A.E.C. WITH NOT TOO MANY MODELS
SH893

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Dedicated to Jouko Väänänen honouring his 60th birthday

Abstract. Consider an a.e.c. (abstract elementary class), that is, a class K of models with a partial order refining $\subseteq$ (submodel) which satisfy the most basic properties of an elementary class. Our test question is trying to show that the function $\dot{I}(\lambda, K)$, counting the number of models in K of cardinality λ up to isomorphism, is "nice", not chaotic, even without assuming it is sometimes 1, i.e. categorical in some λ's. We prove here that for some closed unbounded class $\mathbf{C}$ of cardinals we have (a),(b) or (c) where

- (a) for every $\lambda \in \mathbf{C}$ of cofinality $\aleph_0, \dot{I}(\lambda, K) \geq \lambda$
- (b) for every $\lambda \in \mathbf{C}$ of cofinality $\aleph_0$ and $M \in K_\lambda$, for every cardinal $\kappa \geq \lambda$ there is N_κ of cardinality κ extending M (in the sense of our a.e.c.)
- (c) $\mathfrak{k}$ is bounded; that is, $\dot{I}(\lambda, K) = 0$ for every λ large enough (equivalently $\lambda \geq \beth_{\delta_*}$ where $\delta_* = (2^{\mathrm{LST}(\mathfrak{k})})^+$).

Recall that an important difference of non-elementary classes from the elementary case is the possibility of having models in K, even of large cardinality, which are maximal, or just failing clause (b).

Date: May 7, 2015.

2010 *Mathematics Subject Classification.* Primary: 03C45, 03C48; Secondary: 03C55, 03C75.

Key words and phrases. Model theory, classification theory, a.e.c. (abstract elementary classes), categoricity, number of non-isomorphic models.

The author would like to thank the Israel Science Foundation for partial support of this research (Grant No. 242/03). The author thanks Alice Leonhardt for the beautiful typing. Paper 893 in the author's list. First Version - 2004/June/20.

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§ 0. Introduction to the subject

We would like to have classification theory for non-elementary classes K and more specifically to generalize stability. Naturally we use the function $\dot{I}(\lambda, K) =$ number of models up to isomorphism, as a major test problem. Now "non-elementary" has more than one interpretation, we shall start with the infinitary logics $\mathbb{L}_{\lambda,\kappa}$.

There are other directions; mostly where compactness in some form holds (e.g. a.e.c. with amalgmation, see about those in [She], and on a try to blend with descriptive set theory see [She16]). We had held that for $\kappa > \aleph_0$ the above cannot be developed as, e.g. if $\mathbf{V} = \mathbf{L}$ or just $\mathbf{V} \models$ "$0^\#$ does not exist", then there is $\psi \in \mathbb{L}_{\aleph_1,\aleph_1}$ such that if $\mathrm{cf}(\mu) = \aleph_0 \wedge (\forall \alpha < \mu)(|\alpha|^{\aleph_0} < \mu)$ then $M \models \psi, \|M\| = \mu$ iff $M \cong (\mathbf{L}_\mu, \in)$. However, lately [HS81] gives evidence that for θ a compact cardinal, we can generalize to $\mathbb{L}_{\theta,\theta}$ some theorems of [She90, Ch.VI] on saturation of ultrapowers and Keisler's order. This shows that stability theory for $T \subseteq \mathbb{L}_{\theta,\theta}$ exists, but it is still not clear how far we can go including $A = |N|, N \prec M$ and $\cup\{M_u : u \subset n\}$ when $\langle M_u : u \subset n\rangle$ is a so called stable $\mathscr{P}^-(n)$-system.

Anyhow (for the purposes of this history, and the present paper) we now concentrate on $\mathrm{Mod}_\psi, \psi \in \mathbb{L}_{\lambda^+,\aleph_0}$ so $\kappa = \aleph_0$. Here we have both downward LST theorems, even using $\le \lambda$ finitary Skolem functions. Also we have the upward LST theorem, using EM models.

Naturally all works started with assuming categoricity in some cardinal, except some dealing with the $\aleph_n$'s for $\psi \in \mathbb{L}_{\aleph_1,\aleph_0}$. In this case we may many times deal with $\psi \in \mathbb{L}_{\aleph_1,\aleph_0}(Q)$. Some works apeared in the eighties (see the books [Bal09], and [She09e], [She09f]).

Definition 0.1. Let $\dot{I}(\lambda, K)$ be the cardinality of $\{M/\cong : M \in K$ of cardinality $\lambda\}$ where K is a class of $\tau(K)$-models (e.g. $K = K_{\mathfrak{k}}$ where $\mathfrak{k} = (K_{\mathfrak{k}}, \le_{\mathfrak{k}})$).

First, in ZFC, answering a question of Baldwin, it was proved that ψ cannot be categorical, moreover if $\dot{I}(\aleph_1, \psi) = 1$ then $\dot{I}(\aleph_2, \psi) \ge 1$. Also if $\dot{I}(\aleph_1, \psi) < 2^{\aleph_1}$, then for some countable first order T with an atomic model $K_T = \{M : M$ an atomic model of $T\}$ is $\subseteq \mathrm{Mod}_\psi$, but $1 \le \dot{I}(\aleph_1, K_T)$. Fix T for awhile, now if $2^{\aleph_n} < \aleph_{n+1}, \dot{I}(\aleph_n, T) < \mu_{\mathrm{wd}}(\aleph_{n+1}, 2^{\aleph_n})$ for[1] every n then K_T is excellent which means it is quite similar to the class of models of an $\aleph_0$-stable countable complete first order theory. For this we consider $\mathbf{S}^m(A, M)$ for $A \subseteq M \in K_T$, only for some "nice" A. On the other hand for any n for some such T_n, K_{T_n} is categorical in every $\lambda \le \aleph_n$ but $\dot{I}(\lambda, T) = 2^\lambda$ for λ large enough. However, we do not know:

Conjecture 0.2. (Baldwin) If K_T is categorical in $\aleph_1$, then K_T is $\aleph_0$-stable, equivalently is absolutely categorical.

Related is the:

Conjecture 0.3. If K_T is categorical in $\aleph_1$ but not $\aleph_0$-stable then $\dot{I}(2^{\aleph_0}, K_T) = \beth_2$.

See work in preparation Baldwin-Laskowski-Shelah ([S⁺a]) on such K_T's; it certainly says there is a positive theory for such classes (e.g. pseudo minimal types exist). We recently have changed our mind and now think:

Conjecture 0.4. If K_T is categorical in every $\aleph_n$ then K_T is excellent.

This means that the present counter-examples are best possible. As this seems very far we may consider a weaker conjecture.

[1] note that $\mu_{\mathrm{wd}}(\lambda^+, 2^\lambda)$ is essentially 2^{λ^+}.

Conjecture 0.5. Assume $\mathbb{P}$ is a c.c.c. forcing notion of cardinality λ such that $\Vdash_{\mathbb{P}}$ "MA $+ 2^{\aleph_0} = \lambda$" and $\lambda = \lambda^{<\lambda} > \beth_{\omega_1}$. If K_T is categorical in every $\lambda < 2^{\aleph_0}$ then K_T is excellent.

There is more to be said, see [S⁺b].

* * *

In another direction, the investigation of models of cardinality $\aleph_1$ does not point to a canonical choice of logic for which the theorems on $\dot{I}(\psi, \aleph_1) = 1$ holds. This had motivated the definition of a.e.c. $\mathfrak{k} = (K_{\mathfrak{k}}, \leq_{\mathfrak{k}})$ which has the "bottom" property of elementary class $K = (\mathrm{Mod}_T, \prec), T$ a complete first order theory (i.e. $K_{\mathfrak{k}}$, a class of $\tau_{\mathfrak{k}}$-models, $\leq_{\mathfrak{k}}$ a partial order on it, both closed under isomorphism, union under $\leq_{\mathfrak{k}}$-directed systems of member of $K_{\mathfrak{k}}$ belong to $K_{\mathfrak{k}}$, moreover is a $\leq_{\mathfrak{k}}$-lub (= union of a directed system of $\leq_{\mathfrak{k}}$-submodels of N is a $\leq_{\mathfrak{k}}$-submodel of N), existence of a LST number and $M_1 \subseteq M_2 \wedge M_1 \leq_{\mathfrak{k}} N \wedge M_2 \leq_{\mathfrak{k}} N \Rightarrow M_1 \leq_{\mathfrak{k}} M_2$).

Thesis 0.6. 1) The framework of a.e.c. $\mathfrak{k}$ is wider and not too far and better than the family of $(\mathrm{Mod}_\psi, \prec_{\mathrm{sub}(\psi)})$ where $\psi \in \mathbb{L}_{\lambda^+,\aleph_0}$.
2) The right generalization of types in this context is orbital types.

Why? The "wider" in 0.6(1) is obvious. The "not too far" is by the representation theorem which says that for some vocabulary $\tau_1 \supseteq \tau(\mathfrak{k})$ of cardinality $\leq \lambda, \lambda$ the LST-number $+|\tau(\mathfrak{k})|$ and set Γ of quantifier free 1-types, $K_{\mathfrak{k}} = \mathrm{PC}(\emptyset, \Gamma) = \{M{\restriction}\tau_{\mathfrak{k}} : M \text{ a } \tau_1\text{-model omitting every } p(x) \in \Gamma\}$; similarly $\leq_{\mathfrak{k}}$. We can deduce the upward LST, and so existence of suitable $\Phi \in \Upsilon^{\mathrm{lin}}[\mathfrak{k}]$ so we have EM-models. For $\mathfrak{k}$ with $\mathrm{LST}_{\mathfrak{k}} = \aleph_0$ it is natural to restrict ourselves to the case "Γ is countable" above for both $K_{\mathfrak{k}}$ and $\leq_{\mathfrak{k}}$, then we say $\mathfrak{k}$ is $\aleph_0$-presentable. So we may wonder for such $\mathfrak{k}$ <u>if</u> $n < \omega \Rightarrow 2^{\aleph_n} + \dot{I}(\aleph_{n+1}, K_{\mathfrak{k}}) < \mu_{\mathrm{wd}}(\aleph_{n+1}, 2^{\aleph_n})$ implies $\mathfrak{k}$ satisfies the parallel of being excellent? The answer is yes by [She09e], [She09f], but the way is long. Also, we may replace $\aleph_0$ by any λ provided that $I(\lambda, K_{\mathfrak{k}}) = 1 = I(\lambda^+, K_{\mathfrak{k}})$ and $1 \leq \dot{I}(\lambda^{++}, K_{\mathfrak{k}}) < \mu_{\mathrm{wd}}(\lambda^{++}, 2^{\lambda^+})$, see more in [She].

A central notion there is "$\mathfrak{s}$ is a good λ-frame", $\mathfrak{k}_{\mathfrak{s}} = \mathfrak{k}, \mathrm{LST}_{\mathfrak{k}} \leq \lambda$, this is "bare bones superstable".

This is enough for proving

(∗) if ($\mathfrak{k}$ is an a.e.c.), $\mathrm{LST}_{\mathfrak{k}} \leq \lambda, 2^{\lambda^{+n}} < 2^{\lambda^{+n^+}}$ and $\dot{I}(\lambda^{+n}, K_{\mathfrak{k}}) = 1$ for every n and $K_{\mathfrak{k}}$ has models of cardinality $\geq \beth_{(2^{\mathrm{LST}(\mathfrak{k})})^+}$, <u>then</u> $K_{\mathfrak{k}}$ is categorical in every $\mu \geq \lambda$.

However

Conjecture 0.7. If $\mathfrak{k}$ is an a.e.c., $K_{\mathfrak{k}}$ is categorical in some λ large enough than $\mathrm{LST}_{\mathfrak{k}}$, <u>then</u> $K_{\mathfrak{k}}$ is categorical in every $\mu \geq \lambda$.

Note that [She09b] is a step ahead: in the context of 0.7, for many $\mu = \beth_\mu \in [\mathrm{LST}_{\mathfrak{k}}, \lambda)$, there is a good μ-frame $\mathfrak{s}_\mu$ such that $\mathfrak{k}_{\mathfrak{s}} = K^{\mathfrak{k}}_\mu$. If we have this for ω successive μ's we shall be done by [She09c], but in [She09b] the family of such μ's is scattered; a beginning is [SV].

A much harder conjecture is:

Conjecture 0.8. 1) The main gap theorem holds for a.e.c. $K_{\mathfrak{k}}$ for λ large enough.
2) The class $\sup - \lim_{\mathfrak{k}} = \{\lambda$: there is a super-limit $M \in K^{\mathfrak{k}}_\lambda\}$ is "nice", e.g. contains every large enough λ or contains no large enough λ.

We are continuing this work in [S+c].

* * *

We may wonder

Question 0.9. 1) Maybe there is a natural logic which is the natural framework for categoricity spectrum.
2) Also for the super-limit spectrum.

We expect such logic to be stronger than $\mathbb{L}_{\lambda^+,\aleph_0}$ but weaker than $\mathbb{L}_{\lambda,\lambda}$. This may remind us of [She12]. The logic discovered there is $\mathbb{L}^1_{<\lambda}$ for $\lambda = \beth_\lambda$, it is between $\mathbb{L}^{-1}_{<\lambda} = \cup\{\mathbb{L}_{\mu^+,\aleph_0} : \mu < \lambda\}$ and $L^0_{<\lambda,\mu} = \cup\{\mathbb{L}_{\mu^+,\mu^+} : \mu < \lambda\}$, in a strong way well ordering is not well defined and it can be characterized (as Lindström theorem characterize first order logic) and has interpolation. In addition, for λ a compact cardinal $\mathbb{L}^1_{<\lambda}$-equivalence of M_1, M_2 is equivalent to having isomorphism ω-limit ultra-powers by λ-complete ultrafilters, see [S+d].

However, probably the characterization in [She12] was by "the maximal logic such that ...". So maybe we should restrict the logic further such that "EM model can be constructed".

We conjecture there is a logic characterized by being maximal under this stronger demand, and in it we can say at least something on the function $\dot{I}(\lambda, \psi)$, and maybe much. This is interesting also from the point of view of soft model theory: we conjecture that there are many such intermediate logics with characterization (and the related interpolation theorem).

§ 1. Introduction to the paper

In this section, we begin by motivating our line of investigation. See notation in §(1D) below (and more self contained introduction in §(1B), §(1C)).

§ 1(A). **Motivation/Content.**

We knew of old (see: [She90, Ch.XIII,4.15]):

Theorem 1.1. *For a countable complete* <u>first order</u> *theory T, one of the following holds:*

- (a) *T is categorical in every $\lambda > \aleph_0$*
- (b) *$\dot{I}(\lambda, T) = \beth_2$ for every cardinal $\lambda \geq 2^{\aleph_0}$*
- (c) *$\dot{I}(\aleph_\alpha, T) \geq 1 + |\alpha|$ for every ordinal α.*

For a.e.c. we have something when $\mathfrak{k}$ is categorical in some λ's ([She09b], [She09c]) and something about $\dot{I}(\aleph_1, \mathfrak{k})$, ([She09a], about when $1 \leq \dot{I}(\aleph_1, \mathfrak{k}) < 2^{\aleph_1}$, particularly when $2^{\aleph_0} < 2^{\aleph_1}$ and then on higher cardinals) but nothing for general a.e.c. $\mathfrak{k}$. The current paper is motivated by hopes of finding something like 1.1 for a.e.c.'s. Recall the history.
Our approach here assumes/relies on:

Thesis 1.2. Reasonable to concentrate on cardinals from $\mathbf{C}_{\rm fp} = \{\lambda : \lambda = \beth_\lambda\}$, where fp stands for "fixed points".

Why? If $\lambda \in \mathbf{C}_{\rm fp}, \lambda > {\rm LST}(\mathfrak{k})$ and $M \in K^{\mathfrak{k}}_\lambda$ <u>then</u> for every $\theta \in [{\rm LST}(\mathfrak{k}), \lambda)$ and $N \leq_{\mathfrak{k}} M, \|N\| = \theta$ there is $\Phi \in \Upsilon_{\mathfrak{k},\theta}$ so $|\tau(\Phi)| = \theta$ such that for any linear order I, e.g. $I = \lambda$ we have $N \leq_{\mathfrak{k}} {\rm EM}_{\tau(\mathfrak{k})}(I, \Phi)$. So in $K^{\mathfrak{k}}_\lambda$ we have many models of the form ${\rm EM}_{\tau(\mathfrak{k})}(I, \Phi), \Phi \in \Upsilon_{\mathfrak{k},<\lambda}$. If $\dot{I}(\lambda, \mathfrak{k}) < \lambda$, many of them will be isomorphic. Hence for many $\theta_1 < \theta_2 < \lambda, \theta_1 \geq {\rm LST}(\mathfrak{k})$, every $N \leq_{\mathfrak{k}} M$ of cardinality θ_2 can be $\leq_{\mathfrak{k}}$-embedded into some ${\rm EM}_{\tau(\mathfrak{k})}(\lambda, \Phi), \Phi \in \Upsilon^{\rm or}_\kappa[\mathfrak{k}]$.

Informally, the point is it allows us to use EM models. The key point is finding a suitable template, set Φ of quantifier free types, which requires finding enough indiscernible sequences. When $K_{\mathfrak{k}}$ is an a.e.c. (as opposed to an elementary or pseudo elementary class) we must go through the Presentation Theorem to find an indiscernible sequence, i.e. we require sufficiently large models omitting the types in Γ.

To further motivate our approach, consider a not so strong conjecture, still enough to exemplify "the function $\lambda \mapsto \dot{I}(\lambda, \mathfrak{k})$ cannot be too wild".

Conjecture 1.3. 1) Letting $\mathbf{C}^{\rm fp}_{\aleph_0} = \{\lambda : \lambda = \beth_\lambda$ and ${\rm cf}(\lambda) = \aleph_0\}$ and fixing an a.e.c. $\mathfrak{k}$, not both of the following classes are stationary (or restrict yourself to some strongly inaccessible μ and "stationary" means below it):

- (a) $\mathbf{S}_1 = \{\lambda \in \mathbf{C}^{\rm fp}_{\aleph_0} : \dot{I}(\lambda, \mathfrak{k}) < \lambda\}$
- (b) $\mathbf{S}_2 = \{\lambda \in \mathbf{C}^{\rm fp}_{\aleph_0} : \dot{I}(\lambda, \mathfrak{k}) \geq \lambda\}$.

2) A weaker conjecture (presented in the abstract) is replacing clause (b) by

- $(b)'$ $\mathbf{S}_3 = \{\lambda \in \mathbf{C}^{\rm fp}_{\aleph_0}$: for every $M \in K^{\mathfrak{k}}_\lambda$ has $\leq_{\mathfrak{k}}$-extensions N of any cardinality $> \lambda\}$.

Why "$\mathrm{cf}(\lambda) = \aleph_0$"? First, trying to prove $\lambda \in \mathbf{S}_3$, we can approximate N by $\Phi \in \Upsilon^{\mathrm{or}}_{\lambda_n}[\mathfrak{k}], \lambda_n < \lambda$ as we can approximate M by $N' \leq_{\mathfrak{k}} M, \|N'\| = \lambda_n$ where $\lambda_n < \lambda_{n+1} < \lambda = \Sigma\{\lambda_m : m\}$. Second, for $\lambda \in \mathbf{C}^{\mathrm{fp}}_{\aleph_0}$ it is enough to show that $\{M/\equiv_{\mathbb{L}_{\infty,\lambda}}: M \in K^{\mathfrak{k}}_\lambda\}$ is small because it is well known that if $\mathrm{cf}(\lambda) = \aleph_0$ and M_1, M_2 are of cardinality λ and $\mathbb{L}_{\infty,\lambda}$-equivalent then they are isomorphic; on such logics see, e.g. [Dic85].

Thesis 1.4. There are, for a.e.c. $\mathfrak{k}$, meaningful dichotomy theorems for $\dot{I}(\lambda, K_{\mathfrak{k}})$ when K is a class of $\tau(\mathfrak{k})$-models, $K = K_{\mathfrak{k}}$ and $\mathfrak{k} = (K_{\mathfrak{k}}, \leq_{\mathfrak{k}})$.

This is a more concrete thesis than "considering a.e.c.'s is a good frame for model theory"; even more concrete is the "main gap conjecture". It had been proved that if $K_{\mathfrak{k}}$ is the class of models of a complete countable first order theory <u>then</u> it satisfies the "main gap", i.e. either $\dot{I}(\lambda, K)$ is large, even $= 2^\lambda$ for all uncountable λ or $\dot{I}(\aleph_\alpha, K)$ is small, even $< \beth_{\omega_1}(|\alpha|)$ for all $\alpha > 0$; see [She90, Ch.XII], "The book's main theorem". In general for a class K of τ-models the "main gap" will say that either $\dot{I}(\lambda, K)$ is large (i.e. 2^λ or $\geq \lambda^+$) for every λ large enough or it is small for every λ large enough say $\dot{I}(\aleph_\alpha, K)$ is $\leq \beth_{1,n}(|\alpha|)$ for some $n = n(K) < \omega$.

We are far away from this, still, until now for the a.e.c. the categoricity case was almost alone, i.e. we start assuming $\dot{I}(\lambda, K) = 1$ in some λ, see below, but we try here to look "higher".

The contribution of the present paper is to show that in the much more general context of a.e.c.'s for some $\aleph_0$-closed unbounded class $\mathbf{C}$ of cardinals, we have $\lambda \in \mathbf{C} \Rightarrow \dot{I}(\lambda, K_{\mathfrak{k}}) \geq \lambda$, a non-structure result, <u>or</u> $\lambda \in \mathbf{C} \wedge M \in K^{\mathfrak{k}}_\lambda \Rightarrow M$ has arbitrary large $\leq_{\mathfrak{k}}$-extensions. Note that the latter property is now taken for granted for elementary classes but is a real gain for a.e.c.

As noted in §0, in [She09b] and [She09c] we obtained results on $\dot{I}(\lambda, K)$ for a.e.c.'s assuming categoricity in some λ's. However, nothing was known for general a.e.c.'s under weaker few models assumption.

On abstract elementary classes, see [She09a], [Bal09] and [She]. We will make essential use of the Presentation Theorem, which says that every a.e.c. can be represented as a PC class, say $\mathrm{PC}(T, \Gamma)$, see [She09a, §1].

We thank the audience in the lecture in the Hebrew University seminar 2/2005 for their comments on an earlier version of this paper and Maryanthe Malliaris for helping much in improving §1 and some corrections in fall 2011 - winter 2012 and Will Boney for some further corrections (fall 2013).

§ 1(B). **Discussion.**

We give some further details regarding §(1A).

In Thesis 1.2 the result on EM models needed is: [She99, Claim 0.6], [She99, Claim 8.6], the "a.e.c. omitting types theorem" and [She99, Lemma 8.7,p.46].

Fact 1.5. Let $\mathfrak{k}$ be an a.e.c. If $\lambda \in \mathbf{C}_{\mathrm{fp}}, \lambda > \mathrm{LST}_{\mathfrak{k}}$ and $M \in K^{\mathfrak{k}}_\lambda$ <u>then</u> for every $\theta \in [\mathrm{LST}_{\mathfrak{k}}, \lambda)$ and $N \leq_{\mathfrak{k}} M$ of cardinality θ there is $\Phi \in \Upsilon[\mathfrak{k}]$ such that:

(a) $|\tau(\Phi)| = \theta$

(b) for any linear order I, in particular $I = \lambda$, without loss of generality $N \leq_{\mathfrak{k}} \mathrm{EM}_{\tau(\mathfrak{k})}(I, \Phi)$ where this denotes the reduct of the EM model to the vocabulary of $\mathfrak{k}$.

Comment:
Let us repeat, the two points when $\mathrm{cf}(\lambda) = \aleph_0$ may be as required:

(a) downward large depth in §3,
(b) if we like to find large $N \leq_{\mathfrak{k}}$-extending M for a given $M \in K^{\mathfrak{k}}_{\lambda}$, if $\mathrm{cf}(\lambda) = \aleph_0$ we can get it as an ω-limit of $M' <_{\mathfrak{k}} M, \|M'\| < \lambda$.

Such considerations further lead us to

Question 1.6. Let $\Phi \in \Upsilon_\theta[\mathfrak{k}]$ and κ be a cardinal.
Sort out the functions

(a) $\lambda \mapsto |\{\mathrm{EM}_{\tau(\mathfrak{k})}(I, \Phi)/\cong : I$ a linear order of cardinality $\lambda\}|$
(b) $\lambda \mapsto \dot{I}_{\tau(\mathfrak{k})}(\lambda, \kappa, \Phi) := |\{\mathrm{EM}_{\tau(\mathfrak{k})}(I, \Phi)/\equiv_{\mathbb{L}_{\infty,\kappa}} : I$ a linear order of cardinality $\lambda\}$.

Recall, by [She71] restricting ourselves to cardinals $\lambda = \lambda^{<\kappa}$, that the function in clause (b) of 1.6 is "nice", more specifically: if $\theta \leq \lambda_1 = \lambda_1^{<\kappa} < \lambda_2$ then $\dot{I}_{\tau(\mathfrak{k})}(\lambda_1, \kappa, \mathfrak{k}) \geq \min\{\lambda_1^+, \dot{I}(\lambda_2, \kappa, \mathfrak{k})\}$.

What occurs if $\lambda_1 < \lambda_1^{<\kappa}$? The case $\lambda_1 = \beth_\delta, \mathrm{cf}(\delta) = \aleph_0$ is more approachable than the general case, see 4.2.

Our hope is to get "bare bones superstability", i.e. good λ-frames inside $\mathfrak{k}$, (as in [She09c],[She09b]).

Another point concerning the function $\dot{I}(\lambda, \kappa, \mathfrak{k})$ is: for a model M, cardinal θ and logic $\mathscr{L}$ we can define the depth of M for $(\mathscr{L}, \theta)$ as $\min\{\alpha$: if $\bar{a}, \bar{b} \in {}^{\varepsilon}M, \varepsilon < \theta$ and $\bar{a}, \bar{b}$ realizes the same formulas of $\mathbb{L}_{\infty,\theta}$ (or $\mathbb{L}_{\infty,\theta}[\mathfrak{k}]$) of depth $< \alpha$ then they realize the same $\mathbb{L}_{\infty,\theta}$-formulas$\}$; of course, only formulas in $L_{\|M\|^{<\theta},\theta}$ are relevant. This is a good way to "slice" the equivalence and it is easier for LST considerations.

§ 1(C). **What is Done.**

A phenomena making the investigation of general a.e.c. hard is having $\leq_{\mathfrak{k}}$-maximal models of large cardinality. As with amalgamation, we may consider the property

$(*)^1_\lambda$ if $M \in K^{\mathfrak{k}}_\lambda$ then M is not $\leq_{\mathfrak{k}}$-maximal.

In investigations like [She09d] and [She01], which look at $\cup\{K^{\mathfrak{k}}_{\lambda+\ell} : \ell < 4\}$ this is relevant. But in investigations as in [She09b], looking at $\cup\{K^{\mathfrak{k}}_\lambda : \lambda = \beth_\lambda\}$, it is more natural to consider

$(*)^2_\lambda$ if $M \in K^{\mathfrak{k}}_\lambda$ then for any $\mu > \lambda$ there is $N \in K^{\mathfrak{k}}_\mu$ which $\leq_{\mathfrak{k}}$-extends N.

In §3 we consider a $\lambda = \beth_\lambda$ of cofinality $\aleph_0$ which is more than strong limit and try to prove non-structure from $\neg(*)^2_\lambda$. Given $N \in K^{\mathfrak{k}}_\lambda$ we try to build an EM model (that is construct the Φ) $\le_{\mathfrak{k}}$-extending N by an increasing chain of approximations: given $\lambda_n \to \lambda, M_n \to N, M_n \in K^{\mathfrak{k}}_{\lambda_n}$. The n-th approximation Φ_n to Φ has to have "Φ_n in a suitable sense is represented in N say of size λ_{n+1}".

Being stuck should be a reason for non-structure. For simplicity we consider only cardinals $\mu = \beth_\mu$, the gain without this restriction seems minor.

Concerning the results of §3 it would be nicer to make one more step concerning 3.15, 3.14 and deal also with $\lambda = \beth_\lambda$ instead of $\lambda = \beth_{1,\lambda}$, but a more central question is to get the non-structure result for every $\lambda' > \lambda$. It is natural to try given $\Phi \in \Upsilon^{\rm sor}_\kappa[\mathfrak{k}_M]$ and $M \le_{\mathfrak{k}} N$, to define a "depth" for approximation of the existence of a $\le_{\mathfrak{k}}$-embedding of standard $\mathrm{EM}_{\tau(\mathfrak{k})}(I,\Phi)$ into N (see Definition 2.2(2)), so that depth infinity give existence. But this does not work for us, so Definition 3.2 is a substitute, moreover we need "indirect evidence", see Definition 3.7.
Our main theorem is

Theorem 1.7. *For any a.e.c. for some closed unbounded class of cardinals* $\mathbf{C}$, *if* $(\exists \lambda \in \mathbf{C})[\mathrm{cf}(\lambda) = \aleph_0 \wedge \dot{I}(\lambda, K_{\mathfrak{k}}) < \lambda]$ *and* $M \in K_{\mathfrak{k}}$ *of cardinality* $\mu \in \mathbf{C}$ *of cofinality* $\aleph_0$, *then* M *has a proper* $<_{\mathfrak{k}}$*-extension, and even ones of arbitrarily large cardinality.*

The natural next steps are

Conjecture 1.8. 1) In Theorem 3.16, i.e. what is promised in the abstract we can choose $\mathbf{C}$ as an end segment of $\{\mu : \mu = \beth_{1,\mu}\}$ or just choose $\mathbf{C}$ as $\{\mu : \mu = \beth_{2,\mu} > \mathrm{LST}_{\mathfrak{k}}\}$.
2) For every a.e.c. $\mathfrak{k}$ for some closed unbounded class $\mathbf{C}$ of cardinals, we have $M \in K^{\mathfrak{k}}_\lambda \wedge \lambda \in \mathbf{C} \wedge \mathrm{cf}(\lambda) = \aleph_0 \Rightarrow \Upsilon^{\rm or}_\lambda[\mathfrak{k}_M] \ne \emptyset$ or $\lambda \in \mathbf{C} \wedge \mathrm{cf}(\lambda) = \aleph_0 \Rightarrow \dot{I}(\lambda, K_{\mathfrak{k}}) \ge 2^\lambda$ or at least $\ge \lambda^+$.

We intend to deal with part (1) in a continuation.

§ 1(D). **Recalling Definitions and Notation.**

Notation 1.9. Let Card be the class of infinite cardinals.

Definition 1.10. 1) Let $\beth_{0,\alpha}(\lambda) = \beth_\alpha(\lambda) := \lambda + \Sigma\{2^{\beth_\beta(\lambda)} : \beta < \alpha\}$. Let $\beth_{\varepsilon,\alpha}(\lambda)$ be defined by induction on $\varepsilon > 0$ and for each ε by induction on $\alpha : \beth_{\varepsilon,0}(\lambda) = \lambda$, for limit β we let $\beth_{\varepsilon,\beta} = \sum\limits_{\gamma<\beta} \beth_{\varepsilon,\gamma}$ and for $\varepsilon = \zeta + 1$ let $\beth_{\zeta+1,\beta+1}(\lambda) = \beth_{\zeta,\mu}(\lambda)$ where[2] $\mu = (2^{\beth_{\zeta,\beta}(\lambda)})^+$, lastly for limit ε let $\langle \beth_{\varepsilon,\alpha} : \alpha \in \mathrm{Ord}\rangle$ list in increasing order the closed unbounded class $\bigcap\limits_{\zeta<\varepsilon} \{\beth_{\zeta,\alpha} : \alpha \in \mathrm{Ord}\}$.
2) Let $\lambda \gg \kappa$ mean $(\forall \alpha < \lambda)(|\alpha|^\kappa < \lambda)$.

Convention 1.11. 1) $\mathfrak{k} = (K_{\mathfrak{k}}, \le_{\mathfrak{k}})$ is an a.e.c., with vocabulary $\tau_{\mathfrak{k}} = \tau(\mathfrak{k})$ and $\mathrm{LST}(\mathfrak{k}) = \mathrm{LST}_{\mathfrak{k}}$ its Löwenheim-Skolem-Tarski number, see [She09a, §1]. If not said otherwise, we assume $|\tau_{\mathfrak{k}}| \le \mathrm{LST}_{\mathfrak{k}}$.
2) $K^{\mathfrak{k}}_\lambda = K_{\mathfrak{k},\lambda} = \{M \in K_{\mathfrak{k}} : \|M\| = \lambda\}$.

[2] why not, e.g. $\mu = \beth_{1,\beta}(\lambda)^+$? Not a serious difference as for limit α we shall get the same value and in 1.14(1) this simplifies the notation.

3) If $K = K_{\mathfrak{k}}$ we may write $\mathfrak{k}$ instead of K; also we may write K or K_λ omitting $\mathfrak{k}$ when (as usually here) $\mathfrak{k}$ is clear from the context.

Definition 1.12. For a class K of τ-models:

- (a) for a cardinal λ, let $\dot{I}(\lambda, K)$ be the cardinality of $\{M/\cong : M \in K$ has cardinality $\lambda\}$
- (b) for a cardinal λ and a logic $\mathscr{L}$, let $\dot{I}(\lambda, \mathscr{L}, K) = \{M/\equiv_{\mathscr{L}(\tau)} : M \in K$ has cardinality $\lambda\}$.

Definition 1.13. 1) Φ is a template proper for linear orders <u>when</u>:

- (a) for some vocabulary $\tau = \tau_\Phi = \tau(\Phi), \Phi$ is an ω-sequence, with the n-th element a complete quantifier free n-type in the vocabulary τ,
- (b) for every linear order I there is a τ-model M denoted by $\mathrm{EM}(I, \Phi)$, generated by $\{a_t : t \in I\}$ such that $s \neq t \Rightarrow a_s \neq a_t$ for $s, t \in I$ and $\langle a_{t_0}, \ldots, a_{t_{n-1}}\rangle$ realizes the quantifier free n-type from clause (a) whenever $n < \omega$ and $t_0 <_I \ldots <_I t_{n-1}$. We call $(M, \langle a_t : t \in I\rangle)$ a Φ $-$ EM-pair or EM-pair for Φ; so really M and even $(M, \langle a_t : t \in I\rangle)$ are determined only up to isomorphism but abusing notation we may ignore this and use $I_1 \subseteq J_1 \Rightarrow \mathrm{EM}(I_1, \Phi) \subseteq \mathrm{EM}(I_2, \Phi)$. We call $\langle a_t : t \in I\rangle$ "the" skeleton of M; of course again "the" is an abuse of notation as it is not necessarily unique.

1A) If $\tau \subseteq \tau(\Phi)$ then we let $\mathrm{EM}_\tau(I, \Phi)$ be the τ-reduct of $\mathrm{EM}(I, \Phi)$.
2) $\Upsilon^{\mathrm{or}}_\kappa[\mathfrak{k}]$ is the class of templates Φ proper for linear orders satisfying clauses $(a)(\alpha), (b), (c)$ of Claim 1.14(1) below and $|\tau(\Phi)\backslash\tau_{\mathfrak{k}}| \leq \kappa$; normally we assume $\kappa \geq |\tau_{\mathfrak{k}}| + \mathrm{LST}_{\mathfrak{k}}$ but using $\mathfrak{k}_M$ we do not assume $\kappa \geq \|M\|$, see 2.1. The default value of κ is $\mathrm{LST}_{\mathfrak{k}}$ and then we may write $\Upsilon^{\mathrm{or}}_{\mathfrak{k}}$ or $\Upsilon^{\mathrm{or}}[\mathfrak{k}]$ and for simplicity if not said otherwise $\kappa \geq \mathrm{LST}_{\mathfrak{k}}$ (and so $\kappa \geq |\tau_{\mathfrak{k}}|$). We may omit $\mathfrak{k}$ when clear from the context and may write $\Upsilon_{\mathfrak{k}}$ using 0 as the default value.
3) For a class K of so called index models, we define "Φ proper for K" similarly when in clause (b) of part (1) we demand $I \in K$, so K is a class of τ_K-models, i.e.

- (a) Φ is a function, giving for any complete quantifier free n-type in τ_K realized in some $M \in K$, a quantifier free n-type in τ_Φ
- $(b)'$ in clause (b) of part (1), the quantifier free type which $\langle a_{t_0}, \ldots, a_{t_{n-1}}\rangle$ realizes in M is $\Phi(\mathrm{tp}_{\mathrm{qf}}(\langle t_0, \ldots, t_{n-1}\rangle, \emptyset, I))$ for $n < \omega, t_0, \ldots, t_{n-1} \in I$.

Fact 1.14. 1) Let $\mathfrak{k}$ be an a.e.c. and $M \in K_{\mathfrak{k}}$ be of cardinality $\geq \lambda = \beth_{1,1}(\mathrm{LST}_{\mathfrak{k}})$ recalling we may assume $|\tau_{\mathfrak{k}}| \leq \mathrm{LST}_{\mathfrak{k}}$ as usual.
<u>Then</u> there is a Φ such that Φ is proper for linear orders and:

- (a) (α) $\tau_{\mathfrak{k}} \subseteq \tau_\Phi$,
  (β) $|\tau_\Phi| = \mathrm{LST}_{\mathfrak{k}} + |\tau_{\mathfrak{k}}|$
- (b) for any linear order I the model $\mathrm{EM}(I, \Phi)$ has cardinality $|\tau(\Phi)| + |I|$ and we have $\mathrm{EM}_{\tau(\mathfrak{k})}(I, \Phi) \in K_{\mathfrak{k}}$
- (c) for any linear orders $I \subseteq J$ we have $\mathrm{EM}_{\tau(\mathfrak{k})}(I, \Phi) \leq_{\mathfrak{k}} \mathrm{EM}_{\tau(\mathfrak{k})}(J, \Phi)$; moreover, if $M \subseteq \mathrm{EM}(J, \Phi)$ then $(M{\restriction}\tau_{\mathfrak{k}}) \leq_{\mathfrak{k}} \mathrm{EM}_{\tau(\mathfrak{k})}(J, \Phi)$
- (d) for every finite linear order I, the model $\mathrm{EM}_{\tau(\mathfrak{k})}(I, \Phi)$ can be $\leq_{\mathfrak{k}}$-embedded into M.

1A) Moreover, assume in (1) also $\lambda = \beth_{1,1}(\kappa), \kappa \geq \mathrm{LST}_{\mathfrak{k}} + |\tau_{\mathfrak{k}}|$ so not necessarily assuming $\mathrm{LST}_{\mathfrak{k}} \geq |\tau_{\mathfrak{k}}|, M^+$ is an expansion of M with $\tau(M^+)$ of cardinality $\leq \kappa$ and $b_\alpha \in M$ for $\alpha < \lambda$ are pairwise distinct. Then there is Φ proper for linear orders such that:

- (a) (α) $\tau(M^+) \subseteq \tau_\Phi$ hence $\tau(\mathfrak{k}) \subseteq \tau_\Phi$
  - (β) τ_Φ has cardinality κ
- $(b), (c)$ has in part (1)
- (d) if I is a finite linear order and $t_0 <_I \ldots <_I t_{n-1}$ list its elements and $M_I = \mathrm{EM}(I, \Phi)$ with skeleton $\langle a_{t_i} : t \in I \rangle$, then for some ordinals $\alpha_0 < \ldots < \alpha_{n-1} < \lambda$ there is an embedding of M_I into M^+ mapping a_{t_ℓ} to b_{α_ℓ} for $\ell < n$.

2) If $\mathrm{LST}_{\mathfrak{k}} < |\tau_{\mathfrak{k}}|$ and there is $M \in K_{\mathfrak{k}}$ of cardinality $\geq \beth_{1,1}(2^{\mathrm{LST}_{\mathfrak{k}}})$, then there is $\Phi \in \Upsilon^{\mathrm{or}}_{\mathrm{LST}(\mathfrak{k}) + |\tau(\Phi)|}[\mathfrak{k}]$ such that $\mathrm{EM}(I, \Phi)$ has cardinality $\leq \mathrm{LST}_{\mathfrak{k}}$ for I finite and $\tau_\Phi \backslash \tau(M)$ has cardinality $\mathrm{LST}_{\mathfrak{k}}$. Note that $\mathscr{E}$ has $\leq 2^{\mathrm{LST}_{\mathfrak{k}}}$ equivalence classes where $\mathscr{E} = \{(P_1, P_2) : P_1, P_2 \in \tau_\Phi$ and $P_1^{\mathrm{EM}(I,\Phi)} = P_2^{\mathrm{EM}(I,\Phi)}$ for every linear order $I\}$ hence above "$\geq \beth_{1,1}(2^{\mathrm{LST}(\mathfrak{k})})$" suffice.
3) We can combine parts (1A) and (2). Also in both cases having a model of cardinality $\geq \beth_\alpha$ for every $\alpha < (2^{\mathrm{LST}(\mathfrak{k}) + |\tau(\mathfrak{k})|})^+$ suffice in parts (1),(1A) and for every $\alpha < \beth_2(\mathrm{LST}_{\mathfrak{k}})^+$ suffice in part (2).

We add

Claim 1.15. *For every cardinal μ and strong limit $\chi \leq \mu$ there is a dense κ-saturated linear order $I = I_\mu$ of cardinality μ such that:*

- $(*)$ *if $\theta < \partial = \mathrm{cf}(\partial) < \mu, 2^\theta \leq \chi$ then*
- $(*)_{I,\chi,\partial,\theta}$ *we have $2^\theta \leq \chi$ and $\theta < \partial = \mathrm{cf}(\partial)$ and $(A) \Rightarrow (B)$ where:*
  - (A) (a) *$I_0 \subseteq I$*
    - (b) *I_0 has cardinality $\leq \theta$*
    - (c) *I_1 is a linear order extending I_0*
    - (d) *$u_n \subseteq u_{n+1} \subseteq \theta = \bigcup\limits_n u_n$*
    - (e) *$\bar{t}^1_\alpha \in {}^\theta(I_1)$ for $\alpha < \partial$ and $\langle \bar{t}^1_\alpha : \alpha < \partial \rangle$ is an indiscernible sequence in I_1 over I_0 (for quantifier free formulas)*
    - (f) *for every $n, I_{1,n} = I_1 {\restriction} (\{t^1_{\alpha,i} : i \in u_n, \alpha < \partial\} \cup I_0)$ is embeddable into I over I_0*
  - (B) *there is $\langle \bar{t}_\alpha : \alpha < \mu \rangle$ such that*
    - (a) *$\bar{t}_\alpha \in {}^\theta I$*
    - (b) *$\langle \bar{t}_\alpha : \alpha < \mu \rangle$ is an indiscernible sequence over I_0 into I (for quantifier free formulas)*
    - (c) *the quantifier free type of $\bar{t}_0 \hat{} \ldots \hat{} \bar{t}_n$ over I_0 in I is equal to the quantifier free type of $\bar{t}^1_0 \hat{} \ldots \hat{} \bar{t}^1_n$ over I_0 in I_1 for every n*
  - $(B)^+$ *moreover we can replace $\langle \bar{t}_\alpha : \alpha < \mu \rangle$ by $\langle \bar{t}_s : s \in I \rangle$.*

Remark 1.16. 1) We may consider replacing (A)(e) by (e)′ $\alpha = \beth_2(\theta)^+, u_n \subseteq u_{n+1} \subseteq \theta = \bigcup\limits_n u_n$ and $I_{1,n} = \{t^1_{\alpha,\varepsilon} : \alpha < \partial, \varepsilon \in u_n\}$ and there is $\bar f = \langle f_\eta : \eta \in \Lambda\rangle$ such that f_η embeds $I_{1,\ell g(\eta)}$ into I_1 over I_0 and $\nu \triangleleft \eta \Rightarrow f_\nu \subseteq f_\eta$ where $\Lambda = \{\eta : \eta$ is a decreasing sequence of ordinals $< \alpha\}$.

2) Clauses (A)(d),(e) can be weakened to:

$\oplus$ if $i, j < \theta$ <u>then</u> $I_1{\restriction}(\{t^1_{\alpha,i}\alpha = 0, 1$ and $i < \theta\} \cup I_0)$ can be embedded into I over I_0.

But the present form fits our application.

Proof. First we give a sufficient condition for $(*)_{I,\chi,\partial,\theta}$

$\boxplus$ the linear order I satisfies $(*)_{I,\chi,\partial,\theta}$ when: $\chi > \partial = \mathrm{cf}(\partial) > \theta$ and

- (a) I is a linear order of cardinality μ
- (b) if $I_0 \subseteq I, |I_0| \le \theta$ <u>then</u> the set $I^+_0 = \{t \in I : t \notin I_0$ and there is no $t' \in I \backslash I_0 \backslash \{t\}$ realizing the same cut of I_0 in $I\}$ has cardinality $< \partial$, so if $\partial = (2^\theta)^+$ this holds
- (c) if $a <_I b$ then I is embeddable into $(a, b)_I$
- (d) every linear order of cardinality $\le \theta$ is embeddable into I
- (e) in I there is a decreasing sequence of length μ and an increasing sequence of length μ
- (f) to get $(B)^+$ we need: if J is a linear order of cardinality $\le \theta$ then we can embed $I \times J$ (ordered lexicographically into I).

It is obvious that there is such linear order. It is also easy to see that if I satisfies (a)-(d) then $(*)_{I,\partial,\theta}$. $\square_{1.15}$

§ 2. More on Templates

Why do we need $\Upsilon^{\rm sor}_\kappa[M,\mathfrak{k}]$? Remember that such Φ's are witnesses to M having $\le_{\mathfrak{k}}$-extensions in every $\mu > {\rm LST}_{\mathfrak{k}} + \|M\|$ so proving existence is a major theme here. First, why do we need below $\Upsilon^{\rm sor}_\kappa$? Because "$\Upsilon^{\rm sor}_\kappa[M,\mathfrak{k}] \ne \emptyset$" is equivalent to M being not $\le_{\mathfrak{k}}$-maximal; moreover has $\le_{\mathfrak{k}}$-extensions of arbitrarily large cardinality so proving this for every $M \in K^{\mathfrak{k}}_\lambda$ indicates "$\mathfrak{k}$ is nice, at least in λ". Second, why do we need various partial orders on $\Upsilon^{\rm sor}_\kappa[M,\mathfrak{k}]$'s?

In a major proof here to build $\Phi \in \Upsilon^{\rm sor}_\kappa[M,\mathfrak{k}]$ we use $\le_{\mathfrak{k}}$-increasing M_n with union M and try to choose $\Phi_n \in \Upsilon^{\rm sor}_\kappa[M_n,\mathfrak{k}]$ increasing with n. For this we assume $\|M_n\| = \lambda_n, \lambda_n \ll \lambda_{n+1}$ and we use an induction hypothesis that Φ_n has a say λ_{n+5}-witness in M.

Of course, it is nice if ${\rm EM}_{\tau(\mathfrak{k})}(\lambda_{n+5},\Phi_n)$ is $\le_{\mathfrak{k}}$-embeddable into M over M_n but for this we do not have strong enough existence theorem. To fine tune this and having a limit ($\Phi \in \Upsilon^{\rm sor}_\kappa[M,\mathfrak{k}]$) we need some orders.

Definition 2.1. For $\mathfrak{k}$ an a.e.c. and $M \in K_{\mathfrak{k}}$ let $\mathfrak{k}_M = \mathfrak{k}[M]$ be the following a.e.c.:

- (a) vocabulary $\tau_{\mathfrak{k}} \cup \{c_a : a \in M\}$ where the c_a's are pairwise distinct new individual constants
- (b) $N \in K_{\mathfrak{k}_M}$ <u>iff</u> $N{\restriction}\tau_{\mathfrak{k}} \in K_{\mathfrak{k}}$ and $a \mapsto c^N_a$ is a $\le_{\mathfrak{k}}$-embedding of M into $N{\restriction}\tau_{\mathfrak{k}}$;
- (c) $N_1 \le_{\mathfrak{k}_M} N_2$ <u>iff</u>
  - (α) N_1, N_2 are $\tau_{\mathfrak{k}_M}$-models from $K_{\mathfrak{k}_\mu}$
  - (β) $N_1 \subseteq N_2$
  - (γ) $(N_1{\restriction}\tau_{\mathfrak{k}}) \le_{\mathfrak{k}} (N_2{\restriction}\tau_{\mathfrak{k}})$.

Definition 2.2. 1) We call $N \in K_{\mathfrak{k}_M}$ standard <u>when</u> $M \le_{\mathfrak{k}} N{\restriction}\tau_{\mathfrak{k}}$ and $a \in M \Rightarrow c^N_a = a$.
2) If $N^1 \in K_{\mathfrak{k}_M}$ is standard and $N^0 = N^1{\restriction}\tau_{\mathfrak{k}}$ <u>then</u> we write $N^1 = N^0_{[M]}$.
3) We call $\Phi \in \Upsilon^{\rm or}_{\mathfrak{k}}$ standard <u>when</u> $M = {\rm EM}_{\tau(\mathfrak{k})}(\emptyset,\Phi)$ implies $N \le_{\mathfrak{k}} M{\restriction}\tau_{\mathfrak{k}}$ when N is the submodel[3] of $M{\restriction}\tau_{\mathfrak{k}}$ with universe $\{c^M : c \in \tau(\Phi)$ an individual constant$\}$. We call Φ fully standard when above $N = M{\restriction}\tau_{\mathfrak{k}}$.
4) Let $\Upsilon^{\rm sor}_\kappa[\mathfrak{k}]$ be the class of standard $\Phi \in \Upsilon^{\rm or}_\kappa[\mathfrak{k}]$.
5) For $M \in K_{\mathfrak{k}}$ let $\Upsilon^{\rm sor}_\kappa[\mathfrak{k}_M]$ be the class of κ-standard $\Phi \in \Upsilon^{\rm or}_\kappa[\mathfrak{k}_M]$ which[4] means:

- (a) letting $\kappa_1 = \kappa + \|M\|$, we have $\Phi \in \Upsilon^{\rm sor}_{\kappa_1}[\mathfrak{k}]$
- (b) $\{c_a : a \in M\} = \{c \in \tau(\Phi) : c$ an individual constant$\}$.
- (c) $N = {\rm EM}(\emptyset,\Phi) \Rightarrow |N| = \{c^N : c \in \tau_\Phi\}$
- (d) $\tau'_\Phi := \tau_{\mathfrak{k}} \backslash \{c \in \tau_\Phi$ is an individual constant$\}$ has cardinality $\le \kappa$
- (e) if $N = {\rm EM}(I,\Phi)$ and N_1 is a submodel of $N{\restriction}\tau'_\Phi$ then $N_1{\restriction}\tau_{\mathfrak{k}} \le_{\mathfrak{k}} N{\restriction}\tau_{\mathfrak{k}}$.

5A) We may omit κ in part (5) when $\kappa = {\rm LST}_{\mathfrak{k}} + |\tau_{\mathfrak{k}}|$. We may write $\Upsilon^{\rm sor}_\kappa[M,\mathfrak{k}]$ instead of $\Upsilon^{\rm sor}_\kappa[\mathfrak{k}_M]$, useful when $\mathfrak{k}$ is not clear from the context.

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[3]Note that we have not said "$\Phi \in \Upsilon^{\rm or}_{\mathfrak{k}[N]}$" but by renaming this follows.

[4]So though such Φ belongs to $\Upsilon^{\rm sor}_\kappa[\mathfrak{k}]$, being standard for $\Upsilon^{\rm sor}_\kappa[\mathfrak{k}_M]$ is a different demand than being standard for $\Upsilon^{\rm or}_\kappa[\mathfrak{k}]$ as for the latter possibly $\{c_a : a \in M\} \subsetneqq \{c \in \tau_\Phi : c$ an individual constant$\}$.

Observation 2.3. *1) If $\Phi \in \Upsilon^{\rm sor}_\kappa[\mathfrak{k}, M]$ then $\Phi \in \Upsilon^{\rm or}_{\kappa+\|M\|}[\mathfrak{k}]$ but not necessarily the inverse.*
2) If $\Phi \in \Upsilon^{\rm sor}_\kappa[\mathfrak{k}, M]$ then Φ is a fully standard member of $\Upsilon^{\rm or}_\kappa[\mathfrak{k}_M]$.

Claim 2.4. *Assume $\mathfrak{k}$ is an a.e.c. and $M \in K_{\mathfrak{k}}$ and $\mathfrak{k}_1 = \mathfrak{k}_M$ then:*

- (a) *$\mathfrak{k}_1$ is an a.e.c.*
- (b) $\mathrm{LST}_{\mathfrak{k}_1} = \mathrm{LST}_{\mathfrak{k}} + \|M\|$
- (c) *applying 1.14 to $\mathfrak{k}_1$, we can add "$\Phi \in \Upsilon^{\rm sor}_\kappa[\mathfrak{k}_M]$".*

Proof. Straightforward. $\square_{2.4}$

Definition 2.5. Assume J is a linear order of cardinality λ and $\lambda \to (\mu)^n_\theta$. We define the ideal $\mathscr{I} = \mathrm{ER}^n_{J,\mu,\theta}$ on the set $[J]^\mu$ by:

- $\mathscr{S} \subseteq [J]^\mu$ belongs to $\mathscr{I}$ iff for some $\mathbf{c} : [J]^{\le n} \to \theta$ there is no $s \in \mathscr{S}$ such that $\mathbf{c}{\restriction}[s]^n$ is constant.

Observation 2.6. *1) If $|J| = \lambda$ and $\lambda \to (\mu)^n_\theta$ then $\mathrm{ER}^n_{J,\mu,\theta}$ is indeed an ideal, i.e. $J \notin \mathrm{ER}^n_{J,\mu,\theta}$.*
2) If $\theta = \theta^{<\kappa}$ then this ideal is κ-complete.

Definition 2.7. 1) For vocabularies τ_1, τ_2 we say that $\mathbf{h}$ is an isomorphism from τ_1 onto τ_2 when $\mathbf{h}$ is a one-to-one function from the non-logical symbols of τ_1 (= the predicates and function symbols) onto those of τ_2 such that:

- (a) if $P \in \tau_1$ is a predicate then $\mathbf{h}(P)$ is a predicate of τ_2 and $\mathrm{arity}_{\tau_1}(P) = \mathrm{arity}_{\tau_2}(\mathbf{h}(P))$
- (b) if $F \in \tau_1$ is a function symbol[5] then $\mathbf{h}(F)$ is a function symbol of τ_2 and $\mathrm{arity}_{\tau_1}(F) = \mathrm{arity}_{\tau_2}(\mathbf{h}(F))$.

2) If $\mathbf{h}$ is an isomorphism from the vocabulary τ_1 onto the vocabulary τ_1 and M_1 is a τ_1-model then $M_1^{[\mathbf{h}]}$ is the unique M_2 such that:

- (a) M_2 is a τ_2-model
- (b) $|M_2| = |M_1|$
- (c) $P_2^{M_2} = P_1^{M_1}$ when $P_1 \in \tau_1$ is a predicate and $P_2 = \mathbf{h}(P_1)$
- (d) $F_2^{M_2} = F_1^{M_1}$ when $F_1 \in \tau_1$ is a function symbol and $F_2 = \mathbf{h}(F_1)$.

3) We say $\mathbf{h}$ is an isomorphism from τ_1 onto τ_2 over τ when $\tau \subseteq \tau_1 \cap \tau_2, \mathbf{h}$ is an isomorphism from τ_1 onto τ_2 and $\mathbf{h}{\restriction}\tau$ is the identity.
4) If $\Phi_1 \in \Upsilon^{\rm or}_\kappa$ and $\mathbf{h}$ is an isomorphism from the vocabulary $\tau_1 := \tau(\Phi)$ onto the vocabulary τ_2 then $\Phi^{[\mathbf{h}]}$ is the unique $\Phi_2 \in \Upsilon^{\rm or}_\kappa$ such that: if I is a linear order, $M_1 = \mathrm{EM}(I, \Phi_1)$ with skeleton $\langle a_t : t \in I\rangle$ then $M_1^{[\mathbf{h}]}$ is the model $(\mathrm{EM}(I, \Phi_2))^{[\mathbf{h}]}$ with the same skeleton.

Observation 2.8. *1) In 2.7(2), $M_2 = M_1^{[\mathbf{h}]}$ is indeed a τ_2-model. If in addition $\mathbf{h}$ is over τ (i.e. $\tau \subseteq \tau_1 \cap \tau_2$ and $\mathbf{h}{\restriction}\tau = \mathrm{id}_\tau$) then $M_1{\restriction}\tau = M_2{\restriction}\tau$.*
2) In 2.7(4), indeed $\Phi_2 \in \Upsilon^{\rm or}_\kappa$.
3) If $\mathbf{h}$ is an isomorphism from τ_1 onto τ_2 over $\tau_{\mathfrak{k}}$ so $\tau_{\mathfrak{k}} \subseteq \tau_1 \cap \tau_2$ and $\Phi_1 \in \Upsilon^{\rm or}_\kappa[\mathfrak{k}], \tau_1 = \tau(\Phi_1)$ then $\Phi_2 = \Phi_1^{[\mathbf{h}]}$ belongs to $\Upsilon^{\rm or}_\kappa[\mathfrak{k}]$.

[5] this includes individual constants

4) In part (3) if in addition $M \in K_{\mathfrak{k}}$ and $\Phi_1 \in \Upsilon^{\rm sor}_\kappa[M,\mathfrak{k}]$ and $a \in M \Rightarrow \mathbf{h}(c_a) = c_a$ then $\Phi_2 = \Phi_1^{[\mathbf{h}]}$ belongs to $\Upsilon^{\rm sor}_\kappa[M,\mathfrak{k}]$.

Proof. Straightforward. $\square_{2.8}$

Next we recall the partial orders $\le^1_\kappa, \le^2_\kappa$ and define an equivalence relation and some quasi-orders on $\Upsilon^{\rm or}_\kappa[\mathfrak{k}]$.

Definition 2.9. Fixing $\mathfrak{k}$, we define partial orders $\le^\oplus_\kappa = \le^1_\kappa = \le^1_{\mathfrak{k},\kappa}$ and $\le^\otimes_\kappa = \le^2_\kappa = \le^2_{\mathfrak{k},\kappa}$ on $\Upsilon^{\rm or}_\kappa[\mathfrak{k}]$ (for $\kappa \ge {\rm LST}_{\mathfrak{k}}$):
1) $\Psi_1 \le^\oplus_\kappa \Psi_2$ if $\tau(\Psi_1) \subseteq \tau(\Psi_2)$ and ${\rm EM}_{\tau(\mathfrak{k})}(I,\Psi_1) \le_{\mathfrak{k}} {\rm EM}_{\tau(\mathfrak{k})}(I,\Psi_2)$ and ${\rm EM}(I,\Psi_1) = {\rm EM}_{\tau(\Psi_1)}(I,\Psi_1) \subseteq {\rm EM}_{\tau(\Psi_1)}(I,\Psi_2)$ for any linear order I (so, of course, same a_t's, etc.).
Again for $\kappa = {\rm LST}_{\mathfrak{k}} + |\tau_{\mathfrak{k}}|$ we may drop the κ.
2) For $\Phi_1,\Phi_2 \in \Upsilon^{\rm or}_\kappa$, we say Φ_2 is an inessential extension of Φ_1 and write $\Phi_1 \le^{\rm ie}_\kappa \Phi_2$ if $\Phi_1 \le^\oplus_\kappa \Phi_2$ and for every linear order I, we have

$${\rm EM}_{\tau(\mathfrak{k})}(I,\Phi_1) = {\rm EM}_{\tau(\mathfrak{k})}(I,\Phi_2).$$

(note: there may be more function symbols in $\tau(\Phi_2)$!)

2A) We define the two-place relation $\mathbf{E}^{\rm æ}$ on $\Upsilon^{\rm or}_{\mathfrak{k}}$ as follows $\Phi_1 \mathbf{E}^{\rm æ} \Phi_2$ iff $\tau(\Phi_1) = \tau(\Phi_2)$ and for some unary function symbol $F \in \tau(\Phi_1)$ or F is just a (finite) composition[6] of such function symbols, if $M = {\rm EM}(I,\Phi_1)$ with skeleton $\langle a^1_t : t \in I\rangle$ and we let $a^2_t = F^M(a^1_2)$ for $t \in I$ then:

- $F^M(a^2_t) = a^1_t$
- M is ${\rm EM}(I,\Phi_2)$ with skeleton $\langle a^2_t : t \in I\rangle$;

"æ" stands for almost equal.
2B) Above we say $\Phi_2 \mathbf{E}^{\rm æ} \Phi_2$ is witnessed by F.
2C) We define the two-place relation $\mathbf{E}^{\rm ie}_\kappa$ on $\Upsilon^{\rm or}_{\mathfrak{k}}$ by: $\Phi_1 \mathbf{E}^{\rm ie}_\kappa \Phi_2$ iff for some $\Phi_3, \Phi_1 \le^{\rm ie}_\kappa \Phi_3$ and $\Phi_2 \le^{\rm ie}_K \Phi_3$.
2D) We define a two-place relation $\mathbf{E}^{\rm ai}_\kappa$ on $\Upsilon^{\rm or}_\kappa[\mathfrak{k}]$ by $\Phi_1 \mathbf{E}^{\rm ai}_\kappa \Phi_3$ iff for some $\Phi_2 \in \Upsilon^{\rm ai}_\kappa[\mathfrak{k}]$ we have $\Phi_1 \mathbf{E}^{\rm æ}_\kappa \Phi_2$ and $\Phi_2 \mathbf{E}^{\rm ie}_K \Phi_3$.
3) Let $\Upsilon^{\rm lin}_\kappa$ be the class of Ψ proper for linear order and producing linear orders, that is, such that:

(a) $\tau(\Psi)$ has cardinality $\le \kappa$,
(b) ${\rm EM}_{\{<\}}(I,\Psi)$ is a linear order which is an extension of I which means $s <_I t \Rightarrow {\rm EM}(I,\Psi) \models$ "$a_s < a_t$"; in fact we can have $[t \in I \Rightarrow a_t = t]$.

4) $\Phi_1 \le^\otimes_\kappa \Phi_2$ iff there is Ψ such that:

(a) $\Psi \in \Upsilon^{\rm lin}_\kappa$
(b) $\Phi_\ell \in \Upsilon^{\rm or}_\kappa$ for $\ell = 1,2$
(c) $\Phi'_2 \le^{\rm ie}_\kappa \Phi_2$ where $\Phi'_2 = \Psi \circ \Phi_1$, i.e.

[6] but abusing our notation we may still write $F \in \tau_\Phi$

$$\mathrm{EM}_{\tau(\Phi_1)}(I,\Phi_2') = \mathrm{EM}(\mathrm{EM}_{\{<\}}(I,\Psi),\Phi_1).$$

(So we allow further expansion by functions definable from earlier ones (composition or even definition by cases), as long as the number is $\le \kappa$).

It is not a real loss to restrict ourselves to standard Φ because

Claim 2.10. *1) For every $\Phi_1 \in \Upsilon^{\rm or}_\kappa[\mathfrak{k}]$ there is a standard $\Phi_2 \in \Upsilon^{\rm or}_\kappa[\mathfrak{k}]$ such that $\Phi_1 \le^{\rm ie}_\kappa \Phi_2$; moreover $M = \mathrm{EM}(\emptyset,\Phi_2) \Rightarrow |M| = \{c^M : c \in \tau(\Phi_2)$ an individual constant$\}$, that is Φ_2 is fully standard.*
2) Assume $\Phi_1 \in \Upsilon^{\rm or}_\kappa[\mathfrak{k}], F \in \tau(\Phi)$ is a unary function symbol such that $M = \mathrm{EM}(I,\Phi_1) \wedge t \in I \Rightarrow F^M(F^M(a_t)) = a_t$. Then for a unique $\Phi_2, \Phi_1 \mathbf{E}^{\rm æ}\Phi_2$ as witnessed by F and $\Phi_1 \in \Upsilon^{\rm sor}_\kappa[\mathfrak{k}_M] \Leftrightarrow \Phi_2 \in \Upsilon^{\rm sor}_\kappa[\mathfrak{k}_M]$.
3) $\mathbf{E}^x_\kappa$ is an equivalence relation on $\Upsilon^{\rm or}_K[\mathfrak{k}]$ for $x \in \{$æ$, \mathrm{ie}, \mathrm{ai}\}$ all refining $\mathbf{E}^{\rm ai}_\kappa$.

Proof. Obvious. $\square_{2.10}$

Observation 2.11. *Let $\ell = 1,2$.*
1) The relation $\le^\ell_\kappa$ is a partial order on $\Upsilon^{\rm or}_\kappa[\mathfrak{k}]$.
2) If $\langle \Phi_\alpha : \alpha < \delta\rangle$ is $\le^\ell_\kappa$-increasing with δ a limit ordinal $< \kappa^+$ then $\bigcup_{\alpha<\delta} \Phi_\alpha$ naturally defined is a $\le^\ell_\kappa$-lub.
3) $\mathbf{E}^{\rm æ}$ is an equivalence relation on $\Upsilon^{\rm or}$.
4) If $\Upsilon^{\rm or}_{\kappa_1}[\mathfrak{k}] \subseteq \Upsilon^{\rm or}_{\kappa_2}[\mathfrak{k}]$ then $\kappa_1 \le \kappa_2$. If $\kappa_1 \le \kappa_2$ and $\iota \in \{1,2\}$ and $\Phi,\Psi \in \Upsilon^{\rm or}_{\kappa_1}$ then $[\Phi \le^\iota_{\kappa_1} \Psi \Leftrightarrow \Phi \le^\iota_{\kappa_2} \Psi)$.
5) Similarly for $\Upsilon^{\rm sor}_\kappa[\mathfrak{k}_M]$ defined in 2.2(5).

Definition 2.12. 1) For $\kappa \ge \mathrm{LST}_{\mathfrak{k}} + |\tau_{\mathfrak{k}}|$, we define $\le^{\odot}_\kappa = \le^3_\kappa$, in full $\le^3_{\mathfrak{k},\kappa}$, a two-place relation on $\Upsilon^{\rm sor}_\kappa[\mathfrak{k}]$, recalling Definition 2.2(5) as follows:

Let $\Phi_1 \le^3_\kappa \Phi_2$ mean that: for every linear order I_1 there are a linear order I_2 and $\le_{\mathfrak{k}}$-embedding h of $\mathrm{EM}_{\tau(\mathfrak{k})}(I_1,\Phi_1)$ into $\mathrm{EM}_{\tau(\mathfrak{k})}(I_2,\Phi_2)$, moreover every individual constant c of $\tau(\Phi_1)$ is an individual constant of $\tau(\Phi_2)$ and $h(c^{\mathrm{EM}(I_1,\Phi_1)}) = c^{\mathrm{EM}(I_2,\Phi_2)}$.

2) We define $\le^4_\kappa = \le^4_{\mathfrak{k},\kappa}$; a two-place relation on $\Upsilon^{\rm sor}_\kappa[\mathfrak{k}]$ as follows.

Let $\Phi_1 \le^4_\kappa \Phi_2$ mean that: for some F we have:

- (a) $\Phi_1,\Phi_2 \in \Upsilon^{\rm sor}_\kappa[\mathfrak{k}]$
- (b)
  - $\tau(\Phi_1) \subseteq \tau(\Phi_2)$
  - $F \in \tau(\Phi_2)$ is a unary function symbol or as in 2.9(2A)
- (c) if I is a linear order and $M_2 = \mathrm{EM}(I,\Phi_2)$ with skeleton $\langle a^2_s : s \in I\rangle$ then there is $M_1 = \mathrm{EM}(I,\Phi_1)$ with skeleton $\langle a^1_s : s \in I\rangle$ such that
  - $a^1_s = F^{M_2}(a^2_s)$ for $s \in I$
  - $a^2_s = F^{M_2}(a^1_s)$ for $s \in I$
  - $M_1 \subseteq M_2{\restriction}\tau_{\Phi_1}$ so $\tau(\Phi_1) \subseteq \tau(\Phi_2)$
  - $(M_1{\restriction}\tau_{\mathfrak{k}}) \le_{\mathfrak{k}} (M_2{\restriction}\tau_{\mathfrak{k}})$
  - $c^{M_1} = c^{M_2}$ when $c \in \tau(\Phi_1)$ is an individual constant.

Remark 2.13. So $\le^4_\kappa$ is like $\le^1_\kappa$ but we demand less as $a^1_s = a^2_s$ is weakened by using the function symbol F.

Claim 2.14. *1) $\le^3_\kappa$ is a partial order on $\Upsilon^{\rm sor}_\kappa[\mathfrak{k}]$ as well as $\le^4_\kappa$; also for $\Phi_1,\Phi_2 \in \Upsilon^{\rm sor}_\kappa[\mathfrak{k}]$ and $\ell = 1,2,4$ we have $\Phi_1 \le^2 \Phi_2 \Rightarrow \Phi_1 \le^1 \Phi_2 \Rightarrow \Phi_1 \le^4 \Phi_2 \Rightarrow \Phi_1 \le^3 \Phi_3$.*
2) Assume $\Phi_1,\Phi_2 \in \Upsilon^{\rm sor}_\kappa[\mathfrak{k}]$ have the same individual constants. Then $\Phi_1 \le^3_\kappa \Phi_2$ iff as in 2.12(1) restricting ourselves to $I = \beth_{1,1}(\kappa)$ iff $\Phi_1,\Phi_2 \in \Upsilon^{\rm sor}_\kappa[\mathfrak{k}]$ and for some F and $\Phi'_1,\Phi'_2 \in \Upsilon^{\rm sor}_\kappa[\mathfrak{k}]$ and we have $\Phi_1 \le^4_\kappa \Phi'_1$ witnessed by F and $\Phi'_1 \mathbf{E}^{\rm æ}\Phi'_2$ witnessed by F and for some $\tau_*,\mathbf{h}$ we have $\tau(\mathfrak{k}) \subseteq \tau_* \subseteq \tau(\Phi'_1), h$ is an isomorphism from $\tau(\Phi_2)$ onto τ_* over $\tau(\mathfrak{k}) \cup \{c : c \in \tau(\Phi_1)\}$ and $\Phi_2^{[\mathbf{h}]} \le^{\rm ie}_\kappa \Phi'_2$ iff for some $\Phi' \in \Upsilon^{\rm sor}_\kappa[\mathfrak{k}]$ we have $\Phi_1 \le^3 \Phi'$ and $\Phi'\mathbf{E}^{\rm ai}_\kappa\Phi$, see 2.9(2).
3) If $\Phi_n \in \Upsilon^{\rm sor}_\kappa[\mathfrak{k}]$ and $\Phi_n \le^3_\kappa \Phi_{n+1}$ then there is $\Phi_\omega \in \Upsilon_\kappa[\mathfrak{k}]$ such that $n < \omega \Rightarrow \Phi_n \le^3_\kappa \Phi$; moreover, ${\rm EM}_{\tau(\mathfrak{k})}(\emptyset,\Phi)$ is the union of the $\le_{\mathfrak{k}}$-increasing sequence $\langle {\rm EM}_{\tau(\mathfrak{k})}(\emptyset,\Phi_n) : n < \omega\rangle$.
4) Similarly for $\le^4_\kappa$.

Proof. 1) Obvious.
2) First clause implies second clause
Holds trivially.

Second clause implies the third clause

Let $I_1 = (\lambda,<),\lambda$ large enough, e.g. $\lambda = \beth_{1,1}(\kappa)$. Let $M_1 = {\rm EM}(I_1,\Phi_1)$ be with skeleton $\langle a^1_t : t \in I_1\rangle$. As $\Phi_1 \le^3_\kappa \Phi_2$, there is a linear order I_2 and $M_2 = {\rm EM}(I_2,\Phi_2)$ with skeleton $\langle a^2_t : t \in I_2\rangle$ and $\le_{\mathfrak{k}}$-embedding f from $M_1{\restriction}\tau(\mathfrak{k})$ into $M_2{\restriction}\tau(\mathfrak{k})$ such that $c \in \tau(\Phi_1) \Rightarrow c \in \tau(\Phi_2) \wedge f(c^{M_1}) = c^{M_2}$; so without loss of generality $|I_2| > \lambda$ by renaming $f{\restriction}{\rm Sk}(\emptyset,M_1)$ is the identity and as $\|M_2\| > \|M_1\| \ge \lambda > \kappa \ge |\tau(M_2)|$, clearly we can find pairwise distinct $t_\alpha \in I_2$ for $\alpha < \lambda$ such that $\{a^2_{t_\alpha} : \alpha < \lambda\} \cap \{f(a^1_\alpha) : \alpha < \lambda\} = \emptyset$".

Let $\tau_1 = \tau(\Phi_1)$ and[7] let the pair $(\mathbf{h},\tau_3)$ be such that: $\mathbf{h}$ is an isomorphism from the vocabulary $\tau_2 = \tau(\Phi_2)$ onto τ_3 over $\tau(\mathfrak{k}) \cup \{c : c \in \tau(\Phi_1)\}$ such that $\tau_1 \cap \tau_3 = \tau(\mathfrak{k}) \cup \{c : c \in \tau(\Phi_2)\}$ and let $M_3 = M_2^{[\mathbf{h}]}$, so $\tau(M_3) = \tau_3, \Phi_3 = \Phi_2^{[\mathbf{h}]}$ so $\tau(M_3) = \tau_3 = \tau(\Phi_3)$ and M_3 is an ${\rm EM}(I_2,\Phi_3)$ model with skeleton $\langle a^2_t : t \in I_2\rangle$.

Let $\tau_4 = \tau_3 \cup \tau_1 \cup \{F,P_\ell : \ell = 1,2,3,4\}$ with F a one place function symbol and $P_\ell, F \notin \tau_3 \cup \tau_1$ and P_ℓ one place predicates for $\ell = 1,2,3,4$. We define a τ_4-model M_4:

- $\bullet_1$ it has universe $|M_3|$
- $\bullet_2$ $F^{M_4}(a^2_{t_\alpha}) = f(a^1_\alpha)$ and $F^{M_4}(f(a^1_\alpha)) = a^2_{t_\alpha}$
- $\bullet_3$ $P^{M_4}_1 = \{a^1_t : t \in I_1\}, P^{M_4}_2 = \{a^2_t : t \in I_2\}, P^{M_4}_3 = \{f(a^1_t) : t \in I_1\}, P^{M_4}_4 = {\rm Rang}(f)$
- $\bullet_4$ $M_4{\restriction}\tau_3 = M_3$
- $\bullet_5$ f embeds M_1 into $M_4{\restriction}\tau_1$.

Clearly there is no problem to do this and we apply 1.14(1A) with $M_4{\restriction}\tau(\mathfrak{k}), M_4, \langle a^2_{t_\alpha} : \alpha < \lambda\rangle$, here standing for $M, M^+, \langle b_\alpha : \alpha < \lambda\rangle$ there and get Φ_4 standing for Φ there. Now by inspection (see Definition 2.12(2)):

$(*)_1$ $\Phi_1 \le^4_\kappa \Phi_4$
$(*)_2$ $\Phi_3 \le^\otimes_\kappa \Phi_4$; moreover $\Phi_3 \le^{\rm ie} \Phi_4$.

[7]The reason is that there may be a symbol in $\tau(\Phi_2) \cap \tau(\Phi_c)$ but not from $\tau(\mathfrak{k}_1) \cup \{c : c \in \tau(\Phi_1)\}$. We eliminate this "accidental equality". Only now $\tau_3 \cup \tau_1$ "makes sense".

We derive Φ_5 from Φ_4 by 2.10(2) using our F so $\Phi_4 \mathbf{E}^{æ} \Phi_5$. To show that the third clause of part (2) indeed holds, we just note that $\Phi'_1, \Phi'_2, \mathbf{h}, \tau_*$, there can stand for $\Phi_4, \Phi_5, \mathbf{h}, \tau_3$ here, so we are done.

The third clause implies the first clause:

So we are given F and $\Phi_1, \Phi_2 \in \Upsilon^{\rm sor}_\kappa[\mathfrak{k}], \Phi'_1, \Phi'_2 \in \Upsilon^{\rm sor}_\kappa[\mathfrak{k}], \tau_* \subseteq \tau(\Phi'_2)$ including $\tau(\mathfrak{k})$ and an isomorphism $\mathbf{h}$ from $\tau(\Phi_2)$ onto τ_* over $\tau_{\mathfrak{k}} \cup \{c : c \in \tau(\Phi_1)\}$ such that $\Phi_1 \le^4_\kappa \Phi'_2$ witness by $F, \Phi'_1 \mathbf{E}^{æ} \Phi'_2$ witness by F and $\Phi_2^{[\mathbf{h}]} \le^{\otimes}_\kappa \Phi'_2$.

Let $\Psi \in \Upsilon^{\rm lin}_\kappa$ witness $\Phi_2^{[\mathbf{h}]} \le^{\otimes}_\kappa \Phi'_2$; and for uniformity of notation we let $\Phi_3 := \Phi'_2$.

We have to prove $\Phi_1 \le^3_\kappa \Phi_2$ so let I_1 be a linear order.

Let $M^*_1 = \mathrm{EM}(I_1, \Phi_1)$ be with skeleton $\langle a^1_t : t \in I_1\rangle$, let $I_2 = \mathrm{EM}_{\{<\}}(\Psi, I_1)$ so with skeleton $\langle t : t \in I_1\rangle$. Let $M_1 \subseteq M_2$ be defined by $M_\ell = \mathrm{EM}(I_\ell, \Phi_2)$ with skeleton $\bar a^\ell = \{a^2_t : t \in I_\ell\}$ for $\ell = 1, 2$ and let $M_3 = \mathrm{EM}(I_1, \Phi'_1)$ be with skeleton $\{a^3_t : t \in I_1\}$.

By the choice of Ψ and of I_2 without loss of generality $M_2^{[\mathbf{h}]} = M_3 {\restriction} \tau_*$.

Lastly, there is a unique embedding f of M^*_1 into $M_3{\restriction}\tau(\Phi_1)$ mapping a^1_t to $F^{M_3}(a^2_t)$ for $t \in I_1$. Easily f is a $\le_{\mathfrak{k}}$-embedding of $M_1{\restriction}\tau(\mathfrak{k})$ into $M_3{\restriction}\tau(\mathfrak{k})$ mapping c^{M_1} to c^{M_2} for $c \in \tau(\Phi_1)$ and $M_3{\restriction}\tau(\mathfrak{k}) = M_2{\restriction}\tau(\mathfrak{k})$ and $c \in \tau(\Phi_1) \Rightarrow c \in \tau(\Phi_2) \wedge f(c^{M^*_1}) = c^{M_2}$.

We leave the fourth clause to the reader.

3) By parts (2) and (4) or directly using 1.14(1) and the definition of $\le^3_\kappa$.

4) So assume that $n < \omega \Rightarrow \Phi_n \le^4_\kappa \Phi_{n+1}$ as witnessed by $F_n \in \tau(\Phi_{n+1})$. For any infinite linear order I we can choose $M_n = \mathrm{EM}(I_n, \Phi_n)$ with skeleton $\langle a^n_t : t \in I\rangle$. Let $\tau_\omega = \cup\{\tau(\Phi_n) : n < \omega\}$. Without loss of generality $M_n \subseteq M_{n+1}{\restriction}\tau(\Phi_n), F_n^{M_{n+1}}(a^{n+1}_t) = a^n_t$ and $F_n^{M_{n+1}}(a^n_t) = a^{n+1}_t$. For each n we define $M_{\omega,n} = \cup\{M_{n+k}{\restriction}\tau_n : k \in [n, \omega)\}$, so $n_1 < n_2 \Rightarrow M_{\omega,n_1} = M_{\omega,n_2}{\restriction}\tau(\Phi_{n_1})$. Hence letting $\tau_\omega = \cup\{\tau(\Phi_n) : n < \omega\}$ there is a τ_ω-model M_ω with universe $|M_{\omega,0}|$ such that $M_\omega{\restriction}\tau_n = M_{\omega,n}$ for $n < \omega$. Now define Φ by $\Phi(n) = \mathrm{tp}_{\rm qf}(\langle a^0_{t_0}, \ldots, a^0_{t_{n-1}}\rangle, \emptyset, M_\omega)$ whenever $t_0 <_I \ldots <_I t_n$.

Clearly $M_\omega = \mathrm{EM}(I, \Phi)$ with skeleton $\langle a^0_t : t \in I\rangle$ and $F_{n-1} \circ \ldots \circ F_1 \circ F_0$ witness $\Phi_n \le^4_\kappa \Phi_\omega$, here we need composition of unary functions. $\square_{2.14}$

Claim 2.15. *For $M \in K_{\mathfrak{k}}$ of cardinality $\kappa \ge \mathrm{LST}_{\mathfrak{k}} + |\tau_{\mathfrak{k}}|$ the following conditions are equivalent:*

- (a) $\Upsilon^{\rm or}_\kappa[\mathfrak{k}_M] \ne \emptyset$
- (b) *for every $\lambda \ge \kappa$ there is N such that $M \le_{\mathfrak{k}} N \in K^{\mathfrak{k}}_\lambda$*
- (c) *for every $\alpha < (2^\kappa)^+$ there is $N \in K^{\mathfrak{k}}_{\ge \beth_\alpha}$ which $\le_{\mathfrak{k}}$-extend M*
- (d) *there is $\Phi \in \Upsilon^{\rm or}_\kappa[\mathfrak{k}_M]$ such that if $N = \mathrm{EM}(I, \Phi)$ and $N{\restriction}\tau_{\mathfrak{k}_M}$ is standard then $M = (N{\restriction}\tau_{\mathfrak{k}}){\restriction}\{c^N : c \in \tau_\Phi$ an individual constant$\}$*
- (e) $\Upsilon^{\rm sor}_\kappa[\mathfrak{k}_M]$ *is non-empty.*

Proof. For (d) note that we can replace an individual constant by a unary function which is interpreted as being a constant function. More generally an n-place function F^N by functions F_1, F_2 where

- F_1 is a $(n+1)$-place function
- if $\bar a = \langle a_\ell : \ell \le n\rangle \in {}^{n+1}N \backslash {}^{n+1}M$ then $F_2(\bar a) = F^N(\bar a{\restriction}n)$
- if $\bar a \in {}^{n+1}M$ then $F_1(\bar a) = a_0$

$\square_{2.15}$

Claim 2.16. *If (A) then (B) when:*

(*A*) (*a*) $M_1 \leq_{\mathfrak{k}} M_2$

(*b*) Φ_1, Ψ_1 *are from* $\Upsilon^{\rm sor}_\kappa[\mathfrak{k}_{M_1}]$ *so are* κ*-standard*

(*c*) $\Psi_2 \in \Upsilon^{\rm sor}_\kappa[\mathfrak{k}_{M_2}]$

(*d*) $\Phi_1 \leq^4_\kappa \Psi_1$

(*e*) $\Psi_1 \leq^1_\kappa \Psi_2$

(*f*) $\{c_a : a \in M_2\} \cap \tau(\Psi_1) = \{c_a : a \in M_1\}$

(*B*) *there is* Φ_2 *such that*

(*a*) $\Phi_2 \in \Upsilon^{\rm sor}_\kappa[\mathfrak{k}_{M_2}]$

(*b*) $\Phi_1 \leq^1_\kappa \Phi_2$

(*c*) $\Phi_2 \leq^4_\kappa \Psi_2$.

Proof. Straightforward: let I be an infinite linear order, $M_2 = {\rm EM}(I, \Psi_2)$ be with skeleton $\langle a^2_t : t \in I\rangle$. Let the unary function symbol F witness $\Phi_1 \leq^4_\kappa \Psi_1$ so $F \in \tau(\Psi_1) \subseteq \tau(\Psi_2)$ and let $a^1_t = F^{M_2}(a^2_t)$. Clearly $\langle a^1_t : t \in I\rangle$ is indiscernible for quantifier formulas in M_2 and generate it hence for some $\Phi_2 \in \Upsilon^{\rm or}_\kappa$ we have $M_2 = {\rm EM}(I, \Phi_2)$ with skeleton $\langle a^1_t : t \in I\rangle$. Clearly $\Phi_2 \in \Upsilon^{\rm sor}_\kappa[\mathfrak{k}]$. Also $\Phi_2 \mathbf{E}^{\rm ae}\Phi_2$ hence $\Phi_2 \leq^4_\kappa \Psi_2$ and $\Phi_1 \leq^\oplus_\kappa \Phi_2$ as required. $\square_{2.16}$

* * *

The following will be used when applied to a tree of approximations to embedding of EM-models to a model. In fact, we use only 2.18 for the case $\mathscr{S} = \mathscr{T} \backslash \max(\mathscr{T})$, see background in 2.19.

Definition 2.17. 1) We say $\mathbf{i} = (\mathscr{T}, \bar{\mathbf{I}}) = (\mathscr{T}_{\mathbf{i}}, \bar{\mathbf{I}}_{\mathbf{i}})$ is pit (partially idealized tree) <u>when</u>:

(*a*) $\mathscr{T}$ is a tree with $\leq \omega$ levels and

- for transparency it is a set of finite sequences ordred by $\triangleleft$, closed under initial segments
- let ${\rm lev}(\eta, \mathscr{T}) = {\rm lev}_{\mathscr{T}}(\eta)$ be the level of $\eta \in \mathscr{T}$ in $\mathscr{T}$, that is $|\{\nu \in \mathscr{T} : \nu \triangleleft \eta\}|$
- let ${\rm rt}_{\mathscr{T}}$ be the root
- the n-level of $\mathscr{T}$ is the set $\{\eta : {\rm lev}_{\mathscr{T}}(n) = n\}$ so we have
- ${\rm lev}_{\mathscr{T}}(\eta) = \ell g(\eta)$ and ${\rm rt}_{\mathscr{T}} = \langle\rangle$

(*b*) $\mathbf{I} = \langle \mathbf{I}_\eta : \eta \in \mathscr{S}\rangle$ where $\mathscr{S} \subseteq \mathscr{T} \backslash \max(\mathscr{T})$, we may write $\mathscr{S}_{\mathbf{i}} = \mathscr{S}$

(*c*) $\mathbf{I}_\eta$ is an ideal on ${\rm suc}_{\mathscr{T}}(\eta) := \{\rho : \nu \in \mathscr{T}, \eta <_{\mathscr{T}} \rho$ and there is no $\nu \in \mathscr{T}$ satisfying $\eta <_{\mathscr{T}} \nu <_{\mathscr{T}} \rho\}$ or just an ideal on a set which $\supseteq {\rm suc}_{\mathscr{T}}(n)$ such that ${\rm suc}_{\mathscr{T}}(\eta) \notin \mathbf{I}_\eta$; we may write $\mathbf{I}_{\mathbf{i},\eta}$.

1A) If $\mathbf{I}_\eta = \{\{s : \eta\hat{\ }\langle s\rangle \in X\} : X \in \mathbf{I}'_\eta\}$ for some ideal $\mathbf{I}'_\eta$ on some set <u>then</u> abusing notation we may write $\mathbf{I}'_\eta$ instead of $\mathbf{I}_\eta$.

2) Let $(\mathscr{T}_1, \bar{\mathbf{I}}_1) \leq (\mathscr{T}_2, \bar{\mathbf{I}}_2)$ <u>when</u> (each is a pit and):

(a) $\mathscr{T}_1 \subseteq_{\rm tr} \mathscr{T}_2$ which means:

(α) $\eta \in \mathscr{T}_2 \Rightarrow \eta_1 \in \mathscr{T}_1 \wedge \text{lev}(\eta, \mathscr{T}_2) = \text{lev}(\eta, \mathscr{T}_1) \wedge \text{suc}(\eta, \mathscr{T}_2) \subseteq \text{suc}(\eta, \mathscr{T}_1)$

(β) $\leq_{\mathscr{T}_1} = <_{\mathscr{T}_2} \restriction \mathscr{T}_1$

(b) $\bar{\mathbf{I}}_2 = \bar{\mathbf{I}}_1 \restriction \mathscr{T}_2$, i.e. $\bar{\mathbf{I}}_1 \restriction \{\eta \in \mathscr{T}_1 : \eta \in \text{Dom}(\mathbf{I}_1)$ and $\eta \in \mathscr{T}_2\}$

(c) if $\eta \in \mathscr{T}_2 \backslash \mathscr{S}_{\mathbf{i}_2}$ then $\text{suc}(\eta, \mathscr{T}_2) = \text{suc}(\eta, \mathscr{T}_1)$.

2A) Let $(\mathscr{T}_1, \bar{\mathbf{I}}_1) \leq_{\rm pr} (\mathscr{T}_2, \bar{\mathbf{I}}_2)$ <u>when</u> (each is a pit and)

(a), (b), (c) as above

(d) if $\eta \in \text{Dom}(\bar{\mathbf{I}}_2)$ then $\text{suc}_{\mathscr{T}_1}(\eta) \backslash \text{suc}_{\mathscr{T}_2}(\eta) \in \mathbf{I}_{1,\eta}$.

3) We say $(\mathscr{T}, \bar{\mathbf{I}})$ is κ-complete <u>when</u> every ideal $\mathbf{I}_\eta$ is.
4) For $\mathbf{i} = (\mathscr{T}, \bar{\mathbf{I}})$ we define $\text{Dp}_{\mathbf{i}} = \text{Dp}_{\mathscr{T}, \bar{\mathbf{I}}} : \mathscr{T} \to \text{Ord} \cup \{\infty\}$ by (stipulate $\infty + 1 = \infty$) defining when $\text{Dp}_{\mathscr{T}, \bar{\mathbf{I}}}(\eta) \geq \alpha$ by induction on α as follows:

(a) if $\eta \in \max(\mathscr{T})$ <u>then</u> $D_{\mathscr{T}, \mathbf{I}}(\eta) \geq \alpha$ iff $\alpha = 0$

(b) if $\eta \in \mathscr{T} \backslash \max(\mathscr{T})$ and $\eta \in \mathscr{S}_{\mathbf{i}} = \text{Dom}(\bar{\mathbf{I}})$ <u>then</u> $\text{Dp}_{\mathscr{T}, \mathbf{I}}(\eta) \geq \alpha$ iff $(\forall \beta < \alpha)(\exists X \subseteq \text{suc}_{\mathscr{T}}(\eta))[X \in \mathbf{I}^+_\eta \wedge (\forall \nu \in X)(\text{Dp}_{(\mathscr{T}, \bar{\mathbf{I}})}(\nu) \geq \beta)]$

(c) if $\eta \in \mathscr{T} \backslash \max(\mathscr{T}) \backslash \mathscr{S}_{\mathbf{i}}$ <u>then</u> $\text{Dp}_{\mathbf{i}}(\eta) \geq \alpha$ <u>iff</u> $(\forall \nu)(\nu \in \text{suc}_{\mathscr{T}}(\eta) \Rightarrow \text{Dp}_{\mathbf{i}}(\nu) \geq \alpha)$.

6) If $\mathbf{i} = (\mathscr{T}, \bar{\mathbf{I}})$ is a pit and $\eta \in \mathscr{T}$ let $\text{proj}(\eta, \mathbf{i}) = \text{proj}_{\mathbf{i}}(\eta)$ is the sequence ν of length $\ell g(\eta)$ such that:

- $\ell < \ell g(\eta) \wedge \eta \restriction \ell \in \text{Dom}(\bar{\mathbf{I}}) \Rightarrow \nu(\ell) = -1$
- $\ell < \ell g(\eta) \wedge \eta \restriction \ell \notin \text{Dom}(\bar{\mathbf{I}}) \Rightarrow \nu(\ell) = \eta(\ell)$.

7) For $\mathbf{i} = (\mathscr{T}, \bar{\mathbf{I}})$ a pit let $\text{proj}(n, \mathbf{i}) = \text{proj}_{\mathbf{i}}(n) = \{\text{proj}_{\mathbf{i}}(\eta) : \eta \in \mathscr{T}$ has length $n\}$ and $\text{proj}_{\mathbf{i}} = \text{proj}(\mathbf{i})$ is $\cup\{\text{proj}_{\mathbf{i}}(\eta) : \eta \in \mathscr{T}\}$.
8) If $\mathbf{i}_\ell$ is a pit for $\ell < n$ then

(a) $\prod\limits^{*}_{\ell < \mathbf{n}} \mathscr{T}_{\mathbf{i}_\ell}$ is $\{\bar{\eta} : \bar{\eta} = \langle \eta_\ell : \ell < \mathbf{n} \rangle$ is such that $\ell < \mathbf{n} \Rightarrow \eta_\ell \in \mathscr{T}_{\mathbf{i}_\ell}$ and moreover for some n called $\text{lev}(\bar{\eta})$ we have $(\forall \ell < \mathbf{n})(\text{lev}_{\mathscr{T}_{\mathbf{i}(\ell)}}(\eta_\ell) = n)\}$.

Theorem 2.18. *There are a* pit $\mathbf{i}_2$ *and* $\langle c_\eta : \eta \in \text{proj}(\mathbf{i}_1) \rangle$ *such that:* $\mathbf{i}_1 \leq \mathbf{i}_2, \text{Dp}_{\mathbf{i}_2}(\text{rt}_{\mathbf{i}_2}) \geq \gamma_2$ *and* $\eta \in \mathscr{T}_{\mathbf{i}_2} \Rightarrow \mathbf{c}(\eta) = \mathbf{c}_{\text{proj}(\eta, \mathbf{i}_1)}$ <u>*when*</u>*:*

(a) $\mathbf{i}_1 = (\mathscr{T}_1, \bar{\mathbf{I}}_1)$ *is a* pit

(b) $\mathbf{i}_1$ *is λ-complete* pit

(c) $2^{\kappa^\theta} < \lambda$ *where* $\theta = |\text{proj}_{\mathbf{i}_1}|, \kappa + \theta$ *is infinite for transparency*[8]

(d) $\mathbf{c}$ *is a colouring of* $\mathscr{T}_1$ *by* $\leq \kappa$ *colours*

(e) $\gamma_1 = \gamma_2 = (2^{(\kappa^\theta)})^+$ *or just*

(α) $\gamma_1 \leq \text{Dp}_{\mathbf{i}_1}(\text{rt}_{\mathbf{i}_1}), \gamma_1$ *is a regular cardinal,*

(β) γ_2 *has cofinality* $> \kappa^\theta$ *and* $\gamma < \gamma_2 \Rightarrow |\gamma|^{\kappa^\theta} < \gamma_1$.

[8]If κ and θ are finite, the computations are somewhat different. Note that $\kappa = 0$ is impossible and if $\kappa = 1$ then $\mathbf{i}_2 = \mathbf{i}_1$ will do so, without loss of generality $\kappa \geq 2$.

Remark 2.19. 1) This relates on the one hand to the partition theorem of [She98, Ch.XI] continuing Rubin-Shelah [RS87], Shelah [She98, Ch.XI] and on the other hand to Komjath-Shelah [KS03]; the latter is continued in Gruenhut-Shelah [GS11] but presently this is not used.
2) Now 2.18 is what we use but we can get a somewhat more general result - see 2.21.
3) In 2.18 the case $\gamma_1 = \gamma_2 > |\mathscr{T}_1|$ is equivalent to $\gamma_1 = \gamma_2 = \infty$.

Proof. Let $\mathscr{C} = \{\bar{c} : \bar{c} = \langle c_\varrho : \varrho \in \mathrm{proj}_{\mathbf{i}_1}\rangle, c_{<>} = \mathbf{c}(\mathrm{rt}(\mathscr{T}_1))$ and where $c_\varrho \in \mathrm{Rang}(\mathbf{c})$ or just $(\exists \eta \in \mathscr{T}_1)(\varrho = \mathrm{proj}_{\mathbf{i}_2}(\eta) \wedge c_\varrho = \mathbf{c}(\eta))\}$. For transparency without loss of generality we assume $\mathrm{Rang}(\mathbf{c}{\restriction}\max(\mathscr{T}_{\mathbf{i}_1})), \mathrm{Rang}(\mathbf{c}{\restriction}(\mathscr{T}_{\mathbf{i}_1} \backslash \max(\mathscr{T}_{\mathbf{i}_1}))$ are disjoint. Clearly $|\mathscr{C}| \le \kappa^{|\mathrm{proj}(\mathbf{i}_1)|} = \kappa^\theta < \lambda$.

Fix for a while $\bar{c} \in \mathscr{C}$, first let $\mathscr{T}_{\bar{c}} = \{\eta \in \mathscr{T}_1$: if $\nu \trianglelefteq \eta$ then $\mathbf{c}(\nu) = \mathbf{c}_{\mathrm{proj}(\nu,\mathbf{i}_1)}\}$ so a subtree of $\mathscr{T}_1$, i.e. a downward closed subset noting that $\mathrm{rt}_{\mathscr{T}_1} \in \mathscr{T}_{\bar{c}}$.

Second, for $\eta \in \mathscr{T}_1$, let $X^1_{\bar{c},\eta}$ be $\mathrm{suc}_{\mathscr{T}_{\bar{c}}}(\eta)$ if $\eta \in \mathscr{T}_{\bar{c}} \cap \mathrm{Dom}(\bar{\mathbf{I}}_1)$ and this set is $\in \mathbf{I}_{1,\eta}$ and be $\emptyset$ otherwise. Let $\mathscr{T}'_{\bar{c}} = \{\eta \in \mathscr{T}_{\bar{c}}$: if $\ell < \ell g(\eta)$ and $\eta{\restriction}\ell \in \mathrm{Dom}(\mathbf{I}_1)$ then $\eta{\restriction}(\ell+1) \notin X^1_{\bar{c},\eta}$, i.e. $\mathrm{suc}_{\mathscr{T}_{\bar{c}}}(\eta{\restriction}\ell) := \{\nu \in \mathrm{suc}_{\mathscr{T}_1}(\eta) : \nu \in \mathscr{T}_{\bar{c}}\} \neq \emptyset \mod \mathbf{I}_{1,\eta}\}$, again $\mathscr{T}'_{\bar{c}}$ is a subtree of $\mathscr{T}_{\bar{c}}$, moreover $\mathbf{i}_{2,\bar{c}} = (\mathscr{T}'_{\bar{c}}, \bar{\mathbf{I}}{\restriction}\mathscr{T}'_{\bar{c}})$ is a pit.

Third, for $\eta \in \mathscr{T}'_{\bar{c}}, \mathrm{Dp}_{\mathbf{i}_1}(\eta) \in \mathrm{Ord} \cup \{\infty\}$ is well defined and, now for $\eta \in \mathscr{T}_1$, let $X^2_{\bar{c},\eta}$ be $\{\nu \in \mathrm{suc}_{\mathscr{T}'_{\bar{c}}}(\eta) : \mathrm{Dp}_{\mathbf{i}_{2,\bar{c}}}(\nu) \ge \mathrm{Dp}_{\mathbf{i}_{2,\bar{c}}}(\eta)\} = \emptyset \mod \mathbf{I}_{1,\eta}\}$ if $\eta \in \mathscr{T}'_{\bar{c}} \cap \mathrm{Dom}(\bar{\mathbf{I}}_1), \mathrm{Dp}_{\mathbf{i}_{2,\bar{c}}}(\eta) < \infty$ and be $\emptyset$ otherwise.

If for some $\bar{c} \in \mathscr{C}, \mathrm{Dp}_{\mathbf{i}_{2,\bar{c}}}(\mathrm{rt}_{\mathscr{T}'_{\bar{c}}}) \ge \gamma_2$ easily we are done, so toward a contradiction assume this is not the case, so recalling $\mathrm{cf}(\gamma_2) > |\mathscr{C}|$ clearly $\gamma_* = \sup\{\mathrm{Dp}_{\mathbf{i}_{2,\bar{c}}}(\mathrm{rt}_{\mathscr{T}'_{\bar{c}}})+1 : \bar{c} \in \mathscr{C}\} < \gamma_2$. Now for each $\eta \in \mathrm{Dom}(\bar{\mathbf{I}}_1)$ clearly all $X^1_{\bar{c},\eta}, X^2_{\bar{c},\eta}$ are from $\mathbf{I}_{1,\eta}$ and their number is $\le 2|\mathscr{C}| < \lambda$ hence $X_\eta := \cup\{X^1_{\bar{c},\eta} \cup X^2_{\bar{c},\eta} : \bar{c} \in \mathscr{C}\}$ belong to $\mathbf{I}_{1,\eta}$.

Hence $\mathbf{i}_3$ is an pit and $\mathbf{i}_1 \le \mathbf{i}_3$ where $\mathbf{i}_3 = \mathbf{i}(3) := \mathbf{i}_1{\restriction}\{\eta \in \mathscr{T}_1$: if $\ell < \ell g(\eta)$ and $\eta{\restriction}\ell \in \mathrm{Dom}(\bar{\mathbf{I}}_1)$ then $\eta{\restriction}(\ell+1) \notin X_\eta\}$; moreover by the definition of $\mathrm{Dp}_{\mathbf{i}_3}$ and the choice of $\mathbf{i}_3$, clearly

$(*)_1$ (a) $\mathbf{i}_3$ is a pit; moreover $\mathbf{i}_1 \le_{\mathrm{pr}} \mathbf{i}_3$ hence

(b) $\eta \in \mathscr{T}_{\mathbf{i}_3} \Rightarrow \mathrm{Dp}_{\mathbf{i}_3}(\eta) = \mathrm{Dp}_{\mathbf{i}_1}(\eta)$.

Define h by

$(*)_2$ h is a function from $\mathscr{T}_{\mathbf{i}_1} \times \mathscr{C}$ defined by
- $h(\eta, \bar{c})$ is -1 if $\eta \in \mathscr{T}_{\mathbf{i}_1} \backslash \mathscr{T}'_{\bar{c}}$
- $h(\eta, \bar{c})$ is $\mathrm{Dp}_{\mathbf{i}_{2,\bar{c}}}(\eta)$ if $\eta \in \mathscr{T}'_{\bar{c}}$ and $\mathrm{Dp}_{\mathbf{i}_{2,\bar{c}}}(\eta) < \gamma_*$
- $\mathrm{Dp}(\eta, \bar{c}) = \gamma_*$ if none of the above.

We now choose $(\mathbf{c}_n, h_n, \mathscr{X}_n, \bar{\mathscr{Y}}_n, \mathscr{S}_n)$ by induction on n such that:

⊞ (a)(α) $\mathscr{X}_n$ is a subset of $\cup\{\mathrm{proj}_{\mathbf{i}_1}(m) : m \le n\}$

(β) if $n = k+1$ then $\mathscr{X}_k = \mathscr{X}_n \cap (\cup\{\mathrm{proj}_{\mathbf{i}_1}(m) : m \le k\})$

(γ) $\mathscr{S}_n \subseteq \mathscr{X}_n$

(b)(α) h_n is a function with domain $\mathscr{X}_n \times \mathscr{C}$ to $\gamma_* + 1$

(β) $\mathbf{c}_n$ is a function from $\mathscr{X}_n$ to $\mathrm{Rang}(\mathbf{c})$

(c) $\bar{\mathscr{Y}}_n = \langle \mathscr{Y}_{n,\gamma} : \gamma < \gamma_1\rangle$

(d)(α) $\mathscr{Y}_{n,\gamma}$ is a subset of $\mathscr{T}_{\mathbf{i}_3}$, downward closed of cardinality $\le \theta$

(β) if $\eta \in \mathscr{Y}_{n,\gamma}$ then $\ell g(\eta) \le n$

(γ) if $\eta \in \mathscr{Y}_{n,\gamma}$ then $\mathrm{Dp}_{\mathbf{i}_3}(\eta) = \mathrm{Dp}_{\mathbf{i}_1}(\eta) \ge \gamma$

(δ) if $\eta \in \mathscr{Y}_{n,\gamma}$ and $\ell g(\eta) < n$ and $\eta \notin \mathrm{Dom}(\bar{\mathbf{I}}_1)$ then $\mathrm{suc}_{\mathscr{T}_{\mathbf{i}_3}}(\eta) = \mathrm{suc}_{\mathscr{T}_{\mathbf{i}_1}}(\eta)$ is $\subseteq \mathscr{Y}_{n,\gamma}$

(ε) if $\eta \in \mathscr{Y}_{n,\gamma}$ and $\ell g(\eta) < n$ and $\eta \in \mathrm{Dom}(\bar{\mathbf{I}}_1)$ then $\mathrm{suc}_{\mathscr{T}_{\mathbf{i}_3}}(\eta)$ is a singleton

(ζ) if $\gamma < \gamma_2$ then $\mathscr{X}_n = \{\mathrm{proj}_{\mathbf{i}_1}(\eta) : \eta \in \mathscr{Y}_{n,\gamma}\}$

(η) if $\eta \in \mathscr{Y}_{n,\gamma}$ and $\nu = \mathrm{proj}_{\mathbf{i}_3}(\eta)$ then:

- $\bullet_1$ $\mathbf{c}(\eta) = \mathbf{c}_n(\nu)$
- $\bullet_2$ $h_n(\nu, \bar{c}) = h(\eta, \bar{c})$ for every $\bar{c} \in \mathscr{C}$
- $\bullet_3$ $\eta \in \mathrm{Dom}(\mathbf{I}_1)$ iff $\nu \in \mathscr{S}_n$.

(θ) follows: the function $\eta \mapsto \mathrm{proj}_{\mathbf{i}_3}(\eta)$ on $\mathscr{Y}_{n,\gamma}$ is one to one.

Why this is possible:

For $n = 0$ this is trivial.

For $n = m+1$ for every $\gamma < \gamma_1$, choose $\bar{\varrho}_{n,\gamma} \in \Pi\{\mathrm{suc}_{\mathscr{T}_{\mathbf{i}_3}}(\eta) : \eta \in \mathscr{Y}_{m,\gamma+1}, \ell g(\eta) = m, \eta \in \mathrm{Dom}(\bar{\mathbf{I}}_1)\}$ such that if $\eta \in \mathrm{Dom}(\bar{\varrho}_{n,\gamma})$ then $\mathrm{dp}_{\mathbf{i}_1}(\eta) \ge \gamma$, possible as $\eta \in \mathrm{Dom}(\bar{\varrho}_{n,\gamma}) \Rightarrow \mathrm{dp}_{\mathbf{i}_1}(\eta) \ge \gamma + 1$. Let $\mathscr{Y}'_{n,\gamma} = \mathscr{Y}_{m,\gamma+1} \cup \{\nu$: for some $\eta \in \mathscr{Y}_{n,\gamma+1}$ we have $\ell g(\eta) = m$ and we have $\eta \notin \mathrm{Dom}(\bar{\mathbf{I}}_1) \Rightarrow \nu = \varrho_{n,\gamma}(\eta)$ and $\eta \notin \mathrm{Dom}(\bar{\mathbf{I}}_1) \Rightarrow \nu \in \mathrm{suc}_{\mathscr{T}_{\mathbf{i}_3}}(\eta)\} \cup \mathrm{Rang}(\bar{\varrho}_{n,\gamma})$.

Let $\mathscr{X}'_{n,\gamma} = \{\mathrm{proj}_{\mathbf{i}_1}(\eta) : \eta \in \mathscr{Y}'_{n,\gamma}\}$ and let the function $\mathbf{c}'_{n,\gamma} : \mathscr{X}'_{n,\gamma} \to \mathrm{Rang}(\mathbf{c})$ be defined by $\eta \in \mathscr{Y}'_{n,\gamma} \Rightarrow \mathbf{c}'_{n,\gamma}(\mathrm{proj}_{\mathbf{i}_3}(\eta))) = \mathbf{c}(\eta)$, well defined as in $\boxplus(d)(\eta)$ and let $\mathscr{S}_{n,\gamma} = \{\mathrm{proj}_{\mathbf{i}_1}(\eta) : \eta \in \mathscr{Y}'_{n,\gamma}$ and $\eta \in \mathrm{Dom}(\bar{\mathbf{I}}_1)\}$. Let $h_{n,\gamma} : \mathscr{X}'_{n,\gamma} \to \gamma_* + 1$ be defined by : if $\bar{c} \in \mathscr{C}, \nu = \mathrm{proj}_{\mathbf{i}_{2,\bar{c}}}(\eta)$ and $\eta \in \mathscr{Y}_{n,\gamma}$ then $\eta \notin \mathscr{T}'_{\bar{c}} \Rightarrow h_{n,\gamma}(\nu) = \gamma, \eta \in \mathscr{T}'_{\bar{c}} \Rightarrow h_{n,\gamma}(v) = \mathrm{Dp}_{\mathbf{i}_{2,\bar{c}}}(\eta)$.

Now $\mathscr{X}'_{n,\gamma}$ is a subset of $\mathrm{proj}_{\mathbf{i}_1}$, a set of cardinality $\le \theta$ and $\mathbf{c}'_{n,\gamma}$ is a funtion from $\mathscr{X}'_{n,\gamma}$ into $\mathrm{Rang}(\mathbf{c})$, a set of cardinality $\le \kappa$ and $h_{n,\gamma}$ is a function from $\mathscr{X}'_{n,\gamma} \subseteq \mathrm{proj}_{\mathbf{i}_1}$ into γ_*. But $\gamma_* < \gamma_2, \gamma_* + \kappa < \gamma_1, \gamma_1$ is a regular cardinal (recalling clause (e) of the theorem) and $(|\gamma_*| + \kappa)^\theta < \mathrm{cf}(\gamma_1) = \gamma_1$ hence for every $\gamma < \gamma_1$ we have $|\{(X'_{n,\gamma}, \mathbf{c}_{n,\gamma}, h_{n,\gamma}) : \gamma < \gamma_1\}| \le 2^\theta \cdot \kappa^\theta \cdot |\gamma_*|^\theta < \mathrm{cf}(\gamma_1) = \gamma_1$ hence for some $\mathbf{c}_n, h_n, \mathscr{X}_n$ the set $S_n := \{\gamma < \gamma_1 : \mathbf{c}'_{n,\gamma} = \mathbf{c}_n$ and $h_{n,\gamma} = h_n, \mathscr{X}'_{n,\gamma} = \mathscr{X}_n$ and $\mathscr{S}_{n,\gamma} = \mathscr{S}_n\}$ is unbounded in γ_1.

Lastly, let $\mathscr{Y}_{n,\gamma} = \mathscr{Y}'_{n,\min(S_n \setminus \gamma)}$, clearly $\mathbf{c}_{n+1}, h_{n+1}, \langle \mathscr{Y}_{n,\gamma} : \gamma < \gamma_2 \rangle$ are as required; so we can carry the induction.

Why this is enough:

Let $\mathscr{X} = \cup\{\mathscr{X}_n : n < \omega\} \subseteq \mathrm{proj}(\mathbf{i}_1)$ and $\mathscr{S} = \cup\{\mathscr{S}_n : n < \omega\}$ and $\mathbf{c} = \cup\{\mathbf{c}_n : n < \omega\}$ and $\mathbf{h} = \cup\{h_n : n < \omega\}$ so by $\boxplus(d)(\eta)$ clearly there is $\bar{c}^* \in \mathscr{C}$ such that $c^*_\varrho = \mathbf{c}(\varrho)$ when the latter is defined, so:

$\odot_1$ if $n < \omega, \gamma < \gamma_1, \eta \in \mathscr{Y}_{n,\gamma}$ and $\nu = \mathrm{proj}(\mathbf{i}_1) \in \mathscr{X}$ then

(a) $\mathbf{c}(\eta) = \mathbf{c}_n(\mathrm{proj}_{\mathbf{i}_1}(\eta))$

(b) $\mathrm{Dp}_{\mathbf{i}_{2,\bar{c}}}(\eta) = h(\eta, \bar{c}) = h_n(\nu, \bar{c})$

(c) $\mathrm{Dp}_{\mathbf{i}_1}(\eta) \ge \gamma$

Also

$\odot_2$ $\mathscr{X} \subseteq \mathrm{proj}_{\mathbf{i}_1}$ is a set of finite sequences, closed under initial segments with no $\triangleleft$-maximal member.

[Why? Straight, e.g. if $\nu \in X$ choose $n = \ell g(\nu)+2$ let $\gamma < \gamma_1$ and choose $\eta \in Y_{n,\gamma+1}$ such that $\mathrm{proj}_{\mathbf{i}_1}(\eta) = \nu$, now by clause (c) of $\odot_1$ we know that $\mathrm{Dp}_{\mathbf{i}_1}(\eta) \geq \gamma+1$, hence there is $\eta_1 \in \mathrm{suc}_{\mathscr{T}_{\mathbf{i}_1}}(\eta)$ in $Y_{n,\gamma+1}$ hence $\nu_1 = \mathrm{proj}_{\mathbf{i}_1}(\eta_1)$ is in $\mathrm{suc}_{\mathscr{X}}(\nu)$, i.e. successor of η in $\mathscr{X}_{n+1}$ hence in $\mathscr{X}$.]

$\odot_3$ if $\nu \in \mathscr{X}$ then $\mathbf{h}(\nu, \bar{c}) \neq -1$.

[Why? Let $n > \ell g(\nu)$, let $\gamma < \gamma_2$. Now by $\boxplus(d)(\zeta)$ there is $\eta \in \mathscr{Y}_{n,\gamma}$ such that $\mathrm{proj}_{\mathbf{i}_1}(\eta) = \nu$.

Next by $(*)_2$ we have $h(\eta, \bar{c})$ is -1 iff $\eta \notin \mathscr{T}'_{\bar{c}}$. However, $\eta \in \mathscr{T}_{\bar{c}}$ by the definition of $\mathscr{T}_{\bar{c}}$ and the choice of $\bar{c}$ and $\boxplus(d)(\eta)$; moreover $\eta \in \mathscr{T}'_{\bar{c}}$ by the definition of $\mathscr{T}'_{\bar{c}}$ anda of $\mathbf{i}_3$ and clause $\boxplus(d)(\alpha)$.

By the last two sentences $h(\eta, \bar{c}) \neq -1$ hence by the choice of η, i.e. as $\mathrm{proj}_{\mathbf{i}_1}(\eta) = \nu$, clause $\boxplus(d)(\eta)$ tells us $\mathbf{h}(\nu, \bar{c}) = h(\eta, \bar{c})$ so together $\mathbf{h}(\nu, \bar{c}) \neq -1$ as promised.]

$\odot_4$ $0 \leq \mathrm{Dp}_{\mathbf{i}_{2,\bar{c}}}(\langle\rangle) < \gamma_*$ hence $\mathbf{h}(<>, \bar{c}) < \gamma_*$.

[Why? Similarly using $\boxplus(d)(\eta)\bullet_3$.]

$\odot_5$ if $\nu \in \mathscr{X} \backslash \mathscr{S}$ and $0 \leq \mathbf{h}(\nu, \bar{c}) < \gamma_*$ <u>then</u> for some $\rho \in \mathrm{suc}_{\mathscr{X}}(\nu)$ we have $0 \leq \mathbf{h}(\rho, \bar{c}) < \mathbf{h}(\nu, \bar{c}) < \gamma_*$.

[Why? Similarly using $\boxplus(d)(\delta)$.]

$\odot_6$ if $\nu \in \mathscr{S}(\subseteq \mathscr{X})$ and $0 \leq \mathbf{h}(\nu, \bar{c}) < \gamma_*$ <u>then</u> for the unique $\rho \in \mathrm{suc}_{\mathscr{X}}(\nu)$ we have $0 \leq \mathbf{h}(\rho, \bar{c}) < \mathbf{h}(\nu, \bar{c}) < \gamma_*$.

[Why? Similarly using $\boxplus(d)(\varepsilon)$.]

By $\odot_4, \odot_5, \odot_6$ together we get a contradiction. $\square_{2.18}$

We may prefer the following variant of 2.18.

Definition 2.20. 1) For a pit $\mathbf{i} = (\mathscr{T}, \bar{\mathbf{I}})$ and partition $\bar{\mathscr{S}} = (\mathscr{S}_0, \mathscr{S}_1)$ of $\mathscr{S}_{\mathbf{i}}$ (or just $\bar{\mathscr{S}} = (\mathscr{S}_0, \mathscr{S}_1)$ such that $\mathscr{S}_0 \cap \mathscr{S}_1 = \emptyset$ and $\mathscr{S}_{\mathbf{i}} \subseteq \mathscr{S}_0 \cup \mathscr{S}_1$) we define $\mathrm{Dp}_{\mathbf{i},\mathscr{S}} : \mathscr{T} \to \mathrm{Ord} \cup \{\infty\}$, stipulating $\infty + 1 = \infty$ by defining when $\mathrm{Dp}_{\mathbf{i},\lambda}(\eta) \geq \alpha$ by induction on the ordinal α (compare with 2.17(4)):

- (a) if $\eta \in \max(\mathscr{T})$ <u>then</u> $\mathrm{Dp}_{\mathbf{i},\lambda}(\eta) \geq \alpha$ iff $\alpha = 0$
- $(b)_0$ if $\eta \in \mathscr{S}_0$ hence $\eta \in \mathscr{S}, \eta \notin \max(\mathscr{T})$ <u>then</u> $\mathrm{Dp}_{\mathbf{i},\bar{\mathscr{S}}}(\eta) \geq \alpha$ iff for every $\beta < \alpha$ the set $\{\nu \in \mathrm{suc}_{\mathscr{T}}(\eta) : \mathrm{Dp}_{\mathbf{i},\bar{\mathscr{S}}}(\nu) \geq \beta\}$ belong to $\mathbf{I}^+_\eta$
- $(b)_1$ if $\eta \in \mathscr{S}_1$ <u>then</u> $\mathrm{Dp}_{\mathbf{i},\bar{\mathscr{S}}}(\eta) \geq \alpha$ iff $\{\nu \in \mathrm{suc}_{\mathscr{T}}(\eta) : \mathrm{Dp}_{\mathbf{i},\bar{\mathscr{S}}}(\nu) \geq \alpha\}$ belongs to $\mathbf{I}^+_\eta$
- (c) if $\eta \in \mathscr{T} \backslash \mathscr{S} \backslash \max(\mathscr{T})$ <u>then</u> $\mathrm{Dp}_{\mathbf{i},\bar{\mathscr{S}}}(\eta) \geq \alpha$ iff for every $\nu \in \mathrm{suc}_{\mathscr{T}}(\eta)$ we have $\mathrm{Dp}_{\mathbf{i},\bar{\mathscr{S}}}(\eta)$.

Theorem 2.21. *There are a pit $\mathbf{i}_2$ and $\bar{c} = \langle c_\eta : \eta \in \mathrm{proj}(\mathbf{i}_1)\rangle$ such that $\mathbf{i}_1 \leq \mathbf{i}_2, \mathrm{Dp}_{\mathbf{i}_2,\bar{\mathscr{S}}}(\mathrm{rt}_{\mathbf{i}_2}) \geq \gamma_2$ and $\eta \in \mathscr{T}_{\mathbf{i}_2} \Rightarrow \mathbf{c}(\eta) = \mathbf{c}_{\mathrm{proj}(\eta,\mathbf{i}_1)}$ <u>when</u>:*

- $(a)-(e)$ *as in 2.18 replacing $\mathrm{Dp}_{\mathbf{i}_2}$ by $\mathrm{Dp}_{\mathbf{i}_2,\bar{\mathscr{S}}}$ in $(e)(\alpha)$*
- (f) *$\bar{\mathscr{S}} = (\mathscr{S}_0, \mathscr{S}_1)$ is a partition of $\mathscr{S}_{\mathbf{i}}$.*

Proof. Similarly. $\square_{2.21}$

§ 3. Approximation to EM models

In the game below the protagonist tries to exemplify in a weak form that the standard $\mathrm{EM}_{\tau(\mathfrak{k})}(\lambda,\Phi)$ is $\le_{\mathfrak{k}}$-embeddable into N over M. We may consider games in which the protagonist tries to exemplify a weak form of isomorphism, this is connected to logics which have EM models, continuing [She12], but not for now.

Here we do not try to get the best cardinal bounds; just enough for the result promised in the abstract.

Definition 3.1. Assume $\lambda > \kappa \ge \mathrm{LST}_{\mathfrak{k}} + |\tau_{\mathfrak{k}}|$ and $M \in K^{\mathfrak{k}}_\kappa$ and $M \le_{\mathfrak{k}} N$ and γ is an ordinal.
1) We say Φ is an $(M,\lambda,\kappa,\gamma)$-solution of N or is an $(N,M,\lambda,\kappa,\gamma)$-solution <u>when</u> $\Phi \in \Upsilon^{\mathrm{sor}}_\kappa(\mathfrak{k}_M)$ and in the game $\Game^1_{N,M,\lambda,\Phi,\gamma}$ the protagonist has a winning strategy.
2) Assume $\Phi \in \Upsilon_\kappa(\mathfrak{k}_M)$ recalling Definition 2.1 fixing $M_\lambda = \mathrm{EM}(\lambda,\Phi)$ and $M_I = \mathrm{EM}(I,\Phi)$ for $I \subseteq \lambda$ and without loss of generality every M_I (equivalently some M_I) is standard, hence in particular $M \le_{\mathfrak{k}} M_I{\restriction}\tau(\mathfrak{k})$. We define the game $\Game^1_{N,M,\lambda,\Phi,\gamma}$, a play last $< \omega$ moves, in the n-th move $\lambda_n, J_n, \bar h_n, \gamma_n$ are chosen such that:

$\boxplus_n$ (a) $\lambda_0 = \lambda$
- (b) if $n = m+1$ then $\kappa < \lambda_n < \lambda_m$ moreover $\lambda_m \to (\lambda_n)^n_{2^\kappa}$
- (c) $J_0 = \lambda$, and if $n = m+1$ then $J_n \subseteq J_m$
- (d) $|J_n| = \lambda_n$
- (e) $\bar h_n = \langle h_u : u \in [J_n]^n\rangle$
- (f) if $u \in [J_n]^n$ <u>then</u> h_u is a $\le_{\mathfrak{k}}$-embedding of M_u into N extending h_v whenever $v \subseteq u$

  [Explanation: note if $v \subset u, |v| = m$ then $v \in [J_n]^m \subseteq [J_m]^n$ hence h_v was defined; this says then for $u_1, u_2 \in [J_n]^n, h_{u_2}, h_{u_2}$ are compatible functions]
- (g) $\gamma_0 = \gamma$ and γ_{n+1} is an ordinal $< \gamma_n$.

In the n-th move:

- (A) if $n = 0$ the antagonist chooses $\lambda_0 = \lambda, J_0 = \lambda, \gamma_0 = \gamma$ and the protagonist chooses $\bar h_0$
- (B) if $n = m+1$ then
  - (a) the antagonist chooses an ordinal $\gamma_n < \gamma_m$ and $\lambda_n > \kappa$ such that $\lambda_m \to (\lambda_n)^n_{\beth_2(\kappa)}$
  - (b) the protagonist chooses $\bar h'_n = \langle h_u : u \in [J_m]^n\rangle$ and $\mathscr{S}_n \in (\mathrm{ER}^n_{J_m,\lambda_n,\beth_2(\kappa)})^+$, i.e. $\mathscr{S}_n \subseteq [\lambda_m]^{\lambda_n}$ and $\mathscr{S}_n$ is not from this ideal, see Definition 2.5
  - (c) the antagonist chooses $J_n \in \mathscr{S}_n \subseteq [J_m]^{\lambda_n}$ and we let $\bar h_n = \bar h'_n{\restriction}[J_n]^n$
- (C) the play ends when a player has no legal move and then this player loses.

Another presentation:

Definition 3.2. Assume $M \le_{\mathfrak{k}} N$ and $\mathrm{LST}_{\mathfrak{k}} + |\tau_{\mathfrak{k}}| \le \theta, \|M\| + \theta \le \kappa < \lambda$ and $\Phi \in \Upsilon^{\mathrm{or}}_\theta[M,\mathfrak{k}]$.
1) Below we omit γ if (a) or (b), where:

- (a) $\gamma = \mathrm{cf}(\lambda), \lambda$ strong limit and $\alpha < \mathrm{cf}(\lambda) \Rightarrow |\alpha|^{2^{\kappa+\|M\|}} < \mathrm{cf}(\lambda)$

(b) not (a) but γ is maximal such that $\gamma = \omega\gamma$ is infinite and $\beth_\gamma(\kappa + \|M\|) \le \lambda$ and λ is strong limit of cofinality $> \beth_2(\kappa)$
(similarly in all such definitions).

2) We say that $\mathbf{x}$ is a direct witness for $(N, M, \lambda, \kappa, \gamma, \Phi)$ <u>when</u> $\mathbf{x}$ consists of:

(a) $N, M, \Phi, \lambda, \kappa$ and γ
(b) $\mathscr{T}$ is a non-empty set of finite sequences closed under initial segments
(c) if $\eta \in \mathscr{T}$ <u>then</u>:
  (α) $\eta(2n)$ is a cardinal when $2n < \ell g(\eta)$
  (β) $\eta(2n+1)$ is a subset of λ of cardinality $\eta(2n)$ when $2n+1 < \ell g(\eta)$
  (γ) $\eta(2n+1) \supseteq \eta(2n+3)$ when $2n+3 < \ell g(\eta)$
  (δ) $\eta(2n) \ge \eta(2n+2)$, moreover $\eta(2n) \to (\eta(2n+2))^{2n+1}_{\beth_2(\kappa)}$ when $2n+2 < \ell g(\eta)$
(d) I_η, λ_η for $\eta \in \mathscr{T}$ are defined by:
  (α) if $\ell g(\eta) = 0$ then $I_\eta = \lambda, \lambda_\eta = \lambda$
  (β) if $\ell g(\eta) = 2n+1$ then $I_\eta = I_{\eta \restriction (2n)}$, see ($\alpha$) or ($\gamma$) and $\lambda_\eta = \eta(2n)$
  (γ) if $\ell g(\eta) = 2n+2$ then $I_\eta = \eta(2n+1), \lambda_\eta = \eta(2n) = \lambda_{\eta \restriction (2n+1)}$, see ($\alpha$) or ($\beta$)
(e) if $\eta \in \mathscr{T} \backslash \max(\mathscr{T})$ has length $2n+1$ then: the set $\mathscr{S}_\eta = \{I_\nu : \nu \in \mathrm{suc}_{\mathscr{T}}(\eta)\} \subseteq [I_\eta]^{\lambda_\eta}$ is not from the ideal $\mathrm{ER}^{\lfloor \ell g(\eta)/2 \rfloor}_{I_\eta, \lambda_\eta, \beth_2(\kappa)}$
(f) if $\eta \in \mathscr{T}$ then:
  (α) $\bar{h}_\eta = \langle h_{\eta,u} : u \in [I_\eta]^{\le \lfloor \ell g(\eta)/2 \rfloor} \rangle$
  (β) $h_{\eta,u}$ is a $\le_{\mathfrak{k}}$-embedding of $\mathrm{EM}_{\tau(\mathfrak{k})}(u, \Phi)$ into N for $u \in [I_\eta]^{\le \lfloor \ell g(\eta)/2 \rfloor}$
  (γ) $u_1 \subseteq u_2 \in [I_\eta]^{\le \lfloor \ell g(\eta)/2 \rfloor} \Rightarrow h_{\eta,u_1} \subseteq h_{\eta,u_2}$
  (δ) if $u \in [I_\eta]^{\le \lfloor \ell g(\eta)/2 \rfloor}$ and $\nu \triangleleft \eta$ and $\ell g(\nu) \ge 2|u|$, <u>then</u> $h_{\eta,u} = h_{\nu,u}$
  (ε) if $\ell g(\eta) = 2n+2$ and $u \in [I_\eta]^{\le n}$ then $h_{\eta,u} = h_{\eta \restriction (2n+1),u}$
  (ζ) there is $\bar{a} = \bar{a}_{\mathbf{x}} = \langle a_\alpha : \alpha < \lambda \rangle \in {}^\lambda N$ such that $\alpha \in u \in [I_\eta]^{\le \lfloor \ell g(\eta) \rfloor / 2} + h_{\eta,u}(\alpha) = a_\alpha$ and $\bar{a}$ is with no repetitions
(g) $\mathrm{Dp}_{\mathbf{x}}(<>) \ge \gamma$ where $\mathrm{Dp}_{\mathbf{x}}(\eta)$ is defined as $\mathrm{Dp}_{\mathbf{i}(\mathbf{x})}(\eta)$, see Definition 2.17, where $\mathbf{i} = \mathbf{i}(\mathbf{x}) = \mathbf{i}_{\mathbf{x}}$ is defined by:
  - $\mathscr{T}_{\mathbf{i}} = \mathscr{T}$
  - $\mathscr{S}_{\mathbf{i}} = \{\eta \in \mathscr{T} : \eta$ is not $\triangleleft$-maximal in $\mathscr{T}$ and $\ell g(\eta)$ is odd$\}$
  - if $\eta \in \mathscr{S}_{\mathbf{i}}$ and $\ell g(\eta)$ is odd then $\mathbf{I}_{\mathbf{i},\eta} = \mathrm{ER}^{\lfloor \ell g(\eta) \rfloor}_{I_\eta, \lambda_\eta, \beth_2(\kappa)}$ recalling 2.17(1A)
  - if $\eta \in \mathscr{S}_{\mathbf{i}}$ and $\ell g(\eta)$ is even then $\mathscr{I}_{\mathbf{i},\eta} = \{\emptyset\}$.

Definition 3.3. 1) We say $\mathbf{x}$ is a pre-$\mathfrak{k}$-witness of $(N, M, \lambda, \kappa, \delta)$ <u>when</u> it as in 3.2 omitting $\bar{h}$, i.e. clause (f), so N, M are irrelevant.
2) We say $\mathbf{x}$ is a semi-$\mathfrak{k}$-witness of $(N^+, M, \lambda, \kappa, \delta)$ <u>when</u>: it consists of:

(a) N^+ expands a model from $K_{\mathfrak{k}}, M \le_{\mathfrak{k}} (N^+ \restriction \tau(\mathfrak{k})), \lambda \ge \kappa \ge (\tau(N^+))$
(b) − (e) as in 3.2(2)
(f) $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle$

(*g*) as in 3.2(2).

Claim 3.4. *1) The definitions 3.1, 3.2 are equivalent.*
2) In Definition 3.2, $\mathbf{i}_{\mathbf{x}}$ is indeed a pit.
3) If $\Phi_1 \mathbf{E}^{\rm ai}_\kappa \Phi_2, \Phi_\ell \in \Upsilon^{\rm sor}_\kappa[M,\mathfrak{k}]$ for $\ell = 1,2$ and Φ_1 has a (N,M,λ,κ)-witness <u>then</u> *Φ_2 has a (N,M,λ,κ)-witness.*

Proof. Straightforward. $\square_{3.2}$

Claim 3.5. *1) If $\Phi_\ell \in \Upsilon^{\rm sor}_\kappa[\mathfrak{k}_M], \kappa \ge \tau(\mathfrak{k}) + \|M\|$ and $M_\ell = {\rm EM}_{\tau(\mathfrak{k})}(\lambda,\Phi_\ell)$ for $\ell = 1,2$ and λ is strong limit of cofinality μ where $\mu = (\beth_2(\kappa))^+$ or μ is regular such that $(\forall \alpha < \mu)(|\alpha|^{2^\kappa} < \mu)$ and the protagonist wins in the game $\Game^1_{M_2,M,\lambda,\Phi_1,\mu}$ (equivalently some $\mathbf{x}$ is a witness for $(M_2,M,\lambda,\kappa,\Phi_1)$)* <u>then</u> *$\Phi_1 \le^3_\kappa \Phi_2$, see Definition 2.12.*

Proof. Straightforward by 2.18 and the definitions of the ideal ER in 2.5. See details in a similar case in the proof of 3.6(1) below. $\square_{3.5}$

Claim 3.6. *Assume $M \le_{\mathfrak{k}} N, \kappa \ge \|M\| + \theta, \theta \ge {\rm LST}_{\mathfrak{k}} + |\tau_{\mathfrak{k}}|$ and $\|N\| \ge \lambda, \lambda$ strong limit of cofinality μ and $\mu = (\beth_2(\kappa))^+$ or μ is regular such that $(\forall \alpha < \mu)(|\alpha|^{2^\kappa} < \mu)$.*
1) There are $\mathbf{x}, \Phi$ such that:

- (*a*) $\Phi \in \Upsilon^{\rm sor}_\theta(\mathfrak{k}_M)$
- (*b*) *$\mathbf{x}$ is a direct witness of $(N,M,\lambda,\kappa,\Phi)$.*

2) If $M_1 = M, \Phi_1 \in \Upsilon^{\rm sor}_\theta[\mathfrak{k}_{M_1}]$ and $\mathbf{x}_1$ a direct witness for $(N,M_1,\lambda,\kappa,\Phi_1)$ and $M_1 \le_{\mathfrak{k}} M_2 \le_{\mathfrak{k}} N$ and $\|M_2\| \le \kappa$ <u>then</u> *there are $\Phi_2,\mathbf{x}_2$ such that:*

- (*a*) $\Phi_2 \in \Upsilon^{\rm sor}_\theta[M_2]$
- (*b*) *$\Phi_1 \le^1_\kappa \Phi_2$ and $\Phi_1 \le^4_\kappa \Phi_2$*
- (*c*) *$\mathbf{x}_2$ is a direct witness $(N,M_2,\lambda,\kappa,\Phi_2)$.*

3) If in part (1) we change the assumption on λ to $\lambda = \beth_{\omega\cdot\gamma}(\kappa)$ <u>then</u> *there are $\Phi,\mathbf{x}$ such that:*

- (*a*) $\Phi \in \Upsilon^{\rm sor}_\kappa[M,\mathfrak{k}]$
- (*b*) *$\mathbf{x}$ is a direct witness of $(N,M,\Phi,\lambda,\kappa,\gamma,\Phi)$.*

4) Also part (2) has a version with (γ_1,γ_2) as in 2.18.

Proof. 1) Let $\langle a_\alpha : \alpha < \lambda\rangle$ be a sequence of pairwise distinct members of N.
Now

$(*)_1$ let $\mathscr{T}$ be the set of finite sequences η satisfying clauses (b),(c) of Definition 3.2

$(*)_2$ let $\bar{\mathbf{I}} = \langle \mathbf{I}_\eta : \eta \in \mathscr{S}\rangle$ where
- $\mathscr{S} = \{\eta \in \mathscr{T} : \eta$ is not $\triangleleft$-maximal in $\mathscr{T}\}$
- if $\eta \in \mathscr{S}, \ell g(\eta) = 2n+1$ then $\mathbf{I}_\eta = {\rm ER}^n_{I_\eta,\lambda_\eta,\beth_2(\kappa)}$
- if $\eta \in \mathscr{S}$ and $\ell g(\eta) = 2n$ then $\mathbf{I}_\eta = \{\emptyset\}$, the trivial ideal

$(*)_3$ $\mathbf{i}_1 = \mathbf{i}(1) = (\mathscr{T},\bar{\mathbf{I}})$ is a pit and is $(2^\kappa)^+$-complete and ${\rm Dp}_{\mathbf{i}_1}(<>) \ge (\beth_2(\kappa))^+$.

[Why? Just read Definition 2.17(3) and the ideal ER is from Definition 2.5 and it is $(2^\kappa)^+$-complete by 2.6 and as for the depth recall $\mu = (\beth_2(\kappa))^+$.]

$(*)_4$ Let M^+ be such that:

- (a) M^+ is an expansion of N
- (b) $|\tau(M^+)| \le \kappa$ and $\tau' := \tau(M^+) \backslash \{c_a : a \in M\}$ has cardinality $\le \theta$
- (c) if $M_1^+ {\restriction} \tau' \subseteq M^+ {\restriction} \tau'$ then $M_1^+ {\restriction} \tau(\mathfrak{k}) \le M^+ {\restriction} \tau(\mathfrak{k})$
- (d) $|M| = \{c^{M^+} : c \in \tau(M^+)\}$.

[Why M^+ exists? By the representation theorem, [She09a, §1] except clause (d) which as before is easy.]

We like to apply Theorem 2.18 but before this we need

$(*)_5$ there is a pit $\mathbf{i}_2 = \mathbf{i}(2)$ such that $\mathbf{i}(1) \le_{\rm pr} \mathbf{i}(2)$ (see 2.17(2A)) so $\mathrm{Dp}_{\mathbf{i}(2)}(\eta) = \mathrm{Dp}_{\mathbf{i}(1)}(\eta)$ for $\eta \in \mathscr{T}_{\mathbf{i}(2)}$ and:

- if $\eta \in \mathscr{T}_{\mathbf{i}(2)}, \ell g(\eta) = 2n+1$ and $\nu \in \mathrm{suc}_{\mathscr{T}_{\mathbf{i}(2)}}(\eta)$ then $\langle a_\alpha : \alpha \in \nu(2n+1)\rangle$ is an n-indiscernible sequence in M^+ for quantifier free formulas, may add: and $N {\restriction} \{\sigma_\varepsilon(a_{\alpha_0}, \dots, a_{\alpha_{n-1}}) : \varepsilon < \zeta\} \le_{\mathfrak{k}} N$ where $\zeta < \kappa^+$ and σ_ε is a $\tau(M^+)$-term.

[Why such $\mathbf{i}(2)$ exists? By the definition of the ideal $\mathbf{I}_\eta$, see $(*)_2$ above and by Definition 1.14. That is, for $\eta \in \mathrm{Dom}(\mathbf{I}_{\mathbf{i}_1})$ of length $2n+1$ let $X_\eta = \{\nu : \nu \in \mathrm{suc}_{\mathscr{T}}(\eta), \langle a_\alpha : \alpha \in \nu(2n+1)\rangle$ is n-indiscernible in M^+ for quantifier free formulas$\}$, recalling $\mathrm{Dom}(\mathbf{I}_{\mathbf{i}_1,\eta}) = \{u \subseteq I_\eta : |u| = \eta(2n)\}$. By 2.5 clearly $X_\eta = [\lambda_\eta]^{\eta(2n)}$ mod $\mathrm{ER}^n_{\lambda_\eta,\eta(2),\beth_2(\kappa)}$, see Definition 2.17(1A).

Now let $\mathscr{T}' = \{\eta \in \mathscr{T}$: if $2n+1 < \ell g(\eta)$ then $\eta {\restriction} (2n+2) \in X_\eta\}$ and $\mathbf{i}_2 = \mathbf{i}_1 {\restriction} \mathscr{T}'$, so clearly $\mathbf{i}_1 \le_{\rm pr} \mathbf{i}_2$, see Definition 2.17(2A).]

Next

$(*)_6$ define a function $\mathbf{c}$ with domain $\mathscr{T}_{\mathbf{i}_2}$ as follows:

- if $\eta \in \mathscr{T}, \ell g(\eta) = 2n+2$, <u>then</u> $\mathbf{c}(\eta)$ is the quantifier type in M^+ of $\langle a_\ell : \ell < n\rangle$ for any $\alpha_0 < \alpha_1 < \ldots < \alpha_{n-1}$ from $\eta(2n+1)$
- if $\eta \in \mathscr{T}, \ell g(\eta) = 2n+1$ or $\ell g(\eta) = 0$, <u>then</u> $\mathbf{c}(\eta) = 0$.

Clearly

$(*)_7$ $\mathrm{Rang}(\mathbf{c})$ has cardinality $\le 2^\kappa = 2^\kappa$.

So by 2.18 (with a degenerate projection; so κ, θ there stands for $2^\kappa, \aleph_0$ here):

$(*)_8$ there are $\mathbf{i}(3) = \mathbf{i}_3 \ge \mathbf{i}_2$ and $\langle c_n : n < \omega\rangle$ such that:

- (a) $\eta \in \mathscr{T}_{\mathbf{i}_3} \Rightarrow \mathbf{c}(\eta) = c_{\ell g(\eta)}$
- (b) $\mathrm{Dp}_{\mathbf{i}_3}(<>) \ge \beth_2(\kappa)$.

The rest should be clear.

2) Similar proof, this time in M^+ we have individual constants for every member of M_2 and we start with the witness $\mathbf{x}_1$ so X_η have fewer elements still positive modulo the ideal.

3),4) Similarly. $\square_{3.6}$

Definition 3.7. We say $\mathbf{x}$ is an indirect witness for $(N, M, \lambda, \kappa, \gamma, \Phi)$, recalling 3.2(1), <u>when</u> for some Ψ:

(a) $M, N, \lambda, \kappa, \gamma, \Phi$ are as in Definition 3.2

(b) $\Psi \in \Upsilon^{\rm sor}_\kappa[\mathfrak{k}_M]$ and $\Phi \le^4_\kappa \Psi$, see Definition 2.12

(c) $\mathbf{x}$ is a direct witness of $(N, M, \lambda, \kappa, \gamma, \Psi)$.

Remark 3.8. Why do we need the indirect witnesses? As if we use direct witness only in the proof of 3.14 it is not clear how to get many non-isomorphic models.

Claim 3.9. *Assume $I = I_\chi$ is as in 1.15.*
If (A) then (B) where:

(A) (a) $\mathrm{LST}_{\mathfrak{k}} + |\tau_{\mathfrak{k}}| \le \kappa < \chi_1 < \chi_2 < \chi_3 \le \chi$ *and for* $\ell = 1, 2$, $\chi_{\ell+1}$ *is strong limit of cofinality* $> \beth_2(\chi_\ell)$

(b) $N = \mathrm{EM}_{\tau(\mathfrak{k})}(I, \Phi_1)$ *where* $\Phi_1 \in \Upsilon^{\rm sor}_\kappa[M_1, \mathfrak{k}], \|M_1\| \le \chi_1$

(c) $M_2 \le_{\mathfrak{k}} N$ *and* $\|M_2\| \le \chi_1$

(d) $\Phi_2 \in \Upsilon^{\rm sor}_\kappa[M, \mathfrak{k}]$

(e) Φ_2 *has a witness for* (N, M_2, χ_2, κ)

(B) (a) Φ_2 *has a witness for* (N, M_2, χ_3, κ)

(b) *if in addition* $M_2 \le_{\mathfrak{k}} M_1$ *then* $\Phi_2 \le^3_\kappa \Phi_1$

(c) *we can* $\le_{\mathfrak{k}}$*-embed* $\mathrm{EM}_{\tau(\mathfrak{k})}(I_\chi, \Phi_2)$ *into* N.

Proof. As in the proof of 3.6 recalling the choice of I in 1.15; for (B)(c) we use Clause $(B)^+$ of 3.6. $\square_{3.9}$

Remark 3.10. In fact, in 3.9, $\chi_2 = \beth_{1,1}(\chi_1)$ and $\chi_3 = \beth_{\omega\gamma}(\chi_1)$ suffices so, of course, in (B)(a) we use $(N, M_1, \chi_3, \kappa, \gamma)$.

Claim 3.11. *If (A) then (B) where:*

(A) (a) $M_1 \le_{\mathfrak{k}} M_2 \le_{\mathfrak{k}} N$

(b)(α) M_ℓ *has cardinality* κ_ℓ

(β) $\|N\| \ge \lambda$

(γ) $\kappa_\ell \ge \kappa \ge \mathrm{LST}_{\mathfrak{k}} + |\tau_{\mathfrak{k}}|$

(c) $\Phi_1 \in \Upsilon^{\rm sor}_\kappa(M_1, \mathfrak{k})$

(d) λ *is strong limit and* $\mathrm{cf}(\lambda) = (\beth_2(\kappa_2))^+$ *or just* $(\forall \alpha < \mathrm{cf}(\lambda))(|\alpha|^{2^\kappa} < \mathrm{cf}(\lambda))$

(e) $\mathbf{x}_1$ *is an indirect witness for* $(N, M_1, \lambda, \kappa, \Phi_1)$

(B) *there are* $\Phi_2, \mathbf{x}_2$ *such that:*

(a) $\Phi_2 \in \Upsilon^{\rm sor}_\kappa(\mathfrak{k}_{M_2})$

(b) $\Phi_1 \le^1_{\kappa_2} \Phi_2$

(c) $\mathbf{x}_2$ *is an indirect witness for* $(N, M_2, \lambda, \kappa_2, \Phi_2)$.

Proof. By clause $(A)(e)$ of the assumption and the definition of indirect witness in 3.7 there is Ψ_1 such that:

$(*)_1$ (a) $\Psi_1 \in \Upsilon^{\rm or}_{\kappa_1}[\mathfrak{k}_{M_1}]$ which is standard

(b) $\mathbf{x}_1$ is a direct witness of $(N, M_1, \lambda, \kappa_1, \Psi_1)$

(c) $\Phi_1 \le^4_{\kappa_1} \Psi_1$.

By claim 3.6(2) there are $\mathbf{x}_2, \Psi_2$ such that

$(*)_2$ (a) $\Psi_2 \in \Upsilon^{\rm sor}_{\kappa_2}[\mathfrak{k}_{M_2}]$

(b) $\Psi_1 \le^1_{\kappa_2} \Psi_2$

(c) $\mathbf{x}_2$ is a direct witness of $(N, M_2, \lambda, \kappa_2, \Psi_2)$.

Lastly, by 2.16 applied to our Φ_1, Ψ_1, Ψ_2 and get Φ_2 such that

$(*)_3$ (a) $\Phi_1 \in \Upsilon^{\rm sor}_\kappa[\mathfrak{k}_{M_2}]$

(b) $\Phi_1 \le^1_\kappa \Phi_2$

(c) $\Phi_2 \le^4_\kappa \Psi_2$.

So we have gotten Clause (B) as promised. $\square_{3.11}$

Claim 3.12. *If (A) + (B) then (C) where:*

(A) (a) $\lambda_n \ge \mathrm{LST}_\mathfrak{k}$ *is strong limit,* $\mathrm{cf}(\lambda_n) = (\beth_2(\mathrm{LST}_\mathfrak{k} + \lambda_m))^+$ *if* $n = m+1$

(b) $\lambda = \sum_n \lambda_n$ *and* $\lambda_n < \lambda_{n+1}$

(c) $N \in K^\mathfrak{k}_\lambda$

(d) $M_n \le_\mathfrak{k} M_{n+1} <_\mathfrak{k} N$ *and* $\|M_n\| = \lambda_n$

(e) $N = \cup\{M_n : n < \omega\}$

(B) *there is no* $\Phi \in \Upsilon^{\rm sor}_\lambda[\mathfrak{k}_N]$*, see 2.15*

(C) *for some* n *and* Φ

(a) $\Phi \in \Upsilon^{\rm sor}_{\lambda_n}[\mathfrak{k}_{M_n}]$

(b) *there is an indirect witness*[9] *for* $(N, M_n, \lambda_{n+4}, \lambda_n, \Phi_n)$

(c) *there is no indirect witness for* $(N, M_n, \lambda_{n+5}, \lambda_n, \Phi_n)$.

Remark 3.13. 1) Later we shall weaken $(A)(a)$.
2) We may use $\Upsilon^{\rm sor}_\kappa[\mathfrak{k}_{M_n}]$ where $\lambda_0 \ge \kappa \ge \mathrm{LST}_\mathfrak{k} + |\tau_\mathfrak{k}|$ in 3.11 and in 3.12, also in 3.14.

Proof. We assume $(A) + \neg(C)$ and shall prove $\neg(B)$, this suffices. We try to choose $(\Phi_n, \mathbf{x}_n)$ by induction on n such that:

$\otimes$ (a) $\Phi_n \in \Upsilon^{\rm sor}_{\lambda_n}[\mathfrak{k}_{M_n}]$

(b) $\{c_a : a \in N\} \cap \tau(\Phi_n) = \{c_a : a \in M_n\}$

(c) $\mathbf{x}_n$ is an indirect witness for $(N, M_n, \lambda_{n+4}, \lambda_n, \Phi_n)$

(d) if $n = m+1$ then $\Phi_m \le^1_{\lambda_n} \Phi_n$.

Now

$(*)_1$ if we succeed to carry the induction <u>then</u> there is $\Phi \in \Upsilon^{\rm sor}_\lambda[\mathfrak{k}_N]$.

[Why? Note that $\Phi_n \in \Upsilon^{\rm sor}_{\lambda_n}[\mathfrak{k}_{M_n}] \subseteq \Upsilon^{\rm sor}_{\lambda_n}[\mathfrak{k}]$ and as $\lambda_n \le \lambda$ clearly $\Phi_n \in \Upsilon^{\rm sor}_{\lambda_n}[\mathfrak{k}] \subseteq \Upsilon^{\rm sor}_\lambda[\mathfrak{k}]$ and so by 2.11(2) there is $\Phi \in \Upsilon_\lambda[\mathfrak{k}]$ such that $n < \omega \Rightarrow \Phi_n \le^1_\lambda \Phi$. Easily N is $\le_\mathfrak{k}$-embeddable into every $\mathrm{EM}_{\tau(\mathfrak{k})}(I, \Phi)$, in fact, $\Phi \in \Upsilon_\lambda[\mathfrak{k}_N]$, contradiction to clause (B) of the assumption.]

$(*)_2$ we can choose $(\mathbf{x}_n, \Phi_n)$ for $n = 0$.

[Why? By 3.6(1).]

$(*)_3$ if $n = m+1$ and we have chosen $(\mathbf{x}_m, \Phi_m)$ then we can choose $(\mathbf{x}_n, \Phi_n)$.

[9] hence also a direct one; similarly in $\otimes(d)$ in the proof

[Why? If there is no indirect witness $\mathbf{y}_m$ for $(N, M_m, \lambda_{m+5}, \lambda_m, \Phi_m)$ we have gotten clause (C), so without loss of generality $\mathbf{y}_m$ exists. Now apply 3.11 with $(\mathbf{y}_n, M_m, M_n, \lambda_{n+5}, \lambda_n)$ here standing for $(\mathbf{x}_1, M_1, M_2, \lambda, \kappa, \Phi_1)$ there, so we get $\mathbf{x}_n, \Phi_n$ here stand for $\mathbf{x}_2, \Phi_2$ there.] $\square_{3.12}$

Claim 3.14. *We have $\dot{I}(\mu, K_{\mathfrak{k}}) \geq \chi$ <u>when</u>:*

$\oplus$ (a) $\mathrm{LST}_{\mathfrak{k}} + |\tau_{\mathfrak{k}}| \leq \kappa \leq \chi_1 < \chi_2 < \chi_3 \leq \min\{\lambda, \mu\}$
(b) $M \leq_{\mathfrak{k}} N$
(c) $\|M\| \leq \kappa$ *and* $\|N\| \geq \lambda$
(d) $\Phi \in \Upsilon^{\mathrm{sor}}_{\kappa}[\mathfrak{k}_M]$
(e) $\mathbf{x}$ *is an indirect witness for* $(N, M, \chi_2, \chi_1, \Phi)$
(f) *there is no indirect witness for* $(N, M, \chi_3, \chi_1, \Phi)$
(g) χ_3 *is strong limit of cofinality* $(\beth_2(\chi_2))^+$
(h) $\chi = |\{\theta : \theta = \beth_\theta$ *and* $\theta \in [\chi_1, \chi_2]\}|$

Proof. Let γ_* be maximal such that $\beth_{\omega\cdot\gamma_*}(\chi_1) \leq \chi_2$. Let $\Psi \in \Upsilon^{\mathrm{sor}}_{\kappa}[\mathfrak{k}_M]$ be such that $\Phi \leq^4_\kappa \Psi$ and Ψ has a direct witness for $(N, M, \chi_2, \chi_1, \Psi)$ and choose such a witness $\mathbf{x}$.

Let M_2 be such that $M \leq_{\mathfrak{k}} M_2 \leq_{\mathfrak{k}} N$ and $\|M_2\| = \beth_{\omega\cdot\gamma_*}(\chi_1) \leq \chi_2$ and $\mathbf{x}$ is a direct witness for $(M_2, M, \beth_{\omega\cdot\gamma}(\chi_1), \chi_1, \gamma_*, \Psi)$.

As χ_3 is strong limit of cofinality $> \beth_2(\chi_2)$ there are $\Phi_3 \in \Upsilon^{\mathrm{sor}}_{\kappa}[\mathfrak{k}_{M_2}]$ and $\mathbf{y}$ which is a direct witness for $(N, M_2, \chi_3, \chi_2, \Phi_3)$ and so $\tau'_{\Phi_3} := \tau(\Phi_3) \backslash \{c_a : a \in M_2\}$ has cardinality κ. For each $\gamma < \gamma_*$ there are $M_{2,\gamma}, \mathbf{x}_\gamma$ such that:

$(*)_1$ (a) $M_{2,\gamma} \leq_{\mathfrak{k}} M_2$
(b) $\|M_{2,\gamma}\|$ is $\geq \beth_{\omega\cdot\gamma}(\chi_1)$ but $< \beth_{\omega\cdot\gamma+\omega}(\chi_1)$; can get even $\|M_{2,\gamma}\| = \beth_{\omega\cdot\gamma}(\chi_1)$
(c) $\mathbf{x}_\gamma$ is a direct witness for $(M_{2,\gamma}, M, \beth_{\omega\cdot\gamma}(\chi_1), \chi_1, \gamma, \Psi)$.

[Why? Try by induction on k to choose $\eta_k \in \mathscr{T}_{\mathbf{x}}$ such that $\ell g(\eta_k) = 2k+1, \eta_k(2k) \geq \beth_{\omega\cdot\gamma}(\chi_1)$ and $\ell < k \Rightarrow \eta_\ell \triangleleft \eta_k$. For $k = 0$, clearly $\eta_k = \langle\rangle$ is O.K., and as $\eta_\ell(2\ell) > \eta_{\ell+1}(2\ell+2)$, necessarily for some k we have η_k but cannot choose η_{k+1}; let $A_\gamma = \cup\{\mathrm{Rang}(h^{\mathbf{x}}_{\eta,u}) : \eta_k \trianglelefteq \eta \in \mathscr{T}_{\mathbf{x}}$ and $u \in [I^{\mathbf{x}}_\eta]^{\lfloor \ell g(\eta)/2 \rfloor}\}$ so $A_\gamma \subseteq M$ has cardinality $\eta_k(2k) \in [\beth_{\omega\cdot\gamma}(\chi_1), \beth_{\omega\cdot\gamma+\omega}(\chi_1)$. Without loss of generality if $N_* = \mathrm{EM}(\emptyset, \Phi_3)$ is standard (i.e. $M = N_* \restriction \tau_{M_2}$) then A_γ is closed under the functions of $N_* \restriction \tau'_{\Phi_3}$. Let $M_{2,\gamma} = M_2 \restriction A_\gamma$; it is $\leq_{\mathfrak{k}} M$ and it satisfies clauses (a),(b) and include A_γ. Then we can easily find $\mathbf{x}_\gamma$ as required in clause (c).]

Next we can find $\mathbf{y}_\gamma, \Phi_{3,\gamma}$ such that

$(*)_2$ (a) $\mathbf{y}_\gamma$ is a direct witness of $(N, M_{2,\gamma}, \chi_3, \|M_{2,\gamma}\|, \Phi_{3,\gamma})$
(b) $\Phi_{3,\gamma} \in \Upsilon^{\mathrm{sor}}_{\kappa}[M_{2,\gamma}, \mathfrak{k}]$.

[Why? Recall $\tau(\Phi_3) \backslash \{c_a : a \in M_2\}$ has cardinality κ. Let $\tau_{2,\gamma} = \tau(\Phi_3) \backslash \{c_a : a \in M_2 \backslash M_{2,\gamma}\}$ so has cardinality $\|M_{2,\gamma}\|$, let $\Phi_{3,\gamma} = \Phi_3 \restriction \tau_{2,\gamma}$, is as required in $(*)_2(k)$. As for $\mathbf{y}_\gamma$ we derived it form $\mathbf{y}$.]

Now let $I = I_\mu$ be a linear order of cardinality μ as required in 1.15.

Lastly, let $N_\gamma = \mathrm{EM}_{\tau(\mathfrak{k})}(\mu, \Phi_{3,\gamma})$ be standard hence $M_{2,\gamma} \leq_{\mathfrak{k}} N_\gamma \in K^{\mathfrak{k}}_\mu$.

We choose ∂_i by induction on i such that: if $i = 0$ then $\partial_i = \chi_1$, if i is limit then $\partial_i = \cup\{\partial_j : j < i\}$ and if $i = j + 1$ then $\partial_i = \beth_{\beth_2(\partial_j)^+}$ when it is $\leq \chi_2$ and undefined otherwise. Let ∂_i be defined iff $i < i(*)$ and let $\Theta = \{\partial_{i+1} : i+1 < i(*)\}$. Now $|\Theta| \geq \chi$ so it suffices to prove that $\langle N_\theta : \theta \in \Theta \rangle$ are pairwise non-isomorphic.

So toward contradiction assume

$(*)_3$ $\theta_1 < \theta_2$ are from Θ and π is an isomorphism from N_{θ_2} onto N_{θ_1}.

We can find $M_* \leq_{\mathfrak{k}} N_{\theta_1}$ such that $\|M_*\| = \theta_2$ and $M \cup M_{2,\theta_1} \cup \pi(M_{2,\theta_2}) \subseteq M_*$ and without loss of generality we can find $I_* \subseteq \mu$ of cardinality θ_2 such that $M_* = \mathrm{EM}_{\tau(\mathfrak{k})}(I_*, \Phi_{3,\theta_1})$.

Let $I_1^* \subseteq I_*$ be of cardinality θ_1 such that $M_{2,\theta_1} \cup \pi(M) \subseteq N'_{\theta_1} := \mathrm{EM}_{\tau(\mathfrak{k})}(I_1^*, \Phi_{3,\theta_1})$ and let $N'_{\theta_2} = \pi^{-1}(N'_{\theta_1})$. By 3.6(2) we can find $\Psi' \in \Upsilon^{\mathrm{sor}}_\kappa(N'_{\theta_2}, \mathfrak{k})$ and $\mathbf{x}_{\theta_2}$ a witness for $(M_{2,\theta_2}, N'_{\theta_2}, \theta_2, \kappa, \Psi')$ such that $\Psi \leq^4_\kappa \Psi'$ and $\mathbf{x}_{\theta_2} \leq \mathbf{x}'_{\theta_2}$ where $\theta_2 = \beth_{\omega \cdot \gamma_2}(\chi_1)$.

Now clearly $N'_{\theta_1}, \Psi, \pi(\Psi'), \pi(\mathbf{x}'_{\theta_2})$ satisfies the parallel statements in N_{θ_1}. By 3.9(B)(a) and the choice of I_μ there is a witness for $(N_{\theta_1}, N'_{\theta_1}, \chi_3, \kappa, \pi(\Psi'))$, hence applying π^{-1} there is a witness $\mathbf{x}''_{\theta_2}$ for $(N_{\theta_1}, N'_{\theta_1}, \chi_3, \kappa, \Psi')$.

Hence by 3.9(B)(b), $\Psi' \leq^3_\kappa \Phi_{3,\theta_2}$ but together $\Phi \leq^4_\kappa \Psi \leq^4_\kappa \Psi' \leq^3_\kappa \Phi_{3,\theta_2}$ hence $\Phi \leq^3_{\theta_2} \Phi_{3,\theta_2}$ by 2.14(1) so by 2.14(2), the last clause, there is $\Phi'_{3,\theta_2} \in \Phi_{3,\theta_2}/\mathbf{E}^{\mathrm{ai}}_{\theta_2}$ such that $\Phi \leq^4_{\theta_2} \Phi'_{3,\theta_2}$. But as Φ_{3,θ_2} has a $(N, M_{2,\theta_2}, \chi_3, \theta_2)$ witness by 3.4(3) also Φ'_{3,θ_2} has hence Φ has an indirect witness for (N, M, χ_3, κ), contradiction. $\square_{3.14}$

Conclusion 3.15. *Assume* $\mathrm{cf}(\lambda) = \aleph_0$ *and* $\lambda = \beth_{1,\lambda}$.
1) If $\lambda > \dot{I}(\lambda, K_{\mathfrak{k}})$ *then* $M \in K^{\mathfrak{k}}_\lambda \Rightarrow \Upsilon^{\mathrm{sor}}_\lambda[\mathfrak{k}_M] \neq \emptyset$.
2) If $\mu \geq \lambda > \dot{I}(\mu, K_{\mathfrak{k}})$ *<u>then</u>* $M \in K^{\mathfrak{k}}_\lambda \Rightarrow \Upsilon^{\mathrm{sor}}_\lambda[\mathfrak{k}_M] \neq \emptyset$.
Moreover, at least one of the following holds:

(a) *for some* $\chi_1 < \lambda$ *if* $\chi_1 < \chi_2 = \beth_{2,\delta} \leq \min\{\lambda, \mu\}$ *then* $|\delta| \leq \dot{I}(\mu, K_{\mathfrak{k}})$
(b) $\Upsilon^{\mathrm{sor}}_\lambda[\mathfrak{k}_M] \neq \emptyset$ *for every* $M \in K^{\mathfrak{k}}_\lambda$.

Theorem 3.16. *The result from the abstract holds, that is, for every a.e.c.* $\mathfrak{k}$ *for some closed unbounded class* $\mathbf{C}$ *of cardinals we have (a) or (b) where*

(a) *for every* $\lambda \in \mathbf{C}$ *of cofinality* $\aleph_0, \dot{I}(\lambda, K) \geq \lambda$
(b) *for every* $\lambda \in \mathbf{C}$ *of cofinality* $\aleph_0$ *and* $M \in K_\lambda$, *for every cardinal* $\kappa \geq \lambda$ *there is* N_κ *of cardinality* κ *extending* M *(in the sense of our a.e.c.).*

Proof. Let $\Theta = \{\mu : \mu = \beth_{2,\delta}$ and $|\delta| > \dot{I}(\mu, K_{\mathfrak{k}})$ for some limit ordinal $\delta\}$.

<u>Case 1</u>: Θ is an unbounded class of cardinals.

So $\mathbf{C} = \{\mu : \mu = \sup(\mu \cap \Theta)\}$ is a closed unbounded class of cardinals. Easily $\mu \in \mathbf{C} \Rightarrow \mu = \beth_{1,\mu}$ and by 3.15 + 2.15 for every $\mu \in \mathbf{C}$, clause (b) of 3.16 holds.

<u>Case 2</u>: Θ is a bounded class of cardinals.

So by the definition of $\Theta, \mathbf{C} = \{\mu : \mu > \sup(\Theta), \mu = \beth_{2,\mu}\}$ is as required. $\square_{3.16}$

Also

Theorem 3.17. *For every* aec $\mathfrak{k}$ *one of the following holds:*

(a) *for some* χ *we have* $\chi < \mu = \beth_{2,\mu} \Rightarrow \dot{I}(\mu, K_{\mathfrak{k}}) \geq \mu$ *and* $\chi < \mu = \beth_{1,\omega \cdot \gamma} \Rightarrow \dot{I}(\mu, K_{\mathfrak{k}}) \geq |\gamma|$
(b) *for some closed unbounded class* $\mathscr{C}$ *of cardinals we have* $\mathrm{cf}(\lambda) = \aleph_0 \wedge \lambda \in \mathbf{C} \wedge M \in K^{\mathfrak{k}}_\lambda \Rightarrow \Upsilon^{\mathrm{sor}}[M, \mathfrak{k}] \neq \emptyset$.

Proof. Similarly to 3.16, using Fodor lemma for classes of cardinals. $\square_{3.17}$

§ 4. Concluding Remarks

Definition 4.1. 1) For an ordinal γ, τ-models M_1, M_2 and cardinal λ we define a game $\eth = \eth_{\theta,\gamma}(M_1, M_2)$. A play lasts less than ω models is defined as in [She12, 2.1].

Claim 4.2. *1) Assume* $\mathrm{cf}(\lambda) = \aleph_0$ *and* M_1, M_2 *are* τ*-models of cardinality* λ*. If the isomorphic player wins in* $\eth_{\lambda,\gamma}(M_1, M_2)$ *for every* γ *or just* $\gamma < (2^{<\lambda})^+$ *then* M_1, M_2 *are isomorphism.*
1A) If above λ *is strong limit then* "$(2^{<\lambda})^+ = \lambda^+$".
2) Assume λ *is strong limit of cofinality* $K = K_{\mathfrak{k}}$ *and* $|\tau_{\mathfrak{k}}| + \mathrm{LST}_{\mathfrak{k}} \leq \lambda$ *and* $K = \{M{\restriction}\tau : M \models \psi\}$ *for some* $\psi \in \mathbb{L}_{\lambda^+,\aleph_0}$.

If $\dot{I}(\lambda, K) \leq \lambda$ *then for every* $M_1 \in K$ *there is* $M_2 \in K_{\leq\lambda}$ *such that the isomorphism player wins in* $\eth_{\lambda,\gamma}(M_1, M_2)$ *for every* λ*.*

Conjecture 4.3. For every a.e.c. $\mathfrak{k}$ letting $\kappa = \mathrm{LST}_{\mathfrak{k}} + |\tau_{\mathfrak{k}}|$, at least one of the following occurs:

(a) if $\lambda = \beth_{1,\lambda} > \kappa$ and $\mathrm{cf}(\lambda) = \aleph_0$, then $\Upsilon^{\mathrm{sor}}_{\kappa}[M, \mathfrak{k}] \neq \emptyset$
(b) if $\lambda = \beth_{1,\lambda} > \kappa$ and $\mathrm{cf}(\lambda) = \aleph_0$, then $\dot{I}(\lambda, K_{\mathfrak{k}}) = 2^{\lambda}$.

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