BIGNESS PROPERTIES FOR κ-TREES AND LINEAR ORDERS
E81

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Abstract. This continues [She90, Ch.VIII],[Shee], [Shea], and [Shed], deriving complicated model from complicated index models (which are mainly linear orders and trees with $\omega + 1$ levels, under lexicographic order).

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§ 0. Introduction

Our aim is to get general non-structure theorems as in [She90, Ch.VIII] and [She83] (see also [Shee], [Shea], and [Shed]) but we try to make the paper self-contained. Those papers deal mainly with the class $K^\omega_{\rm tr}$ as the class of index models, where $K^\omega_{\rm tr}$ is the class of trees with $\omega+1$ levels and lexicographic orders. The thesis is that if we can build complicated sequences of index models, we can deduce various non-structure results for many classes.

In particular:

(A) Using K = the class of linear order is suitable for, e.g., dealing with the class of models of an unstable theory via EM models where the skeleton is linearly ordered by some formula $\varphi(x,y)$.

(B) $K = K^\omega_{\rm tr}$ is suitable for, e.g., dealing with the class of models of an unsuperstable theorem T.

(C) Using $K = K^\theta_{\rm tr}$, i.e. trees with $\theta+1$ levels, is suitable for e.g. dealing with the model of T when $\kappa(T) > \theta$.

(D) In all those cases we can apply the methods to

$$\mathrm{PC}(T_1, T) := \{M_1 \restriction \tau_T : M_1 \text{ is a model of } T_2\}$$

where $T_1 \supseteq T$ are complete first order theories and, e.g., T_1 has Skolem functions: and to more general situations.

(E) Sometimes we can deal with models of (or reductions of models of) a sentence $\psi \in \mathbb{L}_{\lambda^+,\aleph_0}$.

Here we have three aims:

(A) Linear Orders (as the class of index models)

In [Shee, 2.29=L2.25] we get the desired properties for the formula $\varphi_{{\rm or},\alpha,\beta,\pi}(\bar x,\bar y)$, but we were able to prove it (i.e., prove so-called *strong* $(2^\lambda,\lambda,\mu,\aleph_0)$*-bigness*) only for the case where λ is a regular cardinal $> \mu$ such that $(\forall\gamma<\lambda)\big[|\gamma|^{|\alpha|} < \lambda\big]$. For λ singular ($> \mu$) we get there only 2^{λ_1} for any $\lambda_1 < \lambda$, rather than 2^λ. We [would] like to get the full results. See §2.

(B) Concerning $K^\omega_{\rm tr}$, consider improving on [Shea, 2.20=L7.11].

The best outcome is to get the so-called *full strong* $(\lambda,\lambda,\mu,\kappa)^{6^+}$*-bigness* property, or similar; it seems that in the problematic case it suffices to have just

$(*)$ $\mu^+ > \|N_n\|^{\aleph_0}$.

See below.

(C) $K^\theta_{\rm tr}$ with θ not necessarily $\aleph_0$.

Do we have the full strong $(\lambda,\lambda,\mu,\kappa)$-bigness property? Using $\mathscr{M}_{\mu,\kappa}$, let us review cases and see what they cover:

(a) $\lambda = \mathrm{cf}(\lambda) > \mu$

(α) $\kappa = \theta = \aleph_0$ (see [Shea, 1.11=L7.6(1),p.12]) [will] get super$^{7^+}$. [CHECK] This implies "super$^{4^{++}}$"; moreover, [Shea, 2.13=L7.8I] for $\ell = 7^+$.

(β) For $\theta \geq \kappa + \aleph_1$ such that $\lambda \gg \kappa$, which means

$$(\forall\alpha<\lambda)\big[|\alpha|^{<\kappa} < \lambda\big].$$

(b) $\lambda > \mu + \mathrm{cf}(\lambda)$, $\partial = \mathrm{cf}(\partial) \in [\kappa,\theta]$, $\lambda \gg \kappa$, $\mathcal{T}$ a tree with ∂ levels and $< \lambda$ nodes, with $|\lim_\partial(\mathcal{T})| \geq \lambda$.

We choose $\chi \in (\mu + \mathrm{cf}(\lambda) + |\mathcal{T}| + \kappa + \partial)^+$ and use a partial square $\overline{C} = \langle C_\alpha : \alpha \in S\rangle$, $S \subseteq S^{\chi^{++}}_{\leq \chi}$, $\mathrm{otp}(C_\delta) \leq \chi$, and $\overline{C} \restriction S^\lambda_\chi$ guesses clubs.

As in [Shea, 2.1=L7.8] we get

(∗) $\ell = 5$ which has a weaker (vii), **[CHECK — is $\ell = 5$ well defined? What is proved?]**

For $\theta = \kappa = \aleph_0$ we get (see 1.1) 4^{++}-bigness (so $M_n \cap \mu = N_n \cap \mu$, $\kappa \subseteq \mu_n$), $\nu \in I \cap \nu_\varepsilon \restriction k \in M_n \Rightarrow \nu \in M_n$.

Question 0.1. When do we [ask / require] $[M_n]^{<\kappa} \subseteq M_n$? (Now $n < \theta$, so $\theta = \aleph_0$ was the old case.)

(c) λ is strong limit of cofinality $> \theta$, or at least equal to

$$\sup\{\chi : \chi \text{ strong limit of cofinality } \kappa\}.$$

If $\theta = \aleph_0$ see [Sheb, 1.11(3)=L7.6] for $\ell = 6$ [Shea, 2.1=L7.8,pg.19] getting $\ell = 4^{++}$ (check). Choose $\langle \chi_i : i < \mathrm{cf}(\lambda)\rangle$ increasing to λ, χ_i strong limit, $\mathrm{cf}(\chi_i) = \kappa$ "or at least," not clear, but if $\mathrm{cf}(\lambda)$.

[If $\mathrm{cf}(\lambda)$ is *what*, then *what*?]

First, try to build $\langle I_\alpha : \alpha < 2^\lambda\rangle$, but choose $I_{\alpha,i} = I_{A_{\alpha,i}}$, $A_{\alpha,i} = A_\alpha \cap \chi_i$, etc.

Second, try to build $\langle I_\alpha : \alpha < \lambda\rangle$:

(d) $\lambda = \sup\{\chi : \mathrm{cf}(\chi) = \kappa < \chi < \lambda,\ \mathrm{pp}(\chi) > \chi^+\}$.
[Is λ being defined in terms of itself?]

Like (a) using non-reflecting for $\theta = \aleph_0$, see [Shea, 1.16=L7.7].

Seems to generalize easily:

(e) $\mathrm{cf}(\lambda) < \chi = \chi^\theta < \lambda \leq 2^\chi$.

If $\theta = \aleph_0$; see [Shea, 1.11(2) = L7.6] we get super^6; what about getting 4^{++}? **[Check?]** If $\kappa = \kappa^{\aleph_0}$ then we get super$^{6^+}$.

Assume $\mathrm{cf}([\chi]^\partial, \subseteq) = \chi$, $\theta < \partial = \mathrm{cf}(\partial)$, and maybe $\mathrm{cf}([\partial]^\theta, \subseteq) = \partial$.

Can we get models of M_i, N_i of cardinality ∂?

Now if $\mu^{2^{\aleph_0}} < \lambda$ then let $\chi = \min\{\chi : \chi^{(2^{\aleph_0})} \geq \lambda\}$, so $\mathrm{cf}(\chi) \leq 2^{\aleph_0}$ we get $\ell = 4^{++}$ or more; find models of cardinality. **[Of *what* cardinality?]**

(f) Assume $\mathrm{cf}(\lambda) \leq \theta < \chi < \lambda$, $\mathrm{cf}(\chi) = \theta, \mathrm{pp}(\chi) = \chi^+$, maybe $\chi = (2^\partial)^{+\kappa}$, $\partial < \lambda$ large enough, and $\partial = \partial^\theta$.

See [Shea, 2.15=L7.9,pg.27] using $\langle \mathfrak{a}_i : i < \mathrm{cf}(\chi)\rangle$ pairwise disjoint unbounded subsets of $\mathrm{Reg} \cap \chi$: say, $\ell = 4^+$. See also [Shea, 3.23=L7.14,pg.47]. Note that without loss of generality, $\chi = \chi_*^{+\omega}$, $\mathrm{pp}(\chi) = \chi_*^+$, and $\chi_*^{\aleph_0} = \chi_*$.

(g) The remaining case.

[It??] $\theta = \aleph_0$; this means $\lambda = \aleph_{\alpha+\omega}$ is strong limit, see [Shea, 2.17=L7.109,pg.32] for $\ell = 6$, [Shea, 2.19=L7.10,pg.35] and $[M_n]^{\aleph_0} \subseteq M_n$ which gives 4^{++}; more?

Discussion 0.2. Can we improve [Shea, 2.15]?

1) E.g., we assume only

(∗) (a) $\lambda > \mathrm{cf}(\lambda)$
(b) $\alpha < \lambda \Rightarrow |\alpha|^{\aleph_0} < \lambda$
(c) $\mathrm{cf}(\lambda) + \mu < \chi_n = \mathrm{cf}(\chi_n) = \chi_n^{\aleph_0} < \chi_{n+1} < \chi = \sum\limits_n \chi_n < \lambda$
(d) $\mathrm{pp}(\chi) = \chi^+$

2) We choose

(∗) $\bar{\mu} = \langle \mu_\varepsilon : \varepsilon < \mathrm{cf}(\lambda)\rangle$ increases $\mu_\varepsilon = \mathrm{cf}(\mu_\varepsilon) = \mu_\varepsilon^{\aleph_0} \in (\chi^+, \lambda)$, $\mu_\varepsilon^{+7} < \mu_{\varepsilon+1}$, $\lambda_\varepsilon = \mu_\varepsilon^{+2}$, or $\lambda_\varepsilon = \mu_\varepsilon^{+(\varepsilon+1)}$.

3) **Question**: $2^{\aleph_0} = \aleph_1$?
If not, then Υ_ε satisfies: every Υ and did not contain a perfect set.
[This is not a sentence.]

4) Another direction:
(∗) (a) Without loss of generality, $\chi_n = \chi_0^{+n}$.
(b) We choose $\bar{\eta} = \langle \eta_\alpha : \alpha < \chi^+\rangle$, $<_{J^{\mathrm{bd}}_\omega}$-increasing cofinal in $\prod\limits_n \chi_n$.
(c) Without loss of generality $\bar{\eta}$ is (χ, χ_0)-free r using $\lambda_\varepsilon = \mu_\varepsilon^{+(\omega+1)}$ (see [She13]).
[What is r? Could've been a comma, I guess.]
(d) Look for models of cardinality χ_0.

* * *

Definition 0.3. We define K^δ_{tr} as the class of trees with $\delta + 1$ levels and lexicographic orders as in [Shee, §2] – **[FILL]**.

Definition 0.4. We define $\check{I}_\theta[\lambda]$ (where $\theta < \lambda$ is regular) as follows:[1] $S \in \check{I}_\theta[\lambda]$ if there is $\langle a_\alpha : \alpha \in S^+\rangle$ which witness it, meaning:

$(*)_1$ (a) $S \subseteq S^+ \subseteq S^\lambda_{\leq\kappa}$
(b) $S^+ = \{\delta \in S^+ : \mathrm{cf}(\delta) = \kappa\} = \{\delta \in S^+ : \mathrm{otp}(a_\alpha) = \kappa\}$
(c) $a_\alpha \subseteq \alpha$ has order type $\leq \kappa$.
(d) $\delta \in S^+ \Rightarrow \delta = \sup(a_\delta)$
(e) For every $\alpha < \lambda$, the set $\{a_\beta \cap \alpha : \beta$ satisfies $\alpha \in a_B$ has cardinality $< \lambda$ **[END OF LINE]**
[There's an open brace and no close brace. I could see it going before 'has cardinality $< \lambda$,' but I have no idea what "β satisfies $\alpha \in a_B$" is supposed to mean.]

By [She93], [Shec], more [She13].

Claim 0.5. ***(Existence and Existence with guessing clubs)***
Let λ be regular uncountable.
1) If $S \in \check{I}[\lambda]$ then we can find a witness $(E, \bar{a})$ for $S \in \check{I}[\lambda]$ such that:

(a) $\delta \in S \cap E \Rightarrow \mathrm{otp}(a_\delta) = \mathrm{cf}(\delta)$
(b) If $\alpha \notin S$ then $\mathrm{otp}(a_\alpha) < \mathrm{cf}(\delta)$ for some $\delta \in S \cap E$.

2) $S \in \check{I}[\lambda]$ iff there is a pair $(E, \overline{\mathscr{P}})$ such that:

(a) E is a club of the regular uncountable λ.
(b) $\overline{\mathscr{P}} = \langle \mathscr{P}_\alpha : \alpha < \lambda\rangle$, where $\mathscr{P}_\alpha \subseteq \{u : u \subseteq \alpha\}$ has cardinality $< \lambda$.
(c) If $\alpha < \beta < \lambda$ and $\alpha \in u \in \mathscr{P}_\beta$ then $u \cap \alpha \in \mathscr{P}_\alpha$.
(d) If $\delta \in E \cap S$ then some $u \in \mathscr{P}_\delta$ is an unbounded subset of δ (and δ is a limit ordinal).

[1] see [She09, §0,y14]

§ 1. On $K^\theta_{\rm tr}$

We consider a strengthening of [Shea, 1.1=L7.1], but we give a self-contained definition.

Definition 1.1. 1) $I \in K^\theta_{\rm tr}$ is (μ,κ)-super$^+$-unembeddable[2] (or super$^{4^{++}}$-unembeddable) into $J \in K^\theta_{\rm tr}$ <u>when</u> as in [Shea, 1.1=L7.1] with $n < \theta$, that is: for every regular large enough χ_* (in particular such that $\{I, J, \mu, \kappa, \theta\} \in \mathcal{H}(\chi^*)$) and well ordering $<^*_{\chi_*}$ of $\mathcal{H}(\chi_*)$ we have:

- $(*)$ There are η, M_i, N_i for $i < \theta$ such that
  - (i) $M_i \prec N_i \prec M_j \prec N_j \prec (\mathcal{H}(\chi^*), \in, <^*_{\chi_*})$ for $i < j < \theta$.
  - (ii) $M_i \cap \mu = M_0 \cap \mu$ for $i < t$.
  - (iii) I, J, μ, κ belong to M_0.
  - (iv) $\eta \in P^I_\theta$
  - (v) For every $i < \theta$, for some $j < \theta$, we have $\eta \restriction j \in M_i$, $\eta(j) \in N_i \setminus M_i$ (hence $\eta \restriction (j+1) \in N_i \setminus M_i$).
  - (vi) If $\nu \in P^J_\theta$ and $\{\nu \restriction j : j < \theta\} \subseteq \bigcup_i M_i$ then $\nu \in \bigcup_i M_i$.

Similarly,

Definition 1.2. We repeat [Shea, 1.4=L7.2], defining [full] super$^+$-bigness. That is:

- (A) $K^\omega_{\rm tr}$ has the $(\chi,\lambda,\mu,\kappa)$-super-bigness property <u>when</u>: there are $I_\alpha \in (K^\omega_{\rm tr})_\lambda$, for $\alpha < \chi$, such that for $\alpha \neq \beta$, I_α is (μ,κ)-super unembeddable into I_β.
- (B) $K^\omega_{\rm tr}$ has the full $(\chi,\lambda,\mu,\kappa)$-super-bigness property <u>when</u>: there are $I_\alpha \in (K^\omega_{\rm tr})_\lambda$, for $\alpha < \chi$, such that I_α is (μ,κ)-super unembeddable into $\sum\limits_{\beta<\chi,\beta\neq\alpha} I_\beta$ (see [Shee]).
- (C) We may omit κ if $\kappa = \aleph_0$.

Exercise 1.3. Put 4^{++} in the diagram from [Shea] and see [Shea, 1.7=L7.4], [Shea, 1.8=L7.6].

Claim 1.4. *If λ is singular $> \mu$ <u>then</u> in* [Shea, 2.20=L7.11]*, not only does $K^\omega_{\rm tr}$ have the $(\lambda,\lambda,\mu,\aleph_0)$-super-bigness property, but we can add the following to Definition* [Sheb, 1.1=L7.1]*:*

- *For every $n < \omega$, for some $\mu' \in [\mu,\lambda)$, we have $\|M_n\| \le \mu'$ and $M_n \cap \mu' = N_n \cap \mu'$.*

Proof. We should check the proof of [Shea, 2.20=L7.11]; that is, the cases each treated by a claim there.

Case 1: λ regular $> \aleph_0$ (see [Shea, 2.13=L7.8,pg.25]).

We can find η, $\langle M_n : n < \omega\rangle$ as in "super$^{7^+}$" of Definition [Shea, 1.1=L7.1] so $M_n = N_{\alpha_{\delta,n}}$ (with guessing clubs) [and] $\eta_\delta := \langle \alpha_{\delta,n} : n < \omega\rangle$ lists C_δ.

Let N'_n be the Skolem hull of $N_{\alpha_{\delta,n}} \cup \{\eta_\delta(n)\}$ in $N_{\alpha_{\delta,n}+1}$.

Case 2: $\lambda = \aleph_1$ (see [Shea, 1.11=L7.6(1),pg.12]).

Similar.

Case 3: λ singular, $(\exists\chi)[\chi^{\aleph_0} < \lambda \le 2^\chi]$.

See [Shea, 1.11(2)=L7.6(2),pg.12]; so, prove the models are countable.

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[2] but below we omit the superscript + because we do not use any other version; similarly in Definition 1.2.

Case 4: λ is singular, $\lambda = \sup\{\chi : \mathrm{cf}(\chi) = \aleph_0 \text{ and } \mathrm{pp}(\chi) > \chi^+\}$.
See [Shea, 1.16=L7.7(1),pg.17].

Case 5: λ is singular and $(\exists \chi)[\chi < \lambda \leq \chi^{\aleph_0}]$.
See [Shea, 2.1=L7.8,pg.19].

Case 6: $\lambda = \aleph_{\alpha+\omega}$ is strong limit.
By [Shea, 2.19=L7.10,pg.35]. $\square_{1.4}$

§ 2. Back to linear orders

We complete [Shee, 2.27=L2.23,2.31=L2.27]; see there.

Definition 2.1. 1) For any $I \in K^{\kappa}_{\rm tr}$ we define $\mathbf{or}(I)$ as the following linear order (See Definition [Shee, 2.24=L2.20]).

The set of elements is chosen as $\big\{(t,\ell) : t \in I,\ \ell \in \{1,-1\}\big\}$.

The order is defined by $(t_1,\ell_1) < (t_2,\ell_2)$ if and only if one of the following holds:

- $t_1 \lhd t_2 \wedge \ell_1 = 1$
- $t_2 \lhd t_1 \wedge \ell_2 = -1$
- $t_1 = t_2 \wedge \ell_1 = -1 \wedge \ell_2 = 1$
- $t_1 <_{\ell x} t_2 \wedge (t_1, t_2$ are $\lhd$-incomparable.)

2) Let $\varphi_{\rm or} = \varphi_{\rm or}(x_0, x_1; y_0, y_1)$ be the formula $x_0 < x_1 \wedge y_1 < y_0$.
3) Let $\varphi^{\kappa}_{\rm tr} = \varphi^{\kappa}_{\rm tr}(x_0, x_1; y_0, y_1)$ be[3]

$$\begin{aligned}\varphi_{\rm tr}(x_0, x_1 : y_0, y_1) := [x_0 = y_0] \text{ and } P_\kappa(x_0) \quad &\wedge \bigvee_{\epsilon<\kappa} [P_{\epsilon+1}(x_1) \\ &\wedge P_{\epsilon+1}(y_1) \wedge P_\epsilon(x_1 \cap y_1)] \\ &\wedge [x_1 \lhd x_0 \wedge \neg(y_1 \lhd y_0)] \text{ and } y_1 <_{\ell x} x_1].\end{aligned}$$

[There are two open brackets and three close brackets after that first conjunction.]

Recall [Shee, 2.28=L2.24].

Definition 2.2. We define the following (quantifier free infinitary) formulas for the vocabulary $\{<\}$. For any ordinal α, β and a one-to-one function π from α onto β, and we let $\varphi_{{\rm or},\alpha,\beta,\pi}(\bar x, \bar y)$ (where $\bar x = \bar x^\alpha = \langle x_i : i < \alpha\rangle$ and $\bar y = \bar y^\alpha = \langle y_i : i < \alpha\rangle$) be

$$\bigwedge\{x_i < x_j : i < j < \alpha\} \text{ and } \bigwedge\{y_i < y_j : i, j < \alpha \text{ and } \pi(i) < \pi(j)\}.$$

Claim 2.3. *Assume $\lambda > \mu$.*
1) For (α, β, π) as in 2.2, such that $\alpha, \beta \le \mu^+$, the class $K_{\rm or}$ has the full strong $(\lambda, \lambda, \mu, \kappa)$-bigness property for $\varphi_{{\rm or},\alpha,\beta,\pi}(\bar x, \bar y)$.
2) For (α, β, π) as in 2.2 such that $\alpha, \beta \le \mu^+$, the class $K_{\rm or}$ has the strong $(2^\lambda, \lambda, \mu, \kappa)$ bigness property for $\varphi_{{\rm or},\alpha,\beta,\pi}$.
3) In fact, in both part (1) and (2) we can find examples which satisfy the conclusion for all triples (α, β, π) as there simultaneously.

Proof. 1) By 2.4 below.
2) By part (1) and [Shee, 2.27=L2.23(1)], [Shee, 2.20=L2.8(1)] or here.
3) Check the proof. $\square_{2.3}$

Claim 2.4. *Assume $\mu < \lambda$.*
If $I, J \in K^{\kappa}_{\rm or}$ satisfies $\circledast$ below, $\alpha_*, \beta_* \le \mu^+$, and π is a one-to-one function from α_* onto β_* then (recalling 2.1) $\mathbf{or}(I)$ is strongly $\varphi_{{\rm or},\alpha_*,\beta_*,\pi}(\bar x^{\alpha_*}, \bar y^{\alpha_*})$-unembeddable for (μ, κ) into $\mathbf{or}(J)$, where

$\circledast$ *(a) $I, J \in K^{\omega}_{\rm tr}$*
*(b) I is $(\mu, \aleph_0)$-super unembeddable into J (see Definition 1.1). **[check]***
*(c) $I \in K^{\kappa}_{\rm tr}$ is **[equal to?]** / **[of the form?]***
$\{\eta_\delta \restriction i : i \le \partial,\ \delta \in S_1\} \cup \{\langle\alpha\rangle : \alpha < \lambda_1\}$
*(d) $J \in K^{\kappa}_{\rm tr}$ is **[equal to?]** / **[of the form?]***
$\{\eta_\delta \restriction i : i \le \partial,\ \delta \in S_1\} \cup \{\langle\alpha\rangle : \alpha < \lambda_1\}$.
[These two lines are character-for-character identical.]

[3] This is for $K^{\kappa}_{\rm tr}$; for $\kappa = \aleph_0$ see example [Shee, 2.9=L2.4A].

Proof. So let f be a function from $\mathbf{or}(I)$ into $\mathscr{M}_{\mu,\kappa}(\mathbf{or}(J))$ (so actually a function from $I \times \{1,-1\}$ into $\mathscr{M}_{\mu,\kappa}(J \times \{1,-1\})$) and let $<_*$ be a well ordering of $\mathscr{M}_{\mu,\kappa}(J)$ but we "forget" to deal with it, as there are no problems, and let χ be large enough. Let χ^* be large enough and let η_*, M_η, N_n (for $n < \omega$) satisfy $(*)$ of Definition **??**, with η_* here standing for η there. In particular, $M_n \prec N_n \prec (\mathcal{H}(\chi), \in)$ such that $I, J, \lambda, \mu, \mathscr{M}_{\mu,\kappa}(J), f, <_*$ all belong to N_0 and $M_n \cap \mu = N_n \cap \mu$. As it happens, "$\alpha_*, \beta_*, \pi \in N_0$" is not needed. For any $\eta \in I$, clearly $f((\eta,1))$ is well defined and $\in \mathscr{M}_{\mu,\kappa}(J)$, so let $f((\eta,1)) = \sigma_\eta(\bar\nu_\eta)$, $\bar\nu_\eta = \langle(\nu_{\eta,\epsilon}, \iota_{\eta,\epsilon}) : \epsilon < \epsilon_\eta\rangle$, $\nu_{\eta,i} \in J$, and $\iota_{\eta,\epsilon} \in \{1,-1\}$ [for] $\epsilon < \omega$.

Let $\epsilon_* = \epsilon_{\eta_*}$, $\iota_\epsilon = \iota_{\eta_*,\epsilon}$, $i^*_\epsilon = \ell g(\nu_{\eta_*,\epsilon})$ for $\epsilon < \epsilon_*$ and let

$$j^*_\epsilon = \sup\{j \le i^*_\epsilon : \sup \mathrm{Rang}(\nu_{\eta_*,\epsilon} \restriction j) < \delta\}.$$

Let n_* be large enough such that:

- If $\varepsilon < \varepsilon_\eta$ and then $\{\nu_{\eta_*,\varepsilon} \restriction j : j \le \ell g(\nu_{\eta_*,\varepsilon})\} \cap N_{n_*} \subseteq M_{n_*}$.

[Why? This is by clause (v) of Definition 1.1, where for $j = \ell g(\nu_{\eta_*,\varepsilon})$, $\varepsilon < \varepsilon_*$ we do it "by hand" as ε_* is finite.]

[You're mixing together ϵ and ε.]

Let $\nu^*_\epsilon = \nu_{\eta_*,\epsilon} \restriction j^*_\epsilon$; it belongs to M_{n_*}.

So $\{\nu^*_\epsilon : \epsilon < \epsilon_*\} \subseteq M_{n_*}$ is finite, hence it follows that $\nu^* = \langle \nu^*_\epsilon : \epsilon < \epsilon_*\rangle \in M_{n_*}$. Let k_* be such that $\eta_* \restriction k_* \in M_{n_*}$, $\eta_* \restriction (k_*+1) \in N_{n_*} \setminus M_{n_*}$ hence $\eta_*(k_*) \in N_{n_*} \cap \lambda \setminus M_{n_*}$ and let $\nu_{\eta_*,\varepsilon_*} = \eta_*$, $j^*_{\varepsilon_*} = k_*$, and $\alpha^*_\lambda = \min(M_{n_*} \cap \lambda \setminus \eta_*(k_*))$ hence $\sup(M_{n_*} \cap \lambda) \le \eta_*(k_*) < \alpha_*$.

Let $u_* = u_1 = \{\epsilon < \epsilon_* : j^*_\epsilon < i^*_\epsilon\}$. For $\epsilon \in u_*$ let[4]

$$\alpha^*_\epsilon = \min\big(N_{n_*} \cap (\lambda+1) \setminus \nu_{\eta_*,\epsilon}(j^*_\epsilon)\big),$$

so also $\bar\alpha^* := \langle \alpha_\epsilon : \epsilon \in u_*\rangle$ belongs to M_{n_*}.

We define $\mathscr{U}_1$ as the set of $\eta \in I$ [such that]:

$(*)_\eta$
- (a) $\eta_* \restriction k_* {}^\frown \langle\beta\rangle \lhd \eta \in \ell g(\eta) = \omega$
- (b) $\sigma_\eta = \sigma_*$, so $\epsilon_\eta = \epsilon_*$.
- (c) $\ell g(\nu_{\eta,\epsilon}) = i^*_\epsilon$ for $\epsilon < \epsilon_*$.
- (d) $\nu^*_\epsilon = \nu_{\eta_\beta,\epsilon} \restriction j^*_\epsilon$ for $\epsilon < \epsilon_*$.
- (e) $\iota_\epsilon = \iota_{\eta,\epsilon}$ for $\epsilon < \epsilon_*$.

Note

$(*)_2$
- (a) $\eta_* \in \mathscr{U}_1$ and $\mathscr{U}_1 \in M_{n_*}$.
- (b) $\mathrm{cf}(\alpha^*_\epsilon) \ge \mu^+$ for $\epsilon \in u_*$
- (c) If $\bar\alpha \in \prod_{\epsilon \in u_*} \alpha^*_\epsilon$ then for some $\eta \in \mathscr{U}_1$, we have

$$\epsilon \in u_* \Rightarrow \nu_{\eta,\epsilon}(j^*_\epsilon) \in (\alpha_\epsilon, \alpha^*_\epsilon).$$

[Why? Clause (a) direct by our choice. If $\mathrm{cf}(\alpha^*_\varepsilon) \le \mu$ then α^*_ε is a limit ordinal and there is in $\mathcal{H}(\chi)$ an increasing function f_ε from $\mathrm{cf}(\alpha^*_\varepsilon)$ into α^*_ε with unbounded range. Without loss of generality, $f_\varepsilon \in M_{n_*}$ so $\{f_\varepsilon(\beta) : \beta \in N_{n_*} \cap \mathrm{cf}(\alpha^*_\varepsilon)\}$ is an unbounded subset of $N_{n_*} \cap \alpha^*_\varepsilon$; but this set is equal to $\{f_\varepsilon(\beta) : \beta \in M_{n_*} \cap \mathrm{cf}(\alpha^*_\varepsilon)\}$, so $\sup(\alpha^*_\varepsilon \cap M^*_{n_*}) = \sup(\alpha^*_\varepsilon \cap N_{n_*})$, but this contradicts the choice α^*_ε via $\nu_{\eta_*,\varepsilon}(j^*_\varepsilon)$. Clause (c) follows.]

[4] Consider $u_2 = \{\varepsilon < \varepsilon_* : j^*_\varepsilon = i^*_\varepsilon = \omega$ but $\nu_{\eta_*,\varepsilon} \notin M_{n_*}\}$. Below we first assume $u_2 = \varnothing$. Second, if λ is regular or $\mu_1 < \lambda \le \mu_1^{\aleph_0}$ for some μ_1, by [Shea, xxx,yyy] this is to justify. If not, then (by xxx) without loss of generality, $\|N_n\|^{\aleph_0}$. See §1.

$(*)_3$ let $\mathscr{U}_2$ be the set of $\beta < \alpha_*$ such that for every $\bar{\alpha} \in \prod_{\varepsilon \in u_*} \alpha^*_\varepsilon$, there is η such that:
(a) $\eta \in \mathscr{U}_1$
(b) $\eta(k_*) = \beta$
(c) $\nu_{\eta,\varepsilon}(j^*_\varepsilon) \in (\alpha_\varepsilon, \alpha^*_\varepsilon)$ for $\varepsilon \in u_*$.

$(*)_4$ $\eta_*(k_*) \in \mathscr{U}_2$.

[Why? Note that $\eta_*(k_*) \in N_{n_*}$, but $\varepsilon \in u_* \Rightarrow \nu_{\eta_*,\varepsilon}(j^*_\varepsilon) \notin N_{n_*}$.]

Let $\alpha, \beta \leq \lambda$ and π be a one-to-one function from α onto β.

Now first we choose $\eta_\zeta \in I$ by induction on $\zeta < \alpha$ such that

$(*)_5$ (a) $\eta_{\zeta,1} \in \mathscr{U}_\eta$
(b) If $\epsilon \in u_*$ then $\nu_{\eta_{\delta(1,\zeta)},\epsilon}(j^*_\epsilon)$ is $< \alpha^*_\epsilon$ but is $> \mathrm{sub}\{\nu_{\eta_{\xi,1},\epsilon}(j^*_\epsilon) : \xi < \zeta\}$.

This is easy.

Second, we choose $\eta_{\zeta,2}$ by induction on $\zeta < \beta$ such that:

$(*)_6$ (a) $\eta_{\zeta,2} \in \mathscr{U}$
(b) if $\epsilon \in u_*$ then $\nu_{\eta_{\zeta,2},\epsilon}(j^*_\epsilon)$ is $< \alpha^*_\epsilon$ but is $> \sup\{\nu_{\eta_{\xi,2},\epsilon}(j^*_\epsilon) : \xi < \zeta\}$.

Let $\bar{a} = \langle a_\zeta : \zeta < \alpha\rangle$, $\bar{b} = \langle b_\zeta : \zeta < \alpha\rangle$ from ${}^\alpha I$ be chosen as follows: $a_\zeta = (\eta_{\delta(1,\zeta)}, 1)$, $b_\zeta = (\eta_{\delta(1,\pi(\zeta))}, 1)$ for $\zeta < \alpha$.

Now check, e.g.:

$(*)_6$ $a_{\zeta(1)} <_{\mathrm{or}(I)} a_{\zeta(2)}$ $\underline{\text{iff}}$ $\gamma_{\zeta(1)} < \gamma_{\zeta(2)}$ $\underline{\text{iff}}$ $\zeta(1) < \zeta(2)$.
$(*)_7$ $b_{\zeta(1)} <_{\mathrm{or}(I)} b_{\zeta(2)}$ $\underline{\text{iff}}$ $\gamma_{\pi\circ\zeta(1)} < \gamma_{\pi\circ\zeta(2)}$ $\underline{\text{iff}}$ $\pi \circ \zeta(1) < \pi \circ \zeta(2)$.

$\square_{2.4}$

Conclusion 2.5. *For $(\mu, \lambda, \alpha_*, \beta_*, \pi)$ as in 2.3(1), the class K_{or} has the full strong $(\lambda, \lambda_1, \mu, \kappa)$-$\varphi_{\mathrm{or},\alpha_*,\beta_*,\pi}$-bigness property and the strong $(2^\lambda, \lambda, \mu, \aleph_0)$-$\varphi_{\mathrm{or},\alpha_*,\beta_*,\pi}$-bigness property.*

Proof. By 2.4 + 1.4. $\square_{2.5}$

§ 3. Toward large "$\dot I(T_1, T, \kappa$-saturated) so large" when $\kappa_r(T) = \kappa$

We return to the non-superstable version (as in [Shee]). That is, we try to deal with $K^\kappa_{\rm tr}$, $\kappa = {\rm cf}(\kappa) > \aleph_0$ using $\mathscr{M}_{\mu,\kappa}(-)$.

Compare with 2.1(3).

Definition 3.1. For the class of $I \in K^\kappa_{\rm tr}$:

$$\varphi^\kappa_{\rm tr}(x_0, x_1 : y_0, y_1) := \Big[x_0 = y_0 \text{ and } P_\kappa(x_0) \text{ and } \bigvee_{i<\kappa} \big[P_{i+1}(x_1) \text{ and } P_{i+1}(y_1) \text{ and } P_i(x_1 \cap y_1)\big] \text{ and } \big[x_1 \lhd x_0 \wedge y_1 \ntriangleleft y_0\big] \text{ and } y_1 <_{\rm lex} x_1\Big].$$

In other words, when for transparency we restrict ourselves to standard $I \subseteq {}^{\kappa \geq}\lambda$ [such that] $x_0 = y_0 \in {}^\kappa\lambda$, and for some $n_i < \kappa$ and $\alpha < \beta < \lambda$ we have

$$x_1 = (x_0 \restriction i)\hat{\ }\langle\alpha\rangle \lhd x_0$$

and

$$y_1 = (x_0 \restriction i)\hat{\ }\langle\beta\rangle.$$

Claim 3.2. *1) Let $\kappa = {\rm cf}(\kappa) > \aleph_0$. There is a sequence $\langle I_\alpha : \alpha < \lambda\rangle$ which witnesses full strong $\bar\varphi^\kappa_{\rm tr} - (\lambda, \lambda, \mu$,bigness) <u>when</u> at least one of the of the following cases occurs (see inside the proof on the super version):*

(A) $\lambda = {\rm cf}(\lambda) > \kappa^{++}, \mu^{<\kappa}$ hence $\lambda \geq (2^{<\kappa})^+$,

(B) (a) $\lambda > \mu + {\rm cf}(\lambda)$

(b) $\lambda > \chi > {\rm cf}(\chi) = \kappa$ and $\lambda \leq \chi^{\langle\kappa\rangle_{\rm tr}}$.

2) We have (A) $\Rightarrow \boxplus \Rightarrow$ the conclusion of part (1), where

$\boxplus$ *There are $I_\varepsilon \in K^\kappa_{\rm tr}$ for $\varepsilon < \lambda$, $|I_\varepsilon| \leq \lambda$ and I_φ is super$^{7^+}$-unembeddable into $J_\varepsilon = \Sigma\{I_\zeta : \zeta \in \lambda \setminus \{\theta\}$, which means*

[Two open braces and only one close brace.]

$\boxplus_{\lambda,\mu,I,J}$ *If $\chi_* \gg \lambda$, $x \in \mathcal{H}(\chi_*)$ then we can find a pair $(\overline{M}, \eta)$ such that*

(i) $\overline{M} = \langle M_i : i < \kappa\rangle$, $M_i \prec (\mathcal{H}(\chi_*), \in, <^*_{\chi_*})$, M_i is $\prec$-increasing continuous, $\kappa + 1 \subseteq M_0$, and $M_i \cap \mu = \mathscr{M}_0 \cap N$.

(ii) $M_i \cap \mu = \mu_0 \cap \mu$

(iii) $\{I, J, \mu, \kappa, x\}$ belongs to M.

(iv) $\eta \in P^I_\kappa$ and $\{\eta \restriction i : i < \kappa\} \subseteq \bigcup_{j<\kappa} M_j$.

(v) $\eta \restriction j_i \in M_i$, $\eta(j_i) \in M_{i+1}$ for $i < \kappa$ successor.

*(vi) If $\eta \in P^J_\kappa$, then for some $j < \kappa$, $\{\eta \restriction i : i < \kappa\} \cap \bigcup_{i<\kappa} M_i \subseteq M_j$ **[or $\in M_j$?]**.*

(vii) See [Shea, 1.5=L7.3(B)(vii)].

Proof. 1) As $\lambda = {\rm cf}(\lambda) > \kappa^+$, there is a stationary $S \subseteq S^\lambda_\kappa$ which belongs to $\check I_\kappa[\theta]$ (see 0.4, 0.5). We can find $\bar a = \langle a_\alpha : \alpha \in S^+\rangle$ which witnesses it (see e.g. [She09, 0.7=L0.5]) so

$(*)_1$ (a) $S \subseteq S^+ \subseteq S^\lambda_{\leq\kappa}$

(b) $S = \{\delta \in S^+ : {\rm cf}(\delta) = \kappa\} = \{\delta \in S^+ : {\rm otp}(a_\alpha) = \kappa\}$

[I don't think a_α depends on δ.]

(c) $a_\alpha \subseteq \alpha$ has order type $\leq \kappa$.

(d) $\delta \in S^+ \Rightarrow \delta = \sup(a_\delta)$

(e) For every $\alpha < \lambda$ the set $\{a_\beta \cap \alpha : \beta$ satisfies $\alpha \in a_\beta$ has cardinality $< \lambda$ **[Same thing happened here.]**

(f) $\bar{a}$ guesses clubs.

[Why? See 0.5 or [She09, 0.8=L0.6].]

$(*)_2$ (a) (α) Let $\langle S_\varepsilon : \varepsilon < \lambda \rangle$ be a division of S to stationary subsets of λ.

(β) Without loss of generality, $\big\langle \bigcup\limits_{\alpha \in S_\varepsilon} a_\alpha : \varepsilon < \lambda \big\rangle$ is a sequence of pairwise disjoint $\eta_\alpha \in {}^\kappa\delta$ list[ing] a_α in increasing order for $\delta \in S^+$.

(b) Let

(α) I_ε be the tree $\{\eta_\alpha : \alpha \in S^+ \setminus S\} \cup \{\eta_\delta : \delta \in S_\varepsilon\}$.

(β) $I^+_\varepsilon = \{\eta : \eta \trianglelefteq \nu$ for some $\nu \in I_\varepsilon\}$

(γ) $I_{\varepsilon,\alpha} = I_\varepsilon \cap {}^{\kappa\geq}\alpha$, $I^+_{\varepsilon,\alpha} = I^+_\varepsilon \cap {}^{\kappa\geq}\alpha$ for $\alpha < \lambda$.

(c) For $\zeta < \lambda$ and $\alpha < \lambda$ let

(α) $J_\zeta = \Sigma\{I_\varepsilon : \varepsilon < \lambda$ and $\varepsilon \neq \zeta\}$

(β) $J_{\zeta,\alpha} = J_\zeta \restriction \big\{\nu :$ for some $\xi \in \alpha \setminus \{\zeta\}$ we have $\nu \in I_\xi \cap {}^{\kappa>}\alpha\big\}$.

[Are these Sigmas supposed to be sums?]

Note: if $\eta \in I^+_\varepsilon \setminus I_\varepsilon$ then $\ell g(\eta)$ is a limit ordinal and member of $I^+_\varepsilon \setminus I_\varepsilon$. **[This may]** cause problems, as $I^+_{\varepsilon,\alpha}$ may have cardinality λ, whereas:

$(*)_{2.1}$ If $\alpha < \lambda$, $\varepsilon < \lambda$ then $I_\varepsilon \cap {}^{\kappa\geq}\alpha$ and $J_\varepsilon \cap {}^{\kappa\geq}\alpha$ have cardinality $< \lambda$. To prove the claim assume **[the following two bullets]**:

$(*)_3$ $\zeta < \lambda$ and $f : I_\zeta \to {}^{\kappa>}\mathscr{M}_{\mu,\kappa}(J_\zeta)$.

$(*)_4$ (a) For $\eta \in I_\zeta$, let $f(\eta) = \sigma_\eta(\bar{\nu}_\eta)$, $\bar{\nu}_\eta \in {}^{\kappa>}(J_\zeta)$.

(b) Let $\bar{\nu} = \langle \nu_{\eta,i} : i < i_\eta \rangle$, $\nu_{\eta,i} \in I_{\varepsilon(\eta,i)}$, and $\varepsilon(\eta,i) \in \lambda \setminus \{\zeta\}$.

(c) For $\delta \in S_\zeta$, let $j_\delta = \sup\big\{\ell g(\eta_\delta \cap \nu_{\eta_\delta,i}) : i < i_{\eta_\delta}\big\}$.

(d) Let

$$E = \big\{\delta < \lambda : \delta \text{ a limit ordinal, } \big[\alpha \in \delta \wedge \eta \in I_\zeta \cap {}^{\kappa>}\alpha \Rightarrow f(\eta) \in {}^{\kappa>}(J_{\zeta,\delta})\big]\big\}$$

$(*)_5$ Now, for every $\nu \in I_\varepsilon$ of length $< \kappa$, let

$$\begin{aligned} u_\nu := \big\{\alpha < \lambda : {} & \text{there is no } v \subseteq \lambda \text{ of cardinality } (2^{<\kappa})^+ \\ & \text{such that } \alpha, \beta \in v \Rightarrow \sigma_{\nu\char`\^\langle\alpha\rangle} = \sigma_{\nu\char`\^\langle\beta\rangle} \\ & \text{and } \langle \bar{\nu}_{\nu\char`\^\langle\beta\rangle} : \beta \in v \rangle \text{ is an indiscernible sequence in } K^\kappa_{\mathrm{tr}}\big\}. \end{aligned}$$

$(*)_6$ Above, u_ν has cardinality $\leq 2^{<\kappa} + \mu^{<\kappa}$.

[Why? Let $E_{\delta,j} = \{(\alpha,\beta) : \alpha,\beta < \lambda$ and $\sigma_{(\eta_\delta\restriction 1)\char`\^\langle\alpha\rangle} = \sigma_{(\eta_\delta\restriction 1)\char`\^\langle\beta\rangle}\}$. **[This]** is an equivalence relation with $\leq \mu^{<\kappa}$ equivalence classes, hence it suffices to prove $u_{\delta,j}$ has $\leq 2^{<\kappa}$ members in each equivalence class. So fix **[an]** equivalence class $\gamma/E_{\delta,j}$. If $v_i \subseteq \gamma/E_{\delta,j}$ has cardinality $(2^{<\kappa})^+$ then some $v \subseteq v_1$ of cardinality $(2^{<\kappa})^+$ satisfies the condition, in the " ... for no ... v ...", so we are done.]

$(*)_7$ (a) E is a club of λ, where

$$E = \big\{\delta < \lambda : \delta \text{ a limit ordinal, } \big[\alpha < \delta \wedge \nu \in I_{\varepsilon,\alpha} \Rightarrow \sup(u_\nu) < \delta\big]\big\}.$$

(b) E' is a club of λ, where $E' = \{\delta \in E : \mathrm{otp}(E \cap \delta) = \delta > \kappa\}$.

Now choose $\delta \in S_\varepsilon \cap E'$ such that $a_\delta \subseteq E'$. Choose a successor ordinal $j \in [j_\delta, \kappa)$ so necessarily $u_{\eta_\delta\restriction j}$ **[*does WHAT?*]**. $\square_{3.2}$

Comment 3.3. For the super version (i.e. as in [Shea, §1] rather than [Shee, §2])

$(*)$ Generalizing [Shea, 1.1], on $(\eta, \overline{M})$ we should add:
- $\eta(i) \in M_{i+1} \setminus M_i$; moreover, $\eta(i) \notin \bigcup\{u \in M_i : |u| \le 2^{<\kappa}\}$.
- The proof above shows how bigness implies the bigness properties.

Claim 3.4. *The conclusion of [3.2 or so] holds when:*

(B) (a) $\lambda > \theta = \mathrm{cf}(\theta) > \kappa^+ + \mu^{<\kappa}$ and λ is singular.
(b) $\lambda = \Sigma\{\lambda_\zeta : \zeta < \mathrm{cf}(\lambda)\}$, λ_η increasing, λ_ζ regular.
(c) $S_\zeta \subseteq S^{\lambda_\zeta}_\kappa$ is stationary, $S_\zeta \in \check{I}_\kappa[\lambda_\zeta]$.
(d) $\bar\eta_\zeta = \langle \eta_{\zeta,\delta} : \delta \in S_\zeta\rangle$, where $\eta_{\zeta,\delta} \in {}^\kappa\delta$ is increasing with limit δ.
(e) If $\zeta < \mathrm{cf}(\lambda)$ and $\alpha < \lambda$ then $\{\eta_{\zeta,\delta} \restriction i : \delta \in S_\zeta$ satisfies $\eta_{\zeta,\delta}(i) = \alpha\}$ has cardinality $< \lambda_\zeta$.
(f) $\bar\eta_\zeta$ guesses clubs.
(g) $\bar\eta_\zeta$ is (θ,θ)-free (see [She20] and [She13]). Moreover, if $\mathcal{T} \subseteq {}^{\kappa>}(\lambda_\zeta)$ [is] a sub-tree [with] $|\mathcal{T}| = \theta$ then

$$\Lambda = \big\{\delta \in S_{\zeta(1)} : \eta_{\zeta(1),\delta} \in \lim(\mathcal{T})\big\}$$

has cardinality $\le \theta$ and there is an $h : \Lambda \to \kappa$ such that

$$(\forall \eta \in \Lambda)(\exists^{<\theta}\nu \in \Lambda)\big[\nu \restriction h(\nu) = \eta \restriction h(\eta)\big].$$

Proof. Without loss of generality,

$(*)_1$ If $\zeta < \mathrm{cf}(\lambda)$, $\delta \in S - \varepsilon$, and $i < \kappa$ then θ^+ divides $\eta_{\zeta,\delta}(i)$.

Let $S_* \in \check{I}_\kappa[\theta]$ be stationary and let $\bar\rho = \langle \rho_\delta : \delta \in S_*\rangle$ (with $\rho_\delta \in {}^\kappa\delta$ increasing with limit δ) guess clubs and $\theta > |\{\rho_\delta \restriction i : \rho_\delta(i) = \alpha\}|$ for every $\alpha < \theta$. Choose $\langle S^*_\varepsilon : \varepsilon < \mathrm{cf}(\lambda)\rangle$ as a sequence of pairwise disjoint stationary subsets of S_i.

For $\delta \in S_\varepsilon$, let $\eta^*_{\varepsilon,\delta,\beta} = \langle \eta_{\varepsilon,\delta}(i) + \rho_{\varepsilon,\beta}(i) : i < \kappa\rangle$. Let $\langle S_{\varepsilon,\alpha} : \alpha < \lambda_\varepsilon\rangle$ be a partition of S_ε to stationary subsets of λ_ε.

Now, if $\alpha \in [\lambda_{<\zeta}, \lambda_\zeta)$ we define

$(*)_2$ (a) $I_\alpha = \{\eta^*_{\zeta,\delta,\alpha} \restriction i : i \le \kappa \text{ and } \delta \in S_{\lambda,\alpha}\} \cup \{\langle\gamma\rangle : \gamma < \lambda\}$
(b) $J_\alpha = \Sigma\big\{I_\gamma : \gamma \in \lambda \setminus \{\alpha\}\big\}$.

So assume

$(*)_3$ χ_* regular $\gg \lambda$, $\alpha(1) \in [\lambda_{<\zeta(1)}, \lambda_{\zeta(1)})$, and $(I_{\alpha(1)}, J_{\alpha(1)}, \mu, \kappa, \dots) \in \mathcal{H}(\chi_*)$.

We can choose N^1_β by induction on $\beta < \lambda_\zeta$ such that:

$(*)_4$ (a) $N'_\beta \prec (\mathcal{H}(\chi_*), \in, <^*_{\chi_*})$ is increasing continuous with β.
(b) If $\gamma' < \beta$ then $\langle N^1_\gamma : \gamma \le \gamma'\rangle \in N_\beta$.
(c) $\{I_\alpha, J_\alpha, \mu, \kappa\} \in N^1_\beta$
(d) $\|N^1_\beta\| < \lambda_{\zeta(1)}$ and $N^1_\beta \cap \lambda_\zeta \in \lambda_\zeta$.

$(*)_5$ Choose $\delta(1) \in S_{\zeta(1),\alpha(1)}$ such that $N^1_{\delta(1)} \cap \lambda_{\zeta(1)} = \delta(1)$; moreover,

$$\{\eta_{\zeta(1),\delta(1)}(i) : i < \kappa\} \subseteq \{\beta : N^1_\beta \cap \lambda = \beta\}.$$

$(*)_6$ we choose $M_{\gamma,i}$ for $i < \kappa$ by induction on $\gamma < \theta$ such that:
(a) $M_{\gamma,i} \prec N_{\eta_{\zeta(1),\delta(1)}}(i+1)$ has cardinality $< \theta$.
(b) $j < i \Rightarrow M_{\gamma,j} \cap N_{\eta_{\zeta(1),\delta(1)}}(i+1) \subseteq M_{\gamma,i}$
(c) $\eta_{\zeta(1),\delta(1)} \restriction i \in M_{\gamma,i}$
(d) $M_{\gamma,i} \supseteq \mu^{<\kappa} + 1$ and $j < i \Rightarrow M_{\gamma,j} \subseteq M_{\gamma,i}$.

Let $M_i = \bigcup\{M_{\gamma,i} : \gamma < \theta\}$, $M = V\}M_i : i < \kappa\}$.

[I assume that's supposed to be $M = \bigcup\limits_{i<\kappa} M_i$?]

Let $\Lambda = \{\eta \in P^J_\kappa : (\forall i < \kappa)[\eta \restriction i \in M]\}$ and **[let it]** be as in clause (B)(f), so $|\Lambda| \leq E = \{\gamma < \theta$: if $\eta \in \Lambda$ and $h(\eta) \in \bigcup\limits_i M_{\gamma,i}$ then $\{\eta \in \Lambda : (\eta \restriction h(\eta) \in \bigcup\limits_{i<\kappa} M_{\gamma,i}\} \subseteq \bigcup\limits_{i<\kappa} M_{\gamma,i}$.

[More open braces than close, and also more open parens than closes.]

So E is a club of θ and we can choose $\gamma(r) \in S^*_{\zeta(1)} \cap E$.

The rest should be filled. $\square_{3.4}$

Claim 3.5. *If $\lambda > \mu^{<\kappa}$ then (λ, λ, μ) has the super bigness property (see 3.2), except possibly when*

(∗) $\lambda \leq (\mu^{<\kappa})^{+\kappa}$.

Proof. **Case 1**: λ is regular.

Use 3.2.

Case 2: $\lambda > (\mu^{<\kappa})^{+\kappa}$.

Let $\theta = (\mu^{<\kappa})^{+4}$. Now by [She13] for arbitrarily large regular $\lambda' \in [\mu^{<\kappa}, \lambda)$ there are $S', \bar\eta'$ as in 3.4(B)(d) for $(S_\zeta, \bar\eta_\zeta)$ **[FILL]**.

Case 3: $\lambda \in \big(\mu^{<\kappa}, (\mu^{<\kappa})^{+\kappa}\big)$ is singular.

If $\kappa = \aleph_0$ this is empty. Can we imitate [Shea, 2.19=L7.10,pg.35]? $\square_{3.5}$

Discussion 3.6. Can 3.4 be improved to get $\overline{N} \restriction (i+1) \in N_{i+1}$ for all/all successor i? Probably

(∗) $\alpha < \lambda_\zeta \Rightarrow \mathrm{cf}([\alpha]^\theta, \subseteq) < \lambda_\zeta$.

Claim 3.7. *1) If (A) then (B), where:*

(A) (a) $\lambda > \theta > \mu^{<\kappa} + \mathrm{cf}(\lambda)$

(b) $\mathrm{cf}(\theta) = \aleph_0$

(c) Optional: there is a sequence $\langle \mathfrak{a}_\varepsilon : \varepsilon < \mathrm{cf}(\lambda)\rangle, \mathfrak{a}_\alpha \subseteq \mathrm{Reg} \cap \theta \setminus (\mu < \kappa)^+$, $\sup(\mathfrak{a}_\varepsilon) = \theta^+$, $\mathrm{otp}(\mathfrak{a}_\varepsilon) = \omega$, $\varepsilon \neq \zeta \Rightarrow \aleph_0 > |\mathfrak{a}_\varepsilon \cap \geq_\zeta |, (\pi\mathfrak{a}_\varepsilon)$

[I have no idea what was intended on that line.]

(B) K^κ_{tr} *has the* $(\lambda, \lambda, \mu, \kappa)$*-bigness property.*

2) Debt: define a super-bigness version.

Proof. **Step A**:

(∗)$_0$ (a) let $\bar{\mathfrak{a}} = \big\langle \mathfrak{a}_\zeta : \zeta < \mathrm{cf}(\lambda)\big\rangle$ be as in 3.7(A)(c).

(b) $S^*_\zeta \subseteq S^{\theta^+}_\kappa$ is stationary and belongs to $\check{I}_\theta[\theta^+]$ for $\zeta < \mathrm{cf}(\lambda)$.

(c) $\rho_\zeta = \langle \rho_{\zeta,\gamma} : \gamma < \theta^+\rangle$ is a $<_{J^{\mathrm{bd}}_{\mathfrak{a}_\zeta}}$-increasing cofinal in $(\pi\mathfrak{a}_\zeta, <_{J^{\mathrm{bd}}_{\mathfrak{a}_\zeta}})$.

(d) $\bar\rho_\zeta = \langle \rho_{\zeta,\gamma} : \gamma \in S^*_\zeta\rangle$, $\rho_{\zeta,\gamma} \in {}^\kappa\gamma$ is increasing, clearly with limit γ.

[Both these guys are a sequence of $\rho_{\zeta,\gamma}$-s. It could be what was intended, but should be verified.]

(∗)$_1$ $\rho_\delta \restriction (i+1) \in N_{\rho_\delta(i+1)}$ for $\delta \in S$, $i < \kappa$.

[Why? Should be clear.]

Stage B:

Now we imitate the proof of 3.4.

(∗)$_2$ (a) We choose $\big\langle (\lambda_\zeta, S_\zeta) : \zeta < \mathrm{cf}(\lambda)\big\rangle$ and as in 3.4(B)(c).

(b) we choose $\bar\eta_\zeta$ such that:

(α) $\bar\eta_\zeta = \langle \eta^1_{\zeta,\delta} : \delta \in S_\zeta \rangle$

(β) $\eta^1_{\zeta,\delta} \in {}^{(\theta^+ \cdot \kappa)}\delta$ is increasing with limit δ.

(γ) $i < \theta^+ \cdot \kappa \wedge \alpha < \lambda_\zeta \Rightarrow \big|\{\eta^1_{\zeta,\delta} \restriction i : \delta \text{ satisfies } \eta_{\zeta,\delta}(i) = \alpha\}\big| < \lambda_\zeta$

(δ) Let $\langle S_{1,\alpha} = S_{\zeta,\alpha} : \alpha \in [\lambda_{<\zeta}, \lambda_\zeta)\rangle$ be a partition of S_ζ to stationary subsets.

(ε) $\bar\eta_\zeta \restriction S_{\zeta,\alpha}$ guesses clubs.

$(*)_3$ (a) If $\zeta < \mathrm{cf}(\lambda)$, $\alpha \in [\lambda_{<\zeta}, \lambda_\zeta)$, $\delta \in S_{\zeta,\alpha}$, [and] $\gamma \in S^*_\zeta$ then we define $\eta^*_{\zeta,\alpha,\delta,\gamma} \in {}^\kappa\delta$ by

$$\eta^*_{\zeta,\alpha,\delta,\gamma}(\omega i + n) = \eta^1_{\zeta,\delta}\big(\theta^+ \cdot i + \theta \cdot \rho_{\zeta,\gamma}(i) + \varrho_{\zeta,\gamma}(n)\big).$$

(b) For $\zeta < \mathrm{cf}(\lambda)$ and $\alpha \in [\lambda_{<\zeta}, \lambda_\zeta)$, let

$$I_\alpha = \big\{\eta^*_{\zeta,\alpha,\delta,\gamma} \restriction i : i \le \kappa,\ \delta \in S_{\zeta,\alpha} \text{ and } \gamma \in S^*_\zeta\big\} \cup \big\{\langle\beta\rangle : \beta < \lambda\big\}.$$

So it suffices to prove

$(*)_4$ If $\zeta(1) < \mathrm{cf}(\lambda)$, $\alpha(1) \in [\lambda_{<\zeta(1)}, \lambda_{\zeta(1)})$, $J_{\alpha(1)} = \Sigma\big\{I_\beta : \beta \in \lambda \setminus \{\alpha(1)\}\big\}$ then I_α is (μ,κ)-unembeddable into $\mathscr{M}_{\mu,\kappa}(J_\alpha)$.

So assume

$\boxplus$ $p : I_{\alpha(1)} \to {}^{\kappa>}(\mathscr{M}_{\mu,\kappa}(J_\alpha))$.

Let $\chi_* > \lambda^+$ be regular.

$\boxplus$ Choose N^1_β by induction on $\beta < \lambda_\alpha$ such that:
(a) $N^1_\beta \prec (\mathcal{H}^*(\chi_*), \in, <^*_{\chi_*})$
(b) If $\gamma^1 < \beta$ then $\langle N^1_\gamma : \gamma \le \gamma^1\rangle \in N_\beta$.
(c) $\{I_\alpha, J_\alpha, f, \mu, \kappa\} \in N^1_\beta$
(d) $\|N^1_\beta\| < \lambda_{\zeta(1)}$ and $N^1_\beta \cap \lambda_{\zeta(1)} \in \lambda_{\zeta(1)}$.

$\boxplus$ Choose $\delta(1) \in S_{\zeta(1),\alpha(1)}$ such that $N^1_{\delta(1)} \cap \lambda_{\zeta(1)} = \delta(1)$, and moreover, $\{\eta^1_{\zeta(1),\delta(1)}(i) : i < \theta^+ \cdot \kappa\} \subseteq \{\beta < \lambda_{\zeta(1)} : N^1_\beta \cap \lambda_{\zeta(1)} = \beta\}$.
$(*)$ Let
(a) $f(\eta^*_{\zeta(1),\alpha(1),\delta(1),\gamma}) = \sigma_\gamma(\bar\nu_\gamma)$
(b) $\bar\nu_\gamma = \langle \nu_i : i < i^*_\gamma\rangle$, so $\nu_{\gamma,i} \in J_{\alpha(1)}$.
(c) $j_\gamma = \sup\big\{\ell g(\eta^*_{\zeta(1),\alpha(1),\delta(1),\gamma} \cap \nu_{\gamma,i}) + 1 : i < i_*\big\}$
(d) $E = \big\{(\gamma_1,\gamma_2) : \gamma_1, \gamma_2 \in S^*_\zeta$ and $\sigma_{\gamma_1} = \sigma_{\gamma_2},\ i^*_{\gamma_1} = i^*_{\gamma_2},\ \ell g(\nu_{\gamma_1,i}) = \ell g(\nu_{\gamma_2,i})$ for $i < i^*_{\gamma_1},\ j_{\gamma_1} = j_{\gamma_2}\big\}$ **[maybe more]**
(e) $\gamma_* = \min\{\gamma : \gamma/E \text{ is stationary}\}$.

Now we can fix an interval of length θ in θ^+ and corresponding considering?? γ_*/E and imitate [Shea, 2.15=L7.9,pg.27]. $\square_{3.7}$

Discussion 3.8. 1) Is

$(*)_1$ $\mu^{<\kappa} + \mathrm{cf}(\lambda) < \theta < \lambda$, $\mathrm{cf}(\theta) = \aleph_0$

enough, or do we need also

$(*)_2$ there are $\langle \mathfrak{a}_\varepsilon : \varepsilon < \mathrm{cf}(\lambda)\rangle$ as in 3.7?

If $(*)_2$ suffices, then $\lambda = (\mu^{<\kappa})^{+\omega}$ is the only open[ing] (for bigness, ignoring the super bigness version). Hence, as $\kappa = \mathrm{cf}(\kappa) > \aleph_0$, we have $(\mu^{<\kappa})^{\aleph_0} = \mu^{<\kappa}$. We may try to combine aspects of the last proof and [Shea, 2.19=L7.10,pg.35].

2) To prove $(*)_2$, we then need to use a fixed $\mathfrak{a}$ for all ζ, but $\bar{\rho}_\zeta = \bar{\rho} \restriction S^*$, $\langle S^*_\zeta : \zeta < \mathrm{cf}(\lambda)\rangle$ are pairwise disjoint stationary subsets of $S^{\theta^+}_{\aleph_0}$.

3) But, if we let $\lambda = (\mu^{<\kappa})^{+\delta}$:

Case 1: $\delta > \kappa$

See [xxx].

Case 2: $\delta \leq \kappa$, and for some $\sigma < \mathrm{cf}(\delta)$ we have $\sigma^{\aleph_0} \geq \mathrm{cf}(\delta)$.

By the version with $(*)_1 + (*)_2$.

Case 3: Neither Case 1 nor Case 2.

So $\delta = \partial = \mathrm{cf}(\partial) \leq \kappa$ and $\alpha < \partial \Rightarrow |\alpha|^{\aleph_0} < \sigma$.

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