

**THERE ARE NOETHERIAN
DOMAIN IN EVERY CARDINALITY
WITH FREE ADDITIVE GROUP
SH217**

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Theorem. *There are Noetherian rings (in fact domains) with a free additive group, in every infinite cardinality.*

Remark. 1) For \aleph_1 this was proved by O'Neill.

2) The work was done in Sept., '83.

3) We thank Fuchs for suggesting to us the problem.

4) This is an expanded version of [SgSh 217] which appears in the Notices of AMS.

Sketch of Proof. Let \mathfrak{Z} be the ring of integers, X a set of distinct variables, $Z[X]$ the ring of polynomials over \mathfrak{Z} , $\mathfrak{Z}(X)$ its field of quotients and R_X the additive subgroup of $\mathfrak{Z}(X)$ generated by $\{p/q : p \in \mathfrak{Z}[X], q \in \mathfrak{Z}[X], p \text{ not divisible (nontrivially) by any integer}\} \subseteq \mathfrak{Z}(X)$. It is known that R_X is a Noetherian domain. Let for a ring R , R^+ be its additive group. For $Y \subseteq X$ we can define $Z[Y]$, $Z(Y)$, R_Y similarly.

Lemma. 1) R_X^+ is a free abelian group.

2) If $n \geq 0$, $Y \subseteq X$, $x(1), \dots, x(n) \in X \setminus Y$ pairwise distinct, $W = \{x(1), \dots, x(n)\}$,

$W(\ell) = W - \{x(\ell)\}$ then $R_{W \cup Y}^+ / \sum_{\ell=1}^n R_{W(\ell) \cup Y}^+$ is a free abelian group.

Proof. 1) Follows by 2) for $n = 0$, $Y = X$.

2) This is phrased because it is the natural way to prove 1) by induction on $|Y|$, for all n simultaneously (a degenerated case of [Sh 87a]). If $|Y| > \aleph_0$, let $Y = \{y(\alpha) : \alpha < \lambda\}$ with no repetitions, so $\lambda = |Y|$, $Y_\alpha = \{y(i) : i < \alpha\}$. It suffices for each

$\alpha < \lambda$ to prove that $G_\alpha =: (R_{Y_{\alpha+1}}^+ + \sum_{\ell=1}^n R_{Y \cup W(\ell)}^+) / (\sum_{\ell=1}^n R_{Y_{\alpha+1} \cup W(\ell)}^+ + R_{Y_\alpha \cup W}^+)$ is free.

We now show that G_α is isomorphic to $G'_\alpha = R_{Y_{\alpha+1}}^+ / (\sum_{\ell=1}^n R_{Y_{\alpha+1} \cup W(\ell)}^+ + R_{Y_\alpha \cup W}^+)$.

For this it is enough to show $(\sum_{\ell=1}^n R_{Y_\alpha \cup W(\ell)}^+) \cap R_{Y_{\alpha+1}}^+ = \sum_{\ell=1}^n R_{Y_{\alpha+1} \cup W(\ell)}^+$, as the right

side is included in the left side trivially we have to show $\sum_{\ell=1}^n \frac{p_\ell}{q_\ell} \in \sum_{\ell=1}^n R_{Y_{\alpha+1} \cup W(\ell)}^+$

if $\frac{p_\ell}{q_\ell} \in R_{Y_1 \cup W(\ell)}^+$ and $\sum_{\ell=1}^n \frac{p_\ell}{q_\ell} \in R_{Y_{\alpha+1} \cup W}^+$ which is easy by projections). But G'_α is free by induction hypothesis.

The next claim completes the case “ y countable”.

Claim. *If $Y \cup \{x(1), \dots, x(n)\} \subseteq X, x(\ell) \in X \setminus Y$ distinct, $G = R_{W \cup Y}^+$, $I = \sum_i I_i, I_i = R_{W(i) \cup Y}^+$ then G/I is free, when Y is countable.*

Proof. It suffices to prove:

- (a) G/I is torsion free
- (b) if $a_1, \dots, a_k \in G/I$ are independent, then $\{m \in Z^+ : \text{there are } \langle q_1, \dots, q_k \rangle \in L \text{ such that } \sum_{i=1, \dots, k} q_i a_i \text{ is divisible by } m \text{ in } G/I\}$ is finite, where $L = \{\langle q_1, \dots, q_k \rangle : q_i \in Z, \text{ not all zero and they are with no common divisor}\}$.

Let $x_1(q) \in X$ for $q = 1, \dots, n$ be new distinct variable and let $V = \{x_1(1), \dots, x_1(n)\}$. For $u \subseteq \{1, \dots, n\}$ let us define $h_u : R_{V \cup W \cup Y} \rightarrow R_{V \cup Y}$ an isomorphism $h_u(y) = y$ for $y \in Y, h_u(x(q)) = x_1(q)$ if $q \in u, h_u(x(q)) = x(q)$ if $q \notin u$. So let $a_1 + I, \dots, a_k + I$ be independent.

Suppose $\langle q_1, \dots, q_k \rangle \in L, m_0 m_1 \in Z \setminus \{0\}, m_0 m_1$ divides $\sum_i m_0 q_i a_i + I$. So for some $s \in R_{W \cup Y}$ and $p_\ell \in I_\ell$ for $\ell = 1, \dots, n$ we have: $\sum_i m_0 q_i a_i = m_0 m_1 s +$

$\sum_{\ell=1, \dots, n} p_\ell$. Let u vary on subsets of $\{1, \dots, n\}, b_u = \sum_u (-1)^{|u|} h_u(a_\ell) \in R_{V \cup W \cup Y}$, so $\sum_i m_0 q_i b_i = \sum_u (\sum_i m_0 q_i (h_u(a_i))) = m_0 m_1 \sum_u h_u(s) + \sum_{\ell=1, \dots, n} \sum_u h_u(p_\ell)$. How-

ever for each $\ell = 1, \dots, n$ we have $\sum_u h_u(p_\ell)$ is zero (as $x(\ell)$ does not appear in it).

So $\sum_i m_0 q_i b_i$ is divisible by $m_0 m_0$ in $R_{V \cup W \cup Y}^+$. As $R_{V \cup W \cup Y}^+$ is free, it suffices to

prove $\{b_i : i = 1, \dots, k\}$ is independent, equivalently they are linearly independent (over the rationals) in $Z(Y \cup W \cup V)$. But, if not, we can substitute suitable numbers for $x_1(1), \dots, x_1(n)$ and get contradiction to “ $\{a_i + I : i = 1, \dots, n\}$ is independent.” That is let R' be a subring of $R_{V \cup W \cup Y}$ generated by $Z[X] \cup \{\frac{1}{q_1}, \dots, \frac{1}{q_m}\}$ for some $m, q, \dots, q_\ell \in Z[X]$ such that $h_u(a_i) \in R'$. Let g be a homomorphism from R' to $R_{W \cup Y}$ which is the identity on $R_{W \cup Y}$ and maps each $x_1(q)$ to an integer (so we require from $\langle g(x_i(q)) : q = 1, \dots, n \rangle$ to make some finitely many polynomials over the integers nonzero which is possible). Now $\ell \in u \subseteq \{1, \dots, n\} \Rightarrow h_u(a_i) \in I_\ell$. So it is enough to show that $\langle g(b_i) : i = 1, \dots, k \rangle$ is linearly independent. But $g(b_i) = \sum_u g(h_u(a_i)) \in gh_\emptyset(a_i) + I = g(a_i) + I = a_i + I$.

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