfrak{x} = \lambda$; $\lambda \geq \kappa$ implies $P \times Q \Vdash \mathfrak{x} \geq \kappa$.
(b) If $\mathfrak{x}$ is $\mathfrak{h}$-like: $P \times Q \Vdash \mathfrak{x} \leq \lambda$.
(c) $P \Vdash \mathrm{LCU}_R(\lambda)$ implies $P \times Q \Vdash \mathrm{LCU}_R(\lambda)$.
(d) $P \Vdash \mathrm{COB}_R(\lambda, \mu)$ implies $P \times Q \Vdash \mathrm{COB}_R(\lambda, \mu)$.

Proof. P, Q satisfy Assumption 6.2 with $\kappa = \theta$ and so we can use Lemma 3.1 and Corollary 3.5. □

The following is shown in [DS]:

Lemma 6.4. *Assume that $\kappa = \kappa^{<\kappa}$ and P is a κ-cc poset that forces $\kappa \leq \mathfrak{p}$. In the P-extension V', let $Q = (\kappa^{<\kappa})^V$. Then,*

(a) $P \times Q = P * Q$ forces $\mathfrak{p} = \kappa$
(b) If in addition P forces $\kappa \leq \mathfrak{p} = \mathfrak{h} = \lambda$ then $P \times Q$ forces $\mathfrak{h} = \lambda$.

Proof. Work in the P-extension V'. Q preserves cardinals and cofinalities, and it forces $\mathfrak{p} \geq \kappa$ by Corollary 6.3.

There is an embedding F from $\langle Q, \subsetneq\rangle$ into $\langle [\omega]^{\aleph_0}, \supsetneq^*\rangle$ preserving the order and incompatibility (using the fact that $\kappa \leq \mathfrak{p} = \mathfrak{t}$ and that every infinite set can be split into κ many almost disjoint sets). Now, Q adds a new sequence $z \in \kappa^\kappa \setminus V'$ and forces that $\dot{T} = \{F(z \restriction \alpha) : \alpha < \kappa\}$ is a tower (hence $\mathfrak{t} \leq \kappa$). If this were not the case, some condition in Q would force that $\dot{T}$ has a pseudo-intersection a, but actually $a \in V'$ and it determines uniquely a branch in κ^κ, and this branch would be in fact z, i.e., $z \in V'$, a contradiction. So we have shown $P \times Q \Vdash \mathfrak{t} = \kappa$.

[22]I.e., P forces that all antichains of Q have size $\leq\theta$.

For (b): We already know that $Q \Vdash \mathfrak{h} \leq \lambda$. To show that $\mathfrak{h}$ does not decrease, again work in V'. Note that $\langle [\omega]^{\aleph_0}, \subseteq^* \rangle$ is $<\lambda$-closed (as $\mathfrak{t} = \lambda$). We claim that Q forces that $\langle [\omega]^{\aleph_0}, \subseteq^* \rangle$ is $<\lambda$-distributive, (which implies $Q \Vdash \mathfrak{h} \geq \lambda$).

If $\lambda = \kappa$ then $\langle [\omega]^{\aleph_0}, \subseteq^* \rangle$ is still $<\kappa$-closed because Q is $<\kappa$-distributive; so assume $\kappa < \lambda$. Then Q is λ-cc (because $|Q| = \kappa$), so $\langle [\omega]^{\aleph_0}, \subseteq^* \rangle$ is forced to be $<\lambda$-distributive by Easton's Lemma (recall that Q does not add new reals). □

In conjunction with the results of Sections 4 and 5, we show that additional arbitrary regular values can be forced for the Martin numbers and $\mathfrak{p}$ in the old constructions of Section 2.

Theorem 6.5. *Assume the hypothesis of Theorem 2.1 and that* $\aleph_1 \leq \lambda_{\mathfrak{m}} \leq \kappa = \kappa^{<\kappa} \leq \lambda_{\mathfrak{b}}$ *are regular such that* $\lambda_{\mathfrak{m}} \leq \lambda_{\mathrm{add}(\mathcal{N})}$.

- *(a) We can modify* $P^{\mathtt{vA}}$ *and* $P^{\mathtt{vB}}$ *of Theorem 2.1 to additionally force* $\mathfrak{p} = \kappa$, $\mathfrak{h} = \lambda_{\mathfrak{b}}$ *and one of the following:*
  - *(M1)* $\mathfrak{m} = \mathfrak{m}(\textit{precaliber}) = \lambda_{\mathfrak{m}}$;
  - *(M2)* $\mathfrak{m}(k\textit{-Knaster}) = \aleph_1$ *and* $\mathfrak{m}(k+1\textit{-Knaster}) = \lambda_{\mathfrak{m}}$ *for a given* $1 \leq k < \omega$;
  - *(M3)* $\mathfrak{m}(k\textit{-Knaster}) = \aleph_1$ *for all* $1 \leq k < \omega$ *and* $\mathfrak{m}(\textit{precaliber}) = \lambda_{\mathfrak{m}}$.
- *(b) In the hypothesis of Theorem 2.3, assume in addition that* $\kappa \leq \kappa_9$. *Then* $P^{\mathtt{vA}*}$ *and* $P^{\mathtt{vB}*}$ *can be modified to additionally force* $\mathfrak{p} = \kappa$ *and one of (M1)–(M3).*

In both situations, the modified forcing is of the form $P \times \kappa^{<\kappa}$ *for some ccc forcing* P, *in particular it is* κ^+*-cc.*

Proof. We show the $\mathtt{vA}$ case. Depending on an instance of (M1)–(M3), let P'' be the modification of $P^{\mathtt{vA}}$ presented in Corollary 4.9 (or in Lemma 5.5 for (M3) when $\lambda_{\mathfrak{m}} > \aleph_1$), so it forces $\mathfrak{p} = \mathfrak{b} = \lambda_{\mathfrak{b}}$ and one of (M1)–(M3). This implies $\mathfrak{h} = \lambda_{\mathfrak{b}}$. Put $P^{+2} := P'' \times \kappa^{<\kappa}$, which is as desired in (a) by Lemma 6.4 and Corollary 6.3 (for the preservation of strong witnesses of the Cichoń-characteristics). This shows (a).

For (b) assume in addition that $\kappa \leq \kappa_9$. We deal with (M1), but the other two alternatives are proved similarly. Let P^{**} be the poset obtained in the proof of Corollary 4.9. If $\lambda_{\mathfrak{m}} < \kappa_9$ then P^{**} already forces (M1) and $\mathfrak{p} \geq \kappa$, but in the case $\lambda_{\mathfrak{m}} = \kappa_9$ it just forces $\mathfrak{m} \geq \kappa_9$. Whichever the case, $P^{**} \times \kappa^{<\kappa}$ is as required (in the latter case $\kappa_9 = \kappa$, so this product forces $\kappa \leq \mathfrak{m} \leq \mathfrak{m}(\text{precaliber}) \leq \mathfrak{p} \leq \kappa$). □

7. Dealing with $\mathfrak{h}$

The following is a very useful tool to deal with $\mathfrak{g}$.

Lemma 7.1 (Blass [Bla89, Thm. 2], see also Brendle [Bre10, Lem. 1.17])**.** *Let* ν *be an uncountable regular cardinal and let* $(V_\alpha)_{\alpha \leq \nu}$ *be an increasing sequence of transitive models of ZFC such that*

- *(i)* $\omega^\omega \cap (V_{\alpha+1} \smallsetminus V_\alpha) \neq \emptyset$,
- *(ii)* $(\omega^\omega \cap V_\alpha)_{\alpha<\nu} \in V_\nu$, *and*
- *(iii)* $\omega^\omega \cap V_\nu = \bigcup_{\alpha<\nu} \omega^\omega \cap V_\alpha$.

Then, in V_ν, $\mathfrak{g} \leq \nu$.

This result gives an alternative proof of the well-known:

Corollary 7.2. $\mathfrak{g} \leq \mathrm{cof}(\mathfrak{c})$.[23]

[23]A more elementary proof can be found in [Bla10, Thm.8.6, Cor. 8.7]

Proof. Put $\nu := \mathrm{cof}(\mathfrak{c})$ and let $(\mu_\alpha)_{\alpha<\nu}$ be a cofinal increasing sequence in $\mathfrak{c}$ formed by limit ordinals. By recursion, we can find an increasing sequence $(V_\alpha)_{\alpha<\nu}$ of transitive models of (a large enough fragment of) ZFC such that (i) of Lemma 7.1 is satisfied, $\mu_\alpha \in V_\alpha$, $|V_\alpha| = |\mu_\alpha|$ and $\bigcup_{\alpha<\nu} \omega^\omega \cap V_\alpha = \omega^\omega$. Set $V_\nu := V$, so Lemma 7.1 applies, i.e., $\mathfrak{g} \leq \nu = \mathrm{cof}(\mathfrak{c})$. □

The following result is the main tool to modify the values of $\mathfrak{g}$ and $\mathfrak{c}$ via a complete subposet of some forcing, while preserving $\mathfrak{m}$-like and Blass-uniform values from the original poset. This is a direct consequence of Lemmas 3.2 and 7.1 and Corollary 3.5. As we are only interested in finitely many characteristics, the index sets I_1, I_2, J and K will always be finite when we apply the lemma.

Lemma 7.3. *Assume the following:*

(1) $\kappa \leq \nu$ *are uncountable regular cardinals,* P *is a* κ*-cc poset.*
(2) $\mu = \mu^{<\kappa} \geq \nu$ *and* P *forces* $\mathfrak{c} > \mu$*.*
(3) For some Borel relations R^1_i $(i \in I_1)$ *on* ω^ω *and some regular* $\lambda^1_i \leq \mu$*:* P *forces* $\mathrm{LCU}_{R^1_i}(\lambda^1_i)$
(4) For some Borel relations R^2_i $(i \in I_2)$ *on* ω^ω*,* $\lambda^2_i \leq \mu$ *regular and a cardinal* $\vartheta^2_i \leq \mu$*:* P *forces* $\mathrm{COB}_{R^2_i}(\lambda^2_i, \vartheta^2_i)$*.*
(5) For some $\mathfrak{m}$*-like characteristics* $\mathfrak{y}_j$ $(j \in J)$ *and* $\lambda_j < \kappa$*:* $P \Vdash \mathfrak{y}_j = \lambda_j$*.*
(6) For some $\mathfrak{m}$*-like characteristics* $\mathfrak{y}'_k$ $(k \in K)$*:* $P \Vdash \mathfrak{y}'_k \geq \kappa$*.*
(7) $|I_1 \cup I_2 \cup J \cup K| \leq \mu$*.*

Then there is a complete subforcing P' *of* P *of size* μ *forcing*

(a) $\mathfrak{y}_j = \lambda_j$*,* $\mathfrak{y}'_k \geq \kappa$*,* $\mathrm{LCU}_{R^1_i}(\lambda^1_i)$ *and* $\mathrm{COB}_{R^2_{i'}}(\lambda^2_{i'}, \vartheta^2_{i'})$ *for all* $i \in I_1$*,* $i' \in I_2$*,* $j \in J$ *and* $k \in K$*;*
(b) $\mathfrak{c} = \mu$ *and* $\mathfrak{g} \leq \nu$*.*

Proof. Construct an increasing sequence of elementary submodels $(M_\alpha : \alpha < \nu)$ of some $(H(\chi), \in)$ for some sufficiently large χ, where each M_α is $<\kappa$-closed with cardinality μ, in a way that $M := M_\nu = \bigcup_{\alpha<\nu} M_\alpha$ satisfies:

(i) $\mu \cup \{\mu\} \subseteq M_0$,
(ii) $I_1 \cup I_2 \cup J \cup K \subseteq M_0$,
(iii) M_0 contains all the definitions of the characteristics we use,
(iv) M_0 contains all the P-names of witnesses of each $\mathrm{LCU}_{R^1_i}(\lambda^1_i)$ $(i \in I_1)$,
(v) for each $i \in I_2$ and some chosen name $(\dot{\trianglelefteq}^i, \dot{\bar{g}}^i)$ of a witness of $\mathrm{COB}_{R^2_i}(\lambda^2_i, \vartheta^2_i)$: for all $(s,t) \in \vartheta^2_i \times \vartheta^2_i$, $\dot{g}^i_s \in M_0$ and the maximal antichain deciding "$s \dot{\trianglelefteq}^i t$" belongs to M_0,
(vi) $M_{\alpha+1}$ contains P-names of reals that are forced not to be in the $P \cap M_\alpha$-extension (this is because P forces $\mathfrak{c} > \mu$).

Note that M is also a $<\kappa$-closed elementary submodel of $H(\chi)$ of size μ, and that $P_\alpha := P \cap M_\alpha$ (for $\alpha \leq \nu$) is a complete subposet of P. Put $P' := P_\nu$.

According to Corollary 3.5, in the P'-extension, each $\mathfrak{m}$-like characteristic below κ is preserved (as in the P-extension) and for the others "$\mathfrak{y}'_k \geq \kappa$" is preserved; and according to Lemma 3.2 the LCU and COB statements are preserved as well. This shows (a).

It is clear that P_α is a complete subposet of P_β for every $\alpha < \beta \leq \nu$, and that P' is the direct limit of the P_α. Therefore, if V' denotes the P'-extension and V_α

denotes the P_α-intermediate extensions, then $\omega^\omega \cap V_{\alpha+1} \smallsetminus V_\alpha \neq \emptyset$ (by (vi)) and $\omega^\omega \cap V' \subseteq \bigcup_{\alpha<\nu} V_\alpha$. Hence, by Lemma 7.1, $V' \models \mathfrak{g} \leq \nu$. Clearly, $V' \models \mathfrak{c} = \mu$. □

Remark 7.4. So we can preserve $\mathrm{COB}(\lambda, \theta)$ provided both λ and θ are $\leq \mu$.

For larger λ or θ this is generally not possible. (E.g., if $\lambda > \mu$, then $\mathrm{COB}_R(\lambda, \theta)$ will fail as it implies $\mathfrak{b}_R \geq \lambda > \mu = \mathfrak{c}$.) However we do get the following (the proof is straightforward):

If we assume, in addition to the conditions of Lemma 7.3, that $P \Vdash \mathrm{COB}_R(\lambda, \theta)$ for some Borel relation R and $\lambda \leq \nu$ (now allowing also $\theta > \mu$), then we can construct P' such that $P' \Vdash \mathrm{COB}_R(\lambda, \mu)$. (But this only gives us $\mathfrak{d}_{R^3_i} \leq \mu = \mathfrak{c}$.)

In the case of $\nu < \lambda$, according to Remark 2.15 we have $P \Vdash \mathrm{COB}_R(\nu, \theta)$, so as we have just seen we can get $P' \Vdash \mathrm{COB}_R(\nu, \min(\mu, \theta))$ (which implies $\mathfrak{b}_R \geq \nu$, which is a bit better than the $\mathfrak{b}_R \geq \kappa$ we get from (6)).

We are now ready to prove the consistency of 13 pairwise different classical characteristics.

Theorem 7.5. *Assume* $\aleph_1 \leq \lambda_{\mathfrak{m}} \leq \lambda_{\mathfrak{p}} \leq \lambda_{\mathfrak{h}} \leq \kappa_9 < \lambda_1 < \kappa_8 < \lambda_2 < \kappa_7 < \lambda_3 \leq \lambda_4 \leq \lambda_5 \leq \lambda_6 \leq \lambda_7 \leq \lambda_8 < \mu$ *such that*

(i) For $j = 7, 8, 9$, κ_i *is strongly compact,*
(ii) $\lambda_j^{\kappa_j} = \lambda_j$ *for* $j = 7, 8$,
(iii) λ_i *is regular for* $i \neq 6$
(iv) $\lambda_{\mathfrak{p}}^{<\lambda_{\mathfrak{p}}} = \lambda_{\mathfrak{p}}$
(v) $\lambda_6^{<\lambda_3} = \lambda_6$,
(vi) $\mu^{<\lambda_{\mathfrak{h}}} = \mu$.

Then there is a $\lambda_{\mathfrak{p}}^+$*-cc poset* P *which preserves cofinalities and forces one of (M1)–(M3) of Theorem 6.5 and*

$$\mathfrak{p} = \lambda_{\mathfrak{p}},\ \mathfrak{h} = \mathfrak{g} = \lambda_{\mathfrak{h}},\ \mathrm{add}(\mathcal{N}) = \lambda_1,\ \mathrm{cov}(\mathcal{N}) = \lambda_2,\ \mathfrak{b} = \lambda_3,\ \mathrm{non}(\mathcal{M}) = \lambda_4,$$
$$\mathrm{cov}(\mathcal{M}) = \lambda_5,\ \mathfrak{d} = \lambda_6,\ \mathrm{non}(\mathcal{N}) = \lambda_7,\ \mathrm{cof}(\mathcal{N}) = \lambda_8,\ \textit{and } \mathfrak{c} = \mu.$$

Proof. We prove the (M1) case. Let P^* be the ccc poset obtained in the proof of Corollary 4.9 for $\lambda_9 := (\mu^{\kappa_9})^+$ (the modification of $P^{\mathtt{vA}*}$). This is a ccc poset of size λ_9 that forces the values of the Cichoń-characteristics as in Theorem 2.3 ($\mathtt{vA}$*) with strong witnesses, and forces $\mathfrak{m} = \mathfrak{m}(\text{precaliber}) = \lambda_{\mathfrak{m}}$ and $\mathfrak{p} \geq \kappa_9$ whenever $\lambda_{\mathfrak{m}} < \kappa_9$, but in the case $\lambda_{\mathfrak{m}} = \kappa_9$ it just forces $\mathfrak{m} \geq \kappa_9$.

By application of Lemma 7.3 to $\kappa = \nu = \lambda_{\mathfrak{h}}$ and to μ, we find a complete subposet P' of P^* forcing $\mathfrak{m} = \lambda_{\mathfrak{m}}$, $\lambda_{\mathfrak{h}} \leq \mathfrak{p} \leq \mathfrak{g} \leq \lambda_{\mathfrak{h}}$ (so they are equalities), $\mathfrak{c} = \mu$ and that the values of the other cardinals in Cichoń's diagram are the same values forced by P^*, even with strong witnesses. This is clear in the case $\lambda_{\mathfrak{m}} < \lambda_{\mathfrak{h}}$, but the case $\lambda_{\mathfrak{m}} = \lambda_{\mathfrak{h}}$ (even $\lambda_{\mathfrak{m}} = \kappa_9$) is also fine because P' would force $\lambda_{\mathfrak{m}} \leq \mathfrak{m} \leq \mathfrak{m}(\text{precaliber}) \leq \mathfrak{p} \leq \mathfrak{g} \leq \lambda_{\mathfrak{m}}$.

If $\lambda_{\mathfrak{p}} = \lambda_{\mathfrak{h}}$ then we would be done, so assume that $\lambda_{\mathfrak{p}} < \lambda_{\mathfrak{h}}$. Hence, by Corollary 6.3 and Lemma 6.4, $P := P' \times (\lambda_{\mathfrak{p}}^{<\lambda_{\mathfrak{p}}})$ is as required. It is clear that P forces $\mathfrak{m} = \mathfrak{m}(\text{precaliber}) = \lambda_{\mathfrak{m}}$ when $\lambda_{\mathfrak{m}} < \lambda_{\mathfrak{p}}$, but the same happens when $\lambda_{\mathfrak{m}} = \lambda_{\mathfrak{p}}$ because P would force $\lambda_{\mathfrak{m}} \leq \mathfrak{m} \leq \mathfrak{m}(\text{precaliber}) \leq \mathfrak{p} \leq \lambda_{\mathfrak{m}}$. □

The same argument works to get a similar version of the previous result for $\mathtt{vB}$*.

Theorem 7.6. *Assume* $\aleph_1 \leq \lambda_{\mathfrak{m}} \leq \lambda_{\mathfrak{p}} \leq \lambda_{\mathfrak{h}} < \kappa_9 < \lambda_1 < \kappa_8 < \lambda_2 < \kappa_7 < \lambda_3 < \kappa_6 < \lambda_4 \leq \lambda_5 \leq \lambda_6 \leq \lambda_7 \leq \lambda_8 < \mu$ *such that*

(i) *for* $j = 6, 7, 8, 9$, κ_j *is strongly compact,*
(ii) $\lambda_j^{\kappa_j} = \lambda_j$ *for* $j = 6, 7, 8$,
(iii) λ_i *is regular for* $i \neq 5$,
(iv) $\lambda_{\mathfrak{p}}^{<\lambda_{\mathfrak{p}}} = \lambda_{\mathfrak{p}}$
(v) $\lambda_2^{<\lambda_2} = \lambda_2$, $\lambda_4^{\aleph_0} = \lambda_4$, $\lambda_5^{<\lambda_4} = \lambda_5$,
(vi) λ_3 *is* $\aleph_1$*-inaccessible, and*
(vii) $\mu^{<\lambda_{\mathfrak{h}}} = \mu$.

Then there is a $\lambda_{\mathfrak{p}}^+$*-cc poset* P*, preserving cofinalities, that forces one of (M1)–(M3) of Theorem 6.5 and*

$$\mathfrak{p} = \lambda_{\mathfrak{p}},\ \mathfrak{h} = \mathfrak{g} = \lambda_{\mathfrak{h}},\ \mathrm{add}(\mathcal{N}) = \lambda_1,\ \mathfrak{b} = \lambda_2,\ \mathrm{cov}(\mathcal{N}) = \lambda_3,\ \mathrm{non}(\mathcal{M}) = \lambda_4,$$
$$\mathrm{cov}(\mathcal{M}) = \lambda_5,\ \mathrm{non}(\mathcal{N}) = \lambda_6,\ \mathfrak{d} = \lambda_7,\ \mathrm{cof}(\mathcal{N}) = \lambda_8,\ \textit{and } \mathfrak{c} = \mu.$$

We can preserve all characteristics only because $\mathrm{cof}(\mathcal{N})$ is smaller than the continuum. In particular, if we use version `vA` without large cardinals, and we cannot further increase the continuum above $\mathrm{cof}(\mathcal{N})$, then the methods of this section only ensures a model of one of (M1)–(M3) plus

$$\mathfrak{p} = \lambda_{\mathfrak{p}},\ \mathfrak{g} = \mathfrak{h} = \lambda_{\mathfrak{h}},$$
$$\min\{\lambda_1, \lambda_{\mathfrak{h}}\} \leq \mathrm{add}(\mathcal{N}) \leq \lambda_1,\ \min\{\lambda_2, \lambda_{\mathfrak{h}}\} \leq \mathrm{cov}(\mathcal{N}) \leq \lambda_2,\ \min\{\lambda_3, \lambda_{\mathfrak{h}}\} \leq \mathfrak{b} \leq \lambda_3,$$
$$\mathrm{non}(\mathcal{M}) = \lambda_4,\ \mathrm{cov}(\mathcal{M}) = \lambda_5,\ \mathfrak{d} = \mathrm{non}(\mathcal{N}) = \mathfrak{c} = \mu,$$

whenever $\aleph_1 \leq \lambda_{\mathfrak{m}} \leq \lambda_{\mathfrak{p}} = \lambda_{\mathfrak{p}}^{<\lambda_{\mathfrak{p}}} \leq \lambda_{\mathfrak{h}} \leq \lambda_3$ are regular, $\lambda_{\mathfrak{m}} \leq \lambda_1$ and $\lambda_5 \leq \mu = \mu^{<\lambda_3} < \lambda_6$, where λ_i (i=1,. . . ,6) are as in `vA` (see Remark 7.4). That is, some left side Cichoń-characteristics do not get decided unless $\lambda_{\mathfrak{x}} \leq \lambda_{\mathfrak{h}}$ Hence, it is unclear whether $\mathfrak{h}$ gets separated from all the left side characteristics, and actually the best instance of the above (when $\lambda_{\mathfrak{h}} = \lambda_3$) is just the same as Theorem 6.5(a). A similar situation occurs with version `vB`: we may lose $\mathfrak{x} \geq \lambda_{\mathfrak{x}}$ for any left side characteristic $\mathfrak{x}$ when $\lambda_{\mathfrak{h}} < \lambda_{\mathfrak{x}}$, since these values $\lambda_{\mathfrak{x}}$ are above the closure of the elementary submodel and the COB witnesses are larger than its size.

8. Reducing gaps (or getting rid of them)

As mentioned in Remark 2.4, we can choose right side Cichoń-characteristics rather arbitrarily or even choose them to be equal (equality allows a construction from fewer compact cardinals). However, large gaps were required between some left side cardinals. We deal with this problem now, and show that we can reasonably assign arbitrary values to all characteristics, and in particular set any "reasonable selection" of them equal.

Let us introduce notation to describe this effect:

Definition 8.1. Let $\bar{\mathfrak{x}} = (\mathfrak{x}_i : i < n)$ be a finite sequence of cardinal characteristics (i.e., of definitions). Say that $\bar{\mathfrak{x}}$ is a $<$-*consistent sequence* if the statement $\mathfrak{x}_0 < \dots < \mathfrak{x}_{n-1}$ is consistent with ZFC (perhaps modulo large cardinals).

A consistent sequence $\bar{\mathfrak{x}}$ is $\leq$-*consistent* if, in the previous chain of inequalities, it is consistent to replace any desired instance of $<$ with $=$. More formally, for any interval partition $(I_k : k < m)$ of $\{0, \dots, n-1\}$, it is consistent that $\mathfrak{x}_i = \mathfrak{x}_j$ for any $i, j \in I_k$, and $\mathfrak{x}_i < \mathfrak{x}_j$ whenever $i \in I_k$, $j \in I_{k'}$ and $k < k' < m$.

For example, the sequence

$$(\aleph_1, \mathrm{add}(\mathcal{N}), \mathrm{cov}(\mathcal{M}), \mathfrak{b}, \mathrm{non}(\mathcal{M}), \mathrm{cov}(\mathcal{M}), \mathfrak{d})$$

is $\leq$-consistent, as well as

$$(\aleph_1, \mathrm{add}(\mathcal{N}), \mathfrak{b}, \mathrm{cov}(\mathcal{N}), \mathrm{non}(\mathcal{M}), \mathrm{cov}(\mathcal{M})),$$

see Theorem 2.1. Previously, it had not been known whether the sequences of ten Cichoń-characteristics from [GKS, BCM18, KST19] are $\leq$-consistent: It is not immediate that cardinals on the left side can be equal while separating everything on the right side. The reason is that, to separate cardinals on the right side, it is necessary to have a strongly compact cardinal between the dual pair of cardinals on the left, thus the left side gets separated as well. But thanks to the collapsing method of this section, we can equalize cardinals on the left as well. As a result, we obtain the following:[24]

Lemma 8.2. *The sequences*

$$(\aleph_1, \mathfrak{m}, \mathfrak{p}, \mathrm{add}(\mathcal{N}), \mathrm{cov}(\mathcal{M}), \mathfrak{b}, \mathrm{non}(\mathcal{M}), \mathrm{cov}(\mathcal{M}), \mathfrak{d}, \mathrm{non}(\mathcal{N}), \mathrm{cof}(\mathcal{N}), \mathfrak{c})\ \textit{and}$$
$$(\aleph_1, \mathfrak{m}, \mathfrak{p}, \mathrm{add}(\mathcal{N}), \mathfrak{b}, \mathrm{cov}(\mathcal{N}), \mathrm{non}(\mathcal{M}), \mathrm{cov}(\mathcal{M}), \mathrm{non}(\mathcal{N}), \mathfrak{d}, \mathrm{cof}(\mathcal{N}), \mathfrak{c})$$

are $\leq$-consistent (modulo large cardinals).

(Note that we lose $\mathfrak{h}$ in the process.)

To prove this claim, recall Assumption 6.2 and its consequences. We now apply it to a collapse:

Lemma 8.3. *Let R be a Borel relation, κ be regular, $\theta > \kappa$, $\theta^{<\kappa} = \theta$, P κ-cc, and set $Q := \mathrm{Coll}(\kappa, \theta)$, i.e., the set of partial functions $f : \kappa \to \theta$ of size $< \kappa$. Then:*

(a) $P \times Q$ forces $|\theta| = \kappa$.

(b) If P forces that λ is a cardinal then

$$P \times Q \Vdash |\lambda| = \begin{cases} \kappa & \textit{if (in } V\textit{) } \kappa \leq \lambda \leq \theta \\ \lambda & \textit{otherwise.} \end{cases}$$

(c) If $\mathfrak{x}$ is $\mathfrak{m}$-like, $\lambda < \kappa$ and $P \Vdash \mathfrak{x} = \lambda$, then $P \times Q \Vdash \mathfrak{x} = \lambda$.

(d) If $\mathfrak{x}$ is $\mathfrak{m}$-like and $P \Vdash \mathfrak{x} \geq \kappa$, then $P \times Q \Vdash \mathfrak{x} \geq \kappa$.

(e) If R is a Borel relation then

(i) $P \Vdash$ "λ regular and $\mathrm{LCU}_R(\lambda)$" implies $P \times Q \Vdash \mathrm{LCU}_R(|\lambda|)$.

(ii) $P \Vdash$ "λ is regular and $\mathrm{COB}_R(\lambda, \mu)$" implies $P \times Q \Vdash \mathrm{COB}_R(|\lambda|, |\mu|)$.

Proof. As mentioned, Assumption 6.2 is met; in particular, P forces that $\check{Q}$ is $<\kappa$-distributive (by 6.2(P2)), we can use Lemma 3.1 and Corollary 3.5. Also note that, whenever $\kappa < \lambda \leq \theta$ and $P \Vdash$ "λ is regular", $P \times Q$ forces $\mathrm{cof}(\lambda) = \kappa = |\lambda|$. $\square$

So we can start, e.g., with a forcing P_0 as in Theorem 7.5: P_0 is $\lambda_{\mathfrak{p}}^+$-cc, and forces strictly increasing values to the characteristics in the first, say, sequence of Lemma 8.2.

We now pick some $\kappa_0 < \theta_0$, satisfying $\lambda_{\mathfrak{p}} < \kappa_0$ and the assumptions of the previous Lemma, i.e., κ_0 is regular and $\theta_0^{<\kappa_0} = \theta_0$. Let Q_0 be the collapse of θ_0 to κ_0, a forcing of size θ_0. So $P_1 := P_0 \times Q_0$ is θ_0^+-cc and, according to the previous

[24]Each sequence yields 2^{11} many consistency results (not all of them new, obviously; CH is one of them).

Lemma, still forces the "same" values (and in fact strong witnesses) to the Cichoń-characteristics (including the case that any value λ_i with $\kappa_0 < \lambda_i \leq \theta_0$ is collapsed to $|\lambda_i| = \kappa_0$). The $\mathfrak{m}$-like invariants below κ_0, i.e., $\mathfrak{m}$ and $\mathfrak{p}$, are also unchanged.

We now pick another pair $\theta_0 < \kappa_1 < \theta_1$ (with the same requirements) and take the product of P_1 with the collapse Q_1 of θ_1 to κ_1, etc.

In the end, we get $P_0 \times Q_0 \times \cdots \times Q_n$. Each characteristic which by P was forced to have value λ now is forced to have value $|\lambda|$, which is κ_m if $\kappa_m \leq \lambda \leq \theta_m$ for some m, and λ otherwise. This immediately gives the

Proof of Lemma 8.2. We start with GCH, and construct the inital forcing to already result in the desirect (in)equalities between $\aleph_1, \mathfrak{m}, \mathfrak{p}$ and to result in pairwise different regular Cichoń values λ_i and $\mathfrak{p} < \text{add}(\mathcal{N})$.

Let $(I_m)_{m \in M}$ be the interval partition of the sequence $(\mathfrak{p}, \text{add}(\mathcal{N}), \ldots, \mathfrak{c})$ indicating which characteristics we want to identify. For each non-singleton I_m, let κ_m be the value of the smallest characteristic in I_m, and θ_m the largest. Note that $\theta_m < \kappa_{m+1} < \theta_{m+1}$. Then $P_0 \times Q_0 \times \cdots \times Q_{M-1}$ forces that all characteristics in I_m have value κ_m, as desired. □

Similarly and easily we get the following:

Lemma 8.4. *We can assign the values $\aleph_1, \aleph_2, \ldots, \aleph_{12}$ to the first sequence of Lemma 8.2 (as in Figure 6).*

We can do the same for the second sequence.

Proof. Again, start with GCH and P_0 forcing the desired values for $\mathfrak{m}$ and $\mathfrak{p}$ (now $\aleph_2$ and $\aleph_3$) and pairwise distinct regular Cichoń values λ_i. Then pick $\kappa_0 = \lambda_{\mathfrak{p}}^+ = \aleph_4$ and $\theta_0 = \lambda_1$ (which then becomes $\aleph_4$ after the collapse). Then set $\kappa_1 = \lambda_1^+$ (which would be $\aleph_5$ after the first collapse), and $\theta_1 = \lambda_2$, etc. □

We can of course just as well assign the values $(\aleph_{\omega \cdot m+1})_{1 \leq m \leq 12}$ instead of $(\aleph_m)_{1 \leq m \leq 12}$. It is a bit awkward to make precise the (not entirely correct) claim "we can assign whatever we want"; nevertheless we will do just that in the rest of this section.

Theorem 8.5. *Assume GCH. Let $1 \leq \alpha_{\mathfrak{m}} \leq \alpha_{\mathfrak{p}} \leq \alpha_1 \leq \cdots \leq \alpha_9$ be a sequence of successor ordinals, and $\kappa_9 < \kappa_8 < \kappa_7$ compact cardinals with $\kappa_9 > \alpha_9$. Then for each of the following three statements*

(M1) $\mathfrak{m} = \mathfrak{m}(\textit{precaliber}) = \aleph_{\alpha_{\mathfrak{m}}}$*;*
(M2) $\mathfrak{m}(k\textit{-Knaster}) = \aleph_1$ *and* $\mathfrak{m}(k+1\textit{-Knaster}) = \aleph_{\alpha_{\mathfrak{m}}}$ *for a given* $1 \leq k < \omega$*;*
(M3) $\mathfrak{m}(k\textit{-Knaster}) = \aleph_1$ *for all* $1 \leq k < \omega$ *and* $\mathfrak{m}(\textit{precaliber}) = \aleph_{\alpha_{\mathfrak{m}}}$.

there is a poset P which forces the statement, and in addition

$$\mathfrak{p} = \aleph_{\alpha_{\mathfrak{p}}},\ \text{add}(\mathcal{N}) = \aleph_{\alpha_1},\ \text{cov}(\mathcal{N}) = \aleph_{\alpha_2},\ \mathfrak{b} = \aleph_{\alpha_3},\ \text{non}(\mathcal{M}) = \aleph_{\alpha_4},$$
$$\text{cov}(\mathcal{M}) = \aleph_{\alpha_5},\ \mathfrak{d} = \aleph_{\alpha_6},\ \text{non}(\mathcal{N}) = \aleph_{\alpha_7},\ \text{cof}(\mathcal{N}) = \aleph_{\alpha_8},\ \textit{and}\ \mathfrak{c} = \aleph_{\alpha_9}.$$

(Note that M1–M3 are from Theorem 6.5.)

Actually, we will prove something more general: We first formulate this more general result for the case $\alpha_1 < \alpha_2 < \alpha_3$; as explained in Remark 8.7, there are variants of the theorem which allow $\alpha_1 = \alpha_2$ and/or $\alpha_2 = \alpha_3$.

Theorem 8.6. *Assume GCH. Let $1 \leq \alpha_{\mathfrak{m}} \leq \alpha_{\mathfrak{p}} \leq \alpha_1 < \alpha_2 < \alpha_3 \leq \alpha_4 \leq \ldots \leq \alpha_9$ be ordinals and assume that there are strongly compact cardinals $\kappa_9 < \kappa_8 < \kappa_7$ such that*

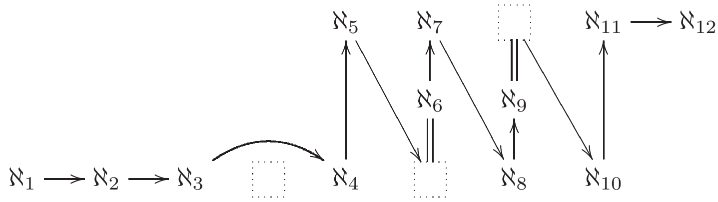

FIGURE 6. A possible assignment for Figure 2 (note that we lose control of $\mathfrak{h}$): $\mathfrak{m} = \aleph_2$, $\mathfrak{p} = \aleph_3$, $\lambda_i = \aleph_{3+i}$ for $i = 1, \ldots, 9$.

(i) $\alpha_{\mathfrak{p}} \leq \kappa_9$*,* $\alpha_1 < \kappa_8$ *and* $\alpha_2 < \kappa_7$*;*
(ii) for $i = 1, 2, 3$*,* $\aleph_{\beta_{i-1}+(\alpha_i - \alpha_{i-1})}$ *is regular,*[25] *where* $\beta_i := \max\{\alpha_i, \kappa_{10-i} + 1\}$ *and* $\alpha_0 = \beta_0 = 0$*;*
(iii) for $i \geq 4$*,* $i \neq 6$*,* $\aleph_{\beta_3+(\alpha_i-\alpha_3)}$ *is regular;*
(iv) $\operatorname{cof}(\aleph_{\beta_3+(\alpha_6-\alpha_3)}) \geq \aleph_{\beta_3}$*; and*
(v) $\aleph_{\alpha_{\mathfrak{m}}}$ *and* $\aleph_{\alpha_{\mathfrak{p}}}$ *are regular.*

Then, for each of (M1)–(M3), we get a poset P *as in the previous Theorem.*

Proof. We show the proof corresponding to (M1). For $4 \leq i \leq 9$ put $\beta_i := \beta_3 + (\alpha_i - \alpha_3)$. Also set $\lambda_{\mathfrak{m}} := \aleph_{\alpha_{\mathfrak{m}}}$, $\lambda_{\mathfrak{p}} := \aleph_{\alpha_{\mathfrak{p}}}$ and $\lambda_i := \aleph_{\beta_i}$ for $1 \leq i \leq 9$. Note that λ_i is regular for $i \neq 6$, $\operatorname{cof}(\lambda_6) \geq \lambda_3$ and $\lambda_{\mathfrak{m}} \leq \lambda_{\mathfrak{p}} \leq \kappa_9 < \lambda_1 < \kappa_8 < \lambda_2 < \kappa_7 < \lambda_3 \leq \lambda_4 \leq \ldots \leq \lambda_9$. In the case $\alpha_{\mathfrak{p}} < \alpha_1$ let P be the $\lambda_{\mathfrak{p}}^+$-cc poset corresponding to Theorem 6.5(b) (the modification of $P^{\mathtt{vA*}}$), otherwise let P be the ccc poset corresponding to Corollary 4.9 and forcing $\mathfrak{p} \geq \kappa_9$ and $\mathfrak{m} = \mathfrak{m}(\text{precaliber}) = \lambda_{\mathfrak{m}}$ (or just $\mathfrak{m} \geq \kappa_9$ when $\alpha_{\mathfrak{m}} = \kappa_9$).

Step 1. We first assume $\alpha_{\mathfrak{p}} < \alpha_1$. In the case $\kappa_9 < \alpha_1$ we have $\beta_1 = \alpha_1$, so let $P_1 := P$; in the case $\alpha_1 \leq \kappa_9$, we have $\beta_1 = \kappa_9 + 1$ and $\lambda_1 = \kappa_9^+$. Put $\kappa_1 := \aleph_{\alpha_1}$ and $P_1 := P \times \operatorname{Coll}(\kappa_1, \lambda_1)$. It is clear that κ_1 is regular and $\kappa_1 \leq \lambda_1$ so, by Lemma 8.3, P_1 forces $\operatorname{add}(\mathcal{N}) = \aleph_{\alpha_1}$ and that the values of the other cardinals are the same as in the P-extension. Even more, for any $\xi \geq \kappa_8$, P_1 forces $\aleph_\xi = \aleph_\xi^V$ because, in the ground model, κ_8 is an $\aleph$-fixed point between $\aleph_{\beta_1}$ and $\aleph_{\beta_2}$ (and thus between β_1 and β_2).

Now assume $\alpha_{\mathfrak{p}} = \alpha_1$ (so P is ccc) and let $\kappa_1 := \lambda_{\mathfrak{p}} = \aleph_{\alpha_1}$. When $\kappa_9 < \alpha_1$ set $P_1 := P \times (\kappa_1^{<\kappa_1})^V$, otherwise set $P_1 := P \times \operatorname{Coll}(\kappa_1, \lambda_1 \times \kappa_1)$. This poset forces the same as the above, but for $\mathfrak{p}$ we just now $\mathfrak{p} \geq \kappa_1$ (or just $\mathfrak{m} \geq \kappa_9$ when $\alpha_p = \kappa_9$), but $\mathfrak{p} \leq \kappa_1$ also holds because $(\kappa_1^{<\kappa_1})^V$ adds a tower of length κ_1 (see the proof of Lemma 6.4), and also because $\operatorname{Coll}(\kappa_1, \lambda_1 \times \kappa_1)$ adds a $\kappa_1^{<\kappa_1}$-generic function.

Step 2. In the case $\kappa_8 < \alpha_2$ put $P_2 := P_1$; otherwise, we have $\beta_2 = \kappa_8 + 1$ and $\lambda_2 = \kappa_8^+$. Set $\kappa_2 := \aleph_{\beta_1+(\alpha_2-\alpha_1)}$ and $P_2 := P_1 \times \operatorname{Coll}(\kappa_2, \lambda_2)$. It is clear that $\kappa_2 < \lambda_2$, so Lemma 8.3 applies, i.e., P_2 forces $\operatorname{cov}(\mathcal{N}) = \kappa_2$ and that the values of the other cardinals are the same as in the P_1-extension. Also note that P_1 forces $\kappa_2 = \aleph_{\alpha_2}$, and this value remains unaltered in the P_2-extension. Furthermore P_2 forces $\aleph_\xi = \aleph_\xi^V$ for any $\xi \geq \kappa_7$.

Step 3. In the case $\kappa_7 < \alpha_3$ put $P_3 := P_2$; otherwise, set $\kappa_3 := \aleph_{\beta_2+(\alpha_3-\alpha_2)}$ and $P_3 := P_2 \times \operatorname{Coll}(\kappa_3, \lambda_3)$. Note that P_3 forces $\mathfrak{b} = \kappa_3 = \aleph_{\alpha_3}$ and that the other values

[25] This is equivalent to say that α_i is either a successor ordinal or a weakly inaccessible larger than β_{i-1}.

are the same as forced by P_2. Hence, P_3 is as desired, e.g., $\mathrm{non}(\mathcal{M}) = \lambda_4 = \aleph^V_{\beta_4} = \aleph_{\alpha_4}$. □

Remark 8.7. Theorem 8.6 also holds when $\alpha_1 \leq \alpha_2 \leq \alpha_3$, but depending on the equalities the hypothesis may change. For example, in the case $\alpha_1 = \alpha_2 < \alpha_3$, hypothesis (ii) is modified by: $\beta_1 = \kappa_9 + 1$, $\beta_2 = \kappa_8 + 1$, $\beta_3 = \max\{\alpha_3, \kappa_7 + 1\}$ and both $\aleph_{\alpha_1}$ and $\kappa_3 := \aleph_{\beta_2+(\alpha_3-\alpha_2)}$ are regular. For the proof, the idea is first collapse $\lambda_2 := \aleph_{\beta_2}$ to $\kappa_1 := \aleph_{\alpha_1}$ (as in step 1 of the proof, considering similar cases for $\alpha_{\mathfrak{p}}$), and then (possibly) collapse $\lambda_3 := \aleph_{\beta_3}$ to κ_3 (as in step 3). This guarantees that the sequence of cardinals in the previous theorem is $\leq$-consistent.

A similar result (and remark about $\leq$-consistency) applies to `vB`*.

Theorem 8.8. *Assume GCH. Let* $1 \leq \alpha_{\mathfrak{m}} \leq \alpha_{\mathfrak{p}} \leq \alpha_1 < \alpha_2 < \alpha_3 \leq \alpha_4 \leq \ldots \leq \alpha_9$ *be ordinals and assume that there are strongly compact cardinals* $\kappa_9 < \kappa_8 < \kappa_7 < \kappa_6$ *such that*

(i) $\alpha_{\mathfrak{p}} \leq \kappa_9$, $\alpha_1 < \kappa_8$, $\alpha_2 < \kappa_7$, *and* $\alpha_3 < \kappa_6$*;*
(ii) for $i = 1, 2, 3, 4$, $\aleph_{\beta_{i-1}+(\alpha_i-\alpha_{i-1})}$ *is regular, where* $\beta_i := \max\{\alpha_i, \kappa_{10-i} + 1\}$ *and* $\alpha_0 = \beta_0 = 0$*;*
(iii) for $i \geq 6$, $\aleph_{\beta_4+(\alpha_i-\alpha_4)}$ *is regular;*
(iv) $\mathrm{cof}(\aleph_{\beta_4+(\alpha_5-\alpha_4)}) \geq \aleph_{\beta_4}$*;*
(v) β_3 *is* <u>*not*</u> *the successor of a cardinal with countable cofinality; and*
(vi) $\aleph_{\alpha_{\mathfrak{m}}}$ *and* $\aleph_{\alpha_{\mathfrak{p}}}$ *are regular.*

Then there is a poset that forces one of (M1)–(M3) of Theorem 6.5 (with value $\aleph_{\alpha_{\mathfrak{m}}}$*) and*

$$\mathfrak{p} = \aleph_{\alpha_{\mathfrak{p}}},\ \mathrm{add}(\mathcal{N}) = \aleph_{\alpha_1},\ \mathfrak{b} = \aleph_{\alpha_2},\ \mathrm{cov}(\mathcal{N}) = \aleph_{\alpha_3},\ \mathrm{non}(\mathcal{M}) = \aleph_{\alpha_4},$$
$$\mathrm{cov}(\mathcal{M}) = \aleph_{\alpha_5},\ \mathrm{non}(\mathcal{N}) = \aleph_{\alpha_6},\ \mathfrak{d} = \aleph_{\alpha_7},\ \mathrm{cof}(\mathcal{N}) = \aleph_{\alpha_8},\ \textit{and}\ \mathfrak{c} = \aleph_{\alpha_9}.$$

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