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Canonical universal locally finite groups

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Abstract – We prove that the existence of universally locally finite groups implies that there is a canonical one in any strong limit singular cardinality of countable cofinality. Moreover, those canonical groups are parallel to the special models for complete first order theories. For showing the existence we rely on the existence of enough indecomposable such groups as we proved in [Rend. Sem. Mat. Univ. Padova 144 (2020), 253–270]. More generally, we also deal with the existence of a universal member in general classes for such cardinals.

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1. Introduction

1.1 – *Background and aims*

Our motivation is to investigate the class $\mathbf{K}_{\mathrm{lf}}$ of locally finite groups. The reader may consider only this case ignoring the general case or may consider universal classes (see Definition 1.4). The present work continues [15]. For historical remarks, see there and [17]; for earlier history, see [11].

The main problem we are facing is the following:

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Problem 1.1. We consider the following questions:

(1) Is there a universal $G \in \mathbf{K}^{\rm lf}_\lambda$ (= the class of members of $\mathbf{K}_{\rm lf}$ of cardinality λ), see Definition 1.3 (1); e.g., for $\lambda = \beth_\omega$? Or just λ a strong limit cardinal of cofinality $\aleph_0$ (which is not above a compact cardinal)?

(2) May there (consistently) be a universal $G \in \mathbf{K}^{\rm lf}_\lambda$, when $\lambda < \lambda^{\aleph_0}$, e.g., for $\lambda = \aleph_1 < 2^{\aleph_0}$?

For general background on the problem of the existence of a universal model for a class in cardinality λ see the classical works by Jonsson [9, 10], Morley–Vaught [13] and the recent surveys by Džamonja [5] and by the author [19].

Returning to locally finite groups, concerning Problem 1.1 (1) recall that by Grossberg–Shelah [7], if $\lambda = \lambda^{\aleph_0}$, then there is no universal member for $\mathbf{K}^{\rm lf}_\lambda$. However, if λ is a strong limit cardinal of cofinality $\aleph_0$ above a compact cardinal κ, then there is $G \in \mathbf{K}^{\rm lf}_\lambda$ which is universal. So Problem 1.1 addresses the remaining main open cases.

Let us consider the model theory of locally finite groups. Recall the following definition.

Definition 1.2. Let G be a group.

(1) G is an lf (*locally finite*) group if every finitely generated subgroup of G is finite.

(2) G is an exlf (*existentially closed locally finite*) group (in [11] it is called ulf, universal locally finite group) if G is a locally finite group and for any finite groups $K \subseteq L$ and embedding of K into G, the embedding can be extended to an embedding of L into G.

(3) Let $\mathbf{K}_{\rm lf}$ be the class of lf (locally finite) groups (partially ordered by $\subseteq$, being a subgroup) and let $\mathbf{K}_{\rm exlf}$ be the class of existentially closed $G \in \mathbf{K}_{\rm lf}$.

Wehrfritz asked about the categoricity of the class of exlf groups in any $\lambda > \aleph_0$. This was answered by Macintyre–Shelah [12] who proved that in every $\lambda > \aleph_0$ there are 2^λ non-isomorphic members of $\mathbf{K}^{\rm exlf}_\lambda$. This was disappointing in some sense: in $\aleph_0$ the class is categorical, so the question was perhaps motivated by the hope that also general structures in the class can be understood to some extent.

The existence of a universal object can be considered as a weak positive answer.

A natural and frequent question on a class of structures is the existence of rigid members, i.e. those with no non-trivial automorphism. Now any exlf group $G \in \mathbf{K}_{\rm exlf}$ has non-trivial automorphisms: the inner automorphisms (recalling it has a trivial center). So the natural question is about complete members where a group is called complete if and only if it has no non-inner automorphism.

Concerning the existence of a complete, locally finite group of cardinality λ: Hickin [8] proved that such group exists in $\aleph_1$ (and more: for example, he found a family of

$2^{\aleph_1}$ such groups pairwise far apart, i.e., no uncountable group is embeddable into two of them). Thomas [22] assumed G.C.H. and built one in every successor cardinal (and more: for example, it has no Abelian or just solvable subgroup of the same cardinality). Related are works by Giorgetta–Shelah [6] and Shelah–Ziegler [21] who investigated $\mathbf{K}_{G_*}$ getting similar results. Dugas–Göbel [4, Thm. 2] proved that for $\lambda = \lambda^{\aleph_0}$ and $G_0 \in \mathbf{K}^{\mathrm{lf}}_{\leq\lambda}$ there is a complete $G \in \mathbf{K}^{\mathrm{exlf}}_{\lambda^+}$ extending G_0; moreover 2^{λ^+} pairwise non-isomorphic ones. Then Braun–Göbel [1] got better results for complete locally finite p-groups.

Now [15] shows that although the class $\mathbf{K}_{\mathrm{exlf}}$ is very "unstable", there is a large enough set of definable types so we can imitate stability theory and have reasonable control in building exlf groups, using quantifier free types. This may be considered as a "correction" to the non-structure results discussed above. This was applied to build a canonical extension of a locally finite group of the same cardinality and also endo-rigid locally finite groups in a more relaxed way.

In the present work, we return to the universality problem for $\mu = \beth_\omega$ or just strong limit of cofinality $\aleph_0$. We prove for $\mathbf{K}_{\mathrm{lf}}$ and similar classes that if there is a universal model of cardinality μ, then there is something like a special model of cardinality μ, in particular, universal and unique up to isomorphism. This relies on [18], which proves the existence and even the density of so-called θ-indecomposable (i.e., θ is not a possible cofinality) models in $\mathbf{K}_{\mathrm{lf}}$ of various cardinalities continuing the work of Corson–Shelah [3] who deal with the class of all groups.

Returning to Problem 1.1 (1), a possible avenue is to try to prove the existence of universal members in μ when $\mu = \sum_{n<\omega} \mu_n$, each μ_n measurable $< \mu$, i.e., maybe for some reasonable classes this holds.

1.2 – *Definitions*

We begin with describing the general context. In the sequel, $\mathbf{K}$ will be one of the following cases:

Case 1 $\mathbf{K} = \mathbf{K}_{\mathrm{lf}}$, the class of locally finite groups, so the submodel relation is just being a subgroup.

Case 2 $\mathbf{K}$ is a universal class, see Definition 1.4 (1) below, the submodel relation is just being a submodel.

Case 3 $\mathbf{K}$ is $\mathfrak{k} = (K_{\mathfrak{k}}, \leq_{\mathfrak{k}})$, an a.e.c. with $\mathrm{LST}_{\mathfrak{k}} < \mu$, see [14, §1]; we shall only comment on it. In particular, in this context, in the definitions, $M \subseteq N$ should be replaced by $M \leq_{\mathfrak{k}} N$.

We now need several definitions.

Definition 1.3. We define the following:

(1) We say that $M \in \mathbf{K}_\mu$ is universal (in $\mathbf{K}$ or in $\mathbf{K}_\mu$) when every member of $\mathbf{K}_\mu$ can be embedded into it.

(2) We say that $M \in \mathbf{K}$ is universal for $\mathbf{K}_{<\mu}$ when every $M \in \mathbf{K}_{<\mu}$ can be embedded into it; see Definition 1.4 (4) below.

(3) We define "$M \in \mathbf{K}$ is universal for $\mathbf{K}_\mu$" and "$M \in \mathbf{K}$ is universal for $\mathbf{K}_{\leq\mu}$" similarly.

Definition 1.4. We define the following:

(1) We shall say that $\mathbf{K}$ is a universal class when for some vocabulary $\tau = \tau_{\mathbf{K}}$:
   (a) $\mathbf{K}$ is a class of τ-models, closed under isomorphisms;
   (b) a τ-model belongs to $\mathbf{K}$ iff every finitely generated sub-model belongs to it.

(2) Let $\mathbf{K}_\mu$ be the class of $M \in \mathbf{K}$ of cardinality μ. We define $\mathbf{K}_{<\mu}, \mathbf{K}_{\leq\mu}$ naturally.

(3) For cardinals $\lambda \leq \mu$ let $\mathbf{K}_{\mu,\lambda}$ be the class of pairs (N, M) such that $N \in \mathbf{K}_\mu$, $M \in \mathbf{K}_\lambda$ and $M \subseteq N$.

(4) Let $(N_1, M_1) \leq_{\mu,\lambda} (N_2, M_2)$ mean that $(N_\ell, M_\ell) \in \mathbf{K}_{\mu,\lambda}$ for $\ell = 1, 2$ and $M_1 \subseteq M_2$, $N_1 \subseteq N_2$.

(5) For $\lambda \leq \mu$ we define $\mathbf{K}_{\mu,<\lambda}$ and $\leq_{\mu,<\lambda}$ similarly.

(6) A universal class $\mathbf{K}$ can be considered as the a.e.c. $\mathfrak{k} = (\mathbf{K}, \subseteq)$.

Also some notation is needed.

Notation 1.5. We introduce the following notation.

(1) Let M, N and also G, H, L denote members of $\mathbf{K}$.

(2) Let $|M|$ be the universe = set of elements of M and $\|M\|$ its cardinality.

(3) Let a, b, c, d denote members of such M, and let $\bar{a}, \bar{b}, \ldots$ denote sequences of such elements.

Finally, we introduce some more definitions.

Definition 1.6. (1) We say that the pair (N, M) is an (χ, μ, κ)-amalgamation base (or amalgamation pair; but we may omit χ when $\chi = \mu$, and we may even omit μ, κ too) when
   (a) $(N, M) \in \mathbf{K}_{\mu,\kappa}$;
   (b) if $N_1 = N$ and $M \subseteq N_2 \in \mathbf{K}_\chi$, then N_1, N_2 can be amalgamated over M, this mean that for some N_3, f_1, f_2 we have $M \subseteq N_3 \in \mathbf{K}$ and f_ℓ-embeds N_ℓ into N_3 over M.

(2) We say that the pair (N, M) is a universal (μ, λ)-amalgamation base (we may omit μ, λ) when

(a) $(N, M) \in \mathbf{K}_{\mu,\lambda}$;

(b) if $N \subseteq N' \in \mathbf{K}_{\mu}$, then N' can be embedded into N over M.

(3) In parts (1) and (2), we may omit μ, κ when $(\mu, \lambda) = (\|N\|, \|M\|)$.

2. Indecomposability

In this section we deal with indecomposability, equivalently $\mathrm{CF}(M)$, see, e.g., [20]. We have $\mathbf{K}_{\mathrm{lf}}$ in mind, but still it is meaningful and of interest also for other classes.

Why do we deal with indecomposable members $\mathbf{K}$? When we shall try to understand universal members M of $\mathbf{K}_{\mu}$, we shall use some θ-indecomposable $N \subseteq M$ of cardinality $< \mu$. How will this help us? The point is that $N \in \mathbf{K}_{<\mu}$ may have too many embeddings into M, but if ($\theta = \mathrm{cf}(\theta) \neq \mathrm{cf}(\mu)$ and $\alpha < \mu \Rightarrow |\alpha|^{\|N\|} < \mu$ and) N is θ-indecomposable and θ is regular uncountable $< \mu$, then this is not the case.

We need indecomposable $\mathbf{c} : [\lambda] \to \theta$ in order to build enough θ-indecomposable locally finite groups (as done in [18]).

Definition 2.1. We define the following notions concerning decomposability.

(1) We say that M is θ-indecomposable or $\theta \in \mathrm{CF}(M)$ when: θ is regular and if $\langle M_i : i < \theta \rangle$ is $\subseteq$-increasing with union M, then $M = M_i$ for some i.

(2) We say that M is Θ-indecomposable when it is θ-indecomposable for every $\theta \in \Theta$. We say that M is Θ^{orth}-indecomposable when it is θ-indecomposable for every regular $\theta \notin \Theta$.

(3) We say that G is θ-indecomposable inside G^{+} when

(a) $\theta = \mathrm{cf}(\theta)$;

(b) $G \subseteq G^{+}$;

(c) if $\langle G_i : i \leq \theta \rangle$ is $\subseteq$-increasing continuous and $G_{\theta} = G^{+}$ (hence $G \subseteq G_{\theta}$), then for some $i < \theta$ we have $G \subseteq G_i$.

(4) For $\theta = \mathrm{cf}(\theta) \leq \lambda \leq \mu$ such that $\theta \notin \Theta_{\lambda}$ (see Theorem 2.2 (1)), we say that $\mathbf{K}$ is (μ, λ, θ)-indecomposable when for every pair $(N, M) \in \mathbf{K}_{\mu,\lambda}$ there is $(N_1, M_1) \in \mathbf{K}_{\mu,\lambda}$ which is $\leq_{\mu,\lambda}$-above it and M_1 is θ-indecomposable (really, not just inside N_1). For $\theta = \mathrm{cf}(\theta) < \lambda \leq \mu$ we say that $\mathbf{K}$ is $(\mu, < \lambda, \theta)$-indecomposable when: if $\theta = \mathrm{cf}(\theta) \leq \lambda_1 < \lambda$, $\theta \notin \Theta_{\lambda_1}$, then $\mathbf{K}$ is (μ, λ_2, θ)-indecomposable for some $\lambda_2 \in [\lambda_1, \mu]$.

(5) We say that $\mathbf{c} : [\lambda]^2 \to S$ is θ-indecomposable when: if $\langle u_i : i < \theta \rangle$ is a $\subseteq$-increasing sequence of sets with union λ, then $S = \{\mathbf{c}\{\alpha, \beta\} : \alpha \neq \beta \in u_i\}$ for some $i < \theta$.

(6) We may replace above the cardinal θ by a set or class Θ of regular cardinals (as done in Definition 2.1 (2)).

A group G may be considered indecomposable as a group or as a semi-group; our default choice is semi-group; but note that for locally finite groups the two interpretations are equivalent. The following was proved in [18].

THEOREM 2.2. *The following holds.*

(1) *If* $\lambda \geq \aleph_1$ *and we let* $\Theta_\lambda = \{\mathrm{cf}(\lambda)\}$ *except that* $\Theta_\lambda = \{\mathrm{cf}(\lambda), \partial\} = \{\lambda, \partial\}$ *when* $(\mathrm{c})_{\lambda,\partial}$ *below holds, then clauses* (a) *and* (b) *hold:*

  (a) *Some* $\mathbf{c} : [\lambda]^2 \to \lambda$ *is* θ*-indecomposable for every* $\theta = \mathrm{cf}(\theta) \notin \Theta_\lambda$.

  (b) *For every* $G_1 \in \mathbf{K}^{\mathrm{lf}}_{\leq\lambda}$ *there is an extension* $G_2 \in \mathbf{K}^{\mathrm{lf}}_\lambda$ *which is* $\Theta^{\mathrm{orth}}_\lambda$*-indecomposable.*

  $(\mathrm{c})_{\lambda,\partial}$ *For some* μ, $\lambda = \mu^+$, $\mu > \partial = \mathrm{cf}(\mu)$ *and* $\mu = \sup\{\theta < \mu : \theta$ *is a regular Jonsson cardinal*$\}$.

(2) *If* $\mu \geq \lambda \geq \theta = \mathrm{cf}(\theta)$ *and* $\theta \notin \Theta_\lambda$, $\lambda \geq \aleph_1$, *then* $\mathbf{K}_{\mathrm{lf}}$ *is* (μ, λ, θ)*-indecomposable.*

(3) *If* $\mu \geq \lambda$ *and* $(H_1, G_1) \in \mathbf{K}_{\leq\mu,\leq\lambda}$, *then we can find a pair* $(H_2, G_2) \in K_{\mu,\lambda}$ *such that*

  (a) G_2 *is* $\Theta^{\mathrm{orth}}_\lambda$*-indecomposable;*

  (b) *if* $\mu > \lambda$, *then the pair* (H_2, G_1) *is* θ*-indecomposable for every regular* θ*;*

  (c) H_2 *is* $\Theta^{\mathrm{orth}}_\mu$*-indecomposable.*

For the convenience of the reader we give some details of the proof.

PROOF. (1) By [18, Thm. 3.5].

(2) The proof will serve also for part (3). Let $(N, M) \in \mathbf{K}_{\mu,\lambda}$ be given. We choose a pair (χ, ∂) of cardinals and $\mathbf{c}$ such that $\lambda \leq \chi \leq \mu$, $\partial = \mathrm{cf}(\partial) \leq \lambda$, $\partial \neq \theta$ and $\mathbf{c} : [\chi]^2 \to \chi$ is θ-indecomposable (possible here as $\theta \notin \Theta_\lambda$, $\lambda \geq \aleph_1$ even for $\chi = \lambda$).

By induction on $\alpha \leq \partial$, we choose H_α, L_α, but L_α is chosen together with $H_{\alpha+1}$ when α is a successor ordinal, such that

(a) $(H_\alpha, L_\alpha) \in \mathbf{K}_{\mu,\lambda}$ is increasing continuous with α;

(b) $(H_0, L_0) = (N, M)$;

(c) if $\alpha = \beta + 1 < \theta$, then L_α is θ-indecomposable.

Why can we carry out the induction? For $\alpha = 0$ this is trivial; similarly for α a limit ordinal. Lastly, by clause (b) of part (1), for $\alpha = \beta + 1 \leq \alpha_*$, recall the proof of [18, Prop. 3.4], pedantically as without loss of generality, H_β, L_β are existentially closed, hence generated by the elements of order 2. Let $\langle a_\alpha : \alpha < \mu \rangle$ list $\{a \in L_\beta : a \text{ of order } 2\}$. By [18, Prop. 3.4 (2)], with $u_\alpha = \{\alpha\}$, we can find $H_{\alpha,1} \in \mathbf{K}^{\rm lf}_\mu$ extending H_β and pairwise commuting $b_\alpha \in H_{\alpha,1}$ each of order 2, for $\alpha < \mu$ (the order 2 was not mentioned but proved) and pairwise commuting $d_\alpha \in H_{\alpha,1}$, each of order 2, for $\alpha < \mu$ such that L_β is included in the subgroup $L_{\alpha,1}$ of $H_{\alpha,1}$ generated by $\{b_\alpha, d_\alpha : \alpha < \lambda\}$.

Now apply [18, Prop. 3.4 (1)] for a θ-indecomposable $\mathbf{c} : [\lambda]^2 \to \lambda$.

(3) We deal with every regular $\theta \leq \mu$ successively. Fixing θ, we can use the proof of part (2). ■

Now comes the central definition. What is its role? We like to sort out when there is a universal member of $\mathbf{K}_\mu$ and when there is a canonical universal member. For reasons explained above we concentrate on the case μ being a strong limit of cofinality $\aleph_0$, for example $\beth_\omega$. To find out the answer to those two questions for every universal class $\mathbf{K}$ seems like too much to hope for. Definition 2.3 accomplishes a more modest task: it gives a large frame satisfied by a large family of pairs $(\mathbf{K}, \mu)$ for which we shall prove an equivalence. In particular, our class $\mathbf{K}_{\rm lf}$ belongs to this family.

Definition 2.3. We say that $\mathbf{K}$ is μ-nice when

(a) $\tau_{\mathbf{K}}$ has cardinality $< \mu$;

(b) for every $M \in \mathbf{K}_{<\mu}$ there is $N \in \mathbf{K}_\mu$ extending M;

(c) $\mathbf{K}$ has the JEP (joint embedding property);

(d) $\mathbf{K}$ is $(\mu, <\mu, \mathrm{cf}(\mu))$-indecomposable;

or just

(d)′ for arbitrarily large $\lambda_2 < \mu$ letting $\theta = \mathrm{cf}(\mu) \leq \lambda_2$ we have that $\mathbf{K}$ is (μ, λ_2, θ)-indecomposable.

Naturally we like to prove that the pair $(\mathbf{K}_{\rm lf}, \beth_\omega)$ falls under the frame of Definition 2.3. This is the role of Claims 2.4 and 2.5. In Section 3 we point out an additional family. For the main case, μ is a strong limit of cofinality $\aleph_0$.

Claim 2.4. *$\mathbf{K}_{\rm lf}$ is μ-nice when $\mu > \aleph_1$.*

Proof. In Definition 2.3 clause (a) is trivial. As $\mathbf{K}_{\rm lf}$ is closed under products, clearly clauses (b) and (c) are clear. For μ regular, clause (d) is trivial (and is not used), and for μ singular, it holds by Theorem 2.2 (3), see also Claim 2.5 (2) below. ■

We give below more than what is strictly needed.

Claim 2.5. *Assume* $\mathbf{K} = \mathbf{K}_{\mathrm{lf}}$.

(1) *We have* (A) $\Rightarrow$ (B) *where:*

(A) (i) $\mu \ge \aleph_1$;

(ii) $\delta_* \le \mu$ *and* $\lambda_\alpha < \mu$ *for* $\alpha < \delta_*$;

(iii) $\lambda_\alpha \ge |\alpha|$ *is non-decreasing;*

(iv) $G_1 \in \mathbf{K}_{\le \mu}$;

(v) $G_{1,\alpha} \in \mathbf{K}_{\le \lambda_\alpha}$ *and* $G_{1,\alpha} \subseteq G_1$ *for* $\alpha < \delta_*$.

(B) *There are* $G_2, \bar{G}_2$ *such that*

(i) $G_2 \in \mathbf{K}_\mu$ *extends* G_1;

(ii) $\bar{G}_2 = \langle G_{2,\alpha} : \alpha < \delta_* \rangle$ *is increasing;*

(iii) $G_{2,\alpha} \in \mathbf{K}_{\lambda_\alpha}$ *extends* $G_{1,\alpha}$;

(iv) G_2 *is* Θ*-indecomposable where* $\Theta = (\Theta_\mu \cup \{\mathrm{cf}(\delta_*)\})^{\mathrm{orth}}$;

(v) $G_{2,\alpha}$ *is* $\Theta^{\mathrm{orth}}_{\lambda_\alpha}$*-indecomposable (not just inside* H_2*) for every* $\alpha < \delta_*$;

(vi) *if* $\mu = \sum\{\lambda_\alpha : \alpha < \delta_*\}$, *then* $G_2 = \bigcup\{G_{2,\alpha} : \alpha < \delta_*\}$.

(2) *If* $\mu > \lambda \ge \aleph_1$, *then* $\aleph_0 \in \Theta^{\mathrm{orth}}_{\mathrm{cf}(\mu)} \cup \Theta^{\mathrm{orth}}_\lambda$ *except possibly when* $\mu = \lambda^+$ *and* $\mathrm{cf}(\lambda) = \aleph_0$.

Proof. (1) We prove the claim step by step. By induction on $\alpha \le \delta_*$ we choose $H_\alpha, \bar{H}_\alpha, L_\alpha$, but L_α is chosen together with $H_{\alpha+1}$ and not chosen for $\alpha = \alpha_*$, such that

(a) H_α is increasing continuous with α;

(b) $H_0 = G_1$ and $\alpha > 0 \Rightarrow H_\alpha \in \mathbf{K}_\mu$;

(c) $(H_\alpha, L_\beta) \in \mathbf{K}_{\lambda,\lambda_\beta}$ when $\alpha = \beta + 1 \le \alpha_*$;

(d) $\bar{H}_\alpha = \langle H_{\alpha,\varepsilon} : \varepsilon < \delta_* \rangle$ such that if $\mu = \sum\{\lambda_\varepsilon : \varepsilon < \delta_*\}$, then this sequence is increasing with union H_α, and $H_{\alpha,\varepsilon}$ has cardinality λ_ε when $\alpha > 0$ and $\le \lambda_\varepsilon$ when $\alpha = 0$;

(e) $G_{1,\beta}, H_{\beta,\varepsilon}, L_\gamma$ are subgroups of L_α when $\beta \le \alpha$, $\varepsilon \le \alpha$, $\gamma < \alpha$;

(f) L_β is $\Theta^{\mathrm{orth}}_{\lambda_\beta}$-indecomposable;

(g) G_2 is Θ-indecomposable where $\Theta = (\Theta_\mu \cup \{\mathrm{cf}(\delta_*)\})^{\mathrm{orth}}$.

Why can we carry out the induction? We choose $\bar{H}_\alpha$ just after H_α was chosen. For $\alpha = 0$ this is trivial (note that L_α is not chosen), similarly for α a limit ordinal. Lastly, for $\alpha = \beta + 1 \le \alpha_*$, Definition 2.1 (4) and Theorem 2.2 (3) give the desired conclusion. In details, first choose $L^+_\beta \subseteq H_\beta$ of cardinality at most λ_α satisfying the desired sets (listed in clause (e)). Then apply Theorem 2.2 (3) to the pair (H_β, L^+_β) to get (H_α, L_α). Lastly, let $G_2 \in \mathbf{K}_\mu$ extend H_{δ_*} and let it satisfy the indecomposability demand. Letting $G_{2,\alpha} = L_\alpha$, we are done.

(2) Easy. ■

The final claim of this section is immediate and we omit its proof.

Claim 2.6. *If μ is strong limit singular and $N \in \mathbf{K}_\mu$, then the set*

$$\mathrm{IDC}_{<\mu}(N) = \{M : M \subseteq N \text{ has cardinality} < \mu \text{ and is } \mathrm{cf}(\mu)\text{-indecomposable}\}$$

has cardinality $\leq \mu$.

3. Universality

For quite many classes, there are universal members in any (large enough) μ which is a strong limit of cofinality $\aleph_0$, see [16] which includes history. Below we investigate "is there a universal member of $\mathbf{K}^{\mathrm{lf}}_\mu$ for such μ". We prove that if there is a universal member, e.g., in $\mathbf{K}^{\mathrm{lf}}_\mu$, then there is a canonical one.

What do we mean by "canonical"? This is not a precise definition, but we mean it is unique up to isomorphism, by a natural definition. Examples we have in mind are the algebraic closure of a field, the saturated model of a complete first-order theory T in cardinality $\mu^+ = 2^\mu > |T|$, and the special model of a complete first-order theory T in a singular strong limit cardinal $\mu > |T|$, see [2]. The last one means:

(∗) For such T, μ we say that M is a special model of T of cardinality μ when some $\bar{M}$ witnesses M, which means
- (a) $\bar{M} = \langle M_i : i < \mathrm{cf}(\mu)\rangle$;
- (b) M_i is $\prec$-increasing with i;
- (c) each M_i has cardinality $< \mu$;
- (d) $M = \bigcup\{M_i : i < \mathrm{cf}(\mu)\}$;
- (e) for every $\lambda < \mu$ and for every large enough $i < \mathrm{cf}(\mu)$ the model M_i is λ^+-saturated.

Considering our main case, $\mathbf{K}_{\mathrm{lf}}$, a major difference between what we prove here (e.g., for $\mathbf{K}_{\mathrm{lf}}$) and (∗) is that here amalgamation fails, so clause (B) of Theorem 3.1 is a poor man's replacement.

THEOREM 3.1. *Assume μ is a strong limit of cofinality $\aleph_0$ and $\mathbf{K}$ is μ-nice.*

(1) *The following conditions are equivalent:*

(A) *There is a universal $G \in \mathbf{K}_\mu$.*

(B) *If $H \in \mathbf{K}_\lambda$ is $\aleph_0$-indecomposable for some $\lambda < \mu$, then there is a sequence $\bar{G} = \langle G_\alpha : \alpha < \alpha_* \le \mu \rangle$ such that*

(a) *$H \subseteq G_\alpha \in \mathbf{K}_\mu$;*

(b) *if $G \in \mathbf{K}_\mu$ extends H, then for some α, G is embeddable into G_α over H.*

(B)$^+$ *We can add in* (B)*:*

(c) *If $\alpha_1 < \alpha_2 < \alpha_*$, then $G_{\alpha_1}, G_{\alpha_2}$ cannot be amalgamated over H, that is, there are no G, f_1, f_2 such that $H \subseteq G \in \mathbf{K}$ and f_ℓ embeds G_{α_ℓ} into G over H for $\ell = 1, 2$.*

(d) *(H, G_α) is an amalgamation pair (see Definition 1.6* (1)*), moreover a universal amalgamation base (see Definition 1.6* (2)*).*

(2) *We can add in part* (1)*:*

(C) *There is G_* such that*

(a) *$G_* \in \mathbf{K}_\mu$ is universal for $\mathbf{K}_{<\mu}$;*

(b) *$\mathcal{E}^{\aleph_0}_{G_*,<\mu}$ (see Definition 3.2 below) is an equivalence relation with $\le \mu$ equivalence classes;*

(c) *G_* is μ-special (see Definition 3.2* (5) *below).*

(C)$^+$ *Like clause* (C) *but we add:*

(d) *If $G, G_* \in \mathbf{K}_\mu$ are μ-special, then G, G_* are isomorphic (that is, uniqueness).*

Before we prove Theorem 3.1, we state the following definition, which is not just used in the proof but also in phrasing Theorem 3.1 (2).

DEFINITION 3.2. For $\theta = \mathrm{cf}(\mu) < \mu$ and $M_* \in \mathbf{K}_\mu$ we define:

(1) $\mathrm{IND}^\theta_{M_*,<\mu} = \{N : N \le_{\mathfrak{k}} M_*$ has cardinality $< \mu$ and is θ-indecomposable$\}$.

(2) $\mathcal{F}^\theta_{M_*,<\mu} = \{f :$ for some θ-indecomposable $N = N_f \in \mathbf{K}_{<\mu}$ with universe an ordinal, f is an embedding of N into $M_*\}$.

(3) $\mathcal{E}^\theta_{M_*,<\mu} = \{(f_1, f_2) : f_1, f_2 \in \mathcal{F}^\theta_{M_*,<\mu}, N_{f_1} = N_{f_2}$ and there are embeddings g_1, g_2 of M_* into some extension $M \in \mathbf{K}_\mu$ of M_* such that $g_1 \circ f_1 = g_2 \circ f_2\}$.

(4) We say that M_* is θ-$\mathcal{E}^\theta_{M_*,<\mu}$-indecomposably homogeneous (or just M_* is θ-indecomposably homogeneous) when some $\bar{M}$ witnesses it, which means:

(a) $\bar{M} = \langle M_i : i < \mathrm{cf}(M) \rangle$ is increasing continuous with limit μ;

(b) if $f_1, f_2 \in \mathcal{F}^{\theta}_{M_*,<\mu}$ and $(f_1, f_2) \in \mathcal{E}^{\theta}_{M_*,<\mu}$ and there exists $i < \theta$ such that $A \subseteq M_i$ has cardinality $< \mu$, then there is $(g_1, g_2) \in \mathcal{E}^{\theta}_{M_*,<\mu}$ such that $f_1 \subseteq g_1 \wedge f_2 \subseteq g_2$ and $A \subseteq \mathrm{Rang}(g_1) \cap \mathrm{Rang}(g_2)$.

It follows that if $\mathrm{cf}(\mu) = \aleph_0$, then for some $g \in \mathrm{aut}(M_*)$ we have $f_2 = g \circ f_1$.

(5) We say that $M_* \in \mathbf{K}_\mu$ is μ-special when it is θ-indecomposably homogeneous and is universal for $\mathbf{K}_{<\mu}$, that is, every $M \in \mathbf{K}_{<\mu}$ is embeddable into it.

It is worth making the following remark.

Remark 3.3. We may consider in Theorem 3.1 also $(\mathrm{A})_0 \Rightarrow (\mathrm{A})$ where $(\mathrm{A})_0$ is as follows:

$(\mathrm{A})_0$ If $\lambda < \mu$, $H \subseteq G_1 \in \mathbf{K}_{<\mu}$ and $|H| \le \lambda$, then for some G_2 we have $G_1 \subseteq G_2 \in \mathbf{K}_{<\mu}$ and (H, G_2) is a (μ, μ, λ)-amalgamation base.

Proof of Theorem 3.1. It suffices to prove the following implications:

(A) $\Rightarrow$ (B). Let $G_* \in \mathbf{K}_\mu$ be universal and choose a sequence $\langle G^*_n : n < \omega \rangle$ such that $G_* = \bigcup_n G^*_n$, $G^*_n \subseteq G^*_{n+1}$, $|G^*_n| < \mu$.

Let H be as in Theorem 3.1 (B) and let $\mathcal{G} = \{g : g \text{ embeds } H \text{ into } G^*_n \text{ for some } n\}$. So clearly $|\mathcal{G}| \le \sum_n |G^*_n|^{|H|} \le \sum_{\lambda<\mu} 2^\lambda = \mu$ (an over-kill).

Let $\langle g^*_\alpha : \alpha < \alpha_* \le \mu \rangle$ list $\mathcal{G}$ and let (G_α, g_α) be such that

$(*)_1$ (a) $H \subseteq G_\alpha \in \mathbf{K}_\mu$;

(b) g_α is an isomorphism from G_α onto G_* extending g^*_α.

Why? Let $\mathcal{U}$ be a set of cardinality μ extending H. As $|\mathcal{U}| = |G_*| = \mu > |H|$, there is a one-to-one function g_α from $\mathcal{U}$ onto G_* extending g^*_α. Let $G_\alpha \in \mathbf{K}$ have universe $\mathcal{U}$ such that g_α is an isomorphism from G_α onto G_*.

It suffices to prove that $\bar{G} = \langle G_\alpha : \alpha < \alpha_* \rangle$ is as required in clause (B). Now clause (B)(a) holds by $(*)_1$ (a) above. As for clause (B)(b), let G satisfy $H \subseteq G \in \mathbf{K}_{\le\mu}$, hence there is an embedding g of G into G_*. We know that $g(H) \subseteq G = \bigcup_n G_n$ hence $\langle g(H) \cap G_n : n < \omega \rangle$ is $\subseteq$-increasing with union $g(H)$; but $g(H)$ by the assumption on H is $\aleph_0$-indecomposable, hence $g(H) = g(H) \cap G^*_n \subseteq G^*_n$ for some n. This implies $g \restriction H \in \mathcal{G}$ and so for some $\alpha < \alpha_*$ we have $g \restriction H = g^*_\alpha$. Hence $g_\alpha^{-1} g$ is an embedding of G into G_* extending $(g_\alpha \restriction H)^{-1}(g \restriction H) = (g^*_\alpha)^{-1}(g^*_\alpha) = \mathrm{id}_H$ as promised.

(B) $\Rightarrow$ (B)$^+$. What about (B)$^+$ (c)? While $\bar{G}$ does not necessarily satisfy it, we can "correct it", e.g., we choose u_α, v_α, and if $\alpha \notin \bigcup\{v_\beta : \beta < \alpha\}$, we also choose G'_α by induction on $\alpha < \alpha_*$ such that[1]

$(*)^2_\alpha$ (a) $G_\alpha \subseteq G'_\alpha \in \mathbf{K}_\mu$ if $\alpha \notin \bigcup\{v_\beta : \beta < \alpha\}$;

(b) $u_\alpha \subseteq \alpha$ and $v_\alpha \subseteq \alpha_* \setminus (\alpha + 1)$;

(c) if $\beta < \alpha$, then $u_\beta = u_\alpha \cap \beta$ and $u_\alpha \cap v_\beta = \emptyset$;

(d) if $\alpha = \beta + 1$, then $\beta \in u_\alpha$ iff $\beta \notin \bigcup\{v_\gamma : \gamma < \beta\}$;

(e) if $\alpha \notin \bigcup\{v_\gamma : \gamma < \alpha\}$, then:

$\bullet_1$ $\gamma \in v_\alpha$ iff ($\gamma > \alpha$ and) G_γ is embeddable into G'_α over H;

$\bullet_2$ if $\gamma \in \alpha_* \setminus (\alpha + 1) \setminus (\bigcup\{v_\beta : \beta \leq \alpha\})$, then G_γ is not embeddable over H into any G' satisfying $G'_\alpha \subseteq G' \in \mathbf{K}$;

(f) if $\alpha = \beta + 1$ and $\beta \notin u_\alpha$, then $v_\beta = \emptyset$.

Why is this sufficient? Because if we let $u_{\alpha_*} = \alpha_* \setminus (\bigcup\{v_\gamma : \gamma < \alpha_*\})$, then $\langle G'_\alpha : \alpha \in u_{\alpha_*}\rangle$ is as required; but we elaborate.

First, for clause $(B)^+$ (c) assume that $\alpha < \beta$ are from u_{α_*}. As $\beta \notin v_\alpha$, by $(*)^2_\alpha$ (e) $\bullet_2$ we know that G_β is not embeddable into any extension of G'_α over H; but as $G_\beta \subseteq G'_\beta$ clearly also G'_β is not embeddable into any extension of G'_α over H. Renaming this means that G'_α, G'_β cannot be amalgamated over H, as promised.

Second, for clause $(B)^+$ (d), let $\alpha \in u_{\alpha_*}$. We have to prove that the pair (G'_α, H) is a universal (μ, κ)-amalgamation base where κ is the cardinality of H. So assume $G' \in \mathbf{K}_\mu$ extends G'_α; recall that we are assuming that $\langle G_\alpha : \alpha < \alpha_*\rangle$ is as in clause (B), hence there are $\beta < \alpha_*$ and an embedding f of G' into G_β over H. We shall prove that $\beta = \alpha$ hence (recalling $G_\alpha \subseteq G'_\alpha$) f embeds G' into G'_α over H, which completes the proof of (B) $\Rightarrow$ $(B)^+$.

If $\beta \in u_{\alpha_*} \setminus \{\alpha\}$, then $f \restriction G'_\alpha$ embeds G'_α into G'_β over H, a contradiction to $(B)^+$ (c) which we have already proved.

If $\beta \in \alpha_* \setminus u_{\alpha_*}$, then for some γ we have $\beta \in v_\gamma$ hence $\gamma < \beta$ and G_β is embeddable into G'_γ over H; hence G' is embeddable into G'_γ over H. As in the previous sentence necessarily $\gamma = \alpha$ and we are done.

Why can we carry out the induction? For $\alpha = 0$ and for α a limit ordinal, we have nothing to do because u_α is determined by $(*)^2_\alpha$ (b) and $(*)^2_\alpha$ (c). For $\alpha = \beta + 1$, if $\beta \in \bigcup_{\gamma<\beta} v_\gamma$, we have nothing to do, in the remaining case we choose $G'_{\beta,i} \in \mathbf{K}_\mu$ by induction on $i \in [\alpha, \alpha_*]$, increasing continuous with i. For $i = 0$ let $G'_{\beta,i} = G_\beta$ and for limit i let $G'_{\beta,i} = \bigcup\{G'_{\beta,j} : j < i\}$. Then choose $G'_{\beta,i+1}$ to make clause (e) true.

(1) The idea is that if $\beta \in v_\alpha$, then $\beta > \alpha$ and G_β is discarded being embeddable into some G'_α, and G'_α is the "corrected" member.

That is, first, if $G'_{\beta,i}$ has an extension into which G_i is embeddable over H, then there is such an extension of cardinality μ; and choose $G'_{\beta,i+1}$ as such an extension.

Second, if $G'_{\beta,i}$ has no extension into which G_i is embeddable over H, then we let $G'_{\beta,i+1} = G'_{\beta,i}$.

Lastly, let $G'_\alpha = G'_{\alpha,\alpha_*}$ and $u_\alpha = u_\beta \cup \{\alpha\}$ and $v_\alpha = \{i : i \in \alpha_*,\ i > \alpha,\ i \notin \bigcup\{v_\gamma : \gamma < \beta\}$ and G_i is embeddable into G'_β over $H\}$.

$(B)^+ \Rightarrow (A)$. We prove below more: there is something like a "special model", i.e. Theorem 3.1 (2), that is, $(B)^+ \Rightarrow (C)^+$.

$(C)^+ \Rightarrow (C) \Rightarrow (A)$. This is trivial so we are left with proving the following.

$(B)^+ \Rightarrow (C)^+$. Let $\mathbf{K}^{\mathrm{slf}}_\mu$ be the class of G such that

$(*)^3_G$ (a) $G \in \mathbf{K}_\mu$;

(b) if $H \subseteq G$, $H \in \mathbf{K}_{<\mu}$, then there are $\aleph_0$-indecomposable $H_n \subseteq G$ for $n < \omega$ with union of cardinality $< \mu$ such that $H \subseteq \bigcup\{H_n : n < \omega\}$, and there are $\aleph_0$-indecomposable $G_n \subseteq G$ for $n < \omega$ such that $G_n \in \mathbf{K}_{<\mu}$, $G_n \subseteq G_{n+1}$ and $G = \bigcup\{G_n : n < \omega\}$;

(c) if $H \subseteq G$ is $\aleph_0$-indecomposable of cardinality $< \mu$, then the pair (G, H) is a universal $(\mu, < \mu)$-amalgamation base (see Definition 1.6 (2));

(d) if $H \subseteq G$ is $\aleph_0$-indecomposable of cardinality $< \mu$, $H \subseteq H' \in \mathbf{K}_{<\mu}$, H' is $\aleph_0$-indecomposable[2], and G, H' are compatible over H (in $\mathbf{K}_{\le\mu}$), then H' is embeddable into G over H.

Now we can finish by proving $(*)_4$ and $(*)_5$ below.

$(*)_4$ If $G \in \mathbf{K}_{\le\mu}$, then some $H \in \mathbf{K}^{\mathrm{slf}}_{\bar\lambda}$ extends G;

We break the proof into four steps; $(*)_{4.3}$ gives the desired conclusion of $(*)_4$.

$(*)_{4.0}$ If $G \in \mathbf{K}_{\le\mu}$, then for some $H, \bar H$ we have

(a) $G \subseteq H \in \mathbf{K}_\mu$;

(b) $\bar H = \langle H_n : n < \omega\rangle$;

(c) $H_n \subseteq H_{n+1} \subseteq H$;

(d) $H = \bigcup\{H_n : n < \omega\}$;

(e) each H_n is $\aleph_0$-indecomposable of cardinality $< \mu$;

(2) The $\aleph_0$-indecomposability is not always necessary, but we need it sometimes.

(f) (not really needed) when $\mathbf{K} = \mathbf{K}_{\rm lf}$, if $K \subseteq H_n$, $|K| \le \partial$ and $2^\partial \le |H_n|$, then there is a subgroup L of H_n extending K which is $\Theta^{\rm orth}_\partial$-indecomposable.

Why? For clauses (a)–(e) by the definition of $\mathbf{K}$ being nice. For clause (f) by Claim 2.5 (1), (2).

$(*)_{4.1}$ If $N_1 \in \mathbf{K}_{\le\mu}$, then there is N_2 such that

(a) $N_2 \in \mathbf{K}_\mu$;

(b) $N_1 \subseteq N_2$;

(c) if $H \in \mathrm{IDC}_{<\mu}(N_1)$, then (N_2, H) is a universal $(\mu, <\mu)$-amalgamation base.

Why? By Claim 2.6 it is enough to deal with one such H, which is okay by clause (d) of Definition 2.3, recalling "universal $(\mu, <\mu)$-amalgamation base" by $(\mathrm{B})^+$ which we are assuming.

$(*)_{4.2}$ Like $(*)_{4.1}$ but clause (c) is replaced by

(c)$'$ if $H_1 \in \mathrm{IDC}_{<\mu}(N_1)$ and $H_1 \subseteq H_2 \in \mathbf{K}_{<\mu}$ (and we may add: H_2 is $\aleph_0$-indecomposable), then either N_2, H_1 are incompatible over H_1 in $\mathbf{K}_{\le\mu}$ or H_2 is embeddable into N_2 over H_1.

Why? Again it is enough to deal with one pair (H_1, H_2), which is done by hand.

$(*)_{4.3}$ If $N_1 \in \mathbf{K}_{\le\mu}$, then there is N_2 such that

(a) $N_2 \in \mathbf{K}_\mu$;

(b) $N_1 \subseteq N_2$;

(c) if $H \in \mathrm{IDC}_{<\mu}(N_2)$, then (N_2, H) is a universal $(\mu, <\mu)$-amalgamation base;

(d) if $H_1 \in \mathrm{IDC}_{<\mu}(N_2)$ and $H_1 \subseteq H_2 \in \mathbf{K}_{<\mu}$ (and we may add: H_2 is $\aleph_0$-indecomposable), then either N_2, H_1 are incompatible over H_1 in $\mathbf{K}_{\le\mu}$ or H_2 is embeddable into N_2 over H_1.

Why? We choose $L_\varepsilon \in \mathbf{K}_\mu$ by induction on $\varepsilon \le \mathrm{cf}(\mu)$, such that

(a) $L_\alpha \in \mathbf{K}_\mu$;

(b) $\langle L_\beta : \beta \le \alpha\rangle$ is increasing continuous;

(c) $G_1 \subseteq L_0$;

(d) if $\alpha = 3\beta + 1$, then L_α relates to $L_{3\beta}$ as N_2 relates to N_1 in $(*)_{4.0}$;

(e) if $\alpha = 3\beta + 2$, then L_α relates to $L_{3\beta+1}$ as N_2 relates to N_1 in $(*)_{4.1}$;

(f) if $\alpha = 3\beta + 3$, then L_α relates to $L_{3\beta+2}$ as N_2 relates to N_1 in $(*)_{4.2}$.

There is no problem to carry out the induction. Note that if $N \subseteq L_{\mathrm{cf}(\mu)}$ is $\mathrm{cf}(\mu)$-indecomposable, then for some $\varepsilon < \mathrm{cf}(\mu)$ we have $N \subseteq L_\varepsilon$. Then $N_2 = L_{\mathrm{cf}(\mu)}$ is as required in $(*)_{4.3}$ hence in $(*)_4$.

$(*)_5$ (a) If $G_1, G_2 \in \mathbf{K}^{\mathrm{slf}}_\mu$, then G_1, G_2 are isomorphic.

(b) If $G_1, G_2 \in \mathbf{K}^{\mathrm{slf}}_\mu$, $H \in \mathbf{K}_{<\mu}$ is $\aleph_0$-indecomposable and f_ℓ embeds H into G_ℓ for $\ell = 1, 2$, and this diagram can be completed (i.e., there are $G \in \mathbf{K}_\mu$ and an embedding $g_\ell : G_\ell \to G_*$ such that $g_1 \circ f_1 = g_2 \circ f_2$), then there is h such that

(α) h is an isomorphism from G_1 onto G_2;

(β) $h \circ f_1 = f_2$.

Why? Clause (a) follows from clause (b) using as H the trivial group. For clause (b), let $\mathcal{F} = \mathcal{F}[G_1, G_2]$ be the set of f such that

(a) f is an isomorphism from $G_{1,f} \in \mathrm{IDC}_{<\mu}(G_1)$ onto $G_{2,f} \in \mathrm{IDC}_{<\mu}(G_2)$;

(b) G_1, G_2 are f-compatible in $\mathbf{K}_\mu$, which means that there are $G \in \mathbf{K}_\mu$ and embeddings g_ℓ of G_ℓ into G for $\ell = 1, 2$ such that $g_2 \circ f = g_1 \restriction G_{1,f}$.

First, $\mathcal{F}$ is non-empty (the function f with domain $f_1(H)$ and range $f_2(H)$ will do). Second, use the hence and forth argument; here we use $\mathrm{cf}(\mu) = \aleph_0$. ■

We make a final remark in this section.

REMARK 3.4. (1) Can we prove for strong limit singular μ of uncountable cofinality κ a parallel result? Well, we have to consider the following game:

$(*)$ The game is defined as follows:

(a) A play last θ moves.

(b) In the ε move, first, player I chooses $M_\varepsilon \in \mathbf{K}_{<\mu}$ and then, player II chooses $N_\varepsilon \in \mathbf{K}_{<\mu}$.

(c) $M_\varepsilon \in \mathbf{K}_{<\mu}$, and if ε is non-limit, then M_ε is $\mathrm{cf}(\mu)$-indecomposable.

(d) $\langle M_\zeta : \zeta \leq \varepsilon \rangle$ is increasing continuous.

(e) $M_\varepsilon \subseteq N_\varepsilon \subseteq M_{\varepsilon+1}$.

(f) In the end of the play, player II wins iff for every limit ordinal $\varepsilon < \mathrm{cf}(\mu)$, M_ε is an amalgamation base inside $\mathbf{K}_{<\mu}$.

Now, if player II does not lose, then we can imitate the proof above; this should be clear. Does the existence of a universal member of $\mathbf{K}_\mu$ implies this? We hope to return to this elsewhere.

(2) Another remark is that the proof works for any a.e.c. $\mathfrak{k}$ with $\mathrm{LST}_{\mathfrak{k}} < \mu$. But we may wonder: can we weaken the demand on $\mathfrak{k}$? Actually we can: there is no need of smoothness; that is, if $\langle M_\alpha : \alpha \leq \delta \rangle$ is $\leq_{\mathfrak{k}}$-increasing, then $\bigcup\{M_\alpha : \alpha < \delta\} \leq_{\mathfrak{k}} M_\delta$. Moreover, while we need the existence of an upper bound for any $\leq_{\mathfrak{k}}$-increasing sequence, we also demand the union being such upper bound only for the cofinality $\mathrm{cf}(\mu)$. Finally, we may add a version fixing $\bar{\lambda}$.

4. Universal in $\beth_\omega$

In Section 2 we have characterized when there are special models in $\mathbf{K}$ of cardinality, e.g., $\beth_\omega$. We try to analyze a related combinatorial problem. Our intention is to first investigate $\mathfrak{k}_{\mathrm{fnq}}$ (the class structures consisting of a set and a directed family of equivalence relations on it, each with a finite bound on the size of equivalence classes). So $\mathfrak{k}_{\mathrm{fnq}}$ is similar to $\mathbf{K}_{\mathrm{lf}}$ but seems easier to analyze. We consider some partial orders on $\mathfrak{k} = \mathfrak{k}_{\mathrm{fnq}}$.

First, under the substructure order, $\leq_1 = \subseteq$, this class fails amalgamation. Second, we have other orders: $\leq_3$ and $\leq_2$, demanding a Tarski–Vaught condition TV (see below). However using $\leq_3$, where we have a similar demand for countably many points and finitely many equivalence relations, we have amalgamation. This is naturally connected to locally finite groups, see Definition 4.6 and Discussion 4.7.

Definition 4.1. Let $\mathbf{K} = \mathbf{K}_{\mathrm{fnq}}$ be the class of structures M such that[3]

(a) P^M, Q^M is a partition of M, P^M non-empty;

(b) $E^M \subseteq P^M \times P^M \times Q^M$ (is a three-place relation) and we write $aE^M_c b$ for $(a, b, c) \in E^M$;

(c) for $c \in Q^M$, E^M_c is an equivalence relation on P^M with $\sup\{|a/E^M_c| : a \in P^M\}$ finite (see more later);

(d) $Q^M_{n,k} \subseteq (Q^M)^n$ for $n, k \geq 1$;

(e) if $\bar{c} = \langle c_\ell : \ell < n \rangle \in {}^n(Q^M)$, we let $E^M_{\bar{c}}$ be the closure of $\bigcup_\ell E_\ell$ to an equivalence relation;

(f) ${}^n(Q^M) = \bigcup_{k \geq 1} Q^M_{n,k}$;

(g) if $\bar{c} \in Q^M_{n,k}$, then $k \geq |a/E^M_{\bar{c}}|$ for every $a \in P^M$.

We need a further definition.

[3] The vocabulary is defined implicitly and is $\tau_{\mathbf{K}}$, i.e. depends just on $\mathbf{K}$.

Definition 4.2. We define some partial orders on $\mathbf{K}$.

(1) $\leq_1 = \leq^1_{\mathbf{K}} = \leq^1_{\rm fnq}$ is a sub-model.

(2) $\leq_3 = \leq^3_{\mathbf{K}} = \leq^3_{\rm fnq}$ is the following: $M \leq_3 N$ iff

(a) $M, N \in \mathbf{K}$;

(b) $M \subseteq N$;

(c) if $A \subseteq N$ is countable and $A \cap Q^N$ is finite, then there is an embedding of $N \restriction A$ into M over $A \cap M$ or just a one-to-one homomorphism.

(3) $\leq_2 = \leq^2_{\mathbf{K}} = \leq^2_{\rm fnq}$ is defined like $\leq_3$ but in clause (c), A is finite.

We first state an easy claim.

Claim 4.3. (1) *$\mathbf{K}$ is a universal class, so $(\mathbf{K}, \subseteq)$ is an a.e.c.*

(2) *$\leq^3_{\mathbf{K}}, \leq^2_{\mathbf{K}}, \leq^1_{\mathbf{K}}$ are partial orders on $\mathbf{K}$.*

(3) *$(\mathbf{K}, \leq^2_{\mathbf{K}})$ is an a.e.c.*

(4) *$(\mathbf{K}, \leq^3_{\mathbf{K}})$ has disjoint amalgamation.*

Proof. (1)–(3) Easy. (4) By Claim 4.4 below. ■

Claim 4.4. *If $M_0 \leq^1_{\mathbf{K}} M_1$, $M_0 \leq^3_{\mathbf{K}} M_2$ and $M_1 \cap M_2 = M_0$, then $M = M_1 + M_2$, the disjoint sum of M_1, M_2 belongs to $\mathbf{K}$ and extends M_ℓ for $\ell = 0, 1, 2$ and even $M_1 \leq^3_{\rm fnq} M$ and $[M_0 \leq^2_{\mathbf{K}} M_1 \Rightarrow M_2 \leq^2_{\mathbf{K}} M]$ when:*

$(*)$ *$M = M_1 +_{M_0} M_2$ means that M is defined as follows:*

(a) $|M| = |M_1| \cup |M_2|$.

(b) $P^M = P^{M_1} \cup P^{M_2}$.

(c) $Q = Q^{M_1} \cup Q^{M_2}$.

(d) *We define E^M by defining E^M_c for $c \in Q^M$ by cases:*

(α) *if $c \in Q^{M_0}$, then E^M_c is the closure of $E^{M_1}_\ell \cup E^{M_2}_\ell$ to an equivalence relation;*

(β) *if $c \in Q^{M_\ell} \setminus Q^{M_0}$ and $\ell \in \{1, 2\}$, then E^M_c is defined by*

- *$a E^M_c b$ iff $a = b \in P^{M_{3-\ell}} \setminus M_0$ or $a E^{M_\ell}_c b$ so $a, b \in P^{M_\ell}$.*

(e) *$Q^M_{n,k}$ is the union of $Q^{M_1}_{n,k}, Q^{M_2}_{n,k}$ and the set of $\bar{c}$ satisfying*

(α) *$\bar{c} \in {}^n(Q^M)$;*

(β) *$\bar{c} \notin {}^n(Q^{M_1}) \cup {}^n(Q^{M_2})$;*

(γ) *$E^M_{\bar{c}}$, which is now well defined, has no equivalence class with more than k members, that is, for some finite A and pairwise distinct $a_0, \ldots, a_k \in A$,* *which are members of $a/E^{\mu}_{\bar{c}}$, the closure of $\bigcup\{E^M_{c_i} \restriction A : i < \lg(\bar{c})\}$ to an equivalence relation satisfies $a_i E' a$ for $i \le k$.*

Proof. Clearly M is a well-defined structure, extends M_0, M_1, M_2 and satisfies clauses (a), (b), (c) of Definition 4.1. There are two points to be checked:

$(*)_1$ If $a \in P^M$ and $\bar{c} \in Q^M_{n,k}$, then $|a/E^M_{\bar{c}}| \le k$.

$(*)_2$ If $\bar{c} \in {}^n(Q^M)$, then $\bar{c} \in \bigcup_k Q^M_{n,k}$.

Proof of $(*)_1$. If $\bar{c} \in Q^M_{n,k} \setminus (Q^{M_1}_{n,k} \cup Q^{M_2}_{n,k})$, this holds by the definition, so assume $\bar{c} \in Q^{M_\iota}_{n,k}$, $\iota \le 2$. If this fails, then there is a finite set $A \subseteq M$ such that $\bar{c} \subseteq A$, $a \in A$ and the closure of $\bigcup\{E^M_{c_\ell} \restriction A : \ell < \lg(\bar{c})\}$ to an equivalence relation satisfies: some equivalence class has $> k$ members. Letting $N = M \restriction A$, we have $|a/E^N_{\bar{c}}| > k$. By $M_0 \le^1_{\mathbf{K}} M_1$, $M_0 \le^3_{\mathbf{K}} M_2$ (really $M_0 \le^2_{\mathbf{K}} M_2$ suffice) there is a one-to-one homomorphism f from $A \cap M_2$ into M_0 over A. Let $B' = (A \cup M_1) \cup f(A \cap M_2)$ and $N' = M \restriction B$ and let $g = f \cup \mathrm{id}_{A \cap M_1}$. So g is a homomorphism from N onto N' and $g(a)/E^{N'}_{g(\bar{c})}$ has $> k$ members, which implies that $g'(a)/E^{M_1}_{g'(\bar{c})}$ has $> k$ members. Moreover, $g(\bar{c}) \in Q^{M_1}_{n,k}$ (Why? Trivially if $\iota = 1$; if $\iota = 2$ by the choice of f), contradiction to $M_1 \in \mathbf{K}$.

Proof of $(*)_2$. If $\bar{c} \in M_1$ or $\bar{c} \subseteq M_2$, this is obvious by the definition of M, so assume that they fail. By the definition of the $Q^M_{n,k}$'s we have to prove that $\sup\{a/E^M_{\bar{c}} : a \in P^M\}$ is finite. Toward contradiction assume this fails for each $k \ge 1$, hence there is $a_k \in P^M$ such that $a_k/E^M_{\bar{c}}$ has $\ge k$ elements, hence there is a finite $A_k \subseteq M$ such that $a_k/E^{M \restriction A_k}_{\bar{c}}$ has $\ge k$ elements. Let $A = \bigcup_{k \ge 1} A_k$, so A is a countable subset of M and we continue as in the proof of $(*)_1$.

Additional points (not really used) are proved like $(*)_1$:

$(*)_3$ $M_1 \le^3_{\mathbf{K}} M$.

$(*)_4$ $M_0 \le^2_{\mathbf{K}} M_1 \Rightarrow M_2 \le^2_{\mathbf{K}} M$.

$(*)_5$ $M_1 +_{M_0} M_2$ is equal to $M_2 +_{M_0} M_1$. ■

Claim 4.5. (1) *If $\lambda = \lambda^{<\mu}$ and $M \in \mathbf{K}$ has cardinality $\le \lambda$, then there is N such that*

(a) *$N \in \mathbf{K}_\lambda$ extends M;*

(b) *if $N_0 \le^3_{\mathbf{K}} N_1$ and N_0 has cardinality $< \mu$ and f_0 embeds N_0 into N, then there is an embedding f_1 of N_1 into N extending f_0.*

(2) *For every $M \in \mathbf{K}$ we can define an equivalence relation $E = E_{\mathbf{K}}$ on the class $\{N \in \mathbf{K} : M \le_2 N\}$ with $\le 2^{\|M\|^{\aleph_0}}$-equivalence classes such that if N_1, N_2 are E-equivalence, then they can be amalgamated over M (in $(\mathbf{K}, \le_2)$).*

(3) *If μ is a strong limit, then* $(\mathbf{K}, \leq_2)$ *is μ-nice.*

What is the connection to $\mathbf{K}_{\mathrm{lf}}$? The following definition explains this (see [11]).

Definition 4.6. (1) For a group $G \in \mathbf{K}_{\mathrm{lf}}$ we define $M = \mathrm{fnq}_G \in \mathbf{K}_{\mathrm{fnq}}$ as follows:

(a) P^M is the set of elements of G.

(b) $Q^M = \{(c, 1) : c \in G\}$, a copy of G.

(c) E^M is the set of triples $(a, b, (c, 1))$ such that $a, b, c \in G$ and for some $n, m \in \mathbb{Z}$ we have $G \models c^n a c^m = b$.

(2) For $M \in \mathbf{K}$ we define $G = \mathrm{grp}_M$ as the subgroup of $\mathrm{sym}(P^M)$ consisting of the permutations π of P^M such that for some finite sequence $\bar{c}$ of elements of Q^M we have $\pi(x) E^M_{\bar{c}} x$ for every $x \in P^M$.

Discussion 4.7. The problem is that cases of amalgamation in $(\mathbf{K}, \leq_2)$ cannot be lifted to one in $\mathbf{K}_{\mathrm{lf}}$. For a related theorem on the existence of universal members in cardinals as above, see [16, Th. 1.16].

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References

[1] G. Braun – R. Göbel, Outer automorphisms of locally finite p-groups. *J. Algebra* **264** (2003), no. 1, 55–67. Zbl 1060.20031 MR 1980685

[2] C.-C. Chang – H. J. Keisler, Model theories with truth values in a uniform space. *Bull. Amer. Math. Soc.* **68** (1962), 107–109. Zbl 0104.24105 MR 136542

[3] S. M. Corson – S. Shelah, Strongly bounded groups of various cardinalities. *Proc. Amer. Math. Soc.* **148** (2020), no. 12, 5045–5057. Zbl 1497.20004 MR 4163821

[4] M. Dugas – R. Göbel, On locally finite p-groups and a problem of Philip Hall's. *J. Algebra* **159** (1993), no. 1, 115–138. Zbl 0795.20019 MR 1231206

[5] M. Džamonja, Club guessing and the universal models, *Notre Dame J. Formal Logic* **46** (2005), no. 3, 283–300. Zbl 1105.03037 MR 2160658

[6] D. Giorgetta – S. Shelah, Existentially closed structures in the power of the continuum. *Ann. Pure Appl. Logic* **26** (1984), no. 2, 123–148. Zbl 0561.03018 MR 739576

[7] R. Grossberg – S. Shelah, On universal locally finite groups. *Israel J. Math.* **44** (1983), no. 4, 289–302. Zbl 0525.20025 MR 710234

[8] K. Hickin, Complete universal locally finite groups. *Trans. Amer. Math. Soc.* **239** (1978), 213–227. Zbl 0386.20014 MR 480750

[9] B. Jónsson, Universal relational systems. *Math. Scand.* **4** (1956), 193–208. Zbl 0077.25302 MR 96608

[10] B. Jónsson, Homogeneous universal relational systems. *Math. Scand.* **8** (1960), 137–142. Zbl 0173.00505 MR 125021

[11] O. H. Kegel – B. A. F. Wehrfritz, *Locally finite groups*. North-Holland Math. Libr. 3, North-Holland Publishing, Amsterdam, 1973. Zbl 0259.20001 MR 0470081

[12] A. Macintyre – S. Shelah, Uncountable universal locally finite groups. *J. Algebra* **43** (1976), no. 1, 168–175. Zbl 0363.20032 MR 439625

[13] M. Morley – R. Vaught, Homogeneous universal models. *Math. Scand.* **11** (1962), 37–57. Zbl 0112.00603 MR 150032

[14] S. Shelah, *Classification theory for elementary abstract classes*. Stud. Log. (Lond.) 18, College Publications, London, 2009. Zbl 1225.03036 MR 2643267

[15] S. Shelah, Existentially closed locally finite groups (Sh312). In *Beyond first order model theory*, pp. 221–298, CRC Press, Boca Raton, FL, 2017. Zbl 1433.03090 MR 3729328

[16] S. Shelah, Universal structures. *Notre Dame J. Form. Log.* **58** (2017), no. 2, 159–177. Zbl 1417.03231 MR 3634974

[17] S. Shelah, LF groups, aec amalgamation, few automorphisms. 2019, arXiv:1901.09747. To appear as Chapter 6 in *Beyond first order model theory, Vol. 2*, CRC Press, Boca Raton, FL.

[18] S. Shelah, Density of indecomposable locally finite groups. *Rend. Semin. Mat. Univ. Padova* **144** (2020), 253–270. Zbl 07367420 MR 4186458

[19] S. Shelah, Divide and conquer: dividing lines and universality. *Theoria* **87** (2021), no. 2, 259–348. MR 4329456

[20] S. Shelah – S. Thomas, The cofinality spectrum of the infinite symmetric group. *J. Symbolic Logic* **62** (1997), no. 3, 902–916. Zbl 0889.03037 MR 1472129

[21] S. Shelah – M. Ziegler, Algebraically closed groups of large cardinality. *J. Symbolic Logic* **44** (1979), no. 4, 522–532. Zbl 0427.03025 MR 550381

[22] S. Thomas, Complete universal locally finite groups of large cardinality. In *Logic colloquium '84* (Manchester, 1984), pp. 277–301, Stud. Logic Found. Math. 120, North-Holland, Amsterdam, 1986. Zbl 0603.20034 MR 861429

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