A Variety with Solvable, but not Uniformly Solvable, Word Problem

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Abstract

In the literature two notions of the word problem for a variety occur. A variety has a *decidable word problem* if every finitely presented algebra in the variety has a decidable word problem. It has a *uniformly decidable word problem* if there is an algorithm which given a finite presentation produces an algorithm for solving the word problem of the algebra so presented. A variety is given with finitely many axioms having a decidable, but not uniformly decidable, word problem. Other related examples are given as well.

§0. INTRODUCTION

The following two options occur in the literature for what is meant by the solvability of the word problem for a variety V :

(1) There is an algorithm which, given a finite presentation $\mathcal{P}$ in finitely many generators and relations, solves the word problem for $\mathcal{P}$ relative to the variety V.

(2) For each finite presentation $\mathcal{P}$ in finitely many generators and relations, there is an algorithm which solves the word problem for $\mathcal{P}$ relative to the variety V.

We say that V has uniformly solvable word problem if (1) holds. It is the first notion that is studied in Evans [2, 3], where it is called just the word problem for V, and the second coincides with the terminology in Burris and Sankapannavar [1]. Benjamin Wells has informed us that Tarski was interested in the existence of varieties with solvable but not uniformly solvable word problem.

Varieties with uniformly solvable word problem include commutative semigroups and abelian groups (each of these are equivalent to the existence of an algorithm for solving systems of linear equations over the integers which is due to Aryabhata, see chapter 5 of [10]), any finitely based locally finite or residually finite variety, and the variety of all algebras of a given finite type (see [4]).

The examples which appear in the literature, of varieties with unsolvable word problem, all provide a finite presentation $\mathcal{P}$ for which the word problem for $\mathcal{P}$ relative to that variety is unsolvable. These include semigroups [9], groups [8], and modular lattices [5].

Here, we present a finitely based variety V of finite type which does not have a uniformly solvable word problem, but which nevertheless has solvable word problem. We also present a recursively based variety of finite type, which is defined by laws involving only constants (i.e., no variables), with solvable but not uniformly solvable word problem. This second result is the best possible one can provide for varieties defined by laws involving no variables: every finitely based such variety has uniformly solvable word problem. If one would be satisfied with varieties with infinitely many operations, then it is relatively easy to produce an example of a recursively based variety with solvable but not uniformly solvable word problem; we present such an example, essentially due to B. Wells [11], at the end of the paper.

Our proof uses the unsolvability of the halting problem for the universal

Turing machine, and the laws defining the variety precisely allow us to model the action of the universal Turing machine in the variety. In the usual proofs that the variety of semigroups has an undecidable word problem, a finitely presented congruence is given so that for any initial Turing machine configuration, the instantaneous descriptions of the Turing machine calculations all lie in the same congruence class. Then laws are added which make the halting state a right and left zero. So, the undecidability of the word problem in this algebra comes from not being able to decide whether a given word (initial configuration) is congruent to the halting state. This algebra contains all possible Turing machine calculations. In our variety each calculation will be modeled by a single algebra.

Our approach is based on a different picture of a Turing machine calculation than the sequence of instantaneous descriptions used in semigroups. We view a Turing machine calculation as taking place on a $\mathbb{Z} \times \omega$ grid, where the copy of $\mathbb{Z}$ with second coordinate n represents the Turing machine tape at time n. To understand the calculation, we must know the alphabet content of each square, which square the head is reading for each time n, and the state the machine is in at time n. There are various possible ways to formalize this insight, so that each Turing machine calculation corresponds to a finitely presented algebra.

To ensure that the word problem doesn't have a uniform solution, we introduce a function which has value 1 when applied to any state the machine reaches and value 0 on the halting state. Then a decision procedure which given a finite presentation, determines whether 1 is congruent to 0 would solve the halting problem. There are considerable technical difficulties in implementing this idea in such a way that we can prove that each finitely presented algebra in the variety has a decidable word problem.

It seems to us that there are two interesting directions that research can follow in light of the results in this paper. There remains the question of whether a finitely based variety of unary algebras with solvable word problem has uniformly solvable word problem. Another direction research could take is to consider subvarieties of interesting natural varieties. This could either be understood as varieties of X where X is a favourite class of algebras or, say, congruence modular varieties. This second problem was suggested to us by the persistent question of everyone to whom we told the result, namely "Is there a natural example?"

The research for this paper was begun out while the latter two authors

were visitors at the Department of Mathematics and Statistics at Simon Fraser University. We gratefully acknowledge financial support from the Natural Science and Engineering Research Council of Canada. This is paper #291 on Shelah's publication list. We also wish to thank the referee for a thorough job of reading the paper.

§1. DEFINITIONS

We assume Σ is some finitary type of algebras (with possibly infinitely many operations). A presentation is a pair $\mathcal{P} = (X, R)$ consisting of a set X (of generators) and a set $R \subseteq FX \times FX$ (of relations), where FX is the (absolutely) free Σ-algebra over X. A finite presentation is a presentation $\mathcal{P} = (X, R)$ where both X and R are finite.

Given a variety V and a presentation $\mathcal{P} = (X, R)$, there is an algebra $A \in V$ and a homomorphism $h : FX \to A$ with $R \subseteq Ker(h)$, such that any homomorphism $g : FX \to B$ with $B \in V$ and $R \subseteq Ker(g)$ factors uniquely through h. The algebra A is unique up to isomorphism, and is called the algebra given by the presentation $\mathcal{P}$ relative to the variety V.

The word problem for $\mathcal{P}$ relative to the variety V is to determine, given s, $t \in FX$, whether $(s, t) \in Ker(h)$. Note that $Ker(h)$ is the congruence on FX generated by $R \cup \theta_V$, where θ_V consists of all equations in variables from X satisfied by the variety V; equivalently, θ_V is the kernel of the unique homomorphism from FX to the V-free algebra on X mapping the elements of X identically.

Next, we introduce the notion of a partial subalgebra, and state one result which will be proved and used in §6. The proof bears a familial resemblance to the more complicated proof in §5.

Definition. A *partial subalgebra* is a pair $(A, \equiv_A)$ where A is a subset of FX which is closed under formation of subterms, and $\equiv_A$ is an equivalence relation on A which is a *partial congruence*, i.e., has the property that for each operation σ of arity n, if $a_i \equiv_A b_i$ for $1 \leq i \leq n$ and $\sigma(a_1, \ldots, a_n)$, $\sigma(b_1, \ldots, b_n) \in A$ then

$$\sigma(a_1, \ldots, a_n) \equiv_A \sigma(b_1, \ldots, b_n).$$

Proposition 1.1 *If for a partial subalgebra* $(A, \equiv_A)$,

(1) *membership in* A *is decidable (for elements of* FX*),*

(2) *membership in* $\equiv_A$ *is decidable (for pairs of elements of* FX*),*

(3) *there is an algorithm which, given an operation* σ *of arity* n *and* $a_1, \ldots,$ $a_n \in A$*, determines whether there exist* $b_1, \ldots, b_n \in A$ *for which* $a_i \equiv_A b_i$ *for* $1 \leq i \leq n$ *and* $\sigma(b_1, \ldots, b_n) \in A$*, then* $\equiv$*, the congruence on* FX *generated by* $\equiv_A$*, is decidable. Further this decision procedure is uniform in the algorithms for deciding* (1), (2), (3).

Remark: The conclusion of this result says that there is a solution to the word problem for the presentation $(X, \equiv_A)$ relative to the variety of all algebras of the given type.

Definition A partial subalgebra satifying the hypotheses of Proposition 1.1 is called *decidable.*

We delay the proof of Proposition 1.1 until the end of §6.

Corollary 1.2. (Evans) *Let* V *be the variety of all algebras in some finite language. Then* V *has uniformly decidable word problem.*

Proof. Suppose we are given some finite presentation. Let A be the finite set consisting of the terms appearing in the presentation and their subterms. By brute search through the finitely many possibilities, we can find $\equiv_A$, the smallest partial congruence on A containing all the relations in the presentation. Now we can apply Proposition 1.1.

Corollary 1.2 implies that any variety in a finite language which is defined by finitely many laws involving only constants has a uniformly decidable word problem. Any presentation can be viewed as a new presentation in the variety without the laws, by viewing each law as a relation in the new presentation.

§2. THE FINITELY BASED VARIETY.

2.1 MODIFICATION OF THE UNIVERSAL TURING MACHINE

Suppose that we are given a universal Turing machine with a unique halting state h which, at each move, prints some letter on the scanned square, moves one square either left or right (denoted respectively by -1 or 1) and enters a new (or the same) state. We are first going to adjust the machine by adding right and left end markers e_R and e_L, (as new members of the alphabet), and adding, for each state, two new states q_L and q_R, and appropriate instructions so that the adjusted machine does the following: if it is scanning e_R in state q, it prints B (blank), moves right (into state q_R), prints e_R and then moves left and returns to state q; if it is scanning e_L in state q it prints B, moves left, prints e_L, and moves right and returns to state q. That is, to the Turing flow chart we add the following

$$q_L \underset{e_L:B:1}{\overset{e_L:B:-1}{\leftrightarrows}} q \underset{e_R:B:-1}{\overset{e_R:B:1}{\rightleftarrows}} q_R$$

The resulting machine, if started on a finite tape inscription with the left and right endmarkers at the appropriate ends, and the rest of the tape blank, does what the original machine would have done if placed on that inscription with the rest of the tape blank, except that whenever the adjusted machine hits an endmarker it first moves it out one square, leaving behind a blank square.

Suppose that the resulting machine has state set Q and alphabet Σ, and that its action is given by the functions σ, μ, and α operating on $Q \times \Sigma$, which specify the next state, the motion (either left or right) and the print instruction, so that

$\sigma : Q \times \Sigma \to Q$
$\mu : Q \times \Sigma \to \{-1, 1\}$
$\alpha : Q \times \Sigma \to \Sigma.$

For simplicity we will assume σ, μ and α are total functions and so defined even if we reach the "halting state".

2.2 DEFINITION OF THE VARIETY

Our variety V has the following operations:

CONSTANTS: c, all elements of $Q \cup \Sigma$, 0, 1, 0_F, 1_F
UNARY: T, S, S^{-1}, H, P, C_Σ, C_Q, U, E
BINARY: F, R, K, K^*, C^*,
TERNARY: N_H, N_Q, N_Σ

The elements of the range of P will be the space-time elements (in the intended interpretation they represent the tape squares); the action of S and S^{-1} represents stepping right and left respectively though space, of T represents moving ahead one time period, and of H represents moving to the head position. C_Σ gives the letter in the square and C_Q gives the state the machine is in while scanning the square. The action of the ternary operation N_H gives head position at the next time instant, while N_Σ gives the letter in the square scanned by the head at the next time position and C_Q gives the state at the next time.

The intended interpretations of $F(x, y)$ and $R(x, y)$ are "y follows x" and "y is to the right of x". K and K^* are comparison functions, and U is a modified addition by 1. The intended interpretation is explained more fully in paranthetical comments below and in the proof of Theorem 3.1.

The laws defining our variety are the following:

I $PP(x) \approx P(x)$
$PT(x) \approx TP(x) \approx T(x)$
$PS(x) \approx SP(x) \approx S(x)$
$PS^{-1}(x) \approx S^{-1}P(x) \approx S^{-1}(x)$
$PH(x) \approx HP(x) \approx H(x)$
$PN_H(x, y, z) \approx N_H(x, y,\, P(z)) \approx N_H(x, y, z)$
$PK(x, y) \approx K(x, P(y)) \approx K(x, y)$

$HS(x) \approx HS^{-1}(x) \approx HH(x) \approx H(x)$
$HTH(x) \approx HT(x)$
$TS(x) \approx ST(x)$
$TS^{-1}(x) \approx S^{-1}T(x)$
$SS^{-1}(x) \approx S^{-1}S(x) \approx P(x)$

$N_H(x, y,\, H(z)) \approx N_H(x, y, z)$
$HN_H(x, y, z) \approx HT(z)$.

II $N_Q(q, a, H(x)) \approx \sigma(q, a)$ for all $q \in Q$, $a \in \Sigma$
$N_\Sigma(q, a, H(x)) \approx \alpha(q, a)$ for all $q \in Q$, $a \in \Sigma$
$N_H(q, a, H(x)) \approx S^{\mu(q,a)}TH(x)$ for all $q \in Q - Q_{LR}$

$N_Q(q_L, C_\Sigma H(x),\, H(x)) \approx q \approx N_Q(q_R, C_\Sigma H(x),\, H(x))$

$N_H(q_L, C_\Sigma H(x),\, H(x)) \approx S^{-1}TH(x)$
$N_H(q_R, C_\Sigma H(x),\, H(x)) \approx STH(x)$

$N_\Sigma(q_L, C_\Sigma H(x),\, H(x)) \approx e_L$
$N_\Sigma(q_R, C_\Sigma H(x),\, H(x)) \approx e_R$

III $C_\Sigma TH(x) \approx N_\Sigma(C_Q H(x), C_\Sigma H(x), H(x))$
$C_Q TH(x) \approx N_Q(C_Q H(x), C_\Sigma H(x), H(x))$
$HT(x) \approx N_H(C_Q H(x), C_\Sigma H(x), H(x))$

IV $C_Q P(x) \approx C_Q H(x) \approx C_Q(x)$

$C_\Sigma(x) \approx C_\Sigma P(x)$
$C_\Sigma TP(x) \approx C^*(P(x), R(P(x), H(x)))$
$C_\Sigma TP(x) \approx C^*(P(x), R(H(x), P(x)))$
$C_\Sigma P(x) \approx C^*(P(x), 1)$

(C^* ensures the symbol in a square, which is either to the "right" or to the "left" of the square being scanned, remains unchanged at the next time.)

V $R(x, y) \approx R(P(x), P(y))$
$F(x, y) \approx F(Px, Py)$
$R(P(x), P(x)) \approx 0$
$R(P(x), SP(y)) \approx UR(P(x), P(y))$

$F(P(x), P(y)) \approx F(H(x), H(y))$
$F(P(x), P(x)) \approx 0_F$
$F(P(x), TP(y)) \approx UF(P(x), P(y))$

$U(0) \approx U(1) \approx 1$
$U(0_F) \approx U(1_F) \approx 1_F$

VI For all operations f except T, S, S^{-1}, H, P, K, and N_H, Pf is constant with value $P(c)$.

VII (i) $K(0, P(x)) \approx P(x)$
$K(1, P(x)) \approx P(c)$
$K(0_F, P(x)) \approx P(x)$
$K(1_F, P(x)) \approx P(c)$

(K ensures that if $0 = 1$ or $0_F = 1_F$ in an algebra then all the space time elements are identical or in other words that space-time is degenerate.)

(ii) $K^*(d, d) \approx 0$ for all constants d
$K^*(d, e) \approx 1$ for all constants d, e with $d \neq e$.
$K^*(P(x), d) \approx 1$ for all constants $d \neq c$
$K^*(C_\Sigma P(x), d) \approx 1$ for all constants $d \notin \Sigma$
$K^*(C_Q P(x), d) \approx 1$ for all constants $d \notin Q$
$K^*(R(P(x), P(y)), d) \approx 1$ for all constants $d \neq 0, 1$
$K^*(F(P(x), P(y)), d) \approx 1$ for all constants $d \neq 0_F, 1_F$
$K^*(t, t) \approx 0$ and $K^*(s, t) \approx 1$ for all $s \neq t$ where both s, t belong to
$\{P(x_1),\ C_\Sigma P(x_2),\ C_Q P(x_3),\ R(P(x_4),\ P(y_4)),\ F(P(x_5),\ P(y_5))\}$

(K^* ensures that space-time is degenerate if there is any undesired equalities between constants or if there is a nonempty intersection between the ranges of certain operations.)

VIII $\quad EC_Q P(x) \approx 1$

$E(h) \approx 0$

(E ensures that space-time is degenerate if the halting state is reached.)

2.3 NORMAL FORM FOR SPACE-TIME ELEMENTS

The terms which are in the image of the operations P, S, S^{-1}, T, H, N_H and K are called space-time terms. For each such term t, $P(t)$ is equivalent, modulo the laws of our variety, to t. For all terms t in the images of the other operations, $P(t)$ is equivalent, modulo the laws of our variety, to $P(c)$. Thus a term t is a space-time term if and only if t and $P(t)$ are equivalent, modulo the laws of our variety.

We are going to develop a normal-form representation for space-time terms.

First, for a space-time term t define

$$\begin{aligned}\Lambda_t &= \{S^nT^m(t) | m,n \in \mathbb{Z}, m \geq 0\} \cup \{S^nT^mHT^k(t) | n,m,k \in \mathbb{Z}, m,k \geq 0\} \\ &\cup \{S^nT^mN_H(s,u,T^k(t)) | n,m,k \in \mathbb{Z}, m,k \geq 0, s,u \text{ arbitrary terms}\}\end{aligned}$$

Now, define the set G of *generating space-time terms* as follows

(i) $P(c) \in G$

(ii) $P(x) \in G$ for each variable x

(iii) For each term t, and each $g \in G$ and $\lambda \in \Lambda_g$, the term $K(t,\lambda) \in G$.

(iv) G is the smallest set of terms satisfying (i), (ii) and (iii).

Further, define

$\Lambda = \cup \Lambda_g (g \in G)$.

The members of Λ_g for $g \in G$ are called the space-time terms in normal form with respect to g and the Λ_g is called the space-time component of g, in particular $g \in \Lambda_g$.

For space-time terms in normal form with respect to g, we define the g-time prefix, g-time coordinate and g-space coordinate as follows:

term in Λ	g-time prefix	g-time coordinate	g-space coordinate
$S^nT^m(g)$	$T^m(g)$	m	n
$S^nT^mHT^k(g)$	$T^mHT^k(g)$	$m+k$	n
$S^nT^mN_H(s,t,T^k(g))$	$T^mN_H(s,t,T^k(g))$	$m+k+1$	n

Proposition 2.1: *There is an effective procedure which, given a space-time term s, produces a term $t \in \Lambda$ (i.e., in normal form) such that the laws* I *entail $s \approx t$.*

Proof. The procedure is described inductively on the complexity of terms. To begin, of course, the normal form of $g \in G$ is g. If $t = P(s)$ for some normal form space-time term $s \in \Lambda_g$ then the normal form for t is the same as that of s.

If $t = H(s)$ or $T(s)$, for some normal form space-time term $s \in \Lambda_g$, then the normal form t' for t is given in the following table:

s	$H(s)$	$T(s)$
$S^nT^m(g)$	$HT^m(g)$	$S^nT^{m+1}(g)$
$S^nT^mHT^k(g)$	$HT^{m+k}(g)$	$S^nT^{m+1}HT^k(g)$
$S^nT^mN_H(t_1, t_2, T^k(g))$	$HT^{m+k+1}(g)$	$S^nT^{m+1}N_H(t_1, t_2, T^k(g))$

If $t = S(s)$ or $S^{-1}(s)$ for some normal form space-time term s then the normal form for t is obtained from s by adding or subtracting 1 respectively to the space component.

If $t = N_H(s_1, s_2, s)$ for a normal form space-time term s, then the normal form t' for t is given in the following table:

s	t'
$S^nT^m(g)$	$N_H(s_1, s_2, T^m(g))$
$S^nT^mHT^k(g)$	$N_H(s_1, s_2, T^{m+k}(g))$
$S^nT^mN_H(t_1, t_2, T^k(g))$	$N_H(s_1, s_2, T^{m+k+1}(g))$

If $t = K(s_1, s)$ for a normal form space-time term s then t is in normal form.

This completes the description of the procedure.

Remark. In the ensuing development, we will always deal only with space-time elements in normal form, and when we write $H(\lambda)$, $S(\lambda)$, etc. for $\lambda \in \Lambda$, we will mean the normal form of $H(\lambda)$, etc.

Proposition 2.2: *For terms $s, t \in \Lambda_g$ with time coordinates m, n respectively, if $m < n$ then the laws V entail $F(s, t) \approx 1_F$.*

Proposition 2.3: *For terms $s, t \in \Lambda_g$ with the same time prefix but different space coordinates, the laws V entail either $R(s, t) \approx 1$ or $R(t, s) \approx 1$.*

§3. NON-UNIFORM SOLVABILITY OF THE WORD PROBLEM

This section is devoted to a proof of the following:

Theorem 3.1. *V does not have uniformly solvable word problem.*

Proof. For any initial tape configuration

...	e_L	a_0	a_1	a_2	...	a_k	e_R	...
		$\uparrow$						

where $\uparrow$ indicates head position) for the universal Turing machine, there is a corresponding finite presentation

$\mathcal{P} : C_Q(c) \approx q_0$ and $C_\Sigma(S^{-1}(c)) \approx e_L$ and $C_\Sigma(c) \approx a_0$ and $\dots C_\Sigma(S^k(c)) \approx a_k$ and $C_\Sigma(S^{k+1}(c)) \approx e_R$

We claim that the universal Turing machine, started on that configuration, eventually halts, if and only if $E(q_0) \approx h$ (equivalently, $0 \approx 1$) follows from the presentation $\mathcal{P}$ in the variety V. Thus, since there is no algorithm which determines, given an initial tape configuration, whether or not the universal Turing machine will halt, this establishes the fact that V does not have uniformly solvable word problem.

$(\rightarrow)$: This direction is clear; $\mathcal{P}$ together with the equations defining V entail the analogous information at each successive configuration. If the machine halts at time n then we obtain $\lambda \in \Lambda_{P(c)}$ such that $C_Q(\lambda) \approx h$ and so

$$0 \approx E(h) \approx E(C_Q(\lambda)) \approx 1 \approx E(q_0)$$

follow from $\mathcal{P}$ in the variety V.

$(\leftarrow)$: Suppose the machine, started on the above configuration, never halts. Then we produce a model $A \in V$ satisfying all the equations in $\mathcal{P}$, in which $0 \neq 1$.

The set of elements of A is $\{*\} \cup \Sigma \cup Q \cup \{S^n T^m(c) | n \in \mathbb{Z}, m \in \mathbb{N}\} \cup \{n | n \in \mathbb{Z}$ and $n \leq 1\} \cup \{n_F | n \in \mathbb{Z}$ and $n \leq 1\}$.

The operations are defined in A as follows:

(i) T, S, S^{-1} are defined on elements of the form $S^n T^m(c)$ according to equations I so as to yield elements again of this form; for other elements y, $T(y) = T(c)$, $S(y) = S(c)$, $S^{-1}(y) = S^{-1}(c)$.

(ii) P maps all elements of the form $S^n T^m(c)$ identically and all other elements to c, in particular $P(c) = c$.

(iii) $U(n) = (n+1)$ and $U(n_F) = (n+1)_F$ for $n \leq 0$, $U(1) = 1$, $U(1_F) = 1_F$. U maps all other elements to $*$.

(iv) $E(q) = 1$ for all $q \in Q$, $q \neq h$

$E(h) = 0$

E maps all other elements to $*$.

(v) $R(S^nT^m(c),\, S^kT^j(c)) = \begin{cases} 1 & \text{if} \quad k > n \\ k - n & \text{if } k \leq n \end{cases}$.

$R(x, y) = R(P(x), P(y))$ otherwise.

(vi) $F(S^nT^m(c), S^kT^j(c)) = \begin{cases} 1_F & \text{if } j > m \\ (j-m)_F & \text{if } j \leq m \end{cases}$

$F(x, y) = F(P(x), P(y))$ otherwise.

(vii) $K(0, S^nT^m(c)) = S^nT^m(c) = K(0_F, S^nT^m(c))$

$K(1, S^nT^m(c)) = c = K(1_F, S^nT^m(c))$

$K(x, y) = c$ otherwise.

(viii) $K^*(d, d) = 0$ for all $d \in \{0, 1, 0_F, 1_F, c\} \cup Q \cup \Sigma$

$K^*(d, e) = 1$ for all d, e as above with $d \neq e$

$K^*(k, d) = 1$ for all $k \in \{n | n \leq 1\}$, all $d \neq 0, 1$

$K^*(k, d) = 1$ for all $k \in \{n_F | n \leq 1\}$, all $d \neq 0_F$, 1_F

$K^*(S^nT^m(c), d) = 1$ for all constants $d \neq c$ (including all n and n_F, with $n \leq 1$)

$K^*(x, y) = *$ else.

The values of $H(x)$, $C_\Sigma(x)$, $C_Q(x)$, $N_\Sigma(q, a, H(x))$, $N_Q(q, a, H(x))$, and $N_H(q, a, H(x))$ for $x \in \{S^nT^m(c) | n \in \mathbb{Z}, m \in \mathbb{N}\}$ are defined by induction on m:

define $H(c) = HS^n(c) = c$

$C_Q(c) = C_Q S^n(c) = q_0$

$C_\Sigma(S^n(c))$ as in $\mathcal{P}$ for $-1 \leq n \leq k+1$

$C_\Sigma(S^n(c)) = B$ for all other values of n

$N_Q(q, a, c)$, $N_H(q, a, c,)$ and $N_\Sigma(q, a, c)$ are defined as in equations II (note that $c = H(c)$).

Suppose we have already defined, for all $n \in \mathbb{Z}$,

$H(S^nT^m(c)) = H(T^m(c)) = S^kT^m c$ for some k

$C_Q(S^nT^m(c)) = C_Q(T^m(c)) \in Q$

$C_\Sigma(S^nT^m(c)) \in \Sigma$

and N_H, N_Q, N_Σ for all triples $(q, a, HT^m(c))$, with appropriate values, i.e., $im(N_\Sigma) \subseteq \Sigma$ etc.

Then define for all $n \in \mathbb{Z}$

$H(S^nT^{m+1}(c)) = N_H(C_Q(HT^m(c)),\, C_\Sigma(TH^m(c)), HT^m(c))$

$C_Q(S^nT^{m+1}(c)) = N_Q(C_Q(HT^m(c)),\, C_\Sigma(HT^m(c)), HT^m(c))$
$C_\Sigma(THT^m(c)) = N_\Sigma(C_Q(HT^m(c)), C_\Sigma(HT^m(c)), HT^m(c))$
$C_\Sigma(S^nTHT^m(c)) = C_\Sigma(S^nHT^m(c))$ for all $n \neq 0$
and then define N_Q, N_Σ, N_H for all triples $(q, a, HT^{m+1}(c))$ according to the rules II.

This completes the inductive definition.

Define $N_H(x, y, z) = N_H(x, y, P(z))$ if the latter has already been defined.
Define N_H and H on all other elements to have value c.
Define $C_Q(y) = C_Q(c) = q_0$ for all y not of the form $S^nT^m(c)$
$C_\Sigma(y) = C_\Sigma(c) = a_0$, for all y not of the form $S^nT^m(c)$
$N_Q(x, y, z) = N_\Sigma(x, y, z) = *$ for all values not defined above.
Define $C^*(S^nT^m(c), n) = \begin{cases} C_\Sigma(S^nT^{m+1}(c)) & \text{if } n \leq 0 \\ C_\Sigma(S^nT^m(c)) & \text{if } n = 1 \end{cases}$, $C^*(x, y) = *$ otherwise.

Then the resulting algebra A satisfies all the laws of the variety and the equations of the presentation $\mathcal{P}$, and $0 \neq 1$ in A.

§4. SOLVABLE WORD PROBLEM IN THE DEGENERATE CASE

This section and the next are devoted to proving that V has solvable word problem.

Let $\mathcal{P}$ be a finite presentation on a generating set $X = \{x_1, \ldots, x_n\}$ and let $\theta_\mathcal{P}$ be the congruence on FX generated by the relations of $\mathcal{P}$ together with the substitution instances of the laws defining our variety V. We must prove that $\theta_\mathcal{P}$ is decidable.

Definition. $\mathcal{P}$ has *degenerate space-time* if $P(t)\theta_\mathcal{P}P(c)$ for all terms t. Note that, by the laws of V if $\mathcal{P}$ has degenerate space-time then $0\theta_\mathcal{P}1$ and $0_F\theta_\mathcal{P}1_F$. In fact, the laws of V allow this conclusion to be drawn from any failure of the operations S, T to behave without loops. Also the laws of VII imply that if either $0\theta_\mathcal{P}1$ or $0_F\theta_\mathcal{P}1_F$, then space-time is degenerate.

We first prove the the following.

Theorem 4.1 *If $\mathcal{P}$ has degenerate space-time then the word problem for $\mathcal{P}$ relative to our variety V is decidable.*

Proof. In this case, in the presented algebra $F(X)/\theta_\mathcal{P}$, all the operations P, T, S, S^{-1}, H and N_H are constant with value $P(c)$. Moreover, R and F are constant, with value 0 and 0_F respectively, C_Q is constant with value $C_Q P(c)$ and C_Σ is constant, with value $C_\Sigma P(c)$.

Now, consider the type obtained from the one with which we are working, by deleting the operations P, T, S, S^{-1}, H, N_H, R, F, C_Q, C_Σ, and adding three constants c_1, c_2, and c_3, which will stand for $P(c)$, $C_Q P(c)$, and $C_\Sigma P(c)$ respectively. Then there is an effective procedure which, given a term of the larger type, produces a term of the smaller type which is equivalent to it modulo the laws for our variety, and the equations $0 \approx 1$, $0_F \approx 1_F$, $c_1 \approx P(c)$, $c_2 \approx C_Q P(c)$, $c_3 = C_\Sigma P(c)$.

Thus if we consider the variety V' of this reduced type defined by the following equations:

equations II for N_Q and N_Σ, with space-time terms, and terms in the image of C_Q and C_Σ replaced by c_1, c_2, c_3 respectively.

$c_3 \approx C^*(c_1, 1)$

$U(0) \approx U(1) \approx 1$

$U(0_F) \approx U(1_F) \approx 1_F$

equations VII, where $P(x)$ is replaced by c_1, $C_Q P(x)$ by c_2, $C_\Sigma P(x)$ by c_3 and $R(P(x), P(y))$ and $F(P(x), P(y))$ by 0.

$E(c_2) \approx 1$

$E(h) \approx 0$.

Then, if the terms in the presentation $\mathcal{P}$ are replaced by their equivalents in the new type we obtain a presentation $\mathcal{P}'$ which relative to the variety described above, is equivalent to $\mathcal{P}$ relative to the original variety. Here by "equivalent" we mean that there is an effective translation between terms so that $\mathcal{P}'$ entails $s' \approx t'$ relative to the variety V' if and only if $\mathcal{P}$ entails $s \approx t$ relative to the variety V. Since $\mathcal{P}'$ is a presentation considered relative to a variety defined by finitely many laws which involve no variables, by Corollary 1.2 this word problem is solvable, which shows that the word problem for $\mathcal{P}$ relative to our variety is solvable too.

§5. SOLVABLE WORD PROBLEM IN THE NON-DEGENERATE CASE

5.1 PLAN OF THE PROOF

In this section we prove the following:

Theorem 5.1 *If $\mathcal{P}$ has non-degenerate space-time then the word problem for $\mathcal{P}$ relative to our variety V is decidable.*

Proof: The proof is presented in the remaining subsections of this section. In the remainder of this subsection we will describe the strategy of the proof. We will define by induction an increasing sequence of partial subalgebras $(A_n, \equiv_{A_n})$. There are three points to be verified. First, for all n every instance of the laws of the variety and the relations of $\mathcal{P}$ with elements of A_n is validated by $\equiv_n$. Second, for $a, b \in A_n$, if $a \equiv_n b$ then the laws of the variety and $\mathcal{P}$ imply $a\theta_{\mathcal{P}}b$. The set of terms will, apart from passing to normal forms, equal $\cup A_n$. Hence $\theta_{\mathcal{P}}$ will essentially equal $\cup \equiv_n$. Third, the construction of the A_n and the $\equiv_n$ is uniformly effective and hence $\cup \equiv_n$ and $\theta_{\mathcal{P}}$ are decidable. We will give a careful definition of the A_n and $\equiv_n$, but we will leave it to the reader to verify the three points mentioned above. One other point which is worth mentioning is that before constructing $(A_0, \equiv_0)$ we will demand more information about $\theta_{\mathcal{P}}$ other than its being non-degenerate. We first define two auxiliary sets A and B.

5.2 DEFINITION OF A

Let $B_{\mathcal{P}}$ consist of all terms appearing in the presentation $\mathcal{P}$ and all their subterms and all constants of the variety V. Let $G_{\mathcal{P}}$ consist of $P(c)$, $P(x)$ for each generator x of the presentation, and all terms of the form $K(s,t) \in B_{\mathcal{P}}$; thus $G_{\mathcal{P}}$ is finite.

For each $g \in G_{\mathcal{P}}$, we let $\Gamma_{g,\mathcal{P}}$ be the set of all terms built from g using the unary operations P, S, S^{-1}, T, H, and $N_H(s,t,-)$ where $s, t \in B_{\mathcal{P}}$. Further let $\Lambda_{g,\mathcal{P}}$ be the set of members of $\Gamma_{g,\mathcal{P}}$ which are in normal form with respect to g, and let $\Lambda_{\mathcal{P}} = \cup\Lambda_{g,\mathcal{P}}(g \in G_{\mathcal{P}})$.

Now, we define A as follows: it contains

(1) the terms in $G_{\mathcal{P}}$ and all constants
(2) the terms in $\Lambda_{\mathcal{P}}$
(3) $U(d)$ for $d = 0, 1, 0_F, 1_F$
(4) $R(\lambda,\gamma)$, $F(\lambda,\gamma)$, $UR(\lambda,\gamma)$, $UF(\lambda,\gamma)$ for λ, $\gamma \in \Lambda_{\mathcal{P}}$
(5) $N_Q(q, a, HT^n(g))$,
$N_\Sigma(q, a, HT^n(g))$,
$N_H(q, a, HT^n(g))$ for all $g \in G_{\mathcal{P}}$, $q \in Q$, $a \in \Sigma$
$N_Q(q_L, C_\Sigma HT^n(g), HT^n(g))$,
$N_Q(q_R, C_\Sigma HT^n(g), HT^n(g))$ for all $g \in G_{\mathcal{P}}$, all $q \in Q - Q_{LR}$, all $n \geq 0$.
(6) $C_\Sigma(\lambda)$, $C_Q(\gamma)$, all λ, $\gamma \in \Lambda_{\mathcal{P}}$
(7) $N_\Sigma(C_Q HT^n(g), C_\Sigma HT^n(g), HT^n(g))$,
$N_Q(C_Q HT^n(g), C_\Sigma HT^n(g), HT^n(g))$,

$N_H(C_Q HT^n(g),\ C_\Sigma HT^n(g),\ HT^n(g))$ for all $g \in G_{\mathcal{P}}$, all $n \geq 0$.
(8) $C^*(\lambda, R(\lambda, H\lambda))$,
$C^*(\lambda, R(H\lambda, \lambda))$,
$C^*(\lambda, 1)$ for $\lambda \in \Lambda_{\mathcal{P}}$
(9) $K^*(d, d)$ for constants d
$K^*(d, e)$ for all constants d, e with $d \neq e$.
All substitution instances of terms in laws VII (i) and (ii) where $P(x)$ and $P(y)$ are replaced by arbitrary $\lambda, \gamma \in \Lambda_{\mathcal{P}}$.

(10) $EC_Q(\lambda)$, for all $\lambda \in \Lambda_{\mathcal{P}}$ and $E(h)$

Note that membership in A is decidable.

5.3 DEFINITION OF B

Before we can define B, we need some preliminary results.

Lemma 5.2 *For any* $\lambda \in \Lambda_{g,\mathcal{P}}$, $\{\gamma \in \Lambda_{g,\mathcal{P}} | \gamma\ \theta_{\mathcal{P}} \lambda\}$ *is finite.*

Proof. If $\lambda, \gamma \in \Lambda_{g,\mathcal{P}}$ and $\lambda\ \theta_{\mathcal{P}} \gamma$ then it follows from laws of V and the non-degeneracy of $\mathcal{P}$ that λ and γ have the same time coordinate. (For example, $S^n T^k g\ \theta_{\mathcal{P}} S^m T^i H T^r g$ implies $HT^k g = H(S^n T^k g) \theta_{\mathcal{P}} H(S^m T^i H T^r g) = HT^{i+r} g$ and this yields $k = i + r$.) Moreover, two terms in $\Lambda_{g,\mathcal{P}}$ with the same time prefix and different space coordinates cannot be congruent modulo $\theta_{\mathcal{P}}$. Since there are only finitely many time prefixes with the same time coordinate as λ, this establishes the result.

Corollary 5.3 *For any term* t, $\{\lambda \in \Lambda_{\mathcal{P}} | \lambda\ \theta_{\mathcal{P}} t\}$ *is finite.*

Definition. For a finite $F \subseteq \Lambda_{\mathcal{P}}$, the *maximum time vector* of F is $(m_g)_{g \in G_{\mathcal{P}}}$ where m_g is the *maximum g-time coordinate* of elements of $F \cap \Lambda_g$ (or 0 if $F \cap \Lambda_g$ is empty). We also make an *ad hoc* definition and say a space-time term s is a *right subterm* of a space-time t by induction on the construction of t. If t is Hu, Su, or $S^{-1}u$ for a space-time term u then s is a right subterm of t if it is either t or a right subterm of u. If t is $N_H(w, v, u)$ where u is a space-time term then s is a right subterm of t if it is either t or a right subterm of u.

Lemma 5.4 *For any finite subset* $F \subseteq \Lambda_{\mathcal{P}}$ *with maximum time vector* $(m_g)_{g \in G_{\mathcal{P}}}$ *there is a finite* $\bar{F} \subseteq \Lambda_{\mathcal{P}}$ *with the same maximum time vector, such that*
(i) $F \subseteq \bar{F}$
(ii) *if* $\lambda \in \Lambda_{\mathcal{P}}$ *is a right subterm of* $\gamma \in \bar{F}$ *then* $\lambda \in \bar{F}$

(iii) *if* $\lambda \in \Lambda_{\mathcal{P}}$ *and* $\lambda\ \theta_{\mathcal{P}}\gamma$ *for* $\gamma \in \bar{F}$ *then* $\lambda \in \bar{F}$

(iv) *if* $\lambda \in \Lambda_{\mathcal{P}}$ *and the normal form of* $T\lambda$ *belongs to* $\bar{F}$ *then* $\lambda \in \bar{F}$.

Proof. We may assume that for each $g \in G_{\mathcal{P}}$, $T^j g \in F$ for all $j \leq m_g$.

Now, let $g \in G_{\mathcal{P}}$ and consider the set $F_g \subseteq \Lambda_{g,\mathcal{P}} \cap F$ which consists of all elements of $\Lambda_{g,\mathcal{P}} \cap F$ whose time coordinate relative to g is m_g. Let $F_g^* \subseteq \Lambda_{g,\mathcal{P}}$ consist of all those $\lambda \in \Lambda_{g,\mathcal{P}}$ for which there exists $\gamma \in F_g$ with $\lambda\theta_{\mathcal{P}}\gamma$. By Lemma 5.2, F_g^* is finite.

Let k_1 and k_2 be the maximum and minimum, respectively, of the space coordinate of members of F_g^*. Note that, since $T^{m_g}(g) \in F$, we have $k_2 \leq 0 \leq k_1$.

Let F'_g consist of those members of Λ_g which are the normal forms of all terms of the form $S^k(\lambda)$ where $-k_1 \leq k \leq -k_2$, and $\lambda \in F_g^*$. Then F'_g is finite and contains F_g. We will show

(a) $\gamma \in F'_g$, $\lambda \in \Lambda_g$, $\gamma\ \theta_{\mathcal{P}}\lambda$ implies $\lambda \in F'_g$

(b) $\gamma \in F'_g$, $\lambda \in \Lambda_g$ a right subterm of γ with time coordinate m_g relative to g implies $\lambda \in F'_g$.

re (a): Suppose γ is the normal form of $S^k(\delta)$ where $-k_1 \leq k \leq -k_2$ and $\delta \in F_g^*$. Then $S^{-k}\gamma\theta_{\mathcal{P}}\delta$ and hence $S^{-k}\lambda\theta_{\mathcal{P}}\delta$ and so the normal form of $S^{-k}\lambda$ belongs to F_g^*. Thus λ, which is the normal form of $S^k S^{-k}\lambda$, belongs to F'_g.

re (b): Suppose $\gamma \in F'_g$; then γ is the normal form of a term $S^k(\delta)$ where $-k_1 \leq k \leq -k_2$ and $\delta \in F_g^*$. Let the space coordinate of δ be n and the time prefix of δ be τ; then $\gamma = S^{n+k}\tau$. Moreover, all terms of the form $S^i\tau$ for $n - k_1 \leq i \leq n - k_2$ belong to F'_g. Since $n - k_1 \leq 0 \leq n - k_2$, it follows that if i is any number between $n + k$ and 0, then $S^i\tau \in F'_g$. Now, any normal form subterm of γ with the same time component has the same time prefix and hence this shows that every right subterm of γ with the same time component belongs to F'_g.

Let $F' = \cup F'_g(g \in G_{\mathcal{P}})$; then F' is finite, $F \subseteq F'$, and F' satisfies (ii) and (iii) for any $\lambda \in \Lambda_g$ with g-time coordinate m_g. Add to F' each term $\lambda \in \Lambda_g$ with g-time coordinate $m_g - 1$ such that the normal form of $T\lambda$ belongs to F_g. The result is still finite. Now repeat the procedure for elements of g-time coordinate $m_g - 1$, etc. to eventually obtain the desired set $\bar{F}$. This completes the proof.

Definition. Define B as follows:

Recall that $B_{\mathcal{P}}$ consists of all terms appearing in the presentation $\mathcal{P}$ and all subterms thereof, and all constants of our variety V.

Enlarge $B_{\mathcal{P}}$ as follows:

(i) For each $b \in B_{\mathcal{P}}$, if there exists $a \in A$ with $a\theta_{\mathcal{P}}b$, add one such a, and choose $a \in \Lambda_{\mathcal{P}}$ whenever possible.

(ii) For each $g \in G_{\mathcal{P}}$ let m_g be the maximum time coordinate of all the elements of Λ_g that we have so far, and add all g-time prefixes up to time m_g.

(iii) Let F consist of all elements of $\Lambda_{\mathcal{P}}$ that we have so far. Add the set $\bar{F} \supseteq F$ given in the above lemma.

(iv) For all $\lambda, \gamma \in \bar{F}$, add $C_Q(\lambda)$, $C_\Sigma(\lambda)$, $R(\lambda,\gamma)$, $F(\lambda,\gamma)$, $U(R(\lambda,\gamma))$, $U(F(\lambda,\gamma))$.

The resulting set B is finite, is closed under taking subterms, and for λ, $\gamma \in \Lambda_{\mathcal{P}}$, if $\gamma \in B$ and $\lambda\ \theta_{\mathcal{P}}\gamma$ then $\lambda \in B$.

Let $\equiv_B = \theta_{\mathcal{P}}|B$; then $\equiv_B$ is finite and hence decidable. Also $\equiv_B$ contains the relations of $\mathcal{P}$.

5.4 DEFINITION OF A_0

Now, define $A_0 = A \cup B$; then membership in A_0 is decidable. We are going to define a partial congruence relation $\equiv_0$ on A_0 so that the pair $(A_0, \equiv_0)$ is a partial subalgebra such that membership in $\equiv_0$ as well as A_0 is decidable. In fact, $\equiv_0$ will be $\theta_{\mathcal{P}}$ restricted to A_0, but we will define $\equiv_0$ by induction on the complexity of terms and the size of the time coordinate for members of $\Lambda_{\mathcal{P}}$.

For $a, b \in B$, $a \equiv_0 b$ if and only if $a \equiv_B b$.

For $a \in A$, $b \in B$, $a \equiv_0 b$ if and only if there exists $c \in A \cap B$ with $a \equiv_0 c$ (as described below) and $c \equiv_0 b$, i.e., $c \equiv_B b$. Since B is finite, we decide whether $a \equiv_0 b$ by searching through all $c \in A \cap B$ and checking the latter two conditions. Thus it is enough to describe $\equiv_0$ between pairs of elements of A.

There are some members of A that we can essentially ignore, because we know they must be in the relation $\equiv_0$ to other elements that we have to deal with anyway.

Thus, to begin, we decree:

$U(0) \equiv_0 U(1) \equiv_0 1$

$U(0_F) \equiv_0 U(1_F) \equiv_0 1_F$

and for all λ, $\gamma \in \Lambda_p$,

$UR(\lambda,\gamma) \equiv_0 R(\lambda,\gamma')$ where γ' is the normal form of $S(\gamma)$

$UF(\lambda,\gamma) \equiv_0 F(\lambda,\gamma')$ where γ' is the normal form of $T(\gamma)$

and so we may ignore, for the purposes of defining $\equiv_0$ between elements of A, all those elements of A which are in the range of U.

Similarly, using the appropriate terms given in laws II and III we may ignore the elements of A in the range of N_Σ or N_Q, by making them $\equiv_0$ congruent to elements in Q or the range of C_Q, and Σ or the range of C_Σ, respectively.

We dispose in the same way of the elements of A that are in the range of C^*, K or K^*.

Thus we need only define $\equiv_0$ between pairs of elements of A that are either constants, Λ-elements, or in the range of the operations C_Σ, C_Q, F and R. Moreover, since $\mathcal{P}$ is non-degenerate, we know by laws VIII that the interpreted images of these latter four operations are disjoint from one another and from all the interpretations of Λ-elements in $F(X)/\theta_{\mathcal{P}}$. In addition, all the constants are pairwise distinct in $F(X)/\theta_{\mathcal{P}}$, and A-elements in the range of C_Σ, C_Q, F, and R can be $\theta_{\mathcal{P}}$-congruent to constants only if they belong to Σ, Q, $\{0, 1\}$, $\{0_F, 1_F\}$, respectively.

5.5 DEFINITION OF $\equiv_0$ FOR ELEMENTS WITH SMALL TIME COMPONENT

We begin by describing $\equiv_0$ for elements λ, $C_\Sigma(\lambda)$, $C_Q(\lambda)$, $R(\lambda,\gamma)$ and $F(\lambda,\gamma)$ for λ, $\gamma \in \Lambda_{\mathcal{P}}$ with time coordinate less than or equal to the maximum occurring in B, relative to whatever space-time component λ and γ are in.

(1) For λ, $\gamma \in \Lambda_{\mathcal{P}}$ with time coordinate less than or equal to the maximum in B, define

$\lambda \equiv_0 \gamma$ if and only if $S^{-n}\lambda \equiv_{\mathcal{B}} S^{-n}\gamma$

where n is the space coordinate of λ.

Note that $S^{-n}\lambda \in B$, and hence if $\lambda\ \theta_{\mathcal{P}}\gamma$ then $S^{-n}\lambda\ \theta_{\mathcal{P}} S^{-n}\gamma$ and hence $S^{-n}\lambda \equiv_B S^{-n}\gamma$. The converse is also true, of course. The point about the definition is that, given λ, we know its space coordinate and so we can decide $\lambda \equiv_0 \gamma$ because $\equiv_B$ is decidable. Moreover (and we will need this later), given λ, we can calculate all (there are only finitely many) $\gamma \in \Lambda_{\mathcal{P}}$ with $\lambda \equiv_0 \gamma$.

(2) For λ, $\gamma \in \Lambda_{\mathcal{P}}$ with time coordinates less than or equal to the maximum in B,

(i) $R(\lambda,\gamma) \equiv_0 0$ if and only if either $\lambda \equiv_0 \gamma$ as in (1) above

or $\lambda, \gamma \in B$ and $R(\lambda,\gamma) \equiv_B 0$.

(ii) $R(\lambda,\gamma) \equiv_0 1$ if and only if

either there exists $n > 0$ with $\gamma \equiv_0 S^n\lambda$

or there exists $\delta \in \Lambda_{\mathcal{P}} \cap B$ and $n \geq 0$ with $\gamma \equiv_0 S^n\delta$ and $R(\lambda, \delta) \equiv_B 0$

or there exists $\delta \in \Lambda_{\mathcal{P}} \cap B$ and $n > 0$ with $\gamma \equiv_0 S^n\delta$ and $R(\lambda, \delta) \equiv_B 1$.

Note that this is decidable: for example, to check whether there exists $n > 0$ with $\gamma \equiv_0 S^n\lambda$, it is enough to determine whether there exists $n > 0$ with $S^{-m}\gamma \equiv_B S^{n-m}\lambda$ where m is the space coordinate of γ, and the latter is decidable because B is finite.

(iii) $R(\lambda_1, \gamma_1) \equiv_0 R(\lambda_2, \gamma_2)$ if and only if

either both are congruent to 0 or 1 by (i) or (ii)

or $\lambda_1 \equiv_0 \lambda_2$ and $\gamma_1 \equiv_0 \gamma_2$

or there exist δ_1 and δ_2 and $n \geq 0$ with $R(\lambda_1, \delta_1) \equiv_B R(\lambda_2, \delta_2)$ and $\gamma_1 \equiv_0 S^n\delta_1$, $\gamma_2 \equiv_0 S^n\delta_2$.

Remark. From the above definition, we have $R(\lambda, \lambda) \equiv_0 0$ for all $\lambda \in \Lambda_{\mathcal{P}}$ with time coordinate less than or equal to the maximum in B, and moreover, if $R(\lambda, \gamma) \equiv_0 0$ or 1 then $R(\lambda, S(\gamma)) \equiv_0 1$, and so the laws V for R and these values of $P(x)$, $P(y)$, are satisfied.

(3) For λ, γ, λ', $\gamma' \in \Lambda_{\mathcal{P}}$ with time coordinate less than or equal to the maximum in B, define

$F(\lambda, \gamma) \equiv_0 0_F$ if and only if $F(H\lambda,\, H\gamma) \equiv_B 0_F$

$F(\lambda, \gamma) \equiv_0 1_F$ if and only if $F(H\gamma, H\delta) \equiv_B 1_F$

$F(\lambda, \gamma) \equiv_0 F(\lambda', \gamma')$ if and only if $F(H\lambda, H\gamma) \equiv_B F(H\lambda', H\gamma')$.

(4) For λ, $\gamma \in \Lambda_{\mathcal{P}}$ with time coordinate less than or equal to the maximum in B, define

$C_Q(\lambda) \equiv_0 q \in Q$ if and only if $C_Q(H\lambda) \equiv_B q$

$C_Q(\lambda) \equiv_0 C_Q(\gamma)$ if and only if either both are $\equiv_0$ the same $q \in Q$

or $C_Q(H\lambda) \equiv_B C_Q(H\gamma)$.

(5) The description of when $C_\Sigma(\lambda) \equiv_0 C_\Sigma(\gamma)$ is somewhat more complicated. First, for space-time elements λ, $\gamma \in \Lambda$, define $\lambda \uparrow \gamma$ to mean $\gamma = T\lambda$ and either $R(\lambda, H\lambda) \equiv_0 1$ or $R(H\lambda, \lambda) \equiv_0 1$. Further, define $\lambda \downarrow \gamma$ to mean $\gamma \uparrow \lambda$.

Note that if $\lambda \downarrow \gamma \uparrow \delta$ then $\lambda = \delta$, and if $\lambda \downarrow \gamma \equiv_0 \delta \uparrow \xi$ then $\lambda \equiv_0 \xi$.

Now define $\lambda \uparrow^* \gamma$ if and only if there is a finite sequence of $\uparrow$-moves from λ to γ, i.e., if and only if there exist $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that $\lambda = \lambda_1 \uparrow \lambda_2 \uparrow \lambda_3 \ldots \uparrow \lambda_k = \gamma$. Similarly define $\downarrow^*$.

Now, define

$$C_\Sigma(\lambda) \equiv_0 C_\Sigma(\gamma)$$

if and only if there exist natural numbers $k \leq n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \Lambda_p$ with time coordinates less than or equal to the maximum in B such that

(i) $\lambda = \lambda_1$, or $C_\Sigma(\lambda) \equiv_B C_\Sigma(\lambda_1)$

and (ii) $\lambda_i \uparrow^* \lambda_{i+1}$ for i odd, $i \leq k$

$\lambda_i \downarrow^* \lambda_{i+1}$ for i odd, $i > k$

$\lambda_i \equiv_0 \lambda_{i+1}$ for i even

and (iii) $\lambda_n = \gamma$ or $C_\Sigma(\lambda_n) \equiv_B C_\Sigma(\gamma)$.

Note that if such a sequence exists then the length of the shortest possible such sequence (including the lengths of the sequences involved in the $\uparrow^*$ and $\downarrow^*$ parts) is bounded above by twice the sum of the maximum time coordinates of elements in B. Hence we can decide, given λ and γ, whether such a sequence exists.

Further, define

$$C_\Sigma(\lambda) \equiv_0 a \in \Sigma$$

if and only if there exists $\gamma \in B$ with $C_\Sigma(\lambda) \equiv_0 C_\Sigma(\gamma)$ as above and $C_\Sigma(\gamma) \equiv_B a$.

With this definition, the congruence $\equiv_0$ (up to the maximum time coordinate in B) satisfies the laws IV. The identities implied by laws II and III for C_Σ are also satisfied because they only involve $C_\Sigma H(\lambda)$, and all the $H(\lambda)$ belong to B.

5.6 COMPLETION OF THE DEFINITION OF $\equiv_0$

Now, we complete the definition of $\equiv_0$ for elements of A which are, or which involve, space-time elements with time coordinate larger than the maximum in B, by induction on the time coordinate.

Suppose we have described $\equiv_0$ as above for pairs (λ, γ), $(R(\lambda, \gamma), R(\lambda', \gamma'))$, etc. whenever the $\Lambda_{\mathcal{P}}$-elements λ, γ, etc. in the space-time coordinate of g have time coordinate less than or equal k_g. The following describes $\equiv_0$ for those elements in the space-time coordinate of g involving time coordinate $k_g + 1$.

1. (i) For $\lambda, \gamma \in A$, in the same space-time component, say that of g, with time coordinate $k_g + 1$, define

$\lambda \equiv_0 \gamma$ if and only if

<u>either</u> $\lambda = \gamma$

<u>or</u> $\lambda = T\lambda'$, $\gamma = T\gamma'$ and $\lambda' \equiv_0 \gamma'$

<u>or</u> $\lambda = S^n H T^{k_g+1}(g)$ and

either there exist $q \in Q$, $a \in \Sigma$ with $C_Q HT^{k_g}(g) \equiv_0 q$ and $C_\Sigma HT^{k_g}(g) \equiv_0 a$

and $\gamma = S^n T\ \gamma'$ for $\gamma' \equiv_0 S^{\mu(q,a)} HT^{k_g}(g)$

or there exists $q \in Q$ with $C_Q HT^{k_g}(g) \equiv_0 q_L$ and $\gamma = S^n T\ \gamma'$ for $\gamma' \equiv_0 S^{-1} HT^{k_g}(g)$

or there exists $q \in Q$ with $C_Q HT^{k_g}(g) \equiv_0 q_R$

and $\gamma = S^n T \gamma'$ for $\gamma' \equiv_0 SHT^{k_g}(g)$

or $\gamma = S^n N_H(s, t, T^{k_g}(g))$ and $s \equiv_0 C_Q T^{k_g}(g)$

and $t \equiv_0 C_\Sigma T^{k_g}(g)$.

<u>or</u> $\lambda = S^n N_H(t_1, t_2, T^{k_g}(g))$, $\gamma = S^n N_H(s_1, s_2, T^{k_g}(g))$

and $t_1 \equiv_B s_1$, $t_2 \equiv_B s_2$

<u>or</u> vice-versa (with λ, γ switched).

(ii) For λ, γ in different space-time components, say λ in the space-time component of g and γ in the space-time component of y, with time coordinates $k_g + 1$ and less than or equal $k_y + 1$ respectively, define $\lambda \equiv_0 \gamma$ if and only if one of the following holds:

Case 1 $\lambda = S^n T^{k_g+1}(g)$ and there exists γ' in the space-time component of y with $\gamma' \equiv_0 T^{k_g}(g)$ and $\gamma \equiv_0 S^n T\ \gamma'$ by the preceeding description for "the same space-time component".

Case 2 $\lambda = S^n T^{m+1} HT^k(g)$ and there exists γ' in the space-time component of y with $\gamma' \equiv_0 T^m HT^k(g)$ and $\gamma \equiv_0 S^n T\ \gamma'$.

Case 3 $\lambda = S^n HT^{k_g+1}(g)$ and there exists γ' in the space-time component of y with $\gamma' \equiv_0 T^{k_g}(g)$ and $\gamma \equiv_0 S^n H\ T\ \gamma'$.

Case 4 $\lambda = N_H(t_1, t_2, T^{k_g}(g))$ and there exists γ' in the space-time component of y with $\gamma' \equiv_0 T^k(g)$ and $\gamma \equiv_0 N_H(t_1, t_2, \gamma')$.

Case 5 $\lambda = S^n T^{m+1} N_H(t_1, t_2, T^k(g))$ and there exists γ' in the space-time component of y with $\gamma' \equiv_0 T^m N_H(t_1, t_2, T^k(g))$ and $\gamma \equiv_0 S^n T \gamma'$.

2 (i) Define $R(\lambda, \gamma) \equiv_0 0$ where λ has time coordinate $k_g + 1$ relative to g and γ has time coordinate less than or equal to $k_y + 1$ relative to y (or vice-versa) if and only if $\lambda \equiv_0 \gamma$ as in 1.

(ii) Define $R(\lambda, \gamma) \equiv_0 1$ for λ, γ as in (i) if and only if there exists $n > 0$ and $\delta \in \Lambda_{\mathcal{P}}$ with $\gamma \equiv_0 S^n \delta$ and $R(\lambda, \delta) \equiv_0 0$ (i.e., $\lambda \equiv_0 \delta$).

(Note that we can find all possible values for δ and hence can decide whether these conditions are satisfied.)

(iii) Define $R(\lambda, \gamma) \equiv_0 R(\lambda', \gamma')$ (where all of $\lambda, \gamma, \lambda', \gamma'$ have time coordinate less than or equal to the relevant $k_g + 1$ and one of them has that time coordinate) if and only if either both $R(\lambda,\gamma)$ and $R(\lambda',\gamma')$ are $\equiv_0 0$ or both are $\equiv_0 1$ by (i) or (ii) respectively, or $\lambda \equiv_0 \lambda'$ and $\gamma \equiv_0 \gamma'$.

3. For λ, γ as in 2 we define

$F(\lambda,\gamma) \equiv_0 0_F$ if and only if $H\lambda \equiv_0 H\gamma$.

Further, we will define $F(\lambda,\gamma) \equiv_0 1_F$ if and only if $F(H\lambda,\, H\gamma) \equiv_0 1_F$, so it is enough to consider $\lambda = HT^k(g)$, $\gamma = HT^m(y)$.

If $k \leq k_g$ and $m = k_y + 1$ then define

$F(\lambda,\gamma) \equiv_0 1_F$ if and only if $F(T^k(g), T^{k_y}(y)) \equiv_0 0_F$ or 1_F.

If $k = k_g + 1$ and $m \leq k_y + 1$ then define

$F(\lambda,\gamma) \equiv_0 1_F$ if and only if there exists n with $0 < n \leq m$ and $HT^k(g) \equiv_0 HT^{m-n}(y)$.

Finally, define $F(\lambda,\gamma) \equiv_0 F(\lambda',\gamma')$ if and only if either both are $\equiv_0$ to 0_F or both are $\equiv_0$ to 1_F by the above, or $\lambda \equiv_0 \lambda'$ and $\gamma \equiv_0 \gamma'$, or both λ and λ' have time coordinate less than the induction step, and there exists $n \geq 0$ with $H\gamma = HT^{n+p}(g)$, $H\gamma' = HT^{n+m}(y)$ and $F(\lambda, HT^p(g)) \equiv_0 F(\gamma', HT^m(y))$.

4. For λ in the space-time component of g with time coordinate $k_g + 1$, we define $C_Q(\lambda) \equiv_0 q \in Q$ if and only if $C_Q(H\lambda) \equiv_0 q \in Q$, and for $H\lambda$, which is just $HT^{k_g+1}(g)$, we define $C_Q(H\lambda) \equiv_0 q \in Q$ if and only if either $C_Q HT^{k_g}(g) \equiv_0 q_R$ or q_L, or there exists $q' \in Q$, $a \in \Sigma$ with $\sigma(q',a) = q$ and $C_Q HT^{k_g}(g) \equiv_0 q'$ and $C_\Sigma HT^{k_g}(g) \equiv_0 a$.

Then, define $C_Q(\lambda) \equiv_0 C_Q(\gamma)$ if and only if either $H\lambda \equiv_0 H\gamma$ or both $C_Q(\lambda)$ and $C_Q(\gamma)$ are $\equiv_0$ to some $q \in Q$ by the preceding paragraph.

5. Now, for λ in the space-time component of g with time coordinate $k_g + 1$, we define

$C_\Sigma(\lambda) \equiv a \in \Sigma$

if and only if

<u>Case 1</u> $\lambda = THT^{k_g}(g)$ and

<u>either</u> there exists $q \in Q$, $b \in \Sigma$ with $\alpha(q,b) = a$ and

$$C_\Sigma(HT^{k_g}(g)) \equiv_0 b \text{ and } C_Q(HT^{k_g}(g)) \equiv_0 q$$

<u>or</u> $a = e_R$ and there exists $q \in Q$ with $C_Q(H\ T^{k_g}(g)) \equiv_0 q_R$

<u>or</u> $a = e_L$ and there exists $q \in Q$ with $C_Q(H\ T^{k_g}(g)) \equiv_0 q_L$

<u>Case 2</u> There exists γ with $\lambda = T\gamma$ and either $R(\gamma, H\gamma) \equiv_0 1$ or $R(H\gamma,\gamma) \equiv_0 1$ and in addition $C_\Sigma(\gamma) \equiv_0 a$

<u>Case 3</u> There exists γ in another space-time component with $C_\Sigma(\gamma) \equiv_0 a$ by the above two cases and $\lambda \equiv_0 \gamma$.

Finally, we define $C_\Sigma(\lambda) \equiv_0 C_\Sigma(\gamma)$ if and only if either both are $\equiv_0$ some $a \in \Sigma$ by the preceding definition, or $\lambda \equiv_0 \gamma$, or, if λ is in the space-time component of g with time coordinate $k_g + 1$ and γ is in the space-time component of y with time coordinate less than or equal to k_y and there exists δ with $\lambda = T\delta$, and either $R(\delta, H\delta)$ or $R(H\delta, \delta) \equiv_0 1$ and in addition $C_\Sigma(\delta) \equiv_0 C_\Sigma(\gamma)$ (or vice-versa with the roles of λ and δ reversed).

This completes the definition of $(A_0, \equiv_0)$. As the notation suggests $\equiv_0$ is an equivalence relation, in fact a partial congruence. In particular transitivity is taken care of in the inductive construction. Furthermore it should be noted that $\equiv_0$ is decidable.

5.7 DEFINITION OF $(A'_n, \equiv'_n)$

Now, suppose we have defined, for each $m \leq n$, a decidable partial subalgebra $(A_m, \equiv_m)$ such that $A_m \subseteq A_{m+1}$, $\equiv_m = \equiv_{m+1} |A_m$, $\equiv_m$ is a partial congruence, the laws of our variety are contained in $\equiv_m$ insofar as they apply to the elements of A_m, and in addition

(i) A_m is closed under P, H, S, S^{-1}, T (modulo normal form) and for all $a \in A_m$ and any operation among P, C_Σ, C_Q, R, F, if a is $\equiv_n$-equivalent to an element in the image of the operation then it is $\equiv_m$-equivalent to an element in the image of that operation.

Further we construct by induction algorithms which

(ii) given space-time elements, λ, $\gamma \in A_m$, determine whether there exists k with $\lambda \equiv_m S^k\gamma$.

(iii) given space-time elements λ, $\gamma \in A_m$, determine whether there exists $k > 0$ with $\lambda \equiv_m T^k\gamma$.

(iv) given a space-time element $\lambda \in A_m$ determine whether it is $\equiv_m$ to some element of A_0, and if so, produces such an element λ'.

(v) given an element $a \in A_m$ and an operation among P, C_Σ, C_Q, R, F, determine if a is $\equiv_m$ to an element in the range of the operation.

The base case of the construction is $n = 0$. Note that $(A_0, \equiv_0)$ has the first four of these five properties: (i) follows from the definition of A_0 and the fact that $P(c) \in A_0$.

(ii) is seen as follows, for $\lambda, \gamma \in A_0$: we can effectively list as $\lambda_1, \ldots, \lambda_n$, the finitely many elements in A_0 to which λ is $\equiv_0$. For some k, $\lambda \equiv_0 S^k\gamma$ if and only if for some i, λ_i and γ have the same time prefix. The latter condition can be effectively checked.

(iii) Let $\lambda_1, \ldots, \lambda_n$ be as above. Then there is k so that $\lambda \equiv_0 T^k\gamma$ if and only if some λ_i is in the space-time component of γ, k_0 is the difference of the time coordinates of λ_i and γ, and $\lambda_i \equiv_0 T^{k_0}\gamma$.

(iv) is trivial for A_0.

Property (v) is a bit trickier. Since an element a is $\equiv_n$-equivalent to an element in the range of P if and only if $a \equiv_n P(a)$, only the other operations present any problems. Below we will state an inductive hypothesis on the equivalence $\equiv_n$. The inductive hypothesis will have two uses. First it will allow us to verify property (v) and the second part of (i) by giving a complete description of which elements are $\equiv_n$-equivalent to an element in the range of a non-space-time operation. Second the hypothesis will determine $\equiv'_n$, the restriction of $\equiv_{n+1}$ to A'_n (defined below). In the remarks after the inductive hypothesis for elements in the image of R, we will expand on these points. The reader will be able to observe the inductive hypotheses hold for the case $n = 0$ and so (v) holds as well.

Let A'_n be A_n together with the image of all the elements of A_n under the operations R, F, U, C_Q, N_Q, C_Σ, N_Σ, C^*, E, K^*.

Extend $\equiv_n$ to a partial congruence $\equiv'_n$ on A'_n by considering each operation in turn. First extend it to A_n together with the image of R by letting it be the unique symmetric relation extending $\equiv_n$ which satisfies the inductive hypothesis. Continuing, given an operation we define $\equiv'_n$ on the image of that operation, A_n and the operations previously considered, by letting it be the unique symmetric relation which satisfies the inductive hypothesis and extends the restriction of $\equiv'_n$ previously defined. Transitivity will be an easy consequence of the definition since we will always link to a previous $\equiv_n$.

(1) Image of R

Inductive Hypothesis

$R(s,t) \equiv_n 0$ if and only if either for some λ_1, $\lambda_2 \in A_0$, $P(s) \equiv_n \lambda_1$, $P(t) \equiv_n \lambda_2$ and $R(\lambda_1, \lambda_2) \equiv_0 0$ or $P(s) \equiv_n P(t)$.

$R(s,t) \equiv_n 1$ if and only if either for some λ_1, $\lambda_2 \in A_0 P(s) \equiv_n \lambda_1$, $P(t) \equiv_n \lambda_2$ and $R(\lambda_1, \lambda_2) \equiv_0 1$ or there is some $k > 0$ such that $P(t) \equiv_n S^k P(s)$.

$R(s,t) \equiv_n R(s',t')$ if and only if either for some λ_1, λ_2, λ_3, $\lambda_4 \in A_0$, $P(s) \equiv_n \lambda_1$, $P(t) \equiv_n \lambda_2$, $P(s') \equiv_n \lambda_3$, $P(t') \equiv_n \lambda_4$ and $R(\lambda_1, \lambda_2) \equiv_0 R(\lambda_3, \lambda_4)$ or $R(s, t) \equiv_n 0 \equiv_n R(s', t')$ (as above) or $R(s,t) \equiv_n 1 \equiv_n R(s',t')$ (as above) or $P(s) \equiv_n P(s')$ and $P(t) \equiv_n P(t')$.

$R(u,v) \equiv_n U(s)$ if and only if either there are t, λ_1, $\lambda_2 \in B$ so that $t \equiv_n s$, $P(u) \equiv_n \lambda_1$, $P(v) \equiv_n \lambda_2$ and $U(t) \equiv_0 R(\lambda_1, \lambda_2)$ or there are λ_1, $\lambda_2 \in A_n$ so that $s \equiv_n R(\lambda_1,\lambda_2)$ and $R(\lambda_1, S\lambda_2) \equiv_n R(u,v)$.

$R(s, t) \equiv_n u$ if and only if either one of the above cases holds or there is $v \in B$ such that $R(s,t) \equiv_n v$ as above, and $v \equiv_n u$.

Remark. Note first that every equivalence on the right hand side of the inductive hypothesis either concerns elements in A_n or is covered in earlier clauses. As to the decidability of the relation, at various points we need to know if there are elements with a certain property. For example in the fourth clause we ask whether "there are λ_1, λ_2 so that $s \equiv_n R(\lambda_1,\lambda_2)$ and $R(\lambda_1, S\lambda_2) \equiv_n R(u,v)$". By property (v) we can tell if there are λ_1', λ_2' so that $s \equiv_n R(\lambda_1',\lambda_2')$. Since $\equiv_n$ is a partial congruence, if there are λ_1, λ_2 so that $s \equiv_n R(\lambda_1,\lambda_2)$ and $R(\lambda_1,S\lambda_2) \equiv_n R(u,v)$ then for all λ_1', λ_2', $s \equiv_n R(\lambda_1',\lambda_2')$ implies that $R(\lambda_1',S\lambda_2') \equiv_n R(u,v)$. Hence we have an algorithm for answering the question. Similar comments apply throughout.

There remains property (v) to consider. It is enough in view of the inductive hypothesis to be able to decide when elements of the form $U(s)$ and members of B are in the range of R. Since B is finite, we can assume we know the answer for elements of B relative to A_0. By property (i), we will then know the answer for all A_n, if we can settle the case $n = 0$. By the fourth clause, we can reduce the question to either elements of B or elements of A_0. In A_0 all the elements of the form $U(s)$ are either in B, $\equiv_0$-equivalent to an element in the range of R or $\equiv_0$-equivalent to an element in the range of F. Since no element in the range of R can be $\equiv_0$-equivalent to an element in the range of F, we can decide whether a given element of A_0 in the range of U is $\equiv_0$-equivalent to an element in the range of R. Such considerations recur throughout.

(2) Image of F

Inductive Hypothesis

$F(s,t) \equiv_n 0_F$ if and only if either there are λ_1, $\lambda_2 \in A_0$ so that $H(s) \equiv_n \lambda_1$, $H(t) \equiv_n \lambda_2$ and $F(\lambda_1,\lambda_2) \equiv_0 0_F$ or $H(s) \equiv_n H(t)$.

$F(s,t) \equiv_n 1_F$ if and only if either there are λ_1, $\lambda_2 \in A_0$ so that $H(s) \equiv_n \lambda_1$, $H(t) \equiv_n \lambda_2$ and $F(\lambda_1,\lambda_2) \equiv_0 1_F$ or there is $k > 0$ $H(t) \equiv_n HT^k(s)$.

$F(s,t) \equiv_n F(s',t')$ if and only if either there are λ_1, λ_2, λ_1', $\lambda_2' \in A_0$ so that $H(s) \equiv_n \lambda_1$, $H(t) \equiv_n \lambda_2$, $H(s') \equiv_n \lambda_1'$, $H(t') \equiv_n \lambda_2'$ and $F(\lambda_1,\lambda_2) \equiv_0 F(\lambda_1',\lambda_2')$ or $F(s,t) \equiv_n 0_F \equiv_n F(s',t')$ (as above) or $F(s,t) \equiv_n 1_F \equiv_n F(s',t')$

(as above) <u>or</u> $H(s) \equiv_n H(s')$ and $H(t) \equiv_n H(t')$.

$F(u,v) \equiv_n U(s)$ if and only if <u>either</u> there are t, λ_1, $\lambda_2 \in B$ so that $t \equiv_n s$, $H(u) \equiv_n \lambda_1$, $H(v) \equiv_n \lambda_2$ and $U(t) \equiv_0 F(\lambda_1, \lambda_2)$ <u>or</u> there are λ_1, λ_2 so that $s \equiv_n F(\lambda_1, \lambda_2)$ and $F(\lambda_1, T\lambda_2) \equiv_n F(u,v)$.

$F(s,t) \equiv_n u$ if and only if <u>either</u> one of the above cases holds <u>or</u> there is $v \in B$ such that $F(s,t) \equiv_n v$ as above, and $v \equiv_n u$.

(3) Image of U

Inductive Hypothesis

$U(s) \equiv_n 0$ if and only if <u>either</u> for some $t \in B$, $s \equiv_n t$ and $U(t) \equiv_0 0$ <u>or</u> there are λ_1, $\lambda_2 \in A_n$ so that $s \equiv_n R(\lambda_1, \lambda_2)$ and $R(\lambda_1, S\lambda_2) \equiv_n 0$

$U(s) \equiv_n 1$ if and only if <u>either</u> for some $t \in B$, $s \equiv_n t$ and $U(t) \equiv_0 1$ <u>or</u> there are $\lambda_1, \lambda_2 \in A_n$ so that $s \equiv_n R(\lambda_1, \lambda_2)$ and $R(\lambda_1, S\lambda_2) \equiv_n 1$

$U(s) \equiv_n 0_F$ if and only if <u>either</u> for some $t \in B$, $s \equiv_n t$ and $U(t) \equiv_0 0_F$ <u>or</u> there are λ_1, $\lambda_2 \in A_n$ so that $s \equiv_n F(\lambda_1, \lambda_2)$ and $F(\lambda_1, T\lambda_2) \equiv_n 0_F$

$U(s) \equiv_n 1_F$ if and only if <u>either</u> for some $t \in B$, $s \equiv_n t$ and $U(t) \equiv_0 1_F$ <u>or</u> there are λ_1, $\lambda_2 \in A_n$ so that $s \equiv_n F(\lambda_1, \lambda_2)$ and $F(\lambda_1, T\lambda_2) \equiv_n 1_F$

$U(s) \equiv_n R(u,v)$ if and only if <u>either</u> there are t, $\lambda_1, \lambda_2 \in B$, so that $t \equiv_n s$, $P(u) \equiv_n \lambda_1$, $P(v) \equiv_n \lambda_2$ and $U(t) \equiv_0 R(\lambda_1, \lambda_2)$ <u>or</u> there are λ_1, $\lambda_2 \in A_n$ so that $s \equiv_n R(\lambda_1, \lambda_2)$ and $R(\lambda_1, S\lambda_2) \equiv_n R(u,v)$.

$U(s) \equiv_n F(u,v)$ if and only if <u>either</u> there are t, λ_1, $\lambda_2 \in B$, so that $t \equiv_n s$, $P(u) \equiv_n \lambda_1$, $P(v) \equiv_n \lambda_2$ and $U(t) \equiv_0 F(\lambda_1, \lambda_2)$ <u>or</u> there are $\lambda_1, \lambda_2 \in A_n$ so that $s \equiv_n F(\lambda_1, \lambda_2)$ and $F(\lambda_1, T\lambda_2) \equiv_n F(u,v)$.

$U(s) \equiv_n U(t)$ if and only if <u>either</u> $U(s) \equiv_n 0 \equiv_n U(t)$ (as above) <u>or</u> $U(s) \equiv_n 1 \equiv_n U(t)$ <u>or</u> $U(s) \equiv_n 0_F \equiv_n U(t)$ <u>or</u> $U(s) \equiv_n 1_F \equiv_n U(t)$ <u>or</u> $s \equiv_n t$ <u>or</u> there are s', $t' \in B$ so that $s \equiv_n s'$, $t \equiv_n t'$ and $U(s') \equiv_B U(t')$.

$U(s) \equiv_n u$ if and only if <u>either</u> one of the above cases holds <u>or</u> there is $v \in B$ such that $U(s) \equiv_n v$ as above, and $v \equiv_n u$.

(4) Image of C_Q

Inductive Hypothesis

$C_Q(s) \equiv_n q$ for $q \in Q$ if and only if there is $\lambda \in A_0$ such that $H(s) \equiv_n \lambda$ and $C_Q(\lambda) \equiv_0 q$.

$C_Q(s) \equiv_n C_Q(t)$ if and only if <u>either</u> there are λ_1, $\lambda_2 \in A_0$ so that $H(s) \equiv_n \lambda_1$, $H(t) \equiv_n \lambda_2$ and $C_Q(\lambda_1) \equiv_n C_Q(\lambda_2)$ <u>or</u> $H(s) \equiv_n H(t)$

$C_Q(s) \equiv_n N_Q(u, v, w)$ if and only if <u>either</u> $H(w) \equiv_n w$, $HT(w) \equiv_n H(s)$, $C_Q(w) \equiv_n u$ and $C_\Sigma(w) \equiv_n v$ <u>or</u> for some $q \in Q$, $C_Q(s) \equiv_n q \equiv_n N_Q(u,v,w)$

or there are $\lambda \in A_0$, and $u_0, v_0, w_0 \in B$ so that $H(s) \equiv_n \lambda$, $u_0 \equiv_n u$, $v_0 \equiv_n v$, $w_0 \equiv_n w$ and $C_Q(\lambda) \equiv_0 N_Q(u_0, v_0, w_0)$.

$C_Q(s) \equiv_n u$ if and only if either one of the above cases holds or there is $v \in B$ such that $C_Q(s) \equiv_n v$ as above and $v \equiv_n u$.

Remark. In the third clause we have to decide whether $C_Q(w) \equiv_n u$ and $C_\Sigma(w) \equiv_n v$. In order that $\equiv_n'$ be well defined we need to know that if $N_Q(u, v, w) \in A_n$ then $C_\Sigma(\lambda) \in A_n$ where λ is the normal form of $H(w)$. This can be verified by induction on n.

(5) Image of N_Q

Inductive Hypothesis

$N_Q(s, t, u) \equiv_n q \in Q$ if and only if either there exist $p \in Q$ and $a \in \Sigma$ such that $s \equiv_n p$, $t \equiv_n a$, $H(u) \equiv_n u$ and $\sigma(p, a) = q$ or $s \equiv_n q_L$ or q_R, $H(u) \equiv_n u$ and $t \equiv_n C_\Sigma(u)$ or $u \equiv_n H(u)$, $s \equiv_n C_Q(u)$, $t \equiv_n C_\Sigma(u)$ and $C_Q(T(u)) \equiv_n q$ or there are s', t', $u' \in B$ so that $s \equiv_n s'$, $t \equiv_n t'$, $u \equiv_n u'$ $N_Q(s, t, u) \in B$ and $N_Q(s', t', u') \equiv_B q$.

$N_Q(s, t, u) \equiv_n N_Q(s', t', u')$ if and only if either for some $q \in Q$, $N_Q(s, t, u) \equiv_n q \equiv_n N_Q(s', t', u')$ or $s \equiv_n s'$, $t \equiv_n t'$ and $u \equiv_n u'$ or $u \equiv_n H(u)$, $u' \equiv_n H(u')$, $s \equiv_n C_Q(u)$, $s' \equiv_n C_Q(u')$, $t \equiv_n C_\Sigma(u)$, $t' \equiv_n C_\Sigma(u')$ and $C_Q T(u) \equiv_n C_Q T(u')$ or there exist $b \equiv_B b'$ such that $N_Q(s, t, u) \equiv_n b$ (as above) and $N_Q(s', t', u') \equiv_n b'$.

$N_Q(u, v, w) \equiv_n C_Q(s)$ if and only if either $H(w) \equiv_n w$, $C_Q(w) \equiv_n u$, $C_\Sigma(w) \equiv_n v$ and $C_Q T(w) \equiv_n C_Q(s)$ or for some $q \in Q$, $C_Q(s) \equiv_n q \equiv_n N_Q(u, v, w)$ or there is some $b \in B$ so that $N_Q(u, v, w) \equiv_n b$ (as in the paragraph above) and $C_Q(s) \equiv_n b$.

$N_Q(s, t, u) \equiv_n v$ if and only if either one of the above cases holds or there is $w \in B$ such that $N_Q(s, t, u) \equiv_n w$ as above, and $v \equiv_n w$.

(6) Image of C_Σ

Inductive Hypothesis

If $R(s, H(s)) \equiv_n 1$ and $P(s)$ is not equivalent to an element of A_0 then $R(T(s), HT(s))$ is not $\equiv_n 1$ and dually for $R(H(s), s)$.

$C_\Sigma(s) \equiv_n a \in \Sigma$ if and only if there are $\lambda \in A_0$ such that $\lambda \equiv_n P(s)$ and $C_\Sigma(\lambda) \equiv_0 a$.

$C_\Sigma(s) \equiv_n C_\Sigma(t)$ if and only if either there are $\lambda_1, \lambda_2 \in A_0$ so that $P(s) \equiv_n \lambda_1$, $P(t) \equiv_n \lambda_2$ and $C_\Sigma(\lambda_1) \equiv_0 C_\Sigma(\lambda_2)$ or $P(s) \equiv_n P(t)$ or $P(s) \equiv_n TP(t)$ and $R(t, H(t)) \equiv_n 1$ or $R(H(t), t) \equiv_n 1$ or the last condition holds with the roles of s and t reversed.

$C_\Sigma(s) \equiv_n N_\Sigma(u,v,w)$ if and only if <u>either</u> there is $a \in \Sigma$, so that $C_\Sigma(s) \equiv_n a \equiv_n N_\Sigma(u,v,w)$ <u>or</u> $u \equiv_n C_Q(w)$ and $v \equiv_n C_\Sigma(w)$, $w \equiv_n H(w)$ and $C_\Sigma(s) \equiv_n C_\Sigma(T(w))$ <u>or</u> there are $\lambda \in A_0$, u', v', $w' \in B$ such that $P(s) \equiv_n \lambda$, $u \equiv_n u'$, $v \equiv_n v'$, $w \equiv_n w'$, $N_\Sigma(u',v',w') \in B$ and $C_\Sigma(\lambda) \equiv_n N_\Sigma(u',v',w')$.

$C_\Sigma(s) \equiv_n C^*(t,\ u)$ if and only if <u>either</u> for some $a \in \Sigma$, $C_\Sigma(s) \equiv_n a \equiv_n C^*(t,u)$ <u>or</u> $u \equiv_n R(H(t),t)$ <u>or</u> $u \equiv_n R(t,H(t))$ and $C_\Sigma(s) \equiv_n C_\Sigma(T(t))$ <u>or</u> $u \equiv_n 1$, $t \equiv_n P(t)$ and $C_\Sigma(s) \equiv_n C_\Sigma(t)$ <u>or</u> there are t', $u' \in B$ and $\lambda \in A_0$ so that $t \equiv_n t'$, $u \equiv_n u'$, $C^*(t',u') \in B$, $P(s) \equiv_n \lambda$ and $C^*(t',u') \equiv_0 C_\Sigma(\lambda)$.

$C_\Sigma(s) \equiv_n u$ if and only if <u>either</u> one of the above cases holds <u>or</u> there is $v \in B$ such that $C_\Sigma(s) \equiv_n v$ as above and $v \equiv_n u$.

(7) Image of N_Σ

Inductive Hypothesis

$N_\Sigma(s,t,u) \equiv_n a \in \Sigma$ if and only if <u>either</u> there are $q \in Q$, $b \in \Sigma$, such that $u \equiv_n H(u)$, $\alpha(q,b) = a$ and $s \equiv_n q$, $t \equiv_n b$ <u>or</u> $a = e_L$ (resp. e_R) and there is $q \in Q$, such that $u \equiv_n H(u)$, $s \equiv_n q_L$ (resp. q_R), $t \equiv_n C_\Sigma(u)$ <u>or</u> $H(u) \equiv_n u$, $C_Q(u) \equiv_n s$, $C_\Sigma(u) \equiv_n t$ and $a \equiv_n C_\Sigma T(u)$ <u>or</u> there are s', t', $u' \in B$ so that $s \equiv_n s'$, $t \equiv_n t'$, $u \equiv_n u'$, $N_\Sigma(s',t',u') \in B$ and $N_\Sigma(s',t',u') \equiv_B a$.

$N_\Sigma(s,t,u) \equiv_n N_\Sigma(v,w,z)$ if and only if <u>either</u> they are both $\equiv_n$-equivalent to the same $a \in \Sigma$ <u>or</u> $s \equiv_n v$, $t \equiv_n w$ and $u \equiv_n z$ <u>or</u> $H(u) \equiv_n u$, $C_Q(u) \equiv_n s$, $C_\Sigma(u) \equiv_n t$, $H(z) \equiv_n z$, $C_Q(z) \equiv_n v$, $C_\Sigma(z) \equiv_n w$ and $C_\Sigma T(u) \equiv_n C_\Sigma T(z)$ <u>or</u> there are $b \equiv_B b' \in B$ so that $N_\Sigma(s,t,u) \equiv_n b$ and $N_\Sigma(v,w,z) \equiv_n b'$ (as above).

$N_\Sigma(u,v,w) \equiv_n C_\Sigma(s)$ if and only if <u>either</u> there is $a \in \Sigma$, so that $C_\Sigma(s) \equiv_n a \equiv_n N_\Sigma(u,v,w)$ <u>or</u> $u \equiv_n C_Q(w)$ and $v \equiv_n C_\Sigma(w)$, $w \equiv_n H(w)$ and $C_\Sigma(s) \equiv_n C_\Sigma(T(w))$ <u>or</u> there is $\lambda \in A_0$, u', v', $w' \in B$ such that $P(s) \equiv_n \lambda$, $u \equiv_n u'$, $v \equiv_n v'$, $w \equiv_n w'$, $N_\Sigma(u',v',w') \in B$ and $C_\Sigma(\lambda) \equiv_n N_\Sigma(u',v',w')$.

$N_\Sigma(s,t,u) \equiv_n C^*(v,w)$ if and only if <u>either</u> there is $a \in \Sigma$ so that $C^*(v,w) \equiv_n a \equiv_n N_\Sigma(s,t,u)$ <u>or</u> there is some z so that $C^*(v,w) \equiv_n C_\Sigma(z) \equiv_n N_\Sigma(s,t,u)$ <u>or</u> there is some $b \in B$ so that $N_\Sigma(s,t,u) \equiv_n b$ (as above) and $b \equiv_n C^*(v,w)$.

$N_\Sigma(s,t,u) \equiv_n v$ if and only if <u>either</u> one of the above cases holds <u>or</u> there is $w \in B$ such that $N_\Sigma(s,t,u) \equiv_n w$ as above and $v \equiv_n w$.

(8) Image of C^*

Inductive Hypothesis

$C^*(u,v) \equiv_n a \in \Sigma$ if and only if <u>either</u> there is $u \equiv_n P(u)$, $v \equiv_n R(u,H(u))$ or $R(H(u),u)$ and $C_\Sigma T(u) \equiv_n a$ <u>or</u> $v \equiv_n 1$, $u \equiv_n P(u)$ and

$P(u) \equiv_n a$ or there exist u', $v' \in B$ so that $u \equiv_n u'$, $v \equiv_n v'$, $C^*(u', v') \in B$ and $C^*(u', v') \equiv_B a$.

$C^*(t, u) \equiv_n C_\Sigma(s)$ if and only if either for some $a \in \Sigma$, $C_\Sigma(s) \equiv_n a \equiv_n C^*(t, u)$ or $u \equiv_n R(H(t), t)$ or $u \equiv_n R(t, H(t))$ and $C_\Sigma(s) \equiv_n C_\Sigma(T(t))$ or $u \equiv_n 1$, $t \equiv_n P(t)$ and $C_\Sigma(s) \equiv_n C_\Sigma(t)$ or there are t', $u' \in B$ and $\lambda \in A_0$ so that $t \equiv_n t'$, $u \equiv_n u'$, $C^*(t', u') \in B$, $P(s) \equiv_n \lambda$ and $C^*(t', u') \equiv_0 C_\Sigma(\lambda)$.

$C^*(s, t) \equiv_n C^*(u, v)$ if and only if either both are $\equiv_n$-equivalent to the same $a \in \Sigma$ or there are λ_1, λ_2 so that $C^*(s, t) \equiv_n C_\Sigma(\lambda_1)$, $C^*(u, v) \equiv_n C_\Sigma(\lambda_2)$ and $C_\Sigma(\lambda_1) \equiv_n C_\Sigma(\lambda_2)$ or $s \equiv_n u$ and $t \equiv_n v$ or there are $b_1 \equiv_B b_2 \in B$ so that $C^*(s, t) \equiv_n b_1$ and $C^*(s, t) \equiv_n b_2$ as above.

$C^*(v, w) \equiv_n N_\Sigma(s, t, u)$ if and only if either there is $a \in \Sigma$ so that $C^*(v, w) \equiv_n a \equiv_n N_\Sigma(s, t, u)$ or there is some z so that $C^*(v, w) \equiv_n C_\Sigma(z) \equiv_n N_\Sigma(s, t, u)$ or there is some $b \in B$ so that $N_\Sigma(s, t, u) \equiv_n b$ and $b \equiv_n C^*(v, w)$ (as above).

$C^*(s, t) \equiv_n u$ if and only if either one of the above cases holds or there is $v \in B$ such that $C^*(s, t) \equiv_n v$ as above and $u \equiv_n v$.

(9) Image of E

Inductive Hypothesis

$E(s) \equiv_n 1$ if and only if either for some λ, $s \equiv_n C_Q(\lambda)$ or there is $t \in B$ so that $s \equiv_n t$ and $E(t) \equiv_B 1$.

$E(s) \equiv_n 0$ if and only if either $s \equiv_n h$ or there is $t \in B$ so that $s \equiv_n t$ and $E(t) \equiv_B 0$.

$E(s) \equiv_n E(t)$ if and only if either $s \equiv_n t$ or $E(s) \equiv_n 1 \equiv_n E(t)$ or $E(s) \equiv_n 0 \equiv_n E(t)$ or there are $b_1 \equiv_B b_2$ so that $E(s) \equiv_n b_1$ and $E(t) \equiv_n b_2$ (as above).

$E(s) \equiv_n t$ if and only if either one of the above cases holds or there is $u \in B$ such that $E(s) \equiv_n u$ (as above) and $u \equiv_n t$.

(10) Image of K^*

Inductive Hypothesis

$K^*(s, t) \equiv_n 0$ if and only if either $s \equiv_n t$ and either s is $\equiv_n$-equivalent to a constant or $P(s) \equiv_n s$ or s is equivalent to an element in the image of C_Σ, C_Q, R or F or there are b_1, $b_2 \in B$ so that $s \equiv_n b_1$, $t \equiv_n b_2$ and $K^*(b_1, b_2) \equiv_B 0$.

$K^*(s, t) \equiv_n 1$ if and only if either $P(s) \equiv_n s$ and there is a constant $d \neq c$ with $t \equiv_n d$ or s is equivalent to an element in the range of C_Σ and there is a constant $d \notin \Sigma$ so that $t \equiv_n d$ or s is equivalent to an element in the range of C_Q and there is a constant $d \notin Q$ so that $t \equiv_n d$ or s is equivalent to an element in the range of R and there is a constant $d \neq 0, 1$ so that $t \equiv_n d$ or s is equivalent to an element in the range of F and there is a constant $d \neq 0_F$, 1_F so that $t \equiv_n d$ or s is equivalent to an element in the range of one of the operations P, C_Σ, C_Q, R, F and t is equivalent to an element in the range of a different one of these operations or there are b_1, $b_2 \in B$ so that $s \equiv_n b_1$, $t \equiv_n b_2$ and $K^*(b_1, b_2) \equiv_B 1$.

$K^*(s,t) \equiv_n K^*(u,\ v)$ if and only if either $K^*(s,t) \equiv_n 0 \equiv_n K^*(u,v)$ or $K^*(s,t) \equiv_n 1 \equiv_n K^*(u,v)$ or $s \equiv_n u$ and $t \equiv_n v$ or there is $b_1 \equiv_B b_2$ so that $K^*(s,t) \equiv_n b_1$ and $K^*(u,v) \equiv_n b_2$ (as above).

$K^*(s,t) \equiv_n u$ if and only if either one of the above cases holds or there is $v \in B$ such that $K^*(s,t) \equiv_n v$ (as above) and $u \equiv_n v$.

5.8 DEFINITION OF $(A''_n, \equiv''_n)$

Define A''_n and $\equiv''_n$ as follows:
for n even:

$A''_n = A'_n \cup \{N_H(s,t,H\lambda) | s,t \in A'_n, \lambda \in A_n \text{ space-time}\}$

for n odd:

$A''_n = A'_n \cup \{K(s,\lambda) | s \in A'_n, \lambda \in A_n \text{ space-time}\}$

We will extend $\equiv'_n$ to a partial congruence $\equiv''_n$ on A''_n. First, for n even, we define $N_H(s,t,H\lambda) \in A''_n - A_n$ to be reducible (to u) if and only if one of the following holds:

(i) there exists s', t', λ' with $u = N_H(s', t', H\lambda') \in B$ and $s \equiv'_n s'$, $t \equiv'_n t'$ and $H\lambda \equiv_n H\lambda'$.

(ii) $s \equiv'_n q \in Q$, and $t \equiv'_n a \in \Sigma$, and $u = S^{\mu(q,a)}TH(\lambda)$

(iii) $s \equiv'_n q_L$ and $t \equiv'_n C_\Sigma H(\lambda)$ and $u = S^{-1}TH(\lambda)$

(iv) $s \equiv'_n q_R$ and $t \equiv'_n C_\Sigma H(\lambda)$, and $u = STH(\lambda)$

(v) $s \equiv'_n C_Q H(\lambda)$, $t \equiv'_n C_\Sigma H(\lambda)$ and $u = HT\lambda$.

Now define, for n even,

$N_H(s,t,H\lambda) \equiv''_n N_H(s',t',H\lambda')$ (for both $N_H(s,t,H\lambda)$ and $N_H(s',t',H\lambda') \in A''_n - A_n$) if and only if either $s \equiv'_n s'$, $t \equiv'_n t'$, $H\lambda \equiv H\lambda$ or both $N_H(s,t,H\lambda)$ and $N_H(s',t',H\lambda')$ are reducible, to u, u' respectively , and $u \equiv'_n u'$

$N_H(s,t,H\lambda) \equiv''_n v \in A_n$ if and only if either $N_H(s,t,H\lambda)$ is reducible to u, and $u \equiv'_n v$, or $v = N_H(s',t',H\lambda')$ and $s \equiv'_n s'$, $t \equiv'_n t'$, $H\lambda \equiv'_n H\lambda'$.

Similarly, for n odd, we define $K(s,\lambda) \in A''_n - A_n$ to be reducible (to u) if and only if one of the following hold

(i) there exist s', λ' with $u = K(s', \lambda') \in B$ and $s \equiv'_n s$, $\lambda \equiv'_n \lambda'$
(ii) $s \equiv'_n 0$ or 0_F and $u = \lambda$
(iii) $s \equiv'_n 1$ or 1_F and $u = P(c)$.

Now , for n odd define $K(s, \lambda) \equiv''_n K(s', \lambda')$ (for both $K(s, \lambda)$ and $K(s', \lambda)$ and $K(s', \lambda')$ in $A''_n - A_n$ if and only if <u>either</u> $s \equiv'_n s'$ and $\lambda \equiv'_n \lambda'$ <u>or</u> both $K(s, \lambda)$ and $K(s', \lambda')$ are reducible to u, u' respectively, and $u \equiv'_n u'$.

$K(s, \lambda) \equiv'_n v \in A_n$ if and only if $K(s, \lambda)$ is reducible to u, and $u \equiv'_n v$ <u>or</u> $v = K(s', \lambda')$ and both $K(s, \lambda)$ and $K(s', \lambda')$ are irreducible, and $s \equiv_n s'$, $\lambda \equiv'_n \lambda$.

Finally, we define A_{n+1} and $\equiv_{n+1}$ to extend A''_n and $\equiv''_n$ as follows:
For n even: $A_{n+1} = A'_n \cup \{S^n T^m \lambda | \lambda \in A''_n - A_n,\ n \in \mathbb{Z}, m \geq 0\}$.
For n odd:

$$\begin{aligned} A_{n+1} = A''_n \ &\cup \ \{S^n T^m \lambda | \lambda \in A''_n - A_n, n \in \mathbb{Z}, m \geq 0\} \\ &\cup \ \{S^n T^m H T^k \lambda | \lambda \in A''_n - A_n, n \in \mathbb{Z}, m, k \geq 0\}. \end{aligned}$$

5.9 DEFINITION OF $(A_{n+1}, \equiv_{n+1})$

We extend $\equiv''_n$ to $\equiv_{n+1}$ on A_{n+1} as follows:

For n even:
$S^n T^m \lambda \equiv_{n+1} S^{n'} T^{m'}$ if and only if $n = n'$, $m = m'$ and $\lambda \equiv''_n \lambda'$ and
$S^n T^m \lambda \equiv_{n+1} v \in A_n$ if and only if λ is reducible, to u, and $S^n T^m u \equiv_n v$.
For n odd:
$S^n T^m \lambda \equiv_{n+1} S^{n'} T^{m'} \lambda'$ if and only if $n = n'$, $m = m'$, and $\lambda \equiv_n \lambda'$.
$S^n T^m H T^k \lambda \equiv_{n+1} S^{n'} T^{m'} H T^{k'} \lambda'$ if and only if $n = n'$. $m = m'$, $k = k'$, and $\lambda = \lambda'$.

Finally, $S^n T^m \lambda \equiv_{n+1} v \in A_n$ if and only if λ reduces to u and $S^n T^m u \equiv_n v$, and similarly with $S^n T^m H T^k \lambda$.

This completes the definition of $(A_{n+1}, \equiv_{n+1})$. It is straightforward to check that $(A_{n+1}, \equiv_{n+1})$ is a decidable partial congruence satisfying (i), (ii), (iii), (iv) above. Property (v) can be verified in the other cases as it was after the definition of the inductive hypothesis on R. Also, as discussed in subsection 5.1, $\bigcup_{n\geq 0} \equiv_n$ is precisely $\theta_{\mathcal{P}}$ restricted to $\bigcup_{n\geq 0} A_n$. The A_n and $\equiv_n$ are uniformly decidable in n. Together this means that $\bigcup_{n\geq 0} A_n$ and $\bigcup_{n\geq 0} \equiv_n$ are decidable. Since $\bigcup_{n\geq 0} A_n$ contains all terms, modulo (effectively) reducing space-time terms to their normal forms, this completes the proof.

If we combine Theoerems 3.1, 4.1 and 5.1, we obtain the following theorem.

Theorem 5.5 *There is a finitely based variety of finite type which has solvable but not uniformly solvable word problem.*

§6. A RECURSIVELY BASED VARIETY DEFINED BY LAWS INVOLVING NO VARIABLES.

6.1 DEFINITION OF THE VARIETY

In this section we will describe a recursively based variety of finite type, defined by laws which involve no variables, which has solvable but not uniformly solvable word problem.

The variety is a modification of the finitely based variety defined in the preceding sections. The use of infinitely many axioms allows us to use a simpler picture of space-time.

The operations are the same, except that P, C^*, and U are deleted and K and K^* are identified. Specifically, the operations are:

constants: c, all $a \in \Sigma$, all $q \in Q$, 0, 1, 0_F, 1_F
unary: T, S, S^{-1}, H, C_Σ, C_Q, E
binary: F, R, K
ternary: N_H, N_Q, N_Σ.

Define, for each k, $H_k = HT^k(c)$, and let $\Lambda = \{S^nT^m(H_k) | n \in \mathbb{Z}, m, k \in \mathbb{N}\}$. These will be the space-time elements.

The laws defining the variety are as follows:

I. $H(c) \approx c$,
$T(S^nT^m(H_k)) \approx S^nT^{m+1}(H_k)$,
$S(S^nT^m(H_k)) \approx S^{n+1}T^m(H_k)$,
$S^{-1}(S^nT^m(H_k)) \approx S^{n-1}T^m(H_k)$,
$H(S^nT^m(H_k)) \approx H_{m+k}$ for all $k, m \geq 0$ and $n \in \mathbb{Z}$.

II. $N_Q(q, a, H_k) \approx \sigma(q, a)$ for all $q \in Q$, $a \in \Sigma$, $k \geq 0$,
$N_H(q, a, H_k) \approx S^{\mu(q,a)}T(H_k)$ for all $q \in Q$, $a \in \Sigma$, $k \geq 0$,
$N_\Sigma(q, a, H_k) \approx \alpha(q, a)$ for all $q \in Q - Q_{LR}$, $a \in \Sigma$, $k \geq 0$.

$N_Q(q_L, C_\Sigma(H_k), H_k) \approx q \approx N_Q(q_R, C_\Sigma(H_k), H_k)$,
$N_H(q_L, C_\Sigma(H_k), H_k) \approx ST(H_k)$,
$N_H(q_R, C_\Sigma(H_k), H_k) \approx S^{-1}T(H_k)$,

$N_\Sigma(q_L, C_\Sigma(H_k), H_k) \approx e_L$,
$N_\Sigma(q_R, C_\Sigma(H_k), H_k) \approx e_R$, for all $q \in Q \setminus Q_{\mathrm{LR}}$ and $k \geq 0$.

III. $C_\Sigma T(H_k) \approx N_\Sigma(C_Q(H_k), C_\Sigma(H_k), H_k)$,
$C_Q T(H_k) \approx N_Q(C_Q(H_k),\ C_\Sigma(H_k), H_k)$,
$H_{k+1} \approx N_H(C_Q(H_k), C_\Sigma(H_k), H_k)$ for all $k \geq 0$.

IV. $C_Q(\lambda) \approx C_Q H(\lambda)$ for all $\lambda \in \Lambda$.

V. $R(\lambda, \lambda) \approx 0$ for all $\lambda \in \Lambda$,
$R(\lambda, S^k(\lambda)) \approx 1$ for all $k > 0$, and $\lambda \in \Lambda$,
$F(\lambda, \lambda) \approx 0_F$ for all $\lambda \in \Lambda$,
$F(\lambda, T^k(\lambda)) \approx 1_F$ for all $\lambda \in \Lambda$ and $k > 0$,
$F(\lambda, \gamma) \approx F(H\lambda, H(\gamma))$ for all $\lambda, \gamma \in \Lambda$.

VI. $C_\Sigma(S^n T H(\lambda)) \approx C_\Sigma(S^n H(\lambda))$ for all $\lambda \in \Lambda$ and $n \neq 0$.

VII. $K(0, \lambda) \approx \lambda$ for all $\lambda \in \Lambda$,
$K(1, \lambda) \approx c$ for all $\lambda \in \Lambda$,
$K(d, d) \approx 0$ for all constants d,
$K(d, e) \approx 1$ for all constants $d \neq e \neq c$,
$K(\lambda, d) \approx 1$, for all $\lambda \in \Lambda$ and constants $d \neq c$,
$K(C_\Sigma(\lambda), d) \approx 1$ for all constants $d \notin \Sigma$, and $\lambda \in \Lambda$,
$K(C_Q(\lambda), d) \approx 1$ for all constants $d \notin Q$, and $\lambda \in \Lambda$,
$K(R(\lambda, \gamma), d) \approx 1$ for all $\lambda,\ \gamma \in \Lambda$, and $d \neq 0, 1$,
$K(F(\lambda, \gamma), d) \approx 1$ for all $\lambda,\ \gamma \in \Lambda$, and $d \neq 0_F, 1_F$.

$K(t, t) = 0$ and $K(s, t) = 1$ for all s, t where s, t belong to different members of the following list of sets: Λ, $\{C_\Sigma(\lambda) | \lambda \in \Lambda\}$, $\{C_Q(\lambda) | \lambda \in \Lambda\}$, $\{R(\lambda, \gamma) | \lambda, \gamma \in \Lambda\}$, $\{F(\lambda, \gamma) | \lambda, \gamma \in \Lambda\}$.

VIII. $E C_Q(\lambda) \approx 1$ for all $\lambda \in \Lambda$
$E(h) \approx 0$.

Note that every term generated from c by the operations S, S^{-1}, T and H is equivalent modulo the above laws I to an element of Λ; in fact there is an effective procedure which, given such a term t, produces $\lambda \in \Lambda$ with t equivalent (modulo I) to λ. Thus we may, and will, ignore all such terms t except those in Λ.

6.2 NON-UNIFORM SOLVABILITY OF THE WORD PROBLEM

Proposition 6.1 V *does not have uniformly solvable word problem.*

Proof. The proof is analogous to the proof of Theorem 3.1: for an initial tape configuration as described there, we have associated the same presentation $\mathcal{P}$ and prove that the universal Turing machine, started on that configuration, eventually halts if and only if $0 \equiv_{\mathcal{P}} 1$.

As in the proof of Theorem 3.1, the "only if" part is clear.

For the "if" direction, if the machine does not halt, we again produce a model $A \in V$ satisfying all the $\mathcal{P}$ equations, in which $0 \neq 1$.

The underlying set of A is as in the proof of Theorem 3.1, and the operations are as defined there, with the following changes:

(ii), (iii) and the definition of C^* are deleted (since we have deleted the operations P, U and C^*).

In (v) $R(x,y) = *$ unless both $x, y \in \Lambda$.

In (vi) $F(x,y) = *$ unless both $x, y \in \Lambda$.

In (vii) $K(x,y) = *$ for x, y not in the form of the first two lines.

In (viii) replace K^* by K.

6.3 SOLVABILITY OF THE WORD PROBLEM

The proof that V has solvable word problem is somewhat different than the proof in section 5.

We again differentiate two cases: whether or not $\mathcal{P}$ has degenerate space-time, which in this case means $\lambda \equiv_{\mathcal{P}} c$ for all $\lambda \in \Lambda$.

In the degenerate case, we have $(\lambda, c) \in \equiv_{\mathcal{P}}$ for all $\lambda \in \Lambda$, and hence the equations defining our variety are equivalent (modulo $\equiv_{\mathcal{P}}$) to finitely many equations, namely $S(c) = T(c) = S^{-1}(c) = c$ together with all instances of the equations defining the variety with c substituted for the arbitrary $\lambda \in \Lambda$ which appear. Thus $\equiv_{\mathcal{P}}$ is finitely generated relative to the variety of all algebras, and hence is decidable by Corollary 1.2.

In the non-degenerate case we proceed, at first, similarly to section 5, bearing in mind that there are fewer space-time elements (see above definition of Λ), but time coordinates, time prefixes, and space prefixes are defined as before, except that all these notions are always relative to $c = H(c)$, i.e. the only space-time component is that of c.

The proof of the next lemma is essentially the same as the proofs of Lemmas 5.2 and 5.3.

Lemma 6.2 *For any finite subset $F \subseteq \Lambda$, with maximum time coordinate m, there is a finite $\bar{F} \subseteq \Lambda$ with the same maximum time coordinate such that*

(i) $F \subseteq \overline{F}$
(ii) *if* λ *is a right subterm of* $\gamma \in \bar{F}$ *then* $\lambda \in \bar{F}$
(iii) *if* $\lambda \in \Lambda$ *and* $\lambda\theta_{\mathcal{P}}\gamma$ *for* $\gamma \in \bar{F}$ *then* $\lambda \in \bar{F}$
(iv) *if* $\lambda \in \Lambda$ *and* $T\lambda \in \bar{F}$ *then* $\lambda \in \bar{F}$.

Definition of A:

A consists of all terms appearing in the equations defining the variety, and all subterms thereof, plus all elements $C_\Sigma(\lambda)$, and $K(\lambda, \gamma)$ for λ, $\gamma \in \Lambda$, (the elements $C_Q(\lambda)$ are already included).

Let $\equiv_A$ be the restriction to A of the congruence defining our variety, then, because we have no non-trivial information about C_Σ and C_Q, rules III cannot be applied in a non-trivial way, and hence $\equiv_A$ is decidable, as in condition (3) of Proposition 1.1. Thus $(A, \equiv_A)$ is a partial subalgebra satisfying the hypotheses of Proposition 1.1.

Definition of B:

Let $B_{\mathcal{P}}$ consist of all terms appearing in the presentation $\mathcal{P}$, and all subterms thereof, and all constants.

Enlarge $B_{\mathcal{P}}$ as follows:

(i) For each $b \in B_{\mathcal{P}}$, if there exists $a \in A$ with $a\theta_{\mathcal{P}}b$, add one such a, and choose $a \in \Lambda$ whenever possible.

(ii) Let F consist of all the elements of Λ we have so far, and add the set $\bar{F}$ of Lemma 6.2.

(iii) For all $\lambda, \gamma \in \bar{F}$, add $C_Q(\lambda)$, $C_\Sigma(\lambda)$, $R(\lambda, \gamma)$, $F(\lambda, \gamma)$.

The resulting set is B. It is closed under taking subterms, and for $\lambda, \gamma \in \Lambda$, if $\gamma \in B$ and $\lambda\theta_{\mathcal{P}}\gamma$ then $\lambda \in B$.

Let $\equiv_B$ be $\theta_{\mathcal{P}}|B$, then B and $\equiv_B$ are both finite, and hence decidable.

Definition of A_0:

Let $A_0 = A \cup B$, and let $\equiv_0$ be the partial congruence on $A \cup B$ generated by $\equiv_A \cup \equiv_B$; we are going to show that $(A_0, \equiv_0)$ is a partial subalgebra satisfying the hypothesis of Proposition 1. Since $\theta_{\mathcal{P}}$ is generated by $\equiv_A \cup \equiv_B$ and hence by $\equiv_0$, the decidability of $\theta_{\mathcal{P}}$ will then follow from Proposition 1.1, the proof of which is deferred to the next subsection.

Since membership in A is decidable, and B is finite, we know that membership in A_0 is decidable.

Next, we need to establish that $\equiv_0$ is decidable; this, however, can be proved analogously to the proof in section 5 that $\equiv_0$ (as defined there) is decidable, deleting from that proof consideration of elements which we do not have in this example, such as elements in the image of U or C^* and space-time elements except those in the presently defined Λ.

It remains to check that $(A_0, \equiv_0)$ satisfies hypothesis (3) of Proposition 1.1, i.e., that there is an algorithm which, given an operation σ of arity n, and $a_1, \dots, a_n \in A_0$, determines whether there exist $b_1, \dots, b_n \in A_0$ with $a_i \equiv_0 b_i$ and $\sigma(b_1, \dots, b_n) \in A$.

Now, B is finite, and hence we can check all elements of the form $\sigma(b_1, \dots, b_n) \in B$, and decide whether $a_i \equiv_0 b_i$.

Thus it is enough to decide whether there exist $b_1, \dots, b_n \in A$ with $\sigma(b_1, \dots, b_n) \in A$ and $a_i \equiv_0 b_i$. Moreover, if some $a_i \in B - A$ then $a_i \in B_{\mathcal{P}}$ and so if there exists $b_i \in A$ with $a_i \equiv_0 b_i$ then there exists $c_i \in A \cap B$ with $a_i \equiv_0 c_i$ and we may replace a_i by c_i. Thus we may assume without loss of generality that all $a_i \in A$.

Hence we have reduced the problem to the following:

> given $a_1, \dots, a_n \in A$ do there exist $b_1, \dots, b_n \in A$ with $a_i \equiv_0 b_i$ and $\sigma(b_1, \dots, b_n) \in A$?

There is an effective procedure which, given $a \in A$, produces $d \in A$ such that $d \equiv_A a$ and either $d \in \Lambda$, d is a constant, or $d \in C_Q(\Lambda)$, $d \in C_\Sigma(\Lambda)$, $d \in R(\Lambda, \Lambda)$, or $d \in F(\Lambda, \Lambda)$. Thus we may assume that each a_i is already of this form.

We consider the operations in turn

T: For $a \in A$, if there exists $b \in A$ with $b \equiv_0 a$ and $T(b) \in A$ then $b \in \Lambda$ hence $a \in \Lambda$. Thus there is such a b if and only if $a \in \Lambda$.

S, S^{-1} and H are the same as T.

R: For a_1, $a_2 \in A$, if there exist $b_i \equiv_0 a_i$ with $b_i \in A$ and $R(b_1, b_2) \in A$ then $b_i \in \Lambda$ and hence $a_i \in \Lambda$, conversely if $a_i \in \Lambda$ then $R(a_1, a_2) \in A$.

F: Is the same as R.

C_Q : For $b \in A$, $C_Q(b) \in A$ if and only if $b \in \Lambda$, hence there exists $b \equiv_0 a$ with $C_Q(b) \in A$ if and only if $a \in \Lambda$

C_Σ: same as C_Q.

N_Q : If a_1, a_2, $a_3 \in A$ and there exist $b_i \equiv_0 a_i$ with $N_Q(b_1, b_2, b_3) \in A$, then $b_3 = H_k$ for some k, then because $a_3 \equiv_0 b_3$ we must have that the time component of a_3 is k so we can just check whether $a_3 \equiv H_k$. If the answer is affirmative, then for b_1 and b_2 we have $N_Q(b_1, b_2, H_k) \in A$ if and only if either $a_1 \equiv_0 q \in Q$ and $a_2 \equiv_0 a \in \Sigma$, or $a_1 \equiv_0 q_L$ or q_L or $C_Q(H_k)$ and $a_2 \equiv_0 C_\Sigma(H_k)$; there are only finitely many cases to check.

N_Σ, N_H: The argument is the same as for N_Q.

E: For $a \in A$, if there exists $b \in A$ with $E(b) \in A$ then either $a \equiv_0 h$ or $a \equiv_0 C_\Sigma(\lambda)$ for some $\lambda \in \Lambda$. The latter occurs if and only if either $a \in im(C_\Sigma)$ or $a \equiv_0$ to some $d \in \Sigma$ or $a \equiv_0 C_\Sigma(b) \in B$; these finitely many cases can be checked.

K: If $a_1 \equiv_0 b_1$ and $a_2 \equiv_0 b_2$ and $K(b_1, b_2) \in A$ then there are various possibilities:

(1) $b_2 \in \Lambda$, hence $a_2 \in \Lambda$, and $a_1 \equiv_0 0$ or $a_1 \equiv_0 1$ or a_1 is in Λ or in the image of C_Σ, C_Q, R or F, or $a_1 \equiv_0$ an element in the image of one of these operations in B. These finitely many cases can be checked.

(2) a_1 and a_2 are each $\equiv_0$ some constant (possibly different ones).

(3) $a_1 \in \Lambda$ and a_2 is either $\equiv_0$ some constant not equal to c, or $a_2 \in \Lambda$, or a_2 is in the image of C_Σ, C_Q, R or F, or a_2 is $\equiv_0$ to an element in the image of one of these operations in B.

(4) a_1 is in the image of C_Σ or is $\equiv_0$ an element of B which is in the image of C_Σ, and $a_2 \equiv_0 d$, a constant $\notin \Sigma$.

(5) Similar to (4), with C_Σ replaced by C_Q, or R, or F respectively, with the appropriate constraint on the constant $d \equiv_0 a_2$.

This completes the proof.

Applying Proposition 1.1, we have proved the following theorem.

Theorem 6.4 *There is a variety in a finite language defined by a recursive set of laws involving only constants which has solvable but not uniformly solvable word problem.*

6.4 PROOF OF PROPOSITION 1.1

We complete this section with the promised proof of Proposition 1.1.

Proof (of Proposition 1.1). We first produce a partial subalgebra $(B, \equiv_B)$ satisfying (1) to (3), such that $A \subseteq B$, and $\equiv_A \subseteq \equiv_B$, which has the feature that if $a_i \equiv_B b_i$ for $1 \leq i \leq n$ and if $\sigma(a_1, \ldots, a_n) \in B$ then $\sigma(b_1, ..., b_n) \in B$. B and $\equiv_B$ are defined by induction on the complexity of terms, as follows.

Let $B_0 = A$ and $\equiv_0$ be $\equiv_A$. For each natural number k, let

$$B_{k+1} = B_k \cup \{\sigma(b_1, \ldots, b_n) | \sigma \in \Sigma, b_i \in B_k \text{ and there exists } a_i \equiv_k b_i \text{ with } \sigma(a_1, \ldots, a_n) \in A\},$$

$R_{k+1} = \{(\sigma(b_1, \ldots, b_n), a) | \sigma \in \Sigma, b_i \in B_k, a \in A$ and there exists $a_i \equiv_k b_i$ with $a = \sigma(a_1, \ldots, a_n)\}$.

Let $\equiv_{k+1} = \equiv_A \cup (\equiv_A \circ R_{k+1}) \cup (R_{k+1}^{-1} \circ \equiv_A) \cup (R_{k+1}^{-1} \circ \equiv_A \circ R_{k+1})$.

Define $B = \cup B_k (k \in \omega)$ and define $\equiv_B = \cup \equiv_k$ $(k \in \omega)$.

Note that each B_k is closed under subterms.

We will prove the following by induction on k:

(i) $\equiv_k \subseteq \equiv_{k+1}$.

(ii) If $b \in B_k$ and $b \equiv_k a, b \equiv_k c$ for $a, c \in A$ then $a \equiv_A c$.

(iii) $\equiv_k |B_i = \equiv_i$ for all $i < k$.

(iv) $\equiv_k$ is transitive.

(v) $\equiv_k$ is a partial congruence on B_k.

$k = 0$: trivial.

Induction Step: Suppose we have (i) to (v) for k.

(i) Then $R_{k+1} \subseteq R_{k+2}$ and hence $\equiv_{k+1} \subseteq \equiv_{k+2}$.

(ii) Suppose $b \in B_{k+1}$ and $b \equiv_{k+1} a$, $b \equiv_{k+1} c$ for $a, c \in A$. Note that $R_{k+1}|A \subseteq \equiv_A$, and hence if $b \in A$ then $b \equiv_A a$ and $b \equiv_A c$ so $a \equiv_A c$.

Assume $b \notin A$. Then $b = \sigma(b_1, \ldots, b_n)$ and there exist $a_i \equiv_k b_i$ with $\sigma(a_1, \ldots, a_n) \in A$ and $\sigma(a_1, \ldots, a_n) \equiv_A a$. Similarly there exist $c_i \equiv_k b_i$ with $\sigma(c_1, ..., c_n) \in A$ and $\sigma(c_1, ..., c_n) \equiv_A c$. But by the induction hypothesis we get $a_i \equiv_A c_i$ and hence $\sigma(a_1, ..., a_n) \equiv_A \sigma(c_1, ..., c_n)$ so $a \equiv_A c$.

(iii) It is enough to prove that $\equiv_{k+1} |B_k = \equiv_k$, and for this it is enough to prove that $R_{k+1}|B_k \subseteq R_k(or \equiv_A$ if $k = 0)$. However, if $(\sigma(b_1, \ldots, b_n), a) \in R_{k+1}$ and $b = \sigma(b_1, \ldots, b_n) \in B_k$ then there exist $a_1, \ldots, a_n \in A$ with $b_i \equiv_k a_i$ and $a = \sigma(a_1, \ldots, a_n)$. If $k = 0$ then we have $b \in A$ and hence $(b, a) \in \equiv_A$. If $k > 0$ then $b \in B_K$ implies that $b_1, \ldots, b_n \in B_{k-1}$ and so the induction hypothesis yields $b_i \equiv_{k-1} a_i$ and hence $(b, a) \in \equiv_k$.

(iv) is an direct consequence of (ii).

(v) For $k = 0$ this is just the hypothesis on $(A, \equiv_A)$, since we have assumed that $\equiv_A$ is a partial congruence on A. Suppose $b_i \equiv_{k+1} d_i$ and $\sigma(b_1, \ldots, b_n) \in B_{k+1}$ and $\sigma(d_1, \ldots, d_n) \in B_{k+1}$. Then b_i, $d_i \in B_k$ for $1 \leq i \leq n$ and hence by (iii), $b_i \equiv_k d_i$. Also, there exist a_i, $c_i \in A$, $1 \leq i \leq n$ with $b_i \equiv_k a_i$ and $d_i \equiv_k c_i$ and $\sigma(a_1, \ldots, a_n)$, $\sigma(c_1, \ldots, c_n) \in A$. By (iv) and (ii) we obtain $a_i \equiv_A c_i$ and hence $\sigma(a_1, \ldots, a_n) \equiv_A \sigma(c_1, \ldots, c_n)$ and thus $\sigma(b_1, \ldots, b_n) \equiv_{k+1} \sigma(d_1, \ldots, d_n)$.

It remains to verify (1), (2), and (3) for $(B, \equiv_B)$.

Note first of all that $\sigma(b_1, \ldots, b_n) \in B$ if and only if there exist $a_1, \ldots, a_n \in A$ with $a_i \equiv_B b_i$ and $\sigma(a_1, \ldots, a_n) \in A$. Moreover, if $\sigma(b_1, \ldots, b_n) \in B$ and the a_i are as above then whenever $b_i \equiv_B c_i \in A$ with $\sigma(c_1, \ldots, c_n) \in A$ then $\sigma(a_1, \ldots, a_n) \equiv_A \sigma(c_1, \ldots, c_n)$.

We show that there is an algorithm which, given $b \in FX$, determines whether $b \in B$ and in the affirmative case produces $a \in A$ with $b \equiv_B a$.

Consider $b \in FX$. It is decidable whether $b \in A$, and in the affirmative case we are finished. If $b \notin A$ then $b = \sigma(b_1, \ldots, b_n)$ for unique $b_1, \ldots, b_n$ and σ. In this case, $b \in B$ if and only if all the $b_i \in B$, and there exist $a_i \in A$ with $\sigma(a_1, \ldots, a_n) \in A$ and $b_i \equiv_B a_i$. The b_i are of lower complexity than b; determine for each whether it belongs to B and if so, produce $c_i \in A$ with $c_i \equiv_B b_i$. Given the c_i it is decidable whether there exist $a_1, \ldots, a_n \in A$ with $a_i \equiv_A c_i$ and $\sigma(a_1, \ldots, a_n) \in A$, and moreover, since membership in A is decidable, we can effectively produce the a_i in the affirmative case, thus yielding an appropriate $a \in A$ with $a \equiv_B b$, namely $a = \sigma(a_1, \ldots, a_n) \equiv_n \sigma(c_1, \ldots, c_n) \equiv_n \sigma(b_1, \ldots, b_n) = b$.

Thus membership in B is decidable, and hence $\equiv_B$ is decidable: given b, $c \in B$ we effectively produce $a, d \in A$ with $b \equiv_B a$, $c \equiv_B d$ and then $b \equiv_B c$ if and only if $a \equiv_A d$, and the latter is decidable.

Finally, given an n-ary operation σ and $b_1, \ldots, b_n \in B$, there exist $a_1, \ldots, a_n \in B$ with $a_i \equiv_B b_i$ and $\sigma(a_1, \ldots, a_n) \in B$, if and only if $\sigma(b_1, \ldots, b_n) \in B$, and we have just proved that the latter is decidable.

Thus $(B, \equiv_B)$ is a partial subalgebra with all the properties claimed above.

Now, each element $s \in FX$ can effectively be written as $s = s'(u_1, \ldots, u_n)$ where the $u_i \in B$ are subterms which are maximal with respect to belonging to B.

Define a relation $\equiv$ on FX as follows: for $s = s'(u_1, \ldots, u_n)$ and $t = t'(v_1, \ldots, v_k)$, where the u_i, v_j are maximal B-subterms,

$s \equiv t$ if and only if $s' = t'$ and $u_i \equiv_B v_i$ for all i.

Then $\equiv$ is a congruence on FX, which extends $\equiv_A$ and is generated by it, and so is the congruence on FX generated by $\equiv_A$. Moreover, the above description of $\equiv$, together with the decidability of $\equiv_B$, yields the decidability of $\equiv$, as required.

§7. A VARIETY WITH INFINITELY MANY OPERATIONS

If we allow infinitely many operations, then it is much easier to obtain a variety with solvable, but not uniformly solvable, word problem. The following example is a modification of an example given in Wells [W, p.161] for a different, although related purpose. He suggested its relevance to our question.

Let V be the variety with a constant, 0, a binary operation denoted by juxtaposition, and countably many unary operations h_n $(n \in \omega)$ satisfying the following laws:

$xy \approx yx$
$x(yz) \approx (xy)z$
$x0 \approx 0$
$x^2 \approx 0$
$xh_n(y) \approx 0$ for all $n \in \omega$
$h_n(h_n(x)) \approx h_n(x)$ for all $n \in \omega$
$h_n(h_k(x)) \approx 0$ for all $n \neq k$

and

$(*)$ ${h_{m_n}}^n(x_1x_2\cdots x_{m_n}) \approx 0.$

where $\{m_n | n \in \mathbb{N}\}$ is a recursive listing of a non-recursive set X.

Thus V is a variety of commutative, square-zero semigroups with countably many idempotent unary operations, and the above is a recursive set of equations defining V. It is worth commenting on why the system of equations is recursive. Obviously the only problem is identifying when an equation is included in the scheme $(*)$. Now the equations in $(*)$ are of the form ${h_j}^k(x_1\cdots x_j) \approx 0$. Such an equation is in $(*)$ if and only if $j = m_k$. The trick of using ${h_{m_n}}^n(x_1\cdots x_{m_n})$ rather than $h_{m_n}(x_1\cdots x_{m_n})$ is a variant of the old trick of pleonasm due to Craig which he used to prove that any theory with a recursively enumerable axiomatization has a recursive axiomatization (see Monk [7, p.262]) .

We will show that V has an undecidable equational theory, and hence does not have uniformly solvable word problem, by establishing that V satisfies the equation $h_k(x_1\cdots x_k) \approx 0$ if and only if $k \in X$. One direction is trivial by the laws above. To complete the proof of undecidibility, we construct an algebra in which for $k \notin X$, h_k is non-zero on a product of k elements.

Let S be the free algebra on countably many generators in the class of commutative semigroups with 0 satisfying $x^2 = 0$. Let $\{a_i | i \in \mathbb{N}\}$ be a countable set disjoint from S, and let $A = S \cup \{a_i | i \in \mathbb{N}\}$. Define the operations in A as follows: the binary multiplication extends that of S, and otherwise is constant with value 0.

For $n \in X$, h_n is constant with value 0. For $n \notin X$,

$$h_n(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or } x = a_i \text{ for some } i \neq n \\ a_n & \text{otherwise} \end{cases} .$$

It is easy to check that this algebra has the desired properties.

Now, to see that V has solvable word problem, consider a finite presentation $\mathcal{P}$ in generators $b_1, \ldots, b_n$. Let m be greater than n, and greater than k for any k such that h_k appears in one of the defining relations of $\mathcal{P}$. Let B be the algebra given by the presentation $\mathcal{P}$ in the variety V', which has operations 0, multiplication, and h_i for $i \leq m$, and is defined by the laws defining V which involve only the h_i for $i \leq m$. Then B is finite, and hence the word problem for $\mathcal{P}$ relative to the variety V' is decidable. Let $C \subseteq B$ consist of all non-zero elements of B which are not the image of any h_i ($i \leq m$). Then the algebra A given by the presentation $\mathcal{P}$ in the variety V has as underlying set $B \cup (C \times \{i \in \mathbb{N} | i > m\})$; the multiplication extends that of B and otherwise has value 0, the h_i for $i \leq m$ extend those of B and otherwise have value 0, and for $i > m$ and $c \in C$, $h_i(c) = (c, i) = h_i((c, i))$ and h_i has value 0 otherwise. The equations for $i > m$ are satisfied in A because all products $x_1 \cdots x_k$ for $k > m$ are 0. (It is a simple exercise to show that the laws imply $h_k(0) \approx 0$.) This explicit description of A yields a solution to the word problem for $\mathcal{P}$ relative to the variety V.

REFERENCES.

[1] S. Burris and H. Sankapannavar. A Course in Universal Algebra Springer-Verlag, 1981.

[2] T. Evans. The Word Problem for Abstract Algebras. J. London Math. Society, 26 (1951) 64–71.

[3] T.Evans. Embeddability and the Word Problem. J. London Math. Society 28 (1953) 76–80.

[4] T. Evans. Some Solvable Word Problems. in Word Problems II ed. by S. I. Adian, W. W. Boone and G. Higman. North Holland (1980) 87–100.

[5] G. Hutchinson. Recursively unsolvable word problems of modular lattices and diagram chasing. J. Alg. 26 (1973) 385–399.

[6] A. Markov. On the impossibility of certain algorithms in the theory of associative systems. Dokl. Akad. Nauk. SSSR (NS) 55 (1947) 583–586.
[7] D. Monk. Mathematical Logic. Springer-Verlag 1976.
[8] P.S. Novikov. On the Algorithmic unsolvability of the word problem in group theory. Trudy Mat. Inst. Steklov 44, English Translation Proc. Steklov Inst. Math. (2) 9 (1958) 1–122.
[9] E. Post. Recursive unsolvability of a problem of Thue. Journal of Symbolic Logic. 12 (1947) 1–11.
[10] B. L. van der Waerden. Geometry and Algebra in Ancient Civilizations. Springer-Verlag 1983.
[11] B. Wells. Pseudorecursive varieties and their implications for word problems Ph.D. thesis, University of California, Berkeley 1982.