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On the bounding, splitting, and distributivity numbers

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Abstract. The cardinal invariants $\mathfrak{h}, \mathfrak{b}, \mathfrak{s}$ of $\mathcal{P}(\omega)$ are known to satisfy that $\omega_1 \leq \mathfrak{h} \leq \min\{\mathfrak{b}, \mathfrak{s}\}$. We prove that all inequalities can be strict. We also introduce a new upper bound for $\mathfrak{h}$ and show that it can be less than $\mathfrak{s}$. The key method is to utilize finite support matrix iterations of ccc posets following paper Ultrafilters with small generating sets by A. Blass and S. Shelah (1989).

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1. Introduction

Of course the cardinal invariants of the continuum discussed in this article are very well known, see [15, page 111], so we just give a brief reminder. They deal with the mod finite ordering of the infinite subsets of the integers. We follow convention and let $[\omega]^\omega$ (or $[\omega]^{\aleph_0}$) denote the family of infinite subsets of ω. A set A is a pseudo-intersection of a family $\mathcal{Y} \subset [\omega]^\omega$ if A is infinite and $A \setminus Y$ is finite for all $Y \in \mathcal{Y}$. The family $\mathcal{Y}$ has the strong finite intersection property (sfip) if every finite subset has infinite intersection and $\mathfrak{p}$ is the minimum cardinal for which there is such a family with no pseudointersection. A family $\mathcal{I} \subset \mathcal{P}(\omega)$ is an ideal if it is closed under finite unions and mod finite subsets. An ideal $\mathcal{I} \subset \mathcal{P}(\omega)$ is dense if every $Y \in [\omega]^\omega$ contains an infinite member of $\mathcal{I}$. A set $S \subset \omega$ is *unsplit* by a family $\mathcal{Y} \subset [\omega]^\omega$ if S is mod finite contained in one member of $\{Y, \omega \setminus Y\}$ for each $Y \in \mathcal{Y}$. The splitting number $\mathfrak{s}$ is the minimum cardinal of a family $\mathcal{Y}$ for which there is no infinite set unsplit by $\mathcal{Y}$ (i.e. every $S \in [\omega]^\omega$ is *split* by some member of $\mathcal{Y}$ and $\mathcal{Y}$ is called a splitting family). The bounding number $\mathfrak{b}$ can easily be defined in these same terms, but it is best defined by the mod finite ordering "$<^*$" on the family of functions ω^ω. The cardinal $\mathfrak{b}$ is the minimum cardinal for which there is a $<^*$-unbounded family $B \subset \omega^\omega$ with $|B| = \mathfrak{b}$.

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The finite support iteration of the standard Hechler poset was shown in [2] to produce models of $\aleph_1 = \mathfrak{s} < \mathfrak{b}$. The consistency of $\aleph_1 = \mathfrak{b} < \mathfrak{s} = \aleph_2$ was established in [17] with a countable support iteration of a special poset we now call $\mathcal{Q}_{\text{Bould}}$. It is shown in [11] that one can use Cohen forcing to select countable chain condition (ccc) subposets of $\mathcal{Q}_{\text{Bould}}$ and finite support iterations to obtain models of $\aleph_1 < \mathfrak{b} < \mathfrak{s} = \mathfrak{b}^+$. This result was improved in [5] to show that the gap between $\mathfrak{b}$ and $\mathfrak{s}$ can be made arbitrarily large. The papers [4], [5] and [6] are able to use ccc versions of the well-known Mathias forcing in their iterations in place of those discovered in [11]. The paper [5] also nicely expands on the method of matrix iterated forcing first introduced in [4], as do a number of more recent papers, see [9], [16] and [10] using template forcing. The distributivity number (degree) $\mathfrak{h}$ was first studied in [1]. It equals the minimum number of dense ideals whose intersection is simply the Fréchet ideal $[\omega]^{<\omega}$. It was shown in [1], that $\mathfrak{p} \leq \mathfrak{h} \leq \min\{\mathfrak{b}, \mathfrak{s}\}$. Our goal is to separate all these cardinals. We succeed but confront a new problem since we use the result, also from [1], that $\mathfrak{h} \leq \operatorname{cof}(\mathfrak{c})$.

2. A new bound on $\mathfrak{h}$

In [1], a family $\mathfrak{A}$ of maximal almost disjoint families of infinite subsets of ω is called a matrix. A matrix $\mathfrak{A}$ is *shattering* if the entire collection $\bigcup \mathfrak{A}$ is splitting. Evidently, if $\{s_\alpha \colon \alpha < \kappa\}$ is a splitting family, then the family $\mathfrak{A} = \{\{s_\alpha, \omega \setminus s_\alpha\} \colon \alpha < \kappa\}$ is a shattering matrix. A shattering matrix $\mathfrak{A} = \{\mathcal{A}_\alpha \colon \alpha < \kappa\}$ is *refining*, if for all $\alpha < \beta < \kappa$, $\mathcal{A}_\beta$ refines $\mathcal{A}_\alpha$ in the natural sense that each member of $\mathcal{A}_\beta$ is mod finite contained in some member of $\mathcal{A}_\alpha$. Finally, a *base matrix* is a refining shattering matrix $\mathfrak{A}$ satisfying that $\bigcup \mathfrak{A}$ is dense in $(\mathcal{P}(\omega)/\operatorname{fin}, \subset^*)$ (i.e. a π-base for ω^*).

We add condition (6) to the following result from [1].

Lemma 2.1. *The value of $\mathfrak{h}$ is the least cardinal κ such that any of the following holds:*

(1) *the Boolean algebra $\mathcal{P}(\omega)/\operatorname{fin}$ is not κ-distributive;*
(2) *there is a shattering matrix of cardinality κ;*
(3) *there is a shattering and refining matrix indexed by κ;*
(4) *there is a base matrix of cardinality κ;*
(5) *there is a family of κ many nowhere dense subsets of ω^* whose union is dense;*
(6) *there is a sequence $\{\mathcal{S}_\alpha \colon \alpha < \kappa\}$ of splitting families satisfying that no 1-to-1 selection $\langle s_\alpha \colon \alpha \in \kappa\rangle \in \Pi\{\mathcal{S}_\alpha \colon \alpha \in \kappa\}$ has a pseudo-intersection.*

Proof: Since (1)–(5) are proven in [1], it is sufficient to prove that for a cardinal κ (3) and (6) are equivalent. First suppose that $\mathfrak{A} = \{\mathcal{A}_\alpha \colon \alpha < \kappa\}$ is a refining and shattering matrix. Since the matrix is refining, it follows easily that $\{\mathcal{A}_\beta \colon \alpha \leq \beta < \kappa\}$ is a shattering matrix for each $\alpha < \kappa$. Therefore, $\mathcal{S}_\alpha = \bigcup\{\mathcal{A}_\beta \colon \alpha \leq \beta\}$ is a splitting family for each $\alpha < \kappa$. Similarly, the refining property ensures that if $\langle a_\alpha \colon \alpha \in \kappa\rangle \in \Pi\{\mathcal{S}_\alpha \colon \alpha \in \kappa\}$, then $\{a_\alpha \colon \alpha \in \kappa\}$ has no pseudo-intersection.

Now assume that $\{\mathcal{S}_\alpha \colon \alpha < \kappa\}$ is a sequence of splitting families as in (6). By [1], it is sufficient to prove that $\mathfrak{h} \leq \kappa$, so let us assume that $\kappa < \mathfrak{h}$. We now make an observation about κ: for each infinite $b \subset \omega$, $\alpha < \kappa$ and family $\mathcal{S}' \subset [\omega]^\omega$ of cardinality less than κ, there is an infinite $a \subset b$ and an $s \in \mathcal{S}_\alpha \setminus \mathcal{S}'$ such that $a \subset s$ and s splits b. We prove this claim. We may ignore all members of $\mathcal{S}'$ that are mod finite disjoint, or mod finite include, b. Since the family $\{\{s' \cap b, b \setminus s'\} \colon s' \in \mathcal{S}'\}$ is not shattering (as a family of subsets of b) there is an infinite $b' \subset b$ that is not split by $\mathcal{S}'$. Choose any $s \in \mathcal{S}_\alpha$ that splits b' and let $a = s \cap b'$. Evidently, s also splits b. Since the ideal generated by a splitting family is dense, we may choose a maximal almost disjoint family $\mathcal{A}_0$ contained in the ideal generated by $\mathcal{S}_0$. Let s_0 denote any mapping from $\mathcal{A}_0$ into $\mathcal{S}_0$ satisfying that $a \subset s_0(a)$ for all $a \in \mathcal{A}_0$. Suppose that $\alpha < \kappa$ and that we have chosen a refining sequence $\{\mathcal{A}_\gamma \colon \gamma < \alpha\}$ of maximal almost disjoint families together with mappings $\{s_\gamma \colon \gamma < \alpha\}$ so that for each $a \in \mathcal{A}_\gamma$, $a \subset s_\gamma(a) \in \mathcal{S}_\gamma$. The extra induction assumption is that for all $a \in \mathcal{A}_\gamma$, $s_\gamma(a)$ is not an element of $\{s_\beta(a') \colon \beta < \gamma$ and $a \subset^* a' \in \mathcal{A}_\beta\}$. The existence of the family $\mathcal{A}_\alpha$ and the mapping s_α satisfying the induction conditions easily follows from the above observation. Now we verify that $\mathfrak{A} = \{\mathcal{A}_\alpha \colon \alpha < \kappa\}$ satisfies that $\bigcup\mathfrak{A}$ is splitting. Fix any infinite $b \subset \omega$ and choose $a_\alpha \in \mathcal{A}_\alpha$ for each $\alpha \in \kappa$ so that $b \cap a_\alpha$ is infinite. By construction, $\{s_\alpha(a_\alpha) \colon \alpha \in \kappa\}$ is a 1-to-1 selection from $\Pi\{\mathcal{S}_\alpha \colon \alpha \in \kappa\}$. Since b is therefore not a pseudo-intersection, there is an $\alpha < \kappa$ such that $b \setminus s_\alpha(a_\alpha) \subset b \setminus a_\alpha$ is infinite. □

The following is an immediate corollary to condition (6) in Lemma 2.1 and provide two approaches to bounding the value of $\mathfrak{h}$.

Corollary 2.2 ([1], [3]). (1) *If $\mathfrak{c}$ is singular, then $\mathfrak{h} \leq \mathrm{cf}(\mathfrak{c})$.*

(2) *A poset $\mathbb{P}$ forces that $\mathfrak{h} \leq \kappa$ if $\mathbb{P}$ preserves κ and can be written as an increasing chain $\{\mathbb{P}_\alpha \colon \alpha < \kappa\}$ of completely embedded posets satisfying that each $\mathbb{P}_{\alpha+1}$ adds a real not added by $\mathbb{P}_\alpha$.*

Proof: For the statement in (1), let $\{\kappa_\alpha \colon \alpha < \mathrm{cf}(\mathfrak{c})\}$ be increasing and cofinal in $\mathfrak{c}$. Let $\{x_\xi \colon \xi \in \mathfrak{c}\}$ be an enumeration of $[\omega]^{\aleph_0}$. To apply (6) from Lemma 2.1, let $\mathcal{S}_\alpha = \{x_\xi \colon (\forall\, \eta < \kappa_\alpha)\, x_\eta \not\subset^* x_\xi\}$. For the statement in (2), let G be a $\mathbb{P}$-generic filter and for each $\alpha \in \kappa$, let $G_\alpha = G \cap \mathbb{P}_\alpha$. To apply (6), let $\mathcal{S}_\alpha$ be the set of $x \in [\omega]^{\aleph_0}$ that contain no infinite $y \in V[G_\alpha]$. To see that $\mathcal{S}_\alpha$ is splitting in either case, given any infinite $x \subset \omega$, consider an enumeration $\{x_t\colon t \in 2^{<\omega}\}$. Then, for all $\alpha \in \kappa$, there is an $f_\alpha \in 2^\omega$ so that $\{x_{f_\alpha \restriction n}\colon n \in \omega\} \in \mathcal{S}_\alpha$. □

Our introduction of condition (6) in Lemma 2.1 is motivated by the fact that it provides us with a new approach to bounding $\mathfrak{h}$. We introduce the following variant of condition (6) in Lemma 2.1 and note that a shattering refining matrix will fail to satisfy the second condition.

Definition 2.3. Let $\kappa < \lambda$ be cardinals and say that a family $\{x_\alpha\colon\ \alpha < \lambda\}$ of infinite subsets of ω is (κ, λ)-shattering if for all infinite $b \subset \omega$

(1) the set $\{\alpha < \lambda\colon\ b \subset^* x_\alpha\}$ has cardinality less than κ; and
(2) the set $\{\alpha < \lambda\colon\ b \cap x_\alpha =^* \emptyset\}$ has cardinality less than λ.

Say that $\{x_\alpha\colon\ \alpha < \lambda\}$ is strongly (κ, λ)-shattering if it contains no splitting family of size less than λ.

Needless to say a (κ, λ)-shattering family is strongly (κ, λ)-shattering if $\lambda = \mathfrak{s}$ and this is the kind of families we are interested in. However it seems likely that producing strongly (κ, λ)-shattering families would be interesting (and as difficult) even without requiring that $\lambda = \mathfrak{s}$. Nevertheless $\mathfrak{s}$ is necessarily less than λ as we show next.

Proposition 2.4. *If there is a (κ, λ)-shattering family, then $\mathfrak{h} \leq \kappa$ and $\mathfrak{s} \leq \lambda$.*

Proof: Let $\mathcal{S} = \{x_\alpha\colon\ \alpha < \lambda\}$ be a (κ, λ)-shattering family. Given any infinite $b \subset \omega$, there is a $\beta < \lambda$ such that each of $b \subset^* x_\beta$ and $b \cap x_\beta =^* \emptyset$ fail. This means that $\mathcal{S}$ is splitting. By condition (1) in Definition 2.3 and applying condition (6) of Lemma 2.1 with $\mathcal{S}_\alpha = \mathcal{S}$ for all $\alpha < \kappa$, it follows that $\mathfrak{h} \leq \kappa$. □

For any index set I the standard poset for adding Cohen reals, $\mathcal{C}_I$, is the set of all finite functions into 2 with domain a subset of I where $p < q$ providing $p \supset q$. If $I = \lambda$ is an ordinal, then we may use $\dot{x}_\alpha$ to be the canonical $\mathcal{C}_\lambda$-name $\{(\check{n}, \{\langle \alpha + n, 1\rangle\})\colon n \in \omega\}$ (i.e., for $s \in \mathcal{C}_\lambda$, $s \Vdash n \in \dot{x}_\alpha$ providing $s(\alpha + n) = 1$).

It is routine to verify that, for any regular cardinal $\lambda > \aleph_1$, forcing with $\mathcal{C}_\lambda$ will naturally add an $(\aleph_1, \lambda)$-shattering family but it is clear that this family would not be strongly $(\aleph_1, \lambda)$-shattering. Nevertheless, it may be possible with further forcing, to have it become strongly (κ, λ)-shattering for some $\aleph_1 \leq \kappa < \mathfrak{s}$.

In Theorem 5.9 we will prove that it is consistent with $\aleph_2 < \kappa^+ < \mathfrak{c}$ that there is a strongly (κ, κ^+)-shattering family.

Question 2.1. Assume that $\kappa < \lambda$ are regular cardinals and that there is a strongly (κ, λ)-shattering family. We pose the following questions.

(1) Is it consistent that $\kappa^+ < \lambda$?

(2) Is it consistent that $\lambda < \mathfrak{b}$?
(3) Is it consistent that $\kappa < \mathfrak{b} < \lambda$?

3. Matrix forcing and distinguishing $\mathfrak{h}, \mathfrak{s}, \mathfrak{b}$

In this section we recall the forcing methods for distinguishing $\mathfrak{b}$ and $\mathfrak{s}$ and apply them to prove the main results. We denote by $\mathbb{D}$ the standard (Hechler) poset for adding a dominating real. The poset $\mathbb{D}$ is an ordering on $\omega^{<\omega}\times\omega^\omega$ where $(s,f) < (t,g)$ providing $g \leq f$ and s extends t by values that are coordinatewise above g. Given a sfip family $\mathcal{F}$ of subsets of ω, there are two main posets for adding a pseudo-intersection. The Mathias–Prikry style poset is $\mathbb{M}(\mathcal{F})$ and consists of pairs (a, A) where A is in the filter base generated by $\mathcal{F}$, $a \subset \min(A)$, and $\mathbb{M}(\mathcal{F})$ is ordered by $(a_1, A_1) < (a_2, A_2)$ providing $a_2 \subset a_1 \subset a_2 \cup A_2$ and $A_1 \subset A_2$. When the context is clear, we will let $\dot{x}_{\mathcal{F}}$ denote the canonical name, $\{(\check{n}, (a, \omega \setminus n+1)) \colon n \in a \subset n+1\}$, which is forced to be the desired pseudo-intersection. When $\mathcal{U}$ is a free ultrafilter on ω, $\mathbb{M}(\mathcal{U})$ was the poset used in [4] and [5] and, in this case, $\dot{x}_{\mathcal{U}}$ is unsplit by the set of ground model subsets of ω. When mixed with matrix iteration methods, the ultrafilter $\mathcal{U}$ can be constructed so as to not add a dominating real.

The Laver style poset, $\mathbb{L}(\mathcal{F})$, is also very useful in matrix iterations and is defined as follows. The members of $\mathbb{L}(\mathcal{F})$ are subtrees T of $\omega^{<\omega}$ with a root or stem, $\mathrm{root}(T)$, and for all $\mathrm{root}(T) \subseteq t \in T$, the set $\mathrm{Br}(T,t) = \{j \in \omega \colon t^\frown j \in T\}$ is an element of the filter generated by $\mathcal{F}$. This poset is ordered by "$\subset$". For each $T \in \mathbb{L}(\mathcal{F})$ and $t \in T$, the subtree $T_t = \{t' \in T \colon t \cup t' \in \omega^{<\omega}\}$ is also a condition. The generic function, $\dot{f}_{\mathbb{L}(\mathcal{F})}$, added by $\mathbb{L}(\mathcal{F})$ can be described by the name of the union of the branch of $\omega^{<\omega}$ named by $\{(\check{t}, (\omega^{<\omega})_t) \colon t \in \omega^{<\omega}\}$. This poset forces that $\dot{f}_{\mathbb{L}(\mathcal{F})}$ dominates the ground model reals and the range of $\dot{f}_{\mathbb{L}(\mathcal{F})}$ is a pseudo-intersection of $\mathcal{F}$. Again, if $\mathcal{F}$ is an ultrafilter, this pseudo-intersection is not split by any ground model set.

For each sfip family $\mathcal{U}$ on ω, each of the posets $\mathbb{D}$, $\mathbb{M}(\mathcal{U})$, and $\mathbb{L}(\mathcal{U})$ is σ-centered. We just need this for the fact that this ensures that they are upwards ccc.

For a poset P and a set X, a canonical P-name for a subset of X will be a name of the form $\bigcup\{\check{x}\times A_x \colon x \in X\}$ where for each $x \in X$, A_x is an antichain of P. Of course if $\dot{Y}$ is any P-name of a subset of X, there is a canonical name that is forced to equal it. When we say that a poset P forces a statement, we intend the meaning that every element (i.e. 1_P) of P forces that statement. We write $P \lessdot Q$ to mean that P is a complete suborder of Q.

The terminology "matrix iterations" is used in [5], see also forthcoming preprint (F1222) from the second author.

Definition 3.1. For an infinite cardinal κ with uncountable cofinality, and an ordinal ζ, a $\kappa{\times}\zeta$-matrix iteration is a family

$$\langle\langle\mathbb{P}_{\alpha,\xi}\colon\ \alpha\le\kappa,\ \xi\le\zeta\rangle,\langle\dot{\mathbb{Q}}_{\alpha,\xi}\colon\ \alpha\le\kappa,\ \xi<\zeta\rangle\rangle$$

where for each $\alpha<\beta\le\kappa$ and $\xi<\eta\le\zeta$:

(1) $\mathbb{P}_{\beta,\xi}$ is a ccc poset;
(2) $\mathbb{P}_{\alpha,\xi}<\cdot\ \mathbb{P}_{\beta,\xi}<\cdot\ \mathbb{P}_{\beta,\eta}$;
(3) $\mathbb{P}_{\kappa,\xi}$ is the union of the chain $\{\mathbb{P}_{\gamma,\xi}\colon\gamma<\kappa\}$;
(4) $\dot{\mathbb{Q}}_{\alpha,\xi}$ is a $\mathbb{P}_{\alpha,\xi}$-name of a ccc poset and $\mathbb{P}_{\alpha,\xi+1}=\mathbb{P}_{\alpha,\xi}*\dot{\mathbb{Q}}_{\alpha,\xi}$;
(5) if η is a limit, then $\mathbb{P}_{\beta,\eta}=\bigcup\{\mathbb{P}_{\beta,\gamma}\colon\gamma<\eta\}$.

One constructs $\kappa{\times}\zeta$-matrices by recursion on ζ and, for successor steps, by careful choice of the component sequence $\{\dot{\mathbb{Q}}_{\alpha,\xi}\colon\ \alpha\le\kappa\}$. An important observation is that all the work is in the successor steps. The following is from [5, Lemma 3.10]

Lemma 3.2. *If ζ is a limit then a family*

$$\langle\langle\mathbb{P}_{\alpha,\xi}\colon\ \alpha\le\kappa,\ \xi\le\zeta\rangle,\langle\dot{\mathbb{Q}}_{\alpha,\xi}\colon\ \alpha\le\kappa,\ \xi<\zeta\rangle\rangle$$

is a $\kappa{\times}\zeta$-matrix iteration provided that for all $\eta<\zeta$ and $\beta\le\kappa$:

(1) $\langle\langle\mathbb{P}_{\alpha,\xi}\colon\ \alpha\le\kappa,\ \xi\le\eta\rangle,\ \langle\dot{\mathbb{Q}}_{\alpha,\xi}\colon\ \alpha\le\kappa,\ \xi<\eta\rangle\rangle$ *is a $\kappa{\times}\eta$-matrix iteration; and*
(2) $\mathbb{P}_{\beta,\zeta}=\bigcup\{\mathbb{P}_{\beta,\xi}\colon\xi<\zeta\}$.

The following is well-known, see for example [16, Section 5] and [13].

Proposition 3.3. *For any ζ and $\kappa{\times}\zeta$-matrix iteration*

$$\langle\langle\mathbb{P}_{\alpha,\xi}\colon\ \alpha\le\kappa,\ \xi\le\zeta\rangle,\langle\dot{\mathbb{Q}}_{\alpha,\xi}\colon\ \alpha\le\kappa,\ \xi<\zeta\rangle\rangle$$

the extension

$$\langle\langle\mathbb{P}_{\alpha,\xi}\colon\ \alpha\le\kappa,\ \xi\le\zeta+1\rangle,\langle\dot{\mathbb{Q}}_{\alpha,\xi}\colon\ \alpha\le\kappa,\ \xi<\zeta+1\rangle\rangle$$

is a $\kappa{\times}(\zeta+1)$-matrix iteration if either the following holds:

$(1)_{\mathbb{Q}}$ *for all $\alpha\le\kappa$, $\dot{\mathbb{Q}}_{\alpha,\zeta}$ is the $\mathbb{P}_{\alpha,\zeta}$-name for $\mathbb{D}$;*
$(2)_{\mathbb{Q}}$ *there is an $\alpha<\kappa$ such that $\dot{\mathbb{Q}}_{\beta,\zeta}$ is the trivial poset for $\beta<\alpha$, $\dot{\mathbb{Q}}_{\alpha,\zeta}$ is a $\mathbb{P}_{\alpha,\zeta}$-name of a σ-centered poset, and $\dot{\mathbb{Q}}_{\beta,\zeta}=\dot{\mathbb{Q}}_{\alpha,\zeta}$ for all $\alpha\le\beta\le\kappa$.*

Notice that if we define the extension as in $(1)_{\mathbb{Q}}$ then we will be adding a dominating real, but even if $\dot{\mathbb{Q}}_{\alpha,\zeta}$ is forced to equal $\mathbb{D}$ in $(2)_{\mathbb{Q}}$, the real added will only dominate the reals added by $\mathbb{P}_{\alpha,\zeta}$.

Proposition 3.4 ([4])**.** *Let M be a model of (a sufficient amount of) set-theory and $P\in M$ be a poset that is also contained in M. Then for any $f\in\omega^\omega$ that is* *not dominated by any $g \in M \cap \omega^\omega$, P forces that $f \not\leq \dot{g}$ for all P-names $\dot{g} \in M$ of elements of ω^ω.*

Proof: Let $p \in P$ and $n \in \omega$. It suffices to prove that there is a $q < p$ in P and a $k > n$ and $m < f(k)$ such that $q \Vdash \dot{g}(k) = m$. Since $p \in M$, we can work in M and define a function $h \in \omega^\omega$ by the rule that, for all $k \in \omega$, there is a $q_k < p$ such that $q_k \Vdash \dot{g}(k) = h(k)$. Choose any $k > n$ so that $h(k) < f(k)$. Then $q_k \Vdash \dot{g}(k) < f(k)$ and proves that $p \not\Vdash f \leq \dot{g}$. □

An analogous result, with the same proof, holds for splitting.

Proposition 3.5. *Let M be a model of (a sufficient amount of) set-theory and $P \in M$ be a poset that is also contained in M. If $x \in [\omega]^\omega$ satisfies that $y \not\subset x$ for all $y \in M \cap [\omega]^\omega$, then P forces that $\dot{y} \not\subset x$ for all P-names $\dot{y} \in M$ for elements of $[\omega]^\omega$.*

We also use the main construction from [4].

Proposition 3.6. *Suppose that*

$$\langle\langle \mathbb{P}_{\alpha,\xi}\colon \alpha \leq \kappa,\ \xi \leq \zeta\rangle, \langle \dot{\mathbb{Q}}_{\alpha,\xi}\colon \alpha \leq \kappa,\ \xi < \zeta\rangle\rangle$$

is a $\kappa \times \zeta$-matrix iteration and that $\{\dot{f}_\alpha\colon \alpha < \kappa\}$ is a sequence satisfying that for all $\alpha < \kappa$:

(1) *$\dot{f}_\alpha$ is a $\mathbb{P}_{\alpha,\zeta}$-name that is forced to be in ω^ω;*
(2) *for all $\beta < \alpha$ and $\mathbb{P}_{\beta,\zeta}$-name $\dot{g}$ of a member of ω^ω, $\mathbb{P}_{\alpha,\zeta}$ forces that $\dot{f}_\alpha \not< \dot{g}$.*

Then there is a sequence $\{\dot{\mathcal{U}}_{\alpha,\zeta}\colon \alpha \leq \kappa\}$ such that for all $\alpha < \kappa$:

(3) *$\dot{\mathcal{U}}_{\alpha,\zeta}$ is a $\mathbb{P}_{\alpha,\zeta}$-name of an ultrafilter on ω;*
(4) *for $\beta < \alpha$, $\dot{\mathcal{U}}_{\beta,\zeta}$ is a subset of $\dot{\mathcal{U}}_{\alpha,\zeta}$;*
(5) *for each $\beta < \alpha$ and each $\mathbb{P}_{\beta,\zeta} * \mathbb{M}(\dot{\mathcal{U}}_{\beta,\zeta})$-name $\dot{g}$ of an element of ω^ω, $\mathbb{P}_{\alpha,\zeta} * \mathbb{M}(\dot{\mathcal{U}}_{\alpha,\zeta})$ forces that $\dot{f}_\alpha \not< \dot{g}$; and*
(6) *$\langle\langle \mathbb{P}_{\alpha,\xi}\colon \alpha \leq \kappa,\ \xi \leq \zeta+1\rangle, \langle \dot{\mathbb{Q}}_{\alpha,\xi}\colon \alpha \leq \kappa,\ \xi < \zeta+1\rangle\rangle$ is a $\kappa\times(\zeta+1)$-matrix iteration, where for each $\alpha \leq \kappa$, $\mathbb{P}_{\alpha,\zeta+1} = \mathbb{P}_{\alpha,\zeta} * \dot{\mathbb{Q}}_{\alpha,\zeta}$ and $\dot{\mathbb{Q}}_{\alpha,\zeta}$ is the $\mathbb{P}_{\alpha,\zeta}$-name for $\mathbb{M}(\dot{\mathcal{U}}_{\alpha,\zeta})$.*

We record two more well-known preparatory preservation results.

Proposition 3.7 ([2])**.** *Suppose that $M \subset N$ are models of (a sufficient amount of) set-theory and that G is $\mathbb{D}$-generic over N. If $x \in N \cap [\omega]^\omega$ does not include any $y \in M \cap [\omega]^\omega$, it will not include any $y \in M[G] \cap [\omega]^\omega$.*

Proposition 3.8. *Assume that $\{P_\alpha\colon \alpha \leq \delta\}$ is a $<\cdot$-increasing chain of ccc posets with $P_\delta = \bigcup\{P_\alpha\colon \alpha < \delta\}$. Let G_δ be P_δ-generic. Let $x \in [\omega]^\omega$ and $f \in \omega^\omega$. Then each of the following holds:*

(1) *If* $f \not\leq^* g$ *for each* $g \in V[G_\alpha]$ *and for all* $\alpha < \delta$, *then* $f \not\leq^* g$ *for each* $g \in V[G_\delta]$.
(2) *If* x *does not contain any* $y \in [\omega]^\omega \cap V[G_\alpha]$ *for all* $\alpha < \kappa$, *then* x *does not contain any* $y \in [\omega]^\omega \cap V[G_\delta]$.

Proof: We prove only (1) since the proof of (2) is similar. If δ has uncountable cofinality, then there is nothing to prove since $V[G_\delta] \cap \omega^\omega$ would then equal $\bigcup\{V[G_\alpha] \cap \omega^\omega \colon \alpha < \delta\}$. Otherwise, consider any P_δ-name $\dot{g}$ and condition $p \in P_\delta$ forcing that $\dot{g} \in \omega^\omega$. We prove that p does not force that $\dot{g}(n) > f(n)$ for all $k < n$. We may assume that $\dot{g}$ is a canonical name, so let $\dot{g} = \bigcup\{\widetilde{(n,m)} \times A_{n,m} \colon n, m \in \omega\times\omega\}$. Choose any $\alpha < \delta$ so that $p \in P_\alpha$ and work in $V[G_\alpha]$. We define a function $h \in \omega^\omega \cap V[G_\alpha]$. For each $n \in \omega$, we set $h(n)$ to be the minimum m such that there is $q_{n,m} \in A_{n,m}$ having a P_α-reduct $p_{n,m} \in G_\alpha$. Since $A_n = \bigcup\{A_{n,m} \colon m \in \omega\}$ is predense in P_κ, the set of P_α-reducts of members of A_n is predense in P_α. By hypothesis, there is a $k < n$ such that $h(n) < f(n)$. Since $q_{n,h(m)}$ is compatible with p, this prove that $p \not\Vdash \dot{g}(n) > f(n)$. □

4. Building the models to distinguish $\mathfrak{h}, \mathfrak{b}, \mathfrak{s}$

For simplicity we assume GCH. Let $\aleph_1 \leq \mu < \kappa < \lambda$ be regular cardinals and assume that $\theta > \lambda$ is a cardinal with cofinality μ. We will need to enumerate names in order to force that $\mathfrak{p} \geq \mu$. For each ccc poset $\tilde{P} \in H(\theta^+)$ let $\{\dot{Y}(\tilde{P}, \xi) \colon \xi < \theta\}$ be an enumeration of the set of all canonical $\tilde{P}$-names of subsets of ω. Also let $\{S_\xi \colon \xi < \theta\}$ be an enumeration of all subsets of θ that have cardinality less than μ. For each $\eta < \lambda$, let ζ_η denote the ordinal product $\theta \cdot \eta$.

Theorem 4.1. *There is a ccc poset that forces* $\mathfrak{p} = \mathfrak{h} = \mu$, $\mathfrak{b} = \kappa$, $\mathfrak{s} = \lambda$ *and* $\mathfrak{c} = \theta$.

Proof: The poset will be obtained by constructing a $\kappa\times\zeta$-matrix iteration where ζ is the ordinal product $\theta\cdot\lambda$ (the lexicographic ordering on $\lambda\times\theta$). We begin with the $\kappa\times\kappa$-matrix iteration

$$\langle\langle \mathbb{P}_{\alpha,\xi} \colon \alpha \leq \kappa,\ \xi \leq \kappa\rangle, \langle \dot{\mathbb{Q}}_{\alpha,\xi} \colon \alpha \leq \kappa,\ \xi < \kappa\rangle\rangle$$

where, for each $\alpha < \kappa$, $\mathbb{P}_{\alpha,\alpha}$ forces that $\dot{\mathbb{Q}}_{\alpha,\alpha}$ is $\mathbb{D}$, for $\beta < \alpha$, $\dot{\mathbb{Q}}_{\beta,\alpha}$ is the trivial poset, and for $\alpha \leq \beta \leq \kappa$, $\dot{\mathbb{Q}}_{\beta,\alpha}$ equals $\dot{\mathbb{Q}}_{\alpha,\alpha}$. By Proposition 3.3, there is such a matrix. For each $\alpha < \kappa$, let $\dot{f}_\alpha$ be the canonical name for the dominating real added by $\mathbb{P}_{\alpha,\alpha+1}$. By Propositions 3.4 and 3.8, it follows that for all $\beta < \alpha < \kappa$, $\mathbb{P}_{\alpha,\kappa}$ forces that $\dot{f}_\alpha \not\leq^* \dot{g}$ for all $\mathbb{P}_{\beta,\kappa}$-names $\dot{g}$ of elements of ω^ω.

We omit the routine enumeration details involved in the recursive construction and state the properties we require of our $\kappa\times\zeta$-matrix iteration. Each step of the construction uses either (2) of Proposition 3.3 or Proposition 3.6 to choose the next sequence $\{\dot{\mathbb{Q}}_{\alpha,\xi}\colon \alpha \le \kappa\}$. In the case of Proposition 3.3 (2), the preservation of inductive condition (1) follows from Proposition 3.4. The preservation through limit steps follows from Proposition 3.8.

There is a matrix-iteration sequence

$$\langle\langle \mathbb{P}_{\alpha,\xi}\colon \alpha \le \kappa,\ \xi \le \zeta\rangle, \langle \dot{\mathbb{Q}}_{\alpha,\xi}\colon \alpha \le \kappa,\ \xi < \zeta\rangle\rangle$$

satisfying each of the following for each $\xi < \zeta$:

(1) for each $\beta < \alpha < \kappa$ and each $\mathbb{P}_{\beta,\xi}$-name $\dot{g}$ for an element of ω^ω, $\mathbb{P}_{\alpha,\xi}$ forces that $\dot{f}_\alpha \not\le^* \dot{g}$;

(2) for each $\beta < \lambda$ with $\zeta_{\beta+1} \le \xi$ and each $\eta < \theta$, if $\mathbb{P}_{\kappa,\zeta_\beta}$ forces that the family $\mathcal{F}_{\beta,\eta} = \{\dot{Y}(\mathbb{P}_{\kappa,\zeta_\beta},\gamma)\colon \gamma \in S_\eta\}$ has the sfip, then there is a $\bar{\eta} < \zeta_{\beta+1}$ and an $\alpha < \kappa$ such that $\dot{\mathbb{Q}}_{\beta,\bar{\eta}}$ equals the $\mathbb{P}_{\alpha,\bar{\eta}}$-name for $\mathbb{M}(\mathcal{F}_{\beta,\eta})$ for all $\alpha \le \beta \le \kappa$;

(3) for each $\beta < \lambda$ such that $\zeta_\beta < \xi$, $\mathbb{P}_{\kappa,\zeta_\beta+1}$ equals $\mathbb{P}_{\kappa,\zeta_\beta} * \mathbb{M}(\dot{\mathcal{U}}_{\kappa,\zeta_\beta})$ and $\dot{\mathcal{U}}_{\kappa,\zeta_\beta}$ is a $\mathbb{P}_{\kappa,\zeta_\beta}$-name of an ultrafilter on ω;

(4) for each $\eta < \lambda$ and each $\alpha < \kappa$ such that $\zeta_\eta < \xi$, then $\dot{\mathbb{Q}}_{\alpha,\zeta_\eta+\alpha}$ is the $\mathbb{P}_{\alpha,\zeta_\eta+\alpha}$-name for $\mathbb{D}$, and $\dot{\mathbb{Q}}_{\beta,\zeta_\eta+\alpha} = \dot{\mathbb{Q}}_{\alpha,\zeta_\eta+\alpha}$ for all $\alpha \le \beta \le \kappa$.

Now we verify that $P = \mathbb{P}_{\kappa,\zeta}$ has the desired properties. Since P is ccc, it preserves cardinals and clearly forces that $\mathfrak{c} = \theta$. It thus follows from Corollary 2.2 that $\mathfrak{p} \le \mathfrak{h} \le \mu = \mathrm{cf}(\mathfrak{c})$. If $\mathcal{Y}$ is a family of fewer than μ many canonical P-names of subsets of ω, then there is an $\alpha < \kappa$ and $\eta < \lambda$ such that $\mathcal{Y}$ is a family of $\mathbb{P}_{\alpha,\zeta_\eta}$-names. It follows that there is a $\beta < \theta$ such that $\mathcal{Y}$ is equal to the set $\{\dot{Y}(\mathbb{P}_{\kappa,\zeta_\beta},\gamma)\colon \gamma \in S_\eta\}$. If $\mathbb{P}_{\kappa,\zeta_\beta}$ forces that $\mathcal{Y}$ has the sfip, then inductive condition 2 ensures that there is a P-name for a pseudo-intersection for $\mathcal{Y}$. This shows that P forces that $\mathfrak{p} \ge \mu$. It is clear that inductive condition 1 ensures that $\mathfrak{b} \le \kappa$. We check that condition 4 ensure that $\mathfrak{b} \ge \kappa$. Suppose that $\mathcal{G}$ is a family of fewer than κ many canonical P-names of members of ω^ω. We again find $\eta < \lambda$ and $\alpha < \kappa$ such that $\mathcal{G}$ is a family of $\mathbb{P}_{\alpha,\zeta_\eta}$-names. Condition 4 forces there is a function that dominates $\mathcal{G}$. Finally we verify that condition 3 ensures that P forces that $\mathfrak{s} = \lambda$. If $\mathcal{S}$ is any family of fewer than λ-many canonical P-names of subsets of ω, then there is an $\eta < \lambda$ such that $\mathcal{S}$ is a family of $\mathbb{P}_{\kappa,\zeta_\eta}$-names. Evidently, $\mathbb{P}_{\kappa,\zeta_\eta+1}$ adds a subset of ω that is not split by $\mathcal{S}$. There are a number of ways to observe that for each $\eta < \lambda$, $\mathbb{P}_{\kappa,\zeta_{\eta+1}}$ adds a real that is Cohen over the extension by $\mathbb{P}_{\kappa,\zeta_\eta}$. This ensures that P forces that $\mathfrak{s} \le \lambda$. □

In the next result we proceed similarly except that we first add κ many Cohen reals and preserve that they are splitting. We then cofinally add dominating reals with Hechler's $\mathbb{D}$ and again use small posets to ensure $\mathfrak{p} \ge \mu$.

Theorem 4.2. *There is a ccc poset that forces* $\mathfrak{p} = \mathfrak{h} = \mu$, $\mathfrak{s} = \kappa$, $\mathfrak{b} = \lambda$ *and* $\mathfrak{c} = \theta$.

Proof: We begin with the $\kappa\times\kappa$-matrix iteration

$$\langle\langle \mathbb{P}_{\alpha,\xi}\colon\ \alpha \leq \kappa,\ \xi \leq \kappa\rangle, \langle \dot{\mathbb{Q}}_{\alpha,\xi}\colon\ \alpha \leq \kappa,\ \xi < \kappa\rangle\rangle$$

where $\mathbb{P}_{\alpha,\alpha}$ forces that $\dot{\mathbb{Q}}_{\alpha,\alpha}$ is $\mathcal{C}_\omega$, for $\beta < \alpha$, $\dot{\mathbb{Q}}_{\beta,\alpha}$ is the trivial poset, and for $\alpha \leq \beta \leq \kappa$, $\dot{\mathbb{Q}}_{\beta,\alpha}$ equals $\dot{\mathbb{Q}}_{\alpha,\alpha}$. We let $\dot{x}_\alpha$ denote the canonical Cohen real added by $\mathbb{P}_{\alpha,\alpha+1}$. Of course $\mathbb{P}_{\alpha,\alpha+1}$ forces that neither $\dot{x}_\alpha$ nor its complement include any infinite subsets of ω that have, for any $\beta < \alpha$, a $\mathbb{P}_{\beta,\alpha+1}$-name. By Proposition 3.8, the inductive condition 4 below holds for $\xi = \kappa$.

Then, proceeding as in the proof of Theorem 4.1, we just assert the existence of a $\kappa\times\zeta$-matrix iteration

$$\langle\langle \mathbb{P}_{\alpha,\xi}\colon\ \alpha \leq \kappa,\ \xi \leq \zeta\rangle, \langle \dot{\mathbb{Q}}_{\alpha,\xi}\colon\ \alpha \leq \kappa,\ \xi < \zeta\rangle\rangle$$

satisfying each of the following for each $\kappa \leq \xi < \zeta$:

(1) for each $\beta < \alpha < \kappa$, $\mathbb{P}_{\alpha,\xi}$ forces that neither $\dot{x}_\alpha$ nor $\omega \setminus \dot{x}_\alpha$ include any infinite subset of ω that has a $\mathbb{P}_{\beta,\xi}$-name;

(2) for each $\eta < \lambda$ with $\zeta_{\eta+1} \leq \xi$ and each $\delta < \theta$, if $\mathbb{P}_{\kappa,\zeta_\eta}$ forces that the family $\mathcal{F}_{\eta,\delta} = \{\dot{Y}(\mathbb{P}_{\kappa,\zeta_\eta},\gamma)\colon\ \gamma \in S_\delta\}$ has the sfip, then there is a $\bar{\delta} < \zeta_{\eta+1}$ and an $\alpha < \kappa$ such that $\dot{\mathbb{Q}}_{\beta,\bar{\delta}}$ equals the $\mathbb{P}_{\alpha,\bar{\delta}}$-name for $\mathbb{M}(\mathcal{F}_{\eta,\delta})$ for all $\alpha \leq \beta \leq \kappa$;

(3) for each $\eta < \lambda$ and each $\alpha < \kappa$ such that $\zeta_\eta < \xi$, then $\dot{\mathbb{Q}}_{\alpha,\zeta_\eta+\alpha}$ is the $\mathbb{P}_{\alpha,\zeta_\eta+\alpha}$-name for $\mathbb{M}(\dot{\mathcal{U}}_{\alpha,\zeta_\beta})$ where $\dot{\mathcal{U}}_{\alpha,\zeta_\beta}$ is a $\mathbb{P}_{\alpha,\zeta_\beta}$-name of an ultrafilter on ω, and $\dot{\mathbb{Q}}_{\beta,\zeta_\eta+\alpha} = \dot{\mathbb{Q}}_{\alpha,\zeta_\eta+\alpha}$ for all $\alpha \leq \beta \leq \kappa$;

(4) for each $\eta < \lambda$ such that $\zeta_\eta < \xi$, $\mathbb{P}_{\kappa,\zeta_\eta+1}$ equals $\mathbb{P}_{\kappa,\zeta_\eta} * \mathbb{D}$.

Evidently conditions (2) and (3) are similar and can be achieved while preserving condition (1) by Proposition 3.3 (2). The fact that $\mathbb{P}_{\kappa,\zeta_\eta} * \mathbb{D}$ preserves condition (1) follows from Proposition 3.7. Condition (1) ensures that $\mathfrak{s} \leq \kappa$, and by arguments similar to those in Theorem 4.1, condition (3) ensures that $\mathfrak{s} \geq \kappa$. The fact that $\mathfrak{b} = \lambda$ (in fact $\mathfrak{d} = \lambda$) follows easily from condition (4). The facts that $\mathfrak{c} = \theta$, $\mathfrak{p} \geq \mu$ and $\mathfrak{h} = \mu$ are proven exactly as in Theorem 4.1. □

5. On (κ, λ)-shattering

In this section we prove, see Theorem 5.9, that it is consistent that strongly (κ, κ^+)-shattering families exist. The method used in this section is the following generalization of matrix iterations used in [8]. A chain $\{P_\alpha\colon\ \alpha < \delta\}$ is continuous if for every limit $\alpha < \delta$, $P_\alpha = \bigcup\{P_\beta\colon\ \beta < \alpha\}$.

Definition 5.1. Let $\kappa > \omega_1$ be a regular cardinal. For an ordinal ζ, a $\kappa\times\zeta$-matrix of posets is a family $\{P_{\alpha,\xi}\colon\ \alpha \leq \kappa,\ \xi < \zeta\}$ of ccc posets satisfying for each $\alpha < \kappa$, and $\xi < \eta < \zeta$:

(1) $P_{\alpha,\xi} <\!\cdot\ P_{\beta,\xi}$ for all $\alpha < \beta \leq \kappa$;
(2) $P_{\beta,\xi} = \bigcup\{P_{\eta,\xi}\colon \eta < \beta\}$ for $\beta \leq \kappa$ with $cf(\beta) > \omega$; and
(3) for some $\gamma < \kappa$, $P_{\beta,\xi} <\!\cdot\ P_{\beta,\eta}$ for all $\gamma \leq \beta \leq \kappa$;
(4) if η is a limit ordinal, there is a cub $C \subset \eta$ and a $\gamma < \kappa$ such that, for all $\gamma \leq \beta < \kappa$, $\{P_{\beta,\delta}\colon\ \delta \in C \cup \{\eta\}\}$ is a continuous $<\!\cdot$-increasing chain.

One must be careful with a $\kappa \times \zeta$-matrix since there is no natural extension or definition of $P_{\alpha,\zeta}$ for $\alpha < \kappa$. However, when $\mathrm{cf}(\zeta) > \omega_1$ the matrix can be viewed as a matrix type construction of a ccc poset $P_{\kappa,\zeta}$.

Lemma 5.2. *If $\{P_{\alpha,\xi}\colon\ \alpha \leq \kappa,\ \xi < \zeta\}$ is a $\kappa\times\zeta$-matrix of posets with $\kappa > \omega_1$ regular and $\mathrm{cf}(\zeta) > \omega_1$, then the poset $P_{\kappa,\zeta} = \bigcup\{P_{\kappa,\xi}\colon\ \xi < \zeta\}$ is ccc and satisfies that $P_{\alpha,\xi} <\!\cdot\ P_{\kappa,\zeta}$ for all $\alpha \leq \kappa$ and $\xi < \zeta$.*

Proof: Let $\alpha < \kappa$ and $\xi < \zeta$. It follows from property (1) in Definition 5.1 that $P_{\alpha,\xi} <\!\cdot\ P_{\kappa,\xi}$. By (3) of Definition 5.1, we have that $\{P_{\kappa,\eta}\colon\ \xi \leq \eta < \zeta\}$ is a $<\!\cdot$-chain. This implies that $P_{\kappa,\xi} <\!\cdot\ P_{\kappa,\zeta}$. Now we check that $P_{\kappa,\zeta}$ is ccc. Assume that $A \subset P_{\kappa,\zeta}$ has cardinality $\aleph_1$. Choose any $\gamma_0 < \kappa$ so that $A \subset \bigcup\{P_{\beta,\xi}\colon \beta < \gamma_0,\ \xi < \zeta\}$. Similarly choose $\eta < \zeta$ minimal so that $A \subset \bigcup\{P_{\beta,\xi}\colon\ \beta < \gamma_0,\ \xi < \eta\}$. By property (2) of Definition 5.1, there is a $\gamma_0 \leq \gamma_1 < \kappa$ such that $A \subset \bigcup\{P_{\gamma_1,\xi}\colon\ \xi < \eta\}$. Now choose a cub $C \subset \eta$ as in condition (4) of Definition 5.1, and, using conditions (2) and (3) of Definition 5.1, we can choose $\zeta_1 \leq \zeta_2 < \kappa$ so that $A \subset \bigcup\{P_{\zeta_2,\delta}\colon\ \delta \in C\} \subset P_{\zeta_2,\eta}$. Since $P_{\zeta_2,\eta}$ is ccc, it follows that A is not an antichain. □

We will use the method of matrix of posets from Definition 5.1 in which our main component posets to raise the value of $\mathfrak{s}$ will be the Laver style posets. Before proceeding it may be helpful to summarize the rough idea of how we generalize the fundamental preservation technique of a matrix iteration. In a $\kappa \times \kappa^+$-matrix iteration, one may introduce a sequence $\{\dot{a}_\alpha\colon\ \alpha < \kappa\}$ of $P_{\kappa,1}$-names that have no infinite pseudointersection. With this fixed enumeration, one then ensures that no $P_{\alpha,\gamma}$-name will be forced to be a subset of $\dot{a}_\beta$ for any $\alpha \leq \beta < \kappa$. In the construction introduced in [8], we instead continually add to the list a $P_{0,\gamma+1}$-name $\dot{a}_\gamma$ and at stage $\mu < \kappa^+$, we adopt a new enumeration of $\{\dot{a}_\alpha\colon\ \alpha < \mu\}$ in order-type κ (coherent with previous listings) and again ensure that no $P_{\alpha,\mu+1}$-name is a subset of any $\dot{a}_\beta$ for β not listed before α in this new μth listing. We utilize a $\square$-principle to make these enumerations sufficiently coherent and to use as the required cub's in condition (4) of Definition 5.1. The greater flexibility in the definition of $\kappa \times \kappa^+$-matrix of posets makes this possible.

We recall some notions and results about these studied in [7], [8].

Proposition 5.3. *If $P \lessdot P'$ are ccc posets, and $\dot{\mathcal{D}} \subset \dot{\mathcal{E}}$ are, respectively, a P-name and a P'-name of ultrafilters on ω, then $P * \mathbb{L}(\dot{\mathcal{D}}) \lessdot P' * \mathbb{L}(\dot{\mathcal{E}})$.*

Definition 5.4. A family $\mathcal{A} \subset [\omega]^\omega$ is thin over a model M if for every I in the ideal generated by $\mathcal{A}$ and every infinite family $\mathcal{F} \in M$ consisting of pairwise disjoint finite sets of bounded size, I is disjoint from some member of $\mathcal{F}$.

It is routine to prove that for each limit ordinal δ, $\mathcal{C}_\delta$ forces that the family $\{\dot{x}_\alpha \colon \alpha \in \delta\}$, as defined above, is thin over the ground model. In fact if $\mathcal{A}$ is thin over some model M, then $\mathcal{C}_\delta$ forces that $\mathcal{A} \cup \{\dot{x}_\alpha \colon \alpha \in \delta\}$ is also thin over M. This is the notion we use to control that property (1) of the definition of a (κ, κ^+)-shattering sequence will be preserved while at the same time raising the value of $\mathfrak{s}$.

We first note that Proposition 3.5 extends to include this concept.

Proposition 5.5. *Suppose that M is a model of a sufficient amount of set-theory and that $\mathcal{A} \subset [\omega]^\omega$ is thin over M. Then for any poset P such that $P \in M$ and $P \subset M$, $\mathcal{A}$ is thin over the forcing extension by P.*

Proof: Let $\{\dot{F}_l \colon l \in \omega\}$ be P-names and suppose that $p \in P$ forces that $\{\dot{F}_l \colon l \in \omega\}$ are pairwise disjoint subsets of $[\omega]^k$, $k \in \omega$. Also let I be any member of the ideal generated by $\mathcal{A}$. Working in M, recursively choose $q_j < p$, $j \in \omega$, and H_j, l_j so that $q_j \Vdash \dot{F}_{l_j} = \check{H}_j$ and $H_j \cap \bigcup\{H_i \colon i < j\} = \emptyset$. The sequence $\{H_j \colon j \in \omega\}$ is a family in M of pairwise disjoint sets of cardinality k. Therefore there is a j with $H_j \cap I = \emptyset$. This proves that p does not force that I meets every member of $\{\dot{F}_l \colon l \in \omega\}$. □

Lemma 5.6 ([8, 3.8])**.** *Let κ be a regular uncountable cardinal and let $\{P_\beta \colon \beta \leq \kappa\}$ be a $\lessdot$-increasing chain of ccc posets with $P_\kappa = \bigcup\{P_\alpha \colon \alpha < \kappa\}$. Assume that, for each $\beta < \kappa$, $\dot{\mathcal{A}}_\beta$ is a $P_{\beta+1}$-name of a subset of $[\omega]^\omega$ that is forced to be thin over the forcing extension by P_β. Also let $\dot{\mathcal{D}}_0$ be a $P_0 * \mathcal{C}_{\{0\}\times\mathfrak{c}}$-name that is forced to be a Ramsey ultrafilter on ω. Then there is a sequence $\langle \dot{\mathcal{D}}_\beta \colon 0 < \beta < \kappa\rangle$ such that for all $\alpha < \beta < \kappa$:*

1. *$\dot{\mathcal{D}}_\beta$ is a $P_\beta * \mathcal{C}_{(\beta+1)\times\mathfrak{c}}$-name;*
2. *$\dot{\mathcal{D}}_\alpha$ is a subset of $\dot{\mathcal{D}}_\beta$;*
3. *$P_\beta * \mathcal{C}_{(\beta+1)\times\mathfrak{c}}$ forces that $\dot{\mathcal{D}}_\beta$ is a Ramsey ultrafilter;*
4. *$P_\alpha * \mathcal{C}_{(\alpha+1)\times\mathfrak{c}} * \mathbb{L}(\dot{\mathcal{D}}_\alpha) \lessdot P_\beta * \mathcal{C}_{(\beta+1)\times\mathfrak{c}} * \mathbb{L}(\dot{\mathcal{D}}_\beta)$; and*
5. *$P_\beta * \mathcal{C}_{(\beta+1)\times\mathfrak{c}} * \mathbb{L}(\dot{\mathcal{D}}_\beta)$ forces that $\dot{\mathcal{A}}_\beta$ is thin over the forcing extension by $P_\alpha * \mathcal{C}_{(\alpha+1)\times\mathfrak{c}} * \mathbb{L}(\dot{\mathcal{D}}_\alpha)$.*

Lemma 5.7 ([8, 2.7])**.** *Assume that $P_{0,0} \lessdot P_{1,0}$ and that $\dot{\mathcal{A}}$ is a $P_{1,0}$-name of a subset of $[\omega]^\omega$. Assume that $\langle P_{0,\xi} \colon \xi < \delta\rangle$ and $\langle P_{1,\xi} \colon \xi < \delta\rangle$ are $\lessdot$-chains such* *that $P_{0,\xi} <\cdot P_{1,\xi}$ for all $\xi < \delta$, and that $P_{1,\xi}$ forces that $\dot{A}$ is thin over the forcing extension by $P_{0,\xi}$ for all $\xi < \delta$. Then $P_{1,\delta} = \bigcup\{P_{1,\xi} \colon \xi < \delta\}$ forces that $\mathcal{A}$ is thin over the forcing extension by $P_{0,\delta} = \bigcup\{P_{0,\xi} \colon \xi < \delta\}$.*

Before proving the next result we recall the notion of a $\square_\kappa$-sequence. For a set C of ordinals, let $\sup(C)$ be the supremum, $\bigcup C$, of C and let C' denote the set of limit ordinals $\alpha < \sup(C)$ such that $C \cap \alpha$ is cofinal in α. For a limit ordinal α, a set C is a cub in α if $C \subset \alpha = \sup(C)$ and $C' \subset C$.

Definition 5.8 ([14])**.** For a cardinal κ, the family $\{C_\alpha \colon \alpha \in (\kappa^+)'\}$ is a $\square_\kappa$-sequence if for each $\alpha \in (\kappa^+)'$:

(1) C_α is a cub in α;
(2) if $\mathrm{cf}(\alpha) < \kappa$, then $|C_\alpha| < \kappa$;
(3) if $\beta \in C'_\alpha$, then $C_\beta = C_\alpha \cap \beta$.

If there is a $\square_\kappa$-sequence, then $\square_\kappa$ is said to hold.

Theorem 5.9. *It is consistent with $\aleph_1 < \mathfrak{h} < \mathfrak{s} < \mathrm{cf}(\mathfrak{c}) = \mathfrak{c}$ that there is an $(\mathfrak{h}, \mathfrak{s})$-shattering family.*

Proof: We start in a model of GCH satisfying $\square_\kappa$ for some regular cardinal $\kappa > \aleph_1$. Choose any regular $\lambda > \kappa^+$. Fix a $\square_\kappa$-sequence $\{C_\alpha \colon \alpha \in (\kappa^+)'\}$. We may assume that $C_\alpha = \alpha$ for all $\alpha \in \kappa'$. For each $\alpha \in (\kappa^+)'$, let $o(C_\alpha)$ denote the order-type of C_α. When C'_α is bounded in α with $\eta = \max(C'_\alpha)$, then let $\{\varphi^\alpha_l \colon l \in \omega\}$ enumerate $C_\alpha \setminus \eta$ in increasing order.

We will construct a $\kappa\times\kappa^+$-matrix of posets, $\langle P_{\alpha,\xi} \colon \alpha \le \kappa,\ \xi < \kappa^+\rangle \in H(\lambda^+)$ and prove that the poset P_{κ,κ^+} as in Lemma 5.2 has the desired properties. For each $\xi < \eta i < \kappa^+$, we will also choose an $\iota(\xi,\eta) < \kappa$ satisfying, as in (3) of the definition of $\kappa\times(\xi+1)$-matrix that $P_{\alpha,\xi} <\cdot P_{\alpha,\eta}$ for all $\iota(\xi,\eta) \le \alpha < \kappa$. We construct this family by constructing $\langle P_{\alpha,\xi} \colon \alpha \le \kappa,\ \xi < \zeta\rangle$ by recursion on limit $\zeta < \kappa^+$.

We will recursively define two other families. For each $\alpha < \kappa$ and $\xi < \kappa^+$, we will define a set $\mathrm{supp}(P_{\alpha,\xi}) \subset \xi$ that can be viewed as the union of the supports of the elements of $P_{\alpha,\xi}$ and will satisfy that $\{\mathrm{supp}(P_{\alpha,\xi}) \colon \alpha < \kappa\}$ is increasing and covers ξ. For each limit $\eta < \kappa^+$ of cofinality less than κ and each $n \in \omega$, we will select a canonical $P_{\kappa,\eta+n+1}$-name, $\dot{a}_{\eta+n}$ of a subset ω that is forced to be Cohen over the forcing extension by $P_{\kappa,\eta}$. While this condition looks awkward, we simply want to avoid this task at limits of cofinality κ. Needing notation for this, let $E = \kappa^+ \setminus \bigcup\{[\eta, \eta+\omega) \colon \mathrm{cf}(\eta) = \kappa\}$.

For each $\alpha < \kappa$ and $\xi < \eta < \kappa^+$, we define $\mathcal{A}_{\alpha,\xi,\eta}$ to be the family $\{\dot{a}_\gamma \colon \gamma \in E \cap \eta \setminus \mathrm{supp}(P_{\alpha,\xi})\}$. The intention is that for all $\alpha < \xi \le \eta$, $\mathcal{A}_{\alpha,\xi,\eta}$ is a family of $P_{\kappa,\eta}$-names which is forced by the poset $P_{\kappa,\eta}$ to be thin over the forcing extension by $P_{\alpha,\xi}$. Let us note that if $\alpha < \beta$ and $\xi \le \eta$, then $\mathcal{A}_{\alpha,\xi,\eta} \setminus \mathcal{A}_{\beta,\eta,\eta}$

should then be a set of $P_{\beta,\eta}$-names. By ensuring that $\mathrm{supp}(P_{\alpha,\xi})$ has cardinality less than κ for all $\alpha < \kappa$ and $\xi < \kappa^+$, this will ensure that the family $\{\dot{a}_\eta \colon \eta \in E\}$ is (κ, κ^+)-shattering. For each $\eta < \kappa^+$ with cofinality κ we will ensure that $P_{\kappa,\eta+1}$ has the form $P_{\kappa,\eta} * \mathcal{C}_{\kappa\times\lambda}$ and that $P_{\kappa,\eta+2} = P_{\kappa,\eta+1} * \mathbb{L}(\dot{\mathcal{D}}_{\kappa,\eta})$ for a $P_{\kappa,\eta+1}$-name $\dot{\mathcal{D}}_{\kappa,\eta}$ of an ultrafilter on ω. This will ensure that $\mathfrak{c} \geq \lambda$ and $\mathfrak{s} = \kappa^+$. The sequence defining $P_{\kappa,\eta+3}$ will be devoted to ensuring that $\mathfrak{p} \geq \kappa$.

We start the recursion in a rather trivial fashion. For each $\alpha < \kappa$, $P_{\alpha,0} = \mathcal{C}_\omega$ and for each $n \in \omega$, $P_{\alpha,n+1} = P_{\alpha,n} * \mathcal{C}_\omega$. We may also let $\iota(n,m) = 0$ for all $n < m < \omega$. For each $n \in \omega$, let $\dot{a}_n$ be the canonical name of the Cohen real added by the second coordinate of $P_{\kappa,n+1} = P_{\kappa,n} * \mathcal{C}_\omega$. For each $\alpha < \kappa$ and $n \in \omega$, define $\mathrm{supp}(P_{\alpha,n})$ to be n.

It should be clear that $P_{\kappa,\omega}$ forces that for each $\alpha < \kappa$ and $n \in \omega$, the family $\{\dot{a}_m \colon n \leq m \in \omega\}$ is thin over the forcing extension by $P_{\alpha,n}$. Assume that P is a poset whose elements are functions with domain a subset of an ordinal ξ. We adopt the notational convention that for a P-name $\dot{Q}$ for a poset, $P *_\xi \dot{Q}$ will denote the representation of $P * \dot{Q}$ whose elements have the form $p \cup \{(\xi, \dot{q})\}$ for $(p, \dot{q}) \in P * \dot{Q}$.

We will prove, by induction on limit $\zeta < \kappa^+$, there is a $\kappa\times(\zeta+1)$-matrix $\{P_{\alpha,\xi} \colon \alpha \leq \kappa,\ \xi \leq \zeta\}$ and families $\{\mathcal{A}_{\alpha,\xi,\eta} \colon \alpha < \kappa,\ \xi \leq \eta \leq \zeta\}$ satisfying conditions (1)–(10).

(1) For all $\alpha < \beta < \kappa$ and $\xi < \eta < \zeta$, if $P_{\alpha,\xi} <\cdot\, P_{\beta,\eta}$, then the poset $P_{\beta,\eta}$ forces that the family $\mathcal{A}_{\alpha,\xi,\eta} \setminus \mathcal{A}_{\beta,\eta,\eta}$ is thin over the forcing extension by $P_{\alpha,\xi}$;

(2) for all $\alpha < \kappa$ and $\xi < \zeta$, the elements p of the poset $P_{\alpha,\xi}$ are functions that have a finite domain, $\mathrm{dom}(p)$, contained in ξ;

(3) if C'_ζ is cub in ζ and $\eta \in C'_\zeta$, then

- (a) $P_{n,\zeta}$ is the trivial poset and $\mathrm{supp}(P_{n,\zeta}) = \emptyset$ for $n \in \omega$;
- (b) $P_{\alpha,\zeta} = P_{\alpha,\eta}$ and $\mathrm{supp}(P_{\alpha,\zeta}) = \mathrm{supp}(P_{\alpha,\eta})$ for all $o(C_\eta) \leq \alpha < o(C_\eta) + \omega$; and
- (c) $P_{\alpha,\zeta} = \bigcup\{P_{\alpha,\eta} \colon \eta \in C'_\zeta\}$ and $\mathrm{supp}(P_{\alpha,\zeta}) = \bigcup\{\mathrm{supp}(P_{\alpha,\eta}) \colon \eta \in C'_\zeta\}$ for all $o(C_\zeta) \leq \alpha < \kappa$;

also, let $\iota(\eta,\zeta) = o(C_\eta)$ for all $\eta \in C'_\zeta$ and for all $\gamma < \zeta \setminus C'_\zeta$ let $\iota(\gamma,\zeta) = \iota(\gamma,\eta)$ where $\eta = \min(C'_\zeta \setminus \gamma)$;

(4) if $\max(C'_\zeta) < \zeta$ then let

$$\iota_\zeta = \max(o(C_\zeta), \sup\{\iota(\varphi^\zeta_l, \varphi^\zeta_{l'} + n) \colon l \leq l' < n < \omega\})$$

and

- (a) set $P_{\alpha,\zeta} = P_{\alpha,\varphi^\zeta_0}$ and $\mathrm{supp}(P_{\alpha,\zeta}) = \mathrm{supp}(P_{\alpha,\varphi^\zeta_0})$ for all $\alpha < \iota_\zeta$;
- (b) set, for $\iota_\zeta \leq \alpha < \kappa$, $P_{\alpha,\zeta} = \bigcup\{P_{\alpha,\varphi^\zeta_l+n} \colon l,n \in \omega\}$ and $\mathrm{supp}(P_{\alpha,\zeta}) = \bigcup\{\mathrm{supp}(P_{\alpha,\varphi^\zeta_l+n}) \colon l,n \in \omega\}$;

(c) for each $\gamma \in \varphi_0^\zeta$ let $\iota(\gamma,\zeta) = \iota(\gamma,\varphi_0^\zeta)$, let $\iota(\varphi_0^\zeta,\zeta) = o(C_\gamma)$, and for each $\varphi_0^\zeta < \gamma < \zeta$, $\iota(\gamma,\zeta)$ is the maximum of ι_ζ and $\min\{\iota(\gamma,\varphi_l^\zeta + n) \colon l,n \in \omega$ and $\gamma < \varphi_l^\zeta + n\}$;

(5) if $o(C_\zeta) < \kappa$, then for all $\alpha < \kappa$ and $n \in \omega$:

(a) $P_{\alpha,\zeta+n+1} = P_{\alpha,\zeta+n} *_{\zeta+n} \mathcal{C}_\omega$;

(b) $\dot{a}_{\zeta+n}$ in the canonical $P_{0,\zeta+n} *_{\zeta+n} \mathcal{C}_\omega$-name for the Cohen real added by the second coordinate copy of $\mathcal{C}_\omega$;

(c) $\mathrm{supp}(P_{\alpha,\zeta+n+1}) = \mathrm{supp}(P_{\alpha,\zeta}) \cup [\zeta,\zeta+n]$; and

(d) $\iota(\zeta+k,\zeta+n+1) = 0$ for all $k \le n$, and for all $\gamma < \zeta$, $\iota(\gamma,\zeta+n+1) = \iota(\gamma,\zeta)$;

(6) if $o(C_\zeta) = \kappa$, then for all $\alpha < \kappa$, $P_{\alpha,\zeta+1} = P_{\alpha,\zeta} *_\zeta \mathcal{C}_{\alpha+1\times\lambda}$;

(7) if $o(C_\zeta) = \kappa$, then for all $n \in \omega$ and all $\alpha < \kappa$, $P_{\alpha,\zeta+3+n} = P_{\alpha,\zeta+3}$;

(8) if $o(C_\zeta) = \kappa$, then there is an $\iota_\zeta < \kappa$ such that $P_{\beta,\zeta+2} = P_{\beta,\zeta+1}$ for all $\beta < \iota_\zeta$, and there is a sequence $\langle \dot{\mathcal{D}}_{\alpha,\zeta} \colon \iota_\zeta \le \alpha < \kappa \rangle$ such that for each $\iota_\zeta \le \alpha < \kappa$:

(a) $\dot{\mathcal{D}}_{\alpha,\zeta}$ is a $P_{\alpha,\kappa+1}$-name of a Ramsey ultrafilter on ω;

(b) for each $\iota_\zeta \le \beta < \alpha$, $\dot{\mathcal{D}}_{\beta,\zeta} \subset \dot{\mathcal{D}}_{\alpha,\zeta}$;

(c) $P_{\alpha,\zeta+2} = P_{\alpha,\zeta+1} *_{\zeta+1} \mathbb{L}(\dot{\mathcal{D}}_{\alpha,\kappa})$;

(9) if $o(C_\zeta) = \kappa$, then for ι_ζ chosen as in (8)

(a) for each $\alpha < \iota_\zeta$, $P_{\alpha,\kappa+3} = P_{\alpha,\kappa+2}$;

(b) $P_{\iota_\zeta,\zeta+3} = P_{\iota_\zeta,\zeta+2} *_{\zeta+2} \dot{Q}_{\iota_\zeta,\zeta+2}$ for some $P_{\iota_\zeta,\zeta}$-name, $\dot{Q}_{\iota_\zeta,\zeta+2}$ in $H(\lambda^+)$ of a finite support product of σ-centered posets;

(c) for each $\iota_\zeta < \alpha < \kappa$, $P_{\alpha,\zeta+3} = P_{\alpha,\zeta+2} *_{\zeta+2} \dot{Q}_{\iota_\zeta,\zeta+2}$;

(10) if $o(C_\zeta) = \kappa$, then for all $\alpha < \kappa$, $n \in \omega$, and $\gamma < \zeta$, $\mathrm{supp}(P_{\alpha,\zeta+n+1}) = \mathrm{supp}(P_{\alpha,\zeta}) \cup [\zeta,\zeta+n]$, $\iota(\gamma,\zeta+n) = \iota(\gamma,\zeta)$, and $\iota(\zeta+k,\zeta+n) = \iota_\zeta$ for all $k < n \in \omega$.

It should be clear from the properties, and by induction on ζ, that for all $\alpha < \kappa$ and $\xi < \zeta$, each $p \in P_{\alpha,\xi}$ is a function with finite domain contained in $\mathrm{supp}(P_{\alpha,\xi})$. Similarly, it is immediate from the hypotheses that $\mathrm{supp}(P_{\alpha,\xi})$ has cardinality less than κ for all $(\alpha,\xi) \in \kappa\times\kappa^+$.

Before verifying the construction, we first prove, by induction on ζ, that the conditions (2)–(10) ensure that for all $\xi \le \zeta$ and $\eta \in C'_\xi$:

Claim (a): $P_{\alpha,\eta} \lessdot P_{\alpha,\xi}$ for all $o(C_\eta) + \omega \le \alpha \in \kappa$.
Claim (b): $P_{\alpha,\eta} = P_{\alpha,\xi}$ for all $\alpha < o(C_\eta) + \omega$.

If $o(C_\xi) \le \alpha$, then $P_{\alpha,\eta} \lessdot P_{\alpha,\xi}$ follows immediately from clause 2 (c) and, by induction, clauses 3 (a). Now assume $\alpha < o(C_\xi) + \omega$. If C'_ξ is not cofinal in ξ, then, by induction, $P_{\alpha,\eta} = P_{\alpha,\varphi_0^\xi}$ and, by clause 3 (a), $P_{\alpha,\varphi_0^\xi} = P_{\alpha,\xi}$. If C'_ξ is cofinal in ξ, then choose $\bar\eta \in C'_\xi$ so that $o(C_{\bar\eta}) \le \alpha < o(C_{\bar\eta}) + \omega$. By clause 2 (b), $P_{\alpha,\xi} = P_{\alpha,\bar\eta}$. By the inductive assumption, $P_{\alpha,\eta} = P_{\alpha,\bar\eta}$ since one of $\eta = \bar\eta$, $\eta \in C'_{\bar\eta}$ or $\bar\eta \in C'_\eta$ must hold.

The second thing we check is that the conditions (2)–(10) also ensure that for each $\zeta < \kappa^+$, $\langle P_{\alpha,\eta} \colon \alpha \le \kappa,\ \eta \le \zeta\rangle$ is a $\kappa\times\zeta$-matrix. We assume, by induction on limit ζ, that for $\gamma < \eta < \zeta$, $\{P_{\alpha,\gamma} \colon \alpha \le \kappa\}$ is a $<\cdot$-chain and that $P_{\alpha,\gamma} <\cdot P_{\alpha,\eta}$ for all η with $\iota(\gamma,\eta) \le \alpha \le \kappa$. Note that clauses 3 (c) and 4 (b) of the construction ensure that condition (4) of Definition 5.1 holds. We check the details for $\zeta+1$ and skip the easy subsequent verification for $\zeta+n$, $n\in\omega$. Suppose first that C'_ζ is cofinal in ζ and let $\iota(\gamma,\zeta) \le \alpha < \kappa$ for some $\gamma < \zeta$. Of course we may assume that $\gamma \notin C'_\zeta$. Since C'_ζ is cofinal in ζ, let $\eta = \min(C'_\zeta \setminus \gamma)$. By induction, $P_{\alpha,\gamma} <\cdot P_{\alpha,\eta} <\cdot P_{\alpha,\zeta}$. Now assume that C'_ζ is not cofinal in ζ. If $\gamma \le \varphi^\zeta_0$, then $\iota(\gamma,\zeta) = \iota(\gamma,\varphi^\zeta_0)$, and so we have that $P_{\alpha,\gamma} <\cdot P_{\alpha,\varphi^\zeta_0} <\cdot P_{\alpha,\zeta}$. If $\varphi^\zeta_0 < \gamma$, then choose any $l\in\omega$ so that $\gamma < \varphi^\zeta_l$. By construction, $\iota(\gamma,\zeta) \ge \iota(\gamma,\varphi^\zeta_l)$ and so for $\iota(\gamma,\zeta) \le \alpha < \kappa$, $P_{\alpha,\gamma} <\cdot P_{\alpha,\varphi^\zeta_l} <\cdot P_{\alpha,\zeta}$.

Now we consider the values of $\mathcal{A}_{\alpha,\xi,\eta}$ for $\alpha<\kappa$ and $\omega\le\xi\le\eta$ by examining the names $\dot a_\gamma$ for $\gamma\in E$.

By clause (5), $\dot a_\gamma$ is a $P_{0,\gamma+1}$-name and γ is in the domain of each $p\in P_{0,\gamma+1}$ appearing in the name. One direction of this next claim is then obvious given that the domain of every element of $P_{\alpha,\xi}$ is a subset of $\mathrm{supp}(P_{\alpha,\xi})$.

Claim (c): $\dot a_\gamma$ is a $P_{\alpha,\xi}$-name if and only if $\gamma\in\mathrm{supp}(P_{\alpha,\xi})$.

Assume that $\gamma\in\mathrm{supp}(P_{\alpha,\xi})$. We prove this by induction on ξ. If ξ is a limit, then $\mathrm{supp}(P_{\alpha,\xi})$ is defined as a union, hence there is an $\eta<\xi$ such that $\gamma\in\mathrm{supp}(P_{\alpha,\eta})$ and $P_{\alpha,\eta} <\cdot P_{\alpha,\xi}$. If $\xi=\eta+n$ for some limit η and $n\in\omega$, then $P_{\alpha,\eta} <\cdot P_{\alpha,\xi}$ and so we may assume that $\eta\le\gamma=\eta+k<\eta+n$ and that $o(C_\eta)<\kappa$. Since $P_{0,\eta+k} <\cdot P_{\alpha,\eta+k} <\cdot P_{\alpha,\eta+n} = P_{\alpha,\xi}$, it follows that $\dot a_\gamma$ is a $P_{\alpha,\xi}$-name.

We prove by induction on ξ, ξ a limit, that for all $\gamma<\xi$:

Claim (d): for all $\alpha < \iota(\gamma+1,\xi)$, γ is not in $\mathrm{supp}(P_{\alpha,\xi})$.

First consider the case that C'_ξ is cofinal in ξ and let η be the minimum element of $C'_\xi\setminus(\gamma+1)$. By definition $\iota(\gamma+1,\xi)$ is equal to $\iota(\gamma+1,\eta)$ and the claim follows since we have that $\mathrm{supp}(P_{\iota(\gamma+1,\xi),\zeta}) = \mathrm{supp}(P_{\iota(\gamma+1,\xi),\eta})$. Now assume that C'_ξ is not cofinal in ξ and assume that $\alpha<\iota(\gamma+1,\xi)$. We break into cases: $\gamma<\varphi^\xi_0$ and $\varphi^\xi_0\le\gamma<\xi$. In the first case $\iota(\gamma,\xi)=\iota(\gamma,\varphi^\xi_0)$ and the claim follows by induction and the fact that $\mathrm{supp}(P_{\alpha,\varphi^\xi_0}) = \mathrm{supp}(P_{\alpha,\xi})$ for all $\alpha<\iota(\gamma,\xi)$. Now consider $\varphi^\xi_0\le\gamma<\xi$. If $\alpha<\iota_\xi$, then $P_{\alpha,\xi}=P_{\alpha,\varphi^\xi_0}$ and, since $\iota_\xi\le\iota(\gamma+1,\xi)$, γ is not in $\mathrm{supp}(P_{\alpha,\varphi^\xi_0})$. Otherwise, choose $l,n\in\omega$ so that $\iota_\xi\le\alpha<\iota(\gamma+1,\xi)=\iota(\gamma+1,\varphi^\xi_l+n)$ as in the definition of $\iota(\gamma,\xi)$. By the minimality in the choice of φ^ξ_l+n, it follows that γ is not in $\mathrm{supp}(P_{\alpha,\varphi^\xi_{l'}+n})$ for all $l', n \in \omega$. Since $\mathrm{supp}(P_{\alpha,\xi})$ is the union of all such sets, it follows that γ is not in $\mathrm{supp}(P_{\alpha,\xi})$.

Next we prove, by induction on ζ, that the matrix so chosen will additionally satisfy condition (1). We first find a reformulation of condition (1). Note that by Claim (c), $\mathcal{A}_{\alpha,\xi,\eta} = \{\dot{a}_\gamma\colon \gamma \in E \cap \eta \setminus \mathrm{supp}(P_{\alpha,\xi})\}$.

Claim (e): For each $\alpha < \kappa$ and $\xi < \eta < \zeta$ and finite subset $\{\gamma_i\colon i < m\}$ of $E \cap \eta \setminus \mathrm{supp}(P_{\alpha,\xi})$ there is a $\beta < \kappa$ such that $\iota(\xi, \eta) \leq \beta$, $\{\gamma_i\colon i < m\} \subset \mathrm{supp}(P_{\beta,\eta})$ and $P_{\beta,\eta}$ forces that $\{\dot{a}_{\gamma_i}\colon i < m\}$ is thin over the forcing extension by $P_{\alpha,\xi}$.

Let us verify that Claim (e) follows from condition (1). Let α, ξ, η and $\{\gamma_i\colon i < m\}$ be as in the statement of Claim (e). Choose $\beta < \kappa$ so that $\iota(\xi, \eta)$ and each $\iota(\gamma_i + 1, \eta)$ is less than β. Then $P_{\alpha,\xi} <\cdot P_{\beta,\eta}$ and $\{\dot{a}_{\gamma_i}\colon i < m\} \subset \mathcal{A}_{\alpha,\xi,\eta} \setminus \mathcal{A}_{\beta,\eta,\eta}$. This value of β satisfies the conclusion of Claim (e).

Now assume that Claim (e) holds and we prove that condition (1) holds. Assume that $P_{\alpha,\xi} <\cdot P_{\delta,\eta}$. To prove that $\mathcal{A}_{\alpha,\xi,\eta} \setminus \mathcal{A}_{\delta,\eta,\eta}$ is forced by $P_{\delta,\eta}$ to be thin over the forcing extension by $P_{\alpha,\xi}$, it suffices to prove this for any finite subset of $\mathcal{A}_{\alpha,\xi,\eta} \setminus \mathcal{A}_{\delta,\eta,\eta}$. Thus, let $\{\gamma_i\colon i < m\}$ be any finite subset of $\mathrm{supp}(P_{\delta,\eta}) \cap E \cap \eta \setminus \mathrm{supp}(P_{\alpha,\xi})$. Choose β as in the conclusion of the claim. If $\beta \leq \delta$, then $P_{\delta,\eta}$ forces that $\{\dot{a}_{\gamma_i}\colon i < m\}$ is thin over the forcing extension because $P_{\beta,\eta} <\cdot P_{\delta,\eta}$ does. Similarly, if $\delta < \beta$, then $P_{\delta,\eta}$ being completely embedded in $P_{\beta,\eta}$ cannot force that $\{\dot{a}_{\gamma_i}\colon i < m\}$ is not thin over the forcing extension by $P_{\alpha,\xi}$.

We assume that $\omega \leq \zeta < \kappa^+$ is a limit and that $\langle P_{\alpha,\xi}\colon \alpha \leq \kappa,\ \xi < \zeta\rangle$ have been chosen so that conditions (1)–(10) are satisfied. We prove, by induction on $n \in \omega$, that there is an extension $\langle P_{\alpha,\xi}\colon \alpha \leq \kappa,\ \xi < \zeta + n\rangle$ that also satisfies conditions (1)–(10).

For $n = 1$, we define the sequence $\langle P_{\alpha,\zeta}\colon \alpha < \kappa\rangle$ according to the requirement of (3) or (4) as appropriate. It follows from Lemma 5.7 that (2) will hold for the extension $\langle P_{\alpha,\xi}\colon \alpha < \kappa,\ \xi < \zeta + 1\rangle$. Conditions (3)–(10) hold since there are no new requirements. We must verify that the condition in Claim (e) holds for $\eta = \zeta$. Let α, ξ and $\{\gamma_i\colon i < m\}$ be as in the statement of Claim (e) with $\eta = \zeta$. Let $C_\zeta = \{\eta_\beta\colon \beta < o(C_\zeta)\}$ be an order-preserving enumeration. We first deal with case that C'_ζ is cofinal in ζ. Choose any $\beta_0 < \kappa$ large enough so that $\gamma_i \in \mathrm{supp}(P_{\beta_0,\zeta})$ for all $i < m$. Choose $\beta_0 < \beta$ so that $\iota(\xi, \eta_{\beta_0}) \leq \beta$. Now we have that $P_{\alpha,\xi} <\cdot P_{\beta,\eta_{\beta_0}}$ and $P_{\beta,\eta_{\beta_0}} <\cdot P_{\beta,\zeta}$. Applying Claim (e) to η_{β_0}, we have that $P_{\beta,\eta_{\beta_0}}$ forces that $\{\dot{a}_{\gamma_i}\colon i < m\}$ is thin over the forcing extension by $P_{\alpha,\xi}$. As in the proof of Claim (e), this implies that $P_{\beta,\zeta}$ forces the same thing.

Now the case that C'_ζ is not cofinal in ζ. If $\alpha < \iota_\zeta$, then apply Claim (e) to choose β so that P_{β,ι_ζ} forces that $\{\dot a_{\gamma_i}\colon i < m\}$ is not thin over the extension by $P_{\alpha,\xi}$. Since $P_{\beta,\iota_\zeta} <\cdot P_{\beta,\zeta}$ holds for all β, $P_{\beta,\zeta}$ also forces that $\{\dot a_{\gamma_i}\colon i < m\}$ is not thin over the extension by $P_{\alpha,\xi}$. If $\iota_\zeta \le \alpha$, first choose $\delta < \kappa$ large enough so that $\iota(\xi,\zeta)$ and each $\iota(\gamma_i + 1, \zeta)$ is less than δ. Since $\{\gamma_i\colon i < m\}$ is a subset of $\mathrm{supp}(P_{\delta,\zeta})$, we can choose $l < \omega$ large enough so that $\{\gamma_i\colon i < \omega\} \subset \mathrm{supp}(P_{\delta,\varphi^\zeta_l})$. Applying Claim (e) to $\eta = \varphi^\zeta_l$, we choose β as in the claim. As we have seen, there is no loss to assuming that $\delta \le \beta$ and, since $P_{\beta,\varphi^\zeta_l} <\cdot P_{\beta,\zeta}$, this completes the proof.

If $o(C_\zeta) < \kappa$, then the construction of $\langle P_{\alpha,\zeta+n}\colon n \in \omega,\ \alpha < \kappa\rangle$ is canonical so that conditions (2)–(10) hold. We again verify that Claim (e) holds for all values of η with $\zeta < \eta < \zeta + \omega$. Let α, ξ and $\{\gamma_i\colon i < m\}$ be as in Claim (e) for $\eta = \zeta + n$. We may assume that $\{\gamma_i\colon i < m\} \cap \zeta = \{\gamma_i\colon i < \overline{m}\}$ for some $\overline{m} \le m$. If $\xi < \zeta$, let $\bar\xi = \xi$, otherwise, choose any $\bar\xi < \zeta$ so that $P_{\alpha,\zeta} = P_{\alpha,\bar\xi}$. Note that $\{\gamma_i\colon \overline{m} \le i < m\}$ is disjoint from the interval $[\zeta, \xi)$. Choose $\beta < \kappa$ to be greater than $\iota(\bar\xi, \zeta)$ and each $\iota(\gamma_i + 1, \zeta)$, $i < \overline{m}$, and so that $P_{\beta,\zeta}$ forces that $\{\dot a_{\gamma_i}\colon i < \overline{m}\}$ is thin over the extension by $P_{\alpha,\bar\xi}$. If $\overline{m} = m$ we are done by the fact that $P_{\alpha,\xi}$ is isomorphic to $P_{\alpha,\bar\xi} * \mathcal{C}_\omega$. In fact, we similarly have that $P_{\beta,\xi}$ forces that $\{\dot a_{\gamma_i}\colon i < \overline{m}\}$ is thin over the forcing extension by $P_{\alpha,\xi}$. Since $P_{\beta,\zeta+n}$ forces that $\bigcup\{\dot a_{\gamma_i}\colon \overline{m} \le i < m\}$ is a Cohen real over the forcing extension by $P_{\beta,\xi}$ it also follows that $P_{\beta,\zeta+n}$ forces that $\{\dot a_{\gamma_i}\colon i < m\}$ is thin over the extension by $P_{\alpha,\xi}$.

Now we come to the final case where $o(C_\zeta) = \kappa$ and the main step to the proof. The fact that Claim (e) will hold for $\eta = \zeta + 1$ is proven as above for the case when $o(C_\zeta) < \kappa$ and C'_ζ is cofinal in ζ. For values of $n > 3$, there is nothing to prove since $P_{\alpha,\zeta+3+k} = P_{\alpha,\zeta+3}$ for all $k \in \omega$. We also note that $\zeta + n \notin E$ for all $n \in \omega$.

At step $\eta = \zeta + 2$ we must take great care to preserve Claim (e) and at step $\zeta + 3$ we make a strategic choice towards ensuring that $\mathfrak{p}$ will equal κ. Indeed, we begin by choosing the lexicographic minimal pair, $(\xi_\zeta, \alpha_\zeta)$, in $\zeta \times \kappa$ with the property that there is a family of fewer than κ many canonical $P_{\alpha_\zeta,\xi_\zeta}$-names of subsets of ω and a $p \in P_{\alpha_\zeta,\xi_\zeta}$ that forces over $P_{\kappa,\zeta}$ that there is no pseudo-intersection. If there is no such pair, then let $(\alpha_\zeta, \xi_\zeta) = (\omega, \zeta + 1)$. Choose ι_ζ so that $P_{\alpha_\zeta,\xi_\zeta} <\cdot P_{\iota_\zeta,\zeta+1}$.

Assume that $\alpha, \xi, \{\gamma_i\colon i < m\}$ are as in Claim (e). We first check that if $\xi < \zeta + 2$, then there is nothing new to prove. Indeed, simply choose $\beta < \kappa$ large enough so that $P_{\beta,\zeta+1}$ has the properties required in Claim (e) for $P_{\alpha,\xi}$. Of course it follows that $P_{\beta,\zeta+2}$ forces that $\{\dot a_{\gamma_i}\colon i < m\}$ is thin over the extension by $P_{\alpha,\xi}$ since $P_{\beta,\zeta+1}$ already forces this.

This means that we need only consider instances of Claim (e) in which $\xi = \zeta+2$. The analogous statement also holds when we move to $\zeta + 3$. For each $\beta < \kappa$, let

$$T_\beta = E \cap \operatorname{supp}(P_{\beta+1,\zeta}) \setminus \operatorname{supp}(P_{\beta,\zeta})$$

and note that $P_{\beta+1,\zeta+1}$ forces that $\{\dot a_\gamma \colon \gamma \in T_\beta\}$ is thin over the extension by $P_{\beta,\zeta+1}$. Most of the work has been done for us in Lemma 5.6. Except for some minor re-indexing, we can assume that the sequence $\{P_\beta \colon \beta < \kappa\}$ in the statement of Lemma 5.6 is the sequence $\{P_{\beta,\zeta} \colon \beta < \kappa\}$. We also have that $P_{\beta,\zeta} * \mathcal{C}_{(\beta+1)\times\mathfrak{c}}$ is isomorphic to $P_{\beta,\zeta+1}$. We can choose any $P_{0,\zeta+1}$-name $\dot{\mathcal{D}}_{0,\zeta}$-name of a Ramsey ultrafilter on ω. The family $\{\dot a_\gamma \colon \gamma \in T_\beta\}$ will play the role of $\dot{\mathcal{A}}_\beta$ in the statement of Lemma 5.6, and we let $\{\dot{\mathcal{D}}_{\beta,\zeta} \colon 0 < \beta < \kappa\}$ be the sequence as supplied in Lemma 5.6.

Now assume that $\alpha < \kappa$ and that $\{\gamma_i \colon i < m\} \subset E \cap \zeta \setminus \operatorname{supp}(P_{\alpha,\zeta+1})$. Let $\{\dot F_l \colon l \in \omega\}$ be any sequence of $P_{\alpha,\zeta+2}$-names of pairwise disjoint elements of $[\omega]^k$ for some $k \in \omega$. We must find a sufficiently large $\beta < \kappa$ so that $P_{\beta,\zeta+2}$ forces that $\dot a_{\gamma_0} \cup \cdots \cup \dot a_{\gamma_{m-1}}$ is disjoint from $\dot F_l$ for some $l \in \omega$. Let $\{\beta_j \colon j < \overline{m}\}$ be the set (listed in increasing order) of $\beta < \kappa$ such that $T_\beta \cap \{\gamma_i \colon i < m\}$ is not empty and let $\beta_m = \beta_{m-1} + 1$. By re-indexing we can assume there is a sequence $\{m_j \colon j \le \overline{m}\} \subset m+1$ so that $\gamma_i \in T_{\beta_j}$ for $m_j \le i < m_{j+1}$. Although $P_{\beta,\zeta+2} = P_{\beta,\zeta+1}$ for values of $\beta < \iota_\zeta$, we will let $\overline{P}_{\beta,\zeta+2} = P_{\beta,\zeta+1} *_{\zeta+1} \mathbb{L}(\dot{\mathcal{D}}_{\beta,\zeta})$ for $\beta < \iota_\zeta$, and for consistent notation, let $\overline{P}_{\beta,\zeta+2} = P_{\beta,\zeta+2}$ for $\iota_\zeta \le \beta < \kappa$. We note that $\{\dot F_l \colon l \in \omega\}$ is also sequence of $\overline{P}_{\alpha,\zeta+2}$-names of pairwise disjoint elements of $[\omega]^k$.

For each $j < \overline{m}$, let $\dot L_{j+1}$ be the $\overline{P}_{\beta_j+1,\zeta+2}$-name of those l such that $\dot F_l$ is disjoint from $\bigcup\{\dot a_{\gamma_i} \colon i < m_{j+1}\}$. It follows, by induction on $j < \overline{m}$, that $\overline{P}_{\beta_j+1,\zeta+2}$ forces that $\dot L_{j+1}$ is infinite since $\overline{P}_{\beta_j+1,\zeta+2}$ forces that $\{\dot a_{\gamma_i} \colon m_j \le i < m_{j+1}\}$ is thin over the forcing extension by $\overline{P}_{\beta_j,\zeta+2}$. It now follows $\overline{P}_{\beta_m,\zeta+2}$ forces that $\{\dot a_{\gamma_i} \colon i < m\}$ is thin over the forcing extension by $\overline{P}_{\alpha,\zeta+2}$. If $\beta_m < \iota_\zeta$, let $\beta = \iota_\zeta$, otherwise, let $\beta = \beta_m$. It follows that $P_{\beta,\zeta+2}$ forces that $\{\dot a_{\gamma_i} : i < m\}$ is thin over the forcing extension by $P_{\alpha,\zeta+2} \lessdot \overline{P}_{\alpha,\zeta+2}$. This completes the verification of Claim (e) for the case $\eta = \zeta + 2$ and we now turn to the final case of $\eta = \zeta + 3$.

We have chosen the pair $(\alpha_\zeta, \xi_\zeta)$ when choosing ι_ζ. Let $\dot Q_{\iota_\zeta,\zeta+2}$ be the $P_{\iota_\zeta,\zeta+2}$-name of the finite support product of all posets of the form $\mathbb{M}(\mathcal{F})$ where $\mathcal{F}$ is a family of fewer than κ canonical $P_{\alpha_\zeta,\xi_\zeta}$-names of subsets of ω that is forced to have the sfip. Since $P_{\alpha_\zeta,\xi_\zeta} \in H(\lambda^+)$ the set of all such families $\mathcal{F}$ is an element of $H(\lambda^+)$. This is our value of $\dot Q_{\iota_\zeta,\zeta+2}$ as in condition (9) for the definition of $P_{\beta,\zeta+3}$ for all $\beta < \kappa$. The fact that Claim (e) holds in this case follows immediately from the induction hypothesis and Proposition 5.5. We also note that $P_{\iota_\zeta,\zeta+3}$

forces that every family of fewer than κ many canonical $P_{\alpha_\zeta,\xi_\zeta}$-names that is forced to have the sfip is also forced by $P_{\kappa,\zeta+3}$ to have a pseudo-intersection. This means that for values of $\zeta' > \zeta$ with $o(C'_\zeta) = \kappa$, the pair $(\alpha_\zeta, \xi_\zeta)$ will be lexicographically strictly smaller than the choice for ζ'. In other words, the family $\{(\xi_\zeta, \alpha_\zeta)\colon \zeta < \kappa^+,\ \mathrm{cf}(\zeta) = \kappa\}$ is strictly increasing in the lexicographic ordering.

Now we can verify that P_{κ,κ^+} forces that $\mathfrak{p} \geq \kappa$. If it does not, then there is a $\delta < \kappa$ and a family, $\{\dot{y}_\gamma\colon \gamma < \delta\}$ of canonical P_{κ,κ^+}-names of subsets of ω with some $p \in P_{\kappa,\kappa^+}$ forcing that the family has sfip but has no pseudo-intersection. By an easy modification of the names, we can assume that every condition in P_{κ,κ^+} forces that the family $\{\dot{y}_\gamma\colon \gamma < \delta\}$ is forced to have sfip. Choose any $\xi < \kappa^+$ so that $p \in P_{\kappa,\xi}$ and every $\dot{y}_\gamma$ is a $P_{\kappa,\xi}$-name. Choose $\alpha < \kappa$ large enough so that $p \in P_{\alpha,\xi}$, $\iota(\bar{\zeta}, \xi)$, and each α_γ, $\gamma < \delta$, is less than α. It follows that $\dot{y}_\gamma$ is a $P_{\alpha,\xi}$-name for all $\gamma < \delta$. Since the family $\{(\xi_\zeta, \alpha_\zeta)\colon \zeta < \kappa^+,\ \mathrm{cf}(\zeta) = \kappa\}$ is strictly increasing in the lexicographic ordering, and this ordering on $\kappa^+ \times \kappa$ has order type κ^+, there is a minimal $\zeta < \kappa^+$ (with $\mathrm{cf}(\zeta) = \kappa$) such that $(\xi, \alpha) \leq (\xi_\zeta, \alpha_\xi)$. By the assumption on (α, ξ), (ξ_ζ, α_ξ) will be chosen to equal (ξ, α). One of the factors of the poset $\dot{Q}_{\iota_\zeta,\zeta+2}$ will be chosen to be $\mathbb{M}(\{\dot{y}_\gamma\colon \gamma < \delta\})$. This proves that $P_{\kappa,\zeta+3}$ forces $\{\dot{y}_\gamma\colon \gamma < \delta\}$ does have a pseudo-intersection.

It should be clear from condition (8) in the construction that P_{κ,κ^+} forces that $\mathfrak{s} \geq \kappa^+$. To finish the proof we must show that P_{κ,κ^+} forces that $\{\dot{a}_\gamma\colon \gamma \in E\}$ is (κ, κ^+)-shattering. Since $\dot{a}_\gamma$ is forced to be a Cohen real over the extension by $P_{\kappa,\gamma}$, condition (2) in Definition 2.3 of (κ, κ^+)-shattering holds. Finally, we verify condition (1) of Definition 2.3. Choose any P_{κ,κ^+}-name $\dot{b}$ of an infinite subset of ω. Choose any $(\alpha, \xi) \in \kappa \times \kappa^+$ so that $\dot{b}$ is a $P_{\alpha,\xi}$-name. The set $E \cap \mathrm{supp}(P_{\alpha,\xi})$ has cardinality less than κ. For any $\gamma \in E \setminus \mathrm{supp}(P_{\alpha,\xi})$, there is a $(\beta, \zeta) \in \kappa \times \kappa^+$ such that $\{\dot{a}_\gamma\}$ is thin over the forcing extension by $P_{\alpha,\xi}$. It follows trivially that $P_{\beta,\zeta}$ forces that $\dot{b}$ is not a (mod finite) subset of $\dot{a}_\gamma$. □

6. Questions

(1) Is it consistent to have $\omega_1 < \mathfrak{h} < \mathfrak{b} < \mathfrak{s}$ and $\mathfrak{c}$ regular?
(2) Is it consistent to have $\omega_1 < \mathfrak{h} < \mathfrak{s} < \mathfrak{b}$ and $\mathfrak{c}$ regular?

Question (2) has been answered in [12].

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