Models with second order properties V: A general principle

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Abstract. We present a general framework for carrying out the constructions in [2-10] and others of the same type. The unifying factor is a combinatorial principle which we present in terms of a game in which the first player challenges the second player to carry out constructions which would be much easier in a generic extension of the universe, and the second player cheats with the aid of ◇. §1 contains an axiomatic framework suitable for the description of a number of related constructions, and the statement of the main theorem 1.9 in terms of this framework. In §2 we illustrate the use of our combinatorial principle. The proof of the main result is then carried out in §§3-5.

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Contents

§1. Uniform partial orders.

We describe a class of partial orderings associated with attempts to manufacture an object of size λ^+ from approximations of size less than λ. We also introduce some related notions motivated by the forcing method. The underlying idea is that a sufficiently generic filter on the given partial ordering should give rise to the desired object of size λ^+.

We describe a game for two players, in which the first player imposes genericity requirements on a construction, and the second player constructs an object which meets the specified requirements. The main theorem (1.9) is that under certain combinatorial conditions the second player has a winning strategy for this game.

§2. Illustrative application.

We illustrate the content of our general principle with an example. We show the completeness of the logic $\mathcal{L}^{<\omega}$, defined by Magidor and Malitz [2] for the λ^+-interpretation assuming the combinatorial principles Dl_λ and $\diamondsuit_{\lambda^+}$.

§3. Commitments.

We give a preliminary sketch of the proof of Theorem 1.9. We then introduce the notion of "basic data" which is a collection of combinatorial objects derived from Dl_λ and an object called a commitment describing the main features of the second player's strategy in a given play of the genericity game. We state the main results concerning commitments, and show how Theorem 1.9 follows from these results.

§4. Proofs.

We prove the propositions stated in §3 except we defer the proof of Propositions 3.6 and 3.7 to section §5. We use Dl_λ to show that a suitable collection of "basic data" exists. Then we verify some continuity properties applying to our strategy at limit ordinals.

§5. Proof of Proposition 3.7.

We prove Proposition 3.7 as well as Proposition 3.6.

Notation

If $(A_\alpha : \alpha < \delta)$ is an increasing sequence of sets we write $A_{<\delta}$ for $\bigcup_{\alpha<\delta} A_\alpha$. (Note the exception arising in lemma 1.3.)

Throughout the paper, λ is a cardinal such that $\lambda^{<\lambda} = \lambda$.

$\mathcal{P}_{<\lambda}(A) = \{B \subseteq A : |B| < \lambda\}$.

otp (u) will mean the order type of u. Trees are well-founded, and if T is a tree, $\eta \in T$, we write len (η) for otp $\{\nu \in T : \nu < \eta\}$ (the level at which η occurs in T).

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1 Uniform partial orders

We will present an axiomatic framework for the construction of objects of size λ^+ from approximations of size less than λ, under suitable set theoretical hypotheses. The basic idea is that we are constructing objects which can fairly easily be forced to exist in a generic extension, and we replace the forcing construction by the explicit construction of a sufficiently generic object in the ground model.

We begin with the description of the class of partial orderings to which our methods apply. Our idea is that an "approximation" to the desired final object is built from a set of ordinals $u \subseteq \lambda^+$ of size less than λ. Furthermore, though there will be many such sets u, there will be at most λ constructions applicable to an arbitrary set u. We do not axiomatize the notion of a "construction" in any detail; we merely assume that the approximations can be coded by pairs (α, u), where $\alpha < \lambda$ is to be thought of as a code for the particular construction applied to u. An additional feature, suggested by the intuition just described, is captured in the "indiscernibility" condition below, which is a critical feature of the situation – though trivially true in any foreseeable application.

Definition 1.1 *A* standard λ^+-uniform *partial order is a partial order* $\le$ *defined on a subset* $\mathbb{P}$ *of* $\lambda \times \mathcal{P}_{<\lambda}(\lambda^+)$ *satisfying the following conditions, where for* $p = (\alpha, u)$ *in* $\mathbb{P}$ *we write* $\operatorname{dom} p = u$*, and call* u *the* domain *of* p*.*

1. *If* $p \le q$ *then* $\operatorname{dom} p \subseteq \operatorname{dom} q$*.*
2. *For all* $p, q, r \in \mathbb{P}$ *with* $p, q \le r$ *there is* $r' \in \mathbb{P}$ *so that* $p, q \le r' \le r$ *and* $\operatorname{dom} r' = \operatorname{dom} p \cup \operatorname{dom} q$*.*
3. *If* $(p_i)_{i<\delta}$ *is an increasing sequence of length less than* λ*, then it has a least upper bound* q*, with domain* $\bigcup_{i<\delta} \operatorname{dom} p_i$*; we will write* $q = \bigcup_{i<\delta} p_i$*, or more succinctly:* $q = p_{<\delta}$*.*
4. *For all* $p \in \mathbb{P}$ *and* $\alpha < \lambda^+$ *there exists a* $q \in \mathbb{P}$ *with* $q \le p$ *and* $\operatorname{dom} q = \operatorname{dom} p \cap \alpha$*; furthermore, there is a unique maximal such* q*, for which we write* $q = p \restriction \alpha$*.*
5. *For limit ordinals* δ*,* $p \restriction \delta = \bigcup_{\alpha<\delta} p \restriction \alpha$*.*

6. *If $(p_i)_{i<\delta}$ is an increasing sequence of length less than λ, then*

$$(\bigcup_{i<\delta} p_i)\restriction\alpha \;=\; \bigcup_{i<\delta}(p_i\restriction\alpha).$$

7. (Indiscernibility) *If $p=(\alpha,v)\in\mathbb{P}$ and $h: v\to v'\subseteq\lambda^+$ is an order-isomorphism then $(\alpha,v')\in\mathbb{P}$. We write $h[p]=(\alpha,h[v])$. Moreover, if $q\le p$ then $h[q]\le h[p]$.*

8. (Amalgamation) *For every $p,q\in\mathbb{P}$ and $\alpha<\lambda^+$, if $p\restriction\alpha\le q$ and* $\operatorname{dom}p\cap\operatorname{dom}q=\operatorname{dom}p\cap\alpha$*, then there exists $r\in\mathbb{P}$ so that $p,q\le r$.*

It should be remarked that a standard λ^+-uniform partial order comes with the additional structure imposed on it by the domain and restriction functions. We will call a partial order λ^+*-uniform* if it is *isomorphic* to a standard λ^+-uniform partial ordering. It follows that although a λ^+-uniform is isomorphic to a standard one *as a partial order*, there will be an induced notion of domain and restriction. The elements of such a partial order will be called *approximations*, rather than "conditions", as we are aiming at a construction in the ground model.

Observe that $p\restriction\alpha=p$ iff $\operatorname{dom}p\subseteq\alpha$. Note also that for $p\le q$ in $\mathbb{P}$, $p\restriction\alpha\le q\restriction\alpha$. (As $p\restriction\alpha, q\restriction\alpha\le q$, there is $r\le q$ in $\mathbb{P}$ with $p\restriction\alpha, q\restriction\alpha\le r$ and $\operatorname{dom}r=\operatorname{dom}p\restriction\alpha\cup\operatorname{dom}q\restriction\alpha=\operatorname{dom}q\restriction\alpha$; hence $r=q\restriction\alpha$ by maximality of $q\restriction\alpha$, and $p\restriction\alpha\le q\restriction\alpha$.)

It is important to realize that in intended applications there will be λ many comparable elements of a λ^+-uniform partial order which have the same domain (see the first example of the next section).

Typically the only condition that requires attention in concrete cases is the amalgamation condition. It is therefore useful to have a weaker version of the amalgamation property available which is sometimes more conveniently verified, and which is equivalent to the full amalgamation condition in the presence of the other (trivial) hypotheses. Such a version is:

> Weak Amalgamation. *For every $p,q\in\mathbb{P}$, and $\alpha<\lambda^+$, if $p\restriction\alpha\le q$,* $\operatorname{dom}p\subseteq\alpha+1$*, and* $\operatorname{dom}q\subseteq\alpha$*, then there exists $r\in\mathbb{P}$ with $p,q\le r$.*

To prove amalgamation from weak amalgamation, we define a continuous increasing chain of elements $r_\beta\in\mathbb{P}$ for $\beta\ge\alpha$ so that

1. $\mathrm{dom}\,(r_\beta) \subseteq \beta$ and

2. $r_\beta \geq p{\restriction}\beta, q{\restriction}\beta$.

Let $r_\alpha = q{\restriction}\alpha$. For limit ordinals, use conditions 3 and 5 of the definition of uniform partial order.

Suppose we have defined r_β and $\beta \notin \mathrm{dom}\,(p) \cup \mathrm{dom}\,(q)$. Let $r_{\beta+1} = r_\beta$.

If $\beta \in \mathrm{dom}\,(q) \setminus \mathrm{dom}\,(p)$ then $p{\restriction}\beta + 1 = p{\restriction}\beta$. Apply weak amalgamation to r_β and $q{\restriction}\beta$. Using condition 2 of the definition now, we can define $r{\restriction}\beta + 1$.

If $\beta \in \mathrm{dom}\,(p) \setminus \mathrm{dom}\,(q)$ then we can apply weak amalgamation to $p{\restriction}\beta + 1$ and r_β.

Since these are all the possibilities, let $\gamma = \sup(\mathrm{dom}\,(p) \cup \mathrm{dom}\,(q))$ and so $r_\gamma \geq p, q$. This verifies amalgamation.

Notation

For $p, q \in \mathbb{P}$ we write $p \leq_{\mathrm{sd}} q$ to mean $p \leq q$ and $\mathrm{dom}\, p = \mathrm{dom}\, q$. (Here "sd" stands for "same domain".) If $p, q \in \mathbb{P}$ then we write $p \perp q$ if p and q are incompatible i.e. there is no r so that $p \leq r$ and $q \leq r$.

We define the *collapse* p^{col} of an approximation as $h[p]$ where h is the canonical order isomorphism between $\mathrm{dom}\, p$ and $\mathrm{otp}\,(\mathrm{dom}\, p)$.

Convention

For the remainder of this section we fix a standard λ^+-uniform partial order $\mathbb{P}$, and we let

$$\mathbb{P}_\alpha = \{p \in \mathbb{P} : \mathrm{dom}\, p \subseteq \alpha\}$$

for $\alpha < \lambda^+$. Note that $\mathbb{P}_{\lambda^+} = \mathbb{P}$.

Be forewarned that the following definition does not follow the standard set theoretic use of the term "ideal".

Definition 1.2 *1. For $\alpha < \lambda^+$, a* λ-generic ideal *G in $\mathbb{P}_\alpha$ is a subset of $\mathbb{P}_\alpha$ satisfying:*

(a) G is closed downward;

(b) if $Q \subseteq G$ and $|Q| < \lambda$ then Q has an upper bound in G; and

(c) for every $p \in \mathbb{P}_\alpha$, if $p \notin G$ then p is incompatible with some $q \in G$.

$\mathrm{Gen}(\mathbb{P}_\alpha)$ *is the set of λ-generic ideals of $\mathbb{P}_\alpha$.*

2. *If $G \in \mathrm{Gen}(\mathbb{P}_\alpha)$ then*

$$\mathbb{P}/G = \{p \in \mathbb{P} : p \text{ is compatible with every } r \in G\}.$$

Note that $p \in \mathbb{P}/G$ iff $p \restriction \alpha \in G$.

3. *We say an increasing sequence $\langle g_i : i < \lambda \rangle$ is* cofinal *in $G \in \mathrm{Gen}(\mathbb{P}_\alpha)$ if $G = \{r \in \mathbb{P}_\alpha :$ for some i, $r \leq g_i\}$. Every $G \in \mathrm{Gen}(\mathbb{P}_\alpha)$ has a cofinal sequence of length λ (possibly constant in degenerate cases). We will often write $(g_\delta)_\delta$ to mean $\langle g_\delta : \delta < \lambda \rangle$.*

4. *We will say that G is generic if $G \in \mathrm{Gen}(\mathbb{P}_\alpha)$ for some α.*

Lemma 1.3 *Let $G_i \in \mathrm{Gen}(\mathbb{P}_{\alpha_i})$ for $i < \delta$ be an increasing sequence of sets, and $\alpha = \sup_i \alpha_i$. Then there is a unique minimal λ-generic ideal of $\mathbb{P}_\alpha$ containing $\bigcup_{i<\delta} G_i$. This ideal will be denoted $G_{<\delta}$.*

Proof: We may suppose that δ is a regular cardinal, $\delta \leq \lambda$. If $\delta = \lambda$ then it is clear that $\bigcup_{i<\delta} G_i \in \mathrm{Gen}(\mathbb{P}_\alpha)$. Suppose now that $\delta < \lambda$. For $i < \delta$ fix an increasing continuous sequence $(g^i_\gamma)_{\gamma<\lambda}$ cofinal in G_i. Fix $i < j < \delta$. There is a club C_{ij} in λ such that for all $\gamma \in C_{ij}$, $g^i_\gamma = g^j_\gamma \restriction \alpha_i$. Let $C = \bigcap_{i<j<\delta} C_{ij}$. If $\beta \in C$ then define $g_\beta = \bigcup_{i<\delta} g^i_\beta \in \mathbb{P}_\alpha$. Then the downward closure of $(g_\beta : \beta < \lambda)$ is the required generic set in $\mathbb{P}_\alpha$. □

The notion of λ-genericity is of course very weak. In order to get a notion adequate for the applications, we need to formalize the notion of a *uniform* family of dense sets.

Definition 1.4

1. *For $\alpha < \lambda^+$ and $G \in \mathrm{Gen}(\mathbb{P}_\alpha)$ or $G = \emptyset$ (in which case, in what follows, read $\mathbb{P}$ for $\mathbb{P}/G$) we say*

$$D : \{(u, w) : u \subseteq w \in \mathcal{P}_{<\lambda}(\lambda^+)\} \to \mathcal{P}(\mathbb{P})$$

is a density system *over G if:*

(a) for every (u,w), $D(u,w) \subseteq \{p \in \mathbb{P}/G : \operatorname{dom} p \subseteq w\}$,

(b) for every $p, q \in \mathbb{P}/G$, *if* $p \in D(u,w)$, $p \le q$ *and* $\operatorname{dom} q \subseteq w$ *then* $q \in D(u,w)$,

(c) (Density) *For every* (u,w) *and every* $p \in \mathbb{P}/G$, *with* $\operatorname{dom} p \subseteq w$, *there is* $q \ge p$ *in* $D(u,w)$; *and*

(d) (Uniformity) *For every* $(u_1,w_1),(u_2,w_2)$, *if* $w_1 \cap \alpha = w_2 \cap \alpha$ *and there is an order-isomorphism* $h : w_1 \to w_2$ *such that* $h[u_1] = u_2$, *then for every* $p \in \mathbb{P}/G$ *with* $\operatorname{dom} p \subseteq w_1$

$$p \in D(u_1,w_1) \text{ iff } h[p] \in D(u_2,w_2).$$

The term "density system" will refer to density systems over some $G \in \mathrm{Gen}(\mathbb{P}_\alpha)$, *for some* α, *and we write "0-density system" for density system over* $\emptyset$.

2. *For* $G \in \mathrm{Gen}(\mathbb{P}_\gamma)$ *and* D *any density system, we say* G meets D *if for all* $u \in \mathcal{P}_{<\lambda}(\gamma)$ *there is* $v \in \mathcal{P}_{<\lambda}(\gamma)$ *so that* $u \subseteq v$ *and* $G \cap D(u,v) \neq \emptyset$.

We give now two examples of density systems which will be important in the proof of Theorem 1.9. Both examples use the following notion. A closed set X of ordinals will be said to be λ-*collapsed* if $0 \in X$ and for any $\alpha \le \sup X$, $[\alpha, \alpha+\lambda] \cap X \neq \emptyset$. An order isomorphism $h : Y \leftrightarrow X$ between closed sets of ordinals will be called a λ-*isometry* if for every pair $\alpha \le \beta$ in Y and every $\delta < \lambda$, $\beta = \alpha + \delta$ iff $h(\beta) = h(\alpha) + \delta$. Every closed set of ordinals is λ-isometric with a unique λ-collapsed closed set; the corresponding λ-isometry will be called the λ-collapse of Y, and more generally the λ-collapse of any set Y of ordinals is defined as the restriction to Y of the λ-collapse of its closure. Observe that a λ-collapsed set of fewer than λ ordinals is bounded below $\lambda \times \lambda$ (ordinal product).

Example 1.5 We shall show that there is a family $\mathcal{D}$ of at most λ 0-density systems such that for any $\alpha < \lambda^+$, if $G \in \mathrm{Gen}(\mathbb{P}_\alpha)$ meets all $D \in \mathcal{D}$ then $\mathbb{P}/G$ is again λ^+-uniform. (The amalgamation property must be verified.)

Construction

For $p, q \in \mathbb{P}_{\lambda\times\lambda}$ and $\delta < \lambda \times \lambda$ (where $\lambda \times \lambda$ is the *ordinal* product), we define a density system $D_{p,q,\delta}$ as follows. Let $u = (\operatorname{dom} p \cup \operatorname{dom} q) \cap \delta$. For

$$u' \subseteq w' \in \mathcal{P}_{<\lambda}(\lambda^+),$$

if there is an order-isomorphism $h : w' \to w \subseteq \delta$ with $h[u'] = u$, then let

$$D_{p,q,\delta}(u', w') = \{r : \operatorname{dom} r \subseteq w' \text{ and either there does not exist } s \geq p, q, h[r], \text{ or there exists } s \geq p, q \text{ so that } s{\restriction}\delta = h[r]\}.$$

This definition is independent of the choice of h.

If there is no such h then let $D_{p,q,\delta}(u', w') = \{r : \operatorname{dom} p \subseteq w'\}$. We claim that $D_{p,q,\delta}$ is a 0-density system. It suffices to check the density condition for $u \subseteq w \subseteq \delta$, and this is immediate.

Application

We will now show that if $G \in \operatorname{Gen}(\mathbb{P}_\alpha)$ meets every density system of the form $D_{p,q,\delta}$ then $\mathbb{P}/G$ is λ^+-uniform. In order to view $\mathbb{P}/G$ as encoded by elements of $\lambda \times \mathcal{P}_{<\lambda}(\lambda^+)$, we let $h : \lambda^+ \backslash \alpha \leftrightarrow \lambda^+$ be an order isomorphism, and replace (β, u) in $\mathbb{P}/G$ by $(\beta', h[u \backslash \alpha])$ where β' is just a code for the pair $(\beta, u \cap \alpha)$. We need only check the amalgamation condition (8) of the definition.

Let $p, q \in \mathbb{P}/G$, $\beta < \lambda^+$ with $p{\restriction}\beta \leq q$ and $\operatorname{dom} q \cap \operatorname{dom} p = \operatorname{dom} p \cap \beta$. We must find $r \geq p, q$ with $r \in \mathbb{P}/G$. Let $X = \operatorname{dom} p \cup \operatorname{dom} q \cup \{\alpha\}$ and let $h_0 : X \to X'$ be the λ-collapse of X. Let $p' = h_0[p], q' = h_0[q]$, $\alpha' = h_0(\alpha)$, and $u = \operatorname{dom} q \cap \alpha$. Now choose $w \subseteq \alpha$ with $|w| < \lambda$ and $r \in G \cap D_{p',q',\alpha'}(u, w)$. Since X' is λ-isomorphic with X, we can extend h_0 to an order-isomorphism

$$h : X \cup w \to X' \cup w' \text{ with } h[w] = w' \subseteq \alpha'.$$

We claim that there is $s \geq p', q', h[r]$. It suffices to find some $s \geq p, q, r$. Since $p{\restriction}\alpha, q{\restriction}\alpha, r$ are all in G, we may take $r' \geq p{\restriction}\alpha, q{\restriction}\alpha, r$ in G. Since $q \in \mathbb{P}/G$ and $r' \in G$ then by amalgamation we can find $\hat{q} \geq q, r'$ with $\operatorname{dom} \hat{q} = \operatorname{dom} q \cup \operatorname{dom} r'$. But now $\operatorname{dom} p \cap \operatorname{dom} \hat{q} = \operatorname{dom} p \cap \beta$ and $p{\restriction}\beta \leq \hat{q}$, so we can find $s \geq p, \hat{q}$. This is the desired s.

As $r \in D_{p',q',\alpha'}(u, w)$, it now follows that there exists $s \geq p', q'$ so that $s{\restriction}\alpha' = h[r]$, and hence $h^{-1}[s] \geq p, q$ and $(h^{-1}[s]){\restriction}\alpha = r$. So $h^{-1}[s] \in \mathbb{P}/G$ and $h^{-1}[s] \geq p, q$, verifying condition (8) for $\mathbb{P}/G$.

Example 1.6 The next example will be useful in the following situation. Suppose we have $G \in \operatorname{Gen}(\mathbb{P}_\alpha)$, $\beta > \alpha$, and we want to build $G' \supseteq G$ with

$G' \in \mathrm{Gen}(\mathbb{P}_\beta)$. To ensure the genericity of G' we must arrange that for all $q \in \mathbb{P}_\beta$, either $q \in G'$ or else q is incompatible with some $g \in G'$. We will find another family of at most λ 0-density systems $D_{p,q,r,\delta}$ which make it possible to construct a suitable $G' \supseteq G$ if G meets all $D_{p,q,\delta}$ (from Example 1.5) and $D_{p,q,r,\delta}$.

Construction

For $p, q, r \in \mathbb{P}_{\lambda\times\lambda}$, $\delta < \lambda \times \lambda$ such that:

$$p\restriction\delta \leq r;\ \mathrm{dom}\, r \subseteq \delta;\ \text{and there does not exist } s \geq p, q, r,$$

we define $D_{p,q,r,\delta}$ as follows

Let $u = (\mathrm{dom}\, p \cup \mathrm{dom}\, q) \cap \delta$. For $u' \subseteq w' \in \mathcal{P}_{<\lambda}(\lambda^+)$, if there is an order-isomorphism $h : w' \to w$ where $w \subseteq \delta$ and $h[u'] = u$ then let

$$\begin{aligned} D_{p,q,r,\delta}(u', w') \ = \ \{ & s : \mathrm{dom}\, s \subseteq w' \text{ and } h[s] \text{ is incompatible with } r, \\ & \text{or } h[s] \geq r \text{ and there is some } t \geq p \text{ so that} \\ & t\restriction\delta \leq h[s] \text{ and } t \text{ is incompatible with } q\}. \end{aligned}$$

If there is no such h then let $D_{p,q,r,\delta}(u', w') = \{s : \mathrm{dom}\, s \subseteq w'\}$.

We claim that $D_{p,q,r,\delta}$ is a 0-density system. Again we check only the density condition for $u \subseteq w \subseteq \delta$. So suppose we have $s \in \mathbb{P}$, $\mathrm{dom}\, s \subseteq w$, and s is compatible with r. We seek $s' \geq s$ in $D_{p,q,r,\delta}(u, w)$.

Choose $s' \geq r, s$ with domain $\mathrm{dom}\, r \cup \mathrm{dom}\, s$; so $\mathrm{dom}\, s' \subseteq \delta$. Then $s' \geq r \geq p\restriction\delta$ and $\mathrm{dom}\, s' \cap \mathrm{dom}\, p = \mathrm{dom}\, p \cap \delta$, so we can choose $t \geq s', p$ so that $\mathrm{dom}\,(t) = \mathrm{dom}\, s' \cup \mathrm{dom}\, p$, and hence t is incompatible with q (since there is no $t' \geq p, q, r$). Now $t\restriction\delta \geq s' \geq r, s$, so if $s'' = t\restriction\delta$ then $s'' \geq r, s$, and $s'' \in D_{p,q,r,\delta}(u', w')$ as desired.

Application

We return to the situation in which we have $G \in \mathrm{Gen}(\mathbb{P}_\alpha)$, $\beta > \alpha$, and we want to build $G' \supseteq G$ with $G' \in \mathrm{Gen}(\mathbb{P}_\beta)$, assuming that G meets all $D_{p,q,\delta}$ and $D_{p,q,r,\delta}$. We will naturally take G' to be the downward closure of a sequence $(g_i)_{i<\lambda}$ which is constructed inductively, taking suprema at limit ordinals. At successor stages, suppose that the i-th term of our sequence has just been constructed, and let $p = g_i$. Suppose $q \in \mathbb{P}_\beta$ is fixed. We wish to "decide" q: that is, we seek $\hat{p} \geq p$ so that either $\hat{p}$ is incompatible with q, or else $\hat{p} \geq q$.

If p is already incompatible with q then let $\hat{p} = p$. Otherwise, let $X = \operatorname{dom} p \cup \operatorname{dom} q \cup \{\alpha\}$ and let $h : X \to X'$ be the λ-collapse of X. Let $p' = h[p]$, $q' = h[q]$, and $\alpha' = h(\alpha)$. If $u = X \cap \alpha$, choose $w \supseteq u$ and $r \in G \cap D_{p',q',\alpha'}(u, w)$. Extend h to an order-preserving function from $X \cup w$ to $X' \cup w' \subseteq \alpha'$, and let $r' = h[r]$.

Suppose first that there is some $s \geq p', q'$ with $s \restriction \alpha' \leq r'$. We may suppose that $\operatorname{dom} s = \operatorname{dom} p' \cup \operatorname{dom} q'$. In this case let $\hat{p} = h^{-1}[s]$. As $\hat{p} \restriction \alpha \leq r$, we have $\hat{p} \in \mathbb{P}/G$, and q is decided by $\hat{p}$.

Now suppose alternatively that there is no $s \geq p', q', r'$. We may assume that $p \restriction \alpha \leq r$ since $p \restriction \alpha \in G$ and G is directed. Let:

$$Y = \operatorname{dom} p \cup \operatorname{dom} q \cup \operatorname{dom} r \cup \{\alpha\},$$

and let $k : Y \to Y''$ be the λ-collapse of Y. Let $p'' = k(p), q'' = k(q), r'' = k(r)$, and $\alpha'' = k(\alpha)$. Then $p'' \restriction \alpha'' \leq r''$, and there is no $s \geq p'', q'', r''$.

Let $v = (\operatorname{dom} q \cup \operatorname{dom} r) \cap \alpha$, and choose $z \supseteq v$ and $s \in G \cap D_{p'',q'',r'',\alpha''}(v, z)$. We can extend k to an order-isomorphism from $Y \cup z$ to $Y'' \cup z''$ with $k[z] = z'' \subseteq \alpha''$. Let $s'' = k[s]$.

Certainly r'' and s'' are compatible since $r, s \in G$. As s belongs to $D_{p'',q'',r'',\alpha''}(u, z)$, we have $k[s] \geq r''$, and there is some $t'' \geq p''$ so that $t'' \restriction \alpha'' \leq s''$ and t'' is incompatible with q''; in other words, $s \geq r$, and there is some $\hat{p} \geq p$ so that $\hat{p} \restriction \alpha \leq s$ and $\hat{p}$ is incompatible with q. Then $\hat{p} \in \mathbb{P}/G$, and $\hat{p}$ decides q. □

We now introduce the *genericity game*. Our main theorem will state that the second player has a winning strategy in this game, under certain combinatorial conditions.

Definition 1.7 *Let $\mathbb{P}$ be a λ^+-uniform partial order. The* genericity game *for $\mathbb{P}$ is the two-player game of length λ^+ played according to the following rules:*

1. *At the α^{th} move, player II will have chosen an increasing sequence of ordinals $\zeta_\beta < \lambda^+$, and will have defined an increasing sequence of λ-generic ideals G_β on $\mathbb{P}_{\zeta_\beta}$ for all $\beta < \alpha$. Player I will choose an element $g_\alpha \in \mathbb{P}/G_{<\alpha}$ and will also choose at most λ density systems D_i^α over $G_{<\alpha}$. Note that $G_{<\alpha} \in \operatorname{Gen}(\mathbb{P}_{\zeta_{<\alpha}})$ by Lemma 1.3.*

2. *After player I has played his* α^{th} *move, player II will pick an ordinal* ζ_α *and a* λ*-generic ideal of* $\mathbb{P}_{\zeta_\alpha}$.

Player II wins the $\mathbb{P}$*-game if the sequences* ζ_α *and* G_α *are increasing, and for all* α*, and all indices* i *occurring at stage* α*:* $g_\alpha \in G_\alpha$*, and for all* $\beta \geq \alpha$*,* G_β *meets* D_i^α.

Our main theorem uses the following combinatorial principle.

Definition 1.8 *Suppose* λ *is a regular cardinal.* Dl_λ *asserts that there are sets* $\mathcal{A}_\alpha \subseteq \mathcal{P}(\alpha)$*,* $|\mathcal{A}_\alpha| < \lambda$ *for every* $\alpha < \lambda$*, such that for all* $A \subseteq \lambda$*:*

$$\{\alpha \in \lambda : A \cap \alpha \in \mathcal{A}_\alpha\} \text{ is stationary.}$$

Easily, $\diamondsuit_\lambda$ or λ strongly inaccessible (or even $\lambda = \aleph_0$) implies Dl_λ. Also, Kunen showed that Dl_{λ^+} implies $\diamondsuit_{\lambda^+}$. Gregory has shown that if GCH holds and $\mathrm{cf}(\kappa) > \aleph_0$ then $\diamondsuit_{\kappa^+}$ holds. It is useful to note that Dl_λ implies $\lambda^{<\lambda} = \lambda$.

Theorem 1.9 Dl_λ *implies that player II has a winning strategy for the* $\mathbb{P}$*-game.*

This theorem will be proved in §§3-5. We illustrate its use in the next section.

2 Illustrative application

In this section we give an example of an application of the combinatorial principle described in section 1.

In [2], Magidor and Malitz introduce a logic $\mathcal{L}^{<\omega}$ which has a new quantifier Q^n for each $n \in \omega$, in addition to the usual first order connectives and quantifiers. The κ-interpretation of the formula $Q^n\varphi(x_1,\ldots,x_n,\bar{y})$ is

> "there is a set A of cardinality κ so that for any $x_1,\ldots,x_n \in A$, $\varphi(x_1,\ldots,x_n,\bar{y})$ holds."

They then give a list of axioms which are sound for the κ-interpretation when κ is regular, and show that these axioms are complete for the $\aleph_1$-interpretation under the assumption of $\diamondsuit_{\aleph_1}$. They ask whether these axioms are complete for the λ^+-interpretation. We will show that their axioms are complete when both Dl_λ and $\diamondsuit_{\lambda^+}(\{\delta < \lambda^+ : \mathrm{cf}(\delta) = \lambda\})$ hold. This will explain a remark at the end of [5]. See Hodges ([1]) for a treatment in the same vein for the $\aleph_1$-interpretation.

Fix a complete $\mathcal{L}^{<\omega}$ theory T, $|T| \leq \lambda$. Let $Q = Q^1$. We may assume that that associated to each formula φ with free variables $x_1,\ldots,x_n,y_1,\ldots,y_m$, $\mathcal{L}$ contains an $(m+1)$-ary function F_φ, so that T proves $Qx(x=x)$ and, for any fixed $y_1,\ldots y_m$, $F_\varphi(-,y_1,\ldots,y_m)$ is one-to-one and

$$Q^n\bar{x}\varphi(x_1,\ldots,x_n,y_1,\ldots,y_m) \to \\ \forall z_1\ldots z_n(\bigwedge_{i<j} z_i \neq z_j \to \bigwedge_{\sigma\in Sym(n)} \varphi(F_\varphi(z_{\sigma(1)},\bar{y}),\ldots,F_\varphi(z_{\sigma(n)},\bar{y}),\bar{y})).$$

Strictly, it is not necessary to make this conservative extension to our language and theory but it is convenient when handling the inductive step corresponding to Q^n.

We add new constants $\{y_\alpha : \alpha < \lambda^+\}$ and $\{x_i^\alpha : \alpha < \lambda^+, i < \lambda\}$ to $\mathcal{L}$, to obtain a language $\mathcal{L}_1$. The set of constants $\{y_\alpha\} \cup \{x_i^\alpha : i < \lambda\}$ is called the set of α-constants and y_α is called the *special* α-constant. A constant is said to be a w-constant if it is a β-constant for some $\beta \in w$; in particular a constant is a $(<\alpha)$-constant if it is a β-constant for some $\beta < \alpha$.

We define a partial order $\mathbb{P}$ as follows: $p \in \mathbb{P}$ iff

1. p is a set of $\mathcal{L}_1^{<\omega}$ sentences consistent with T;

2. $|p| < \lambda$;

3. p is closed under conjunction and existential quantification; and

4. if $\varphi(y_\alpha, \bar{z}) \in p$ and the $\bar{z}$ are $(< \alpha)$-constants, then $Qy\varphi(y, \bar{z}) \in p$.

We now indicate how $\mathbb{P}$ may be viewed as a standard λ^+-uniform partial order. We order $\mathbb{P}$ by inclusion. Let $\mathbb{P}_\alpha$ be

$$\{p \in \mathbb{P} : \text{all constants occurring in a formula of } p \text{ are } (< \alpha)\text{-constants}\}.$$

The elements of $\mathbb{P}_\lambda$ will be called *templates*. For any template p, there is a least β so that all formulas in p use only constants from $\{y_i : i < \beta\} \cup \{x^i_j : i < \beta, j < \lambda\}$. Call this β_p.

For any template p and any $w \subseteq \lambda^+$ so that $\operatorname{otp}(w) = \beta_p$, fix an order-isomorphism $h : \beta_p \to w$. Define $p(w)$ as the set of formulas obtained by replacing x^i_j and y_i by $x^{h(i)}_j$ and $y_{h(i)}$ respectively for $i < \beta_p$. Every element of $\mathbb{P}$ can be obtained in this way from a template.

Let ι be any bijection between the set of templates and λ. Identify $\mathbb{P}$ with the set $\{(\iota[p], w) : p \in \mathbb{P}_\lambda, w \in \mathcal{P}_{<\lambda}(\lambda^+) \text{ where } \operatorname{otp}(w) = \beta_p\}$ by sending $(\iota[p], w)$ to $p(w)$. Throughout the rest of this section we will treat $\mathbb{P}$ as if it were in standard for although in practice we will use its original definition. We claim that $\mathbb{P}$ is λ^+-uniform; it suffices to check the amalgamation condition 8.

The following notation will be convenient. If $\varphi(y_{\alpha_1}, \bar{z}_1, y_{\alpha_2}, \bar{z}_2, \ldots, y_{\alpha_n}, \bar{z}_n)$ is a formula with $\alpha_1 > \alpha_2 > \ldots > \alpha_n$, and $\bar{z}_i$ is a collection of $[\alpha_{i+1}, \alpha_i)$-constants, then the string S of quantifiers:

$$\exists \bar{x}_n Q y_n \ldots \exists \bar{x}_1 Q y_1$$

is called *standard* for φ where the x's quantify over the z's. Its dual is denoted S^*:

$$\forall \bar{x}_n \neg Q y_n \neg \ldots \forall \bar{x}_1 \neg Q y_1 \neg$$

If p is a set of fewer than λ formulas of $\mathcal{L}_1^{<\omega}$ which is closed under conjunction, then the following are equivalent:

1. $p \subseteq q$ for some $q \in \mathbb{P}$;

2. $S\varphi \in T$ for all $\varphi \in p$ where S is standard for φ.

For $p \in \mathbb{P}$ and $\alpha < \lambda^+$, we have:

$$p \restriction \alpha = \{\varphi \in p : \text{ all constants in } \varphi \text{ are } < \alpha\text{-constants }\}.$$

To show that $\mathbb{P}$ satisfies amalgamation, we will show that it satisfies weak amalgamation. Suppose $p \in \mathbb{P}_{\alpha+\nVdash}$, $q \in \mathbb{P}_\alpha$ and $p{\restriction}\alpha \le q$.

Suppose $\varphi(\bar{x}, y_\alpha, \bar{z}) \in p$ where $\bar{x}$ is all the α-variables except y_α and $\bar{z}$ is the $< \alpha$-variables. Then

$$Qy\exists\bar{x}\varphi(\bar{x}, y, \bar{z}) \in p{\restriction}_\alpha.$$

If $\psi \in q$ then

$$SQy\exists\bar{x}(\psi \wedge \varphi)$$

where S is a standard sequence, is equivalent to

$$S(\psi \wedge Qy\exists\bar{x}\varphi).$$

Since both of the conjuncts are in q, this last sentence is in T. This verifies weak amalgamation.

Now the strategy is to build a set G which is the union of generics so that the constant structure derived from G will form a model of T under the λ^+-interpretation. More precisely, we introduce an equivalence relation $\sim$ on the set of nonspecial constants $A = \{x^\alpha_j : \alpha < \lambda^+, j < \lambda\}$ by:

$$a \sim b \text{ iff } \text{“}a = b\text{”} \in G.$$

Let $\bar{G} = \{a/{\sim} : a \in A\}$ and define the functions and relations on $\bar{G}$ in the usual manner. We want to ensure that for any formula φ in $\mathcal{L}_1^{<\omega}$ we will have:

$$\bar{G} \models \varphi(a_1/{\sim}, \ldots, a_n/{\sim}) \text{ iff } \varphi(a_1, \ldots, a_n) \in G. \tag{1}$$

If (1) is true, its proof naturally proceeds by induction on the complexity of formulas. We now describe a strategy for Player I in the genericity game which can only be defeated by achieving (1). In other words, we will specify density systems and elements $g \in \mathbb{P}$, to be played by Player I, such that a proper response by Player II ensures that G allows an inductive argument for (1) to be carried out. Our discussion will be somewhat informal, stopping well short of actually writing down the density systems in many cases.

We begin with the treatment of the ordinary existential quantifier. Whenever $\exists x \varphi(x, \bar{z}) \in G$ we will want (eventually) to have some $a \in A$ so that $\varphi(a, \bar{z}) \in G$. In particular, for every α there will be some $a \in A$ so that $y_\alpha = a \in G$. The density systems which ensure this condition is met will in fact be 0-density systems.

Next we consider the quantifier Q. For each formula $Qx\varphi(x)$ which is put into G, at cofinally many subsequent stages we wish to add the formula $\varphi(y)$ for an unused special constant y. The first player will play such formulas as "g_α" from time to time. We will also have to deal with the case in which $\neg Qx\varphi(x)$, and we will return to this in a moment.

We now consider the quantifier Q^n. Suppose that the formula $Q^n \bar{x} \varphi(\bar{x}, \bar{y})$ is in G at some stage. This is where we use the function F_φ. If $\bar{G}$ is a model of T then it has cardinality λ^+. Moreover, $F_\varphi(-, \bar{y})$ is one-to-one. Since $Q^n \bar{x} \varphi(\bar{x}, \bar{y})$ is in G, so is

$$\forall z_1 \ldots z_n (\bigwedge_{i<j} z_i \neq z_j \rightarrow \bigwedge_{\sigma \in Sym(n)} \varphi(F_\varphi(z_{\sigma(1)}, \bar{y}), \ldots, F_\varphi(z_{\sigma(n)}, \bar{y}), \bar{y})).$$

It follows that the range of $F_\varphi(-, \bar{y})$ is homogeneous for φ.

We are now left with the cases in which formulas of the form $\neg Q^n x \varphi$ $(n \geq 1)$ are placed in G. We deal first with the case $n = 1$. For this case, we define a number of density systems depending on the following parameters:

1. $j, j_0, \ldots, j_{m-1} < \lambda$;
2. a formula $\varphi(x, y_0, \ldots, y_{m-1})$; and
3. a function $f : m \rightarrow m$.

We associate with these data a density system D. If $\operatorname{otp}(u) \neq m + 1$, we let $D(u, w)$ be degenerate:

$$D(u, w) = \{p \in \mathbb{P} : \operatorname{dom}(\imath) \subseteq \precapprox\}.$$

If $\operatorname{otp}(u) = m + 1$ then let $g : m + 1 \rightarrow u$ be an order preserving map, let $h = gf$ and set $\beta = g(m) = \max u$, and:

$$\psi(x) = \varphi(x, x^{h(0)}_{j_0}, \ldots, x^{h(m-1)}_{j_{m-1}}).$$

We will then let $D(u, w)$ consist of those $p \in \mathbb{P}$ for which, setting $\alpha = \min(w)$, we have:

1. $\mathrm{dom}\,(p) \subseteq w$;
2. If $\neg Qx\psi(x) \in p$, then either $\neg\psi(x_j^\beta) \in p$ or $x_j^\beta = x_i^\alpha \in p$ for some $i < \lambda$.

We shall verify the density condition on D. Suppose $q \in \mathbb{P}$ and $\neg Qx\psi(x) \in q$. The extension of q we are about to construct will only involve the adjunction of formulas with $(\leq \beta)$-constants, so we may assume that q itself contains only $(\leq \beta)$-constants.

If we cannot complete $q \cup \{\neg\psi(x_j^\beta)\}$ to an element of $\mathbb{P}$, then there is some $\chi \in q$ so that:

$$S\exists x(\chi \wedge \neg\psi(x)) \notin T$$

where $S\exists x$ is a standard sequence for the formula $\chi \wedge \neg\psi$. Note that by the assumption that β is the maximal element of $\mathrm{dom}\,(q)$, we may assume that the final quantifier in the standard sequence is an existential quantifier on the constant x in ψ.

By the axioms for the Q-quantifier, for any $\theta \in q$ such that $T \vdash \theta \to \chi$,

$$S\exists x(\theta \wedge \psi(x)) \in T.$$

As $\neg Qx\psi(x) \in q$, repeated use of the Q-quantifier axiom:

$$Qx\exists y\Delta(x,y) \to \exists xQy\Delta(x,y) \vee Qy\exists x\Delta(x,y)$$

shows that $\exists xS(\theta \wedge \psi(x)) \in T$.

If we now choose a constant x_i^α not occurring in q, where $\alpha = \min(w)$, one can conclude that $q \cup \{x_j^\beta = x_i^\alpha\}$ can be completed to an element of $\mathbb{P}$.

It is easy to see that if the foregoing density systems are met, then we can carry out the argument from right to left in condition (1) above for $\varphi = Qx\psi(x)$. We turn now to the treatment of the quantifiers Q^n for $n > 1$.

By applying Fodor's Lemma to the map sending δ to $\mathrm{dom}\,(f(\delta)) \cap \delta$ we obtain:

Lemma 2.1 *If $S \subseteq \{\delta : cf(\delta) = \lambda\}$ is stationary and $f : S \to \mathbb{P}$ then there is a stationary $S' \subseteq S$, a template p and $\sigma < \lambda^+$ so that for $\delta \in S'$, $f(\delta) = p(w_\delta)$ where $w_\delta = \mathrm{dom}\,(f(\delta))$ and $w_\delta \cap \delta \subseteq \sigma$.*

It will be convenient to treat conditions as if they were single formulas. Extending our previous notation, for $p \in \mathbb{P}$ and S a standard sequence covering some of the variables in p, we will write $S(p)$ for the set:

$$\{S_\varphi\varphi : \varphi \in p\}$$

where S_φ is the standard sequence for φ which we think of as a subsequence of the possibly infinite standard sequence S.

Let $(A_\delta)_{\mathrm{cf}\,(\delta)=\lambda}$ be a $\diamondsuit$-sequence. For u, v sets of ordinals, we write $u < v$ if for all $\beta \in u$, $\beta < \min v$.

If $\mathrm{cf}\,(\delta) = \lambda$ and $G_\delta \in \mathrm{Gen}(\mathbb{P}_\delta)$, we will define certain associated density systems over G_δ which depend on the following additional parameters:

1. an $i < \lambda$;
2. a formula $\varphi(x_1, \ldots, x_n, \bar{y})$ (we will suppress the $\bar{y}$);
3. some k with $0 \leq k < n$;
4. templates $p_1, \ldots, p_k$; and
5. ordinals $\gamma_j < \beta_{p_j}$ for $1 \leq j \leq k$.

The density system D that depends on this particular set of parameters will be taken to have $D(u, w)$ degenerate unless:

1. $u = \{\zeta\} \cup \bigcup_{1\leq j\leq k} w_j$;
2. $\delta < \zeta < w_k \setminus \delta < \ldots < w_1 \setminus \delta$;
3. $w_j \cong \beta_{p_j}$;
4. $\bigcup_{1\leq j\leq k} p_j(w_j)$ can be extended to a member of $\mathbb{P}$;

in which case we adopt the following notation. Let ζ_j be the γ_j^{th} element of w_j, and write z^β for x_i^β. Note that since $\gamma_j < \beta_{p_j}$ we will have $\zeta_j > \delta$ and hence $\zeta < \zeta_k < \ldots < \zeta_1$. Define the set $r(\alpha_1, \ldots, \alpha_{n-k-1})$ for $\alpha_1 < \ldots < \alpha_{n-k-1} \in A_\delta$ to be

$$S\Big(\bigwedge_{1\leq j\leq k} p_j(w_j) \wedge \neg\varphi(z^{\alpha_1}, \ldots, z^{\alpha_{n-k-1}}, z^\zeta, z^{\zeta_k}, \ldots, z^{\zeta_1})\Big)$$

where S covers all the $(> \zeta)$-variables.

We now define $D(u, w)$ as the set of $q \in \mathbb{P}/G$ with $\mathrm{dom}\,(q) \subseteq w$ which satisfy one of the following three conditions:

1. $q \perp \bigcup_{1 \leq j \leq k} p_j(w_j)$; or

2. $\bigcup_{1 \leq j \leq k} p_j(w_j) \subseteq q$ and for some $\alpha_1 < \ldots < \alpha_{n-k-1} \in A_\delta$, $r(\alpha_1, \ldots, \alpha_{n-k-1}) \subseteq q$; or

3. $\bigcup_{1 \leq j \leq k} p_j(w_j) \subseteq q$ and for all $\alpha_1 < \ldots < \alpha_{n-k-1} \in A_\delta$:

$$S_0^*(q \to S_k^*(p_k(w_k) \to \ldots S_1^*(p_1(w_1) \to \varphi(z^{\alpha_1}, \ldots, z^{\alpha_{n-k-1}}, z^\zeta, z^{\zeta_k}, \ldots, z^{\zeta_1}))\ldots)) \in G_\delta\}$$

The third condition means that for every $\alpha_1 < \ldots < \alpha_{n-k-1} \in A_\delta$, there is a $\chi \in q$ and $\psi_j \in p_j(w_j)$ so that

$$S_0^*(\chi \to S_k^*(\psi_k \to \ldots S_1^*(\psi_1 \to \varphi(z^{\alpha_1}, \ldots, z^{\alpha_{n-k-1}}, z^\zeta, z^{\zeta_k}, \ldots, z^{\zeta_1}))\ldots)) \in G_\delta$$

where S_j covers all the $(\geq \delta)$-variables in ψ_j for $j > 0$, and S_0 covers all the $(\geq \delta)$-variables in χ. Notice that the only overlap among the variables occur in the $(< \delta)$-variables.

Now suppose $\bar{G} \models Q^n \bar{x} \varphi(\bar{x}, \bar{a}/\sim)$. We would like to argue that $Q^n \bar{x} \varphi(\bar{x}, \bar{a}) \in G$. For convenience we will suppress the parameters $\bar{a}$. We may also assume that $T \vdash \varphi(\bar{x}) \to \bigwedge_{i<j} x_i \neq x_j$.

Since $\bar{G} \models Q^n \bar{x} \varphi(\bar{x})$, there is a λ^+-homogeneous subset $B \subseteq \bar{G}$ for φ. We may assume there is an $i < \lambda$ so that every $b \in B$ is of the form $x_i^\alpha/\sim$ for some α. Let $A = \{\alpha : x_i^\alpha/\sim \in B$ and α is the least such in a given $\sim$-class $\}$. For any δ so that $\mathrm{cf}(\delta) = \lambda$, let $\zeta_\delta = \min(A \setminus A_\delta)$. Note that if $A \cap \delta = A_\delta$ then $\zeta_\delta > \delta$.

We will now produce the following data. There will be:

1. stationary sets S_k for $0 \leq k \leq n$ with $S_{k+1} \subseteq S_k$ for all $k < n$ and
$$S_0 = \{\delta : \mathrm{cf}(\delta) = \lambda, \text{ and } A \cap \delta = A_\delta\};$$

2. templates p_k for $1 \leq k \leq n$, and ordinals γ_k so that $\gamma_k < \beta_{p_k}$;

3. a domain w_δ^k of the same order type as β_{p_k} for each $\delta \in S_k$; a $\sigma_k < S_k$ so that if $\delta \in S_k$ then $w_\delta^k \cap \delta \subseteq \sigma_k$; let ζ_δ^k be the γ_k^{th} element of w_δ^k;

4. For $0 \leq k < n$, if $\delta < \delta_k < \ldots < \delta_1 \in S_k$ are chosen so that

$$\zeta_\delta^{k+1} < w_{\delta_k}^k \setminus \delta_k < \ldots < w_{\delta_1}^1 \setminus \delta_1$$

and D is the density system over G_δ corresponding to $i, \varphi, k, p_1, \ldots, p_k$ and $\gamma_1, \ldots, \gamma_k$, then

$$p_{k+1}(w_\delta^{k+1}) \in D(\{\zeta_\delta^{k+1}\} \cup w_{\delta_k}^k \cup \ldots \cup w_{\delta_1}^1, w_\delta^{k+1}) \cap G$$

Using lemma 2.1 and the fact that G meets all the density systems introduced at stages $\delta \in S_0$, this is straightforward.

Now suppose $\delta_n < \ldots < \delta_1 \in S_n$, so that $w_{\delta_n}^n \setminus \delta_n < \ldots < w_{\delta_1}^1 \setminus \delta_1$. Let $q_k = p_k(w_{\delta_k}^k)$.

Since B is a homogeneous set for φ, it follows that $\varphi(z^{\alpha_{\delta_n}}, \ldots, z^{\alpha_2}, z^{\zeta_{\delta_1}}) \in G$, for every $\alpha_n < \ldots < \alpha_2 < \zeta_{\delta_1}$ in A_δ. Since $q_1 \in G$, using the density systems defined before, we conclude that

$$S_1^*(q_1 \rightarrow \varphi(z^{\zeta_{\delta_n}}, \ldots, z^{\zeta_{\delta_1}})) \in G$$

where S_1 covers the $w_{\delta_1}^1$-variables. Proceeding by induction and using the definition of the density systems, we conclude that

$$S_n^*(q_n \rightarrow S_{n-1}^*(q_{n-1} \rightarrow \ldots S_1^*(q_1 \rightarrow \varphi(z^{\zeta_{\delta_n}}, \ldots, z^{\zeta_{\delta_1}})))) \in G$$

where S_j covers the $w_{\delta_j}^j$-variables.

Of course, $S_n q_n \subseteq G$, so by the Magidor-Malitz axioms, $Q^n \bar{x} \varphi(\bar{x}) \in G$ and we finish.

3 Commitments

In this section we begin the proof of Theorem 1.9. Our main goal at present is to formulate a precise notion of a "commitment" (that is, a commitment to enter a dense set – or in model theoretic terms, to omit a type). We will also formulate the main properties of these commitments, to be proved in §§4-5, and we show how to derive Theorem 1.9 from these facts.

Before getting into the details, we give an outline of the proof of Theorem 1.9.

General overview

Suppose that we wish to meet only the following very simple constraints. We are given some 0-density systems D_i over for $i < \lambda$, and some $g_0 \in \mathbb{P}$, and we seek a λ-generic ideal G_0 containing g_0, and meeting each D_i. Let $\beta = \lambda \cup \sup(\mathrm{dom}\,(g_0))$, and enumerate $\mathbb{P}_\beta$ as $(r_i : i < \lambda)$. Then we may construct G_0 by generating an increasing sequence $(g_\delta)_{\delta<\lambda}$ beginning with the specified g_0, and taking G_0 to be the downward closure of (g_δ). We will run through this in some detail.

Our first obligation is to make G_0 λ-generic in $\mathbb{P}_\beta$. We will say that $r \in \mathbb{P}_\beta$ has been *decided* if we have chosen some $g_\delta \in \mathbb{P}_\beta$ so that either $r \perp g_\delta$ or else $r \leq g_\delta$. If the sequence $(g_\delta)_{\delta<\lambda}$ ultimately decides every $r \in \mathbb{P}_\beta$, then G_0 will be λ-generic in $\mathbb{P}_\beta$. At stage $\delta + 1$ we will ensure that r_δ is decided. This takes care of the basic λ-genericity requirement. At limit stages we can let g_δ be anything greater than $g_{<\delta}$. We will also take pains at limit stages to meet the specified density systems D_i. We enumerate the pairs (u, D_i) with $u \in \mathcal{P}_{<\lambda}(\beta)$, using $\lambda^{<\lambda} = \lambda$, assigning one such pair to each limit ordinal $\delta < \lambda$. Suppose that (u, D_i) is considered at stage δ. Let $v = \mathrm{dom}\, g_{<\delta}$. By the density condition on D_i, we can find $g_\delta \geq g_{<\delta}$ with $g_\delta \in D_i(u, v)$.

Thus it is easy to deal with λ constraints of the type arising in one play of our genericity game. Our strategy in that game will rely on this sort of straightforward "do what you must when you have the time" approach, but will encounter difficulties in "keeping up" at limit stages in the game. We will use Dl_λ to "guess" what additional commitments should be made with regard to various density systems D_i, so that any generic set which we construct subsequently which meets these commitments will meet each D_i. The commitments themselves retain the feature that each of them can easily be met when necessary; deciding when these commitments should be met requires another use of Dl_λ.

At stage 0, player I selects some density systems, to which we may add all the density systems from examples 1.5 and 1.6. From these we construct some stage 0 commitments ${}^0\mathbf{p}$, and a G_0 meeting ${}^0\mathbf{p}$.

At stage δ in the play of the game, Player II is attempting to extend $G_{<\delta}$ to a suitable G_δ. (At limit stages we also will need to check that $G_{<\delta}$ continues to meet suitable commitments). Since $G_{<\delta}$ meets all the previous commitments, in particular it meets all the density systems of examples 1.5 and 1.6, and therefore $\mathbb{P}/G_{<\delta}$ is λ^+-uniform. Consequently the construction of G_0 described at the outset also works in $\mathbb{P}/G_{<\delta}$. Hence we need only construct new commitments ${}^\delta\mathbf{p}$, add them to our previous commitments, and construct G_δ meeting ${}^\delta\mathbf{p}$ as above. In this way, Player II wins the game.

There is a certain difficulty involved in coping with the freedom enjoyed by Player I (in terms of obligations accumulating at limit stages in the game). There are a priori λ^+ sets $u \in \mathcal{P}_{<\lambda}(\lambda^+)$ that may require attention. On the other hand, at a given stage δ we are only prepared to consider fewer than λ such sets. However, by uniformity, it will be sufficient to consider pairs $(u, w) \in \mathcal{P}_{<\lambda}(\beta + \lambda) \times \mathcal{P}_{<\lambda}(\beta + \lambda)$, and hence λ such pairs suffice. This still leaves Player II at a disadvantage, but with the aid of Dl_λ, at limit stages we will guess a relevant set of u's of size less than λ.

It remains to show that this strategy can be implemented, and works.

We introduce the notion of basic data which will be provided by Dl_λ.

Definition 3.1 *A* collection of basic data *will contain*

1. *trees T_δ, subsets of $\mathbb{P}_\lambda$ (but not suborders), with orders $<_\delta$, for every $\delta < \lambda$;*

2. *for every generic set $G \in \mathrm{Gen}(\mathbb{P}_\alpha)$ for some $\alpha < \lambda^+$, two stationary subsets of λ, $S(G)$ and $S'(G)$ and a club C so that $C \cap S'(G) \subseteq S(G)$; and*

3. *for every $\delta < \lambda$, a set $U_\delta \subseteq \mathcal{P}_{<\lambda}(\lambda)$*

with the following properties

1. *$|T_\delta| < \lambda$, $|U_\delta| < \lambda$ for every $\delta < \lambda$,*

2. *if* $p \in T_\delta$ *then* $\mathrm{len}\,(p) = \alpha$*,*

3. *if* $p \leq_\delta q$ *and* $\mathrm{len}\,(p) = \alpha$ *then* $p = q\restriction\alpha$*,*

4. *if* $p \in T_\delta$ *and* $\alpha \leq \mathrm{dom}\,(p)$ *then* $p\restriction\alpha \in T_\delta$*,*

5. *if* $(g_\delta)_\delta$ *is a cofinal sequence for a generic set* $G \in \mathrm{Gen}(\mathbb{P}_\alpha)$ *then there is a club* C *so that for* $\delta \in C \cap S(G)$*,* $(g_{\lessdot\delta})^{col} \in T_\delta$*,*

6. *if* G *and* G' *are generic sets so that* $G \subseteq G'$ *then there is a club* C *so that* $C \cap S(G') \subseteq S(G)$ *and*

7. *(oracle property) for* $\alpha < \lambda^+$ *and* $G \in \mathrm{Gen}(\mathbb{P}_\alpha)$*,* $u \in \mathcal{P}_{<\lambda}(\alpha)$ *and* $\alpha = \bigcup_{\delta<\lambda} w_\delta$ *a continuous increasing union with* $w_\delta \in \mathcal{P}_{<\lambda}(\alpha)$ *and* $u \subseteq w_0$ *then there is a club* C *so that for every* $\delta \in C \cap S'(G)$ *there is* $u' \in U_\delta$ *so that* $(w_\delta, u) \cong (otp\,(w_\delta), u')$*.*

Remarks:

1. Although there is the possibility of confusion between the orders $<_\delta$ and $<$ on $\mathbb{P}_\lambda$, we will use $<$ for both and the context should usually make it clear which we mean.

2. The following will be true of the trees that we eventually build although this property will not be needed in the proof: if $q \in T_\delta$ then there is a generic set G with a cofinal sequence $(g_\delta)_\delta$ so that $q = (g^{col}_{<\alpha})\restriction\beta$ for some α and β

3. If $(g_\delta)_\delta$ and $(g'_\delta)_\delta$ are cofinal sequences for G and G' then there is a club C so that if $\delta \in C$ and $\eta = \mathrm{dom}\,(g^{col}_{\lessdot\delta})$ then $g^{col}_{\lessdot\delta} = (g'_{\lessdot\delta})^{col}\restriction\eta$. In condition 6, we may assume that for particular cofinal sequences, C satisfies this property as well as $C \cap S(G') \subseteq S(G)$. We will often use this version of condition 6.

4. It is important to notice the following about $p \in T_\delta$ for which $\mathrm{dom}\,(p)$ is a limit ordinal. If $\alpha < \mathrm{dom}\,(p)$ then $p\restriction\alpha \in T_\delta$ and $p\restriction\alpha < p$. Hence, any such p is the limit of those elements of T_δ which are less than it.

Lemma 3.2 *(*Dl_λ*) There is a collection of basic data.*

We leave the proof of this until the next section. For the rest of the paper except for the proof of Lemma 3.2, we will fix a particular choice of basic data using the notation of definition 3.1.

Definition 3.3 *A* weak commitment *is a sequence* $\mathbf{p} = \langle p^\delta : \delta < \lambda \rangle$ *where* $p^\delta : T_\delta \to \mathbb{P}_\lambda$ *with the following properties:*

1. $p^\delta(\eta) \in \mathbb{P}_{\mathrm{len}\,(\eta)}$ *(We usually write* p^δ_η *for* $p^\delta(\eta)$.*)*

2. *if* $\eta \leq \nu \in T_\delta$ *then* $p^\delta_\eta \geq p^\delta_\nu \restriction \mathrm{len}\,(\eta)$.

We define an order on weak commitments by $\mathbf{p} \leq \mathbf{q}$ *if for almost all* δ *(i.e. on a club),* $p^\delta \leq q^\delta$ *pointwise. We say that* $\mathbf{q}$ *is stronger than* $\mathbf{p}$.

We will identify two weak commitments $\mathbf{p}$ and $\mathbf{q}$ if $\mathbf{p} \leq \mathbf{q}$ and $\mathbf{q} \leq \mathbf{p}$.
Notation: From the fixed collection of basic data one can extract a critical weak commitment. Define ${}^*\mathbf{p} = \langle {}^*p^\delta : \delta < \lambda \rangle$ where ${}^*p^\delta_\eta = \eta$.

Definition 3.4 *A* commitment *is a weak commitment which is stronger than* ${}^*\mathbf{p}$.

Definition 3.5 *Suppose* G *is generic with a cofinal sequence* $(g_\delta)_\delta$ *and* $\mathbf{p}$ *is a commitment. We say* G meets $\mathbf{p}$ *if there is a club* C *in* λ *so that for every* $\delta \in C \cap S(G)$, $\eta_\delta =: (g_{\lessdot\delta})^{col} \in T_\delta$ *and there is* $r_\delta \in G$ *so that* $\mathrm{dom}\,(r_\delta) = \mathrm{dom}\,(g_{\lessdot\delta})$ *and*

$$r^{col}_\delta \geq p^\delta_\eta \textit{ for all } \eta \leq \eta_\delta.$$

Remark: If $h : \mathrm{len}\,(\eta_\delta) \to \mathrm{dom}\,(g_{\lessdot\delta})$ is an order isomorphism then in the above definition the existence of r_δ is equivalent to saying that $h[p^\delta_\eta] \in G$ for all $\eta \leq \eta_\delta$.

Proposition 3.6 *If* D_i, $i < \lambda$, *are 0-density systems and* $g \in \mathbb{P}_\gamma$ *then there is a commitment* $\mathbf{q}(\geq {}^*\mathbf{p})$ *and some* $G \in \mathrm{Gen}(\mathbb{P}_\gamma)$, *so that:*

1. $g \in G$,

2. G *meets* $\mathbf{q}$ *and*

3. *if* $\gamma \leq \gamma' < \lambda^+$ *and* $G' \in \mathrm{Gen}(\mathbb{P}_{\gamma'})$ *meets* $\mathbf{q}$, *then* G' *meets each* D_i.

Proposition 3.7 *Suppose $G \in \mathrm{Gen}(\mathbb{P}_\alpha)$ and G satisfies*

1. *for all $g \in \mathbb{P}/\mathbb{G}, \eqsim \in \mathbb{P}$ there is $\eth' \in \mathbb{P}/\mathbb{G}$ with $\eth' \geq \eth$ and either $\eth' \geq \eqsim$ or $\eth' \perp \eqsim$ and*

2. *$\mathbb{P}/G$ is λ^+-uniform.*

For $i < \lambda$, let D_i be a density system over G, and suppose $g \in \mathbb{P}_\gamma/G$ where $\alpha \leq \gamma < \lambda^+$ and $\mathbf{p}$ is some commitment that is met by G. Then there is a commitment $\mathbf{q} \geq \mathbf{p}$, and some $G^* \in \mathrm{Gen}(\mathbb{P}_\gamma)$, so that:

1. *$G \subseteq G^*$, $g \in G^*$;*

2. *G^* meets $\mathbf{q}$;*

3. *If $\gamma \leq \gamma' < \lambda^+$ and $G' \in \mathrm{Gen}(\mathbb{P}_{\gamma'})$ contains G and meets $\mathbf{q}$, then G' meets each D_i.*

Lemma 3.8 *Let $({}^\alpha\mathbf{p})_{\alpha<\kappa}$ be an increasing sequence of commitments with $\kappa < \lambda^+$. Then the sequence has a least upper bound.*

Notation 3.9

With the notation of the preceding lemma, we write

$$\bigcup_{\alpha<\delta} {}^\alpha\mathbf{p} \text{ or } {}^{<\delta}\mathbf{p}$$

for the least upper bound of the commitments ${}^\alpha\mathbf{p}$.

Proposition 3.10 *Suppose $G_\alpha \in \mathrm{Gen}(\mathbb{P}_{\zeta_\alpha})$ meets ${}^\alpha\mathbf{p}$ for all $\alpha < \delta$, where $\delta < \lambda^+$, and the G_α and ${}^\alpha\mathbf{p}$ are increasing. Then $G_{<\delta}$ meets ${}^{<\delta}\mathbf{p}$.*

By combining these results we immediately obtain a proof of Theorem 1.9.

Proof of Theorem 1.9: We define a sequence of ordinals ζ_α, a sequence of commitments ${}^\alpha\mathbf{p}$, and a sequence of λ-generic ideals $G_\alpha \in \mathrm{Gen}(\mathbb{P}_{\zeta_\alpha})$, so that:

1. $\zeta_{<\alpha} \leq \zeta_\alpha$; ${}^{<\delta}\mathbf{p} \leq {}^\delta\mathbf{p}$; $G_{<\delta} \subseteq G_\delta$ for $\delta < \lambda^+$

2. G_α meets the commitment ${}^\alpha\mathbf{p}$

3. If $\zeta_\alpha \leq \beta$ and $G \in \mathrm{Gen}(\mathbb{P}_\beta)$ contains G_α and meets the commitment ${}^\alpha\mathbf{p}$, then G meets each α-density system D_i over G_α proposed by Player I at stage α of the genericity game.

At stage 0, Player I provides some $g_0 \in \mathbb{P}$ and at most λ many 0-density systems. To these Player II adds all the 0-density systems mentioned in examples 1.5 and 1.6. We now apply Proposition 3.6 to all these 0-density systems and g_0. This will provide us with G_0, ζ_0 and ${}^0\mathbf{p}$.

At stage δ, we will have $\zeta_{\lhd\delta}$, $G_{\lhd\delta}$, ${}^{\lhd\delta}\mathbf{p}$ defined, and by Proposition 3.10 $G_{\lhd\delta}$ meets ${}^{\lhd\delta}\mathbf{p}$. Now since $G_0 \subseteq G_{\lhd\delta}$, $G_{\lhd\delta}$ meets ${}^0\mathbf{p}$ and hence meets each of the 0-density systems from examples 1.5 and 1.6. It follows that $G_{\lhd\delta}$ satisfies the conditions on the generic set in Proposition 3.7. By Proposition 3.7 a suitable choice of ζ_δ, G_δ, ${}^\delta\mathbf{p}$ can then be made.

Now we verify that Player II wins the genericity game using this strategy. By construction, $g_\alpha \in G_\alpha$ for all α. Suppose that D is a density system over $G_{<\alpha}$ selected by Player I at stage α of the genericity game, and $\beta \geq \alpha$. As G_β meets the commitment ${}^\beta\mathbf{p}$, ${}^\beta\mathbf{p} \geq {}^\alpha\mathbf{p}$, and $G_\beta \supseteq G_\alpha$, it follows that G_β meets D. $\square$

4 Proofs

In this section we give the proofs of the results stated in the previous section except the proofs of Propositions 3.6 and 3.7 which are deferred to the next section.

Proof of Lemma 3.2

The Lemma states that there is a collection of basic data.

Let $(v_\delta)_{\delta<\lambda}$ be an enumeration of $\mathcal{P}_{<\lambda}(\lambda)$ so that $v_\delta \subseteq \delta$ for all δ. Let $V_\delta = \{v_\beta : \beta < \delta\}$ for $\delta < \lambda$. Using Dl_λ and an encoding of

$$(\lambda \times \lambda) \cup (\lambda \times \mathcal{P}_{<\lambda}(\lambda))$$

by λ, we can find sets $\mathcal{R}_\delta \subseteq \mathcal{P}(\delta \times \delta)$ and $\mathcal{G}_\delta \subseteq \mathcal{P}(\delta \times V_\delta)$, such that $|\mathcal{R}_\delta|, |\mathcal{G}_\delta| < \lambda$ for all $\delta < \lambda$, and for any $R \subseteq \lambda \times \lambda$, $G \subseteq \lambda \times \mathcal{P}_{<\lambda}(\lambda)$, the set:

$$\{\delta : R \cap (\delta \times \delta) \in \mathcal{R}_\delta \text{ and } G \cap (\delta \times V_\delta) \in \mathcal{G}_\delta\}$$

is stationary.

Before defining the basic data we establish some notation. For each $\alpha < \lambda^+$, we select a bijection $i_\alpha : \alpha \leftrightarrow |\alpha|$. For simplicity we assume $|\alpha| = \lambda$ throughout in our notation below. For $\delta < \lambda$, let α_δ be the order type of $i_\alpha^{-1}[\delta]$, and let $\pi_{\alpha\delta} : \alpha \cap i_\alpha^{-1}[\delta] \simeq \alpha_\delta$, $j_{\alpha\delta} = \pi_{\alpha\delta} \circ i_\alpha^{-1} : \delta \leftrightarrow \alpha_\delta$.

Let

$$\begin{aligned} R_\alpha &= \{(i_\alpha(\beta), i_\alpha(\gamma)) : \beta < \gamma < \alpha\}; \\ R_{\alpha\delta} &= R_\alpha \cap (\delta \times \delta) \quad (\delta < \lambda). \end{aligned}$$

Then $j_{\alpha\delta} : (\delta, R_{\alpha\delta}) \simeq (\alpha_\delta, <)$. It will be important that $j_{\alpha\delta}$ is determined by $R_{\alpha\delta}$.

If G is a λ-generic ideal in $\mathbb{P}_\alpha$ with $\alpha < \lambda^+$, let:

$$\begin{aligned} \hat{G} &= \{(\beta, i_\alpha[u]) : (\beta, u) \in G\} \\ \hat{G}_\delta &= \hat{G} \cap (\delta \times V_\delta) \\ G_\delta &= \{(\beta, u) \in G : (\beta, i_\alpha[u]) \in (\delta \times V_\delta)\} \\ \bar{G}_\delta &= \{(\beta, \pi_{\alpha\delta}[u]) : (\beta, u) \in G_\delta\} \end{aligned}$$

Again, we can go directly from $\hat{G}_\delta$ to $\bar{G}_\delta$ by applying $j_{\alpha\delta}$. Observe also that $\pi_{\alpha\delta}$ induces an isomorphism $\pi^*_{\alpha\delta} : G_\delta \simeq \bar{G}_\delta$. We are primarily interested in this collapsing map $\pi^*_{\alpha\delta}$, but $\hat{G}$ provides a better "encoding" of G because the sets $\hat{G}_\delta$ increase with δ, while the sets $\bar{G}_\delta$ do not.

Let $C(G)$ be the set of $\delta < \lambda$ for which G_δ contains a cofinal increasing subsequence. Then $C(G)$ is a club in λ. For $\delta \in C(G)$, $\bar{G}_\delta$ has a least upper bound, which will be denoted $\cup \bar{G}_\delta$.

We are now ready to define the basic data. For $\delta < \lambda$ we define T_δ as:

$$\begin{aligned}\{p \in \mathbb{P} : \ & \exists \alpha < \lambda^+ \, \exists G \in \mathrm{Gen}(\mathbb{P}_\alpha) \, \exists \gamma : \quad \delta \in C(G), \\ & \hat{G}_\delta \in \mathcal{G}_\delta, R_{\alpha\delta} \in \mathcal{R}_\delta, \text{ and } p = [\bigcup \bar{G}_\delta] \restriction \gamma.\}\end{aligned}$$

Notice that $\mathrm{dom}\,(p)$ is an ordinal for every $p \in T_\delta$. To see that $|T_\delta| < \lambda$, we use the fact that $\hat{G}_\delta, R_{\alpha\delta}$ together determine $\bar{G}_\delta$, and also that any p in $\mathbb{P}$ has fewer than λ distinct restrictions. For $p, q \in T_\delta$, define the order by; $p \le q$ iff $p = q \restriction \mathrm{dom}\,(p)$.

Now for G a λ-generic ideal in $\mathbb{P}_\alpha$ with $\alpha < \lambda^+$, fix a cofinal sequence $(g_i^G)_{i<\lambda}$ in G, and set:

$$\begin{aligned} S(G) &= \{\delta < \lambda : [g^G_{<\delta}]^{col} \in T_\delta\}; \\ S'(G) &= \{\delta < \lambda : \hat{G}_\delta \in \mathcal{G}_\delta \text{ and } R_{\alpha\delta} \in \mathcal{R}_\delta\}; \\ U_\delta &= \{u : \exists v \in V_\delta \, \exists R \in \mathcal{R}_\delta \, \exists \alpha < \lambda^+ \quad (\delta, v, R) \simeq (\alpha, u, <)\} \end{aligned}$$

Clearly $U_\delta \subseteq \mathcal{P}_{<\lambda}(\lambda)$ and $|U_\delta| < \lambda$. It is also straightforward to see that $S'(G)$ is stationary.

Let C_1 be

$$\{\delta \in C(G) : g^G_{<\delta} = \cup G_\delta\}$$

Then C_1 is a club in λ, and if $\delta \in S'(G) \cap C_1$ then $(g^G_{<\delta})^{col} \in T_\delta$ so $S(G)$ is stationary. If $(g'_\delta)_\delta$ is any other cofinal sequence for G then there is a club

$$C = \{\delta : g'_{<\delta} = g^G_{<\delta}\}$$

and for every $\delta \in C \cap S(G)$, $(g'_{<\delta})^{col} \in T_\delta$.

Let $G \subseteq G^*$ be two λ-generic ideals in $\mathbb{P}_\alpha, \mathbb{P}_{\alpha^*}$ with $S(G), S(G^*)$ determined by cofinal sequences $(g^G_\delta)_\delta, (g^{G^*}_\delta)_\delta$ respectively. If one considers $C = \{\delta : g^{G^*}_{<\delta} \restriction \alpha = g^G_{<\delta}\}$, it is easy to see that $C \cap S(G^*) \subseteq S(G)$.

It remains to verify the oracle property (7) of Definition 3.1. We fix $\alpha < \lambda^+$, G λ-generic in $\mathbb{P}_\alpha$, $u \in \mathcal{P}_{<\lambda}(\alpha)$, and we let $\alpha = \cup_{\delta<\lambda} w_\delta$ be a continuous increasing union with each $|w_\delta| < \lambda$ and $u \subseteq w_0$. On some club C, otp $w_\delta = \alpha_\delta$ and if $v_\alpha \subseteq \delta$ then $\alpha < \delta$. So $(w_\delta, u) \simeq (\alpha_\delta, \pi_{\alpha\delta}[u])$. For $\delta \in C \cap S'(G)$ we have

$$(\delta, i_\alpha[u], R_{\alpha\delta}) \simeq (\alpha_\delta, \pi_{\alpha\delta}[u], <) \simeq (w_\delta, u, <).$$

Hence, $\pi_{\alpha\delta}[u] \in U_\delta$. □

Notation 4.1 In the next few results we make systematic use of the diagonal intersection of clubs. If $(C_\alpha)_{\alpha<\lambda}$ is a sequence of clubs in λ, the diagonal intersection is defined correspondingly as:

$$\Delta_\alpha C_\alpha = \{\delta < \lambda : \delta \in \bigcap_{\alpha<\delta} C_\alpha\}.$$

The diagonal intersection of such a sequence of clubs is again a club.

Proof of Lemma 3.8 Let $({}^\alpha\mathbf{p})_{\alpha<\kappa}$ be an increasing sequence of commitments with $\kappa < \lambda^+$. We claim that the sequence has a least upper bound. We may take κ to be a regular cardinal, with $\kappa \le \lambda$. We deal with the case $\kappa = \lambda$; for $\kappa < \lambda$ our use of a diagonal intersection below would reduce to an ordinary intersection.

For $\beta < \lambda$ let C_β be a club such that for all $\alpha < \beta$:

$${}^\alpha p^\delta \le {}^\beta p^\delta \text{ pointwise for } \delta \in C_\beta.$$

Let $C = \Delta_\beta C_\beta$. For $\delta \in C$ and $\eta \in T_\delta$, let $p^\delta_\eta = \cup_{\alpha<\delta} {}^\alpha p^\delta_\eta$. Then $\mathbf{p}$ is a commitment. We have ${}^\alpha\mathbf{p} \le \mathbf{p}$ since ${}^\alpha p^\delta \le p^\delta$ pointwise for $\delta \in C \setminus \alpha$.

Now we will check that $\mathbf{p}$ is the least upper bound of the sequence as a commitment Let $\mathbf{q}$ be a second upper bound. Let

$$C^*_\alpha = \{\delta < \lambda : q^\delta \ge {}^\alpha p^\delta \text{ pointwise}\},$$

and let $C^* = \Delta_\alpha C^*_\alpha$. For $\delta \in C \cap C^*$, and $\eta \in T_\delta$, we have:

$$q^\delta_\eta \ge \cup_{\alpha<\delta} {}^\alpha p^\delta_\eta = p^\delta_\eta.$$

It follows that $\mathbf{p}$ is the least upper bound. □

We divide Proposition 3.10 into two parts.

Proposition 4.2 *Suppose that $(G_i)_{i<\kappa}$ is an increasing sequence with $G_i \in \mathrm{Gen}(\mathbb{P}_{\alpha_i})$, $\kappa < \lambda^+$, and that each G_i meets a fixed commitment $\mathbf{p}$. Then $\cup_i G_i$ also meets the commitment $\mathbf{p}$.*

Proof: Let $G = \bigcup_{i<\kappa} G_i$. By consulting the proof of Lemma 1.3, we can see that there is a cofinal sequence $(g_j)_{j<\lambda}$ for G so that if $g^i_j = g_j \restriction \alpha_i$ then $(g^i_j)_{j<\lambda}$ is a cofinal sequence in G_i. For each $i < \lambda$ let C_i be a club demonstrating that G_i meets $\mathbf{p}$. In other words, for $\delta \in C_i \cap S(G_i)$ we have $r^i_\delta \in G$ with:

1. $\mathrm{dom}\, r^i_\delta = \mathrm{dom}\, g^i_{\lessdot\delta}$, $\eta^i_\delta = (g^i_{\lessdot\delta})^{col} \in T_\delta$; and
2. $[r^i_\delta]^{col} \geq p^\delta_\eta$ for all $\eta \leq \eta^i_\delta$.

By Definition 3.1.5, we may also suppose that $C_i \cap S(G) \subseteq S(G_i)$.

We consider the case $\kappa = \lambda$ (Use ordinary intersection instead of diagonal intersection when $\kappa < \lambda$).

Let

$$C = \Delta_i C_i \cap \{\delta < \lambda : \sup \mathrm{dom}\, g_{\lessdot\delta} = \sup_{i<\delta} \alpha_i\}$$

If $\delta \in C \cap S(G)$ then we can find $r_\delta \in G$ so that $r_\delta \geq r^i_\delta$ for $i < \delta$ and

$$\mathrm{dom}\,(r_\delta) = \bigcup_{i<\delta} \mathrm{dom}\,(g^i_{\lessdot\delta}) = \bigcup_{i<\delta} \mathrm{dom}\,(g_{\lessdot\delta} \restriction \alpha_i) = \mathrm{dom}\,(g_{\lessdot\delta}).$$

Clearly, $r^{col}_\delta \geq p^\delta_\eta$ for all $\eta < (g_{\lessdot\delta})^{col} = \eta_\delta \in T_\delta$. However, since $p^\delta_{\eta_\delta} \restriction \mathrm{len}\,(\eta) \leq p^\delta_\eta$ for all $\eta \leq \eta_\delta$ and $p^\delta_{\eta_\delta} = \bigcup_{\eta<\eta_\delta} p^\delta_{\eta_\delta} \restriction \mathrm{len}\,(\eta)$, $r^{col}_\delta \geq p^\delta_{\eta_\delta}$. $\square$

Proposition 4.3 *Suppose $G \in \mathrm{Gen}(\mathbb{P}_\alpha)$ meets an increasing sequence of commitments $({}^\gamma\mathbf{p} : \gamma < \kappa)$ where $\kappa < \lambda^+$. Then G meets $\bigcup_{\gamma<\kappa} {}^\gamma\mathbf{p}$.*

Proof: Again, we treat only the case when $\kappa = \lambda$. Let $(g_i)_{i<\lambda}$ be a cofinal sequence in G. For each γ, let C_γ witness the fact that G meets ${}^\gamma\mathbf{p}$. That is, for all $\delta \in C_\gamma \cap S(G)$ there is $r^\gamma_\delta \in G$ so that:

1. $\mathrm{dom}\, r^\gamma_\delta = \mathrm{dom}\, g_{\lessdot\delta}$, $\eta_\delta = g^{col}_{\lessdot\delta} \in T_\delta$; and
2. $(r^\gamma_\delta)^{col} \geq {}^\gamma p^\delta_\eta$ for all $\eta \leq \eta_\delta$.

We may also suppose that C_γ witnesses the relation ${}^i\mathbf{p} \leq {}^\gamma\mathbf{p}$ for $i < \gamma$. Hence we may assume $r^i_\delta \leq r^\gamma_\delta$ for $\delta \in C_\gamma \cap S(G)$. Let $C = \Delta_\gamma C_\gamma$.

For $\delta \in C \cap S(G)$, let $r_\delta = \bigcup_{i<\delta} r^i_\delta$. Then on $C \cap S(G)$, $\operatorname{dom} r_\delta = \operatorname{dom} g_{<\delta}$ and $r^{col}_\delta = \bigcup_{i<\delta}[r^i_\delta]^{col} \geq \bigcup_{i<\delta} {}^i p^\delta_\eta = {}^{<\kappa}p^\delta_\eta$ for all $\eta \leq \eta_\delta$. The last equality follows from the proof of Lemma 3.8. □

Proposition 3.10 is an immediate consequence of the preceding two propositions.

5 Proof of Proposition 3.7

We recall the statement of Proposition 3.7.

Proposition 3.7

Suppose $G \in \mathrm{Gen}(\mathbb{P}_\alpha)$ and G satisfies

1. *for all $g \in \mathbb{P}/\mathbb{G}, \eqsim \in \mathbb{P}$ there is $\eth' \in \mathbb{P}/\mathbb{G}$ with $\eth' \geq \eth$ and either $\eth' \geq \eqsim$ or $\eth' \perp \eqsim$ and*
2. *$\mathbb{P}/G$ is λ^+-uniform.*

For $i < \lambda$, let D_i be a density system over G, and suppose $g \in \mathbb{P}_\gamma/G$ where $\alpha \leq \gamma < \lambda^+$ and $\mathbf{p}$ is some commitment that is met by G. Then there is a commitment $\mathbf{q} \geq \mathbf{p}$, and some $G^* \in \mathrm{Gen}(\mathbb{P}_\gamma)$, so that:

1. *$G \subseteq G^*$, $g \in G^*$;*
2. *G^* meets $\mathbf{q}$;*
3. *If $\gamma \leq \gamma' < \lambda^+$ and $G' \in \mathrm{Gen}(\mathbb{P}_{\gamma'})$ contains G and meets $\mathbf{q}$, then G' meets each D_i.*

We are also obliged to prove Proposition 3.6 as well. The proof is very similar to the proof of Proposition 3.7 and so we will only highlight the formal differences at the end of the proof.

Proof of Proposition 3.7: Let $\gamma = \bigcup_{\delta<\lambda} w_\delta$ be a continuous increasing union with $|w_\delta| < \lambda$. Set $\gamma_\delta = \mathrm{otp}\,(w_\delta)$ and choose ζ_δ so that $\gamma_\delta + \zeta_\delta \geq \mathrm{ht}\,(T_\delta)$. Let

$$h_\delta : \gamma_\delta + \zeta_\delta \to w_\delta \cup [\gamma, \gamma + \zeta_\delta)$$

be an order isomorphism.

Fix a cofinal sequence $(g_\delta)_\delta$ for G. Since G meets $\mathbf{p}$, there is a club C so that for all $\delta \in C \cap S(G)$ we have $g^{col}_{<\delta} =: \eta_\delta \in T_\delta$, and there is $r_\delta \in G$ so that $\mathrm{dom}\,(r_\delta) = w_\delta \cap \alpha$ and $r^{col}_\delta \geq p^\delta_\eta$ for all $\eta \leq \eta_\delta$. We may also assume that $\mathrm{dom}\,(g_{<\delta}) = w_\delta \cap \alpha$ for all $\delta \in C$.

Now we build the commitment $\mathbf{q}$. If $\delta \notin C \cap S(G)$ then let $q^\delta = p^\delta$. Fix then $\delta \in C \cap S(G)$. For $i < \delta$, $\zeta \leq \zeta_\delta$ and $u \in U_\delta$, $u \subseteq \gamma_\delta + \zeta_\delta$, let

$$D^\zeta_i(u) = \{r \in \mathbb{P}_{\gamma_\delta+\zeta} : h_\delta[r] \in D_i(h_\delta[u], w_\delta \cup [\gamma, \gamma + \zeta))\}$$

Let $\mathbb{P}_G[T_\delta]$ be the set of functions $\bar{p} : T_\delta \to \mathbb{P}_\lambda$ so that

1. $\bar{p}(\eta) \in \mathbb{P}_{\mathrm{len}\,(\eta)}$ for all $\eta \in T_\delta$,
2. if $\eta \leq \nu$ then $\bar{p}(\eta) \geq \bar{p}(\nu)\restriction \mathrm{len}\,(\eta)$ and
3. if η is comparable with η_δ then $h_\delta[\bar{p}(\eta)] \in \mathbb{P}/G$.

We will write $\bar{p}_\eta$ for $\bar{p}(\eta)$.
Remark: Since G meets $\mathbf{p}$, if $\delta \in C \cap S(G)$ then $p^\delta \in \mathbb{P}_G[T_\delta]$. To see this, we must show that if η is comparable to η_δ then $h_\delta[p^\delta_\eta] \in \mathbb{P}/G$. If $\eta \leq \eta_\delta$ then since $\delta \in C \cap S(G)$, $h_\delta[p^\delta_\eta] \in G$. Suppose $\eta \geq \eta_\delta$. Now $p^\delta_\eta\restriction \mathrm{len}\,(\eta_\delta) \leq p^\delta_{\eta_\delta}$ and since $w_\delta \cap \alpha = \mathrm{dom}\,(g_{<\delta})$ we have $h_\delta[p^\delta_\eta]\restriction \alpha \leq h_\delta[p^\delta_{\eta_\delta}]$ so $h_\delta[p^\delta_\eta] \in \mathbb{P}/G$.

Proposition 5.1 *There is a $q^\delta \in \mathbb{P}_G[T_\delta]$ with $q^\delta \geq p^\delta$ pointwise and so that for every $u \in U_\delta$, $i < \delta$ if $\eta' \in T_\delta$, $\eta' \geq \eta_\delta$ with* $\mathrm{len}\,(\eta') = \gamma_\delta + \zeta$ *and $u \subseteq \gamma_\delta + \zeta$ then $q^\delta_{\eta'} \in D^\zeta_i(u)$.*

To obtain this q^δ we use a claim whose proof we postpone.

Claim 5.2 *If $\bar{q} \in \mathbb{P}_G[T_\delta]$, $u \in U_\delta$, $i < \delta$, $\zeta \leq \zeta_\delta$ and $\eta^* \in T_\delta$, $\eta^* \geq \eta_\delta$ with* $\mathrm{len}\,(\eta^*) = \gamma_\delta + \zeta$ *and $u \subseteq \gamma_\delta + \zeta$ then there is $\bar{r} \in \mathbb{P}_G[T_\delta]$ so that $\bar{r} \geq \bar{q}$ pointwise and $\bar{r}_{\eta^*} \in D^\zeta_i(u)$.*

Proof of Proposition 5.1 To get the required q^δ, one starts with p^δ, at limit stages take unions and at successor stages use the claim applied to some particular $i < \delta$, $\zeta \leq \zeta_\delta$, $u \in U_\delta$ and $\eta^* \in T_\delta$. After at most $|U_\delta| \cdot |\delta| \cdot |\zeta_\delta| \cdot |T_\delta|$ stages we will have produced q^δ. □

Now we turn to the construction of G^*, a λ-generic ideal in $\mathbb{P}_\gamma$ meeting $\mathbf{q}$ with $G \subseteq G^*$ and $g \in G^*$, hence completing the proof of Proposition 3.7.

Fix an enumeration $(s_\delta)_\delta$ of $\mathbb{P}_\gamma$. G^* will be the downward closure of an increasing sequence $(g^*_\delta)_\delta$ which is constructed inductively starting with $g^*_0 = g$. We shall guarantee that $g^*_\delta \in \mathbb{P}_\gamma/G$ for each δ.

At stage δ, if $\delta \in C \cap S(G)$, $\mathrm{dom}\,(g^*_{<\delta}) = w_\delta$, $(g^*_{<\delta})\restriction \alpha = g_{<\delta}$ and $(g^*_{<\delta})^{col} = \eta^* \in T_\delta$ then let $h : \mathrm{len}\,(\eta^*) \to \mathrm{dom}\,(g^*_{<\delta})$ be an order isomorphism and let $\hat{g}_\delta \in \mathbb{P}/G$ be chosen so that $\mathrm{dom}\,(\hat{g}_\delta) = \mathrm{dom}\,(g^*_{<\delta}), \hat{g}_\delta \geq g^*_{<\delta}$ and $\hat{g}_\delta \geq h_\delta[q^\delta_\eta]$ for every $\eta \leq \eta^*$. This can be accomplished because $\mathbb{P}/G$ is λ^+-uniform by assumption and we guaranteed that $h_\delta[q^\delta_\eta] \in \mathbb{P}/G$ when we built the commitment $\mathbf{q}$.

If any of the above conditions fail, let $\hat{g}_\delta = g^*_{\lessdot\delta}$. In either case, use the assumption on G to find g^*_δ so that $g^*_\delta \in \mathbb{P}_\gamma/G$ with $\hat{g}_\delta \geq g^*_{\lessdot\delta}$ and either $g^*_\delta \geq s_\delta$ or $g^*_\delta \perp s_\delta$.

It follows easily now that $G^* \in \mathrm{Gen}(\mathbb{P}_\gamma)$, $G \subseteq G^*$ and $g \in G^*$. We now show that G^* meets $\mathbf{q}$. Choose C_1 so that $C_1 \cap S(G^*) \subseteq C \cap S(G)$ and for all $\delta \in C_1$, $\mathrm{dom}\,(g^*_{\lessdot\delta}) = w_\delta$ and $(g^*_{\lessdot\delta}) \restriction \alpha = g_{\lessdot\delta}$. If $\delta \in C_1 \cap S(G^*)$ then $(g^*_{\lessdot\delta})^{col} = \eta^* \in T_\delta$ so from considerations at stage δ, $\hat{g}_\delta \in G^*$ and $(\hat{g}_\delta)^{col} \geq q^\delta_\eta$ for all $\eta \leq \eta^*$. It follows that G^* meets $\mathbf{q}$.

Now suppose G' meets $\mathbf{q}$, $G' \in \mathrm{Gen}(\mathbb{P}_{\gamma'})$ contains G with $\gamma \leq \gamma' < \lambda^+$. Fix a cofinal sequence $(g'_\delta)_\delta$ for G', a density system D_i and $u \in \mathcal{P}_{\lessdot\lambda}(\gamma')$. We want to find w so that $u \subseteq w$ and $D_i(u, w) \cap G' \neq \emptyset$. Write $\gamma' \setminus \gamma$ as a continuous increasing union $\bigcup_{\delta<\lambda} w'_\delta$ with $w'_\delta \in \mathcal{P}_{\lessdot\lambda}(\gamma' \setminus \gamma)$.

There is a club C_2 with the following properties:

1. if $\delta \in C_2$, $(g'_{\lessdot\delta})\restriction\alpha = g_{\lessdot\delta}$ and $\mathrm{dom}\,(g'_{\lessdot\delta}) = w_\delta \cup w'_\delta$;

2. $C_2 \cap S'(G') \subseteq C_2 \cap S(G') \subseteq C \cap S(G)$;

3. for $\delta \in C_2 \cap S(G')$ there is $r_\delta \in G'$ with $\mathrm{dom}\,(g'_{\lessdot\delta}) = \mathrm{dom}\,(r_\delta)$, $g'_{\lessdot\delta} \leq r_\delta$, $(g'_{\lessdot\delta})^{col} = \eta'_\delta \in T_\delta$ and $r^{col}_\delta \geq q^\delta_\eta$ for all $\eta \leq \eta'_\delta$; and

4. for $\delta \in C_2 \cap S'(G')$ there is $u' \in U_\delta$ so that
$$f_\delta : (\mathrm{otp}\,(w_\delta \cup w'_\delta), u') \simeq (w_\delta \cup w'_\delta, u).$$

This can be obtained by refering to the definition of basic data, Definition 3.1 and Lemma 3.2. In particular, condition 4 follows from the oracle property.

Now choose $\delta \in C_2 \cap S'(G')$ with $i < \delta$. Let $\zeta = \mathrm{otp}\,(w'_\delta)$. Since $\mathrm{dom}\,(g'_{\lessdot\delta}) = w_\delta \cup w'_\delta$ and $\eta'_\delta = (g'_{\lessdot\delta})^{col} \in T_\delta$, we have that $\gamma_\delta + \zeta \leq \mathrm{ht}\,(T_\delta)$. Moreover, $\eta_\delta = g^{col}_{\lessdot\delta} \in T_\delta$ and $\eta_\delta \leq \eta'_\delta$. Hence,

$$h_\delta[q^\delta_{\eta'_\delta}] \in D_i(h_\delta[u'], w_\delta \cup [\gamma, \gamma + \zeta))$$

and $r^{col}_\delta \geq q^\delta_{\eta'_\delta}$ with $r_\delta \in G'$.

By the indiscernibility of the density systems, we have

$$f_\delta[q^\delta_{\eta'_\delta}] \in D_i(u, w_\delta \cup w'_\delta)$$

since
$$f_\delta h_\delta^{-1} : (w_\delta \cup [\gamma, \gamma+\zeta), h_\delta[u']) \simeq (w_\delta \cup w'_\delta, u).$$

By the indiscernibility of $\mathbb{P}$, we have $r_\delta \geq f_\delta[q^\delta_{\eta'_\delta}]$. Since $r_\delta \in G'$,

$$f_\delta[q^\delta_{\eta'_\delta}] \in G' \cap D_i(u, w_\delta \cup w'_\delta)$$

so G meets D_i. □

It remains to prove Claim 5.2.

Proof of Claim 5.2: Consider the set

$$S = \{h_\delta[\bar{q}_\eta] : \eta \leq \eta^*\}$$

which is a subset of $\mathbb{P}/G$. By the compatibility condition in the definition of $\mathbb{P}_G[T_\delta]$, S is also a compatible set so we can choose $r'_{\eta^*} \in \mathbb{P}_{\mathrm{len}\,(\eta^*)}$ so that $r'_{\eta^*} \geq \bar{q}_\eta$ for all $\eta \leq \eta^*$ and $h_\delta[r'_{\eta^*}] \in \mathbb{P}/G$ since $\mathbb{P}/G$ is λ^+-uniform.

Now choose $r_{\eta^*} \in D^\zeta_i(u)$ so that $r_{\eta^*} \geq r'_{\eta^*}$. This is possible since D_i is a density system over G. Define

$$\bar{r}_\eta = \begin{cases} r_{\eta^*} \restriction \mathrm{len}\,(\eta) & \text{if } \eta \leq \eta^* \\ \bar{q}_\eta & \text{otherwise.} \end{cases}$$

It is easy to check that $\bar{r} \in \mathbb{P}_G[T_\delta]$. □

To obtain a proof of 3.6, make the following changes in the above proof. In the statement of 3.6, there is no G or $\mathbf{p}$ so at the start of the proof, one must consider all $\delta < \lambda$. The definition of $D^\zeta_i(u)$ is the same. We replace $\mathbb{P}_G[T_\delta]$ with $\mathbb{P}[T_\delta]$ which is the same as $\mathbb{P}_G[T_\delta]$ but there is no third condition. With few formal changes, Claim 5.2 can be proved which allows one to build the required $q^\delta \geq {}^*p^\delta$.

The rest of the proof is almost identical except that instead of referring to the two conditions on G in the statement of Proposition 3.7, one uses the fact that $\mathbb{P}$ already possesses these qualities by virtue of being λ^+-uniform.

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