CANONICAL UNIVERSAL LOCALLY FINITE GROUPS SH1175

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ABSTRACT. We prove that for $\lambda = \beth_\omega$ or just λ strong limit singular of cofinality $\aleph_0$, if there is a universal member in the class $\mathbf{K}^{\rm lf}_\lambda$ of locally finite groups of cardinality λ, then there is a canonical one (parallel to special models for elementary classes, which is the replacement of universal homogeneous ones and saturated ones in cardinals $\lambda = \lambda^{<\lambda}$).

For this, we rely on the existence of enough indecomposable such groups, as proved in ". Density of indecomposable locally finite groups". We also more generally deal with the existence of universal members in general classes for such cardinals.

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§ 0. Introduction

§ 0(A). Background and aims.

Our motivation is investigating the class $\mathbf{K}_{\rm lf}$ of locally finite groups (that is, groups such that every finitely generated sub-group is finite), so the reader may consider only this case ignoring the general case; or consider universal classes (see Def. 0.6). We continue [She17a], see history there and see more in [She], and on earlier history the book [KW73].

We wonder:

Problem 0.1. 1) Is there a universal $G \in \mathbf{K}^{\rm lf}_\lambda$ (= the class of members of $\mathbf{K}_{\rm lf}$ of cardinality λ), see 0.5(1); e.g. for $\lambda = \beth_\omega$? Or just λ strong limit of cofinality $\aleph_0$ (which is not above a compact cardinal)?

2) May there be a universal $G \in \mathbf{K}^{\rm lf}_\lambda$, when $\lambda < \lambda^{\aleph_0}$, e.g. for $\lambda = \aleph_1 < 2^{\aleph_0}$, i.e. consistently?

Generally, on the problem of the existence of a universal model of a class in cardinality λ see the classical Jonsson [Jón56], [Jón60], Morley-Vaught [MV62] and the recent survey [Dža05] and later [She21].

Returning to locally finite groups, concerning 0.1(1) recall that by Grossberg-Shelah [GS83], if $\lambda = \lambda^{\aleph_0}$ then there is no universal member of $\mathbf{K}^{\rm lf}_\lambda$. But if λ, a strong limit cardinal of cofinality, $\aleph_0$ is above a compact cardinal κ, then there is $G \in \mathbf{K}^{\rm lf}_\lambda$ which is universal. So Problem 0.1 address the main open cases.

More fully by [She16, 2.17 = L s56] we have:

Conclusion 0.2. *1) $\mu = {\rm cf}(\mu)$, $\mu^+ < \lambda = {\rm cf}(\lambda) < 2^\mu$ then there is no group of cardinality λ universal for the class of locally finite groups.*

2) For example, if $\aleph_2 \le \lambda = {\rm cf}(\lambda) < 2^{\aleph_0}$ this applies.

So the only other cases left are:

(A) ${\rm cf}(\lambda) < \lambda < 2^{\aleph_0}$,

(B) $\beth_\delta < \lambda < 2^{\beth_\delta}$, ${\rm cf}(\delta) = \aleph_0$ and λ singular.

Question: Are there partial long orders on $\{{}^\theta G \colon G \in \mathbf{K}_{\rm lf}\}$?

Above, if $\aleph_0 < \mu = {\rm cf}(\mu) < \lambda$, $2^\mu > \lambda$ such that $\mathbf{T}_{J^{\rm bd}_\mu}(\lambda) < 2^\mu$, then there is no universal in λ, as in [KS92], see [S$^+$].

Let us consider the model theory of locally finite groups.

Recall

Definition 0.3. 1) G is a lf (locally finite) group if G is a group and every finitely generated subgroup is finite.

2) G is an exlf (existentially closed lf) group (in [KW73] it is called ulf, universal locally finite group) when G is a locally finite group and for any finite groups $K \subseteq L$ and embedding of K into G, the embedding can be extended to an embedding of L into G.

3) Let $\mathbf{K}_{\rm lf}$ be the class of lf (locally finite) groups (partially ordered by $\subseteq$, being a subgroup) and let $\mathbf{K}_{\rm exlf}$ be the class of existentially closed $G \in \mathbf{K}_{\rm lf}$.

Wehrfritz asked about the categoricity of the class of exlf groups in any $\lambda > \aleph_0$. This was answered by Macintyre-Shelah [MS76] which proved that in every $\lambda > \aleph_0$ there are 2^λ non-isomorphic members of $\mathbf{K}^{\rm exlf}_\lambda$. This was disappointing in some sense: in $\aleph_0$ the class is categorical, so the question was perhaps motivated by the hope that also general structures in the class can be understood to some extent.

The existence of a universal can be considered as a weak positive answer.

A natural and frequent question on a class of structures is the existence of rigid members, i.e. ones with no non-trivial automorphism. Now any exlf group $G \in \mathbf{K}_{\rm exlf}$ has non-trivial automorphisms - the inner automorphisms (recalling it has a trivial center). So the natural question is about complete members where a group is called complete iff it has no non-inner automorphism.

Concerning the existence of a complete, locally finite group of cardinality λ: Hickin [Hic78] proved one exists in $\aleph_1$ (and more, e.g. he finds a family of $2^{\aleph_1}$ such groups pairwise far apart, i.e. no uncountable group is embeddable into two of them). Thomas [Tho86] assumed G.C.H. and built one in every successor cardinal (and more, e.g. it has no Abelian or just solvable subgroup of the same cardinality). Related are Giorgetta-Shelah [GS84], Shelah-Zigler [SZ79], which investigate $\mathbf{K}_{G_*}$ getting similar results.

Dugas-Göbel [DG93, Th.2] prove that for $\lambda = \lambda^{\aleph_0}$ and $G_0 \in \mathbf{K}^{\rm lf}_{\le\lambda}$ there is a complete $G \in \mathbf{K}^{\rm exlf}_{\lambda^+}$ extending G_0; moreover 2^{λ^+} pairwise non-isomorphic ones. Then Braun-Göbel [BG03] got better results for complete locally finite p-groups.

Now [She17a] show that though the class $\mathbf{K}_{\rm exlf}$ is very "unstable" there is a large enough set of definable types so we can imitate stability theory and have reasonable control in building exlf groups, using quantifier-free types. This may be considered a "correction" to the non-structure results discussed above. This was applied to build a canonical extension of a locally finite group of the same cardinality which is existentially closed (it was known to exist in the power set, see [KW73]). Also, there are endo-rigid locally finite groups in a more relaxed way.

Returning to the present work, here we deal with the universality problem for $\mu = \beth_\omega$ or just strong limit of cofinality $\aleph_0$. We prove for $\mathbf{K}_{\rm lf}$ and similar classes that if there is a universal model of cardinality μ, then there is something like a special model of cardinality μ, in particular, universal, and unique up to isomorphism. This relies on [She20], which proves the existence and even the density of so-called θ-indecomposable (i.e. θ is not a possible cofinality) models in $\mathbf{K}_{\rm lf}$ of various cardinalities continuing Carson-Shelah [CS20] which deal with the class of groups.

Returning to Question 0.1(1), a possible avenue is to try to prove the existence of universal members in μ when $\mu = \Sigma_{n<\omega}\mu_n$ each μ_n measurable $< \mu$, i.e. maybe for some reasonable classes this holds.

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§ 0(B). **Definitions.**

Context 0.4. $\mathbf{K}$ will be one of the following cases:

Case 1: $\mathbf{K} = \mathbf{K}_{\rm lf}$, the class of locally finite groups, so the submodel relation is just a subgroup,

Case 2: $\mathbf{K}$ is a universal class, see Def 0.6(1) below, the submodel relation means just a submodel,

Case 3 $\mathbf{K}$ is $\mathfrak{k} = (K_{\mathfrak{k}}, \le_{\mathfrak{k}})$ an a.e.c. with $\mathrm{LST}_{\mathfrak{k}} < \mu$, see [She09, §1]; we shall only comment on it. In particular, in this context, in the definitions, $M \subseteq N$ should be replaced by $M \le_{\mathfrak{k}} N$.

Definition 0.5. 1) We say that $M \in \mathbf{K}_\mu$ is universal (in $\mathbf{K}$ or in $\mathbf{K}_\mu$, see 0.6) when every member of $\mathbf{K}_\mu$ can be embedded into it.

2) We say that $M \in \mathbf{K}$ is universal for $\mathbf{K}_{<\mu}$ when every $M \in \mathbf{K}_{<\mu}$ can be embedded into it; see Def 0.6(4) below.

3) We define similarly "$M \in \mathbf{K}$ is universal for $\mathbf{K}_\mu$" and "$M \in \mathbf{K}$ is universal for $\mathbf{K}_{\leq\mu}$".

Definition 0.6. 1) We shall say that $\mathbf{K}$ is a universal class when for some vocabulary $\tau = \tau_{\mathbf{K}}$:

(a) $\mathbf{K}$ is a class of τ-models, closed under isomorphisms,
(b) a τ-model belongs to $\mathbf{K}$ iff every finitely generated sub-model belongs to it,

3) Let $\mathbf{K}_\mu$ be the class of $M \in \mathbf{K}$ of cardinality μ. We define $\mathbf{K}_{<\mu}, \mathbf{K}_{\leq\mu}$ naturally.

4) For cardinals $\lambda \leq \mu$ let $\mathbf{K}_{\mu,\lambda}$ be the class of pairs (N, M) such that $N \in \mathbf{K}_\mu, M \in \mathbf{K}_\lambda$ and $M \subseteq N$.

5) Let $(N_1, M_1) \leq_{\mu,\lambda} (N_2, M_2)$ mean that $(N_\ell, M_\ell) \in \mathbf{K}_{\mu,\lambda}$ for $\ell = 1, 2$ and $M_1 \subseteq M_2, N_1 \subseteq N_2$.

6) For $\lambda \leq \mu$ we define $\mathbf{K}_{\mu,<\lambda}$ and $\leq_{\mu,<\lambda}$ similarly.

7) A universal class $\mathbf{K}$ can be considered as the a.e.c. $\mathfrak{k} = (\mathbf{K}, \subseteq)$

Notation 0.7. 1) Let M, N and also G, H, L denote members of $\mathbf{K}$.

2) Let $|M|$ be the universe = set of elements of M and $\|M\|$ its cardinality.

3) Let a, b, c, d denote members of such M, and $\bar{a}, \bar{b} \ldots$ denote sequences of such elements.

Definition 0.8. 1) We say the pair (N, M) is an (χ, μ, κ)-amalgamation base (or amalgamation pair, but we may omit χ when $\chi = \mu$, and we may even omit μ, κ too when clear from the context) when:

(a) $(N, M) \in \mathbf{K}_{\mu,\kappa}$,
(b) if $N_1 = N$ and $M \subseteq N_2 \in \mathbf{K}_\chi$
then N_1, N_2 can be amalgamated over M, this mean that for some N_3, f_1, f_2 we have $M \subseteq N_3 \in \mathbf{K}$ and f_ℓ-embeds N_ℓ into N_3 over M for $\ell = 1, 2$.

2) We say that the pair (N, M) is a universal (μ, λ)-amalgamation base (we may omit μ, λ) when:

(a) $(N, M) \in \mathbf{K}_{\mu,\lambda}$,
(b) if $N \subseteq N' \in \mathbf{K}_\mu$ then N' can be embedded into N over M.

3) We may in parts (1),(2) omit μ, κ when $(\mu, \lambda) = (\|N\|, \|M\|)$.

4) We say $M \in \mathbf{K}_{<\mu}$ is an amalgamation base inside $\mathbf{K}_{<\mu}$ when: if $M \leq N_\ell \in \mathbf{K}_{<\mu}$, then N_1, N_2 can be amalgamated over M (see 0.8(1)(b)) but $N_3 \in \mathbf{K}_{<\mu}$.

§ 1. Indecomposability

In this section we deal with indecomposability, equivalently $\mathrm{CF}(M)$, see e.g. [ST97]; we have $\mathbf{K}_{\mathrm{lf}}$ in mind but still it is meaningful and of interest also for other classes.

Why do we deal with indecomposable members $\mathbf{K}$? When we shall try to understand universal members M of $\mathbf{K}_\mu$ we shall use some θ-indecomposable $N \subseteq M$ of cardinality $< \mu$. How will this help us? The point is that $N \in \mathbf{K}_{<\mu}$ may have too many embeddings into M, but if $\theta = \mathrm{cf}(\theta) \neq \mathrm{cf}(\mu)$, $\alpha < \mu \Rightarrow |\alpha|^{\|N\|} < \mu$, N is θ-indecomposable and θ is regular uncountable $< \mu$, then this is not the case.

We need indecomposable $\mathbf{c} : [\lambda] \to \theta$ in order to build enough θ-indecomposable locally finite groups (as done in [She20]).

Definition 1.1. 1) We say M is θ-indecomposable or $\theta \in \mathrm{CF}(M)$ <u>when</u>: θ is regular and if $\langle M_i : i < \theta\rangle$ is $\subseteq$-increasing with union M, <u>then</u> $M = M_i$ for some i.

2) We say M is Θ-indecomposable <u>when</u> it is θ-indecomposable for every $\theta \in \Theta$. We say M is Θ^{orth}-indecomposable <u>when</u> it is θ-indecomposable for every regular $\theta \notin \Theta$.

3) We say G is θ-indecomposable inside G^+ <u>when</u>:

- (a) $\theta = \mathrm{cf}(\theta)$;
- (b) $G \subseteq G^+$;
- (c) if $\langle G_i : i \leq \theta\rangle$ is $\subseteq$-increasing continuous and $G_\theta = G^+$ (hence $G \subseteq G_\theta$) <u>then</u> for some $i < \theta$ we have $G \subseteq G_i$.

4) For $\theta = \mathrm{cf}(\theta) \leq \lambda \leq \mu$ such that $\theta \notin \Theta_\lambda$ (see 1.2(1)) we say $\mathbf{K}$ is (μ, λ, θ)-indecomposable <u>when</u> for every pair $(N, M) \in \mathbf{K}_{\mu,\lambda}$ there is $(N_1, M_1) \in \mathbf{K}_{\mu,\lambda}$ which is $\leq_{\mu,\lambda}$-above it and M_1 is θ-indecomposable (really, not just inside N_1). For $\theta = \mathrm{cf}(\theta) < \lambda \leq \mu$ we say $\mathbf{K}$ is $(\mu, <\lambda, \theta)$-indecomposable <u>when</u>:

if $\theta = \mathrm{cf}(\theta) \leq \lambda_1 < \lambda, \theta \notin \Theta_{\lambda_1}$ then $\mathbf{K}$ is (μ, λ_2, θ)-indecomposable for some $\lambda_2 \in [\lambda_1, \mu)$.

5) We say $\mathbf{c} : [\lambda]^2 \to S$ is θ-indecomposable <u>when</u>: if $\langle u_i : i < \theta\rangle$ is $\subseteq$-increasing sequence of sets with union λ then $S = \{\mathbf{c}\{\alpha, \beta\} : \alpha \neq \beta \in u_i\}$ for some $i < \theta$;

6) We may replace above the cardinal θ by a set or class Θ of regular cardinals, (as done in 1.1(2)).

A group G may be considered indecomposable as a group or as a semi-group; our default choice is semi-group; but note that for locally finite groups the two interpretations are equivalent.

The following was proved in [She20].

Theorem 1.2. *1) If $\lambda \geq \aleph_1$ and we let $\Theta_\lambda = \{\mathrm{cf}(\lambda)\}$ except that $\Theta_\lambda = \{\mathrm{cf}(\lambda), \partial\} = \{\lambda, \partial\}$ when $(c)_{\lambda,\partial}$ below holds, <u>then</u> clauses (a),(b) hold, where:*

- (a) *some $\mathbf{c} : [\lambda]^2 \to \lambda$ is θ-indecomposable for every $\theta = \mathrm{cf}(\theta) \notin \Theta_\lambda$*
- (b) *for every $G_1 \in \mathbf{K}^{\mathrm{lf}}_{\leq\lambda}$ there is an extension $G_2 \in \mathbf{K}^{\mathrm{lf}}_\lambda$ which is $\Theta^{\mathrm{orth}}_\lambda$-indecomposable*
- $(c)_{\lambda,\partial}$ *for some $\mu, \lambda = \mu^+, \mu > \partial = \mathrm{cf}(\mu)$ and $\mu = \sup\{\theta < \mu : \theta$ is a regular Jonsson cardinal$\}$.*

2) If $\mu \geq \lambda \geq \theta = \mathrm{cf}(\theta)$, and $\theta \notin \Theta_\lambda, \lambda \geq \aleph_1$ <u>then</u> $\mathbf{K}_{\mathrm{lf}}$ is (μ, λ, θ)-indecomposable.

2A) In fact (on part (2)) it suffice to assume $(\exists\lambda_1)(\lambda \leq \lambda_1 \leq \mu \wedge \theta \notin \Theta_{\lambda_1})$.

3) If $\mu \geq \lambda$ and $(H_1, G_1) \in \mathbf{K}_{\leq\mu,\leq\lambda}$ <u>then</u> we can find a pair $(H_2, G_2) \in K_{\mu,\lambda}$ such that:

(a) *G_2 is $\Theta^{\rm orth}_\lambda$-indecomposable,*
(b) *if $\mu > \lambda$ then G_1 is θ-indecomposable inside H_2 for every regular θ,*
(c) *H_2 is $\Theta^{\rm orth}_\mu$-indecomposable.*

Proof. 1) By[1] [She20, Th. 3.5].

2), 2A) The proof will serve also for part (3). Let $(N, M) \in \mathbf{K}_{\mu,\lambda}$ be given. We choose a pair (χ, ∂) of cardinals and $\mathbf{c}$ such that $\lambda \leq \chi \leq \mu$ $\partial = \mathrm{cf}(\partial) \leq \lambda, \partial \neq \theta$ and $\mathbf{c} \colon [\chi]^2 \to \chi$ is θ-indecomposable; (possible here as $\theta \notin \Theta_\lambda, \lambda \geq \aleph_1$ even for $\chi = \lambda$).

By induction of $\alpha \leq \partial$ we choose H_α, L_α, but L_α is chosen together with $H_{\alpha+1}$ when α is a successor ordinal, such that:

(a) $(H_\alpha, L_\alpha) \in \mathbf{K}_{\mu,\lambda}$ is increasing continuous with α
(b) $(H_0, L_0) = (N, M)$,
(c) if $\alpha = \beta + 1 < \theta$ then and L_α is θ-indecomposable.

Why can we carry the induction? For $\alpha = 0$ this is trivial; similarly for α a limit ordinal. Lastly by clause (b) of part (1), for $\alpha = \beta + 1 \leq \alpha_*$, recall the proof of [She20, 3.4], pedantically as without loss of generality H_β, L_β are existentially closed hence generated by the elements of order 2, let $\langle a_\alpha \colon \alpha < \mu \rangle$ list $\{a \in L_\beta \colon a \text{ of order } 2\}$. By [She20, Prop. 3.4(2)], with $u_\alpha = \{\alpha\}$, we can find $H_{\alpha,1} \in \mathbf{K}^{\rm lf}_\mu$ extending H_B and pairwise commuting $b_\alpha \in H_{\alpha,1}$ each of order 2, for $\alpha < \mu$ (the order 2 was not mention but proved) and pairwise commuting $d_\alpha \in H_{\alpha,1}$, each of order 2, for $\alpha < \mu$ such that, L_β is included in the subgroup $L_{\alpha,1}$ of $H_{\alpha,1}$ generated by $\{b_\alpha, d_\alpha \colon \alpha < \lambda\}$.

Now apply [She20, Prop. 3.4(1)] for a θ-indecomposable $\mathbf{c} \colon [\lambda]^2 \to \lambda$.

3) We deal with every regular $\theta \leq \mu$ successively. Fixing θ we can use the proof of part (2). $\square_{1.2}$

Now comes the central definition, what is its role?

We like to sort out when there is a universal member of $\mathbf{K}_\mu$ and when there is a canonical universal member. For reasons explained above we concentrate on the case μ is strong limit of cofinality $\aleph_0$, for example $\beth_\omega$. To find out the answer to those two questions for every universal class $\mathbf{K}$ seem too much to hope for. The Def 1.3 accomplishes a more modest task: it gives a large frame satisfied by a large family of pairs $(\mathbf{K}, \mu)$ for which we shall prove an equivalence. In particular our class $\mathbf{K}_{\rm lf}$ belongs to this family.

Definition 1.3. We say that $\mathbf{K}$ is μ-nice <u>when</u>:

(a) $\tau_{\mathbf{K}}$ has cardinality $< \mu$,
(b) for every $M \in \mathbf{K}_{<\mu}$ there is $N \in \mathbf{K}_\mu$ extending M,
(c) $\mathbf{K}$ has the JEP (joint embedding property),
(d) $\mathbf{K}$ is $(\mu, <\mu, \mathrm{cf}(\mu))$-indecomposable or just,
(d)′ for arbitrarily large $\lambda_2 < \mu$ letting $\theta = \mathrm{cf}(\mu) \leq \lambda_2$ we have $\mathbf{K}$ is (μ, λ_2, θ)-indecomposable.

Naturally we like to prove that the pair $(\mathbf{K}_{\rm lf}, \beth_\omega)$ falls under the frame of Def 1.3. This is the role of 1.4, 1.5. In §3 we point out an additional family. For the main case, μ is a strong limit of cofinality $\aleph_0$.

Claim 1.4. $\mathbf{K}_{\rm lf}$ *is μ -nice when $\mu \geq \aleph_1$.*

[1] But Theorem 1.5 in the author's archive version, similarly 3.4 is 1.4

Proof. In Def 1.3 clause (a) is trivial and as $\mathbf{K}_{\rm lf}$ is closed under products clearly clauses (b),(c) are clear and clause (d), for μ regular is trivial (and is not used) and for μ singular it holds by 1.2(3), see also 1.5(2) below. $\square_{1.4}$

We give below more than what is strictly needed.

Claim 1.5. *Assume $\mathbf{K} = \mathbf{K}_{\rm lf}$.*

1) We have (A) $\Rightarrow$ (B) where:

- (A)
  - (a) $\mu \ge \aleph_1$,
  - (b) $\delta_* \le \mu$ *and* $\lambda_\alpha < \mu$ *for* $\alpha < \delta_*$,
  - (c) $\lambda_\alpha \ge |\alpha|$ *is non-decreasing,*
  - (d) $G_1 \in \mathbf{K}_{\le \mu}$,
  - (e) $G_{1,\alpha} \in \mathbf{K}_{\le \lambda_\alpha}$ *is a subgroup of* G_1 *for* $\alpha < \delta_*$,
- (B) *There are* $G_2, \bar{G}_2$ *such that:*
  - (a) $G_2 \in \mathbf{K}_\mu$ *extends* G_1,
  - (b) $\bar{G}_2 = \langle G_{2,\alpha} \colon \alpha < \delta_* \rangle$ *is increasing,*
  - (c) $G_{2,\alpha} \in \mathbf{K}_{\lambda_\alpha}$ *extend* $G_{1,\alpha}$,
  - (d) G_2 *is* Θ*-indecomposable where*[2] $\Theta = (\Theta_\mu \cup \{{\rm cf}(\delta_*)\})^{\rm orth}$
  - (e) $G_{2,\alpha}$ *is* $\Theta^{\rm orth}_{\lambda_\alpha}$*-indecomposable (not just inside* H_2*) for every* $\alpha < \delta_*$
  - (f) *if* $\mu = \Sigma\{\lambda_\alpha \colon \alpha < \delta_*\}$ *the* $G_2 = \cup\{G_{2,\alpha} \colon \alpha < \delta_*\}$,
  - (g) *if* $\mu > \sum\{\lambda_\alpha \colon \alpha < \delta_*\}$ *then* G_2 *is* $\Theta^{\rm orth}_M$*-indecomposable.*

2) If $\mu > \lambda \ge \aleph_1$ <u>*then*</u> $\aleph_0 \in \Theta^{\rm orth}_{{\rm cf}(\mu)} \cup \Theta^{\rm orth}_\lambda$ *except possibly when* $\mu = \lambda^+, {\rm cf}(\lambda) = \aleph_0$.

Proof. 1) By induction of $\alpha \le \delta_*$ we choose $H_\alpha, \bar{H}_\alpha, L_\alpha$, but L_α is chosen together with $H_{\alpha+1}$ and not chosen for $\alpha = \alpha_*$, such that:

- (a) H_α is increasing continuous with α
- (b) $H_0 = G_1$ and $\alpha > 0 \Rightarrow H_\alpha \in \mathbf{K}_\mu$
- (c) $(H_\alpha, L_\beta) \in \mathbf{K}_{\lambda,\lambda_\beta}$ when $\alpha = \beta + 1 \le \alpha_*$
- (d) $\bar{H}_\alpha = \langle H_{\alpha,\varepsilon} : \varepsilon < \delta_* \rangle$ such that if $\mu = \Sigma\{\lambda_\varepsilon : \varepsilon < \delta_*\}$ then this sequence is increasing with union H_α and $H_{\alpha,\varepsilon}$ has cardinality λ_ε when $\alpha > 0$ and $\le \lambda_\varepsilon$ when $\alpha = 0$
- (e) $G_{1,\beta}, H_{\beta,\varepsilon}, L_\gamma$ are sub-groups of L_α when $\beta \le \alpha, \varepsilon \le \alpha, \gamma < \alpha$
- (f) L_β is $\Theta^{\rm orth}_{\lambda_\beta}$-indecomposable,
- (g) G_2 is Θ-indecomposable where $\Theta = (\Theta_\mu \cup \{{\rm cf}(\delta_*)\})^{\rm orth}$

Why can we carry the induction? We choose $\bar{H}_\alpha$ just after H_α was chosen. For $\alpha = 0$ this is trivial (note that L_α is not chosen), similarly for α a limit ordinal. Lastly for $\alpha = \beta + 1 \le \alpha_*$, Definition 1.1(4) 1.2(3) gives the desired conclusion. In details, first choose $L^+_\beta \subseteq H_\beta$ of cardinality at most λ_α containing the desired sets (listed in clause (e)). Then apply 1.2(3) to the pair (H_β, L^+_β) to get (H_α, L_α). Lastly, if $\mu > \sum\{\lambda^+_\alpha \colon \alpha < \delta_*\}$, let $G_2 \in \mathbf{K}_\lambda$ extend $H_{\alpha_{\delta_*}}$ and satisfies the indecomposablity demand and, if $\mu > \sum\{\lambda_\alpha \colon \alpha < \delta_*\}$, let $G_2 = H_{\delta_*}$. Now, letting $G_{2,\alpha} = L_\alpha$ we are done.

2) Easy. $\square_{1.5}$

Claim 1.6. *If μ is strong limit singular and $N \in \mathbf{K}_\mu$ <u>then</u> the set ${\rm IDC}_{<\mu}(N)$ has cardinality $\le \mu$ where, for $N \in \mathbf{K}_\mu$,*

[2] If $\mu = \sum\{\lambda_\alpha \colon \alpha < \delta_*\}$ then ${\rm cf}(\mu) = {\rm cf}(\delta_*)$, hence the "$\cup\{\delta_*\}$" is not necessary.

(∗) $\mathrm{IDC}_{<\mu}(N) = \{M : M \subseteq N$ *has cardinality* $< \mu$ *and is* $\mathrm{cf}(\mu)-$*indecomposable* $\}$.

Proof. Easy. $\square_{1.6}$

§ 2. Universality

For quite many classes, there are universal members in any (large enough) μ which is strong limit of cofinality $\aleph_0$, see [She17b] which includes history. Below we investigate "is there a universal member of $\mathbf{K}^{\rm lf}_\mu$ for such μ". We prove that if there is a universal member, e.g. in $\mathbf{K}^{\rm lf}_\mu$, then there is a canonical one.

What do we mean by "canonical"? This is not a precise definition, but we mean it is unique up to isomorphism, by a natural definition. Examples we have in mind are the algebraic closure of a field, the saturated model of a complete first order theory T in cardinality $\mu^+ = 2^\mu > |T|$ and the special model of a complete first order theory T in a singular strong limit cardinal $\mu > |T|$, see [CK62]. The last one means:

- $(*)$ for such T, μ we say that M is a special model of T of cardinality μ when some $\bar M$ witness it which means:
  - (a) $\bar M = \langle M_i : i < \mathrm{cf}(\mu)\rangle$,
  - (b) M_i is $\prec$-increasing with i,
  - (c) each M_i has cardinality $< \mu$,
  - (d) $M = \cup\{M_i : i < \mathrm{cf}(\mu)\}$,
  - (e) for every $\lambda < \mu$ for every large enough $i < \mathrm{cf}(\mu)$ the model M_i is λ^+-saturated.

Considering our main case, $\mathbf{K}_{\rm lf}$, a major difference (between what we prove here (e.g. for $\mathbf{K}_{\rm lf}$) and $(*)$ is that here amalgamation fail, so clause (B) of 2.1 is a poor man replacement.

Theorem 2.1. *Assume that μ is a strong limit of cofinality $\aleph_0$ and $\mathbf{K}$ is μ-nice.*

1) The following conditions are equivalent:

- (A) *There is a universal $G \in \mathbf{K}_\mu$.*
- (B) *If $H \in \mathbf{K}_\lambda$ is $\aleph_0$-indecomposable for some $\lambda < \mu$, <u>then</u> there is a sequence $\bar G = \langle G_\alpha : \alpha < \alpha_* \le \mu\rangle$ such that:*
  - (a) *$H \subseteq G_\alpha \in \mathbf{K}_\mu$,*
  - (b) *if $G \in \mathbf{K}_\mu$ extend H, <u>then</u> for some α, G is embeddable into G_α over H.*
- $(B)^+$ *We can add in (B)*
  - (c) *if $\alpha_1 < \alpha_2 < \alpha_*$, <u>then</u> $G_{\alpha_1}, G_{\alpha_2}$ cannot be amalgamated over H, that is there are no G, f_1, f_2 such that $H \subseteq G \in \mathbf{K}$ and f_ℓ embeds G_{α_ℓ} into G over H for $\ell = 1, 2$,*
  - (d) *(H, G_α) is an amalgamation pair (see Definition 0.8(1)), moreover a universal amalgamation base (see 0.8(2)).*

2) We can add in part (1):

- (C) *there is G_* such that:*
  - (a) *$G_* \in \mathbf{K}_\mu$ is universal for $\mathbf{K}_{<\mu}$;*
  - (b) *$\mathscr{E}^{\aleph_0}_{G_*,<\mu}$, see see Def. 2.2 below, is an equivalence relation with $\le \mu$ equivalence classes;*
  - (c) *G_* is μ-special, see 2.2(E) below.*
- $(C)^+$ *like clause (C) but we add*
  - (d) *If $G, G_* \in \mathbf{K}_\mu$ are μ-special <u>then</u> G, G_* are isomorphic, (that is uniqueness).*

Before we prove 2.1, we define (this definition is not just used in the proof but also in phrasing 2.1(2)).

Definition 2.2. For $\theta = \mathrm{cf}(\mu) < \mu$ and $M_* \in \mathbf{K}_\mu$: we define:

(A) $\mathrm{IND}^\theta_{M_*,<\mu} = \{N : N \leq_{\mathfrak{k}} M_*$ has cardinality $< \mu$ and is θ-indecomposable$\}$.

(B) $\mathscr{F}^\theta_{M_*,<\mu} = \{f$: for some θ-indecomposable $N = N_f \in K_{<\mu}$ with universe an ordinal, f is an embedding of N into $M_*\}$.

(C) $\mathscr{E}^\theta_{M_*,<\mu} = \{(f_1, f_2)\colon f_1, f_2 \in \mathscr{F}^\theta_{M_*,<\mu}, N_{f_1} = N_{f_2}$ and there are embeddings g_1, g_2 of M_* into some extension $M \in \mathbf{K}_\mu$ of M_* such that $g_1 \circ f_1 = g_2 \circ f_2\}$.

(D) We say M_* is $\theta - \mathscr{E}^\theta_{M_*,<\mu}$-indecomposably homogeneous (or just M_* is θ-indecomposably homogeneous) <u>when</u> some $\bar{M}$ witness it, which mean:
  (a) $\bar{M} = \langle M_i\colon i < \mathrm{cf}(\mu)\rangle$ is increasing continuous with limit M,
  (b) if $f_1, f_2 \in \mathscr{F}^\theta_{M_*,<\mu}$, $(f_1, f_2) \in \mathscr{E}^\theta_{M_*,<\mu}$ and $(\exists i < \theta)(A \subseteq M_i), A$ of cardinality $< \mu$, <u>then</u> there is $(g_1, g_2) \in \mathscr{E}^\theta_{M_*,<\mu}$ such that $f_1 \subseteq g_1 \wedge f_2 \subseteq g_2$ and $A \subseteq \mathrm{Rang}(g_1) \cap \mathrm{Rang}(g_2)$; it follows that if $\mathrm{cf}(\mu) = \aleph_0$ then for some $g \in \mathrm{aut}(M_*)$ we have $f_2 = g \circ f_1$.

(E) We say that $M_* \in \mathbf{K}_\mu$ is μ-special <u>when</u> it is θ-indecomposably homogeneous and is universal for $\mathbf{K}_{<\mu}$, that is every $M \in \mathbf{K}_{<\mu}$ is embeddable into it.

Remark 2.3. We may consider in 2.1 also $(A)_0 \Rightarrow (A)$ where

$(A)_0$ if $\lambda < \mu, H \subseteq G_1 \in \mathbf{K}_{<\mu}$ and $|H| \leq \lambda$, <u>then</u> for some G_2 we have $G_1 \subseteq G_2 \in \mathbf{K}_{<\mu}$ and (H, G_2) is a (μ, μ, λ)-amalgamation base.

Proof. It suffices to prove the following implications:

<u>$(A) \Rightarrow (B)$</u>:

Let $G_* \in \mathbf{K}_\mu$ be universal and choose a sequence $\langle G^*_n\colon n < \omega\rangle$ such that $G_* = \bigcup_n G^*_n, G^*_n \subseteq G^*_{n+1}, |G^*_n| < \mu$.

Let H be as in 2.1(B) and let $\mathscr{G} = \{g\colon g$ embed H into G^*_n for some $n\}$. So clearly $|\mathscr{G}| \leq \sum_n |G^*_n|^{|H|} \leq \sum_{\lambda<\mu} 2^\lambda = \mu$, (an over-kill).

Let $\langle g^*_\alpha : \alpha < \alpha_* \leq \mu\rangle$ list $\mathscr{G}$ and let (G_α, g_α) be such that (exist by renaming):

$(*)_1$ (a) $H \subseteq G_\alpha \in \mathbf{K}_\mu$;
  (b) g_α is an isomorphism from G_α onto G_* extending g^*_α.

It suffices to prove that $\bar{G} = \langle G_\alpha\colon \alpha < \alpha_*\rangle$ is as required in clause (B). Now clause (B)(a) holds by $(*)_1(a)$ above. As for clause (B)(b), let G satisfy $H \subseteq G \in \mathbf{K}_{\leq\mu}$, so there is $G' \in \mathbf{K}_\mu$ extending G, hence we can find an embedding g of G' into G_*. We know that $g(H) \subseteq G = \bigcup_n G_n$ hence $\langle g(H) \cap G_n\colon n < \omega\rangle$ is $\subseteq$-increasing with union $g(H)$; but $g(H)$ by the assumption on H is $\aleph_0$-indecomposable, hence $g(H) = g(H) \cap G^*_n \subseteq G^*_n$ for some n. This implies $g{\restriction}H \in \mathscr{G}$ and so for some $\alpha < \alpha_*$ we have $g \restriction H = g^*_\alpha$. Hence $g^{-1}_\alpha g$ is an embedding of G into G_* extending $(g_\alpha{\restriction}H)^{-1}(g{\restriction}H) = (g^*_\alpha)^{-1}(g^*_\alpha) = \mathrm{id}_H$ as promised.

<u>$(B) \Rightarrow (B)^+$</u>:

What about $(B)^+(c)$? while $\bar{G}$ does not necessarily satisfy it, we can "correct it", e.g. we choose u_α, v_α and if $\alpha \notin \cup\{v_\beta\colon \beta < \alpha\}$ also G'_α by induction on $\alpha < \alpha_*$ such that (the idea is that if $\beta \in v_\alpha$ then $\beta > \alpha$ and G_β is discarded being embeddable into some G'_α and G'_α is the "corrected" member):

$(*)^2_\alpha$ (a) $G_\alpha \subseteq G'_\alpha \in \mathbf{K}_\mu$ if $\alpha \notin \cup\{v_\beta : \beta < \alpha\}$;

(b) $u_\alpha \subseteq \alpha$ and $v_\alpha \subseteq \alpha_* \backslash (\alpha+1)$;
(c) if $\beta < \alpha$ then $u_\beta = u_\alpha \cap \beta$ and $u_\alpha \cap v_\beta = \emptyset$;
(d) if $\alpha = \beta+1$ then $\beta \in u_\alpha$ iff $\beta \notin \cup\{v_\gamma : \gamma < \beta\}$;
(e) if $\alpha \notin \cup\{v_\gamma \colon \gamma < \alpha\}$, then:
- $\bullet_1$ $\gamma \in v_\alpha$ iff ($\gamma > \alpha$ and) G_γ is embeddable into G'_α over H
- $\bullet_2$ if $\gamma \in \alpha_* \backslash (\alpha+1) \backslash (\cup\{v_\beta : \beta \leq \alpha$) then G_γ is not embeddable over H into any G' satisfying $G'_\alpha \subseteq G' \in \mathbf{K}$;

(f) if $\alpha = \beta+1$ and $\beta \notin u_\alpha$ then $v_\beta = \emptyset$.

Why this suffices?

Because if we let $u_{\alpha_*} = \alpha_* \backslash (\cup\{v_\gamma \colon \gamma < \alpha_*\})$, then $\langle G'_\alpha \colon \alpha \in u_{\alpha_*}\rangle$ is as required; but we elaborate.

First, for clause $(B)^+$(c) assume that $\alpha < \beta$ are from u_{α_*}. As $\beta \notin v_\alpha$, by $(*)^2_\alpha(e)\bullet_2$ we know that G_β is not embeddable into any extension of G'_α over H; but as $G_\beta \subseteq G'_\beta$ clearly also G'_β is not embeddable into any extension of G'_α over H. Renaming this means that G'_α, G'_β cannot be amalgamated over H, as promised.

Second for clause $(B)^+$(d), let $\alpha \in u_{\alpha_*}$ and we have to prove that the pair (G'_α, H) is a universal (μ,κ)-amalgamation base where κ is the cardinality of H. So assume $G' \in \mathbf{K}_\mu$ extends G'_α; recall that we are assuming that $\langle G_\alpha \colon \alpha < \alpha_*\rangle$ is as in clause (B), hence there are $\beta < \alpha_*$ and an embedding f of G' into G_β over H; we shall prove that $\beta = \alpha$ hence (recalling $G_\alpha \subseteq G'_\alpha$) f embeds G' into G'_α over H thus finishing proving (B) $\Rightarrow$ (B)$^+$.

If $\beta \in u_{\alpha_*} \setminus \{\alpha\}$ then $f \restriction G'_\alpha$ embed G'_α into G'_β over H, a contradiction to $(B)^+$(c) which we have already proved.

If $\beta \in \alpha_* \backslash u_{\alpha_*}$ then for some γ we have $\beta \in v_\gamma$ hence $\gamma < \beta$ and G_β is embeddable into G'_γ over H; hence G' is embeddable into G'_γ over H. As in the previous sentence necessarily $\gamma = \alpha$ and we are done.

Why can we carry out the induction?

For $\alpha = 0, \alpha$ limit we have nothing to do because u_α is determined by $(*)^2_\alpha(b)$ and $(*)^2_\alpha(c)$. For $\alpha = \beta+1$, if $\beta \in \bigcup\limits_{\gamma<\beta} v_\gamma$ we have nothing to do, in the remaining case we choose $G'_{\beta,i} \in \mathbf{K}_\mu$ by induction on $i \in [\alpha, \alpha_*]$, increasing continuous with i. For $i = 0$ let $G'_{\beta,i} = G_\beta$ and for limit i let $G'_{\beta,i} = \cup\{G'_{\beta,j} : j < i\}$. Then choose $G'_{\beta,i+1}$ to make clause (e) true. That is, first if $G'_{\beta,i}$ has an extension into which G_i is embeddable over H, then there is such an extension of cardinality μ; and choose $G'_{\beta,i+1}$ as such an extension.

Second, if $G'_{\beta,i}$ has no extension into which G_i is embeddable over H, then we let $G'_{\beta,i+1} = G'_{\beta,i}$.

Lastly, let $G'_\alpha = G'_{\beta,\alpha_*}$ and $u_\alpha = u_\beta \cup \{\beta\}$ and $v_\alpha = \{i \colon i < \alpha_*, i \geq \alpha, i \notin \cup\{v_\gamma : \gamma < \beta\}$ and G_i is embeddable into G'_β over $H\}$.

$(B)^+ \Rightarrow (A)$:

We prove below more: there is something like "special model", i.e. part (2) of 2.1, that is $(B)^+ \Rightarrow (C)^+$.

$(C)^+ \Rightarrow (C) \Rightarrow (A)$

This is trivial so we are left with proving the following.

$(B)^+ \Rightarrow (C)^+$:

Let $\mathbf{K}^{\rm spc}_\mu$ be the class of G such that:

$(*)^3_G$ (a) $G \in \mathbf{K}_\mu$;

(b) if $H \subseteq G, H \in \mathbf{K}_{<\mu}$ then there are $\aleph_0$-indecomposable $H_n \subseteq G$ increasing with n for $n < \omega$ with union of cardinality $< \mu$ such that $H \subseteq \cup\{H_n \colon n < \omega\}$; and[3] there are $\aleph_0$-indecomposable $G_n \subseteq G$ for $n < \omega$ such that $G_n \in \mathbf{K}_{<\mu}$, $G_n \subseteq G_{n+1}$ and $G = \cup\{G_n \colon n < \omega\}$;

(c) if $H \subseteq G$ is $\aleph_0$-indecomposable of cardinality $< \mu$ <u>then</u> the pair (G, H) is an universal $(\mu, < \mu)$-amalgamation base (see Definition 0.8(2));

(d) if $H \subseteq G$ is $\aleph_0$-indecomposable of cardinality $< \mu, H \subseteq H' \in \mathbf{K}_{<\mu}, H'$ is $\aleph_0$-indecomposable[4], and G, H' are compatible over H (in $\mathbf{K}_{\leq\mu}$), <u>then</u> H' is embeddable into G over H.

Now we can finish by proving $(*)_4 + (*)_5$ below.

$(*)_4$ If $G \in \mathbf{K}_{\leq\mu}$ <u>then</u> some $H \in \mathbf{K}^{\rm spc}_{\mu}$ extends G

We break the proof to some stages, $(*)_{4.3}$ gives the desired conclusion of $(*)_4$.

$(*)_{4.0}$ If $G \in \mathbf{K}_{\leq\mu}$ then for some $H, \bar{H}$ we have:

(a) $G \subseteq H \in \mathbf{K}_\mu$;

(b) $\bar{H} = \langle H_n \colon n < \omega\rangle$;

(c) $H_n \subseteq H_{n+1} \subseteq H$;

(d) $H = \cup\{H_n : n < \omega\}$;

(e) each H_n is $\aleph_0$-indecomposable of cardinality μ;

(f) (not really needed) when $\mathbf{K} = \mathbf{K}_{\rm lf}$, if $K \subseteq H_n, |K| \leq \partial$ and $2^\partial \leq |H_n|$ then there is a subgroup L of H_n extending K which is $\Theta^{\rm orth}_\partial$-indecomposable.

[Why? For clauses (a)-(e) by the definition of $\mathbf{K}$ being nice. For clause (f) by 1.5(1),(2)].

$(*)_{4.1}$ if $N_1 \in \mathbf{K}_{\leq\mu}$ <u>then</u> there is N_2 such that

(a) $N_2 \in \mathbf{K}_\mu$;

(b) $N_1 \subseteq N_2$;

(c) if $H \in {\rm IDC}_{<\mu}(N_1)$ <u>then</u> (N_2, H) is a universal $(\mu, <\mu)$-amalgamation base.

Why? by 1.6 it is enough to deal with one such H, which is O.K. by clause (d) of Def 1.3, recalling "universal $(\mu, < \mu)$-amalgamation base" by $(B)^+$ which we are assuming.

$(*)_{4.2}$ like $(*)_{4.1}$ but in clause (c) is replaced by:

(c)' if $H_1 \in {\rm IDC}_{<\mu}(N_1)$ and $H_1 \subseteq H_2 \in \mathbf{K}_{<\mu}$ (and, we may add, H_2 is $\aleph_0$-indecomposable) then <u>either</u> N_2, H_1 are incompatible over H_1 in $\mathbf{K}_{\leq\mu}$ <u>or</u> H_2 is embeddable into N_2 over H_1.

[Why? Again it is enough to deal with one pair (H_1, H_2)] which is done by hand.]

$(*)_{4.3}$ If $N_1 \in \mathbf{K}_{\leq\mu}$ then there is N_2 such that

(a) $N_2 \in \mathbf{K}_\mu$;

(b) $N_1 \subseteq N_2$;

(c) if $H \in {\rm IDC}_{<\,\mu}(N_2)$ <u>then</u> (N_2, H) is a universal $(\mu, < \mu)$-amalgamation base;

(d) if $H_1 \in {\rm IDC}_{<\,\mu}(N_2)$ and $H_1 \subseteq H_2 \in \mathbf{K}_{<\mu}$ (and, we may add, H_2 is $\aleph_0$-indecomposable) then <u>either</u> N_2, H_1 are incompatible over H_1 in $\mathbf{K}_{\leq\mu}$ <u>or</u> H_2 is embeddable into N_2 over H_1.

[3]For universal classes the "and" can be replaced by "hence".

[4]The $\aleph_0$-indecomposability is not always necessary, but we need it sometimes.

[Why? We choose $L_\varepsilon \in \mathbf{K}_\mu$ by induction on $\varepsilon \leq \mathrm{cf}((\mu)$, such that

(a) $L_\alpha \in \mathbf{K}_\mu$;
(b) $\langle L_\beta : \beta \leq \alpha\rangle$ is increasing continuous;
(c) $G_1 \subseteq L_0$;
(d) if $\alpha = 3\beta + 1$ then L_α relate to $L_{3\beta}$ as N_2 relate to N_1 in $(*)_{4.0}$;
(e) if $\alpha = 3\beta + 2$ then L_α relate to $L_{3\beta+1}$ as N_2 relate to N_1 in $(*)_{4.1}$;
(f) if $\alpha = 3\beta + 3$ then L_α relate to $L_{3\beta+2}$ as N_2 relate to N_1 in $(*)_{4.2}$.

There is no problem to carry the induction and note that: if $N \subseteq L_{\mathrm{cf}(\mu)}$ is $\mathrm{cf}(\mu)$-indecomposable the for some $\varepsilon < \mathrm{cf}(\mu)$ we have $N \subseteq L_\varepsilon$. Now then $N_2 = L_{\mathrm{cf}(\mu)}$ is as required in $(*)_{4.3}$ hence in $(*)_4$.

$(*)_5$ (a) if $G_1, G_2 \in \mathbf{K}^{\mathrm{spc}}_\mu$ then G_1, G_2 are isomorphic;
(b) if $G_1, G_2 \in \mathbf{K}^{\mathrm{spc}}_\mu, H \in \mathbf{K}$ is $\aleph_0$-indecomposable and f_ℓ embeds H into G_ℓ, for $\ell = 1, 2$, and this diagram can be completed, (i.e. there are $G \in \mathbf{K}_\mu$ and embedding $g_\ell : G_\ell \to G_*$ such that $g_1 \circ f_1 = g_2 \circ f_2$) <u>then</u> there is h such that:
 (α) h is an isomorphism from G_1 onto G_2;
 (β) $h \circ f_1 = f_2$;

Why? Clause (a) follows from clause (b) using as H the trivial group. For clause (b), let $\mathscr{F} = \mathscr{F}[G_1, G_2]$ be the set of f such that:

(a) f is an isomorphism from $G_{1,f} \in \mathrm{IDC}_{<\mu}(G_1)$ onto $G_{2,f} \in \mathrm{IDC}_{<\mu}(G_2)$;
(b) G_1, G_2 are f-compatible in $\mathbf{K}_\mu$ which means that there is $G \in \mathbf{K}_\mu$ and embeddings g_ℓ of G_ℓ into G for $\ell = 1, 2$ such that $g_2 \circ f = g_1 {\restriction} G_{1,f}$.

First $\mathscr{F}$ is non-empty (the function f with domain $f_1(H)$ and range $f_2(H)$ will do). Second use the hence and forth argument; here we use $\mathrm{cf}(\mu) = \aleph_0$. $\square_{2.1}$

Remark 2.4. 1) Can we prove for strong limit singular μ of uncountable cofinality κ a parallel result? Well, we have to consider the following game:

($*$) the game is defined by:
 (a) a play last θ moves,
 (b) in the ε-th move, first Player I choose $M_\varepsilon \in \mathbf{K}_{<\mu}$ and then player II choose $N_\varepsilon \in \mathbf{K}_{<\mu}$,
 (c) $M_\varepsilon \in \mathbf{K}_{<\mu}$ and if ε is non-limit then M_ε is $\mathrm{cf}(\mu)$-indecomposable,
 (d) $\langle M_\zeta : \zeta \leq \varepsilon\rangle$ is increasing continuous,
 (e) $M_\varepsilon \subseteq N_\varepsilon \subseteq M_{\varepsilon+1}$,
 (f) in the end of the play, the player II wins <u>iff</u> for every limit ordinal $\varepsilon < \mathrm{cf}(\mu)$, M_ε is an amalgamation base inside $\mathbf{K}_{<\mu}$.

Now, if player II does not lose then we can imitate the proof above; this should be clear. Does the existence of a universal member of $\mathbf{K}_\mu$ implies this? we hope to return to this elsewhere.

See below.

2) The proof works for any a.e.c. $\mathfrak{k}$ with $\mathrm{LST}_\mathfrak{k} < \mu$. But we may wonder: can we weaken the demand on $\mathfrak{k}$. Actually, we can: there is no need of smoothness (that is: if $\langle M_\alpha : \alpha \leq \delta\rangle$ is $\leq_\mathfrak{k}$-increasing then $\cup\{M_\alpha : \alpha < \delta\} \leq_\mathfrak{k} M_\delta$). Moreover, while we need the existence of an upper bound for any $\leq_\mathfrak{k}$-increasing sequence, also we demand the union being such upper bound, only for the cofinality $\mathrm{cf}(\mu)$.

3) We may add a version fixing $\bar\lambda$

We may add (after the journal version):

Definition 2.5. We say $\mathbf{K}$ is *μ-very nice* when:

(A) $\mathbf{K}$ is μ-nice (see Definition 1.3),
(B) if $\mathrm{cf}(\mu) > \aleph_0$, <u>then</u> for a club of $\chi < \mu$ we have that $\mathbf{K}$ is χ-nice.

Claim 2.6. *If $\mathbf{K}$ is μ-very nice then the parallel of Theorem 2.1 holds.*

§ 3. Universal in $\beth_\omega$

In §2 we have characterized when there are special models in $\mathbf{K}$ of cardinality, e.g., $\beth_\omega$. We try to analyze a related combinatorial problem. Our intention is to first investigate $\mathfrak{k}_{\rm fnq}$ (the class structures consisting of a set and a directed family of equivalence relations on it, each with a finite bound on the size of equivalence classes). So $\mathfrak{k}_{\rm fnq}$ is similar to $\mathbf{K}_{\rm lf}$ but seems easier to analyze. We consider some partial orders on $\mathfrak{k} = \mathfrak{k}_{\rm fnq}$.

First, under the substructure order, $\le_1 = \subseteq$, this class fails amalgamation. Second, we have other orders: $\le_3$ demanding a Tarski-Vaught condition (see below TV). However, using $\le_3$ where we have a similar demand for countably many points, finitely many equivalence relations, we have amalgamation.

This is naturally connected to locally finite groups, see 3.6, 3.7.

Definition 3.1. Let $\mathbf{K} = \mathbf{K}_{\rm fnq}$ be the class of structures M such that (the vocabulary is defined implicitly and is $\tau_{\mathbf{K}}$, i.e. depends just on $\mathbf{K}$):

(a) P^M, Q^M is a partition of M, P^M non-empty;
(b) $E^M \subseteq P^M \times P^M \times Q^M$ (is a three-place relation) and we write $aE^M_c b$ for $(a,b,c) \in E^M$;
(c) for $c \in Q^M, E^M_c$ is an equivalence relation on P^M with $\sup\{|a/E^M_c| : a \in P^M\}$ finite (see more later);
(d) $Q^M_{n,k} \subseteq (Q^M)^n$ for $n,k \ge 1$
(e) if $\bar c = \langle c_\ell : \ell < n\rangle \in {}^n(Q^M)$ we let $E^M_{\bar c}$ be the closure of $\bigcup\limits_\ell E_\ell$ to an equivalence relation; but $E^M_{\bar c}$ is not part of the vocabulary;
(f) ${}^n(Q^M) = \bigcup\limits_{k\ge 1} Q^M_{n,k}$;
(g) if $\bar c \in Q^M_{n,k}$ then $k \ge |a/E^M_{\bar c}|$ for every $a \in P^M$.

Definition 3.2. We define some partial order on $\mathbf{K}$.

1) $\le_1 = \le^1_{\mathbf{K}} = \le^1_{\rm fnq}$ is being a sub-model.

2) $\le_3 = \le^3_{\mathbf{K}} = \le^3_{\rm fnq}$ is the following: $M \le_3 N$ iff:

(a) $M, N \in \mathbf{K}$,
(b) $M \subseteq N$,
(c) if $A \subseteq N$ is countable and $A \cap Q^N$ is finite, <u>then</u> there is an embedding of $N{\restriction}A$ into M over $A \cap M$ or just a one-to-one homomorphism.

3) $\le_2 = \le^2_{\mathbf{K}} = \le^2_{\rm fnq}$ is defined like $\le_3$ but in clause (c), A is finite.

Claim 3.3. *1) $\mathbf{K}$ is a universal class, so $(\mathbf{K}, \subseteq)$ is an a.e.c.*

2) $\le^3_{\mathbf{K}}, \le^2_{\mathbf{K}}, \le^1_{\mathbf{K}}$ are partial orders on $\mathbf{K}$.

3) $(\mathbf{K}, \le^2_{\mathbf{K}})$ is an a.e.c.

4) $(\mathbf{K}, \le^3_{\mathbf{K}})$ has disjoint amalgamation.

5) If $M \le_2 N$, $c \in Q^M$ and $a \in P^M$, then a/E^N_c is included in M.

6) For every n,k there is an existential first order sentence defining, for $M \in K$, the set $\{\bar a \in {}^{n+2}M : a_n, a_{n+1}$ are $E^M_{\bar a}$-equivalent$\}$.

Proof. 1),2),3) Easy.

4) By 3.4 below. $\square_{3.3}$

Claim 3.4. *If $M_0 \le^1_{\mathbf{K}} M_1, M_0 \le^3_{\mathbf{K}} M_2$ and $M_1 \cap M_2 = M_0$, <u>then</u> $M = M_1 + M_2$, the disjoint sum of M_1, M_2 belongs to $\mathbf{K}$ and extends M_ℓ for $\ell = 0,1,2$ and even $M_1 \le^3_{\rm fnq} M$ and $M_0 \le^2_{\mathbf{K}} M_1 \Rightarrow M_2 \le^2_{\mathbf{K}} M$ <u>when</u>:*

(∗) $M = M_1 +_{M_0} M_2$ *means* M *is defined by:*
  (a) $|M| = |M_1| \cup |M_2|$;
  (b) $P^M = P^{M_1} \cup P^{M_2}$;
  (c) $Q = Q^{M_1} \cup Q^{M_2}$;
  (d) *we define* E^M *by defining* E^M_c *for* $c \in Q^M$ *by cases:*
    (α) *if* $c \in Q^{M_0}$ *then* E^M_c *is the closure of* $E^{M_1}_\ell \cup E^{M_2}_\ell$ *to an equivalence relation;*
    (β) *if* $c \in Q^{M_\ell} \backslash Q^{M_0}$ *and* $\ell \in \{1,2\}$ *then* E^M_c *is defined by*
      • $aE^M_c b$ *iff* $a = b \in P^{M_{3-\ell}} \backslash M_0$ *or* $aE^{M_\ell}_c b$ *so* $a, b \in P^{M_\ell}$;
  (e) $Q^M_{n,k}$ *is the union of* $Q^{M_1}_{n,k}, Q^{M_2}_{n,k}$ *and the set of* $\bar c$ *satisfying*
    (α) $\bar c \in {}^n(Q^M)$,
    (β) $\bar c \notin ({}^n(Q^{M_1})) \cup {}^n(Q^{M_2})\}$,
    (γ) $E^M_{\bar c}$ *which is now well defined, has no equivalence classes with more than* k *members, that is, for some finite set* A *and pairwise distinct* $a_0, \dots, a_k \in A$ *which are members of* $a/E^M_{\bar c}$ *and the closure of* $\bigcup\{E^M_{c_i} \restriction A \colon i < \lg(\bar c)\}$ *to an equivalence relation satisfies* $a_i E' a$ *for* $i \le k$.

Proof. Clearly M is a well defined structure, extends M_0, M_1, M_2 and satisfies clauses (a),(b),(c) of Definition 3.1. There are two points to be checked: $a \in P^M, \bar c \in Q^M_{n,k} \Rightarrow |a/E^M_{\bar c}| \le k$ and ${}^n(Q^M) = \bigcup\limits_{k \ge 1} Q^M_{n,k}$

$(*)_1$ if $a \in P^M$ and $\bar c \in Q^M_{n,k}$ then $|a/E^M_{\bar c}| \le k$.

[Why? If $\bar c \in Q^{M_1}_{n,k} \cup Q^{M_2}_{n,k}$ this holds by the definition, so assume $\bar c \in Q^{M_\iota}_{n,k}, \iota \le 2$. If this fails, then there is a finite set $A \subseteq M$ such that $\bar c \subseteq A, a \in A$ and the closure of $\bigcup\{E^M_{c_\ell} \colon \ell < \lg(\bar c)\}$ to an equivalence relation satisfies: every equivalence class has $\le k$ members. $N = M \restriction A$ we have $|a/E^N_{\bar c}| > k$. By $M_0 \le^1_{\mathbf K} M_1, M_0 \le^3_{\mathbf K} M_2$ (really $M_0 \le^2_{\mathbf K} M_2$ suffice) there is a one-to-one homomorphism f from $A \cap M_2$ into M_0 over $M_0 \cap A$. Let $B' = (A \cup M_1) \cup f(A \cap M_2)$ and $N' = M \restriction B$ and let $g = f \cup \mathrm{id}_{A \cap M_1}$. So g is a homomorphism from N onto N' and $g(a)/E^{N'}_{g(\bar c)}$ has $> k$ members, which implies $g'(a)/E^{M_1}_{g'(\bar c)}$ has $> k$ members. Also $g(\bar c) \in Q^{M_1}_{n,k}$. (Why? If $\iota = 1$ trivially, if $\iota = 2$ by the choice of f), contradiction to $M_1 \in \mathbf K$.]

$(*)_2$ if $\bar c \in {}^n(Q^M)$ then $\bar c \in \bigcup\limits_k Q^M_{n,k}$.

Why? If $\bar c \in M_1$ or $\bar c \subseteq M_2$, this is obvious by the definition of M, so assume that they fail. By the definition of the $Q^M_{n,k}$'s we have to prove that $\sup\{a/E^M_{\bar c} : a \in P^M\}$ is finite. Toward contradiction assume this fails for each $k \ge 1$ hence there is $a_k \in P^M$ such that $a_k/E^M_{\bar c}$ has $\ge k$ elements hence there is a finite $A_k \subseteq M$ such that $a_k/E^{M \restriction A_k}_{\bar c}$ has $\ge k$ elements. Let $A = \bigcup\limits_{k \ge 1} A_k$, so A_k is a countable subset of M and we continue as in the proof of $(*)_1$.

Additional points (not really used) are proved like $(*)_2$:

$(*)_3$ $M_1 \le^3_{\mathbf K} M$;
$(*)_4$ $M_0 \le^2_{\mathbf K} M_1 \Rightarrow M_2 \le^2_{\mathbf K} M$;
$(*)_5$ $M_1 +_{M_0} M_2$ is equal to $M_2 +_{M_0} M_1$. $\square_{3.4}$

Claim 3.5. *1) If* $\lambda = \lambda^{<\mu}$ *and* $M \in \mathbf K$ *has cardinality* $\le \lambda$ *then there is* N *such that:*

(a) *$N \in \mathbf{K}_\lambda$ extend M;*
(b) *if $N_0 \leq^3_{\mathbf{K}} N_1$ and N_0 has cardinality $< \mu$ and f_0 $\leq_2$-embeds N_0 into N, <u>then</u> there is an embedding f_1 of N_1 into N extending f_0.*

2) For every $M \in \mathbf{K}$ we can define an equivalence relation $E = E_{\mathbf{K}}$ on the class $\{N \in \mathbf{K} \colon M \leq_2 N\}$ with $\leq 2^{\|M\|^{\aleph_0}}$-equivalence classes such that; if N_1, N_2 are E -equivalence then they can be amalgamated over M (in $(\mathbf{K}, \leq_2)$.

3) If μ is strong limit then $(\mathbf{K}, \leq_2)$ is μ-nice.

What is the connection to $\mathbf{K}_{\rm lf}$? the following explain (see [KW73])

Definition 3.6. 1) For a group $G \in \mathbf{K}_{\rm lf}$ we define $M = {\rm fnq}_G \in \mathbf{K}_{\rm fnq}$ as follows:
(a) P^M is the set of elements of G,
(b) $Q^M = \{(c, 1) \colon c \in G\}$, a copy of G,
(c) E^M is the set of triples $(a, b, (c, 1)$ such that $a, b, c \in G$ and for some $n, m \in \mathbb{Z}$ we have $G \models c^n a c^m = b$.

2) For $M \in \mathbf{K}$ we define $G = {\rm grp}_M$ as the subgroup of ${\rm sym}(P^M)$ consisting of the permutations π of P^M such that for some finite sequence $\bar{c}$ of elements of Q^M we have: for every $x \in P^M$ we have $\pi(x) E^M_{\bar{c}} x$.

Discussion 3.7. The problem is that cases of amalgamation in $(\mathbf{K}, \leq_2)$ cannot be lifted to one in $\mathbf{K}_{\rm lf}$, that is, in 3.4, for $c \in M_\ell \setminus M_0$, we can choose $E^M_c \restriction (M_{3-\ell} \setminus M_0)$ as the equality but the parallel demand for groups fail.

After publication we add:

Claim 3.8. *Assume that μ is a strong limit singular cardinal of cofinality $\aleph_0$. Then $(\mathbf{K}_{\rm fnq}, \leq_{2/3})$ has a universal member in μ.*

Proof. By the general criterion [She17b, 1.16 = L a34] + JEP, but we elaborate.

$(*)_1$ fix $\bar{\lambda} = \langle \lambda_n \colon n < \omega \rangle$, $2^{\lambda_n} < \lambda_{n+1} < \mu = \sum\{\lambda_k \colon k < \omega\}$ and $\lambda_n = \lambda_n^{\aleph_0}$.

$(*)_2$ For $\xi \leq \omega$, let:
  (a) $\mathbf{K}_\xi = K_{\bar{\lambda},\xi}$ is the class of $\bar{M}$ such that $\bar{M} = \langle M_n \colon n < \xi \rangle$, $M_n \in \mathbf{K}^{\rm fnq}_{\lambda_n}$ is $\leq_3$-increasing with n,
  (b) $\mathbf{K}_\xi = \mathbf{K}_{\bar{\lambda},\xi} = \{\bar{M} \in \mathbf{K}_\xi \colon$ the universe of M_n is λ_n for any $n < \xi\}$.

$(*)_3$ if $M \in \mathbf{K}^{\rm fnq}_M$ then there is $\bar{M} \in \mathbf{K}_{\bar{\lambda},\omega}$ whose union is M.

$(*)_4$ We can find $\bar{N}$ such that:
  (a) $\bar{N} = \left\langle N_{\alpha,\eta} \colon \alpha < 2^{\lambda_0}, \eta \in \prod_{\ell<n} 2^{\lambda_{\ell+1}} \text{ for some } n < \omega \right\rangle$,
  (b) $\langle N_{\alpha,\langle\rangle} \colon \alpha < 2^{\lambda_0} \rangle$ list the $M \in \mathbf{K}_{\rm fnq}$ with universe λ_0,
  (c) for $\alpha < 2^{\lambda_0}$, $\eta \in \prod_{\ell<n} 2^{\lambda_{\ell+1}}$, the sequence $\langle N_{\alpha,\eta^\frown\langle\beta\rangle} \colon \beta < 2^{\lambda_{n+1}}$ list the $M \in \mathbf{K}_{\rm fnq}$ with universe λ_{n+1} which $<_3$-extend $N_{\alpha+1}$.

$(*)_5$ We can find $N_*, \bar{h}$ such that:
  (a) $N_* \in \mathbf{K}_{\rm fnq}$ has cardinality μ,
  (b) $\bar{h} = \left\langle h_{\alpha,\eta} \colon \alpha < 2^{\lambda_0}, \eta \in \prod_{\ell<n} 2^{\lambda_{\ell+1}} \text{ for some } n < \omega \right\rangle$,
  (c) $h_{\alpha,\eta}$ embeds $N_{\alpha,\eta}$ into N_*,
  (d) if $\nu \triangleleft \eta \in \prod_{\ell<n} 2^{\lambda_{\ell+1}}$ and $\alpha < 2^{\lambda_0}$, then $h_{\alpha,\eta} \subseteq h_{\alpha,\nu}$.

$(*)_6$ N_* is a universal member of $\mathbf{K}_{\rm fnq}$ in μ.

[Why? By $(*)_3$ and $(*)_5$.] $\square_{3.8}$

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