STRONG PARTITION RELATIONS BELOW THE POWER SET: CONSISTENCY — WAS SIERPINSKI RIGHT? VOL. II

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Abstract. We continue here [She88] (see the introduction there) but we do not rely on it. The motivation was a conjecture of Galvin stating that $\big(2^\omega \geq \omega_2\big) + \big(\omega_2 \to [\omega_1]^n_{h(n)}\big)$ is consistent for a suitable $h : \omega \to \omega$. In section 5 we disprove this and give similar negative results. In section 3 we prove the consistency of the conjecture replacing ω_2 by 2^ω, which is quite large, starting with an Erdős cardinal. In section 1 we present iteration lemmas which are neededwhen we replace ω by a larger λ, and in section 4 we generalize a theorem of Halpern and Lauchli replacing ω by a larger λ.

This is a slightly corrected version of an old work.

§ 0. Preliminaries

Let $<^*_\chi$ be a well ordering of $\mathcal{H}(\chi)$, where

$$\mathcal{H}(\chi) = \{x : \text{the transitive closure of } x \text{ has cardinality} < \chi\}$$

agreeing with the usual well-ordering of the ordinals. $\mathbb{P}$ (and $\mathbb{Q}$, $\mathbb{R}$) will denote forcing notions; i.e. partial orders (really, quasiorders) with a minimal element $\varnothing = \varnothing_{\mathbb{P}}$.

A forcing notion $\mathbb{P}$ is λ-closed if every increasing sequence of members of $\mathbb{P}$ of length less than λ has an upper bound.

If $\mathbb{P} \in \mathcal{H}(\chi)$, then for a sequence $\bar{p} = \langle p_i : i < \gamma\rangle$ of members of $\mathbb{P}$, let

$$\underset{\sim}{\alpha} = \underset{\sim}{\alpha}_{\bar{p}} = \sup\big\{\underset{\sim}{j} : \{\beta_j : j < \underset{\sim}{j}\} \text{ has an upper bound in } \mathbb{P}\big\}$$

and define $\&\bar{p}$, the *canonical upper bound* of $\bar{p}$, as follows:

(a) It is the least upper bound of $\{p_i : i < \underset{\sim}{\alpha}\}$ in $\mathbb{P}$, if there exists such an element.

(b) If upper bounds of $\bar{p}$ exist but are not unique, we choose the $<^*_\chi$-first upper bound.

(c) p_0, if (a) and (b) fail and $\gamma > 0$.

(d) $\varnothing_{\mathbb{P}}$, if $\gamma = 0$.

Let p_0 & p_1 be the canonical upper bound of $\langle p_\ell : \ell < 2\rangle$.

Notation 0.1. 1) Take $[a]^\kappa := \{b \subseteq a : |b| = \kappa\}$ and $[a]^{<\kappa} := \bigcup\limits_{\theta<\kappa} [a]^\theta$.

2) For sets of ordinals A and B, define $H^{\mathrm{OP}}_{A,B}$ as the maximal order preserving bijection between initial segments of A and B: i.e. it is the function with domain $\{\alpha \in A : \mathrm{otp}(\alpha \cap A) < \mathrm{otp}(B)\}$ such that $H^{\mathrm{OP}}_{A,B}(\alpha) = \beta$ <u>iff</u> $\alpha \in A$, $\beta \in B$, and $\mathrm{otp}(\alpha \cap A) = \mathrm{otp}(\beta \cap B)$.

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Definition 0.2. $\lambda \to^+ (\alpha)^{<\aleph_0}_\mu$ holds if whenever F is a function from $[\lambda]^{<\aleph_0}$ to μ and $C \subseteq \lambda$ is a club, then there is $A \subseteq C$ of order type α such that for any $w_1, w_2 \in [A]^{<\aleph_0}$, $|w_1| = |w_2| \Rightarrow F(w_1) = F(w_2)$.

Definition 0.3. $\lambda \to [\alpha]^n_{\kappa,\theta}$ if for every function F from $[\lambda]^n$ to κ there is $A \subseteq \lambda$ of order type α such that $\{F(w) : w \in [A]^n\}$ has cardinality$\leq \theta$.

Definition 0.4. A forcing notion $\mathbb{P}$ satisfies the Knaster condition (or 'has property K') if for any $\{p_i : i < \omega_1\} \subset \mathbb{P}$ there is an uncountable $A \subset \omega_1$ such that the conditions p_i and p_j are compatible whenever $i, j \in A$.

§ 1. Introduction

Concerning 1.1–1.3, see Shelah [She78] and Shelah and Stanley [SS82], [SS86].

Definition 1.1. A forcing notion $\mathbb{Q}$ satisfies $*^\varepsilon_\mu$, where ε is a limit ordinal $< \mu$, if Player **I** has a winning strategy in the following game:

Playing: the play finishes after ε moves. In the α^{th} move:

Player **I** – If $\alpha \neq 0$ he chooses $\langle q^\alpha_\zeta : \zeta < \mu^+ \rangle$ such that $q^\alpha_\zeta \in \mathbb{Q}$ and

$$(\forall \beta < \alpha)(\forall \zeta < \mu^+)[p^\beta_\zeta \leq q^\alpha_\zeta]$$

and he chooses a regressive function $f_\alpha : \mu^+ \to \mu^+$ (i.e. $f_\alpha(i) < 1 + i$). If $\alpha = 0$ let $q^\alpha_\zeta = \varnothing_{\mathbb{Q}}$ and $f_\alpha = \varnothing$.

Player **II** – He chooses $\langle p^\alpha_\zeta : \zeta < \mu^+ \rangle$ such that $q^\alpha_\zeta \leq p^\alpha_\zeta \in \mathbb{Q}$.

The outcome: Player **I** wins provided whenever $\mu < \zeta < \xi < \mu^+$, $\mathrm{cf}(\zeta) = \mathrm{cf}(\xi) = \mu$, and $\bigwedge\limits_{\beta<\varepsilon} f_\beta(\zeta) = f_\beta(\xi)$, the set $\{p^\alpha_\zeta : \alpha < \varepsilon\} \cup \{p^\alpha_\xi : \alpha < \varepsilon\}$ has an upper bound in $\mathbb{Q}$.

Definition 1.2. We call $\langle \mathbb{P}_i, \mathbb{Q}_j : i \leq i(*),\ j < i(*) \rangle$ a $*^\varepsilon_\mu$-*iteration* provided that:

(a) It is a $(< \mu)$-support iteration (μ is a regular cardinal).

(b) If $i_1 < i_2 \leq i(*)$ and $\mathrm{cf}(i_1) \neq \mu$ then $\mathbb{P}_{i_2}/\mathbb{P}_{i_1}$ satisfies $*^\varepsilon_\mu$.

Lemma 1.3. *If $\mathbf{q} = \langle \mathbb{P}_i, \mathbb{Q}_j : i \leq i(*),\ j < i(*) \rangle$ is a $(< \mu)$-support iteration and (a) or (b) or (c) below hold, then it is a $*^\varepsilon_\mu$-iteration.*

(a) $i(*)$ is limit and $\mathbf{q} \restriction j(*)$ is a $*^\varepsilon_\mu$-iteration for every $j(*) < i(*)$.

(b) $i(*) = j(*) + 1$, $\mathbf{q} \restriction j(*)$ is a $*^\varepsilon_\mu$-iteration, and $\mathbb{Q}_{j(*)}$ satisfies $*^\varepsilon_\mu$ in $\mathbf{V}^{\mathbb{P}_{j(*)}}$.

(c) $i(*) = j(*) + 1$, $\mathrm{cf}(j(*)) = \mu^+$, $\mathbf{q} \restriction j(*)$ is a $*^\varepsilon_\mu$-iteration, and for every successor $i < j(*)$, $\mathbb{P}_{i(*)}/\mathbb{P}_i$ satisfies $*^\varepsilon_\mu$.

Proof. Left to the reader (after reading [She78] or [SS86]). $\square_{1.3}$

Theorem 1.4. *Suppose $\mu = \mu^{<\mu} < \chi < \lambda$ and λ is a strongly inaccessible k^2_2-Mahlo cardinal, where k^2_2 is a suitable natural number (see* [She89, 3.6(2)]*), and assume $\mathbf{V} = \mathbf{L}$ for simplicity.*

<u>Then</u> for some forcing notion $\mathbb{P}$:

(A) $\mathbb{P}$ is μ-complete, satisfies the μ^+-c.c., has cardinality λ, and $\mathbf{V}^{\mathbb{P}} \models$ "$2^\mu = \lambda$".

(B) $\Vdash_{\mathbb{P}} \lambda \to [\mu^+]^2_3$ and even $\lambda \to [\mu^+]^2_{\kappa,2}$ for $\kappa < \mu$.

(C) If $\mu = \aleph_0$ then $\Vdash$ "MA_χ".

(D) If $\mu > \aleph_0$ <u>then</u> $\Vdash_{\mathbb{P}}$ "for every μ-complete forcing notion $\mathbb{Q}$ of cardinality $\leq \chi$ satisfying $*^\varepsilon_\mu$, and for any dense sets $D_i \subseteq \mathbb{Q}$, for $i < i_0 < \lambda$, there is a directed $G \subseteq \mathbb{Q}$ with $\bigwedge\limits_i G \cap D_i \neq \varnothing$".

As the proof[1] is very similar to [She88] (particularly after reading section 3), we do not give details. We shall define below only the systems needed to complete the proof. More general ones are implicit in [She89].

Convention 1.5. We fix a one to one function $\mathrm{Cd} = \mathrm{Cd}_{\lambda,\mu}$ from ${}^{\mu >}\lambda$ onto λ.

Remark 1.6. Below we could have $\mathrm{otp}(B_x) = \mu^+ + 1$ with little change.

Definition 1.7. Let $\mu < \chi < \kappa \le \lambda$, $\lambda = \lambda^{<\mu}$, $\chi = \chi^{<\mu}$, $\mu = \mu^{<\mu}$.

1) We call x a $(\lambda, \kappa, \chi, \mu)$-*pre-candidate* if $x = \langle a^x_u : u \in I_x \rangle$, where for some set B_x (unique, in fact):
   (i) $I_x = [B_x]^{\le 2}$
   (ii) B_x is a subset of κ of order type μ^+.
   (iii) a^x_u is a subset of λ of cardinality $\le \chi$ closed under Cd.
   (iv) $a^x_u \cap B_x = u$
   (v) $a^x_u \cap a^x_v \subseteq a^x_{u \cap v}$
   (vi) If $u, v \in I_x$ and $|u| = |v|$ then a^x_u and a^x_v have the same order type (and so $H^{\mathrm{OP}}_{a^x_u, a^x_v}$ maps a^x_u onto a^x_v).
   (vii) If $u_\ell, v_\ell \in I_x$ and $|u_\ell| = |v_\ell|$ for $\ell = 1, 2$, $|u_1 \cup u_2| = |v_1 \cup v_2|$, and $H^{\mathrm{OP}}_{a^x_{u_1} \cup a^x_{u_2}, a^x_{v_1} \cup a^x_{v_2}}$ maps u_ℓ onto v_ℓ for $\ell = 1, 2$ then $H^{\mathrm{OP}}_{a^x_{u_1}, a^x_{v_1}}$ and $H^{\mathrm{OP}}_{a^x_{u_2}, a^x_{v_2}}$ are compatible.

2) We say x is a $(\lambda, \kappa, \chi, \mu)$-candidate if it has the form $\langle M^x_u : u \in I_x \rangle$, where
   (α) (i) $\langle |M^x_u| : u \in I_x \rangle$ is a $(\lambda, \kappa, \chi, \mu)$-precandidate (with B_x defined as $\bigcup I_x$).
   (ii) τ_x is a vocabulary with $(\le \chi)$-many $(< \mu)$-ary place predicates and function symbols.
   (iii) Each M^x_u is a τ_x-model.
   (iv) For u, $v \in I_x$ with $|u| = |v|$, $M^x_u \restriction (|M^x_u| \cap |M^x_v|)$ is a model, and in fact an elementary submodel of M^x_v, M^x_u and $M^x_{u \cap v}$.
   (β) For u, $v \in I_x$ with $|u| = |v|$, the function $H^{\mathrm{OP}}_{|M^x_u|, |M^x_v|}$ is an isomorphism from M^x_u onto M^x_v.

3) We say the set $\mathfrak{A}$ is a $(\lambda, \kappa, \chi, \mu)$-*system* if
   (A) Each $x \in \mathfrak{A}$ is a $(\lambda, \kappa, \chi, \mu)$-candidate.
   (B) **Guessing**: if τ is as in $(2)(\alpha)(\mathrm{ii})$ and M^* is a τ-model with universe λ, then for some $x \in \mathfrak{A}$, $s \in B_x \Rightarrow M^x_s \prec M^*$.

Definition 1.8. 1) We call the system $\mathfrak{A}$ *disjoint* when:
   ($*$) If $x \ne y$ are from $\mathfrak{A}$ and $\mathrm{otp}(|M^x_\varnothing|) \le \mathrm{otp}(|M^y_\varnothing|)$ <u>then</u> for some $B_1 \subseteq B_x$, $B_2 \subseteq B_y$ we have
   (a) $|B_1| + |B_2| < \mu^+$
   (b) The sets

$$\bigcup \{|M^x_s| : s \in [B_x \setminus B_1]^{\le 2}\} \text{ and } \bigcup \{|M^y_s| : s \in [B_y \setminus B_2]^{\le 2}\}$$

   have intersection $\subseteq M^y_\varnothing$.

2) We call the system $\mathfrak{A}$ *almost disjoint* when:
   ($**$) If x, $y \in \mathfrak{A}$ and $\mathrm{otp}(|M^x_\varnothing|) \le \mathrm{otp}(|M^y_\varnothing|)$ <u>then</u> for some $B_1 \subseteq B_x$ and $B_2 \subseteq B_y$ we have:
   (a) $|B_1| + |B_2| < \mu^+$
   (b) If $s \in [B_x \setminus B_1]^{\le 2}$, $t \in [B_y \setminus B_2]^{\le 2}$ then $|M^x_s| \cap |M^x_t| \subseteq |M^y_\varnothing|$.

[1] In [She00], full details are given for stronger theorems.

§ 2. Introducing the partition on trees

Definition 2.1. Let

1) $\mathrm{Per}({}^{\mu>}2)$ be the set of T such that
   (A) $T \subseteq {}^{\mu>}2$, $\langle\ \rangle \in T$.
   (B) $(\forall \eta \in T)\,(\forall \alpha < \ell g(\eta))\,[\eta \restriction \alpha \in T]$
   (C) If $\eta \in T \cap {}^{\alpha}2$ and $\alpha < \beta < \mu$ then for some $\nu \in T \cap {}^{\beta}2$ we have $\eta \triangleleft \nu$.
   (D) If $\eta \in T$ then for some ν we have $\eta \triangleleft \nu$, $\nu\hat{\ }\langle 0\rangle \in T$, and $\nu\hat{\ }\langle 1\rangle \in T$.
   (E) If $\eta \in {}^{\delta}2$, $\delta < \mu$ is a limit ordinal, and $\{\eta \restriction \alpha : \alpha < \delta\} \subseteq T$ then $\eta \in T$.

2) $\mathrm{Per}_{\mathrm{fe}}({}^{\mu>}2) =$

$$\Big\{T \in \mathrm{Per}({}^{\mu>}2) : \alpha < \mu,\ \nu_1, \nu_2 \in {}^{\alpha}2 \cap T \Rightarrow \Big[\bigwedge_{\ell=0}^{1} \nu_1\hat{\ }\langle \ell\rangle \in T \Leftrightarrow \bigwedge_{\ell=0}^{1} \nu_2\hat{\ }\langle \ell\rangle \in T\Big]\Big\}.$$

3) $\mathrm{Per}_{\mathrm{uq}}({}^{\mu>}2) =$

$$\Big\{T \in \mathrm{Per}({}^{\mu>}2) : \alpha < \mu,\ \nu_1 \neq \nu_2 \text{ from } {}^{\alpha}2 \cap T\ \Rightarrow \bigvee_{\ell=0}^{1} \bigvee_{m=1}^{2} \nu_m\hat{\ }\langle \ell\rangle \notin T\Big\}$$

4) For $T \in \mathrm{Per}({}^{\mu>}2)$, let $\lim T = \big\{\eta \in {}^{\mu}2 : (\forall \alpha < \mu)[\eta \restriction \alpha \in T]\big\}$.

5) For $T \in \mathrm{Per}_{\mathrm{fe}}({}^{\mu>}2)$ let $\mathrm{clp}_T : T \to {}^{\mu>}2$ be the unique one-to-one function from $\mathrm{sp}(T) := \{\eta \in T : \eta\hat{\ }\langle 0\rangle, \eta\hat{\ }\langle 1\rangle \in T\}$ onto ${}^{\mu>}2$ which preserves $\triangleleft$ and lexicographic order.

6) Let $\mathrm{SP}(T) = \{\ell g(\eta) : \eta \in \mathrm{sp}(T)\}$, and for $\eta, \nu \in T$ let

$$\mathrm{sp}(\eta, \nu) = \min\{i : \eta(i) \neq \nu(i) \vee i = \ell g(\eta) \vee i = \ell g(\nu)\}$$

(hence $\mathrm{sp}(\eta, \eta) = \ell g(\eta)$).

Definition 2.2. For cardinals μ, σ and $n < \omega$ and $T \in \mathrm{Per}({}^{\mu>}2)$, let

1) $\mathrm{Col}^n_\sigma(T) = \big\{d : d$ is a function from $\bigcup_{\alpha<\mu} [{}^{\alpha}2]^n \cap T$ to $\sigma\big\}$. We may write $d(\nu_0, \dots, \nu_{n-1})$ for $d\big(\{\nu_0, \dots, \nu_{n-1}\}\big)$.

2) Let $<^*_\alpha$ denote a well ordering of ${}^{\alpha}2$ (in this section it is arbitrary). We call $d \in \mathrm{Col}^n_\sigma(T)$ *end-homogeneous* for $\big\langle <^*_\alpha : \alpha < \mu\big\rangle$ provided that if $\alpha < \beta$ are from $\mathrm{SP}(T)$, $\{\nu_0, \dots, \nu_{n-1}\} \subseteq {}^{\beta}2 \cap T$, $\langle \nu_\ell \restriction \alpha : \ell < n\rangle$ are pairwise distinct, and $\bigwedge_{\ell,m} [\nu_\ell <^*_\beta \nu_m \Leftrightarrow \nu_\ell \restriction \alpha <^*_\alpha \nu_m \restriction \alpha]$ then

$$d(\nu_0, \dots, \nu_{n-1}) = d(\nu_0 \restriction \alpha, \dots, \nu_{n-1} \restriction \alpha).$$

3) Let $\mathrm{EhCol}^n_\sigma(T) =$

$$\big\{d \in \mathrm{Col}^n_\sigma(T) : d \text{ is end-homogeneous for some } \langle <^*_\alpha : \alpha < \mu\rangle\big\}$$

(see above).

4) For $\nu_0, \dots, \nu_{n-1}, \eta_0, \dots, \eta_{n-1}$ from ${}^{\mu>}2$, we say $\bar\nu = \langle \nu_0, \dots, \nu_{n-1}\rangle$ and $\bar\eta = \langle \eta_o, \dots, \eta_{n-1}\rangle$ are *strongly similar* for $\big\langle <^*_\alpha : \alpha < \mu\big\rangle$ if:
   (i) $\ell g(\nu_\ell) = \ell g(\eta_\ell)$
   (ii) $\mathrm{sp}(\nu_\ell, \nu_m) = \mathrm{sp}(\eta_\ell, \eta_m)$ (equivalently, $\ell g(\nu_\ell \cap \nu_m) = \ell g(\eta_\ell \cap \eta_m)$).
   (iii) If $\ell_1, \ell_2, \ell_3, \ell_4 < n$ and $\alpha = \mathrm{sp}(\nu_{\ell_1}, \nu_{\ell_2})$, $\alpha \leq \ell g(\nu_{\ell_3}), \ell g(\nu_{\ell_4})$, then

$$\nu_{\ell_3} \restriction \alpha <^*_\alpha \nu_{\ell_4} \restriction \alpha \Leftrightarrow \eta_{\ell_3} \restriction \alpha <^*_\alpha \eta_{\ell_4} \restriction \alpha \text{ and } \nu_{\ell_3}(\alpha) = \eta_{\ell_3}(\alpha).$$

5) For $\nu^a_0, \dots, \nu^a_{n-1}$, $\nu^b_0, \dots, \nu^b_{n-1}$ from ${}^{\mu>}2$, we say $\bar\nu^a = \langle \nu^a_0, \dots, \nu^a_{n-1}\rangle$ and $\bar\nu^b = \langle \nu^b_0, \dots, \nu^b_{n-1}\rangle$ are *similar* if the truth values of (i)–(iii) below do not depend on $t \in \{a, b\}$ for any $\ell(1), \ell(2), \ell(3), \ell(4) < n$:
   (i) $\ell g(\nu^t_{\ell(1)}) < \ell g(\nu^t_{\ell(2)})$

(ii) $\mathrm{sp}(\nu^t_{\ell(1)}, \nu^t_{\ell(2)}) < \mathrm{sp}(\nu^t_{\ell(3)}, \nu^t_{\ell(4)})$

(iii) for $\alpha = \mathrm{sp}(\nu^t_{\ell(1)}, \nu^t_{\ell(2)})$ and $\ell g(\nu^t_{\ell(3)}), \ell g(\nu^t_{\ell(4)}) \geq \alpha$, the truth value of the following does not depend on ℓ:

$$\nu^t_{\ell(3)} \restriction \alpha <^*_\alpha \nu^t_{\ell(4)} \restriction \alpha \text{ and } \nu^t_{\ell(3)}(\alpha) = 0.$$

6) We say $d \in \mathrm{Col}^n_\sigma(T)$ is almost homogeneous [homogeneous] on $T_1 \subseteq T$ (for $\langle <^*_\alpha : \alpha < \mu \rangle$) <u>if</u> for every $\alpha \in \mathrm{SP}(T_1)$, $\bar\nu$, $\bar\eta \in [{}^\alpha 2]^n \cap T_1$ which are strongly similar [similar] we have $d(\bar\nu) = d(\bar\eta)$.

7) We say $\langle <^*_\alpha : \alpha < \mu \rangle$ is *nice* to $T \in \mathrm{Per}({}^{\mu>}2)$, provided that: if $\alpha < \beta$ are from $\mathrm{SP}(T)$, $(\alpha, \beta) \cap \mathrm{SP}(T) = \varnothing$, $\eta_1 \neq \eta_2 \in {}^\beta 2 \cap T$, $[\eta_1 \restriction \alpha <^*_\alpha \eta_2 \restriction \alpha$ or $\eta_1 \restriction \alpha = \eta_2 \restriction \alpha$, $\eta_1(\alpha) < \eta_2(\alpha)]$ then $\eta_1 <^*_\beta \eta_2$.

Definition 2.3. 1) $\mathrm{Pr}_{\mathrm{eht}}(\mu, n, \sigma)$ means "for every $d \in \mathrm{Col}^n_\sigma({}^{\mu>}2)$, for some $T \in \mathrm{Per}({}^{\mu>}2)$, d is end homogeneous on T."

2) $\mathrm{Pr}_{\mathrm{aht}}(\mu, n, \sigma)$ means "for every $d \in \mathrm{Col}^n_\sigma({}^{\mu>}2)$, for some $T \in \mathrm{Per}({}^{\mu>}2)$, d is almost homogeneous on T."

3) $\mathrm{Pr}_{\mathrm{ht}}(\mu, n, \sigma)$ means for every $d \in \mathrm{Col}^n_\sigma({}^{\mu>}2)$, for some $T \in \mathrm{Per}({}^{\mu>}2)$, d is homogeneous on T.

4) For $x \in \{\mathrm{eht}, \mathrm{aht}, \mathrm{ht}\}$, $\mathrm{Pr}^{\mathrm{fe}}_x(\mu, n, \sigma)$ is defined like $\mathrm{Pr}_x(\mu, n, \sigma)$ but we demand $T \in \mathrm{Per}_{\mathrm{fe}}({}^{\mu>}2)$.

5) If above we replace $\mathrm{eht}, \mathrm{aht}, \mathrm{ht}$ by $\mathrm{ehtn}, \mathrm{ahtn}, \mathrm{htn}$, respectively, this means $\langle <^*_\alpha : \alpha < \mu \rangle$ is fixed *a priori*.

6) Replacing n by "$< \kappa$" and σ by $\bar\sigma = \langle \sigma_\ell : \ell < \kappa \rangle$ for $\kappa \leq \aleph_0$ means that $\langle d_n : n < \kappa \rangle$ are given, $d_n \in \mathrm{Col}^n_\sigma({}^{\mu>}2)$, and the conclusion holds for all d_n with $n < \kappa$ simultaneously. Replacing "σ" by "$< \sigma$" means that the assertion holds for every $\sigma_1 < \sigma$.

Definition 2.4. 1) $\mathrm{Pr}_{\mathrm{aht}}(\mu, n, \sigma(1), \sigma(2))$ means: for every $d \in \mathrm{Col}^n_{\sigma(1)}({}^{\mu>}2)$, for some $T \in \mathrm{Per}({}^{\mu>}2)$ and $\langle <^*_\alpha : \alpha < \mu \rangle$, for every $\bar\eta \in \bigcup \{[{}^\alpha 2]^n \cap T : \alpha \in \mathrm{SP}(T)\}$, the set

$$\Big\{d(\bar\nu) : \bar\nu \in \bigcup \{[{}^\alpha 2]^n \cap T_1 : \alpha \in \mathrm{SP}(T_1)\},\ \bar\eta \text{ and } \bar\nu \text{ are strongly similar for } \langle <^*_\alpha : \alpha < \mu \rangle \Big\}$$

has cardinality $< \sigma(2)$.

2) $\mathrm{Pr}_{\mathrm{ht}}(\mu, n, \sigma(1), \sigma(2))$ is defined similarly with "similar" instead of "strongly similar".

3) $\mathrm{Pr}_x(\mu, < \kappa, \langle \sigma^1_\ell : \ell < \kappa \rangle, \langle \sigma^2_\ell : \ell < \kappa \rangle)$, $\mathrm{Pr}^{\mathrm{fe}}_x(\mu, n, \sigma(1), \sigma(2))$, $\mathrm{Pr}^{\mathrm{fe}}_x(\mu, < \aleph_0, \bar\sigma^1, \bar\sigma^2)$ are defined in the same way.

There are many obvious implications.

Fact 2.5. 1) For every $T \in \mathrm{Per}({}^{\mu>}2)$ there is a $T_1 \subseteq T$ with $T_1 \in \mathrm{Per}_{\mathrm{uq}}({}^{\mu>}2)$.

2) In defining $\mathrm{Pr}^{\mathrm{fe}}_x(\mu, n, \sigma)$ we can demand $T \subseteq T_0$ for any $T_0 \in \mathrm{Per}_{\mathrm{fe}}({}^{\mu>}2)$; similarly for $\mathrm{Pr}^{\mathrm{fe}}_x(\mu, < \kappa, \sigma)$.

3) The obvious monotonicity holds.

Claim 2.6. *1) Suppose μ is regular, $\sigma \geq \aleph_0$, and $\mathrm{Pr}^{\mathrm{fe}}_{\mathrm{eht}}(\mu, n, <\sigma)$. Then $\mathrm{Pr}^{\mathrm{fe}}_{\mathrm{aht}}(\mu, n, <\sigma)$ holds.*

1A) Similarly for $\mathrm{Pr}^{\mathrm{fe}}_{\mathrm{ehtn}}$ and $\mathrm{Pr}^{\mathrm{fe}}_{\mathrm{ahtn}}$.

2) If μ is weakly compact and $\mathrm{Pr}^{\mathrm{fe}}_{\mathrm{aht}}(\mu, n, <\sigma)$ with $\sigma < \mu$, then $\mathrm{Pr}^{\mathrm{fe}}_{\mathrm{ht}}(\mu, n, <\sigma)$ holds.

3) If μ is Ramsey and $\mathrm{Pr}^{\mathrm{fe}}_{\mathrm{aht}}(\mu, < \aleph_0, <\sigma)$ with $\sigma < \mu$, then $\mathrm{Pr}^{\mathrm{fe}}_{\mathrm{ht}}(\mu, < \aleph_0, <\sigma)$.

4) If $\mu = \omega$, in the "nice" version, the orders $\langle <^*_\alpha : \alpha < \mu \rangle$ disappear.

5) In parts (1)-(3), we can replace aht, eht, ht *by* ahtn, ehtn, htn *respectively.*

6) In $\mathrm{Pr}^{\mathrm{fe}}_{\mathrm{eht}}(\mu, n, \sigma)$*, we can strengthen the conclusion to:*

(∗) *If* $\alpha < \beta$ *are from* $\mathrm{SP}(T)$, $\langle \eta_\ell : \ell < n\rangle \in {}^n(2^\alpha)$ *is* $<_\alpha$*-increasing, and* $\eta_\ell \lhd \nu^\iota_\ell \in 2^\beta$ *(for* $\ell < n$ *and* $\iota \in \{1,2\}$*) then*

$$d(\{\nu^1_\ell : \ell < n\}) = d(\{\nu^2_\ell : \ell < n\}).$$

Proof. Easy; e.g. for (1A) we can use (6).

We induct on n; for $n+1$ and given $d_{n+1} : \bigcup\{[{}^\alpha 2]^{n+1} : \alpha < \mu\} \to \sigma$ and $\overline{<}^{n+1} = \langle <^{n+1}_\alpha : \alpha < \mu\rangle$, we apply $\mathrm{Pr}^{\mathrm{fe}}_{\mathrm{ehtn}}(\mu, n, <\sigma)$. We get T.

Let $f = \mathrm{clp}_T : T \to {}^{\mu>}2$ be as in 2.1(5). Define $\overline{<}^* = \langle <^*_\alpha : \alpha < \mu\rangle$ and d_n as follows:

(A) For $\alpha < \mu$ and $\eta_0, \eta_1 \in {}^\alpha 2$, $\mathrm{clp}_T(\nu_\ell) = \eta_\ell$, $\ell g(\nu_\ell) = \beta$ then

$$\eta_0 <^n_\alpha \eta_1 \Leftrightarrow \nu_0 <^{n+1}_\alpha \nu_1$$

(B) for $\alpha < \mu$ and $\eta_0 <^n_\alpha \ldots <^n_\alpha \eta_{n-1}$, $\mathrm{clp}_T(\nu_\ell) = \eta_\ell$, $\ell g(\nu_\ell) = \beta$ and for $k < n$, $\rho < 2$ we have $\nu_k \hat{}\langle \ell\rangle \lhd \rho_{k,\ell} \in \mathrm{sp}(T_{n+1}) \cap {}^\gamma 2$. If γ is minimal then $d_n(\{\eta_0, \ldots \eta_{n-1}\})$ codes the set of the following objects $\mathbf{t}$:
- For some $\gamma > \alpha$ there are $\rho_{k,\ell} \in \mathrm{sp}(T_{n+1}) \cap {}^\gamma 2$ such that $\nu_k \hat{}\langle \ell\rangle \trianglelefteq \rho_{k,\ell}$ for $k < n$, $\ell < 2$ and $\mathbf{t}$ codes all the information on the sequence $\langle \rho_{k,\ell} : k < n, \ell < 2\rangle$ (i.e. the order $<^{n+1}_\gamma$ and instances of $\mathbf{d}_{n+1}$). $\square_{2.6}$

The following theorem is a quite strong positive result for $\mu = \omega$. Halpern-Lauchli proved 2.7(1), Laver proved 2.7(2) (and hence (3)), Pincus pointed out that Halpern-Lauchli's proof can be modified to get 2.7(2), and then $\mathrm{Pr}^{\mathrm{fe}}_{\mathrm{eht}}(\omega, n, <\sigma)$ and (by it) $\mathrm{Pr}^{\mathrm{fe}}_{\mathrm{ht}}(\omega, n, <\sigma)$ are easy.

Theorem 2.7. 1) *If* $d \in \mathrm{Col}^n_\sigma({}^{\omega>}2)$ *and* $\sigma < \aleph_0$, *then there are* $T_0, \ldots, T_{n-1} \in \mathrm{Per}_{\mathrm{fe}}({}^{\omega>}2)$ *and* $k_0 < k_1 < \ldots < k_\ell < \ldots$ *and* $s < \sigma$ *such that for every* $\ell < \omega$, *if* $\mu_0 \in T_0$, $\mu_1 \in T_1, \ldots, \nu_{n-1} \in T_{n-1}$, $\bigwedge_{m<n} \ell g(\nu_m) = k_\ell$, *then* $d(\nu_0, \ldots, \nu_{n-1}) = s$.

2) *We can demand in 1) that*

$$\mathrm{SP}(T_\ell) = \{k_0, k_1, \ldots\}.$$

3) $\mathrm{Pr}^{\mathrm{fe}}_{\mathrm{htn}}(\omega, n, \sigma)$ *for* $\sigma < \aleph_0$.

4) $\mathrm{Pr}^{\mathrm{fe}}_{\mathrm{htn}}\big(\omega, <\aleph_0, \langle \sigma^1_n : n < \omega\rangle, \langle \sigma^2_n : n < \omega\rangle\big)$ *if* $\sigma^1_n < \aleph_0$ *and* $\langle \sigma^2_n : n < \omega\rangle$ *diverge to infinity.*

Definition 2.8. Let d be a function with domain $\supseteq [A]^n$, A be a set of ordinals, F be a one-to-one function from A to ${}^{\alpha(*)}2$, $<^*_\alpha$ be a well ordering of ${}^\alpha 2$ for $\alpha \le \alpha(*)$ such that $F(\alpha) <^*_\alpha F(\beta) \Leftrightarrow \alpha < \beta$, and σ be a cardinal.

1) We say d is (F, σ)-*canonical* on A if for any $\alpha_1 < \ldots < \alpha_n \in A$,

$$\big|\big\{d(\beta_1, \ldots, \beta_n) : \big\langle F(\beta_1), \ldots, F(\beta_n)\big\rangle \text{ similar to } \big\langle F(\alpha_1), \ldots, F(\alpha_n)\big\rangle\big\}\big| \le \sigma$$

2) We define "almost (F, σ)-canonical" similarly using strongly similar instead of "similar".

§ 3. Consistency of a strong partition below the continuum

This section is dedicated to the proof of

Theorem 3.1. *Suppose λ is the first Erdős cardinal (i.e. the first such that $\lambda \to (\omega_1)_2^{<\omega}$). Then, if A is a Cohen subset of λ, in $\mathbf{V}[A]$ for some $\aleph_1$–c.c. forcing notion $\mathbb{P}$ of cardinality λ, $\Vdash_{\mathbb{P}}$ "$\mathrm{MA}_{\aleph_1}(Knaster) + 2^{\aleph_0} = \lambda$" and:*

1) *$\Vdash_{\mathbb{P}}$ "$\lambda \to [\aleph_1]^n_{h(n)}$" for suitable $h : \omega \to \omega$ (explicitly defined below).*

2) *In $\mathbf{V}^{\mathbb{P}}$, for any colorings d_n of λ where d_n is n-place, and for any divergent $\langle \sigma_n : n < \omega \rangle$ (see below), there is a $W \subseteq \lambda$, $|W| = \aleph_1$ and a function $F : W \to {}^\omega 2$ such that d_n is (F, σ_n)-canonical on W for each n. (See Definition 2.8 above.)*

Remark 3.2. 1) $h(n)$ is $n!$ times the number of $u \in [{}^\omega 2]^n$ satisfying "if $\eta_1, \eta_2, \eta_3, \eta_4 \in u$ are distinct and $\eta_1 \cap \eta_2 \neq \eta_3 \cap \eta_4$ then $\mathrm{sp}(\eta_1, \eta_2)$, $\mathrm{sp}(\eta_3, \eta_4)$ are distinct" up to strong similarity for any nice $\langle <^*_\alpha : \alpha < \omega \rangle$.
2) A sequence $\langle \sigma_n : n < \omega \rangle$ is *divergent* if $(\forall m)(\exists k)(\forall n \geq k)[\sigma_n \geq m]$.

Notation 3.3. For a sequence $a = \langle a_i, e^*_i : i < \alpha \rangle$ with $a_i \subseteq i$ and $e_i \in \{1, 2\}$, we call $b \subseteq \alpha$ *closed* (or 'a-closed') if

(i) $i \in b \Rightarrow a_i \subseteq b$
(ii) If $i < \alpha$, $e^*_i = 1$, and $\sup(b \cap i) = i$ then $i \in b$.

Definition 3.4. Let $\mathfrak{K}$ be the family of $\mathbf{q} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_j, a_j, e^*_j : j < \alpha,\ i \leq \alpha \rangle$ such that:

(a) $a_i \subseteq i$, $|a_i| \leq \aleph_1$, and $e^*_i \in \{0, 1\}$.
(b) a_i is closed for $\langle a_j, e^*_j : j < i \rangle$ and $[e^*_i = 1 \Rightarrow \mathrm{cf}(i) = \aleph_1]$.
(c) $\mathbb{P}_i$ is a forcing notion, $\underset{\sim}{\mathbb{Q}}_j$ is a $\mathbb{P}_j$-name of a forcing notion of cardinality $\aleph_1$ with minimal element $\underset{\sim}{\varnothing}$ or $\varnothing_j$, and for simplicity the underlying set of $\underset{\sim}{\mathbb{Q}}_j$ is $\subseteq [\omega_1]^{<\aleph_0}$ (we do not lose anything by this).
(d) $\mathbb{P}_\beta = \{p : p$ is a function whose domain is a finite subset of β and for $i \in \mathrm{dom}(p)$, $\Vdash_{\mathbb{P}_i}$ "$f(i) \in \underset{\sim}{\mathbb{Q}}_i$"$\}$ with the order $p \leq q$ if and only if for $i \in \mathrm{dom}(p)$, $q \restriction i \Vdash_{\mathbb{P}_i}$ "$p(i) \underset{\sim}{\leq} q(i)$".
(e) For $j < i$, $\underset{\sim}{\mathbb{Q}}_j$ is a $\mathbb{P}_j$-name involving only antichains contained in $\{p \in \mathbb{P}_j : \mathrm{dom}(p) \subseteq a_j\}$.

Notation: For $p \in \mathbb{P}_i$, $j < i$, $j \notin \mathrm{dom}(p)$ we let $p(j) = \varnothing$. Note that for $p \in \mathbb{P}_i$ and $j \leq i$, we have $p \restriction j \in \mathbb{P}_j$.

Definition 3.5. For $\mathbf{q} \in \mathfrak{K}$ as above (so $\alpha = \ell g(\mathbf{q})$):
1) for any $b \subseteq \beta \leq \alpha$ closed for $\langle a_i, e^*_i : i < \beta \rangle$, we define $\mathbb{P}^{\mathrm{cn}}_b$ [by simultaneous induction on β]:

$$\mathbb{P}^{\mathrm{cn}}_b = \{p \in \mathbb{P}_\beta : \mathrm{dom}(p) \subseteq b, \text{ and for } i \in \mathrm{dom}(p),\ p(i) \text{ is a canonical name}\}.$$

I.e. for any x, $\{p \in \mathbb{P}^{\mathrm{cn}}_{a_i} : p \Vdash_{\mathbb{P}_i}$ "$p(i) = x$" or $p \Vdash_{\mathbb{P}_i}$ "$p(i) \neq x$"$\}$ is a predense subset of $\mathbb{P}_i$.
2) For $\mathbf{q}$ as above, $\alpha = \ell g(\mathbf{q})$, take $\mathbf{q} \restriction \beta = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_j, a_j : i \leq \beta,\ j < \beta \rangle$ for $\beta \leq \alpha$ and the order is the order in $\mathbb{P}_\alpha$ (if $\beta \geq \alpha$, $\mathbf{q} \restriction \beta = \mathbf{q}$).
3) "b *closed for* $\mathbf{q}$ means "b closed for $\langle a_i, e^*_i : i < \ell g(\mathbf{q}) \rangle$".

Fact 3.6. 1) if $\mathbf{q} \in \mathfrak{K}$ then $\mathbf{q} \restriction \beta \in \mathfrak{K}$.
2) Suppose $b \subseteq c \subseteq \beta \leq \ell g(\bar{\theta})$, b and c are closed for $\mathbf{q} \in \mathfrak{K}$.

(i) If $p \in \mathbb{P}^{\mathrm{cn}}_c$ then $p \restriction b \in \mathbb{P}^{\mathrm{cn}}_b$.
(ii) If $p, q \in \mathbb{P}^{\mathrm{cn}}_c$ and $p \leq q$ then $p \restriction b \leq q \restriction c$.

(iii) $\mathbb{P}^{\rm cn}_c \lessdot \mathbb{P}_\beta$.

3) $\ell g(\mathbf{q})$ is closed for $\mathbf{q}$.

4) If $\mathbf{q} \in \mathfrak{K}$, $\alpha = \ell g(\mathbf{q})$ then $\mathbb{P}^{\rm cn}_\alpha$ is a dense subset of $\mathbb{P}_\alpha$.

5) If b is closed for $\mathbf{q}$, $p, q \in \mathbb{P}^{\rm cn}_{\ell g(\mathbf{q})}$, $p \le q$ in $\mathbb{P}_{\ell g(\mathbf{q})}$ and $i \in \mathrm{dom}(p)$ then $q \restriction a_i \Vdash_{\mathbb{P}_i}$ "$p(i) \le q(i)$" hence $\Vdash_{\mathbb{P}^{\rm cn}_{a_i}}$ "$p(i) \le_{\mathbb{Q}_i} q(i)$".

Definition 3.7. Suppose $W = (W, \le)$ is a finite partial order and $\mathbf{q} \in \mathfrak{K}$.

1) $\mathrm{IN}_W(\mathbf{q})$ is the set of $\bar b$-s satisfying (α)–(γ) below:

(α) $\bar b = \langle b_w : w \in W\rangle$ is an indexed set of $\mathbf{q}$-closed subsets of $\ell g(\mathbf{q})$.

(β) $W \models w_1 \le w_2 \Rightarrow b_{w_1} \subseteq b_{w_2}$.

(γ) If $\zeta \in b_{w_1} \cap b_{w_2}$, $w_1 \le w$, and $w_2 \le w$ then
$$(\exists u \in W)[\zeta \in b_u \wedge u \le w_1 \wedge u \le w_2].$$

We assume $\bar b$ codes $(W, \le)$.

2) For $\bar b \in \mathrm{IN}_W(\mathbf{q})$, let
$$\mathbf{q}[\bar b] = \big\{\langle p_w : w \in W\rangle : p_w \in P^{\rm cn}_{b_w},\ [W \models w_1 \le w_2 \Rightarrow p_{w_2} \restriction b_{w_1} = p_{w_1}]\big\}$$
with ordering $\mathbf{q}[\bar b] \models \bar p^1 \le \bar p^2$ iff $\bigwedge\limits_{w \in W} p^1_w \le p^2_w$.

3) Let $\mathfrak{K}^1$ be the family of $\mathbf{q} \in \mathfrak{K}$ such that for every $\beta \le \ell g(\mathbf{q})$ and $(\mathbf{q} \restriction \beta)$-closed set b, $\mathbb{P}_\beta$ and $\mathbb{P}_\beta / \mathbb{P}^{\rm cn}_b$ satisfy the Knaster condition.

Fact 3.8. Suppose $\mathbf{q} \in \mathfrak{K}^1$, $(W, \le)$ is a finite partial order, $\bar b \in \mathrm{IN}_W(\mathbf{q})$ and $\bar p \in \mathbf{q}[\bar b]$.

1) If $w \in W$, $p_w \le q \in \mathbb{P}^{\rm cn}_{b_w}$ <u>then</u> there is $\bar r \in \mathbf{q}[\bar b]$, $q \le r_w$, $\bar p \le \bar r$. In fact,
$$r_u(\gamma) = \begin{cases} p_u(\gamma) & \text{if } \gamma \in \mathrm{dom}\, p_u \setminus \mathrm{dom}\, q, \\ p_u(\gamma)\ \&\ q(\gamma) & \text{if } \gamma \in b_u \cap \mathrm{dom}\, q \text{ and for some } v \in W, \\ & \quad u \le v \le w \text{ and } \gamma \in b_v, \\ p_u(\gamma) & \text{if } \gamma \in b_u \cap \mathrm{dom}\, q \text{ but the previous case fails.} \end{cases}$$

2) Suppose $(W_1, \le)$ is a submodel of $(W_2, \le)$, both finite partial orders, $\bar b^l \in \mathrm{IN}_{W_l}(\mathbf{q})$, $\bar b^1_w = \bar b^2_w$ for $w \in W_1$.

(α) If $\bar q \in \mathbf{q}[\bar b^2]$ then $\langle q_w : w \in W_1\rangle \in \mathbf{q}[\bar b^1]$.

(β) If $\bar p \in \mathbf{q}[\bar b^1]$ <u>then</u> there is $\bar q \in \mathbf{q}[\bar b^2]$ with $\bar q \restriction W_1 = \bar p$; in fact, $q_w(\gamma)$ is $p_u(\gamma)$ if $u \in W_1$, $\gamma \in b_u$,and $u \le w$, provided that

$(**)$ If $w_1, w_2 \in W_1$, $w \in W_2$, $w_1 \le w$, $w_2 \le w$ and $\zeta \in b_{w_1} \cap b_{w_2}$ then for some $v \in W_1$, $\zeta \in b_v$, $v \le w_1$, $v \le w_2$.

(This guarantees that if there are several u-s as above we shall get the same value.)

3) If $\mathbf{q} \in \mathfrak{K}^1$ then $\mathbf{q}[\bar b]$ satisfies the Knaster condition. If $\varnothing$ is the minimal element of W (i.e. $u \in W \Rightarrow W \models \varnothing \le u$) then $\mathbf{q}[\bar b]/\mathbb{P}^{\rm cn}_{b_\varnothing}$ also satisfies the Knaster condition and so is $\lessdot\, \mathbf{q}[\bar b]$, when we identify $p \in \mathbb{P}^{\rm cn}_b$ with $\langle p : w \in W\rangle$.

Proof. 1) It is easy to check that each $r_u(\gamma)$ is in $\mathbb{P}^{\rm cn}_{b_u}$. So, in order to prove $\bar r \in \mathbf{q}[\bar b]$, we assume $W \models u_1 \le u_2$ and have to prove that $r_{u_2} \restriction b_{u_1} = r_{u_1}$. Let $\zeta \in b_{u_1}$.

First case: $\zeta \notin \mathrm{dom}(p_{u_1}) \cup \mathrm{dom}(q)$.

So $\zeta \notin \mathrm{dom}(r_{u_1})$ (by the definition of r_{u_1}) and $\zeta \notin \mathrm{dom}(p_{u_2})$ (as $\bar p \in \mathbf{q}[\bar b]$) hence $\zeta \notin \mathrm{dom}(p_{u_2}) \cup \mathrm{dom}(q)$ hence $\zeta \notin \mathrm{dom}(r_{u_2})$ by the choice of r_{u_2}, so we have finished.

Second case: $\zeta \in \mathrm{dom}(p_{u_1}) \setminus \mathrm{dom}(q)$.

As $\bar p \in \mathbf{q}[\bar b]$ we have $p_{u_1}(\zeta) = p_{u_2}(\zeta)$, and by their definition, $r_{u_1}(\zeta) = p_{u_1}(\zeta)$, $r_{u_2}(\zeta) = p_{u_2}(\zeta)$.

Third case: $\zeta \in \mathrm{dom}(q)$ and $(\exists v \in W)$ $[\zeta \in b_v \wedge v \leq u_1 \wedge v \leq w]$.

By the definition of $r_{u_1}(\zeta)$, we have $r_{u_1}(\zeta) = p_{u_1}(\zeta)$ & $q(\zeta)$; also, the same v witnesses $r_{u_2}(\zeta) = p_{u_2}(\zeta)$ & $q(\zeta)$

$$(\text{as } \zeta \in b_v \wedge v \leq u_1 \wedge v \leq w \Rightarrow \zeta \in b_v \wedge v \leq u_2 \wedge v \leq w),$$

and of course $p_{u_1}(\zeta) = p_{u_2}(\zeta)$ (as $\bar{p} \in \mathbf{q}[\bar{b}]$).

Fourth case: $\zeta \in \mathrm{dom}(q)$ and $\neg(\exists v \in W)[\zeta \in b_v \wedge v \leq u_1 \wedge v \leq w]$.

By the definition of $r_{u_1}(\zeta)$ we have $r_{u_1}(\zeta) = p_{u_1}(\zeta)$. It is enough to prove that $r_{u_2}(\zeta) = p_{u_2}(\zeta)$ as we know that $p_{u_1}(\zeta) = p_{u_2}(\zeta)$ (because $\bar{p} \in \mathbf{q}[\bar{b}]$, $u_1 \leq u_2$). If not, then for some $v_0 \in W$, $\zeta \in b_{v_0} \wedge v_0 \leq u_2 \wedge v_0 \leq w$. But $\bar{b} \in \mathrm{IN}_W(\mathbf{q})$, hence (see condition (γ) of Definition 3.7(1), applied with ζ, w_1, w_2, w there standing for ζ, v_0, u_1, u_2 here) we know that for some $v \in W$, $\zeta \in v \wedge v \leq v_0 \wedge v \leq u_1$. As $(W, \leq)$ is a partial order, $v \leq v_0$ and $v_0 \leq w$, we can conclude $v \leq w$. So v contradicts our being in the fourth case. So we have finished the fourth case.

Hence we have finished proving $\bar{r} \in \mathbf{q}[\bar{b}]$. We also have to prove $q \leq r_w$, but for $\zeta \in \mathrm{dom}(q)$ we have $\zeta \in b_w$ (as $q \in \mathbb{P}^{\mathrm{cn}}_w$ is on assumption) and $r_w(\zeta) = q(\zeta)$ because $r_w(\zeta)$ is defined by the second case of the definition as

$$(\exists v \in W)[\zeta \in b_w \wedge v \leq w \wedge v \geq w]$$

i.e. $v = w$.

Lastly, we have to prove that $\bar{p} \leq \bar{r}$ (in $\mathbf{q}[\bar{b}]$). So let $u \in W$, $\zeta \in \mathrm{dom}(p_u)$ and we have to prove $r_u \restriction \zeta \Vdash_{\mathbb{P}_\zeta}$ "$p_u(\zeta) \leq_{\mathbb{P}_\zeta} r_u(\zeta)$". As $r_u(\zeta)$ is $p_u(\zeta)$ or $p_u(\zeta)$ & $q(\zeta)$ this is obvious.

2) Immediate.

3) We prove this by induction on $|W|$.

For $|W| = 0$ this is totally trivial.

For $|W| = 1, 2$ this is assumed.

For $|W| > 2$ fix $\bar{p}^i \in \mathbf{q}[\bar{b}]$ for $i < \omega_1$. Choose a maximal element $v \in W$ and let $c = \bigcup\{b_w : W \models w < v\}$. Clearly c is closed for $\mathbf{q}$.

We know that $\mathbb{P}^{\mathrm{cn}}_c$, $\mathbb{P}^{\mathrm{cn}}_{b_v}/\mathbb{P}^{\mathrm{cn}}_c$ are Knaster by the induction hypothesis. We also know that $p^i_v \restriction c \in \mathbb{P}^{\mathrm{cn}}_c$ for $i < \omega_1$, hence for some $r \in \mathbb{P}^{\mathrm{cn}}_c$,

$$r \Vdash \text{"}\underset{\sim}{A} = \left\{i < \omega_1 : p^i_v \restriction c \in \underset{\sim}{G}_{\mathbb{P}^{\mathrm{cn}}_c}\right\} \text{ is uncountable"}$$

hence

$$\begin{aligned}\Vdash \text{"there is an uncountable } A^1 \subseteq \underset{\sim}{A} \text{ such that}\\ \left[i, j \in A^1 \Rightarrow p^i_v, p^j_v \text{ are compatible in } \mathbb{P}^{\mathrm{cn}}_{b_v}/\underset{\sim}{G}_{\mathbb{P}^{\mathrm{cn}}_c}\right]\text{"}.\end{aligned}$$

Fix a $\mathbb{P}^{\mathrm{cn}}_c$-name $\underset{\sim}{A}^1$ for such an A^1.

Let $A^2 = \left\{i < \omega_1 : (\exists q \in P^{\mathrm{cn}}_c)[q \Vdash i \in \underset{\sim}{A}^1]\right\}$. Necessarily $|A_2| = \aleph_1$, and for $i \in A^2$ there is $q^i \in \mathbb{P}^{\mathrm{cn}}_c$, $q^i \Vdash i \in A^1$, and without loss of generality $p^i_v \restriction c \leq q^i$. Note that p^i_v & $q^i \in \mathbb{P}^{\mathrm{cn}}_c$.

For $i \in A^2$, let $\bar{r}^i$ be defined using 3.8(1) (with $\bar{p}^i$, p^i_v & q^i). Let $W_1 = W \setminus \{v\}$, $\bar{b}' = \langle b_w : w \in W_1\rangle$.

By the induction hypothesis applied to W_1, $\bar{b}'$, $\bar{r}^i \restriction W_1$, for $i \in A^2$ there is an uncountable $A^3 \subseteq A^2$ and for $i < j$ in A^3, there is $\bar{r}^{i,j} \in \mathbf{q}[\bar{b}']$ with $\bar{r}^i \restriction W_1 \leq \bar{r}^{i,j}$ and $\bar{r}^j \restriction W_1 \leq \bar{r}^{i,j}$. Now define $r^{i,j}_c \in \mathbb{P}^{\mathrm{cn}}_c$ as follows: its domain is $\bigcup\left\{\mathrm{dom}(r^{i,j}_w) : W \models w < v\right\}$ and$r^{i,j}_c \restriction \mathrm{dom}(r^{i,j}_w) = r^{i,j}_w$ whenever $W \models w < v$.

Why is this a definition? As $W \models w_1 \leq v \wedge w_2 \leq v$, $\zeta \in b_{w_1} \wedge \zeta \in b_{w_2}$ implies that for some $u \in W$, $u \leq w_1 \wedge u \leq w_2$ and $\zeta \in u$. It is easy to check that $r^{i,j}_c \in \mathbb{P}^{\mathrm{cn}}_c$. Now $r^{i,j}_c \Vdash_{\mathbb{P}^{\mathrm{cn}}_c}$ "$p^i_{b_v}, p^j_{b_v}$ are compatible in $\mathbb{P}^{\mathrm{cn}}_{b_v}/\mathbb{P}^{\mathrm{cn}}_c$".

So there is $r \in \mathbb{P}^{\rm cn}_{b_v}$ such that $r^{i,j}_c \le r$, $p^i_{b_v} \le r$, $p^j_{b_v} \le r$. As in part (1) of 3.8, we can combine r and $\bar r^{i,j}$ to a common upper bound of $\bar p^i$, $\bar p^j$ in $\mathbf{q}[\bar b]$. $\square_{3.8}$

Claim 3.9. *If $e = 0,1$ and δ is a limit ordinal, and $\mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i$, α_i, e^*_i (for $i < \delta$) are such that for each $\alpha < \delta$, $\mathbf{q}^\alpha = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_j, \alpha_j, e^*_j : i \le \alpha,\ j < \alpha\rangle$ belongs to $\mathfrak{K}^\ell$, then for a unique $\mathbb{P}_\delta$, $\mathbf{q} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_j, \alpha_j, e^*_j : i \le \delta,\ j < \delta\rangle$ belongs to $\mathfrak{K}^\ell$.*

Proof. We define $\mathbb{P}_\delta$ by Definition 3.4(d). The least easy problem is to verify the Knaster conditions (for $\mathbf{q} \in \mathfrak{K}^1$). The proof is like the preservation of the c.c.c. under iteration for limit stages. $\square_{3.9}$

Convention 3.10. In 3.9, we shall not make a strict distinction between $\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_j, \alpha_j, e^*_j : i \le \delta,\ j < \delta\rangle$ and $\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i, \alpha_i, e^*_i : i < \delta\rangle$.

Claim 3.11. *If $\mathbf{q} \in \mathfrak{K}^\ell$, $\alpha = \ell g(\mathbf{q})$, $a \subset \alpha$ is closed for $\mathbf{q}$, $|a| \le \aleph_1$, and $\underset{\sim}{\mathbb{Q}}_1$ is a $\mathbb{P}^{\rm cn}_a$-name of a forcing notion satisfying (in $\mathbf{V}^{\mathbb{P}_\alpha}$) the Knaster condition whose underlying set is a subset of $[\omega_1]^{<\aleph_0}$, <u>then</u> there is a unique $\mathbf{q}^1 \in \mathfrak{K}^\ell$ with $\ell g(\mathbf{q}_1) = \alpha + 1$, $\mathbb{Q}^1_\alpha = \underset{\sim}{\mathbb{Q}}$, and $\mathbf{q} \restriction \alpha = \mathbf{q}$.*

Proof. Left to the reader. $\square_{3.11}$

We are now ready to prove 3.1.

Proof. **Stage A:** We force by $\mathfrak{K}^1_{<\lambda} = \{\mathbf{q} \in \mathfrak{K}^1 : \ell g(\mathbf{q}) < \lambda,\ \mathbf{q} \in \mathcal{H}(\lambda)\}$ ordered by being an initial segment (which is equivalent to forcing a Cohen subset of λ). The generic object is essentially $\mathbf{q}^* \in \mathfrak{K}^1_\lambda$, $\ell g(\mathbf{q}^*) = \lambda$, and then we force by $\mathbb{P}_\lambda = \lim \mathbf{q}^*$. Clearly $\mathfrak{K}^\ell_{<\lambda}$ is a λ-complete forcing notion of cardinality λ, and $\mathbb{P}_\lambda$ satisfies the c.c.c. Clearly it suffices to prove part (2) of 3.1.

Suppose $\underset{\sim}{d}_n$ is a name of a function from $[\lambda]^n$ to $\underset{\sim}{k}_n$ for $n < \omega$, $\underset{\sim}{\sigma}_n < \omega$, $\langle \sigma_n : n < \omega\rangle$ diverges[2] and for some $\mathbf{q}^0 \in \mathfrak{K}^1_{<\lambda}$, we have

$$\mathbf{q}^0 \Vdash_{\mathfrak{K}^1_{<\lambda}} (\exists p \in \underset{\sim}{\mathbb{P}}_\lambda)\big[p \Vdash_{\mathbb{P}_\lambda} \text{“}\langle \underset{\sim}{d}_n : n < \omega\rangle \text{ is a counterexample to 3.1(2)”}\big].$$

In $\mathbf{V}$ we can define $\langle \mathbf{q}^\zeta : \zeta < \lambda\rangle$ with $\mathbf{q}^\zeta \in \mathfrak{K}^1_{<\lambda}$ such that

$$\zeta < \xi \Rightarrow \mathbf{q}^\zeta = \mathbf{q}^\xi \restriction \ell g(\mathbf{q}^\zeta).$$

In$\mathbf{q}^{\zeta+1}$, $e^*_{\ell g(\mathbf{q}_\zeta)} = 1$, $\mathbf{q}^{\zeta+1}$ forces (in $\mathfrak{K}^1_{<\lambda}$) a value to p and the $\underset{\sim}{\mathbb{P}}_\lambda$-names $\underset{\sim}{d}_n \restriction \zeta$, $\underset{\sim}{\sigma}_n$, $\underset{\sim}{k}_n$ for $n < \omega$; i.e. the values here are still $\mathbb{P}_\lambda$-names. Let $\mathbf{q}^*$ be the limit of the $\mathbf{q}^\xi$-s. So $\mathbf{q}^* \in \mathfrak{K}^1$, $\ell g(\mathbf{q}^*) = \lambda$, $\mathbf{q}^* = \langle \mathbb{P}^*_i, \underset{\sim}{\mathbb{Q}}^*_j, \alpha^*_j, e^*_j : i \le \lambda,\ j < \lambda\rangle$, and the $\mathbb{P}^*_\lambda$-names $\underset{\sim}{d}_n$, $\underset{\sim}{\sigma}_n$, $\underset{\sim}{k}_n$ are defined such that in $\mathbf{V}^{\mathbb{P}^*_\lambda}$, $\underset{\sim}{d}_n$, $\underset{\sim}{\sigma}_n$, $\underset{\sim}{k}_n$ contradict clause (2) (as any $\mathbb{P}^*_\lambda$-name of a bounded subset of λ is a $\mathbb{P}^*_{\ell g(\mathbf{q}^\xi)}$-name for some $\xi < \lambda$).

Stage B: Let $\chi = \kappa^+$ and $<^*_\chi$ be a well-ordering of $\mathcal{H}(\chi)$. Now we can apply $\lambda \to (\omega_1)^{<\omega}_2$ to get δ, B, N_s and $\mathbf{h}_{s,t}$ (for $s,t \in [B]^{<\aleph_0}$ with $|s| = |t|$) such that:

(a) $B \subseteq \lambda$ with $\mathrm{otp}(B) = \omega_1$ and $\sup B = \delta$.
(b) $N_s \prec (\mathcal{H}(\chi), \in, <^*_\chi)$, $\mathbf{q}^* \in N_s$, $\langle \underset{\sim}{d}_n, \underset{\sim}{\sigma}_n, \underset{\sim}{k}_n : n < \omega\rangle \in N_s$.
(c) $N_s \cap N_t = N_{s\cap t}$
(d) $N_s \cap B = s$
(e) If $s = t \cap \alpha$, $t \in [B]^{<\aleph_0}$ then $N_s \cap \lambda$ is an initial segment of N_t.
(f) $\mathbf{h}_{s,t}$ is an isomorphism from N_t onto N_s (when defined).
(g) $\mathbf{h}_{t,s} = \mathbf{h}^{-1}_{s,t}$

[2]I.e. $(\forall m)(\exists k)(\forall n \ge k)[\sigma_n \ge m]$.

(h) $p_0 \in N_s$, $p_0 \Vdash_{\mathbb{P}_\lambda}$ "$\langle \underset{\sim}{d}_n, \underset{\sim}{\sigma}_n, \underset{\sim}{k}_n : n < \omega\rangle$ is a counterexample to the conclusion of 3.1".

(i) $\omega_1 \subseteq N_s$, $|N_s| = \aleph_1$ and if $\gamma \in N_s$, $\mathrm{cf}(\gamma) > \aleph_1$ then $\mathrm{cf}(\sup(\gamma \cap N_s)) = \omega_1$.

Let $\mathbf{q} = \mathbf{q}^* \restriction \delta$, $\mathbb{P} = \mathbb{P}^*_\delta$ and $\mathbb{P}_a = \mathbb{P}^{\mathrm{cn}}_a$ (for $\mathbf{q}$), where a is closed for $\mathbf{q}$.
Note: $\mathbb{P}^*_\lambda \cap N_s = \mathbb{P}^*_\delta \cap N_s = \mathbb{P}_{\sup \lambda \cap N_s} \cap N_s = \mathbb{P}_s \cap N_s$. Note also

$$\gamma \in \lambda \cap N_s \Rightarrow a^*_\gamma \subseteq \lambda \cap N_s.$$

Stage C: It suffices to show that we can define $\underset{\sim}{\mathbb{Q}}_\delta$ in $\mathbf{V}^{\mathbb{P}_\delta}$ which forces a subset $\underset{\sim}{W}$ of B of cardinality $\aleph_1$ and an $\underset{\sim}{F} : W \to {}^\omega 2$ which exemplify the desired conclusion in (2), and prove that $\underset{\sim}{\mathbb{Q}}_\delta$ satisfies the $\aleph_1$-c.c.c. in $\mathbf{V}^{\mathbb{P}_\delta}$ (and has cardinality $\aleph_1$). Moreover (see Definitions 3.4 and 3.7(3)), we also define $a_\delta = \bigcup_{s \in [B]^{<\aleph_0}} N_s$, $e_\delta = 1$, $\mathbf{q}' = \mathbf{q}\hat{\ }\langle \mathbb{P}^*_\delta, \underset{\sim}{\mathbb{Q}}_\delta, a_\delta, e_\delta\rangle$ and prove $\mathbf{q}' \in \mathfrak{K}^1$. We let $\underset{\sim}{d}(u) := \underset{\sim}{d}_{|u|}(u)$.

Let $F : \omega_1 \to {}^\omega 2$ be one-to-one such that $(\forall \eta \in {}^{\omega >}2)(\exists^{\aleph_1} \alpha < \omega_1)[\eta \lhd F(\alpha)]$. (This will not be the needed $\underset{\sim}{F}$, just notation).

For $s, t \in [B]^{<\aleph_0}$, we say $s \equiv^n_F t$ if $|s| = |t|$ and

$$(\forall \xi \in s)(\forall \zeta \in t)[\xi = \mathbf{h}_{s,t}(\zeta) \Rightarrow F(\xi) \restriction n = F(\zeta) \restriction n].$$

Let

$$I_n = I_n(F) := \big\{ s \in [B]^{<\aleph_0} : (\forall \zeta \neq \xi \in s)[F(\zeta) \restriction n \neq F(\xi) \restriction n]\big\}.$$

We define $\mathbb{R}_n$ as follows: a sequence $\langle p_s : s \in I_n\rangle \in \mathbb{R}_n$ if and only if

(i) for $s \in I_n$, $p_s \in \mathbb{P}^*_\lambda \cap N_s$,

(ii) for some c_s we have $p_s \Vdash$ "$\underset{\sim}{d}(s) = c_s$",

(iii) for $s, t \in I_n$, $s \equiv^n_F t \Rightarrow \mathbf{h}_{s,t}(p_t) = p_s$,

(iv) for $s, t \in I_n$, $p_s \restriction N_{s \cap t} = p_t \restriction N_{s \cap t}$.

$\mathbb{R}^-_n$ is defined similarly, omitting (ii).

For $x = \langle p_s : s \in I_n\rangle$ let $n(x) = n$, $p^x_s = p_s$, and (if defined) $c^x_s = c_s$. Note that we could replace $x \in \mathbb{R}_n$ by a finite subsequence. Let $\mathbb{R} = \bigcup_{n<\omega} \mathbb{R}_n$, $\mathbb{R}^- = \bigcup_{n<\omega} \mathbb{R}^-_n$. We define an order on $\mathbb{R}^-$: $x \leq y$ if and only if $n(x) \leq n(y)$ and

$$s \in I_{n(x)} \wedge t \in I_{n(y)} \wedge s \subseteq t \Rightarrow p^x_s \leq p^y_t.$$

Stage D: Note the following facts:
Subfact D(α): If $x \in \mathbb{R}^-_n$, $t \in I_n$ and $p^x_t \leq p^1 \in \mathbb{P}^*_\delta \cap N_t$, <u>then</u> there is y such that $x \leq y \in \mathbb{R}^-_n$ and $p^y_t = p^1$.

Proof. For $s \in I_n$, we let

$$p^y_s = \&\big\{\mathbf{h}_{s_1,t_1}(p^1 \restriction N_{t_1}) : s_1 \subseteq s,\ t_1 \subseteq t,\ s_1 \equiv^n_F t_1\big\} \,\&\, p^x_s.$$

(This notation means that p^y_s is a function whose domain is the union of the domains of the conditions mentioned, and for each coordinate we take the canonical upper bound; see preliminaries.)

Why is p^y_s well defined? Suppose $\beta \in N_s \cap \lambda$ (for $\beta \in \lambda \setminus N_s$, clearly $p^y_s(\beta) = \varnothing_\beta$), $s_\ell \subseteq s$, $t_\ell \subseteq t$, $s_\ell \equiv^n_F t_\ell$ for $\ell = 1, 2$ and $\beta \in \mathrm{dom}\big(\mathbf{h}_{s_\ell,t_\ell}(p^1 \restriction N_{t_\ell})\big)$, and it suffices to show that $p^x_s(\beta)$, $\mathbf{h}_{s_1,t_1}(p^1 \restriction N_{t_1})(\beta)$, and $\mathbf{h}_{s_2,t_2}(p^1 \restriction N_{t_2})(\beta)$ are pairwise comparable. Let $u = \bigcap\big\{v \in [B]^{<\aleph_0} : \beta \in N_v\big\}$; necessarily $u \subseteq s_1 \cap s_2$, and let $u_\ell = \mathbf{h}^{-1}_{s_\ell,t_\ell}(u)$. As $s_\ell, t_\ell, t \in I_n$, $s_\ell \equiv^n_F t_\ell$ and $u_\ell \subseteq t_\ell \subseteq t$, necessarily $u_1 = u_2$. Thus $\gamma = \mathbf{h}^{-1}_{u,v}(\beta) = \mathbf{h}^{-1}_{s_\ell,t_\ell}(\beta)$ and so the last two conditions are equal.

Now

$$p^x_s(\beta) = p^x_u(\beta) = \mathbf{h}_{u,v}(p^x_s(\gamma)) \leq \mathbf{h}_{s_\ell,t_\ell}\big((p^x_t \restriction N_{t_\ell})(\gamma)\big) = \big(\mathbf{h}_{s_\ell,t_\ell}(p^x_t \restriction N_{t_\ell})\big)(\beta).$$

We leave to the reader checking the other requirements. $\square_{\mathbf{D}(\alpha)}$

Subfact $\mathbf{D}(\beta)$: If $x \in \mathbb{R}_n^-$, $t \in I$ then $\bigcup\{p_s^x : s \in I_n, s \subseteq t\}$ (as a union of functions) exists and belongs to $\mathbb{P}_\lambda^* \cap N_t$.

Proof. See (iv) in the definition of $\mathbb{R}_n^-$. $\square_{\mathbf{D}(\beta)}$

Subfact $\mathbf{D}(\gamma)$: If $x \leq y$, $x \in \mathbb{R}_n$, $y \in \mathbb{R}_n^-$, then $y \in \mathbb{R}_n$.

Proof. Check it. $\square_{\mathbf{D}(\gamma)}$

Subfact $\mathbf{D}(\delta)$: If $x \in \mathbb{R}_n^-$, $n < m$, then there is $y \in \mathbb{R}_m$ with $x \leq y$.

Proof. By subfact $\mathbf{D}(\beta)$ we can find $x^1 = \langle p_t^1 : t \in I_m \rangle \in \mathbb{R}_m^-$ with $x \leq x^1$. Repeatedly using subfact $\mathbf{D}(\alpha)$, we can increase x^1 (finitely many times) to get $y \in \mathbb{R}_m$. $\square_{\mathbf{D}(\delta)}$

Subfact $\mathbf{D}(\varepsilon)$: If $x \in \mathbb{R}_n^-$, $s, t \in I_n$, $s \equiv_F^n t$,

$$p_s^x \leq r_1 \in \mathbb{P}_\lambda^* \cap N_s, \quad p_t^x \leq r_2 \in \mathbb{P}_\lambda^* \cap N_t,$$

$(\forall \zeta \in t)\left[F(\zeta)(n) \neq F\big(\mathbf{h}_{s,t}(\zeta)\big)(n)\right]$ (or just $p_{s_1}^x \restriction s_1 = \mathbf{h}_{s,t}(p_{t_1}^x \restriction t_1)$, where $t_1 = \left\{\xi \in t : F(\xi)(n) = F(\mathbf{h}_{s,t}(\xi))(n)\right\}$ and $s_1 = \{\mathbf{h}_{s,t}(\xi) : \xi \in t_1\}$), then there is $y \in \mathbb{R}_{n+1}$ with $x \leq y$ such that $r_1 = p_s^y$ and $r_2 = p_t^y$.

Proof. Left to the reader. $\square_{\mathbf{D}(\varepsilon)}$

Stage E:[3]

We define $T_k^* \subseteq {}^{2^k \geq}2$ by induction on k as follows:

$$\begin{aligned} T_0^* &= \{\langle\, \rangle, \langle 1 \rangle\} \\ T_{k+1}^* &= T_k^* \cup \big\{\nu : 2^k < \ell g(\nu) \leq 2^{k+1},\ \nu \restriction 2^k \in T_k^*, \text{ and} \\ &\qquad [2^k \leq i < 2^{k+1} \wedge \nu(i) = 1] \Rightarrow i = 2^k + \big(\sum_{m<2^k} \nu(i) 2^m\big)\big\}. \end{aligned}$$

We define

$$\begin{aligned} \mathrm{TrEmb}(k,n) := \big\{h :\ & h \text{ a is function from } T_k^* \text{ into } {}^{n\geq}2 \\ & \text{such that for } \nu, \rho \in T_k^* \text{ we have} \\ & \eta = \nu \Leftrightarrow h(\eta) = h(\nu), \\ & \eta \triangleleft \nu \Leftrightarrow h(\eta) \triangleleft h(\nu), \\ & \ell g(\eta) = \ell g(\nu) \Rightarrow \ell g(h(\eta)) = \ell g(h(\nu)), \\ & \nu = \eta\hat{\ }\langle i \rangle \Rightarrow h(\nu)\big(\ell g(h(\eta))\big) = i, \\ & \ell g(\eta) = k \Rightarrow \ell g(h(\eta)) = n\big\}. \end{aligned}$$

$$\begin{aligned} \mathbf{T}(k,n) &:= \big\{\mathrm{Rang}(h) : h \in \mathrm{TrEmb}(k,n)\big\}, \\ \mathbf{T}(*,n) &= \bigcup_k \mathbf{T}(k,n), \\ \mathbf{T}(k,*) &= \bigcup_n \mathbf{T}(k,n). \end{aligned}$$

[3] We will have $T \subset {}^{\omega>}2$ from 2.7(2) and then want to get a subtree with as few colors as possible; we can find one isomorphic to ${}^{\omega>}2$, and there restrict ourselves to $\bigcup_n T_n^*$.

For $T \in \mathbf{T}(k,*)$ let $n(T)$ be the unique n such that $T \in \mathbf{T}(k,n)$ and let

$$B_T = \big\{\alpha \in B : F(\alpha) \restriction n(T) \text{ is a maximal member of } T\big\},$$
$$\mathrm{fs}_T = \big\{t \subseteq B_T : \eta \in t \wedge \nu \in t \wedge \eta \neq \nu \Rightarrow \eta \restriction n(T) \neq \nu \restriction n(T)\big\},$$
$$\Theta_T = \Big\{\langle p_s : s \in \mathrm{fs}_T\rangle : p_s \in \mathbb{P} \cap N_s, \big[s \subseteq t \wedge \{s,t\} \subseteq \mathrm{fs}_T \Rightarrow p_s = p_t \restriction N_s\big]\Big\}.$$

Furthermore, let

$$\Theta_k = \bigcup \{\Theta_T : T \in \mathbf{T}(k,*)\}$$
$$\Theta = \bigcup_k \Theta_k.$$

For $\bar{p} \in \Theta$, $\mathbf{n}_{\bar{p}} = \mathbf{n}(\bar{p})$ and $T_{\bar{p}}$ are defined naturally.

For $\bar{p}, \bar{q} \in \Theta$, $\bar{p} \leq \bar{q}$ iff $\mathbf{n}_{\bar{p}} \leq \mathbf{n}_{\bar{q}}$ and for every $s \in \mathrm{fs}_{T_{\bar{p}}}$ we have $p_s \leq q_s$.

Stage F: Let $\underset{\sim}{g} : \omega \to \omega$, $\underset{\sim}{g} \in N_s$, $\underset{\sim}{g}$ grows fast enough relative to$\langle \sigma_n : n < \omega\rangle$. We define a game Gm. A play of the game lasts ω moves: in the n^{th} move Player **I** chooses $\bar{p}^n \in \Theta_n$ and a function h_n satisfying the restrictions below, and then Player **II** chooses $\bar{q}_n \in \Theta_n$ such that $\bar{p}_n \leq \bar{q}_n$ (so $T_{\bar{p}_n} = T_{\bar{q}_n}$). Player **I** loses the play if at any time he has no legal move; if he never loses, he wins. The restrictions Player **I** has to satisfy are:

(a) For $m < n$, $\bar{q}_m \leq \bar{p}_n$, p^n_s forces a value to $\underset{\sim}{g} \restriction (n+1)$.

(b) h_n is a function from $[B_{T_{\bar{p}_n}}]^{\leq g(n)}$ to ω.

(c) $m < n \Rightarrow h_n, h_m$ are compatible.

(d) If $m < n$, $\ell < g(m)$, and $s \in [B_{T_{\bar{p}_n}}]^\ell$ then $p^n_s \Vdash \underset{\sim}{d}(s) = h_n(s)$.

(e) Let $s_1, s_2 \in \mathrm{dom}(h_n)$. Then $h_n(s_1) = h_n(s_2)$ whenever s_1, s_2 are similar over n, which means:
  (i) $F\big(H^{\mathrm{OP}}_{s_2,s_1}(\zeta)\big) \restriction \mathbf{n}[\bar{p}^n] = F(\zeta) \restriction \mathbf{n}[\bar{p}^n]$ for $\zeta \in s_1$.
  (ii) $H^{\mathrm{OP}}_{s_2,s_1}$ preserves the relations $\mathrm{sp}\big(F(\zeta_1), F(\zeta_2)\big) < \mathrm{sp}\big(F(\zeta_3), F(\zeta_4)\big)$ and $F(\zeta_3)\big(\mathrm{sp}(F(\zeta_1), F(\zeta_2))\big) = i$ (in the interesting case $\zeta_3 \neq \zeta_1$, we have ζ_2 implies $i = 0$).

Stage G/Claim: Player **I** has a winning strategy in this game.

Proof. As the game is closed, it is determined, so we assume Player **II** has a winning strategy , and eventually we shall get a contradiction. We define by induction on n, $\bar{r}^n$ and Φ^n such that

(a) $\bar{r}^n \in \mathbb{R}_n$, $\bar{r}^n \leq \bar{r}^{n+1}$.

(b) Φ^n is a finite set of initial segments of plays of the game.

(c) In each member of Φ^n, Player **II** uses his winning strategy.

(d) If y belongs to Φ^n then it has the form $\langle \bar{p}^{y,\ell}, h^{y,\ell}, \bar{q}^{y,\ell} : \ell \leq m(y)\rangle$; let $h_y = h^{y,n_y}$ and $T_y = T_{\bar{q}^{y},m(y)}$. Also, $T_y \subseteq^{n\geq} 2$ and $q^{y,\ell}_s \leq r^n_s$ for $s \in \mathrm{fs}_{T_y}$.

(e) $\Phi_n \subseteq \Phi_{n+1}$, Φ_n is closed under taking the initial segments and the empty sequence (which too is an initial segment of a play) belongs to Φ_0.

(f) For any $y \in \Phi_n$ and T, h, either for some $z \in \Phi_{n+1}$, $n_z = n_y + 1$, $y = z \restriction (n_y+1)$, $T_z = T$, and $h_z = h$ or Player **I** has no legal $(n_y+1)^{\text{th}}$ move $\bar{p}^n, h^n$ (after y was played) such that $T_{\bar{p}^n} = T$, $h^n = h$, and $p^n_s = r^n_s$ for $s \in \mathrm{fs}_T$ (or always $\leq$ or always $\geq$).

There is no problem to carry the definition. Now $\langle \bar{r}^n_s : n < \omega\rangle$ defines a function d^*: if $\eta_1, \ldots, \eta_k \in {}^m 2$ are distinct then $d^*\big(\langle \eta_1, \ldots, \eta_k\rangle\big) = c$ iff for every (equivalently, 'some') $\zeta_1 < \cdots < \zeta_k$ from B, $\eta_\ell \lhd F(\zeta_\ell)$ and

$$r^k_{\{\zeta_1,\ldots,\zeta_k\}} \Vdash \text{“}\underset{\sim}{d}_k\big(\{\zeta_1, \ldots, \zeta_k\}\big) = c\text{”}.$$

Now apply 2.7(2) to this coloring and get $T^* \subseteq {}^{\omega>}2$ as there. Now Player **I** could have chosen initial segments of this T^* (in the n^{th} move in Φ_n), and we easily get a contradiction. $\square_{\mathbf{G}}$

Stage H: We fix a winning strategy for Player **I** (whose existence is guaranteed by stage **G**).

We define a forcing notion $\mathbb{Q}^*$. We have $(r, y, f) \in \mathbb{Q}^*$ iff

(i) $r \in \mathbb{P}^{\text{cn}}_{a_\delta}$

(ii) $y = \langle \bar{p}^\ell, h^\ell, \bar{q}^\ell : \ell \le m(y) \rangle$ is an initial segment of a play of Gm in which Player **I** uses his winning strategy.

(iii) f is a finite function from B to $\{0,1\}$ such that $f^{-1}(\{1\}) \in \text{fs}_{T_y}$ (where $T_y = T_{\bar{q}^{m(y)}}$).

(iv) $r = q^{y,m(y)}_{f^{-1}(\{1\})}$.

(The order is the natural one.)

Stage I: If $\underset{\sim}{J} \subseteq \mathbb{P}^{\text{cn}}_{a_\delta}$ is dense open then $\{(r, y, f) \in \mathbb{Q}^* : r \in \underset{\sim}{J}\}$ is dense in $\mathbb{Q}^*$.

Proof. By 3.8(1) (by the appropriate renaming). $\square_{\mathbf{I}}$

Stage J: We define $\mathbb{Q}_\delta$ in $\mathbf{V}^{\mathbb{P}_\delta}$ as $\{(r, y, f) \in \mathbb{Q}^* : r \in \underset{\sim}{G}_{\mathbb{P}_\delta}\}$, the order is as in $\mathbb{Q}^*$.

The main point left is to prove the Knaster condition for the partial ordered set $\mathbf{q}^* = \mathbf{q}\hat{\ }\langle \mathbb{P}_\delta, \underset{\sim}{\mathbb{Q}}_\delta, a_\delta, e_\delta \rangle$ demanded in the definition of $\mathfrak{K}^1$. This will follow by 3.8(3) (after you choose meaning and renamings) as done in stages **K** and **L** below.

Stage K: So let $i < \delta$, $\text{cf}(i) \ne \aleph_1$, and we shall prove that $\mathbb{P}^+_{\delta+1}/\mathbb{P}_i$ satisfies the Knaster condition. Let $p_\alpha \in \mathbb{P}^*_{\delta+1}$ for $\alpha < \omega_1$, and we should find $p \in \mathbb{P}_i$, $p \Vdash_{\mathbb{P}_i}$ "there is an unbounded $A \subseteq \{\alpha : p_\alpha \restriction i \in \underset{\sim}{G}_{\mathbb{P}_i}\}$ such that for any $\alpha, \beta \in A$, p_α, p_β are compatible in $\mathbb{P}^*_{\delta+1}/\underset{\sim}{G}_{\mathbb{P}_i}$".

Proof. Without loss of generality:

(a) $p_\alpha \in \mathbb{P}^{\text{cn}}_{\delta+1}$

(b) For some $\langle i_\alpha : \alpha < \omega_1 \rangle$ increasing continuous with limit δ we have $i_0 > i$, $\text{cf}(i_\alpha) \ne \aleph_1$, $p_\alpha \restriction \delta \in \mathbb{P}_{i_{\alpha+1}}$, and $p_\alpha \restriction i_\alpha \in \mathbb{P}_{i_0}$. Let $p^0_\alpha = p^\alpha \restriction i_0$, $p^1_\alpha = p_\alpha \restriction \delta = p_\alpha \restriction i_{\alpha+1}$, and $p_\alpha(\delta) = (r_\alpha, y_\alpha, f_\alpha)$.

(c) $r_\alpha \in \mathbb{P}_{i_{\alpha+1}}$, $r_\alpha \restriction i_\alpha \in \mathbb{P}_{i_0}$, and $m(y_\alpha) = m^*$.

(d) $\text{dom}(f_\alpha) \subseteq i_0 \cup [i_\alpha, i_{\alpha+1})$,

(e) $f_\alpha \restriction i_0$ is constant. (Remember, $\text{otp}(B) = \omega_1$.)

(f) If $\text{dom}(f_\alpha) = \{j^\alpha_0, \dots j^\alpha_{k_\alpha - 1}\}$ then $k_\alpha = k$, $[j^\alpha_\ell < i_\alpha \Leftrightarrow \ell < k^*]$, $\bigwedge_{\ell < k^*} j^\alpha_\ell = j^\ell$, $f(j^\alpha_\ell) = f(j^\beta_\ell)$, and $F(j^\alpha_\ell)) \restriction m(y_\alpha) = F(j^\beta_\ell) \restriction m(y_\beta)$.

The main problem is the compatibility of the $q^{y_\alpha, m(y_\alpha)}$. Now by the definition of Θ_α (in stage **E**) and 3.8(3) this holds. $\square_{\mathbf{K}}$

Stage L: If $c \subset \delta+1$ is closed for $\mathbf{q}^*$, then $\mathbb{P}^*_{\delta+1}/\mathbb{P}^{\text{cn}}_c$ satisfies the Knaster condition.

If c is bounded in δ, choose a successor $i \in (\sup c, \delta)$ for $\mathbf{q} \restriction i \in \mathfrak{K}_1$. We know that $\mathbb{P}_i/\mathbb{P}^{\text{cn}}_c$ satisfies the Knaster condition and by stage K, $\mathbb{P}^*_{\delta+1}/\mathbb{P}_i$ also satisfies the Knaster condition; as it is preserved by composition we have finished the stage.

So assume c is unbounded in δ and it is easy too. So as seen in stage **J**, we have finished the proof of 3.1. $\square_{3.1}$

Theorem 3.12. *If $\lambda \ge \beth_\omega$ and $\mathbb{P}$ is the forcing notion which adds λ Cohen reals, then:*

$(*)_1$ *In* $\mathbf{V}^{\mathbb{P}}$*, if* $n < \omega$ *and* $d : [\lambda]^{\leq n} \to \sigma$ *with* $\sigma < \aleph_0$*, then for some c.c.c. forcing notion* $\mathbb{Q}$ *we have* $\Vdash_{\mathbb{Q}}$ "there are an uncountable $A \subseteq \lambda$ and a one-to-one $F : A \to {}^{\omega}2$ such that d is F-canonical on A" *(see notation in §2).*

$(*)_2$ *If* $\lambda \geq \mu \to_{\text{wsp}} (\kappa)_{\aleph_0}$ *in* $\mathbf{V}$ *(see* [She89]*) and* $d : [\mu]^{\leq n} \to \sigma$ *in* $\mathbf{V}^{\mathbb{P}}$ *(with* $\sigma < \aleph_0$*) then, in* $\mathbf{V}^{\mathbb{P}}$*, for some c.c.c. forcing notion* $\mathbb{Q}$ *we have* $\Vdash_{\mathbb{Q}}$ "there are $A \in [\mu]^{\kappa}$ and one-to-one $F : A \to {}^{\omega}2$ such that d is F-canonical on A" *(see §2).*

$(*)_3$ *If* $\lambda \geq \mu \to_{\text{wsp}} (\aleph_1)^n_{\aleph_2}$ *in* $\mathbf{V}$ *and* $d : [\mu]^{\leq n} \to \sigma$ *in* $\mathbf{V}^{\mathbb{P}}$ *(with* $\sigma < \aleph_0$*) then, in* $\mathbf{V}^{\mathbb{P}}$*, for every* $\alpha < \omega_1$ *and* $F : \alpha \to {}^{\omega}2$*, for some* $A \subseteq \mu$ *of order type* α *and* $F' : A \to {}^{\omega}2$*,* $F'(\beta) = F(\text{otp}(A \cap \beta))$*,* d *is* F'*-canonical on* A*.*

$(*)_4$ *In* $\mathbf{V}^{\mathbb{P}}$*,* $2^{\aleph_0} \to (\alpha, n)^3$ *for every* $\alpha < \omega_1$ *and* $n < \omega$*. Really, assuming* $\mathbf{V} \models \mathsf{GCH}$ *we have* $\aleph_{n^1_3} \to (\alpha, n)$ *(see* [She89]*).*

Proof. Similar to the proof of 3.1. Superficially we need more indiscernibility then we get, but getting $\langle M_u : u \in [B]^{\leq n} \rangle$ we ignore $d(\{\alpha, \beta\})$ when there is no u with $\{\alpha, \beta\} \in M_u$. $\square_{3.12}$

Theorem 3.13. *If* λ *is strongly inaccessible* ω*-Mahlo and* $\mu < \lambda$*, then for some c.c.c. forcing notion* $\mathbb{P}$ *of cardinality* λ*,* $\mathbf{V}^{\mathbb{P}}$ *satisfies*

(*a*) MA_{μ}

(*b*) $2^{\aleph_0} = \lambda = 2^{\kappa}$ *for* $\kappa < \lambda$*.*

(*c*) $\lambda \to [\aleph_1]^n_{\sigma, h(n)}$ *for* $n < \omega$*,* $\sigma < \aleph_0$*, and* $h(n)$ *as in 3.1.*

Proof. Again, like 3.1. $\square_{3.13}$

§ 4. Partition theorem for trees on large cardinals

Lemma 4.1. *Suppose $\mu > \sigma + \aleph_0$ and*

$(*)_\mu$ *for every μ-complete forcing notion $\mathbb{P}$, in $\mathbf{V}^{\mathbb{P}}$, μ is measurable.*

Then

(1) *We have $\mathrm{Pr}^{\mathrm{fe}}_{\mathrm{eht}}(\mu, n, \sigma)$ for all $n < \omega$.*

(2) $\mathrm{Pr}^{\mathrm{fe}}_{\mathrm{eht}}(\mu, < \aleph_0, \sigma)$, *if there is $\lambda > \mu$ such that $\lambda \to \left(\mu^+\right)^{<\omega}_2$.*

(3) *In both cases we can have the $\mathrm{Pr}^{\mathrm{fe}}_{\mathrm{ehtn}}$ version, and even choose the $\langle <^*_\alpha : \alpha < \mu\rangle$ in any of the following ways.*
   (a) *We are given $\langle <^0_\alpha : \alpha < \mu\rangle$, and (for $\eta, \nu \in {}^\alpha 2 \cap T$, $\alpha \in \mathrm{SP}(T)$, and T the subtree we consider) we let:*
   - *$\eta <^*_\alpha \nu$ if and only if $\mathrm{clp}_T(\eta) <^0_\beta \mathrm{clp}_T(\nu)$, where $\beta = \mathrm{otp}(\alpha \cap \mathrm{SP}(T))$ and $\mathrm{clp}_T(\eta) = \big\langle \eta(j) : j \in \ell g(\eta),\ j \in \mathrm{SP}(T)\big\rangle$.*

   (b) *We are given $\langle <^0_\alpha : \alpha < \mu\rangle$, and we say $\eta <^*_\alpha \nu$ if and only if $n \restriction (\beta+1) <^0_{\beta+1} \nu \restriction (\beta+1)$, where $\beta = \sup(\alpha \cap \mathrm{SP}(T))$.*

Remark 4.2. 1) $(*)_\mu$ holds for a supercompact after Laver treatment. On hyper-measurable, see Gitik-Shelah [GS89].
2) We can in $(*)_\mu$ restrict ourselves to the forcing notion $\mathbb{P}$ actually used. For that, by Gitik [Git10] much smaller large cardinals suffice.
3) The proof of 4.1 is a generalization of a proof of Harrington to the Halpern-Lauchli theorem from 1978.

Conclusion 4.3. *In 4.1 we can get $\mathrm{Pr}^{\mathrm{fe}}_{\mathrm{ht}}(\mu, n, \sigma)$ (even with (3)).*

Proof. We do the parallel to 4.1(1). By $(*)_\mu$, μ is weakly compact hence by 2.6(2) it is enough to prove $\mathrm{Pr}^{\mathrm{fe}}_{\mathrm{aht}}(\mu, n, \sigma)$. This follows from 4.1(1) by 2.6(1). $\square_{4.3}$

Proof. **Proof of 4.1:**
1), 2). Let $\kappa \le \omega$, $\sigma(n) < \mu$, $d_n \in \mathrm{Col}^n_{\sigma(n)}({}^{\mu>}2)$ for $n < \kappa$.

Choose λ such that $\lambda \to (\mu^+)^{<2\kappa}_{2^\mu}$ (there is such a λ by assumption for (2) and by $\kappa < \omega$ for (1)). Let $\mathbb{Q}$ be the forcing notion $({}^{\mu>}2, \triangleleft)$, and $\mathbb{P} = \mathbb{P}_\lambda$ be

$$\big\{f : \mathrm{dom}(f) \text{ is a subset of } \lambda \text{ of cardinality} < \mu,\ f(i) \in \mathbb{Q}\big\},$$

ordered naturally. For $i \notin \mathrm{dom}(f)$, take $f(i) = \langle\ \rangle$. Let $\underset{\sim}{\eta}_i$ be the $\mathbb{P}$-name for $\bigcup\{f(i) : f \in \underset{\sim}{G}_{\mathbb{P}}\}$. Let $\underset{\sim}{D}$ be a $\mathbb{P}$-name of a normal ultrafilter over μ. For each $n < \omega$, $d \in \mathrm{Col}^n_{\sigma(n)}({}^{\mu>}2)$, $j < \sigma(n)$ and $u = \{\alpha_0, \ldots, \alpha_{n-1}\}$, where $\alpha_0 < \cdots < \alpha_{n-1} < \lambda$, let $\underset{\sim}{A}^j_d(u)$ be the $\mathbb{P}_\lambda$-name of the set

$$A^j_d(u) = \Big\{i < \mu : \langle \underset{\sim}{\eta}_{\alpha_\ell} \restriction i : \ell < n\rangle \text{ are pairwise distinct, } j = d(\eta_{\alpha_0} \restriction i, \ldots, \eta_{\alpha_{n-1}} \restriction i)\Big\}.$$

So $\underset{\sim}{A}^j_d(u)$ is a $\mathbb{P}_\lambda$-name of a subset of μ, and for $j(1) < j(2) < \sigma(n)$ we have $\Vdash_{\mathbb{P}_\lambda}$ "$\underset{\sim}{A}^{j(1)}_d(u) \cap \underset{\sim}{A}^{j(2)}_d(u) = \varnothing$, and $\bigcup_{j<\sigma(n)} \underset{\sim}{A}^j_d(u)$ is a co-bounded subset of μ". As $\Vdash_{\mathbb{P}}$ "$\mathfrak{D}$ is μ-complete uniform ultrafilter on μ", in $\mathbf{V}^{\mathbb{P}}$ there is exactly one $j < \sigma(n)$ with $A^j_d(u) \in \mathfrak{D}$. Let $\underset{\sim}{j}_d(u)$ be the $\mathbb{P}$-name of this j.

Let $I_d(u) \subseteq \mathbb{P}$ be a maximal antichain of $\mathbb{P}$, each member of $I_d(u)$ forces a value to $\underset{\sim}{j}_d(u)$. Let $W_d(u) = \bigcup\{\mathrm{dom}(p) : p \in I_d(u)\}$ and $W(u) = \bigcup\{W_{d_n}(u) : n < \kappa\}$. So $W_d(u)$ is a subset of λ of cardinality $\le \mu$ as well as $W(u)$ (as $\mathbb{P}$ satisfies the μ^+-c.c. and $p \in P \Rightarrow |\mathrm{dom}(p)| < \mu$).

As $\lambda \to (\mu^{++})^{<2\kappa}_{2^\mu}$, $d_n \in \mathrm{Col}^n_{\sigma_n}({}^{\mu>}2)$ there is a subset Z of λ of cardinality μ^{++} and set $W^+(u)$ for each $u \in [Z]^{<\kappa}$ such that:

(i) $W^+(u_1) \cap W^+(u_2) = W^+(u_1 \cap u_2)$

(ii) $W(u) \subseteq W^+(u)$ if $u \in [Z]^{<\kappa}$.

(iii) If $|u_1| = |u_2| < \kappa$ and $u_1, u_2 \subseteq Z$ then $W^+(u_1)$ and $W^+(u_2)$ have the same order type.

(Note that $H[u_1, u_2] = H^{\mathrm{OP}}_{W^+(u_1), W^+(u_2)}$ naturally induces a map from $\mathbb{P} \restriction u_1 = \{p \in \mathbb{P} : \mathrm{dom}(p) \subseteq W^+(u_1)\}$ to $\mathbb{P} \restriction u_2 = \{p \in \mathbb{P} : \mathrm{dom}(p) \subseteq W^+(u_2)\}$.)

(iv) If $u_1, u_2 \in [Z]^{<\kappa}$ and $|u_1| = |u_2|$ then $H[u_1, u_2]$ maps $I_{d_n}(u_1)$ onto $I_{d_n}(u_2)$ and

$$q \Vdash \text{“}\underset{\sim}{j}_d(u_1) = j\text{”} \Leftrightarrow H[u_1, u_2](q) \Vdash \text{“}\underset{\sim}{j}_d(u_2) = j\text{”}.$$

(v) If $u_1 \subseteq u_2 \in [Z]^{<\kappa}$, $u_3 \subseteq u_4 \in [Z]^{<\kappa}$, $|u_4| = |u_2|$, and $H^{\mathrm{OP}}_{u_2, u_4}$ maps u_1 onto u_3, <u>then</u> $H[u_1, u_3] \subseteq H[u_2, u_4]$.

Let $\gamma(i)$ be the i^{th} member of Z.

Let $s(m)$ be the set of the first m members of Z and

$$\mathbb{R}_n = \big\{p \in \mathbb{P} : \mathrm{dom}(p) \subseteq W^+(s(n)) \setminus \bigcup_{t \subset s(n)} W^+(t)\big\}.$$

We define, by induction on $\alpha < \mu$, a function F_α and $p_u \in \mathbb{R}_{|u|}$ for $u \in \bigcup_{\beta<\alpha} [{}^\beta 2]^{<\kappa}$ where we let $\varnothing_\beta$ be the empty subset of $[{}^\beta 2]$, we behave as if $[\beta \neq \gamma \Rightarrow \varnothing_\beta \neq \varnothing_\gamma]$, and we also define $\zeta(\beta) < \mu$ such that:

(i) F_α is a function from ${}^{\alpha>}2$ into ${}^{\mu>}2$, extending F_β for each $\beta < \alpha$.

(ii) F_α maps ${}^\beta 2$ to ${}^{\zeta(\beta)}2$ for some $\zeta(\beta) < \mu$, and

$$\beta_1 < \beta_2 < \alpha \Rightarrow \zeta(\beta_1) < \zeta(\beta_2).$$

(iii) $\eta \triangleleft \nu \in {}^{\alpha>}2$ implies $F_\alpha(\eta) \triangleleft F_\alpha(\nu)$.

(iv) For $\eta \in^\beta 2$, $\beta + 1 < \alpha$, and $\ell < 2$, we have $F_\alpha(\eta)\hat{}\langle \ell \rangle \trianglelefteq F_\alpha(\eta\hat{}\langle \ell \rangle)$.

(v) $p_u \in \mathbb{R}_m$ whenever $u \in [{}^\beta 2]^m$, $m < \kappa$, $\beta < \alpha$ and for $u(1) \in [Z]^m$ let $p_{u,u(1)} = H[s(|u|), u(1)](p_u)$.

(vi) $\eta \in {}^\beta 2$, $\beta < \alpha$, then $p_{\{\eta\}}(\min Z) = F_\alpha(\eta)$.

(vii) If $\beta < \alpha$, $u \in [{}^\beta 2]^n$, $n < \kappa$, and $h : u \to s(n)$ is one-to-one and onto (but not necessarily order preserving) <u>then</u> for some $c(u, h) < \sigma(n)$,

$$\bigcup_{t \subseteq u} p_{t, h''(t)} \Vdash_{\mathbb{P}_\lambda} \text{“}\underset{\sim}{d}_n(\underset{\sim}{\eta}_{\gamma(0)}, \ldots, \underset{\sim}{\eta}_{\gamma(n-1)}) = c(u, h)\text{”}.$$

(Note: as $p_u \in \mathbb{R}_{|u|}$, the domains of the conditions in this union are pairwise disjoint.)

(viii) If n, u, β, h are as in (vii), $u = \{\nu_0, \ldots, \nu_{n-1}\}$, $\nu_\ell \triangleleft \rho_\ell \in {}^\gamma 2$, and $\beta \leq \gamma < \alpha$, then $d_n(F_\alpha(\rho_0), \ldots, F_\alpha(\rho_{n-1})) = c(u, h)$, where h is the unique function from u onto $s(n)$ such that $[h(\nu_\ell) \leq h(\nu_m) \Rightarrow \rho_\ell <^*_\gamma \rho_m]$.

(ix) If $\beta < \gamma < \alpha$, $\nu_1, \ldots, \nu_{n-1} \in {}^\gamma 2$, $n < \kappa$, and $\nu_0 \restriction \beta, \ldots, \nu_{n-1} \restriction \beta$ are pairwise distinct, then: $p_{\{\nu_0 \restriction \beta, \ldots, \nu_n \restriction \beta\}} \subseteq p_{\{\nu_0, \ldots, \nu_{n-1}\}}$.

For α limit: no problem.

For $\alpha + 1$ with α limit: we try to define $F_\alpha(\eta)$ for $\eta \in {}^\alpha 2$ such that

$$\bigcup_{\beta<\alpha} F_{\beta+1}(\eta \restriction \beta) \trianglelefteq F_\alpha(\eta)$$

and (viii) holds. Let $\zeta = \bigcup_{\beta<\alpha} \zeta(\beta)$. For $\eta \in {}^\alpha 2$, we define

$$F^0_\alpha(\eta) := \bigcup_{\beta<\alpha} F_\alpha(\eta \restriction \beta)$$

and for $u \in [{}^{\alpha}2]^{<\kappa}$,

$$p^0_u = \bigcup \{p^0_{\{\nu \restriction \beta : \nu \in u\}} : \beta < \alpha \wedge \left|\{\nu \restriction \beta : \nu \in u\}\right| = |u|\}.$$

Clearly $p^0_u \in \mathbb{R}_{|u|}$.

Then let $h : {}^{\alpha}2 \to Z$ be one-to-one such that $\eta <^*_\alpha \nu \Leftrightarrow h(\eta) < h(\nu)$ and let

$$p = \bigcup \{p^0_{u,u(1)} : u(1) \in [Z]^{<\kappa},\ u \in [{}^{\alpha}2]^{<\kappa},\ |u(1)| = |u|,\ h''(u) = u(1)\}.$$

For any generic $G \subseteq \mathbb{P}_\lambda$ to which p belongs, for $\beta < \alpha$, $n < \omega$, and ordinals $i_0 < \cdots < i_{n-1}$ from Z such that $\langle h^{-1}(i_\ell) \restriction \beta : \ell < n\rangle$ are pairwise distinct, we have that

$$B_{\{i_\ell : \ell < n\},\beta} := \Big\{\xi < \mu : d_n(\eta_{i_0} \restriction \xi, \dots, \eta_{i_{n-1}} \restriction \xi) = c(u, h^*)\Big\}$$

belongs to $\mathfrak{D}[G]$, where $u = \{h^{-1}(i_\ell) \restriction \beta : \ell < n\}$ and $h^* : u \to s(|u|)$ is defined by $h^*(h^{-1}(i_\ell) \restriction \beta) = H^{\mathrm{OP}}_{\{i_\ell : \ell < n\}, s(n)}(i_\ell)$. Really every large enough $\beta < \mu$ can serve so we omit it. As $\mathfrak{D}[G]$ is μ-complete uniform ultrafilter on μ, we can find $\xi \in (\zeta, \kappa)$ such that $\xi \in B_u$ for every $u \in [{}^{\alpha}2]^{<\omega}$.

For $\nu \in {}^{\alpha}2$, we let $F_\alpha(\nu) = \underset{\sim}{\eta}_{h(i)}[G] \restriction \xi$, and we let $p_u = p^0_u$ except when $u = \{\nu\}$. In that case:

$$p_u(i) = \begin{cases} p^0_u(i) & \text{if } i \neq \gamma(0) \\ F_{\alpha+1}(\nu) & \text{if } i = \gamma(0). \end{cases}$$

For $\alpha + 1$, with α a successor:

First, for $\eta \in {}^{\alpha-1}2$ define $F\big(\eta\char`^\langle \ell\rangle\big) = F_\alpha(\eta)\char`^\langle \ell\rangle$. Next we let $\big\{(u_i, h_i) : i < i^*\big\}$ list all pairs (u, h) with $u \in [{}^{\alpha}2]^{\leq n}$ and $h : u \to s(|u|)$ one-to-one and onto. Now, by induction on $i \leq i^*$, we define p^i_u (for $u \in [{}^{\alpha}2]^{<\kappa}$) such that:

(a) $p^i_u \in \mathbb{R}_{|u|}$

(b) p^i_u increases with i.

(c) For $i + 1$, clause (vii) above holds (with α, u_i, h_i here standing in for β, u, h there).

(d) If $\nu_m \in {}^{\alpha}2$ for $m < n < \kappa$ and $\langle \nu_m \restriction (\alpha - 1) : m < n\rangle$ are pairwise distinct, <u>then</u> $p_{\{\nu_m \restriction (\alpha-1) : m < n\}} \leq p^0_{\{\nu_m : m < n\}}$.

(e) If $\nu \in {}^{\alpha}2$ and $\nu(\alpha - 1) = \ell$ <u>then</u> $p^0_{\{\nu\}}(0) = F_\alpha(\nu \restriction (\alpha - 1))\char`^\langle \ell\rangle$.

There is no problem to carry the induction.

Now $F_{\alpha+1} \restriction {}^{\alpha}2$ is to be defined as in the second case, starting with $\eta \to p^{i^*}_{\{\eta\}}(\eta)$.

For $\alpha = 0, 1$: Left to the reader.

So we have finished the induction hence the proof of 4.1(1), (2).

3) Left to the reader (the only influence is the choice of h in stage of the induction). $\square_{4.1}$

§ 5. Somewhat complementary negative partition relation in ZFC

The negative results here suffice to show that the value we have for $2^{\aleph_0}$ in §3 is reasonable. In particular, the Galvin conjecture is wrong and that for every $n < \omega$, for some $m < \omega$, $\aleph_n \nrightarrow [\aleph_1]^m_{\aleph_0}$.

See Erdős-Hajnal-Máté-Rado [EHMR84] for

Fact 5.1. If $2^{<\mu} < \lambda \le 2^\mu$ and $\mu \nrightarrow [\mu]^n_\sigma$ then $\lambda \nrightarrow [(2^{<\mu})^+]^{n+1}_\sigma$.

This shows that if e.g. in 1.4 we want to increase the exponents to 3 (and still $\mu = \mu^{<\mu}$) then μ cannot be successor (when $\sigma \le \aleph_0$; by [She88, 3.5(2)]).

Definition 5.2. $\mathrm{Pr}_{\mathrm{np}}(\lambda, \mu, \bar\sigma)$ (where $\bar\sigma = \langle \sigma_n : n < \omega\rangle$) means that there are functions $F_n : [\lambda]^n \to \sigma_n$ such that for every $W \in [\lambda]^\mu$, for some n, $F''_n([W]^n) = \sigma(n)$. The negation of this property is denoted by $\mathrm{NPr}_{\mathrm{np}}(\lambda, \mu, \bar\sigma)$.

If the sequence in constantly σ we may write σ instead of $\langle \sigma_n : n < \omega\rangle$.

Remark 5.3. 1) Note that $\lambda \to [\mu]^{<\omega}_\sigma$ means "if $F : [\lambda]^{<\omega} \to \sigma$ then for some $A \in [\lambda]^\mu$, $F''([A]^{<\omega}) \ne \sigma$." So for $\lambda \ge \mu \ge \sigma = \aleph_0$, we have $\lambda \nrightarrow [\mu]^{<\omega}_\sigma$ (use $\alpha \mapsto |\alpha|$ for F), and $\mathrm{Pr}_{\mathrm{np}}(\lambda, \mu, \sigma)$ is stronger than $\lambda \nrightarrow [\mu]^{<\omega}_\sigma$.
2) We do not write down the monotonicity properties of $\mathrm{Pr}_{\mathrm{np}}$: they are obvious.

Claim 5.4. *1) Without loss of generality we can (in 5.2) use $F_{n,m} : [\lambda]^n \to \sigma_m$ for $n, m < \omega$ and obvious monotonicity properties holds, and $\lambda \ge \mu \ge n$.*
2) Suppose $\mathrm{NPr}_{\mathrm{np}}(\lambda, \mu, \kappa)$ and $\kappa \nrightarrow [\kappa]^n_\sigma$, or even $\kappa \nrightarrow [\kappa]^{<\omega}_\sigma$. Then the following case of the Chang conjecture holds:

- $(*)$ *For every model M with universe λ and countable vocabulary, there is an elementary submodel N of M of cardinality μ with*

3) If $\mathrm{NPr}_{\mathrm{np}}(\lambda, \aleph_1, \aleph_0)$ then $(\lambda, \aleph_1) \to (\aleph_1, \aleph_0)$.

Proof. Easy. $\square_{5.4}$

Theorem 5.5. *Suppose $\mathrm{Pr}_{\mathrm{np}}(\lambda_0, \mu, \aleph_0)$, μ is regular $> \aleph_0$ and $\lambda_1 \ge \lambda_0$, and no $\mu' \in (\lambda_0, \lambda_1)$ is μ'-Mahlo. Then $\mathrm{Pr}_{\mathrm{np}}(\lambda_1, \mu, \aleph_0)$.*

Proof. Let $\chi = \beth_8(\lambda_1)^+$, let $\{F^0_{n,m} : m < \omega\}$ list the definable n-place functions in the model $(\mathcal{H}(\chi), \in, <^*_\chi)$ with $\lambda_0, \mu, \lambda_1$ as parameters, let $F^1_{n,m}(\alpha_0, \dots, \alpha_{n-1})$ (for $\alpha_0, \dots, \alpha_{n-1} < \lambda_1$) be equal to $F^0_{n,m}(\alpha_0, \dots, \alpha_{n-1})$ if it is an ordinal $< \lambda_1$ and zero otherwise. Let $F_{n,m}(\alpha_0, \dots, \alpha_{n-1})$ (for $\alpha_0, \dots, \alpha_{n-1} < \lambda_1$) be $F^0_{n,m}(\alpha_0, \dots, \alpha_{n-1})$ if it is an ordinal $< \omega$ and zero otherwise. We shall show that the $F_{n,m}$ (for $n, m < \omega$) exemplify $\mathrm{Pr}_{\mathrm{np}}(\lambda_1, \mu, \aleph_0)$ (see 5.3(1)).

So suppose $W \in [\lambda_1]^\mu$ is a counterexample to $\mathrm{Pr}(\lambda_1, \mu, \aleph_0)$: i.e. for no n, m is $F''_{n,m}([W]^n) = \omega$. Let W^* be the closure of W under $\{F^1_{n,m} : n, m < \omega\}$. Let N be the Skolem Hull of W in $(\mathcal{H}(\chi), \in, <^*_\chi)$, so clearly $N \cap \lambda_1 = W^*$. (Note $W^* \subseteq \lambda_1$ and$|W^*| = \mu$.) Also, as $\mathrm{cf}(\mu) > \aleph_0$, if $A \subseteq W^*$ with $|A| = \mu$ then for some $n, m < \omega$ and $u_i \in [W]^n$ (for $i < \mu$) we have $F^1_{n,m}(u_i) \in A$ and

$$i < j < \mu \Rightarrow F^1_{n,m}(u_i) \ne F^1_{n,m}(u_j).$$

It is easy to check that also $W^1 := \{F^1_{n,m}(u_i) : i < \mu\}$ is a counterexample to $\mathrm{Pr}(\lambda_1, \mu, \sigma)$. In particular, for $n, m < \omega$, $W_{n,m} = \{F^1_{n,m}(u) : u \in [W]^n\}$ is a counterexample if it has power μ. Without loss of generality W is a counterexample with minimal $\delta := \sup(W) = \bigcup\{\alpha + 1 : \alpha \in W\}$. The above discussion shows that $|W^* \cap \alpha| < \mu$ for $\alpha < \delta$. Obviously $\mathrm{cf}(\delta) = \mu^+$. Let $\langle \alpha_i : i < \mu\rangle$ be a strictly increasing sequence of members of W^*, converging to δ, such that for limit i we have $\alpha_i = \min\big(W^* \setminus \bigcup\limits_{j<i} (\alpha_j + 1)\big)$. Let $N = \bigcup\limits_{i<\mu} N_i$ where $N_i \prec N$, $|N_i| < \mu$, N_i increasing continuous, and without loss of generality $N_i \cap \delta = N \cap \alpha_i$.

Fact (α): $\delta > \lambda_0$.

Proof. Otherwise we then get an easy contradiction to $\Pr(\lambda_0, \mu, \sigma)$, as when choosing the $F^0_{n,m}$ we allowed λ_0 as a parameter. $\square_\alpha$

Fact (β): If F is a unary function definable in N, $F(\alpha)$ is a club of α for every limit ordinal α ($< \lambda_1$) then for some club C of μ we have

$$\big(\forall j \in C \setminus \{\min C\}\big)\big(\exists i_1 < j\big)\big(\forall i \in (i_1, j)\big)\big[i \in C \Rightarrow \alpha_i \in F(\alpha_j)\big].$$

Proof. For some club C_0 of μ we have

$$j \in C_0 \Rightarrow \big(N_j, \{\alpha_i : i < j\}, W\big) \prec \big(N, \{\alpha_i : i < \mu\}, W\big).$$

We let $C = C_0' = \mathrm{acc}(C)$ (= set of accumulation points of C_0).

We check C is as required; suppose j is a counterexample. So $j = \sup(j \cap C)$ (otherwise choose $i_1 = \max(j \cap C)$). So we can define, by induction on n, a sequence of i_n such that:

(a) $i_n < i_{n+1} < j$
(b) $\alpha_{i_n} \notin F(\alpha_j)$
(c) $(\alpha_{i_n}, \alpha_{i_{n+1}}) \cap F(\alpha_j) \neq \varnothing$.

Why (C_0')? $\models$ "$F(\alpha_j)$ is unbounded below α_j" hence $N \models$ "$F(\alpha_j)$ is unbounded below α_j", but in N, $\{\alpha_i : i \in C_0,\ i < j\}$ is unbounded below α_j.

Clearly, for some n, m we have $\alpha_j \in W_{n,m}$ (see above). Now we can repeat the proof of [She88, 3.3(2)][4] using only members of $W_{n,m}$.
Note: here we set the number of colors to be $\aleph_0$. $\square_\beta$

Fact $(\beta)^+$: Without loss of generality, the club C in Fact (β) is μ.
Proof: By renaming.

Fact (γ): δ is a limit cardinal.

Proof. Suppose not. Now δ cannot be a successor cardinal (as $\mathrm{cf}(\delta) = \mu \leq \lambda_0 < \delta$) hence for every large enough i, $|\alpha_i| = |\delta|$, so $|\delta| \in W^* \subseteq N$ and $|\delta|^+ \in W^*$.

So $W^* \cap |\delta|$ has cardinality $< \mu$ hence order-type equal to some $\gamma^* < \mu$. Choose $i^* < \mu$ limit such that $[j < i^* \Rightarrow j + \gamma^* < i^*]$. There is a definable function F of $(\mathcal{H}(\chi), \in, <^*_\chi)$ such that for every limit ordinal α, $F(\alpha)$ is a club of α, such that if $|\alpha| < \alpha$ then $F(\alpha) \cap |\alpha| = \varnothing$ and $\mathrm{otp}(F(\alpha)) = \mathrm{cf}(\alpha)$.

So in N there is a closed unbounded subset $C_{\alpha_j} = F(\alpha_j)$ of α_j of order type $\leq \mathrm{cf}(\alpha_j) \leq |\delta|$, hence $C_{\alpha_j} \cap N$ has order type $\leq \gamma^*$, hence for i^* chosen above unboundedly many $i < i^*$, $\alpha_i \notin C_{\alpha_{i^*}}$. We can finish by Fact $(\beta)^+$. $\square_\gamma$

Fact (δ): For each $i < \mu$, α_i is a cardinal.

Proof. If $|\alpha_i| < i$ then $|\alpha_i| \in N_i$, but then $|\alpha_i|^+ \in N_i$ contradicting Fact (γ), by which $|\alpha_i|^+ < \delta$, as we have assumed $N_i \cap \delta = N \cap \alpha_i$. $\square_\delta$

Fact (ε): For a club of $i < \mu$, α_i is a regular cardinal.

Proof. If $S = \{i : \alpha_i \text{ singular}\}$ is stationary, then the function $\alpha_i \mapsto \mathrm{cf}(\alpha_i)$ is regressive on S. By Fodor's lemma, for some $\alpha^* < \delta$, $\{i < \mu : \mathrm{cf}(\alpha_i) < \alpha^*\}$ is stationary. As $|N \cap \alpha^*| < \mu$ for some β^*, $\{i < \mu : \mathrm{cf}(\alpha_i) = \beta^*\}$ is stationary. Let $F_{1,m}(\alpha)$ be a club of α of order type $\mathrm{cf}(\alpha)$, and by Fact (β) we get a contradiction as in Fact (γ). $\square_\varepsilon$

Fact (ζ): For a club of $i < \mu$, α_i is Mahlo.

[4] See mainly the end.

Proof. Use $F_{1,m}(\alpha) =$ a club of α which, if α is a successor cardinal or inaccessible not Mahlo, then it contains no inaccessible, and continue as in Fact (γ). $\square_\zeta$

Fact (ξ): For a club of $i < \mu$, α_i is α_i-Mahlo.

Proof. Let $F_{1,m(0)}(\alpha) = \sup\{\zeta : \alpha \text{ is } \zeta\text{-Mahlo}\}$. If the set $\{i < \mu : \alpha_i \text{ is not } \alpha_i\text{-Mahlo}\}$ is stationary then as before, for some $\gamma \in N$ we have $\{i : F_{1,m(0)}(\alpha_i) = \gamma\}$ is stationary. Let $F_{1,m(1)}(\alpha)$ — a club of α such that if α is not $(\gamma+1)$-Mahlo then the club has no γ-Mahlo member. Finish as in the proof of Fact (δ). $\square_\xi$

Together we are done. $\square_{5.5}$

Remark 5.6. We can continue, and say more.

Lemma 5.7. 1) *Suppose $\lambda > \mu > \theta$ are regular cardinals, $n \geq 2$, and*

(*i*) *For every regular cardinal κ, if $\lambda > \kappa \geq \theta$ then $\kappa \not\to [\theta]^{<\omega}_{\sigma(1)}$.*

(*ii*) *For some $\alpha(*) < \mu$, for every regular $\kappa \in (\alpha(*), \lambda)$, $\kappa \not\to [\alpha(*)]^n_{\sigma(2)}$.*

<u>*Then*</u>

(*a*) *$\lambda \not\to [\mu]^{n+1}_\sigma$, where $\sigma = \min\{\sigma(1), \sigma(2)\}$.*

(*b*) *There are functions $d_2 : [\lambda]^{n+1} \to \sigma(2)$ and $d_1 : [\lambda]^3 \to \sigma(1)$ such that for every $W \in [\lambda]^\mu$ we have $d_1''([W]^3) = \sigma(1)$ or $d_2''([W]^{n+1}) = \sigma(2)$.*

2) *Suppose $\lambda > \mu > \theta$ are regular cardinals, and*

(*i*) *For every regular $\kappa \in [\theta, \lambda)$ we have $\kappa \not\to [\theta]^{<\omega}_{\sigma(1)}$.*

(*ii*) *$\sup\{\kappa < \lambda : \kappa \text{ regular}\} \not\to [\mu]^n_{\sigma(2)}$.*

Then

(*a*) *$\lambda \not\to [\mu]^{2n}_\sigma$, where $\sigma = \min\{\sigma(1), \sigma(2)\}$.*

(*b*) *There are functions $d_1 : [\lambda]^3 \to \sigma(1)$, $d_2 : [\lambda]^{2n} \to \sigma(2)$ such that for every $W \in [\lambda]^\mu$ we have $d_1''([W]^3) = \sigma(1)$ or $d_2''([W]^{2n}) = \sigma(2)$.*

The proof is similar to that of [She88, 3.3,3.2].

Proof. 1) For each i, $0 < i < \lambda_i$, we choose C_i such that if i is a successor ordinal then $C_i = \{i-1, 0\}$, and if i is a limit ordinal then C_i is a club of i of order type $\mathrm{cf}(i)$ containing 0 such that $[\mathrm{cf}(i) < i \Rightarrow \mathrm{cf}(i) < \min(C_i \setminus \{0\})]$ and $C_i \setminus \mathrm{acc}(C_i)$ contains only successor ordinals.

Now for $\alpha < \beta$, $\alpha > 0$ we define $\gamma^+_\ell(\beta,\alpha)$, $\gamma^-_\ell(\beta,\alpha)$ by induction on ℓ, and then $\kappa(\beta,\alpha)$, $\varepsilon(\beta,\alpha)$.

(A) $\gamma^+_0(\beta,\alpha) = \beta$, $\gamma^-_0(\beta,\alpha) = 0$.

(B) If $\gamma^+_\ell(\beta,\alpha)$ is defined and $> \alpha$ and α is not an accumulation point of $C_{\gamma^+_\ell(\beta,\alpha)}$ then we let $\gamma^-_{\ell+1}(\beta,\alpha)$ be the maximal member of $C_{\gamma^+_\ell(\beta,\alpha)}$ which is $< \alpha$ and $\gamma^+_{\ell+1}(\beta,\alpha)$ is the minimal member of $C_{\gamma^+_\ell(\beta,\alpha)}$ which is $\geq \alpha$ (by the choice of $C_{\gamma^+_\ell(\beta,\alpha)}$ and the demands on $\gamma^+_\ell(\beta,\alpha)$ they are well defined).

So

(B1) (a) $\gamma^-_\ell(\beta,\alpha) < \alpha \leq \gamma^+_\ell(\beta,\alpha)$, and if the equality holds then $\gamma^+_{\ell+1}(\beta,\alpha)$ is not defined.

(b) $\gamma^+_{\ell+1}(\beta,\alpha) < \gamma^+_\ell(\beta,\alpha)$ when both are defined.

(C) Let $k = k(\beta,\alpha)$ be the maximal number k such that $\gamma^+_k(\beta,\alpha)$ is defined (it is well defined as $\langle \gamma^+_\ell(\beta,\alpha) : \ell < \omega\rangle$ is strictly decreasing). So

(C1) $\gamma^+_{k(\beta,\alpha)}(\beta,\alpha) = \alpha$ or $\gamma^+_{k(\beta,\alpha)} > \alpha$, $\gamma^+_{k(\beta,\alpha)}$ is a limit ordinal and α is an accumulation point of $C_{\gamma^+_{k(\beta,\alpha)}}(\beta,\alpha)$.

(D) For $m \le k(\beta,\alpha)$ let us define
$$\varepsilon_m(\beta,\alpha) = \max\{\gamma_\ell^-(\beta,\alpha)+1 : \ell \le m\}.$$

Note

(D1) (a) $\varepsilon_m(\beta,\alpha) \le \alpha$ (if defined).

(b) If α is limit then $\varepsilon_m(\beta,\alpha) < \alpha$ (if defined).

(c) If $\varepsilon_m(\beta,\alpha) \le \xi \le \alpha$ then for every $\ell \le m$ we have
$$\gamma_\ell^+(\beta,\alpha) = \gamma_\ell^+(\beta,\xi), \quad \gamma_\ell^-(\beta,\alpha) = \gamma_\ell^-(\beta,\xi), \quad \varepsilon_\ell(\beta,\alpha) = \varepsilon_\ell(\beta,\xi).$$
(Explanation for (c): if $\varepsilon_m(\beta,\alpha) < \alpha$ this is easy (check the definition) and if $\varepsilon_m(\beta,\alpha) = \alpha$, necessarily $\xi = \alpha$ and it is trivial.)

(d) If $\ell \le m$ then $\varepsilon_\ell(\beta,\alpha) \le \varepsilon_m(\beta,\alpha)$.

For a regular $\kappa \in (\alpha(*),\lambda)$ let $g^1_\kappa : [\kappa]^{<\omega} \to \sigma(2)$ exemplify $\kappa \not\to [\theta]^{<\omega}_{\sigma(1)}$, and for every regular cardinal $\kappa \in [\theta,\lambda)$ let $g^2_\kappa : [\kappa]^n \to \sigma(2)$ exemplify $\kappa \not\to [\alpha(*)]^n_{\sigma(2)}$.

Let us define the colourings:

Let $\alpha_0 > \alpha_1 > \ldots > \alpha_n$. (Remember $n \ge 2$.)

Let $n = n(\alpha_0,\alpha_1,\alpha_2)$ be the maximal natural number such that:

(i) $\varepsilon_n(\alpha_0,\alpha_1) < \alpha_0$ is well defined.

(ii) $\gamma_\ell^-(\alpha_0,\alpha_1) = \gamma_\ell^-(\alpha_0,\alpha_2)$ for $\ell \le n$.

We define $d_2(\alpha_0,\alpha_1,\ldots,\alpha_n)$ as $g^2_\kappa(\beta_1,\ldots,\beta_n)$, where
$$\kappa = \mathrm{cf}\big(\gamma^+_{n(\alpha_0,\alpha_1,\alpha_2)}(\alpha_0,\alpha_1)\big),$$
$$\beta_\ell = \mathrm{otp}\big(\alpha_\ell \cap C_{\gamma^+_{n(\alpha_0,\alpha_1,\alpha_2)}(\alpha_0,\alpha_1)}\big).$$

Next we define $d_1(\alpha_0,\alpha_1,\alpha_2)$.

Let $i(*) = \sup\big(C_{\gamma_n^+(\alpha_0,\alpha_2)} \cap C_{\gamma_n^+(\alpha_1,\alpha_2)}\big)$, where $n = n(\alpha_0,\alpha_1,\alpha_2)$. Let E be the equivalence relation on $C_{\gamma_n^+(\alpha_0,\alpha_1)} \setminus i(*)$ defined by
$$\gamma_1 \mathrel{E} \gamma_2 \Leftrightarrow \big(\forall \gamma \in C_{\gamma_n^+(\alpha_0,\alpha_2)}\big)[\gamma_1 < \gamma \Leftrightarrow \gamma_2 < \gamma].$$

If the set $w = \big\{\gamma \in C_{\gamma_n^+(\alpha_0,\alpha_1)} : \gamma > i(*),\ \gamma = \min \gamma/E\big\}$ is finite, we let $d_1(\alpha_0,\alpha_1,\alpha_2)$ be $g^1_\kappa\big(\{\beta_\gamma : \gamma \in w\}\big)$, where $\kappa = \big|C_{\gamma_n^+(\alpha_0,\alpha_1)}\big|$ and
$$\beta_\gamma = \mathrm{otp}\big(\gamma \cap C_{\gamma_n^+(\alpha_0,\alpha_1)}\big).$$

We have defined d_1, d_2 required in condition (b) (though have not yet proved that they work) We still have to define d (exemplifying $\lambda \not\to [\mu]^{n+1}_\ell$). Let $n \ge 3$: for $\alpha_0 > \alpha_1 > \ldots > \alpha_n$, we let $d(\alpha_0,\ldots,\alpha_n)$ be $d_1(\alpha_0,\alpha_1,\alpha_2)$ if w defined during the definition has odd number of members and $d_2(\alpha_0,\ldots,\alpha_n)$ otherwise.

Now suppose Y is a subset of λ of order type μ, and let $\delta = \sup Y$. Let M be a model with universe λ and with relations Y and $\{(i,j) : i \in C_j\}$. Let $\langle N_i : i < \mu\rangle$ be an increasing continuous sequence of elementary submodels of M of cardinality $< \mu$ such that $\alpha(i) = \alpha_i = \min(Y \setminus N_i)$ belongs to N_{i+1}, $\sup(N \cap \alpha_i) = \sup(N \cap \delta)$. Let $N = \bigcup_{i<\mu} N_i$. Let $\delta(i) = \delta_i = \sup(N_i \cap \alpha_i)$, so $0 < \delta_i \le \alpha_i$, and let $n = n_i$ be the first natural number such that δ_i an accumulation point of $C^i = C_{\gamma_n^+(\alpha_i,\delta(i))}$, let $\varepsilon_i = \varepsilon_{n(i)}(\alpha_i,\delta_i)$. Note that $\gamma_n^+(\alpha_i,\delta_i) = \gamma_n^+(\alpha_i,\varepsilon_i)$ hence it belongs to N.

Case I: For some (limit) $i < \mu$, $\mathrm{cf}(i) \ge \theta$ and $(\forall \gamma < i)[\gamma + \alpha(*) < i]$ such that for arbitrarily large $j < i$, $C^i \cap N_j$ is bounded in $N_j \cap \delta = N_j \cap \delta_j$.

This is just like the last part in the proof of [She88, 3.3], using g^1_κ and d_1 for $\kappa = \mathrm{cf}(\gamma^+_{n_i}(\alpha_i,\delta_i))$.

Case II: Not case I.

Let $S_0 = \{i < \mu : (\forall \alpha < i)[\gamma + \alpha(*) < i], \operatorname{cf}(i) = \theta\}$. So for every $i \in S_0$, for some $j(i) < i$,

$$(\forall j)\big[j \in (j(i), i) \Rightarrow C^i \cap N_j \text{ is unbounded in } \delta_j\big].$$

But as $C^i \cap \delta_i$ is a club of δ_i, clearly $(\forall j)\big[j \in (j(i), i) \Rightarrow \delta_j \in C^i\big]$.

We can also demand $j(i) > \varepsilon_{n(\alpha(i),\delta(i))}(\alpha(i), \delta(i))$.

As S_0 is stationary, by 'not case I,' for some stationary $S_1 \subseteq S_0$ and $n(*)$, $j(*)$ we have $(\forall i \in S_1)\big[j(i) = j(*) \wedge n(\alpha(i), \delta_i) = n(*)\big]$.

Choose $i(*) \in S_1$, $i(*) = \sup(i(*) \cap S_1)$, such that the order type of $S_1 \cap i(*)$ is $i(*) > \alpha(*)$. Now if $i_2 < i_1 \in S_1 \cap i(*)$ then $n(\alpha_{i(*)}, \alpha_{i_1}, \alpha_{i_2}) = n(*)$. Now $L_{i(*)} = \big\{\operatorname{otp}(\alpha_i \cap C^{i(*)}) : i \in S_1 \cap i(*)\big\}$ are pairwise distinct and are ordinals $< \kappa = |C^{i(*)}|$, and the set has order type $\alpha(*)$. Now apply the definitions of d_2 and g^2_κ on $L_{i(*)}$. 2) The proof is like the proof of part (1),but for $\alpha_0 > \alpha_1 > \cdots$ we let $d_2(\alpha_0, \ldots, \alpha_{2n-1}) = g^2_\kappa(\beta_0, \ldots, \beta_n)$, where

$$\beta_\ell = \operatorname{otp}\big(C_{\gamma^+_n(\beta_{2\ell},\beta_{2\ell+1})}(\beta_{2\ell}, \beta_{2\ell+1}) \cap \beta_{2\ell+1}\big)$$

and in case II note that the analysis gives μ possible β_ℓ-s so that we can apply the definition of g^2_κ. $\square_{5.7}$

Definition 5.8. Let $\lambda \not\to_{\mathsf{stg}} [\mu]^n_\theta$ mean: if $d : [\lambda]^n \to \theta$, $\langle \alpha_i : i < \mu\rangle$ is strictly increasing continuous, and for $i < j < \mu$, $\gamma_{i,j} \in [\alpha_i, \alpha_{i+1})$ then

$$\theta = \big\{d(w) : \text{for some } j < \mu,\ w \in \big[\{\gamma_{i,j} : i < j\}\big]^n\big\}.$$

Lemma 5.9. *1)* $\aleph_t \not\to [\aleph_1]^{n+1}_{\aleph_0}$ *for* $n \geq 1$.

2) $\aleph_n \not\to_{\mathsf{stg}} [\aleph_1]^{n+1}_{\aleph_0}$ *for* $n \geq 1$.

Proof. 1) For $n = 2$ this is a theorem of Todorčevič [Tod87], and if it holds for $n \geq 2$ by 5.7(1) we get that it holds for n+1 (with n, λ, μ, θ, $\alpha(*)$, $\sigma(1)$, $\sigma(2)$ there corresponding to $n+1$, $\aleph_{n+1}$, $\aleph_1$, $\aleph_0$, $\aleph_0$, $\aleph_0$, $\aleph_0$ here).

2) Similar. $\square_{5.9}$

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