Abstract. We shall deal comprehensively with Black Boxes, the intention being that provably in ZFC we have a sequence of guesses of extra structure on small subsets, the guesses are pairwise with quite little interaction, are far but together are “dense”. We first deal with the simplest case, were the existence comes from winning a game by just writing down the opponent’s moves. We show how it help when instead orders we have trees with boundedly many levels, having freedom in the last. After this we quite systematically look at existence of black boxes, and make connection to non-saturation of natural ideals and diamonds on them.
§ 0. Introduction

The non-structure theorems we have discussed in [Sh:E59] rests usually on some freedom on finite sequences and on a kind of order. When our freedom is related to infinite sequences, and to trees, our work is sometimes harder. In particular, we may consider, for \( \lambda \geq \chi, \chi \) regular, and \( \varphi = \varphi(\bar{x}_0, \ldots, \bar{x}_\alpha, \ldots)_{\alpha < \chi} \) in a vocabulary \( \tau \):

\[ (*) \text{ For any } I \subseteq \chi \geq \lambda \text{ we have a } \tau \text{-model } M_I \text{ and sequences } \bar{a}_\eta \text{ (for } \eta \in \chi > \lambda), \]

such that for \( \eta \in \chi \lambda \) we have:

\[ M_I \models \varphi(\ldots, \bar{a}_\eta \upharpoonright \alpha, \ldots)_{\alpha < \chi} \text{ if and only if } \eta \in I. \]

(Usually, \( M_I \) is to some extend “simply defined” from \( I \)). Of course, if we do not ask more from \( M_I \), we can get nowhere: we certainly restrict its cardinality and/or usually demand it is \( \varphi \)-representable (see Definition [Sh:E59, 2.4] clauses (c),(d)) in (a variant of) \( \mathcal{M}_{\mu, \kappa}(I) \) (for suitable \( \mu, \kappa \)). Certainly for \( T \) unsuperstable we have such a formula \( \varphi \):

\[ \varphi(\ldots, \bar{a}_{\eta \upharpoonright n}, \ldots) = (\exists \bar{x}) \bigcup_n \varphi_n(\bar{x}, \bar{a}_{\eta \upharpoonright n}). \]

There are many natural examples.

Formulated in terms of the existence of \( I \) for which our favorite “anti-isomorphism” player has a winning strategy, we prove this in 1969/70 (in proofs of lower bounds of \( \aleph^\omega (\lambda, T_1, T) \), \( T \) unsuperstable), but it was shortly superseded. However, eventually the method was used in one of the cases in [Sh:a, Ch.VIII,§2]: for strong limit singular [Sh:a, Ch.VIII,2.6]. It was developed in [Sh:172], [Sh:227] for constructing Abelian groups with prescribed endomorphism groups. See further a representation of one of the results here in Eklof-Mekler [EM90], [EM02] a version which was developed for a proof of the existence of Abelian (torsion free \( \aleph_1 \)-free) group \( G \) with

\[ G^{\omega \omega} = G \oplus A \quad (G^* := \text{Hom}(G, \mathbb{Z})) \]

in a work by Mekler and Shelah. A preliminary version of this paper appeared in [Sh:300, Ch.III,§4,§5] but §3 here was just almost ready, and §4 on partitions of stationary sets and \( \Diamond_D \) was written up as a letter to Foreman in the late nineties answering his question on what I know on this.
§ 1. The Easy Black Box and an Easy Application

In this section we do not try to get the strongest results, just provide some examples (i.e., we do not present the results when \( \lambda = \lambda^\chi \) is replaced by \( \lambda = \lambda^{<\chi} \)). By the proof of [Sh:a, Ch.VIII,2.5] (see later for a complete proof):

**Theorem 1.1.** Suppose that

\[
\begin{align*}
(\ast) \quad & (a) \; \lambda = \lambda^\chi \\
& (b) \; \text{a vocabulary } \varphi = \varphi(x_0, x_1, \ldots, x_{\alpha < \chi}) \text{ is a formula in } \mathcal{L}(\tau) \text{ for some logic } \mathcal{L} \\
& (c) \; \text{For any } I \subseteq \chi^>\lambda \text{ we have a } \tau\text{-model } M_I \text{ and sequences } \bar{a}_\eta \text{ (for } \eta \in \chi^{>\lambda}), \text{ where} \\
& \quad \quad [\eta \triangleleft \nu \Rightarrow \bar{a}_\eta \neq \bar{a}_\nu], \quad \ell g(\bar{a}_\eta) = \ell g(\bar{x}_{\ell g(\eta)}), \\
& \quad \quad \text{such that for } \eta \in \chi^\lambda \text{ we have:} \\
& \quad \quad M_I \models \varphi(\ldots, \bar{a}_\eta|\alpha, \ldots)_{\alpha < \chi} \text{ if and only if } \eta \subseteq I \\
& (c) \; \| M_I \| = \lambda \text{ for every } I \text{ satisfying } \chi^{>\lambda} \subseteq I \subseteq \chi^{\leq \lambda}, \text{ and } \ell g(\bar{a}_\eta) \leq \chi \text{ or just } \lambda^{g(\eta)} = \lambda.
\end{align*}
\]

Then (using \( \chi^{>\lambda} \subseteq I \subseteq \chi^{\geq \lambda} \)):
1) There is no model \( M \) of cardinality \( \lambda \) into which every \( M_I \) can be \( (\pm \varphi) \)-embedded (i.e., by a function preserving \( \varphi \) and \( \neg \varphi \)).
2) For any \( M_i \) (for \( i < \lambda \)), \( \| M_i \| \leq \lambda \), for some \( I \) satisfying \( \chi^{>\lambda} \subseteq I \subseteq \chi^{\geq \lambda} \), the model \( M_I \) cannot be \( (\pm \varphi) \)-embedded into any \( M_i \).

**Example 1.2.** Consider the class of Boolean algebras and the formula

\[
\varphi(\ldots, x_n, \ldots) := (\bigcup_n x_n) = 1
\]

(i.e., there is no \( x \neq 0 \) such that \( x \cap x_n = 0 \) for each \( n \)). For \( \omega^{>\lambda} \subseteq I \subseteq \omega^{\geq \lambda} \), let \( M_I \) be the Boolean algebra generated freely by \( x_\eta \) (for \( \eta \in I \)) except the relations: for \( \eta \in I \), if \( n < \ell g(\eta) = \omega \) then \( x_\eta \cap x_{n|\eta} = 0 \).

So \( \| M_I \| = |I| \in [\lambda, \lambda^{\aleph_0}] \) and in \( M_I \) for \( \eta \in \omega^{>\lambda} \) we have: \( M_I \models (\bigcup_n x_{n|\eta}) = 1 \) if and only if \( \eta \notin I \) (work a little in Boolean algebras).

**Conclusion 1.3.** If \( \lambda = \lambda^{\aleph_0} \), then there is no Boolean algebra \( B \) of cardinality \( \lambda \) universal under \( \sigma \)-embeddings (i.e., ones preserving countable unions).

**Remark 1.4.** This is from [Sh:a, Ch.VIII,Ex.2.5,pg.464].

**Proof of the Theorem 1.1.** First we recall the simple black box (and a variant) in 1.5, 1.6 below:

**The Simple B.B. Lemma 1.5.** There are functions \( f_\eta \) (for \( \eta \in \chi^\lambda \)) such that:

\[
\begin{align*}
(i) \; & \text{Dom}(f_\eta) = \{ \eta \mid \alpha : \alpha < \chi \}, \\
(ii) \; & \text{Rang}(f_\eta) \subseteq \lambda,
\end{align*}
\]
(iii) if \( f : \kappa^+ \lambda \to \lambda \), then for some \( \eta \in \kappa^\lambda \) we have \( f_\eta \subseteq f \).

Proof. For \( \eta \in \kappa^\lambda \) let \( f_\eta \) be the function (with domain \( \{ \eta|\alpha : \alpha < \chi \} \)) such that:

\[
f_\eta(\eta|\alpha) = \eta|\alpha.
\]

So \( \{ f_\eta : \eta \in \kappa^\lambda \} \) is well defined. Properties (i), (ii) are straightforward, so let us prove (iii). Let \( f : \kappa^+ \lambda \to \lambda \). We define \( \eta_\alpha = \{ \beta_i : i < \alpha \} \) by induction on \( \alpha \).

For \( \alpha = 0 \) or \( \alpha \) limit — no problem.

For \( \alpha + 1 \): let \( \beta_\alpha \) be the ordinal such that \( \beta_\alpha = f(\eta_\alpha) \).

So \( \eta = : \{ \beta_i : i < \chi \} \) is as required. \( \square \)

**Fact 1.6.** In 1.5:

(a) we can replace the range of \( f, f_\eta \) by any fixed set of power \( \lambda \),

(b) we can replace the domains of \( f, f_\eta \) by \( \{ \tilde{a}_\eta : \eta \in \kappa^\lambda \}, \{ \tilde{a}_\eta|\alpha : \alpha < \chi \} \), respectively, as long as

\[
\alpha < \beta < \chi \land \eta \in \kappa^\lambda \implies \tilde{a}_\eta|\alpha \neq \tilde{a}_\eta|\beta.
\]

Remark 1.7. We can present it as a game. (As in the book [Sh:a, Ch.VIII,2.5]).

**Continuation of the Proof of Theorem 1.1.**

It suffices to prove 1.1(2). Without loss of generality \( (|M_i| : i < \lambda) \) are pairwise disjoint. Now we use 1.6; for the domain we use \( \Gamma \). By the choice of 1.6.

\[1.7\]

Remark. We define

\[
I = (\kappa^\lambda) \cup \{ \eta \in \kappa^\lambda : \text{for some } i < \lambda, \text{Rang}(f_\eta) \text{ is a set of sequences from } |M_i| \text{ and } M_i \models \neg\phi(\ldots, f_\eta(\tilde{a}_\eta|\alpha), \ldots)_{\alpha < \chi} \}.
\]

Look at \( M_I \). It suffices to show:

\( \otimes \) for \( i < \lambda \) there is no \( (\pm \varphi) \)-embedding of \( M_I \) into \( M_i \).

Why does \( \otimes \) hold?

If \( f : M_I \to M_i \) is a \( (\pm \varphi) \)-embedding, then by Fact 1.6, for some \( \eta \in \kappa^\lambda \) we have

\[
f|\{ \tilde{a}_\eta|\alpha : \alpha < \kappa \} = f_\eta.
\]

By the choice of \( f \),

\[
M_I \models \varphi[\ldots, \tilde{a}_\eta|\alpha, \ldots]_{\alpha < \chi} \iff M_i \models \varphi[\ldots, f(\tilde{a}_\eta|\alpha), \ldots]_{\alpha < \chi},
\]

but by the choice of \( I \) and \( M_I \) we have

\[
M_I \models \varphi[\ldots, \tilde{a}_\eta|\alpha, \ldots]_{\alpha < \chi} \iff M_i \models \neg\varphi[\ldots, f_\eta(\tilde{a}_\eta|\alpha), \ldots]_{\alpha < \chi}.
\]

A contradiction, as by the choice of \( \eta \),

\[
\bigwedge_{\alpha < \chi} f(\tilde{a}_\eta|\alpha) = f_\eta(\tilde{a}_\eta|\alpha).
\]
Discussion 1.8. We may be interested whether in 1.1, when $\lambda^+ < 2^\lambda$, we may allow in (1) $\|M\| = \lambda^+$, and/or

(a) get $\geq \lambda^{++}$ non-isomorphic models of the form $M_I$, assuming $2^\lambda > \lambda^+$.

The following lemma shows that we cannot prove those better statements in ZFC, though (see 1.11) in some universes of set theory we can. So this require (elementary) knowledge of forcing, but is not used later. It is here just to justify the limitations of what we can prove and the reader can skip it.

Lemma 1.9. Suppose that in the universe $V$ we have $\kappa < \lambda = \text{cf}(\lambda) = \lambda^{<\lambda}$ and $(\forall \lambda_1 < \lambda)[\lambda_1^\kappa < \lambda]$ and $\lambda < \mu = \mu^\kappa$.

Then for some notion forcing $\mathbb{P}$:

(a) $\mathbb{P}$ is $\lambda$–complete and satisfies the $\lambda^+–c.c.$, and $\|\mathbb{P}\| = \mu$, $\|\mathbb{P} \| \not

\text{2}\lambda = \mu^\kappa$ (so forcing with $\mathbb{P}$ collapses no cardinals, changes no cofinalities, adds no new sequences of ordinals of length $< \lambda$, and $\|\mathbb{P} \| \not= \lambda^{<\lambda} = \lambda^+$).

(b) We can find $\varphi, M_I$ (for $\kappa > \lambda \subseteq I \subseteq \kappa^\geq \lambda$) as in (a) of ??, so with $\|M_I\| = \lambda(\tau–models with $|\tau| = \kappa$ for simplicity) such that:

+ there are up to isomorphism exactly $\lambda^+$ models of the form $M_I$ ($\kappa > \lambda \subseteq I \subseteq \lambda^\geq \lambda$).

(c) In (b), there is a model $M$ such that $\|M\| = \lambda^+$ and every model $M_I$ can be ($\pm \varphi$)–embedded into $M$.

Remark 1.10. 1) Essentially $M_I$ is $(I^+, <)$, the addition of level predicates is immaterial, where $I^+$ extends $I$ “nicely” so that we can let $a_0 = \eta$ for $\eta \in I$.

2) Clearly clause (c) also shows that weakening $\|M\| = \lambda$, even when $\lambda^+ < 2^\lambda$ may make 1.1 false.

Proof. Let $\tau = \{ R_\zeta : \zeta \leq \kappa \} \cup \{ < \}$ with $R_\zeta$ being a monadic predicate, and $<$ being a binary predicate. For a set $I$, $\kappa^> I \subseteq I \subseteq \kappa^\geq \lambda$ let $N_I$ be the $\tau$–model:

$$|N_I| = I, R_\zeta^{N_I} = I \cap \zeta, <^{N_I} = \{ (\eta, \nu) : \eta \in I, \nu \in I, \eta < \nu \},$$

and

$$\varphi(\ldots, x_\zeta, \ldots)_{\zeta<\kappa} = \bigwedge_{\zeta<\xi<\kappa} (x_\xi < x_\zeta & R_\xi(x_\zeta)) \wedge (\exists y)[R_\xi(y) \& \bigwedge_{\zeta<\kappa} x_\zeta < y].$$

Now we define the forcing notion $\mathbb{P}$. It is $\mathbb{P}^\lambda+$, where

$$(\mathbb{P}_i, Q_j : i \leq \lambda^+, j < \lambda^+)$$

is an iteration with support $< \lambda$, of $\lambda$–complete forcing notions, where $Q_j$ is defined as follows.

For $j = 0$ we add $\mu$ many Cohen subsets to $\lambda$:

$$Q_0 = \{ f : f \text{ is a partial function from } \mu \text{ to } \{0, 1\}, |\text{Dom}(f)| < \lambda \},$$

the order is the inclusion.
For $j > 0$, we define $Q_j$ in $V^{P_j}$. Let $(I(j, \alpha) : \alpha < \alpha(j))$ list all sets $I \in V^{P_j}$, $\kappa > \lambda \subseteq I \subseteq \kappa \lambda$ (note that the interpretation of $\kappa \lambda$ does not change from $V$ to $V^{P_j}$ as $\kappa < \lambda$ but the family of such $I$-s increases).

Now

$$Q_j = \left\{ f : f = \langle f_\alpha : \alpha < \alpha(j) \rangle, f_\alpha \text{ is a partial isomorphism from } N_{I(j, \alpha)} \text{ into } N_{\kappa \gamma \lambda}, \right.$$ \[w(f) = \{ \alpha : f_\alpha \neq \emptyset \} \text{ has cardinality } \lambda, \]
$$\text{Dom}(f_\alpha) \text{ has the form } \bigcup_{\beta < \gamma} \kappa \beta \cap N_{I(j, \alpha)} \text{ for some } \gamma \leq \lambda;$$ \[\text{and if } \alpha_1, \alpha_2 < \alpha(j) \text{ and } \eta_1, \eta_2 \subseteq \kappa \lambda, \text{ and for every } \zeta < \kappa, \]
$$f_{\alpha_1}(\eta_1 \zeta \zeta), f_{\alpha_2}(\eta_2 \zeta \zeta) \text{ are defined and equal, then }$$
$$\eta_1 \in I(j, \alpha_1) \iff \eta_2 \in I(j, \alpha_2) \right\}.$$

The order is:

$$f^1 \leq f^2 \quad \text{if and only if} \quad (\forall \alpha < \alpha(j))(f^1_\alpha \subseteq f^2_\alpha) \text{ and}$$
$$\text{for all } \alpha < \beta < \alpha(j), f^1_\alpha \neq \emptyset \land f^2_\beta \neq \emptyset \implies$$
$$\text{Rang}(f^1_\alpha) \cap \text{Rang}(f^2_\beta) = \text{Rang}(f^1_\alpha) \cap \text{Rang}(f^2_\beta).$$

Then, $Q_j$ is $\lambda$-complete and it satisfies the version of $\lambda^\tau$-c.c. from [Sh:80] (see more [Sh:546]), hence each $P_j$ satisfies the $\lambda^\tau$-c.c. (by [Sh:80]).

Now the $P_{j+1}$-name $I_j$, (interpreting it in $V^{P_{j+1}}$ we get $I^* e_j$) is:

$$I^*_j = \kappa \lambda \cup \{ \eta \in \kappa \lambda : \text{for some } f \in G_{Q_j}, \alpha < \alpha(j) \text{ and } \nu \in N_{I(j, \alpha)},$$
$$\ell f(\nu) = \kappa \text{ and } f_\alpha(\nu) = \eta \}. \phantom{\text{for some } f \in G_{Q_j}, \alpha < \alpha(j) \text{ and } \nu \in N_{I(j, \alpha)},}$$
$$\text{Rang}(f^1_\alpha) \cap \text{Rang}(f^2_\beta) = \text{Rang}(f^1_\alpha) \cap \text{Rang}(f^2_\beta).$$

This defines also $f^1_\alpha : I(j, \alpha) \rightarrow I^*_j$, which is forced to be a $(\pm \varphi)$-embedding and also just an embedding.

So now we shall define for every $I, \kappa \lambda \subseteq I \subseteq \kappa \lambda$, a $\tau$–model $M_I$: clearly $I$ belongs to some $V^{P_j}$. Let $j = j(I)$ be the first such $j$, and let $\alpha = \alpha(I)$ be such that $I = I(j, \alpha)$. Let $M_{I(j, \alpha)} = M^*_j$ (and $a_\rho = f^1_\alpha(\rho)$ for $\rho \in I(j, \alpha)$).

We leave the details to the reader. \qquad \Box_{1.9}

On the other hand, consistently we may easily have a better result.

Lemma 1.11. Suppose that, in the universe $V$,

$$\lambda = \text{cf}(\lambda) = \lambda^\kappa = \lambda^{< \lambda}, \quad \lambda < \mu = \mu^\lambda.$$

For some forcing notion $P$:

(a) as in 1.9

(b) in $V^P$, assume that $\varphi$ and the function $I \mapsto (M_I, \langle a^I_\eta : \eta \in \kappa^\mu \lambda \rangle)$ are as required in clauses (a),(b),(c) of (1.1), $\zeta(\ast) < \mu$, and $N_\zeta$ (for $\zeta < \zeta(\ast)$) is a model in the relevant vocabulary, \[\sum \| N_\zeta \|^\mu < \mu \text{ (if the vocabulary}\]
$$\text{is of cardinality } < \lambda \text{ and each predicate or relation symbol has finite arity, then requiring just } \sum \{ | N_\zeta | : \zeta < \zeta(\ast) \} < \mu \text{ suffices}. \text{ Then for some } I, \text{ the model } M_I \text{ cannot be } (\pm \varphi)–\text{embedded into any } N_\zeta.$$

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\bibitem{1.9} (309)

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(c) Assume $\mu_1 = \text{cf}(\mu_1), \lambda < \mu_1 \leq \mu$ and $V \models (\forall \chi < \mu_1)[\chi^\lambda < \mu_1]$. Then in $V^P$, if $\langle M_i : i < \mu_1 \rangle$ are pairwise non-isomorphic, $\kappa_1^\lambda \subseteq I_i \subseteq \kappa_2^\lambda$, and $M_i, a^i_\eta (\eta \in I_i)$ are as in $(*)$ of ??, then for some $i \neq j, M_i$ is not embeddable into $M_j$.

(d) In $V^P$ we can find a sequence $\langle I_\zeta : \zeta < \mu \rangle$ (so $\kappa_1^\lambda \subseteq I_\zeta \subseteq \kappa_2^\lambda$) such that the $M_i$’s satisfy that no one is $(\pm \varphi)$-embeddable into another.

Proof. $P$ is $\mathbb{Q}_0$ from the proof of 1.9. Let $F$ be the generic function that is $\cup \{ f : f \in G_{\mathbb{Q}_0} \}$, clearly it is a function from $\mu$ to $\{0, 1\}$. Now clause (a) is trivial.

Next, concerning clause (b), we are given $\langle N_\zeta : \zeta < \zeta(*) \rangle$. Clearly for some $A \in V$ of size smaller than $\mu, A \subseteq \mu$, to compute the isomorphism types of $N_\zeta$ (for $\zeta < \zeta(*)$) it is enough to know $F | A$. We can force by $\{ f \in \mathbb{Q}_0 : \text{Dom}(f) \subseteq A \}$, then $f | B$ for any $B \subseteq \lambda \setminus A$ of cardinality $\lambda$, (from $V$) gives us an $I$ as required.

To prove clause (c) use $\Delta$-system argument for the names of various $M_i$’s.

The proof of (d) is like that of (c). \[\square_{1.11}\]
§ 2. AN APPLICATION FOR MANY MODELS IN $\lambda$

Discussion 2.1. Next we consider the following:

Assume $\lambda$ is regular, $(\forall \mu < \lambda)[\mu < \chi < \lambda]$. Let $\mathcal{U}_\alpha \subseteq \{\delta < \lambda : \text{cf}(\delta) = \chi\}$ for $\alpha < \lambda$ be pairwise disjoint stationary sets.

For $A \subseteq \lambda$, let

$$\mathcal{U}_A = \bigcup_{i \in A} \mathcal{U}_i.$$ 

We want to define $I_A$ such that $\chi > \lambda \subseteq I_A \subseteq \chi \geq \lambda$ and

$$A \not\subseteq B \Rightarrow M_{I_A} \not\cong M_{I_B}.$$ 

We choose $\langle M_{i_{I_A}} : i < \lambda \rangle$ with $M_{i_{I_A}} = \bigcup_{i < \lambda} M_{i_{I_A}}$, $\|M_{i_{I_A}}\| < \lambda$, $M_{i_{I_A}}$ increasing continuous.

Of course, we have to strengthen the restrictions on $M_{i_{I_A}}$. For $\eta \in I_A \cap \chi$, let $\delta(\eta) = \{\eta(i) + 1 : i < \chi\}$, we are specially interested in $\eta$ such that $\eta$ is strictly increasing converging to some $\delta(\eta) \in \mathcal{U}_A$; we shall put only such $\eta$'s in $I_A$. The decision whether $\eta \in I_A$ will be done by induction on $\delta(\eta)$ for all sets $A$.

Arriving to $\eta$, we assume we know quite a lot on the isomorphism $f : M_{i_{I_A}} \rightarrow M_{i_{I_B}}$, specially we know $f|\bigcup_{\alpha < \chi} \bar{x}_{i_{I_A}}|\alpha$, which we are trying to “kill”, and we can assume $\delta(\eta) \notin \mathcal{U}_B$ and $\delta$ belongs to a thin enough club of $\lambda$ and using all this information we can “compute” what to do.

Note: though this is the typical case, we do not always follow it.

Notation 2.2. 1) For an ordinal $\alpha$ and a regular $\theta \geq \aleph_0$, let $\mathcal{H}_{<\theta}(\alpha)$ be the smallest set $Y$ such that:

(i) $i \in Y$ for $i < \alpha$,

(ii) $x \in Y$ for $x \subseteq Y$ of cardinality $< \theta$.

2) We can agree that $\mathcal{M}_{\alpha}(\alpha)$ from [Sh:E59, §2] is interpretable in $(\mathcal{H}_{<\theta}(\alpha), \in)$ when $\alpha \geq \lambda$, and in particular its universe is a definable subset of $\mathcal{H}_{<\theta}(\alpha)$, and also $R$ is, where:

$$R = \{(\sigma^*, (t_i : i < \gamma_\alpha), x) : x \in \mathcal{M}_{\alpha, \theta}^\alpha, \sigma^* \text{ is a } \tau_{\alpha, \kappa} - \text{term and } \theta \leq \lambda \leq \alpha, x = \sigma^*(\langle t_i : i < \gamma_\alpha\rangle)\}.$$ 

Similarly $\mathcal{M}_{\alpha, \theta}(I)$, where $I \subseteq ^\kappa \lambda$ is interpretable in $(\mathcal{H}_{<\chi}(\lambda^*), \in)$ if $\lambda \leq \lambda^*$, $\theta \leq \chi, \kappa \leq \chi$.

The main theorem of this section is:

Theorem 2.3. $\hat{I}E_{\kappa, \varphi}(\lambda, K) = 2^\lambda$, provided that:

(a) $\lambda = \lambda^\chi$,

(b) $\varphi = \varphi(\ldots, \bar{x}_{\alpha}, \ldots)_{\alpha < \chi}$ is a formula in the vocabulary $\tau_K$. 


(c) for every $I$ such that $\chi^\beta \subseteq I \subseteq \chi^\lambda$ we have a model $M_I \in K_\lambda$ and a function $f_I$, and $\bar{a}_\eta \in \chi^\lambda | M_I|$ for $\eta \in \chi^\beta$ with $\ell g(\bar{a}_\eta) = \ell g(\bar{a}_{\ell g(\eta)})$ such that:

$(\alpha)$ for $\eta \in \chi^\lambda$ we have $M_I \models \varphi(\ldots, \bar{a}_{\ell g(\eta)}, \ldots)$ if and only if $\eta \in I$,

$(\beta)$ $f_I : M_I \to \mathcal{A}_{\mu, \alpha}(I)$, where $\mu \leq \lambda$, $\kappa = \chi^\beta$, and:

$(d)$ for $I$, $\chi^\beta \subseteq I \subseteq \chi^\lambda$ and $b_\alpha \in M_I$, $\ell g(\bar{a}_\eta) = \ell g(b_\alpha)$ for $\alpha < \chi$, $f_I(b_\alpha) = \sigma_\alpha(f_\alpha)$ we have: the truth value of $M_I \models \varphi(\ldots, b_\alpha, \ldots)$ can be computed from $\langle \sigma_\alpha : \alpha < \chi \rangle$, $\langle f_\alpha : \alpha < \chi \rangle$ (not just its q.f. type in $I$) and the truth values of statements of the form

$$\left( \exists \nu \in I \cap \chi^\lambda \right) \left[ \bigwedge_{\eta < \chi} \nu | \epsilon_i = \ell g(\eta_i) | \epsilon_i \right]$$

for $\alpha_i, \beta_i, \gamma_i, \epsilon_i < \chi$ (i.e., in a way not depending on $I, f_I$) [we can weaken this].

We shall first prove 2.3 under stronger assumptions.

**Fact 2.4.** Suppose

$(*) \lambda = \lambda^\chi$, (so $\text{cf}(\lambda) > \chi)$ and $\chi \geq \kappa$. Then there are $\langle (M^\alpha, \eta^\alpha) : \alpha < \alpha(*) \rangle$ such that:

$(i)$ for every model $M$ with universe $\mathcal{H}_{\chi^+}(\lambda)$ such that $|\tau(M)| \leq \chi$ (and, e.g., $\tau \subseteq \mathcal{H}_{\chi^+}(\lambda)$), for some $\alpha$ we have $M^\alpha \prec M$,

$(ii)$ $\eta^\alpha \in \chi^\lambda$, $(\forall i < \chi) |\eta^\alpha[i] \in M^\alpha|$, $\eta^\alpha \notin M^\alpha$, and $\alpha \neq \beta \Rightarrow \eta^\alpha \neq \eta^\beta$,

$(iii)$ for every $\beta < \alpha(*)$ we have: $\{\eta^\alpha[i] : i < \chi\} \not\subseteq M^\beta$,

$(iv)$ for $\beta < \alpha$ if $\{\eta^\beta[i] : i < \chi\} \subseteq M^\alpha$, then $|M^\beta| \subseteq |M^\alpha|$, and

$(v)$ $||M^\alpha|| = \chi$.

**Proof.** By 3.20 + 3.21 below with $\lambda, \lambda^\chi, \chi$ here standing for $\lambda, \chi(*)$, $\theta$ there.

**Proof of 2.3 from the Conclusion of 2.4.**

Without loss of generality the universe of $M_I$ is $\lambda$ in 2.3.

We shall define for every $A \subseteq \lambda$ a set $I[A]$ satisfying $\chi^\beta \lambda \subseteq I[A] \subseteq \chi^\lambda$, moreover

$$I[A] \setminus \chi^\beta \lambda \subseteq \{ \eta^\alpha : \alpha < \alpha(*) \}.$$  

For $\alpha < \alpha(*)$, let $\mathcal{U}_\alpha = \{ \eta \in \chi^\lambda : \{ \eta[i] : i < \chi \} \not\subseteq M^\alpha \}$. We shall define by induction on $\alpha$, for every $A \subseteq \lambda$ the set $I[A] \cap \mathcal{U}_\alpha$ so that on the one hand those restrictions are compatible (so that we can define $I[A]$ in the end, for each $A \subseteq \lambda$), and on the other hand they guarantee the non $\pm\varphi$-embeddability.

For each $\alpha$: (essentially we decide whether $\eta^\alpha \in I[A]$ assuming $M^\alpha$ “guesses” rightly a function $g : M_{I_1} \to M_{I_2}$ ($I_1 = I[A_i]$), and $A_\ell \cap M^\alpha$ for $\ell = 1, 2$, and we make our decision to prevent this)

Case I: there are distinct subsets $A_1, A_2$ of $\lambda$ and $I_1, I_2$ satisfying $\chi^\beta \lambda \subseteq I_\ell \subseteq \chi^\lambda$, and a $\pm\varphi$-embedding $g$ of $M_{I_1}$ into $M_{I_2}$ and

$$M^\alpha < (\mathcal{H}_{\chi^+}(\lambda), \in, R, A_1, A_2, I_1, I_2, M_{I_1}, M_{I_2}, f_{I_1}, f_{I_2}, g),$$

where
\[ R = \{ \{(0, \sigma_x, x), (1 + i, t^x_i, x) \} : i < x \text{ and } x \text{ has the form } \sigma_x(\{t^x_i : i < x\}) \} \]

(we choose for each \( x \) a unique such term \( \sigma_x \)), and \( I_2 \cap \mathfrak{U}_\alpha \subseteq I_2 \cap (\bigcup_{\beta < \alpha} \mathfrak{U}_\beta) \), and \( I_1, I_2 \) satisfy the restrictions we already have imposed on \( I[A_1], I[A_2] \), respectively for each \( \beta < \alpha \). Computing according to clause (d) of 2.3 the truth value for \( M_{I_2} \models \varphi[\ldots, f(\bar{a}_{\varphi x_i})\ldots]_{i<\chi} \) we get \( t^\alpha \).

Then we restrict:

\( (i) \) if \( B \subseteq \lambda, B \cap |M^\alpha| = A_2 \cap |M^\alpha| \), then \( I[B] \cap (\mathfrak{U}^\alpha \setminus \bigcup_{\beta < \alpha} \mathfrak{U}^\beta) = \emptyset \),

\( (ii) \) if \( B \subseteq \lambda, B \cap |M^\alpha| = A_1 \cap |M^\alpha| \) and \( t^\alpha \) is true, then

\[ I[B] \cap (\mathfrak{U}^\alpha \setminus \bigcup_{\beta < \alpha} \mathfrak{U}^\beta) = \emptyset, \]

or just

\[ \eta^\alpha \notin I[B] \]

\( (iii) \) if \( B \subseteq \lambda, B \cap |M^\alpha| = A_1 \cap |M^\alpha| \) and \( t^\alpha \) is false, then

\[ I[B] \cap (\mathfrak{U}^\alpha \setminus \bigcup_{\beta < \alpha} \mathfrak{U}^\beta) = \{ \eta^\alpha \} \]

or just

\[ \eta^\alpha \in I[B] \]

Case II: quad Not I.

No restriction is imposed.

The point is the two facts below which should be clear. \[ \square_{2.4} \]

**Fact 2.5.** The choice of \( A_1, A_2, I_1, I_2, g \) is immaterial (any two candidates lead to the same decision).

**Proof.** Use clause (d) of 2.3. \[ \square_{2.5} \]

**Fact 2.6.** \( M_{I[A]} \) (for \( A \subseteq \lambda \)) are pairwise non-isomorphic. Moreover, for \( A \neq B \) (subsets of \( \lambda \)) there is no \((\pm \varphi)\)-embedding of \( M_{I[A]} \) into \( M_{I[B]} \).

**Proof.** By the choice of the \( I[A] \)’s and (i) of 2.4. \[ \square_{2.6} \]

\[ \ast \ast \ast \]

Still the assumption of 2.4 is too strong: it does not cover all the desirable cases, though it cover many of them. However, a statement weaker than the conclusion of 2.4 holds under weaker cardinality restrictions and the proof of 2.3 above works using it, thus we will finish the proof of 2.3.

**Fact 2.7.** Suppose \( \lambda = \lambda^\chi \).

Then there are \( \{(M^\alpha, A^\alpha_1, A^\alpha_2, \eta^\alpha) : \alpha < \alpha(\ast)\} \) such that:
\(\alpha = \frac{\lambda}{\lambda} \)

Proof. 1) By \([\text{Sh:E59, 1.10}]\) there is a template \(\Phi\) proper for (2.4) but now we have to use the "or just" version in (ii),(iii) there, \(\square\).

There is some information in \([\text{Sh:a, Ch.VIII, 3.46}]\).

Proof of 2.3: Should be clear, we act as in the proof of 2.3 from the conclusion of 2.4 but now we have to use the "or just" version in (ii),(iii) there, \(\square\).

Conclusion 2.8. 1) If \(T \subseteq T_1\) are complete first order theories, \(T\) in the vocabulary \(\tau, \kappa = \text{cf}(\kappa) < \kappa(T)\), hence \(T\) unsuperstable and \(\lambda = \lambda^{\aleph_0} \geq |T_1|\), then \(\tilde{I}_\tau(\lambda, T_1) = 2^\lambda\). \(\hat{I}_\tau\) — see Definition \([\text{Sh:E59, 1.2(2)}]\).

2) Assume \(\kappa = \text{cf}(\kappa)\), \(\Phi\) proper and almost nice for \(K^\kappa_\tau\), see \([\text{Sh:E59, 1.7}]\), \(\sigma^i (i \leq \kappa)\) finite sequence of terms, \(\tau \subseteq \tau_0, \varphi_1(x, y)\) first order in \(\mathbb{L}[\tau]\) and for \(\nu \in ^i \lambda, \eta \in \kappa\lambda, \nu < \eta\) we have \(\text{EM}(^{\nu(\lambda, \Phi)} \models \varphi_1(\sigma^\nu_1(x, y), \sigma^{i+1}(x, y, \nu)))\) holds if and only if \(\alpha = \eta(i)\).

Then

\[2^\lambda = |\{EM_\tau(S, \Phi) / \equiv; \kappa > \lambda \subseteq S \subseteq \kappa \lambda\}|.\]

Proof. 1) By \([\text{Sh:E59, 1.10}]\) there is a template \(\Phi\) proper for \(K^\kappa_\tau\), as required in part (2).

2) By 2.3. \(\square\).

Discussion 2.9. What about Theorem 2.3 in the case we assume only \(\lambda = \lambda^{<\chi}\)? There is some information in \([\text{Sh:a, Ch.VIII, 2}]\).

Of course, concerning unsuperstable \(T\), that is 2.8, more is done there: the assumption is just \(\lambda > |T|\).

Claim 2.10. In 2.3, we can restrict ourselves to \(I\) such that \(I_{\lambda, \chi}^0 \subseteq I \subseteq \chi \geq \lambda\), where

\[I_{\lambda, \chi}^0 = \chi \sup set \{\eta \in \chi \lambda : \eta(i) = 0 \text{ for every } i < \chi \text{ large enough}\}.\]

Proof. By renaming. \(\square\)
§ 3. BLACK BOXES

We try to give comprehensive treatment of black boxes, not few of them are useful in some contexts and some parts are redone here, as explained in §0, §1.

Note that “omitting countable types” is a very useful device for building models of cardinality $\aleph_0$ and $\aleph_1$. The generalization to models of higher cardinality, $\lambda$ or $\lambda^+$, usually requires us to increase the cardinality of the types to $\lambda$, and even so we may encounter problems (see [Sh:E60] and background there). Note we do not look mainly at the omitting type theorem per se, but its applications.

Jensen defined square and proved existence in $L$: in Facts 3.1 — 3.8, we deal with related just weaker principles which can be proved in ZFC. E.g., for $\lambda$ regular $> \aleph_1$, $\{\delta < \lambda^+ : \text{cf}(\delta) < \lambda\}$ is the union of $\lambda$ sets, each has square (as defined there).

You can skip them in first reading, particularly 3.1 (and later take references on belief).

Then we deal with black boxes. In 3.12 we give the simplest case: $\lambda$ regular $> \aleph_0$, $\lambda = \lambda^+ < \chi(\ast)$; really $\lambda < \theta = \lambda^+ < \chi(\ast)$ is almost the same. In 3.12 we also assume $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \theta\}$ is a good stationary set. In 3.16 we weaken this demand such that enough sets $S$ as required exists (provably in ZFC!). The strength of the cardinality hypothesis ($\lambda = \lambda < \chi(\ast)$, $\lambda^+ = \lambda < \chi(\ast)$) vary the conclusion.

In 3.14 – 3.17 we prepare the ground for replacing “$\lambda$ regular” by “$\text{cf}(\lambda) \geq \chi(\ast)$”, which is done in 3.18.

As we noted in §2, it is much nicer to deal with $(\bar{M}^\beta, \eta^\beta)$, this is the first time we deal with $\eta^\beta$, i.e., for no $\alpha < \beta$,

$$\{\eta^\beta | i < \theta\} \subseteq \bigcup_{i<\theta} M^\alpha_i.$$

In 3.20, 3.21 (parallel to 3.12, 3.18, respectively) we guarantee this, at the price of strengthening $\lambda^+ = \lambda < \chi(\ast)$ to

$$\lambda^+ = \lambda^{(1)} \chi(1) = \chi(\ast) + (< \chi(\ast))^\theta.$$

Later, in 3.46, we draw the conclusion necessary for section 2 (in its proof the function $h$, which may look redundant, plays the major role). This (as well as 3.20, 3.21) exemplifies how those principles are self propagating — better ones follow from the old variant (possibly with other parameters).

In 3.22 — 3.27 we deal with the black boxes when $\theta$ (the length of the game) is $\aleph_0$. We use a generalization of the $\Delta$-system lemma for trees and partition theorems on trees (see Rubin-Shelah [RuSh:117, §4], [Shb, Ch.XI] = [Shf, Ch.XI],[Sh:E62, 1.10=L1.7],[Sh:E62, 1.16=L1.15] and here the proof of 3.24; see history there, and 3.6). We get several versions of the black box — as the cardinality restriction becomes more severe, we get a stronger principle.

It would be better if we can use for a strong limit $\kappa > \aleph_0 = \text{cf}(\kappa)$,

$$\kappa^\aleph_0 = \sup \{\lambda : \text{for some } \kappa_n < \kappa \text{ and uniform ultrafilter } D \text{ on } \omega, \text{ cf}(\prod_{n<\omega} \kappa_n/D) = \lambda\}.$$

We know this for the uncountable cofinality case (see [Sh:111] or [Sh:g]), but then there are other obstacles. Now [Sh:355] gives a partial remedy, but lately by [Sh:400] there are many such cardinals.
In 3.41, 3.42 we deal with the case \( \mathrm{cf}(\lambda) \leq \theta \). Note that \( \mathrm{cf}(\lambda^{<\chi(\ast)}) \geq \chi(\ast) \) is always true, so you may wonder why wouldn’t we replace \( \lambda \) by \( \lambda^{<\chi(\ast)} \)? This is true in quite many applications, but is not true, for example, when we want to construct structures with density character \( \lambda \).

Several times, we use results quoted from [Sh:331, §2], but no vicious circle. Also, several times we quote results on pcf quoting [Sh:E62, §3]. We end with various remarks and exercises.

**Fact 3.1.** 1) If \( \mu^\chi = \mu < \lambda \leq 2^\mu, \chi \) and \( \lambda \) are regular uncountable cardinals, and \( S \subseteq \{ \delta < \lambda : \mathrm{cf}(\delta) = \chi \} \) is a stationary set, then there are a stationary set \( W \subseteq \chi \) and functions \( h_a, h_b : \lambda \rightarrow \mu \) and \( \{ S_\xi : 0 < \xi < \lambda \} \) such that:

- \( S_\xi \subseteq S \) is stationary,
- \( \xi \neq \zeta \Rightarrow S_\xi \cap S_\zeta = \emptyset \),
- if \( \delta \in S_\xi \), then for some increasing continuous sequence \( \langle \alpha_i : i < \chi \rangle \) we have \( \delta = \bigcup_{i < \chi} \alpha_i, h_b(\alpha_i) = i, h_a(\alpha_i) \in \{ \xi, 0 \} \), and the set \( \{ i < \chi : h_a(\alpha_i) = \xi \} \) is stationary, in fact is \( W \).

2) If in (1), a sequence \( \langle C_\delta : \delta < \lambda, \mathrm{cf}(\delta) \leq \chi \rangle \) satisfying

\[
(\forall \alpha \in C_\delta)[\alpha \text{ limit } \Rightarrow \alpha = \sup(\alpha \cap C_\delta)]
\]

is given, \( C_\delta \) is closed unbounded subset of \( \delta \) of order type \( \mathrm{cf}(\delta) \), then in the conclusion we can get also \( S^*, \langle C_\delta^* : \delta \in S^* \rangle \) such that (a), (b), (c) hold, and

- \( \cap \langle \alpha_i : i < \chi \rangle \),
- \( \bigcup_{0 < \xi < \lambda} S_\xi \subseteq S^* \subseteq \bigcup_{0 < \xi < \lambda} S_\xi \cup \{ \delta < \lambda : \mathrm{cf}(\delta) < \chi \} \),
- \( W \) is a \( (\aleph_0, \aleph_0) \)-closed, stationary in cofinality \( \aleph_0 \), subset of \( \chi \), which means:

- \( \{ i \in W : \mathrm{cf}(i) = \aleph_0 \} \) is a stationary\(^1\) subset of \( \chi \),
- for \( \delta \in \bigcup_{0 < \xi < \lambda} S_\xi \) we have

\[
C_\delta^* = \{ \alpha \in C_\delta : \mathrm{otp}(\alpha \cap C_\delta) = \sup(W \cap \mathrm{otp}(\alpha \cap C_\delta)) \}
\]

- \( C_\delta^* \) is a club of \( \delta \) included in \( C_\delta \) for \( \delta \in S^* \), and if \( \delta(1) \in C_\delta^* \), \( \delta \in S^*, \delta \in \bigcup_{0 < \xi < \lambda} S_\xi, \delta(1) = \sup(\delta(1) \cap C_\delta^*) \) and \( \mathrm{cf}(\delta(1)) > \aleph_0 \) then \( C_{\delta(1)}^* \subseteq C_\delta^* \),
- if \( C \) is a closed unbounded subset of \( \lambda \), and \( 0 < \xi < \lambda \) then the set \( \{ \delta \in S_\xi : C_\delta^* \subseteq C \} \) is stationary.

**Proof.** 1) We can find \( \{ (h_1^\xi, h_2^\xi) : \xi < \mu \} \) such that:

- for every \( \xi \) we have \( h_1^\xi : \lambda \rightarrow \mu \) and \( h_2^\xi : \lambda \rightarrow \mu \),
- if \( A \subseteq \lambda, |A| \leq \chi \), and \( h_1, h_2 : A \rightarrow \mu \), then for some \( \xi, h_1^\xi | A = h_1 \), and \( h_2^\xi | A = h_2 \).

\(^1\)we can ask \( \xi \) if \( I \) is any normal ideal on \( \{ i < \chi : \mathrm{cf}(i) = \aleph_0 \} \)
This holds by Engelking-Karłowicz [EK65] (see for example [Sh:c, AP]).

For $\delta < \lambda$ let $C_\delta$ be a closed unbounded subset of $\delta$ of order type $\text{cf}(\delta)$. Now for each $\xi < \mu$ and a stationary $a \subseteq \chi$ ask whether for every $i < \lambda$ for some $j < \lambda$ we have

$(\ast)_{i,j}^{\xi,a}$ the following subset of $\lambda$ is stationary:

$$S_{i,j}^{\xi,a} = \{\delta \in S : \ (i) \text{ if } \alpha \in C_\delta, \text{otp}(\alpha \cap C_\delta) \notin a \text{ then } h^1_\xi(\alpha) = 0, \n \ (ii) \text{ if } \alpha \in C_\delta, \text{otp}(\alpha \cap C_\delta) \in a \text{ then the } h^2_\xi(\alpha)-\text{th member of } C_\alpha \text{ belongs to } [i,j), \n \ (iii) \text{ if } \alpha \in C_\delta \text{ then } h^2_\xi(\alpha) = \text{otp}(\alpha \cap C_\delta)\}$$

\[ \square_{3.1} \]

**Subfact 3.2.** For some $\xi < \mu$ and a stationary set $a \subseteq \chi$, for every $i < \lambda$ for some $j \in (i, \lambda)$, the statement $(\ast)_{i,j}^{\xi,a}$ holds.

**Proof.** If not, then for every $\xi < \mu$ and a stationary $a \subseteq \chi$, for some $i = i(\xi, a) < \lambda$, for every $j < \lambda$, $j > i(\xi,a)$, there is a closed unbounded subset $C(\xi, a, i, j)$ of $\lambda$ disjoint from $S_{i,j}^{\xi,a}$.

Let

$$i(*) = \bigcup\{i(\xi,a) + \omega : \xi < \mu \text{ and } a \subseteq \chi \text{ is stationary}\}.$$ 

Clearly $i(*) < \lambda$.

For $i(*) \leq j < \lambda$ let $C(j) = \bigcap\{C(\xi, a, i(\xi,a), j) : \ a \subseteq \chi \text{ is stationary and } \xi < \mu \} \cap (i(*) + \omega, \lambda)$, clearly it is a closed unbounded subset of $\lambda$.

Let

$$C^* = \{\delta < \lambda : \delta > i(*) \text{ and } (\forall j < \delta)[\delta \in C(j)]\}.$$ 

So $C^*$ is a closed unbounded subset of $\lambda$, too. Let $C^+$ be the set of accumulation points of $C^*$. Choose $\delta(*) \in C^+ \cap S$, and we shall define

$$h^1 : C_{\delta(*)} \to \mu, \ h^2 : C_{\delta(*)} \to \mu.$$ 

For $\alpha \in C_{\delta(*)}$ let $h^0(\alpha)$ be:

$$\text{Min}\{\gamma < \chi : \gamma > 0 \text{ and the } \gamma-\text{th member of } C_\alpha \text{ is } > i(*)\}$$

if $\alpha = \text{sup}(C_{\delta(*)} \cap \alpha) > i(*)$, and zero otherwise. Clearly the set

$$\{\alpha \in C_{\delta(*)} : h^0(\alpha) = 0\}$$

is not stationary. Now we can define $g : C_{\delta(*)} \to \delta(*)$ by:

$$g(\alpha) \text{ is the } h^0(\alpha)-\text{th member of } C_\alpha.$$ 

Note that $g$ is pressing down and $\{\alpha \in C_{\delta(*)} : g(\alpha) \leq i(*)\}$ is not stationary. So (by the variant of Fodor’s Lemma speaking on an ordinal of uncountable cofinality) for some $j < \text{sup}(C_{\delta(*)}) = \delta(*)$ the set

$$a := \{\alpha \in C_{\delta(*)} \cap C^* : i(*) < g(\alpha) < j\}$$

is not stationary. Clearly it is a closed unbounded subset of $\delta(*)$. But $a \subseteq \chi$ is stationary and

$$h^1 : a \to \mu, \ h^2 : a \to \mu.$$
is a stationary subset of $\delta(*)$, and let $h^1 : C_{\delta(*)} \to \mu$ be

$$h^1(\alpha) = \begin{cases} 0 & \text{if } \text{otp}(\alpha \cap C^*_\delta) \not\in a, \\ h^0(\alpha) & \text{if } \text{otp}(\alpha \cap C^*_\delta) \in a. \end{cases}$$

Let $h^2 : C_{\delta(*)} \to \mu$ be $h^2(\alpha) = \text{otp}(\alpha \cap C^*_\delta)$. By the choice of $(h^1_\xi, h^2_\xi) : \xi < \mu$, for some $\xi$, we have $h^1_\xi(C_{\delta(*)}) = h^1$ and $h^2_\xi(C_{\delta(*)}) = h^2$. Easily, $\delta(*) \in S^\xi_{\delta,j}$ which is disjoint to $C(\xi, a, i(*), j)$, a contradiction to $\delta(*) \in C^*$ by the definition of $C(j)$ and $C^*$.

So we have proved the subfact 3.2. $\square_{3.2} \{6.1A\}$

Having chosen $\xi$, $a$ we define by induction on $\zeta < \lambda$ an ordinal $i(\zeta) < \lambda$ such that $(i(\zeta) : \zeta < \lambda)$ is increasing continuous, $i(0) = 0$, and $(*)_{i(\zeta), i(\zeta + 1)}$ holds.

Now, for $\alpha < \lambda$ we define $h_\alpha(\alpha)$ as follows: it is $\zeta$ if $h^1_\xi(\alpha) > 0$ and the $h^1_\xi(\alpha)$-th member of $C_{\alpha}$ belongs to $\{i(1 + \zeta), i(1 + \zeta + 1)\}$, and it is zero otherwise. Lastly, let $h_\alpha(\alpha) =: h^2_\xi(\alpha)$ and $W = a$ and

$$S_\zeta := \{\delta \in S : \begin{array}{l} (i) \text{ for } \alpha \in C_{\delta}, \text{ otp}(\alpha \cap C_{\delta}) = h_\alpha(\alpha), \\ (ii) \text{ for } \alpha \in C_{\delta}, h_\alpha(i) \in a \Rightarrow h_\alpha(\alpha) = \zeta, \\ (iii) \text{ for } \alpha \in C_{\delta}, h_\alpha(i) \not\in a \Rightarrow h_\alpha(\alpha) = 0 \}. \{6.1\}$$

Now, it is easy to check that $a, h_\alpha, h_\beta$, and $(S_\zeta : 0 < \zeta < \lambda)$ are as required.

2) In the proof of 3.1(1) we shall now consider only sets $a \subseteq \chi$ which satisfy the demand in clause (e) of 3.1(2) on $W$ [i.e., in the definition of $C(j)$ during the proof of Subfact 3.2 this makes a difference]. Also in $(*)_{i,j}^{\xi,a}$ in the definition of $S^\xi_{i,j}$ we change (iii) to:

$$(iii)' \text{ if } \alpha \in C_{\delta}, h^2_\xi(\alpha) \text{ codes the isomorphism type of, for example,}$$

$$(C_{\delta} \cup \bigcup_{\beta \in C_{\delta}} C_{\beta}, <, \alpha, C_{\delta}, \{i, \beta : i \in C_{\beta}\}). \{6.1A\}$$

In the end, having chosen $\xi, a$ we can define $C^*_\delta$ and $S^*$ in the natural way.

**Fact 3.3.** 1) If $\lambda$ is regular > $2^\kappa$, $\kappa$ regular, $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$ is stationary and for $\delta \in S, C^0_{\delta}$, is a club of $\delta$ of order type $\kappa \ (= \text{cf}(\delta))$, then we can find a club $c^*$ of $\kappa$ (see 3.4(1)) such that letting for $\delta \in S, C_{\delta} = C^0_{\delta} := \{\alpha \in C^0_{\delta} : \text{otp}(\delta \cap \alpha) \in c^*\}$, it is a club of $\delta$ and

$$(*) \text{ for every club } C \subseteq \lambda \text{ we have:}$$

(a) if $\kappa > \aleph_0$, $\{\delta \in S : C_{\delta} \subseteq C\}$ is stationary,

(b) if $\kappa = \aleph_0$, then the set

$$\{\delta \in S : (\exists \alpha, \beta)[\alpha < \beta \wedge \alpha \in C_{\delta} \wedge \beta \in C_{\delta} \Rightarrow (\alpha, \beta) \cap C \neq \emptyset]\}$$

is stationary.

2) If $\lambda$ is a regular cardinal > $2^\kappa$, then we can find $\langle C^\omega_{\delta} : \delta \in S^\omega_{\zeta} \rangle : \zeta < 2^\kappa$ such that:

(a) $\bigcup \{S_{\zeta} : \zeta < 2^\kappa\} = \{\delta < \lambda : \aleph_0 < \text{cf}(\delta) \leq \kappa\}$,

(b) $C^\omega_{\delta}$ is a club of $\delta$ of order type $\text{cf}(\delta)$,
(c) if $\alpha \in S_\zeta$, $\text{cf}(\alpha) > \theta > \aleph_0$, then
\[ \{ \beta \in C_\alpha^\zeta : \text{cf}(\beta) = \theta, \beta \in S_\zeta \text{ and } C_\beta^\zeta \subseteq C_\alpha^\zeta \} \]
is a stationary subset of $\alpha$.

3) If $\lambda$ is regular, $2^\mu \geq \lambda > \mu^\kappa$, then we can find $\langle C_\delta^\zeta : \delta \in S_\zeta \rangle : \zeta < \mu \rangle$ such that:

(a) $\bigcup \{ S_\xi : \zeta < 2^\kappa \} = \{ \delta < \lambda : \aleph_0 < \text{cf}(\delta) \leq \kappa \}$,
(b) $C_\delta^\zeta$ is a club of $\delta$ of order type $\text{cf}(\delta)$,
(c) if $\alpha \in S_\zeta$, $\beta \in C_\delta^\zeta$, $\text{cf}(\beta) > \aleph_0$, then $\beta \in S_\zeta$ and $C_\beta^\zeta \subseteq C_\delta^\zeta$,
(d) moreover, if $\alpha, \beta \in S_\zeta$, $\beta \in C_\alpha^\zeta$, then
\[ \{ \text{otp}(\gamma \cap C_\delta^\zeta), \text{otp}(\gamma \cap C_\beta^\zeta) : \gamma \in C_\delta \} \]
depends on $(\text{otp}(\beta \cap C_\alpha), \text{otp}(C_\alpha))$ only.

4) We can replace in (1)(a) and (b) of (*) “stationary” by “$\notin I$” for any normal ideal $I$ on $\lambda$.

\section{6.2}

\textbf{Remark 3.4.} 1) A club $C$ of $\delta$ where $\text{cf}(\delta) = \aleph_0$ means here just an unbounded subset of $\delta$.

2) In 3.3(1) instead of $2^\kappa$, the cardinal
\[ \text{Min}\{ |S| : S \subseteq \kappa \} \]
suffices.

3) In (b) above, it is equivalent to ask
\[ \{ \delta \in S : (\forall \alpha, \beta)[\alpha < \beta \land \alpha \in C_\delta \land \beta \in C_\delta \Rightarrow \text{otp}(\langle \alpha, \beta \rangle \cap C) > \alpha] \} \]
is stationary.

\textbf{Proof.} 1) If 3.3(1) fails, for each club $c^*$ of $\kappa$ there is a club $C[c^*]$ of $\lambda$ exemplifying its failure. So $C^* = \bigcap \{ C[c^*] : c^* \subseteq \kappa \text{ a club} \}$ is a club of $\lambda$. Choose $\delta \in S$ which is an accumulation point of $C^*$ and get contradiction easily.

2) Let $\lambda = \text{cf}(\lambda) > 2^\kappa$, $C_\alpha$ be a club of $\alpha$ of order type $\text{cf}(\alpha)$, for each limit $\alpha < \lambda$. Without loss of generality
\[ \beta \in C_\alpha \land \beta > \text{sup}(\beta \cap C_\alpha) \Rightarrow \beta \text{ is a successor ordinal}. \]

For any sequence $\bar{e} = \langle e_\theta : \aleph_0 < \theta = \text{cf}(\theta) \leq \kappa \rangle$ such that each $e_\theta$ is a club of $\theta$, for $\delta \in S^* = \{ \alpha < \lambda : \aleph_0 < \text{cf}(\alpha) \leq \kappa \}$ we let:
\[ C_\delta^\alpha = \{ \alpha \in C_\delta : \text{otp}(C_\delta \cap \alpha) \in e_{\text{cf}(\delta)} \}. \]

Now we define $S_\bar{e}$, by defining by induction on $\delta < \lambda$, the set $S_\bar{e} \cap \delta$; the only problem is to define whether $\alpha \in S_\bar{e}$ knowing $S_\bar{e} \cap \delta$; now
\[ \alpha \in S_\bar{e} \text{ if and only if } \]
\begin{itemize}
  \item[(i)] $\aleph_0 < \text{cf}(\alpha) \leq \kappa$,
  \item[(ii)] if $\aleph_0 < \theta = \text{cf}(\theta) < \text{cf}(\alpha)$ then the set $\{ \beta \in C_\alpha^\zeta : \text{cf}(\beta) = \theta, \beta \in S_\zeta \cap \alpha \}$ is stationary in $\alpha$.
\end{itemize}
Let \( \langle \vec{c} : \zeta < 2^\omega \rangle \) list the possible sequences \( \vec{c} \), and let \( S_\zeta = S_{\zeta \in \omega} \) and \( C^c_\zeta = C^c_\zeta \). To finish, note that for each \( \delta < \lambda \) satisfying \( \aleph_0 < \text{cf}(\delta) \leq \kappa \), for some \( \zeta, \delta \in S_\zeta \).

3) Combine the proof of (2) and of 3.1.

4) Similarly.

\[ \square_{3.4} \]

We may remark

**Fact 3.5.** Suppose that \( \lambda \) is a regular cardinal \( > 2^\omega, \kappa = \text{cf}(\kappa) > \aleph_0 \), a set

\[ S \subseteq \{ \delta < \lambda : \text{cf}(\delta) = \kappa \} \]

is stationary, and \( I \) is a normal ideal on \( \lambda \) and \( S \notin I \). If \( I \) is \( \lambda^+ \)-saturated (i.e., in the Boolean algebra \( \mathcal{P}(\lambda)/I \), there is no family of \( \lambda^+ \) pairwise disjoint elements), then we can find \( \langle C_\delta : \delta \in S \rangle \), \( C_\delta \) a club of \( \delta \) of order type \( \text{cf}(\delta) \), such that:

\[ \ast \] for every club \( C \) of \( \lambda \) we have \( \{ \delta \in S : C_\delta \setminus C \text{ is unbounded in } \delta \} \in I \).

\[ \text{Proof.} \]

For \( \delta \in S \), let \( C^*_\delta \) be a club of \( \delta \) of order type \( \text{cf}(\delta) \). Call \( C = \langle C_\delta : \delta \in S^* \rangle \) (where \( S^* \subseteq \lambda \) stationary, \( S^* \notin I \), \( C_\delta \) a club of \( \delta \) \( I \)-large if: for every club \( C \) of \( \lambda \) the set

\[ \{ \delta < \lambda : \delta \in S^* \text{ and } C_\delta \setminus C \text{ is bounded in } \delta \} \]

does not belong to \( I \).

We call \( C \) \( I \)-full if above \( \{ \delta \in S^* : C_\delta \setminus C \text{ unbounded in } \delta \} \in I \).

3.3(4), for every stationary \( S' \subseteq S, S' \notin I \) there is a club \( c^* \) of \( \kappa \) such that

\[ \langle C^*_\delta = c^* : \delta \in S' \rangle \text{ is } I \text{-large.} \]

Now note:

\[ \ast \] if \( \langle C_\delta : \delta \in S' \rangle \text{ is } I \text{-large, } S' \subseteq S \text{, then for some } S'' \subseteq S', S'' \notin I, \langle C^*_\delta : \delta \in S'' \rangle \text{ is } I \text{-full (hence } S'' \notin I). \]

[Proof of \( \ast \): Choose by induction on \( \alpha < \lambda^+ \), a club \( C^\alpha \) of \( \lambda^+ \) such that:

\( a \) for \( \beta < \alpha, C^\beta \setminus C^\alpha \text{ is bounded in } \lambda, \)

\( b \) if \( \beta = \alpha + 1 \) then \( A_\beta \setminus A_\alpha \in I^+, \) where

\[ A_\gamma = : \{ \delta \in S' : C_\delta \setminus C^\gamma \text{ is unbounded in } \delta \}. \]

As clearly

\[ \beta < \alpha \implies A_\beta \setminus A_\alpha \text{ is bounded in } \lambda \]

(by \( a \) and the definition of \( A_\alpha, A_\beta \) and as \( I \) is \( \lambda^+ \)-saturated, clearly for some \( \alpha \) we cannot define \( C^\alpha \). This cannot be true for \( \alpha = 0 \) or a limit \( \alpha \), so necessarily \( \alpha = \beta + 1 \). Now \( S' \setminus A_\beta \) is not in \( I \) as \( C \) was assumed to be \( I \)-large. Check that \( S'' =: S' \setminus A_\beta \) as is required.]

Using repeatedly 3.3(4) and \( \ast \) we get the conclusion. \[ \square_{3.5} \]

**Claim 3.6.** Suppose \( \lambda = \mu^+, \mu = \mu^\chi, \chi \) is a regular cardinal and \( S \subseteq \{ \delta < \lambda : \text{cf}(\delta) = \chi \} \) is stationary. Then we can find \( S^*, \langle C_\delta : \delta \in S^* \rangle \) and \( \langle S_\xi : \xi < \lambda \rangle \) such that:

\[ (a) \bigcup_{\zeta < \mu} S_\zeta \subseteq S^* \subseteq S \cup \{ \delta < \lambda : \text{cf}(\delta) < \chi \}, \]

\[ (b) \]
(b) $S \cap S$ is a stationary subset of $\lambda$ for each $\zeta < \mu$,
(c) for $\alpha \in S^*, C_\alpha$ is a closed subset of $\alpha$ of order type $\leq \chi$, if $\alpha \in S^*$ is a limit then $C_\alpha$ is unbounded in $\alpha$ (so is a club of $\alpha$),
(d) $\langle C_\alpha : \alpha \in S \rangle$ is a square on $S$, i.e., $(S)$ is stationary in $\sup(S)$ and:
   (i) $C_\alpha$ is a closed subset of $\alpha$, unbounded if $\alpha$ is limit,
   (ii) if $\alpha \in S, \alpha(1) \in C_\alpha$ then $\alpha(1) \in S$ and $C(1) = C_\alpha \cap \alpha(1)$,
(e) for each club $C$ of $\lambda$ and $\zeta < \mu$, for some $\delta \in S$, $C_\delta \subseteq C$.

\section{Proof.} Similar to the proof of 3.1 (or see \cite{Sh:237e}). \hfill \Box_{3.6}

\section{We shall use in 3.27

\section{Claim 3.7.} Suppose $\lambda = \mu^+$, $\gamma$ a limit ordinal of cofinality $\chi$,

\section{h : $\gamma \rightarrow \{ \theta : \theta = 1$ or $\theta = \text{cf}(\theta) \leq \mu \}$,

\section{\mu = \mu^{\gamma}, and $S \subseteq \{ \delta < \lambda : \text{cf}(\delta) = \chi \}$ is stationary. Then we can find $S^*, \langle C_\delta : \delta \in S^* \rangle$ and $\langle S_\zeta : \zeta < \lambda \rangle$ such that:

\section{(a) $\bigcup_{\zeta < \lambda} S_\zeta \subseteq S^* \subseteq \{ \delta < \lambda : \text{cf}(\delta) \leq \chi \}$,
(b) $S \cap S$ is stationary for each $\zeta < \lambda$,
(c) for $\delta \in S^*$,
   (i) $\delta$ is a club of $\delta$ of order type $\leq \gamma$ and
   (ii) $\text{otp}(\delta) = \gamma$, if $\delta \in S \cap S^*$,
   (iii) $\alpha \in C_\delta \land \text{sup}(C_\delta \cap \alpha) < \alpha$ $\Rightarrow$ $\alpha$ has cofinality $\text{h[otp}(C_\delta \cap \alpha)]$.
(d) if $\delta \in S$, $\delta(1)$ a limit ordinal in $C_\delta$ then $\delta(1) \in S_\zeta$ and $C_{\delta(1)} = C_\delta \cap \delta(1)$,
(e) for each club $C$ of $\lambda$ and $\zeta < \lambda$ for some $\delta \in S_\zeta$, $C_\delta \subseteq C$.

\section{Proof.} Like 3.6. \hfill \Box_{3.7}

\section{Claim 3.8.} 1) Suppose $\lambda$ is regular $> \aleph_1$, then $\{ \delta < \lambda^+ : \text{cf}(\delta) < \lambda \}$ is a good stationary subset of $\lambda^+$ (i.e., it is in $I[\lambda^+]$, see \cite{Sh:E62, 3.4=Lcd1.1} or \cite{Sh:88r, 0.6.0.7} or 3.9(2) below).

\section{2) Suppose $\lambda$ is regular $> \aleph_1$. Then we can find $\langle S_\zeta : \zeta < \lambda \rangle$ such that:

\section{(a) $\bigcup_{\zeta < \lambda} S_\zeta \subseteq \{ \alpha < \lambda^+ : \text{cf}(\alpha) < \lambda \}$,
(b) on each $S_\zeta$ there is a square (see 3.6 clause (d)), say it is $\langle C_\delta : \alpha \in S_\zeta \rangle$ with $|C_\delta^\alpha| < \lambda$,
(c) if $\delta(*) < \lambda$, and $\kappa = \text{cf}(\kappa)$, then: for some $\zeta < \lambda$ for every club $C$ of $\lambda^+$, for some accumulation point $\delta$ of $C$, $\text{cf}(\delta) = \kappa$ and $\text{otp}(C_\delta \cap C)$ is divisible by $\delta(*)$,
(d) if $\text{cf}(\delta(*)) = \kappa$, we can add in (c)’s conclusion:

$C_\delta \subseteq C$ and $\text{otp}(C_\delta^\alpha) = \delta(*)$.}
\{6.4\}

Remark 3.9. 1) For \( \lambda = \aleph_1 \) the conclusion of 3.8(1), (2)(a),(b) becomes totally trivial; but for \( \delta < \omega_1 \), it means something if we add: \( \{ \alpha \in S_\delta : \text{otp}(C_\delta^\alpha) = \delta \} \) is stationary and for every club \( C \subseteq \lambda \) the set \( \{ \alpha \in S_\delta : \text{otp}(C_\delta^\alpha) = \delta, C_\delta^\alpha \subseteq C \} \) is stationary. So 3.8(2)(c,d) are not so trivial, but still true. Their proofs are similar so we leave them to the reader (used only in [Sh:331, 2.7]).

2) Recall that for a regular uncountable cardinal \( \mu \), the family \( \tilde{I}[\mu] \) of good subsets of \( \mu \) is the family of every club \( C \subseteq \mu \) satisfying: \( a_\alpha \subseteq \alpha \) of order type \( \alpha \) when \( \mu \) is a successor cardinal, \( \beta \in a_\alpha \Rightarrow a_\beta = a_\alpha \cap \beta \) and

\[ (\forall \delta \in S \cap C)(\sup(a_\delta) = \delta \& \text{otp}(a_\alpha) = \text{cf}(\delta)). \]

We may say that the sequence \( \bar{a} \) as above exemplifies that \( S \) is good; if \( C = \mu \) we say “explicitly exemplifies”.

Proof. Appears also in detail in [Sh:351] (originally proved for this work but as its appearance was delayed we put it there, too). Of course, 1) follows from (2).

2) Let \( S = \{ \alpha < \lambda^+ : \text{cf}(\alpha) < \lambda \} \). For each \( \alpha \in S \) choose \( \bar{A}_\alpha \) such that:

- \( (\alpha) \) \( \bar{A}_\alpha = \{ A_\alpha^i : i < \lambda \} \) is an increasing continuous sequence of subsets of \( \alpha \) of cardinality \( \lambda \), such that \( \bigcup_{i<\lambda} A_\alpha^i = \alpha \cap S \),
- \( (\beta) \) if \( \beta \in A_\alpha^\alpha \cup \{ \alpha \} \), \( \beta \) is a limit ordinal and \( \text{cf}(\beta) < \lambda \) (the last actually follows), then \( \beta = \sup(A_\alpha^\alpha \cap \beta) \),
- \( (\gamma) \) if \( \beta \in A_\alpha^\alpha \cup \{ \alpha \} \) is limit and \( \text{cf}(\beta) < \lambda \) then \( A_\alpha^\beta \) contains a club of \( \beta \),
- \( (\delta) \) 0 \( \in A_\alpha^\alpha \) and \( \beta \in S \& \beta + 1 \in A_\alpha^\alpha \cup \{ \alpha \} \Rightarrow \beta \in A_\alpha^\beta \),
- \( (\varepsilon) \) the closure of \( A_\alpha^\alpha \) in \( \alpha \) (in the order topology) is included in \( A_\alpha^{\lambda+1} \).

There are no problems with choosing \( \bar{A}_\alpha \) as required.

We define \( B_\alpha^\alpha \) (for \( i < \lambda, \alpha \in S \)) by induction on \( \alpha \) as follows:

\[ B_\alpha^\alpha = \begin{cases} \text{closure}(A_\alpha^\alpha) \cap \alpha & \text{if } \text{cf}(\alpha) \neq \aleph_1, \\ \bigcap \{ \bigcup_{\beta \in C} B_\beta^\alpha : C \text{ a club of } \alpha \} & \text{if } \text{cf}(\alpha) = \aleph_1. \end{cases} \]

For \( \zeta < \lambda \) we let:

\[ S_\zeta = \{ \alpha \in S : \alpha \text{ satisfies } \begin{cases} (i) & B_\alpha^\zeta \text{ is a closed subset of } \alpha, \\ (ii) & \text{if } \beta \in B_\alpha^\zeta, \text{ then } B_\zeta^\beta = B_\alpha^\beta \cap \beta \text{ and } \\ (iii) & \text{if } \alpha \text{ is limit, then } \alpha = \sup(B_\alpha^\zeta) \end{cases} \} \]

and for \( \alpha \in S_\zeta \) let \( C_\zeta^\alpha = B_\zeta^\alpha \).

Now, demand (b) holds by the choice of \( S_\zeta \). To prove clause (a) we shall show that for any \( \alpha \in S \), for some \( \zeta < \lambda, \alpha \in S_\zeta \); moreover we shall prove

\[ (+)_\alpha^0 E_\alpha := \{ \zeta < \lambda : \text{cf}(\zeta) = \aleph_1 \text{ then } \alpha \in S_\zeta \} \text{ contains a club of } \lambda. \]
For $\alpha \in S$ define $E_\alpha^0 = \{ \zeta < \lambda : \text{if } \text{cf}(\zeta) = \aleph_1 \text{ then } B_\zeta^\alpha = \text{closure}(A_\zeta^\alpha \cap \alpha) \}$. We prove by induction on $\alpha \in S$ that $E_\alpha \cap E_\alpha^0$ contains a club of $\lambda$ and we then choose such a club $E_\alpha^1$.

Arriving to $\alpha$, let

$$E = \{ \zeta < \lambda : \text{ if } \beta \in A^\zeta_\alpha \text{ then } \zeta \in E_\beta^1 \text{ and } A^\zeta_\beta = A^\zeta_\alpha \cap \beta \}.$$

Clearly $E$ is a club of $\lambda$. Let $\zeta \in E$; cf($\zeta$) = $\aleph_1$, and we shall prove that $\alpha \in S_\zeta \cap E_\alpha \cap E_\alpha^0$, this clearly suffices. By the choice of $\zeta$ (and the definition of $E$) we have: if $\beta$ belongs to $A^\zeta_\alpha$ then $A^\zeta_\beta = A^\zeta_\alpha \cap A$ and $B^\beta_\zeta = \text{closure}(A^\zeta_\alpha \cap \beta)$, so

$$(*)_1 \beta \in A^\zeta_\alpha \Rightarrow B^\beta_\zeta = \text{closure}(A^\zeta_\alpha \cap \beta).$$

Let us check the three conditions for \(\alpha \in S_\zeta\) this will suffice for clause (a) of the claim.

**Clause (i):** $B^\beta_\zeta$ is a closed subset of $\alpha$.

- If cf($\alpha$) $\neq \aleph_1$ then $B^\beta_\zeta = \text{closure}(A^\zeta_\alpha \cap \alpha)$, hence necessarily it is a closed subset of $\alpha$.
- If cf($\alpha$) = $\aleph_1$ then $B^\beta_\zeta = \bigcap \{ \bigcup \{ B^\beta_\zeta : C \text{ is a club of } \beta \} \}$. Now, for any club $C$ of $\beta$, $C \cap A^\zeta_\alpha$ is a club of $\alpha$ (see clause (γ) above). By $(*)_1$ above,

$$\bigcup_{\beta \in C} B^\beta_\zeta \supseteq \bigcup_{\beta \in C \cap A^\zeta_\alpha} B^\beta_\zeta = \text{closure}(A^\zeta_\alpha \cap \beta).$$

Note that we have gotten

$$(*)_2 \alpha \in E^0_\zeta;$$

[Why? If cf($\alpha$) = $\aleph_1$ see above, if cf($\alpha$) $\neq \aleph_1$ this is trivial.]

- Indent Clause (ii): If $\beta \in B^\alpha_\zeta$ then $B^\beta_\zeta = B^\alpha_\zeta \cap \beta$.

We know that $B^\alpha_\zeta = \text{closure}(A^\alpha_\zeta \cap \alpha)$, by $(*)_2$ above. If $\beta \in A^\alpha_\zeta$ then (by $(*)_1$) we have $B^\beta_\zeta = \text{closure}(A^\alpha_\zeta \cap \beta)$, so we are done. So assume $\beta \notin A^\alpha_\zeta$. Then, by clause (ε) necessarily

$$\varepsilon < \zeta \Rightarrow \beta > \sup(A^\zeta_\alpha \cap \beta) \text{ and } \sup(A^\zeta_\alpha \cap \beta) \in A^\zeta_{\alpha+1} \subseteq A^\zeta_\alpha.$$

But $\beta \in B^\alpha_\zeta = \text{closure}(A^\alpha_\zeta \cap \alpha)$ by $(*)_2$, hence together $A^\alpha_\zeta$ contains a club of $\beta$ and cf($\beta$) = cf($\zeta$), but cf($\zeta$) = $\aleph_1$, so cf($\beta$) = $\aleph_1$. Now, as in the proof of clause (i), we get $B^\beta_\zeta = \bigcup \{ B^\beta_\zeta : \gamma \in A^\alpha_\zeta \cap \beta \}$, so by the induction hypothesis we are done.

**Clause (iii):** If $\alpha$ is limit then $\alpha = \sup(A^\alpha_\zeta)$.

- By clause (γ) we know $A^\alpha_\zeta$ is unbounded in $\alpha$, but $A^\alpha_\zeta \subseteq B^\alpha_\zeta$ (by $(*)_2$) and we are done.

So we have finished proving $(*)_1$ by induction on $\alpha$ hence clause (a) of the claim.

For proving (c) of 3.8(2), note that above, if $\alpha$ is limit, $C$ is a club of $\alpha$, $C \subseteq S$, and $|C| < \lambda$, then for every $i$ large enough, $C \subseteq A^\alpha_\zeta$, and even $C \subseteq B^\alpha_\zeta$.

Now assume that the conclusion of (c) fails (for fixed $\delta(\ast)$ and $\kappa$). Then for each $\zeta < \lambda$ we have a club $E^0_\zeta$ exemplifying it. Now, $E^0 = \bigcap_{\zeta < \lambda} E^0_\zeta$ is a club of $\lambda^+$.
We define \( \ast \) if \( \text{otp}(\delta) \) is divisible by \( \delta(\ast) \). Choose an unbounded in \( \delta \) set \( e \subseteq E^0 \) of order type divisible by \( \delta(\ast) \). Then, for a final segment of \( \zeta < \lambda \) we have \( e \cap \delta \subseteq C^\delta_3 \).

Note that for any set \( C_1 \) of ordinals, \( \text{otp}(C_1) \) is divisible by \( \delta(\ast) \) if \( C_1 \) has an unbounded subset of order type divisible by \( \delta(\ast) \), so we get a contradiction because by \((\ast)_{\delta(\ast)}^0\) for some \( \zeta < E_{\delta(\ast)} \) (so \( \delta(\ast) \leq S^\zeta \)) by \( E^0 \cap C^\delta_3 \subseteq E^0 \cap \delta \geq e \), \( \text{sup}(e) = \delta \) and \( e \) has order type divisible by \( \delta(\ast) \).

We are left with clause (d) of 3.8(2). Fix \( \kappa, \delta(\ast) \), and \( \zeta \) as above, we may add \( \leq \lambda \) new sequences of the form \( \langle C_\alpha : \alpha \in S_\zeta \rangle \) as long as each is a square. First assume that for every \( \gamma, \beta < \lambda \), such that \( \text{cf}(\beta) = \kappa = \text{cf}(\gamma) \), \( \gamma \) divisible by \( \delta(\ast) \) we have

\[
(\ast)_{\beta,\gamma} \quad \text{there is a club} \ E_{\beta,\gamma} \text{ of } \lambda^+ \text{ such that for no } \delta \in S_\zeta \text{ do we have } \text{otp}(C^\delta_{\beta,\gamma}) = \beta \quad \text{and } \text{otp}(C^\delta_{\beta,\gamma} \cap E_{\beta,\gamma}) = \gamma,
\]

then let

\[
E := \bigcap \{ E_{\beta,\gamma} : \gamma < \lambda, \beta < \lambda, \text{cf}(\beta) = \kappa = \text{cf}(\gamma), \gamma \text{ divisible by } \delta(\ast) \}.
\]

Applying part (c) we get a contradiction.

So for some \( \gamma, \beta < \lambda \), \( \text{cf}(\beta) = \kappa = \text{cf}(\gamma) \), \( \gamma \) divisible by \( \delta(\ast) \) and \((\ast)_{\beta,\gamma}^0\) fails. Also there is a club \( E^* \) of \( \lambda^+ \) such that for every club \( E \subseteq E^* \) for some \( \delta \in S_\zeta \), \( \text{otp}(C^\delta_{E^*}) = \beta \), \( \text{otp}(C^\delta_{E^*} \cap E) = \gamma \) and \( C^\delta_{E^*} \cap E = C^\delta_{E^*} \cap E^* \) (by 3.10 below). Let \( e \subseteq \gamma = \text{sup}(e) \) be closed and such that \( \text{otp}(e) = \delta(\ast) \) and

\[
\epsilon \in e \text{ is limit } \Rightarrow \epsilon = \text{sup}(e \cap \epsilon).
\]

We define \( *C^\delta_{\zeta} \) (for \( \delta \in S_\zeta \)) as follows: if \( \delta \notin E^* \)

\[
*C^\delta_{\zeta} := C^\delta_{\zeta} \setminus (\text{max}(\delta \cap E^*) + 1),
\]

if \( \delta \in E^* \), \( \text{otp}(C^\delta_{E^*} \cap E^*) \in e \cup \{ \gamma \} \) then

\[
*C^\delta_{\zeta} = \{ \alpha \in C^\delta_{E^*} \cap E^* : \text{otp}(\alpha \cap C^\delta_{E^*} \cap E^*) \in e \}
\]

and if \( \delta \in E^* \), \( \text{otp}(C^\delta_{E^*} \cap E^*) \notin e \cup \{ \gamma \} \) let

\[
*C^\delta_{\zeta} = C^\delta_{\zeta} \setminus \left( \text{max}\{ \alpha : \text{otp}(C^\delta_{E^*} \cap E^* \cap \alpha) \in e \cup \{ \gamma \} \} + 1 \right).
\]

One easily checks that (d) and square hold for \( \langle *C^\delta_{\zeta} : \delta \in S_\zeta \rangle \). So, we just have to add \( \langle *C^\delta_{\zeta} : \delta \in S_\zeta \rangle \) to \( \langle C^\delta_{\zeta} : \delta \in S_\zeta \rangle \) for any \( \zeta, \delta(\ast), \kappa \) (for which we choose \( \zeta \) and \( E^* \)).

\[\square_{3.9} \]

Claim 3.10. 1) Assume that \( \kappa_0 < \kappa = \text{cf}(\kappa) \), \( \kappa^+ < \lambda = \text{cf}(\lambda) \), \( S \subseteq \{ \delta < \lambda : \text{cf}(\delta) = \kappa \} \) is stationary, \( C_\delta \) is a club of \( \delta \) (for \( \delta \in S \)), and \( \langle \forall \delta \in S \rangle \{ |C_\delta| = \kappa \} \), or at least \( \text{sup} \{ |C_\delta| \}^+ < \lambda \). \textbf{Then} for some club \( E^* \subseteq \lambda \), for every club \( E \subseteq E^* \), the set \( \{ \delta \in S^* : C_\delta \cap E^* \subseteq E \} \) is stationary, where

\[
S^* := \{ \delta \in S : \delta \in \text{acc}(E^*) \}.
\]
2) Assume that \( \kappa = \text{cf}(\kappa), \kappa^+ < \lambda = \text{cf}(\lambda), S \subseteq \{ \delta < \lambda : \text{cf}(\delta) = \kappa \} \text{ is stationary,} \)
\( C_\delta \text{ is a club of } \delta \) (for \( \delta \in S \)), \( \sup \{ C_\delta \}^+ < \lambda, I_\delta \text{ is an ideal on } C_\delta \text{ including the} \)
bounded subsets, and for every club \( E \) of \( \lambda \) for stationarily many \( \delta \in S \), \( C_\delta \cap E \notin I_\delta \)
(or \( C_\delta \setminus E \in I_\delta \)).
Then for some club \( E^* \) of \( \lambda \), for every club \( E \subseteq E^* \) of \( \lambda \) the set \( \{ \delta \in S^* : C_\delta \cap E^* \subseteq E \} \text{ is stationary, where} \)
\[
S^* := \{ \delta \in S : \delta \in \text{acc}(E^*), \delta = \sup(C_\delta \cap E^*) \text{ and } C_\delta \cap E^* \notin I_\delta \text{ (or } C_\delta \setminus E \in I_\delta \} \}.
\]

Remark 3.11. This also was written in [Sh:365].
Proof. 1) If not, choose by induction on \( i < \mu =: \sup(\{ C_\delta \}^+) \) a club \( E_i^* \subseteq \lambda \),
decreasing with \( i, E_{i+1}^* \) exemplifies that \( E_i^* \) is not as required, i.e.,
\[
\{ \delta \in S^*(E_i^*) : C_\delta \cap E_i^* \subseteq E_{i+1}^* \} = \emptyset.
\]
Now, \( \text{acc}( \bigcap_{i<\mu} E_i^* ) \) is a club of \( \lambda \), so there is \( \delta \in S \cap \text{acc}( \bigcap_{i<\mu} E_i^* ) \). The sequence
\( \langle C_\delta \cap E_i^* : i < \mu \rangle \) is necessarily strictly decreasing, and we get an easy contradiction.
2) Similarly.
\[\square_{3.10}\]

\section{6.5}

Now we turn to the main issue: black boxes.

\begin{lemma}
\label{Lemma6.12}
Suppose that \( \lambda, \theta \) and \( \chi(*) \) are regular cardinals and \( \lambda^\theta = \lambda^{< \chi(*)}, \theta < \)
\( \chi(*) \leq \lambda \), and a set \( S \subseteq \{ \delta < \lambda : \text{cf}(\lambda) = \theta \} \text{ is stationary and in } I[\lambda] \) (if \( \theta = \aleph_0 \)
this holds trivially; see [Sh:E62, 3.4=Lcd1.1] or [Sh:88r, 0.6.0.7] or just 3.9(2)).

Then we can find
\[
W = \{ (M^\alpha, \eta^\alpha) : \alpha < \alpha(*) \}
\]
(pedantically, \( W \) is a sequence) and functions \( \zeta : \alpha(*) \rightarrow S, \) and \( h : \alpha(*) \rightarrow \lambda \)
such that:

\begin{itemize}
  \item[(a0)] \( h(\alpha) \) depends on \( \zeta(\alpha) \) only, and \( \zeta \) is non-decreasing function (but not necessarily strictly increasing)
  \item[(a1)] We have
    \begin{itemize}
      \item[(\alpha))] \( M^\alpha =\langle M^\alpha_i : i \leq \theta \rangle \) is an increasing continuous chain, \( \tau(M^\alpha_i), \) the
      vocabulary, may be increasing,
      \item[(\beta)] each \( M^\alpha_i \) is an expansion of a submodel of \( (\mathcal{H}_{< \chi(*)}(\lambda), \in, <) \) belonging
      to \( \mathcal{H}_{< \chi(*)}(\lambda) \) (so necessarily has cardinality \( < \chi(*) \), of course the order mean the order on the ordinals and, for transparency, the vocabulary
      belongs to \( \mathcal{H}_{< \chi(*)}(\chi(*)) \)),
      \item[(\gamma)] \( M^\alpha_i \cap \chi(*) \) is an ordinal, \( [\chi(*)] = \chi^+ \Rightarrow \chi + 1 \subseteq M^\alpha_i \), and \( M^\alpha_i \in \)
      \( \mathcal{H}_{< \chi(*)}(\eta^\alpha(i)) \),
      \item[(\delta)] \( M^\alpha_i \cap \lambda \subseteq \eta^\alpha(i) \),
      \item[(\epsilon)] \( \langle M^\alpha_j : j \leq i \rangle \in M^\alpha_{i+1} \).
    \end{itemize}
  \end{itemize}
\end{lemma}
(c) \( \eta^\alpha \in \theta \lambda \) is increasing with limit \( \zeta(\alpha) \in S, \eta^\alpha[i+1] \in M^\alpha_{i+1} \).

(a2) In the following game, \( \mathcal{O}(\theta, \lambda, \chi(*)_S, W, h) \), player I has no winning strategy. A play lasts \( \theta \) moves, in the \( i \)-th move player I chooses a model \( M_i \in \mathcal{H}_\subset \chi(*)_S(\lambda) \), and then player II chooses \( \gamma_i < \lambda \). In the first move player I also chooses \( \beta < \lambda \). In the end player II wins the play if \( (\alpha) \Rightarrow (\beta) \) where

(\alpha) the pair \((M_i : i < \theta), (\gamma_i : i < \theta)\) satisfies the relevant demands on \( \eta^\alpha \) and \( M_i \) is a model of \( (\mathcal{H}_\subset \chi(*)_S, \in, <) (M^\alpha \upharpoonright \theta, \eta^\alpha) \) in clause (a1)

(\beta) for some \( \alpha < \alpha(*)_\theta, \eta^\alpha = (\gamma_i : i < \theta), M_i = M^\alpha_i \) (for \( i < \theta \) and \( h(\alpha) = \beta \).

(b0) \( \eta^\alpha \neq \eta^\beta \) for \( \alpha \neq \beta \),

(b1) if \( \{\eta^\alpha[i : i < \theta] \subseteq M^\beta_\theta \ then \alpha < \beta + (< \chi(*))^{\theta, see below, and \( \zeta(\alpha) \leq \zeta(\beta) \),

(b2) if also \( \lambda < \theta = \lambda < \chi(*)_S \), then for every \( \alpha < \alpha(*)_\theta \) and \( i < \theta \), there is \( j < \theta \) such that: \( \eta^\alpha[j \in M^\beta_\delta \ implies M^\alpha_i \subseteq M^\beta_\delta \) (hence \( M^\alpha_i \subseteq M^\beta_\delta \)),

(b3) if \( \lambda = \lambda < \chi(*)_S \) and \( \eta^\alpha[i+1] \in M^\beta_j \ then \ M^\alpha_i \subseteq M^\beta_j \) (and hence \( x \in M^\alpha_j \Rightarrow x \in M^\beta_j \)) and

\[ \{\eta^\alpha[i \neq \eta^\beta[i] \Rightarrow \eta^\alpha(i) \neq \eta^\beta(i)\} \]  

(6.5A)

Remark 3.13. 1) If \( W \) (with \( \zeta, h, \lambda, \theta, \chi(*)_S \)) satisfies (a0), (a1), (a2), (b0), (b1) we call it a barrier.

2) Remember, \( (< \chi)^\theta =: \sum_{\mu < \chi} \mu^\theta \).

3) The existence of a good stationary set \( S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \theta \} \) follows, for example, from \( \lambda = \lambda < \theta \) (see [Sh:E62, 3.4-Lcd1.1] or [Sh:88r, 0.6,0.7]) and from \( \lambda \) is the successor of a regular cardinal and \( \lambda > \theta^+ \). But see 3.16(1),(2),(3).

4) Compare the proof below with [Sh:227, Lemma 1.13,pg.49] and [Sh:140].

\[ \{\eta^\alpha[i : i < \theta] \subseteq M^\beta_\theta \ then \alpha < \beta + (< \chi(*))^{\theta, see below, and \( \zeta(\alpha) \leq \zeta(\beta) \),

(b2) if also \( \lambda < \theta = \lambda < \chi(*)_S \), then for every \( \alpha < \alpha(*)_\theta \) and \( i < \theta \), there is \( j < \theta \) such that: \( \eta^\alpha[j \in M^\beta_\delta \ implies M^\alpha_i \subseteq M^\beta_\delta \) (hence \( M^\alpha_i \subseteq M^\beta_\delta \)),

(b3) if \( \lambda = \lambda < \chi(*)_S \) and \( \eta^\alpha[i+1] \in M^\beta_j \ then \ M^\alpha_i \subseteq M^\beta_j \) (and hence \( x \in M^\alpha_j \Rightarrow x \in M^\beta_j \)) and

\[ \{\eta^\alpha[i \neq \eta^\beta[i] \Rightarrow \eta^\alpha(i) \neq \eta^\beta(i)\} \]  

(6.7)

Proof. First assume \( \lambda = \lambda < \chi(*)_S \).

Let \( \langle S_\gamma : \gamma < \lambda \rangle \) be a sequence of pairwise disjoint stationary subsets of \( S, S = \bigcup S_\gamma \) and without loss of generality \( \gamma < \text{Min}(S_\gamma) \). We define \( h^* : S \rightarrow \lambda \) by \( h^*(\alpha) = \text{“the unique } \gamma \text{ such that } \alpha \in S_\gamma, \text{ and below we shall let } h(\alpha) := h^*(\zeta(\alpha)) \).

Let \( cd = cd_{\lambda, \chi(*)} \) be a one-to-one function from \( \mathcal{H}_\subset \chi(*)_S(\lambda) \) onto \( \lambda \) such that:

\( cd(\langle \alpha, \beta \rangle) \) is an ordinal \( > \alpha, \beta \) but \( < |\alpha + \beta|_+ \) or \( < \omega \), and \( x \in \mathcal{H}_\subset \chi(*)_S(cd(x)) \) for every relevant \( x \). For \( \xi \in S \) let:

\[ W_\xi := \{(M, \eta) : \text{the pair } (M, \eta) \text{ satisfies (a1) of 3.12, sup}\{\eta[i) : i < \theta]\} = \xi \]

and for every \( i < \theta \) for some \( y \in \mathcal{H}_\subset \chi(*)_S(\lambda), \eta(i) = cd(\langle M[i, \eta[i, y]\rangle) \} \)

So (a0), (a1), (b0), (b3) (hence (b2)) should be clear.
We can choose \(((\bar{M}^\alpha, \eta^\alpha) : \alpha < \alpha(\ast))\) an enumeration of \(\bigcup_{\xi \in S} W^0_{\xi}\) to satisfy (b1)
(and \(\dot{\zeta}(\alpha) = \sup \text{ran}(\eta^\alpha)\), of course) because:
\[(\ast)\text{ if } (\bar{M}^\ast, \eta^\ast) \in \bigcup_{\xi} W^0_{\xi}, \text{ then}
\{|\{\eta \in \theta \lambda : \eta[i : i < \theta] \subseteq M^*_{\delta}\}| \leq \|M^\ast\|^\theta \leq (\langle \chi(\ast) \rangle)^\theta\}.
\]
This, in fact, defines the function \(\dot{\zeta}\) as follows: we have \(\dot{\zeta}(\alpha) = \xi \text{ if and only if} (M^\alpha, \eta^\alpha) \in W^0_{\xi}\).

We are left with proving (a2). Let \(G\) be a strategy for player I.

Let \((C_\delta : \delta < \lambda)\) exemplify “\(S\) is a good stationary subset of \(\lambda^n\),” see 3.9(2), and let \(R = \{(i, \alpha) : i \in C_\alpha, \alpha < \lambda\}\).

Let \(\langle \alpha_i : i < \lambda\rangle\) be a representation of the model \(\mathcal{A} = (\mathcal{M}_{< \chi(\ast)}(\lambda), \in, G, R, \text{cd})\), i.e. it is increasing continuous, \(\|\alpha_i\| < \lambda\), and \(\bigcup_i \alpha_i = \mathcal{A}\); without loss of generality \(\alpha_i < \mathcal{A}\) and \(\|\alpha_i\| \cap \lambda\) is an ordinal for \(i < \lambda\).

Let \(G\) “tell” player I to choose \(\beta^* < \lambda\) in his first move. So there is \(\delta \in S_{\beta^*}\) (hence \(\delta > \beta^*\)) such that \(|\alpha_i| \cap \lambda = \delta\). Now, necessarily \(C_\delta \cap \alpha \in \mathcal{A}\) for \(\alpha < \delta\).

Let \(\{\alpha_i : i < \text{cf}(\delta)\}\) list \(C_\delta\) in increasing order.

Lastly, by induction on \(i\), we choose \(M_i, \eta(i)\) as follows:
\[\eta(i) = \text{cd}(\langle M_j : j \leq i\rangle, \langle \eta(j) : j < i\rangle, \langle \alpha_j : j < i\rangle)),\]
and \(M_i\) is what the strategy \(G\) “tells” player I to choose in his \(i\)-th move if player II have chosen \(\eta(j) : j < i\) so far.

Now, for each \(i < \theta\) the sequences \(\langle M_j : j \leq i\rangle, \langle \eta(j) : j < i\rangle\) are definable in \(\mathcal{A}\) with \(\langle \alpha_j : j \leq i\rangle\) as the only parameter, hence they belong to \(\mathcal{A}\). So sup\(|\eta(j) : j < \theta|\) \leq \delta; however, by the choice of \(\eta(i)\) (and cd), \(\eta(i) \supseteq \text{sup}(\alpha_j : j < i)\) and hence sup\(|\eta(j) : j < \theta|\) is necessarily \(\delta\). Now check.

The case \(\lambda < \lambda^{< \theta} = \lambda^{< \chi(\ast)}\) is similar. For a set \(A \subseteq \theta\) of cardinality \(\theta\) we let \(\text{cd}^A = \text{cd}^A_{\lambda^{< \chi(\ast)}}\) be a one-to-one function from \(\mathcal{M}_{< \chi(\ast)}(\lambda)\) onto \(A_\lambda\) where:
\[A_\lambda = \{h : h \text{ is a function from } A \text{ to } \lambda\}.\]
We strengthen (b2) to
\[(b2)' \text{ let } A_i := \{\text{cd}(i, j) : j < \theta\} \text{ for } i \in [1, \theta) \text{ and } A_0 := \theta \setminus \{A_{1+i} : i < \theta\} \text{ so}
\langle A_i : i < \theta\rangle \text{ is a sequence of pairwise disjoint subsets of } \theta \text{ each of cardinality} \theta\text{ with min}(A_i) \geq i \text{ and we have}
\[(\ast) \eta^\ast | A_i = \text{cd}^A_{\lambda^{< \chi(\ast)}}((\bar{M}^\ast | i, \eta^\ast | i)).\]
\[\square_{3.13}\]

\[\ast \ast \ast \ast\]

What can we do when \(S\) is not good? As we say in 3.13(3), in many cases a good \(S\) exists, note that for singular \(\lambda\) we will not have one.
The following rectifies the situation in the other cases (but is interesting mainly for \( \lambda \) singular). We shall, for a regular cardinal \( \lambda \), remove this assumption in 3.16(1)–(3), while 3.17 helps for singular \( \lambda \). (This is carried in 3.18).

**Definition 3.14.** Let \( \vartheta \) be an ordinal and for \( \alpha < \vartheta \) let \( \kappa_\alpha \) be a regular uncountable cardinal, \( S_\alpha \subseteq \{ \delta < \kappa_\alpha : \text{cf}(\delta) = \theta \} \) be a stationary set. Assume \( \theta, \chi \) are regular cardinals such that for every \( \alpha < \vartheta \) we have \( \theta < \chi \leq \kappa_\alpha \). Let \( S = \{ S_\alpha : \alpha < \vartheta \} \), \( \bar{\kappa} = (\kappa_\alpha : \alpha < \vartheta) \). If \( \vartheta = 1 \) we may write \( S_0, \kappa_0 \).

We say that \( \bar{S} \) is good for \((\bar{\kappa}, \theta, \chi)\) when: for every large enough \( \mu \) and model \( \mathcal{A} \) expanding \( \mathcal{M}_{<\chi}(\mu), \mathcal{E}\), \( |\tau(\mathcal{A})| \leq \aleph_0 \), there are \( M_i \) for \( i < \vartheta \) such that:

1. \( M_i < \mathcal{A} \) and \( S \subseteq M_i \)
2. \( \langle M_j : j \leq i \rangle \subseteq M_{i+1}, \| M_i \| < \chi, M_i \cap \chi \subseteq \chi, \chi = \chi_1^+ \Rightarrow \chi_1 + 1 \subseteq M_i \), and
3. \( \alpha < \vartheta, \alpha \in \bigcup_{j < \vartheta} M_j \) implies that \( \sup[\kappa_\alpha \cap (\bigcup M_j)] \) belongs to \( S_\alpha \).

If \( \vartheta = 1 \), we may write \( S_0, \kappa_0 \) instead \( \bar{S}, \bar{\kappa} \). If \( \vartheta < \chi \) then we can demand \( \vartheta \subseteq \mathcal{M}_\vartheta \).

**Definition 3.15.** For regular uncountable cardinal \( \lambda \) and regular \( \theta < \lambda \) let \( J_\theta[\lambda] \) be the family of subsets \( S \) of \( \mathcal{L}_\lambda \) such that \( \{ \delta \in S : \text{cf}(\delta) = \theta \} \) is not good for \((\lambda, \lambda, \theta)\).

**Claim 3.16.** Assume \( \theta = \text{cf}(\theta) < \chi = \text{cf}(\chi) \leq \kappa = \text{cf}(\kappa) \).
1. Then \( \{ \delta < \kappa : \text{cf}(\delta) = \theta \} \) is good for \((\kappa, \theta, \chi)\), i.e. is not in \( J_\theta[\kappa] \).
2. Any \( S \subseteq \kappa \) good for \((\kappa, \theta, \chi)\) is the union of \( \kappa \) pairwise disjoint such sets.
3. In 3.12 it suffices to assume that \( S \) is good for \((\lambda, \theta, \chi)\).
4. \( J_\theta[\lambda] \) is a normal ideal on \( \lambda \) and there is no stationary \( S \subseteq \{ \delta < \kappa : \text{cf}(\delta) = \theta \} \) which belongs to \( J_\theta[\lambda] \cap [\lambda] \).
5. In Definition 3.14, any \( \mu > \lambda^{<\chi} \) is O.K.; and we can preassign \( x \in \mathcal{M}_{<\chi}(\mu) \) and demand \( x \in M_\vartheta \).
6. In 3.12 we can replace the assumption “\( S \subseteq \{ \delta < \lambda : \text{cf}(\delta) = \theta \} \) is stationary and in \([\lambda]^{<\chi} \)” by “\( S \subseteq \{ \delta < \lambda : \text{cf}(\delta) = \theta \} \) is stationary not in \( J_\theta[\lambda] \)” (which holds for \( S = \{ \delta < \kappa : \text{cf}(\delta) = \theta \} \)).

**Proof.** 1) Straightforward (play the game).
2) Similar to the proof of 3.1.
3) Obvious.
4) Easy.
5) Easy.
6) Follows. \( \square_{3.16} \)

**Claim 3.17.** Assume that \( \bar{\kappa}, \theta, \chi \) are as in 3.14 with \( |\vartheta| \leq \chi \).
1. Then the sequence \( \langle \{ \delta < \kappa_i : \text{cf}(\delta) = \theta \} : i < \vartheta \rangle \) is good for \((\bar{\kappa}, \theta, \chi)\).
2. If \( \vartheta_1 < \vartheta \) and \( \langle S_i : i < \vartheta_1 \rangle \) is good for \((\bar{\kappa}|\vartheta_1, \theta, \chi)\) then
   \[ \langle S_i : i < \vartheta_1 \rangle \dashv \langle \{ \delta < \kappa_i : \text{cf}(\delta) = \theta \} : \vartheta_1 \leq i < \vartheta \rangle \]
   is good for \((\bar{\kappa}, \theta, \chi)\).
3. If \( \langle S_i : i < \vartheta_1 \rangle \) is good for \((\bar{\kappa}, \theta, \chi)\) and \( i(*) < \vartheta \), then we can partition \( S_{i(*)} \) to pairwise disjoint sets \( \langle S_{i(*)} : \epsilon < \kappa_i \rangle \) such that for each \( \epsilon < \kappa_i \), the sequence
   \[ \langle S_i : i < i(*) \rangle \dashv \langle S_{i(*)} : \epsilon < \kappa_i \rangle \dashv \langle \{ \delta : \delta < \kappa_i, \text{cf}(\delta) = \theta \} : i(*) < i < \vartheta \rangle \]
is good for \((\bar{\kappa}, \theta, \chi)\).
4) \(S\) good for \((\bar{\kappa}, \theta, \chi)\) implies that \(S_i\) is a stationary subset of \(\kappa_i\) for each \(i < \lg(\bar{\kappa})\).

\[\{\text{67X}\}\]

Proof. Like 3.16 [in 3.17(3) we choose for \(\delta \in S_{\kappa_i}\), a club \(C_\delta\) of \(\delta\) of order type \(\text{cf}(\delta)\); for \(j < \theta, \epsilon < \kappa_{(\alpha)}, \) let \(S^j_{\kappa_i, \epsilon} = \{\delta \in S_{\kappa_i} : \epsilon\) is the \(j\)-th member of \(C_\delta\}\); for some \(j\) and unbounded \(A \subseteq \kappa_{(\alpha)}\), \((S^j_{\kappa_i, \epsilon} : \epsilon \in A\) are as required). \(\square_{3.17}\)

\[\{\text{6.8}\}\]

Now we remove from 3.12 (and subsequently 3.20) the hypothesis “\(\lambda\) is regular” when \(\text{cf}(\lambda) \geq \chi(*)\).

\[\{\text{6.9}\}\]

Lemma 3.18. Suppose \(\lambda^\theta = \lambda^\chi(*)\), \(\lambda\) is singular, \(\theta < \chi(*)\) and \(\text{cf}(\lambda) \geq \chi(*)\). Suppose further that \(\lambda = \sum \mu_i\), each \(\mu_i\) is regular

\(\lambda^\theta + \theta^+\). Then we can find \(W = \{(\check{M}^\alpha, \eta^\alpha) : \alpha < \alpha(*)\}\) and functions \(\check{\zeta} : \alpha(*) \rightarrow \text{cf}(\lambda), \check{\xi} : \alpha(*) \rightarrow \lambda, \) and \(h : \alpha(*) \rightarrow \lambda, \) and \(\{\mu_i^j : i < \text{cf}(\lambda)\}\) such that \(\{\mu_i^j : i < \text{cf}(\lambda)\} = \{\mu_i : i < \text{cf}(\lambda)\}\) and:

\(a0\) \(b(\alpha)\) depends only on \(\check{\zeta}(\alpha), dot\check{\xi}(\alpha)\),
\(\alpha < \beta \Rightarrow \check{\zeta}(\alpha) \leq \check{\zeta}(\beta), [\alpha < \beta \wedge \check{\zeta}(\alpha) = \check{\zeta}(\beta) \Rightarrow \check{\xi}(\alpha) \leq \check{\xi}(\beta)],\)
\(\alpha \in \alpha < \mu_i^j(\check{\zeta}(\alpha))\)

\(a1\) as in 3.12 except that: \(\eta^\alpha(3i) : i < \theta\) is strictly increasing with limit \(\check{\zeta}(\alpha)\)
and \(\eta^\alpha(3i + 1) : i < \theta\) is strictly increasing with limit \(\check{\xi}(\alpha)\) for \(i < \theta, \)
\sup(\{|M_i^\alpha| \cap \mu_i^j(\check{\zeta}(\alpha))\}) < \check{\zeta}(\alpha) = \sup(|M_i^\alpha| \cap \mu_i^j(\check{\zeta}(\alpha))\)
and for every \(i < \theta, \)
\sup(|M_i^\alpha| \cap \text{cf}(\lambda)) < \check{\zeta}(\alpha) = \sup(|M_i^\alpha| \cap \text{cf}(\lambda))\)

\(a2\) as in 3.12

\(b1, b2\) as in 3.12 but in clause \(b3\) we demand \(i = 2 \bmod 3\).

Remark 3.19. To make it similar to 3.12, we can fix \(S^a, S^b_i, S^b_i, S^b_{i, \alpha}, \mu_i^j\) as in the
first paragraph of the proof below.

Proof. First, by 3.16 [(1) + (2)], we can find pairwise disjoint \(S^a_i \subseteq \text{cf}(\lambda)\) for \(i < \text{cf}(\lambda), \) each good for \((\text{cf}(\lambda), \theta, \chi(*)\) (and \(\alpha \in S^a_i \Rightarrow \alpha > i \& \text{cf}(\alpha) = \theta)\), and
let \(S^a = \bigcup_{i < \text{cf}(\lambda)} S^a_i\). We define \(\mu_i^j \in \{\mu_j : j < i\} \) such that for each \(i < \text{cf}(\lambda) : [j \in S^a_i \Rightarrow \mu_j = \mu_i^j]\).

Then for each \(i\), by 3.17 parts (2) (3) (with \(1, 2, S_0, S_0, S_0, \sigma, S^a_i, \text{cf}(\lambda), \mu_i^j\) of \(\delta < \mu_i^j \) \(\text{cf}(\delta) = \theta\) such that for each \(\alpha < \mu_i^j, \) \(S^a_i, S^b_{i, \alpha}\) is good for \(\check{\zeta}(\lambda), \mu_i^j, \chi)\). Let
\(S^b_i = \bigcup_{\alpha < \mu_i^j} S^b_{i, \alpha}\).

Let \(\text{cd} \) be as in 3.12’s proof coding only for ordinals \(i = 2 \bmod 3, \) and for \(\zeta \in S^b_i, \xi \in S^b_{i, j}\) let
The following Lemma improves 3.12 when $\lambda$ satisfies a stronger requirement making the distinct $(M^\alpha, \eta^\alpha)$ interact less. Lemmas 3.20 + 3.18 were used in the proof of 2.4 (and 2.3).

**Lemma 3.20.** In 3.12, if $\lambda = \lambda^{\chi(*)}$, $\chi(*)^\theta = \chi(*)$, then we can strengthen clause (b1) to

(b1)$^+$ if $\alpha \neq \beta$ and $\{\eta^\alpha[i : i < \theta] \subseteq M^\beta$ then $\alpha < \beta$ and $x \in M^\beta \Rightarrow x \in M^\alpha$.

**Proof.** Apply 3.12 (actually, its proof) but using $\lambda$, $\chi(*)^\theta$, $\eta^\beta$, instead of $\lambda, \chi(*)$, $\theta$; and get $W = \{(M^\alpha, \eta^\alpha) : \alpha < \alpha(*)\}$, and the functions $\xi, h$.

Let $\mathcal{U}$ be as in the proof of 3.12. Let $<^*$ be some well ordering of $\mathcal{H}_{\chi(*)}(\lambda)$, and let $\mathcal{W}$ be the set of ordinals $\alpha < \alpha(*)$ such that for $i < \theta$, $M^\alpha_i$ has the form $(N^\alpha_i, c^\alpha_i, <^\alpha_i) \land (|N^\alpha_i|, c^\alpha_i, <^\alpha_i) \times (\mathcal{H}_{\chi(*)}(\lambda), c_i, <^*)$.

Let $\alpha \in \mathcal{W}$, by induction on $\epsilon < \chi(*)$ we define $M^\epsilon, \eta^\epsilon$ as follows:

(A) $\eta^\epsilon(i)$ is $\text{cd}(\eta^\alpha(i), \epsilon)$, (which is an ordinal $< \lambda$ but $> \eta^\alpha(i)$ and $> \epsilon$)

(B) $M^\epsilon_i \preceq N^\alpha_i$ is the Skolem Hull of $\{\eta^\epsilon(i) : j < i\}$ inside $N^\alpha_i$ using as Skolem functions the choice of the $<^*$-first element and making $M^\epsilon_i \cap \chi(*)$ an ordinal [if we want we can use $\eta^\epsilon(i)$ such that it fits the definition in the proof of 3.12].

Note that $\chi(*) = \chi^+ \Rightarrow \chi + 1 \subseteq M^\alpha$ and $M^\epsilon_i$ is definable in $M^\epsilon_i$, as $M^\epsilon_i \subseteq M^\epsilon_{i+1}$ (by the definition of $W^\alpha_i$ in the proof of 3.12). Similarly, $M^\epsilon_j : j < i$ is definable in $M^\epsilon_{i+1}$. It is easy to check that the pair $(M^\epsilon, \eta^\epsilon)$ satisfies condition (a1) of 3.12.

Next we choose by induction on $\alpha \in \mathcal{W}, \epsilon(\alpha) < \chi(*)$ as follows:

(C) $\epsilon(\alpha)$ is the first $\epsilon < \chi(*)$ such that: if $\beta < \alpha$ but $\beta + \chi(*) > \alpha$ then:

\[
(\epsilon) \quad \eta^\epsilon[j : j < \theta] \notin M^\delta_{\epsilon(\beta)}.
\]

This is possible and easy, as for (C) it suffices to have for each suitable $\beta, \epsilon \notin M^\delta_{\epsilon(\beta)}$, so each $\beta$ “disqualifies” $< \chi(*)$ ordinals as candidates for $\epsilon(\alpha)$, and there are $< \chi(*)$ such $\beta$s, and $\chi(*)$ is by the assumptions (see 3.12) regular.

Now

\[
W' = \{(N^\alpha, \eta^\alpha, \epsilon(\alpha)) : \alpha \in \mathcal{W}, \epsilon(\alpha) < \chi(*)\}
\]

are as required except that we should replace $\mathcal{W}$ by an ordinal (and adjust $\zeta, h$ accordingly). In the end replace $N^\alpha_i$ by $N^\alpha_i \cap \mathcal{H}_{\chi(*)}(\lambda)$.
Claim 3.21. If in 3.18 we add “$\lambda = \kappa^{(\theta)}$” (or the condition from 3.20) then $\theta$ can replace (b1) by

(b1) + if $[\eta^\alpha : i < \theta] \subseteq \mathcal{M}_\theta^\beta$ then $\alpha \leq \beta$.

Proof. The same as the proof of 3.20 combined with the proof of 3.18. □

Next we turn to the case (of black boxes with) $\theta = \aleph_0$. We shall deal with several cases.

Lemma 3.22. Suppose that

$\lambda$ is a regular cardinal, $\theta = \aleph_0$, $\mu = \mu^{<\chi(\lambda)} < \lambda \leq 2^\mu$, $S \subseteq \{ \delta < \lambda : \text{cf}(S) = \aleph_0 \}$ is stationary and $\aleph_0 < \chi(*) = \text{cf}(\chi(*))$.

Then we can find

$$W = \{ (\bar{\mathcal{M}}^\alpha, \eta^\alpha) : \alpha < \alpha(*) \}$$

and functions

$$\hat{\zeta} : \alpha(*) \to S \text{ and } h : \alpha(*) \to \lambda$$

such that:

1. $\alpha \neq \beta, (\bar{\mathcal{M}}^\alpha, \eta^\alpha \upharpoonright n < \omega) \subseteq \mathcal{M}_\theta^\beta$ implies $\alpha < \beta$ and even $\hat{\zeta}(\alpha) < \hat{\zeta}(\beta)$,
2. if $\hat{\zeta}(\alpha) = \hat{\zeta}(\beta)$ then $\mathcal{M}_\theta^\beta \cap \mu = \mathcal{M}_\theta^\beta \cap \mu$ and there is an isomorphism $h_{\alpha, \beta}$ from $\mathcal{M}_\theta^\alpha$ onto $\mathcal{M}_\theta^\beta$ mapping $\eta^\alpha(n)$ to $\eta^\beta(n)$, and $\mathcal{M}_\theta^\alpha$ to $\mathcal{M}_\theta^\beta$ for $n < \omega$, and $h_{\alpha, \beta}[\mathcal{M}_\theta^\alpha \cap \mathcal{M}_\theta^\beta]$ is the identity,
3. there is $C = (C_\delta : \delta \in S)$, $C_\delta$ an $\omega$-sequence converging to $\delta$, $0 \notin C_\delta$, and letting $\langle \gamma^\delta : n < \omega \rangle$ enumerate $\{ 0 \} \cup C_\delta$ we have, when $\hat{\zeta}(\alpha) = \delta$:
   (i) $\lambda \cap |\mathcal{M}_\theta^\alpha| \subseteq \gamma^\delta_{n+1}$ but $\lambda \cap |\mathcal{M}_\theta^\alpha|$ is not a subset of $\gamma^\alpha_n$, (hence $\mathcal{M}_\theta^\alpha \cap |\gamma^\alpha_n \setminus \gamma^\delta_{n+1}| \neq \emptyset$);
   (ii) $C_\delta \cap |\mathcal{M}_\theta^\alpha| = \emptyset$;
   (iii) if $\hat{\zeta}(\beta) = \delta$ too then, for each $n$, $h_{\alpha, \beta}$ maps $|\mathcal{M}_\theta^\alpha| \cap |\gamma^\delta_n \setminus \gamma^\delta_{n+1}|$ onto $|\mathcal{M}_\theta^\beta| \cap |\gamma^\alpha_n \setminus \gamma^\delta_{n+1}|$;
   (iv) if $\hat{\zeta}(\beta) = \delta = \hat{\zeta}(\alpha)$ and $\lambda = \lambda^{<\chi(*)}$, then $|\mathcal{M}_\theta^\alpha| \cap \gamma^\delta_\theta = |\mathcal{M}_\theta^\beta| \cap \gamma^\delta_\theta$.

Remark 3.23. 1) We use $\lambda \leq 2^\mu$ only to get “$h_{\alpha, \beta}[(|\mathcal{M}_\theta^\alpha| \cap |\mathcal{M}_\theta^\beta|)] = \text{id}”$ in condition (c1).
2) Below we quote “guessing of clubs” that is clause (ii) in the proof, without this we just get a somewhat weaker conclusion.
Proof. Let \( S \) be the disjoint union of stationary

\[
S_{\alpha, \beta, \gamma} \quad (\alpha < \mu, \beta < \lambda, \gamma < \lambda).
\]

For each \( \alpha, \beta, \gamma \) let \( (C_\delta : \delta \in S_{\alpha, \beta, \gamma}) \) satisfy

\[
\exists (i) \quad C_\delta \text{ is an unbounded subset of } \delta \text{ of order type } \omega, \text{ and}
\]

\[
(ii) \quad \text{for every club } C \text{ of } \lambda, \text{ for stationarily many } \delta \in S_{\alpha, \beta, \gamma}, \text{ we have}
\]

\[
C_\delta \subseteq C
\]

\[
(iii) \quad 0 \notin C_\delta
\]

(exists by [Sh:331, 2.2] or [Sh:365]).

Let \( W^* \) be the family of quadruples \((\delta, M, \eta, C)\) such that:

\[
(a) \quad (M, \eta) \text{ satisfies the requirement (a1) (so } M = \langle M_n : n < \omega \rangle);\]

\[
(b) \quad 0 \notin C, \text{ and letting } \{\gamma_n : n < \omega\} \text{ enumerate in increasing order } C \cup \{0\}
\]

we have \( \lambda \cap M_n \) is a subset of \( \gamma_{n+1} \) but not of \( \gamma_n \), and \( \bigcup_{n<\omega} \gamma_n = \delta \) and

\[
C \cap (\bigcup_n M_n) = \emptyset;
\]

\[
(\gamma) \quad \bigcup_n |M_n| \subseteq \mathcal{H}_{\lambda^+(\star)}(\mu + \mu);
\]

\[
(\delta) \quad \text{in } \tau(M_n) \text{ there are a two place relation } R \text{ and a one place function } \text{cd} \text{ (not necessarily } \text{cd}[M_n = \text{cd}_{M_n^\prime}, \text{ similarly for } R, \text{ see below recall that as usual, } \tau(M_n)) \in \mathcal{H}_{\lambda^+(\star)}(\lambda^+(\star)) \text{ for transparency.)}
\]

As \( \mu^{<\lambda^+(\star)} = \mu \) clearly \( |W^*| = \mu \), so let

\[
W^* = \{(\delta, (M_j, n : n < \omega), \eta_j, C^j) : j < \mu\}.
\]

If \( \lambda = \lambda^{<\lambda^+(\star)} \) let \( \{N_\beta : \beta < \lambda\} \) list the models \( N \in \mathcal{H}_{\lambda^+(\star)}(\lambda) \) with \( \tau(N) \in \mathcal{H}_{\lambda^+(\star)}(\lambda(\star)) \).

Also, let \( (A_\alpha : \alpha < \lambda) \) be a sequence of pairwise distinct subsets of \( \mu \), and define the two place relation \( R \) on \( \lambda \) by

\[
[\gamma_1 R \gamma_2 \iff \gamma_1 < \mu \& \gamma_1 \in A_{\gamma_2}]\]

Lastly, for \( \delta \in S_{\alpha, \beta, \gamma} \) let

\[
W^\delta_0 := \{(M, \eta) : M = \langle M_n : n < \omega \rangle, \eta \in \omega^\omega, \text{ satisfy (a1), so} \}
\]

\( \eta \) is increasing with limit \( \delta \) and there is an isomorphism

\[
h \text{ from } \bigcup_{n<\omega} M_n \text{ onto } \bigcup_{n<\omega} M_{\alpha, n}, \text{ mapping } \eta(n) \text{ to } \eta^\alpha(n) \text{ and}
\]

\( M_n \text{ onto } M_{\alpha, n}, \text{ preserving } \epsilon, R, \text{cd}(x) = y \text{ and their negations; (for } R \text{ and cd :}
\]

\( \in \cup \bigcup_{n<\omega} M_n \text{ we mean the standard cd over } \bigcup_{n<\omega} M_{\alpha, n} \text{ as in (\delta) above); and}
\]

\( (\forall \epsilon < \lambda)[\epsilon \in \bigcup_{n<\omega} M_n \Rightarrow \text{otp}(C_\delta \cap \epsilon) = \text{otp}(C^\alpha \cap \epsilon(\epsilon))].
\]

Also, if \( \lambda = \lambda^{<\lambda^+(\star)} \) then

\[
N_\beta = (\bigcup_n M_n)|\{x \in \bigcup_n M_n : \text{cd}(x) < \text{Min}(C_\delta)\}.
\]

We proceed as in the proof of 3.12 after \( W^\delta_0 \) was defined (only \( \hat{\zeta}(\alpha) = \delta \in \{6.5\} \)

\( S_{\alpha_1, \beta_1, \gamma_1} \Rightarrow h(\alpha) = \gamma_1)\).
Suppose $G$ is a winning strategy for player I. So suppose that if player II has chosen $\eta(0), \eta(1), \ldots, \eta(n - 1)$, player I will choose $M_n$. So $|M_n|$ is a subset of $\mathcal{H}_{\prec \chi(*)}(\lambda)$ of cardinality $< \chi(*)$ and $\text{Rang}(\eta) \subseteq M_n$. For $\eta \in \mathcal{L}$ we define $M_\eta = \bigcup_{i < \omega} M_{\eta[i]}$.

Let $\mathcal{T}_n$ be the set of $\eta \in ^n \lambda$ such that $M_\eta$ is well defined; so $\bigcup \{ \mathcal{T}_n : n < \omega \}$ is a subtree of $(^\omega \lambda, \subseteq)$ with each node having $\lambda$ immediate successors.

We can find a function $c_n$ from $\mathcal{T}_n$ into $\mu$ such that $c_n(\eta) = c_n(\nu)$ iff there is an isomorphism $h$ from $M_\eta$ onto $M_\nu$ mapping $M_{\eta[k]}$ onto $M_{\nu[k]}$ for every $k < n$. By [Sh:E62, 1.10=1.L.7] or the proof of 3.24 below, there is $\mathcal{T}$ such that

$$\mathcal{T} \subseteq \omega^{>\lambda}, \mathcal{T} \text{ is closed under initial segments,}$$

$$\langle \rangle \in \mathcal{T}, \ [\eta \in \mathcal{T} \Rightarrow (\exists \alpha)[\eta^*(\alpha) \in \mathcal{T}]],$$

$$c_n[\langle \mathcal{T} \cap \mathcal{T}_n \rangle] \text{ is constant.}$$

It follows that fixing any $\nu_* \in \text{lim}(\mathcal{T})$ we can find $(h_\eta; \eta \in \mathcal{T})$ such that $h_\eta$ is an isomorphism from $M_{\nu_*[\eta(\eta)]}$ onto $M_\eta$ increasing with $\eta$.

Note that above all those isomorphisms are unique as the interpretation of $\in$ satisfies comprehension. Also clause (c1) follows from the use of $R$.

The rest should be clear. $\square_{3.22}$

**Lemma 3.24.** Let $S$, $\lambda$, $\mu$, $\theta$ be as in (*) of 3.22 and in addition:

$$N_0 \leq \kappa = \text{cf}(\kappa) < \chi(*) \quad (\forall \chi < \chi(*)[\chi^{<\chi(*)} < \chi(*)]).$$

Then we can find $W = \{ (\bar{M}^\alpha, \eta^\alpha) : \alpha < \alpha(*) \}$ and functions $\zeta : \alpha(*) \rightarrow S$ and $h : \alpha(*) \rightarrow \lambda$ such that:

- (b0), (b2) as in 3.22 (i.e. as in 3.12),
- (c1), (c2) as in 3.22,
- (a1) as (a1) in 3.12 except that we omit “$(M_j : j \leq i) \in M_{i+1}$” and add:
  $[\alpha \subseteq |M_i| \& |a| < \kappa \Rightarrow a \in M_i]$ and for $i < j$, $M_i \cap \lambda$ is an initial segment of $M_j \cap \lambda$,
  $[a2]$ for every expansion $\mathcal{A}$ of $(\mathcal{H}_{\prec \chi(*)}(\lambda), \in, <)$ by $\chi < \chi(*)$ relations, for some $\alpha < \alpha(*)$, for every $n$, $M_n^\alpha \subset \mathcal{A}$ in fact, for stationarily many $\zeta \in S$, there is such $\alpha$ satisfying $\zeta(\alpha) = \zeta$.

- (6.11d)

**Remark 3.25.** We can retain (a1) and add $a \subseteq M_i \land |a| < \kappa \Rightarrow a \in M_i$.

**Proof.** Similar to 3.22, use the proof of [Sh:247], but for completeness we give details.

We choose $(S_{\alpha, \beta, \gamma} : \alpha < \mu, \beta < \lambda, \gamma < \lambda)$ as there. The main point is that defining $W^*$ we have one additional demand:

- (c) if $n < \omega$ and $u \subseteq M_n$ has cardinality $< \kappa$ then $u \in M_n$.

We then define $W^*_\eta$ and $(\alpha_n : \alpha < \lambda)$ as there.

This gives the changed demand in (a1)*, but it give extra work in verifying the demand (a2)*.

So let a model $\mathcal{A}$ and cardinal $\chi = \chi^{<\kappa} < \chi(*)$ as there be given; as usual, $\tau(\mathcal{A}) \in \mathcal{H}_{\chi(*)}(\chi(*)$) and $\mathcal{A}$ expand $(\mathcal{H}_{\chi(*)}(\lambda), \in, <)$. For every $\mathbf{x} = (\delta_x, M_x, \eta_x, C_x) \in W^*$ we define a family $\mathcal{F}_x$, a function $n : \mathcal{F} \rightarrow \omega$ and a function $\text{rank}_x$ from $\mathcal{F}_x$ into $\text{Ord} \cup \{ \infty \}$ as follows:
(α) \( \mathcal{F}_x = \emptyset \{ \mathcal{F}_{x,n} : n < \omega \} \)

(β) \( \mathcal{F}_{x,n} = \{ f : f \text{ is an elementary embedding of } M_{x,n} \text{ into } \mathcal{A} \} \)

(γ) \( n(f) = k \) if and only if \( f \in \mathcal{F}_{x,k} \)

(δ) \( \text{rank}(f) = \cup \{ \epsilon + 1 : \text{for every } \alpha < \lambda \text{ there is } g \in \mathcal{F}_{x,n(f)} \text{ extending } f, \text{ such that } \beta = \text{rank}_x(g) \text{ and } \text{Rang}(g) \cap \alpha = \text{Rang}(f) \cap \lambda \} \).

Now

Case 1: for no \( x \in W^* \) and \( f \in \mathcal{F}_{x,0} \) do we have \( \text{rank}_x(f) = \infty \).

For every \( x \in W^* \) and \( f \in \mathcal{F}_x \) let \( \beta(f,x) \) be the first ordinal \( \alpha < \lambda \) such that if \( \text{rank}_x(f) = \epsilon \) then there is no \( g \in \mathcal{F}_{x,n(f) + 1} \) extending \( f \) with \( \text{rank}_x(g) = \epsilon \) and \( \text{Rang}(g) \cap \alpha = \text{Rang}(f) \cap \lambda \).

Next let \( \langle \mathcal{A}_i : i < \lambda \rangle \) be an increasing continuous sequence of elementary submodels of \( \mathcal{A} \), each of cardinality \( < \lambda \) such that \( \langle \mathcal{A}_j : j \leq i \rangle \in \mathcal{A}_{i+1} \).

Easily the set \( E = \{ i < \lambda : \mathcal{A}_i \cap \lambda = i > \mu \} \) is a club of \( \lambda \).

Choose by induction on \( n < \omega \) an ordinal \( i_n \) increasing with \( n \) such that \( i_n \in E \) of cofinality \( \kappa \), possible as \( 2 < \kappa \) as \( \kappa < \chi(*) \) and \( \alpha < \lambda \to |\alpha|^{|\chi(*)|} < \lambda \) hence \( \mathcal{A}_{i_n} \) is an elementary submodel of \( \mathcal{A} \) of cardinality \( < \lambda \).

Choose \( M < \mathcal{A} \) of cardinality \( \chi, \) including \( \{ i_n : n < \omega \} \) such that every \( u \subseteq M \) of cardinality \( < \kappa \) belongs to \( M \).

Note that, if \( u \subseteq \mathcal{A}_{i_n} \) has cardinality \( < \kappa \) then \( u \in \mathcal{A}_{i_n} \) because \( i_n \in E \) and \( cf(i_n) = \kappa \).

Let \( M^*_n = \mathcal{A} \upharpoonright (\mathcal{A}_{i_n} \cap M) \), easily \( M^*_n \in \mathcal{A}_{i_n} \), so \( [u \subseteq M^*_n \land |u| < \kappa \Rightarrow u \in M^*_n] \).

We can find \( x \in W^* \) and isomorphism \( f_n \) from \( M^*_n \) onto \( M^*_n \) increasing with \( n \).

Now clearly \( x \in \mathcal{A}_{i_n} \), (why? as \( sW^* \in \mathcal{A}_{i_n} \) and \( |W^*| \leq \mu + 1 \subseteq \mathcal{A}_{i_n} \)). Also \( f_n \in \mathcal{F}_{x,n} \) and \( f_n \in \mathcal{A}_{i_n} \), (as \( M^*_n, M_{x,n} \in \mathcal{A}_{i_n} \)) and the uniqueness of \( f_n \) as those models expand a submodel of \( (\mathcal{H}_\chi(x), \varepsilon, \lessdot) \) and necessarily are transitive over the ordinals).

Similarly by the choice of \( x \), we have \( f_n \subseteq f_{n+1} \). So \( \text{rank}_x(f_n) : n < \omega \) is constantly \( \infty \) as otherwise we get infinite decreasing sequence of ordinals.

But this contradict our case assumption.

Case 2: Not case 1

So we choose \( x \in W^* \) and \( f \in \mathcal{F}_{x,0} \) such that \( \text{rank}_x(f) = \infty \).

We easily get the desired contradiction and even a \( \Delta \)-system tree of models.

How? Let \( \eta_\alpha : \alpha < \lambda \) list \( \omega \to \lambda \) such that \( \eta_\alpha \upharpoonright \eta_\beta \) implies \( \alpha < \beta \).

Now we choose a pair \( (f_{\eta_\alpha}, \gamma_\alpha) \) by induction on \( \alpha < \lambda \) such that

\begin{itemize}
  \item[(i)] \( f_{\eta_\alpha} \in \mathcal{F}_{x,\ell(\eta_\alpha)} \)
  \item[(ii)] \( \gamma_\alpha = \sup \{ \lambda \cap \text{Rang}(f_{\eta_\alpha}) : \beta < \alpha \} \)
  \item[(iii)] if \( \eta_\beta \upharpoonright \eta_\alpha \) and \( \ell(\eta_\beta) = (\ell(\eta_\beta) + 1) \) then \( \gamma_\alpha \cap \text{Rang}(f_{\eta_\alpha}) = \lambda \cap \text{Rang}(f_{\eta_\alpha}) \).
\end{itemize}

There is no problem to carry the induction. This finishes the proof. \( \square \)

Lemma 3.26. 1) In 3.24 if in addition \( \lambda = \mu^+ \) then we can add:

\begin{itemize}
  \item[(c3)] if \( \zeta(\alpha) = \zeta(\beta) \), then \( |M^*_\alpha \cap M^*_\beta| \cap \lambda \) is an initial segment of \( |M^*_\alpha \cap \lambda| \) and of \( |M^*_\beta \cap \lambda| \), so when \( \alpha \neq \beta \) it is a bounded subset of \( \zeta(\alpha) \).
\end{itemize}

2) In 3.24 (and 3.26), when \( \lambda \geq \aleph_0 \) then it follows that:

\begin{itemize}
  \item[(c4)*] if \( \alpha \neq \beta \) and \( \{ \eta^\alpha : n : n < \omega \} \subseteq M^\beta \) then \( \bar{M}^\alpha, \bar{\eta}^\alpha \in M^\beta \).
\end{itemize}
3) Assume $\lambda = \mu^+$ and $\mu = \kappa^+$ and $S \subseteq \{\delta : \delta < \lambda, \text{cf}(\delta) = \aleph_0\}$ is a stationary subset of $\lambda$ and $(C_\delta : \delta \in S)$ guess clubs (and $C_\delta$ is an unbounded subset of $\delta$ of order type $\omega$, of course).

Then we can find $(\hat{N}_\eta : \eta \in \Gamma)$ such that:

(a) $\Gamma = \bigcup \{ \Gamma_\delta : \delta \in S \}$ where $\Gamma_\delta \subseteq \{ \eta : \eta \text{ in an increasing } \omega\text{-sequence of ordinals } < \delta \text{ with limit } \delta \}$ and $\delta(\eta) = \delta$ when $\eta \in \Gamma_\delta, \delta \in S$

(b) $N_\eta$ is $\langle N_{\eta,n} : n \leq \omega \rangle$ in $\prec$-increasing, and we let $N_\eta = N_{\eta,\omega}$

(c) each $N_\eta$ is a model of cardinality $\kappa$ with vocabulary $\subseteq \mathcal{H}(\kappa^+)$ for notational simplicity, and universe $\subseteq \delta := \delta(\eta)$ and $N_{\eta,n} = N_\eta \upharpoonright \gamma^\delta_n$ where $\gamma^\delta_n$ is the $n$-the member of $C_\delta$ and $N_\eta \cap (\gamma^\delta_n, \gamma^\delta_{n+1}) \neq \emptyset$

(d) for every distinct $\eta, \nu \in \Gamma_\delta$ where $\delta \in S$, for some $n < \omega$ we have $N_\eta \cap N_\nu = N_{\eta,n} = N_{\nu,n}$

(e) for every $\eta, \nu \in \Gamma_\delta$ the models $N_\eta, N_\nu$ are isomorphic, moreover there is such isomorphism $f$ which preserve the order of the ordinals and maps $N_{\eta,n}$ onto $N_{\nu,n}$

(f) if $\mathcal{A}$ is a model with universe $\lambda$ and vocabulary $\subseteq \mathcal{H}(\kappa^+)$ then for stationarily many $\delta \in S$ for some $\eta \in \Gamma_\delta \subseteq \Gamma$ we have $N_\eta \prec \mathcal{A}$. Moreover, if $\kappa^3 = \kappa$ and $h$ is a one to one function from $^0\lambda$ into $\lambda$ then we can add: if $\rho \in ^0(\kappa, n)$ then $h(\rho) \in N_{\eta,n}$.

Proof. 1) Let $g^0, g^1$ be two place functions from $\lambda \times \lambda$ to $\lambda$ such that for $\alpha \in [\mu, \lambda]$:

\[ \langle g^0(\alpha, i) : i < \mu \rangle \text{ enumerate } \{ j : j < \mu \} \text{ without repetitions, and } g^1(\alpha, g^0(\alpha, i)) = i \text{ for } i < \lambda. \]

Now we can restrict ourselves to $\bar{M}^\alpha$ such that each $M_i^\alpha$ (for $i < \omega$) is closed under $g^0, g^1$. Then (c3) follows immediately from

\[ [\hat{\zeta}(\alpha) = \hat{\zeta}(\beta) \Rightarrow |M_\alpha^\delta| \cap \mu = |M_\beta^\delta| \cap \mu] \]

(required in (c1)).

2) Should be clear.

3) This just rephrase what we have proved above. \(\square_{3.26}\)

Lemma 3.27. Suppose that $\lambda = \mu^+$, $\mu = \kappa^\aleph_0 = 2^\kappa > 2^{\aleph_0}$, $\text{cf}(\kappa) = \aleph_0$ and $S \subseteq \{ \delta < \lambda : \text{cf}(\delta) = \aleph_0 \}$ is stationary, $\theta = \aleph_0$, $\aleph_0 < \chi(*) = \text{cf}(\chi(*)) < \kappa$. Then we can find $W = \{ \langle M^\alpha, \eta^\delta \rangle : \alpha < \alpha(*) \}$ and functions

\[ \hat{\zeta} : \alpha(*) \rightarrow S, \quad h : \alpha(*) \rightarrow \lambda \]

and $(C_\delta : \delta \in S)$ with $(\gamma^\delta_n : n < \omega)$ listing $C_\delta$ in increasing order such that:

\begin{itemize}
    \item \{6.\#0\} – (a1) as in 3.12,
    \item \{6.11\} – (a2)* as in 3.24,
    \item \{6.\#0\} – (b2) as in 3.12 and even
    \item \{6.\#0\} – (b1)* $\alpha \neq \beta$, $\{\eta^\alpha[n : n < \omega]\} \subseteq M_\beta^\delta$ implies $\alpha < \beta$ and even $\zeta(\alpha) < \zeta(\beta),$
    \item \{6.\#1\} – (c3) as in 3.22 + 3.26(1),
\end{itemize}
(c4) if $\zeta(\alpha) = \zeta(\beta) = \delta$ but $\alpha \neq \beta$ then for some $n_0 \geq 1$, there are no $n > n_0$ and $\alpha_1 \leq \beta_2 \leq \alpha_3$ satisfying:

$$\begin{align*}
\alpha_1 &\in [M^\mathcal{O}_\omega] \cap [\gamma^\delta_n, \omega^\delta n+1), \\
\beta_2 &\in [M^\mathcal{O}_\omega] \cap [\gamma^\delta_n, \omega^\delta n+1), \\
\alpha_3 &\in [M^\mathcal{O}_\omega] \cap [\gamma^\delta_n, \omega^\delta n+1),
\end{align*}$$

i.e., either $\sup([\gamma^\delta_n, \omega^\delta n+1) \cap [M^\mathcal{O}_\omega]) < \min([\gamma^\delta_n, \gamma^\delta n+1) \cap [M^\mathcal{O}_\omega])$

or $\sup([\gamma^\delta_n, \gamma^\delta n+1) \cap [M^\mathcal{O}_\omega]) < \min([\gamma^\delta_n, \gamma^\delta n+1) \cap [M^\mathcal{O}_\omega]).$

(c5) if $\Upsilon < \kappa$ and there is $B \subseteq \omega_\kappa, |B| = k^{\kappa_0}$ which contains no perfect set with density $\Upsilon$ (holds trivially if $\kappa$ is strong limit), then also $\{\eta^\alpha : \alpha < \alpha(*)\}$ does not contain such a set. (See 3.28).

\begin{equation}
\text{(6.13a)}
\end{equation}

Proof. We repeat the proof of 3.22 with some changes.

Let $\langle S_{\alpha, \beta, \gamma} : \alpha < \mu, \beta < \lambda, \gamma < \lambda \rangle$ be pairwise disjoint stationary subsets of $S$.

Let $g^0, g^1$ be as in the proof of 3.26. By 3.7 there is a sequence $\langle C_\delta : \delta \in S \rangle$ such that:

(i) $C_\delta$ is a club of $\delta$ of order type $\kappa$, not $\omega_1, 0 \notin C_\delta,$

(ii) for $\alpha < \mu, \beta < \lambda, \gamma < \lambda$, for every club $C$ of $\lambda$, the set

$$\{\delta : S_{\alpha, \beta, \gamma} : C_\delta \subseteq C \}$$

is stationary.

We then define $W^*, \langle \delta^j, (M_{j,n} : n < \omega), \eta_j, \omega^j \rangle$ for $j < \mu, A_\nu$ for $\alpha < \lambda$, and $R$ as in the proof of 3.22.

Now, for $\delta \in S_{\alpha, \beta, \gamma}$ let $W^\delta_\alpha$ be the collection of all systems $\langle M_\rho, \eta_\rho : \rho \in \omega^{\omega+}\rangle$

such that:

(i) $\eta_\rho$ is an increasing sequence of ordinals of length $\lg(\rho)$,

(ii) otp $(C_\delta \cap \eta_\rho(\ell)) = 1 + \rho(\ell)$ for $\ell < \lg(\rho)$,

(iii) there are isomorphisms $(h_\rho : \rho \in \omega^{\omega+})$ such that $h_\rho$ maps $M_\rho$ onto $M_{\alpha, \lg(\rho)}$

preserving $\in, R, cd(x) = y, g^0(x_1, x_2) = y, g^1(y) = y$ and their negations,

(iv) if $\rho < \nu$ then $h_\rho \subseteq h_\nu, M_\rho < M_\nu, M_\rho \in M_\nu$,

(v) $M_\rho \cap C_\delta = \emptyset$, and $M_\rho \cap \lambda \subseteq \bigcup_{\ell} \langle \gamma^\rho(\ell), \gamma^\rho(\ell+1) \rangle$, where $\gamma^\rho_\ell$ is the $\ell$-th member of $C_\delta$,

(vi) if $\rho < \omega^{\omega^\kappa}, \ell < \lg(\rho), \gamma$ is the $(1 + \rho(\ell))$-th member of $C_\delta$ then $M_\ell \cap \gamma$

depends only on $\rho[\ell, \gamma \cap M_\rho, \gamma \searrow M_\rho$,

(vii) $N_\delta = M_0$.

Now clearly $|W^\delta_\alpha| \leq \mu$, so let $W^\delta_\alpha = \{(M^\mu_\rho, \eta^\mu_\rho) : \rho \in \omega^{\omega^\kappa} : j < \mu\}$. Let $\langle \rho_j : j < \mu \rangle$

be a list of distinct members of $\omega^{\omega^\kappa}$; for (c5) — choose as there.

Let $M^\delta_\rho = \bigcup_{\ell < \omega} M^\delta_\rho(\ell), \eta^\delta_\rho(\ell + 1)(\ell + 1) : \ell \leq \omega)$.

Now,

$$\{M^\delta_\rho : \ell < \omega) : j < \mu\}$$
is as required in (c4). Also (c5) is straightforward (as taking union for all \( \delta \)'s change little), (of course, we are omitting \( \delta \)'s where we get unreasonable pairs).

The rest is as before. \( \square_{3.27} \)

Remark 3.28. The existence of \( B \) as in (c5) is proved, for some \( \Upsilon \) for all strong limit \( \kappa \) of cofinality \( \aleph_0 \) in \([Sh:gf, \text{Ch II.6.9, pg.104}]\), really stronger conclusions hold. If \( 2^\kappa \) is regular and belongs to \( \{ \text{cf}(\Pi_{\kappa}D) : D \text{ an ultrafilter on } \omega, \kappa_0 < \kappa \} \) or \( 2^\kappa \) is singular and is the supremum of this set, then it exists for \( \Upsilon = (2^{\aleph_0})^+ \). Now, if above we replace \( D \) by the filter of co-bounded subsets of \( \omega \), then we get it even for \( \Upsilon = \aleph_0 \); by \([Sh:E12, \text{Part D}]\) the requirement holds, e.g., for \( \square_3 \) for a club of \( \delta < \omega_1 \).

Moreover, under this assumption on \( \kappa \) we can demand (essentially, this is expanded in 3.33)

(c4)* we strengthen clause (c4) to: if \( \zeta(\alpha) = \zeta(\beta) = \delta \) but \( \alpha \neq \beta \) then for some \( \nu_0 \geq 1 \), we have either for every \( n \in [\nu_1, \omega) \) we have sup(\( [\gamma^\delta_n, \gamma^\delta_{n+1}) \cap |M^\alpha_\beta| \) < min(\( [\gamma^\delta_n, \gamma^\delta_{n+1}) \cap |M^\alpha_\beta| \)) or for every \( n \in [\nu_1, \omega) \) we have sup(\( [\gamma^\delta_n, \gamma^\delta_{n+1}) \cap |M^\alpha_\beta| \) = min(\( [\gamma^\delta_n, \gamma^\delta_{n+1}) \cap |M^\alpha_\beta| \)).

Lemma 3.29. We can combine 3.27 with 3.24.

Proof. Left to the reader. \( \square_{3.29} \)

Lemma 3.30. Suppose \( \aleph_0 = \theta < \chi(\ast) = \text{cf}(\chi(\ast)) \) and: \( \chi_0 = \lambda < \chi(\ast) \), \( \chi(\ast) \leq \lambda \) and: \( \lambda = \lambda_\ast^\ast \), and \( \ast \lambda_\ast \) (see below) holds.

Then \( \ast \lambda \) we can find \( W = \{ (\tilde{M}_\alpha, \eta^\alpha) : \alpha < \ast(\ast) \} \) and functions \( \tilde{\zeta} : \alpha(\ast) \to S \) and \( h : \alpha(\ast) \to \lambda \) such that:

(a0) - (a2) as in 3.12,
(b0) - (b2) as in 3.12, and even
(c3) if \( \tilde{\zeta}(\alpha) = \tilde{\zeta}(\beta) \) then \( |M_\alpha| \cap |M_\beta| \) is a bounded subset of \( \tilde{\zeta}(\alpha) \).

Proof. Left to the reader. \( \square_{3.30} \)

Lemma 3.31. Suppose that \( \lambda \) is a strongly inaccessible uncountable cardinal,

\[ \text{cf}(\lambda) \geq \chi(\ast) = \text{cf}(\chi(\ast)) > \theta = \aleph_0, \]

and let \( S \subseteq \lambda \) consist of strong limit singular cardinals of cofinality \( \aleph_0 \) and be stationary. Then we can find \( W = \{ (\tilde{M}_\alpha, \eta^\alpha) : \alpha < \ast(\ast) \} \) and functions \( \tilde{\zeta} : \alpha(\ast) \to S \) and \( h : \alpha(\ast) \to \lambda \) such that:

(a0) - (a2) of 3.12 (except that \( h(\alpha) \) depends not only on \( \tilde{\zeta}(\alpha) \)),
(b0), (b3) of 3.12,
(b1) + of 3.20,
(c3) - if \( \tilde{\zeta}(\alpha) = \delta \) then \( |M_\alpha| \cap |M_\beta| \cap \delta \) is a bounded subset of \( \delta \).

Remark 3.32. 1) See \([Sh:45]\) for essentially a use of a weaker version.
2) We can generalize 3.24.

Proof. See the proof of \([Sh:331, 1.10(3)]\) but there \( \text{sup}(N_\theta \cap \lambda) < \delta \). \( \square_{3.31} \)
Lemma 3.33. 1) Suppose that $\lambda = \mu^+$, $\mu = \kappa^0 = 2^\kappa$, $\theta < \text{cf}(\chi(\ast)) = \chi(\ast) < \kappa$, $\kappa$ is strong limit, $\kappa > \text{cf}(\kappa) = \theta > \aleph_0$, $S \subseteq \{ \delta < \lambda : \text{cf}(\delta) = \theta \}$ is stationary. Then we can find $W = \{ (\mathcal{M}^\alpha, \eta^\alpha) : \alpha < \alpha(\ast) \}$ (actually, a sequence), functions $\zeta : \alpha(\ast) \rightarrow S$ and $h : \alpha(\ast) \rightarrow \lambda$ and $\langle C_\delta : \delta \in S \rangle$ such that:

(a1) as in 3.12,
(b0) $\eta^\alpha \neq \eta^\beta$ for $\alpha \neq \beta$,
(b1) if $\{ \eta^\alpha[i : i < \theta] \subseteq M^\alpha_\theta$ and $\alpha \neq \beta$ then $\alpha < \beta$ and even $\zeta(\alpha) < \zeta(\beta)$,
(b2) if $\eta^\alpha(j + 1) \in M^\alpha_\theta$ then $M^\alpha_\theta \in M^\beta_\theta$,
(c2) $C = \{ C_\delta : \delta \in S \}$, $C_\delta$ a club of $\delta$ of order type $\theta$, and every club of $\lambda$ contains $C_\delta$ for stationarily many $\delta \in S$,
(c3) if $\delta \in S$, $C_\delta = \{ \gamma_\delta,i : i < \theta \}$ is the increasing enumeration, $\alpha < \alpha^*$ satisfies $\zeta(\alpha) = \delta$, then there is $\langle (\gamma_\alpha,i,\gamma_{\alpha^i}) : i < \theta(\text{odd}) \rangle$ such that $\gamma_\alpha,i \in M^\alpha_\delta$, $M^\alpha_\delta \cap \lambda \subseteq \gamma_\alpha$, $\gamma_\delta,i < \gamma_{\alpha^i} < \gamma_{\alpha^i} < \gamma_{\delta,i+1}$ and

(*) if $\zeta(\alpha) = \zeta(\beta)$, $\alpha < \beta$ then for every large enough odd $i < \theta$, $\gamma_{\alpha,i} < \gamma_{\beta,i}$ (hence $\gamma_\alpha,i,\gamma_{\alpha^i} \cap \gamma_{\beta,i} = \emptyset$) and $\gamma_{\delta,i} = \emptyset$.

2) In part (1), assume $\theta = \aleph_0$ and $\text{pp}(\kappa) = 2^\kappa$. Then the conclusion holds; moreover, (c3) (from 3.26).

Remark 3.34. The assumption $\text{pp}(\kappa) = 2^\kappa$ holds, for example, for $\kappa = \beth_\delta$ for a club of $\delta < \omega$ (see [? , §5]).

Proof. 1) By 3.6 we can find $C = \{ C_\delta : \delta \in S \}$, $C_\delta$ a club of $\delta$, of order type $\kappa$ such that for any club $C$ of $\lambda$ for stationarily many $\delta \in S$, we have: $C_\delta \subseteq C$.

First Case: assume $\mu(= 2^\kappa)$ is regular.

By [Shg, Ch.II.5.9], we can find an increasing sequence $\langle \kappa_i : i < \theta \rangle$ of regular cardinals $> \chi(\ast)$ such that $\kappa = \sum \kappa_i$, and $\prod \kappa_i/j^\text{bd}_\theta$ has true cofinality $\mu$, and let $(f_\varepsilon : \varepsilon < \mu)$ exemplify this, which means:

$$\varepsilon < \zeta < \mu \quad \Rightarrow \quad f_\varepsilon < f_\zeta \mod j^\text{bd}_\theta,$$

and for every $\varepsilon \in \prod \kappa_i$, for some $\varepsilon < \mu$ we have $f_\varepsilon < f_\zeta \mod j^\text{bd}_\theta$. We may assume that if $\varepsilon$ is limit and $f_\varepsilon$ has $<j^\text{bd}_\theta$-l.u.b., then $f_\varepsilon$ is a $<j^\text{bd}_\theta$-l.u.b., and we know that if $\text{cf}(\varepsilon) > 2^\theta$ then this holds, and that without loss of generality $\bigwedge_{\varepsilon < \theta} \text{cf}(f_\varepsilon(\varepsilon)) = \text{cf}(\varepsilon)$.

Without loss of generality $\kappa_i > f_\varepsilon(i)$ for $\varepsilon < \theta$.

We shall define $W$ later. Let $S$ be a strategy for player I. By the choice of $C$, for some $\delta \in S$, for every $\alpha \in C_\delta$ of cofinality $> \theta$, $\mathcal{H}_{\mathcal{X}(\ast)}(\alpha) \subseteq C_\delta$ is closed under the strategy $S$. Let $C_\delta = \{ \alpha_i : i < \kappa \}$ be increasing continuous. For each $\varepsilon < \mu$ we choose a play of the game, player I using $S$, $(M^\varepsilon_j, \eta^\varepsilon_j : j < \theta)$ such that:

$$\langle M^\varepsilon_j : j \leq j \rangle \in \mathcal{H}_{\mathcal{X}(\ast)}(\alpha_{f_\varepsilon(\varepsilon)+1})$$,

$$\eta^\varepsilon_j = \langle \text{cd}(\alpha_{f_\varepsilon(\varepsilon)}, (M^\varepsilon_i : i \leq j)) : j < \gamma \rangle$$, and

$$\eta^\varepsilon_{j+1} \in M^\varepsilon_{j+1}.$$
Then let \( g_\varepsilon \in \prod_{i < \theta} \kappa_i \) be:

\[
g_\varepsilon(i) = \sup(\kappa_i \cap \bigcup_{j < \theta} M_j^i),
\]

so for some \( \beta_\varepsilon \in (\varepsilon, \mu) \), we have \( g_\varepsilon < f_{\beta_\varepsilon} \mod J_\theta^{bd} \).

On the other hand, if \( \text{cf}(\varepsilon) = (2^\theta)^+ \), without loss of generality, \( \text{cf}(f_\varepsilon(i)) = \text{cf}(\varepsilon) \) for every \( i < \theta \) (see [Sh:g, Ch.II, \S 1]), so there is \( \gamma_\varepsilon < \varepsilon \) such that

\[
h_\varepsilon < f_{\gamma_\varepsilon} \mod J_\theta^{bd} \quad \text{where} \quad h_\varepsilon(i) = \sup(f_\varepsilon(i) \cap \bigcup_{j < \theta} M_j^i).
\]

So for some \( \gamma(*) < \mu \) we have:

\[
S_\delta[\text{St}] = \{\varepsilon < \mu : \text{cf}(\varepsilon) = (2^\theta)^+, \text{and } \gamma_\varepsilon = \gamma(*)\} \quad \text{is stationary.}
\]

Now, for each \( \delta \in S \) we can consider the set \( C_\delta \) of all possible such \( \langle M^\varepsilon, \eta^\varepsilon : \varepsilon < \mu \rangle \), where \( M^\varepsilon = (M_j^\varepsilon : j < i) \), \( \eta^\varepsilon_i \) are as above (letting \( \text{St} \) vary on all strategies of player I for which \( \alpha \in C_\delta \) & \( \text{cf}(\alpha) > \theta \) \( \Rightarrow \mathcal{H}_{\xi<\nu}(\alpha) \) is closed under \( \text{St} \)).

A better way to write the members of \( C_\delta \) is \( \langle (M_j^\varepsilon, \eta^\varepsilon_j : j < \theta) : \varepsilon < \mu \rangle \), but for \( j < \theta, f_{\xi(1)} J_j = f_{\xi(2)} j \Rightarrow M_j^{(1)} = M_j^{(2)} \text{ & } \eta^\xi_j^{(1)} = \eta^\xi_j^{(2)} \); actually it is a function from \( \{f, j : \varepsilon < \mu, j < \theta\} \) to \( \mathcal{H}_{\xi<\nu}(\delta) \). But the domain has power \( \kappa \), the range has power \( |\delta| \leq \mu \). So \( |C_\delta| \leq \mu^\kappa = (2^\kappa)^+ = 2^\kappa = \mu \).

So we can well order \( C_\delta \) in a sequence of length \( \mu \), and choose by induction on \( \varepsilon < \mu \) a representative of each for \( W \) satisfying the requirements.

Second case: assume \( \mu \) is singular.

So let \( \mu = \sum_{\xi < \text{cf}(\mu)} \mu_\xi, \mu_\xi \) regular, without loss of generality \( \mu_\xi > (\sum_\xi \mu_\xi : \xi < \xi_\xi)^+ + (\text{cf}(\mu))^+ \). We know that \( \text{cf}(\mu) > \kappa \), and again by [Sh:g, Ch.VIII, \S 1] there are \( \langle \kappa_\xi, i : i < \theta \rangle, \langle \kappa_i : i < \theta \rangle \) such that:

\[
\text{tcf}(\prod_{i < \theta} \kappa_\xi, i / J_\theta^{bd}) = \mu_\xi, \quad \text{tcf}(\prod_{i < \theta} \kappa_i / J_\theta^{bd}) = \text{cf}(\mu),
\]

\[
k_\xi^a < \kappa_\xi < k_\xi^b, \quad k_i^a < k_i < k_i^b \quad \text{and} \quad i < j \Rightarrow k_i^b < k_j^a
\]

(we can even get \( k_i^a > \prod_{j < i} k_j^b \) as we can uniformize on \( \xi \)).

Let \( \langle f_\varepsilon : \varepsilon < \mu_\xi \rangle, \langle f_\varepsilon : \varepsilon < \text{cf}(\mu) \rangle \) witness the true cofinalities. Now, for every \( f \in \prod_i \kappa_i \) (for simplicity such that \( f(i) > \sum_{j < i} \kappa_j \), \( \sum_{i} \text{cf}(f(i)) = (2^\theta)^+ \)) and \( \xi \) we can repeat the previous argument for \( \langle f + f_\varepsilon : \varepsilon < \mu_\xi \rangle \). After “cleaning inside”, replacing by a subset of power \( \mu_\xi \) we find a common bound below \( \prod_i \kappa_i \), and below \( \prod f \), and we can uniformize on \( \xi \).

Thus we apply on every \( f_\varepsilon, \text{cf}(\varepsilon) = (2^\theta)^+ \) and use the same argument on the bound we have just gotten.

2) Should be clear. \( \square_{3.33} \)

{6.11} Similarly to 3.22 with \( \omega^2 \) for \( \theta \), (not a cardinal!) we have
Claim 3.35. Suppose that
\[ (*) \lambda \text{ is a regular cardinal, } \theta = \aleph_0, \mu = \mu^{<\chi(*)} < \lambda \leq 2^\mu, S \subseteq \{ \delta < \lambda : \text{cf}(\delta) = \aleph_0 \} \text{ is stationary and } \aleph_0 < \chi(*) = \text{cf}(\chi(*)). \]

Then we can find
\[ W = \{ (M^\alpha, \eta^\alpha) : \alpha < \alpha(*) \} \]
and functions
\[ \tilde{\zeta} : \alpha(*) \rightarrow S \text{ and } h : \alpha(*) \rightarrow \lambda \]
such that:

(a0) like 3.12,
\[ M^\alpha = (M^\alpha : i \leq \omega^2) \text{ is an increasing continuous elementary chain } (\tau(M^\alpha_i), \text{ the vocabulary, may be increasing too and belongs to } \mathcal{H}_{<\chi(*)}(\chi(*)), \text{ each } M^\alpha_i \text{ is a model belonging to } \mathcal{H}_{<\chi(*)}(\lambda) \text{ [so necessarily has cardinality } < \chi(*)], \]
\[ M^\alpha \cap \chi(*) \text{ is an ordinal, } [\chi(*)] = \chi^+ \Rightarrow \chi + 1 \subseteq M^\alpha, \]
\[ \eta^\alpha \in \omega^2 \lambda \text{ is increasing with limit } \tilde{\zeta}(\alpha) \in S, \eta^\alpha[i] \in M^\alpha_i, M^\alpha_i \text{ belongs to } \mathcal{H}_{<\chi(*)}(\eta^\alpha(i)) \text{ and } \langle M^\alpha_i : i \leq j \rangle \text{ belongs to } M^\alpha_{j+1}. \]

\[ (a2) \text{ like 3.12 (with } \omega^2 \text{ instead } \theta), \]
\[ (b0), (b1), (b2) \text{ as in 3.12}, \]
\[ (b1)^* \text{ as in 3.22}, \]
\[ (c1) \text{ if } \tilde{\zeta}(\alpha) = \tilde{\zeta}(\beta) \text{ then } M^\alpha_\omega \cap \mu = M^\beta_\omega \cap \mu \text{ and there is an isomorphism } \]
\[ h_{\alpha, \beta} \text{ from } M^\alpha_\omega \text{ onto } M^\beta_\omega \text{ mapping } \eta^\alpha(i) \text{ to } \eta^\beta(i), M^\alpha_i \text{ to } M^\beta_i \text{ for } i < \omega^2, h_{\alpha, \beta}|(\bigcup [M^\alpha_i \cap |M^\beta_\omega|]) \text{ is the identity}, \]
\[ (c2) \text{ as in 3.22 using } \langle M^\alpha_n : n < \omega \rangle, \]
\[ (c3) \text{ as in 3.26 assuming } \lambda = \mu^+, \]
\[ (c4) \eta^\alpha(i) > \sup(|M^\alpha_\omega| \cap \lambda) \text{ (so } \sup(|M^\alpha_{\omega(n+1)}| \cap \lambda) = \bigcup _\ell \eta^\alpha(\omega n + \ell)). \]

Proof. We use \( \tilde{\alpha} = \alpha < \alpha(*) \) which we got in 3.22. Now for each \( \alpha \) we look at \( \bigcup _{n<\omega} M^\alpha_n \) as an elementary submodel of \( (\mathcal{H}_{<\chi(*)}(\lambda), \in) \) with a function \( St \) (intended as strategy for player I, in the play for (a2) above).

Play in \( \bigcup _{n<\omega} M^\alpha_n \) and get
\[ \langle M^\alpha_i, \eta^\alpha(i) : i < \omega n \rangle \in M^\alpha_n, \]
\[ \sup(\eta^\alpha(i) : i < \omega n) \in M^\alpha_{n+1}, \]
\[ \eta^\alpha(\omega n) > \sup(M^\alpha_n \cap \lambda). \]

Proof. By \( [Sh:331, 2.10(2)] \) or see \( [Sh:365] \), we know

Lemma 3.36. Assume that \( \lambda \geq \chi(*) > \theta \) are regular cardinals, \( S \subseteq \{ \delta < \lambda : \text{cf}(\delta) = \theta \} \) is a stationary set, \( \lambda^{<\chi(*)} = \lambda \), and the conclusion of 3.33 holds for them. Then it holds also for \( \lambda^+ \) instead of \( \lambda \).

Proof. By \( [Sh:331, 2.10(2)] \) or see \( [Sh:365] \), we know
there are \( \langle C_\delta : \delta < \lambda^+ \text{ and } \text{cf}(\delta) = \theta \rangle, \langle e_\alpha : \alpha < \lambda^+ \rangle \) such that:

(i) \( C_\delta \) is a club of \( \delta \) of order type \( \theta \), \( \alpha \in C_\delta \text{ and } \alpha > \sup(C_\delta \cap \alpha) \Rightarrow \text{cf}(\alpha) = \lambda \),

(ii) \( e_\alpha \) is a club of \( \alpha \) of order type \( \text{cf}(\alpha) \), \( e_\alpha = \{ \beta_\alpha^i : i < \text{cf}(\alpha) \} \) (increasing continuous),

(iii) if \( E \) is a club of \( \lambda^+ \) then for stationarily many \( \delta < \lambda^+ \), \( \text{cf}(\delta) = \theta \), \( C_\delta \subseteq E \) and the set
\[
\{ i < \lambda : \text{ for every } \alpha \in C_\delta, \text{ cf}(\alpha) = \lambda \Rightarrow \beta_\alpha^{i+1} \in E \}
\]
is unbounded in \( \lambda \).

Now copying the black box of \( \lambda \) on each \( \delta < \lambda^+ \), \( \text{cf}(\delta) = \theta \), we can finish easily. \( \square \)

\textbf{Lemma 3.37.} If \( \lambda, \mu, \kappa, \chi(\ast), S \) are as in 3.33, and

\[ \alpha < \chi(\ast) \Rightarrow |\alpha|^\theta < \chi(\ast) \]

then there is a stationary \( S^* \subseteq \{ A \subseteq \lambda : |A| < \chi(\ast) \} \) and a one-to-one function \( \text{cd} \) from \( S^* \) to \( \lambda \) such that:

\[ A \in S^* \text{ and } B \in S^* \text{ and } A \neq B \text{ and } A \subseteq B \Rightarrow \text{cd}(A) \in B. \]

\textbf{Remark 3.38.} This gives another positive instance to a problem of Zwicker. (See [Sh:247].)

\textbf{Proof.} Similar to the proof of 3.33 only choose

\[ \text{cd} : \{ A : A \subseteq \lambda \text{ and } |A| < \chi(\ast) \} \rightarrow \lambda \]

one-to-one, and then define

\[ S^* \cap \{ A : A \subseteq \alpha, |A| < \chi(\ast) \} \]

by induction on \( \alpha \). \( \square \)

\textbf{Problem 3.39.} 1) Can we prove in ZFC that for some regular \( \lambda > \theta \)

(\( \ast \)) we can define for \( \alpha \in S_\theta^\lambda = \{ \delta < \lambda : \aleph_0 \leq \text{cf}(\delta) = \theta \} \) a model \( M_\alpha \) with a countable vocabulary and universe an unbounded subset of \( \alpha \) of power \( < \chi(\ast) \), \( M_\delta \cap \chi(\ast) \) is an ordinal such that: for every model \( M \) with countable vocabulary and universe \( \lambda \), for some (equivalently: stationarily many) \( \delta \in S_\theta^\lambda, M_\delta \subseteq M \).

2) The same dealing with relational vocabularies only (we call it (\( \ast \)) rel, \( \lambda, \theta, \kappa \)).

\textbf{Remark 3.40.} Note that by 3.8 if (\( \ast \)) rel, \( \lambda, \theta, \kappa \), \( \mu = \text{cf}(\mu) > \lambda \) then (\( \ast \)) el, \( \theta, \kappa \).

\textbf{Remark 3.41—3.45} we return to black boxes for singular \( \lambda \), i.e., we deal with the case \( \text{cf}(\lambda) \leq \theta \).
Lemma 3.41. Suppose that $\lambda^\theta = \lambda^{< \chi(*)}$, $\lambda$ is a singular cardinal, $\theta$ is regular, and $\chi(*)$ is regular $> \theta$.

Assume further

(a) $\operatorname{cf}(\lambda) \leq \theta$, 
(b) $\lambda = \sum_{i \in w} \mu_i$, $|w| \leq \theta$, $w \subseteq \theta^+$ (usually $w = \operatorname{cf}(\lambda)$) and $[i < j \Rightarrow \mu_i < \mu_j]$, and each $\mu_i$ is regular $< \lambda$ and $\operatorname{cf}(\lambda) > \aleph_0 \land \operatorname{cf}(\lambda) = \theta \Rightarrow w = \operatorname{cf}(\lambda)$,

(\gamma) $\mu > \lambda$, $\mu$ is a regular cardinal, $D$ is a uniform filter on $w$ (so $\{\alpha \in w : \alpha > \beta\} \in D$ for each $\beta \in w$), $\mu$ is the true cofinality of $\prod_{i \in w} (\mu_i, <) / D$ (see [Sh:E62, 3.6(2)]=Lc18 or [Sh:g]),

(\delta) $f = (f_i / D : i < \mu)$ exemplifies “the true cofinality of $\prod_i (\mu_i, <) / D$ is $\mu$”, i.e.,

\[
\begin{align*}
\alpha &< \beta < \lambda \Rightarrow f_\alpha / D < f_\beta / D, \\
f &\in \prod_i \mu_i \Rightarrow \bigvee_\alpha f / D < f_\alpha / D,
\end{align*}
\]

(c) $S \subseteq \{\delta < \mu : \operatorname{cf}(\delta) = \theta\}$ is good for $(\mu, \theta, \chi(\cdot))$, and

(\zeta) if $\theta > \operatorname{cf}(\lambda)$, $\delta \in S$, then for some $A_\delta \in D$ and unbounded $B_\delta \subseteq \delta$ we have

\[
\alpha \in B_\delta \land \beta \in B_\delta \land \alpha < \beta \land i \in A_\delta \Rightarrow f_\alpha(i) < f_\beta(i),
\]

i.e., $(f_\alpha \upharpoonright A_\delta : \alpha \in B_\delta)$ is $\prec$-increasing.

Then we can find $W = \{\langle M^\alpha, \eta^\alpha \rangle : \alpha < \alpha(*)\}$ (pedantically a sequence) and functions $\hat{\zeta}$ from $\alpha(*)$ to $S$ and $h$ from $\alpha(*)$ to $\mu$ such that:

(a0), (a1), (a2) as in 3.12 except that we replace (\*) of (a1) by

\[
(*') \quad (i) \quad \eta^\alpha \in \theta^\alpha,
(ii) \quad \text{if } i < \operatorname{cf}(\lambda) \text{ then } \sup(\mu_i \cap \text{Rang}(\eta^\alpha)) = \sup(\mu_i \cap M^\alpha_i),
(iii) \quad \text{if } \xi < \zeta(\alpha) \text{ then } f_\xi / E < \langle \sup(\mu_i \cap M^\alpha_i) : i < \operatorname{cf}(\lambda) \rangle / E \leq f_{\zeta(\alpha)}(\lambda) / E,
\]

(b0) – (b3) as in 3.12.

Proof. For $A \subseteq \theta$ of cardinality $\theta$ let $\operatorname{cd}^4_A X(\cdot) : \mathcal{H}_X(\cdot)(\lambda) \rightarrow \mathcal{A}_X(\lambda)$ be one-to-one, and $G : \lambda \rightarrow \lambda$ be such that for $\gamma$ divisible by $|\gamma|$, $\alpha < \gamma \leq \lambda$ ($\mu \geq \aleph_0$), the set $\{\beta < \gamma : G(\beta) = \alpha\}$ is unbounded in $\gamma$ and of order type $\gamma$. Let $A = \{A_i : i < \theta\}$ be a sequence of pairwise disjoint subsets of $\theta$ each of cardinality $\theta$.

Let for $\delta \in S$

\[
W_\delta^0 = \{(\bar{M}, \eta) : \bar{M}, \eta \text{ satisfy (a1)}\},
\]

and for some $y \in \mathcal{H}_X(\cdot)(\lambda)$, for every $i < \theta$ we have

\[
\langle G(\eta(i)) : i \in A_i \rangle = \operatorname{cd}^4_A X(\cdot)(\langle \bar{M}[j, \eta[j, y]\rangle),
\]

The rest is as before.
Claim 3.42. Suppose that $\lambda^0 = \lambda^{<\chi(*)}$, $\lambda$ is singular, $\theta, \chi(*)$ are regular, $\chi(*) > \theta$.
1) If $(\forall \alpha < \lambda)[|\alpha|^{<\chi(*)} < \lambda]$ then by $\lambda^0 = \lambda^{<\chi(*)}$ we know that either $\text{cf}(\lambda) \geq \chi(*)$ (and so lemma 3.18 applies) or $\text{cf}(\lambda) \leq \theta$.
2) We can find regular $\mu_i$ $(i < \text{cf}(\lambda))$ increasing with $i$, $\lambda = \sum_{i < \text{cf}(\lambda)} \mu_i$.
3) For $(\mu_i : i \in w)$ as in 3.41(3) we can find $D, \mu, \bar{f}$ as in 3.41(γ),(δ), $D$ the co-bounded filter plus one unbounded subset of $\omega$.
4) For $(\mu_i : i \in w)$, $D, \mu, \bar{f}$ as in $(\beta), (\gamma), (\delta)$ of 3.41 we can find $\mu$ and pairwise disjoint $S \subseteq \mu$ as required in $(\epsilon)$, $(\delta)$ of 3.41 provided that $\theta > \text{cf}(\lambda) = 2^\theta < \mu$ [equivalently $\mu$]
5) If $\text{cf}(\lambda) > \aleph_0$, $(\forall \alpha < \lambda)[|\alpha|^{|\alpha|} < \lambda]$, $\lambda < \mu = \text{cf}(\mu) \leq \text{cf}(\lambda)$ then we can find $(\mu_i : i < \text{cf}(\lambda))$, and the co-bounded filter $D$ on $\text{cf}(\lambda)$ as required in $(\beta), (\gamma)$ of 3.31.

Proof. Now 1),2),3) are trivial, for 5) see [Sh:345 §9]. As for 4), we should recall [Sh:345, §5] actually say:

Fact 3.43. If $(\mu_i : i \in w), \bar{f}, D$ are as in 3.41, then

$$S = \{\delta < \mu : \text{cf}(\delta) = \theta \text{ and there are } A_\delta \in D, \text{ and unbounded } B_\delta \subseteq \delta$$

such that $\{a \in B_\delta \land \beta \in B_\delta \land \alpha < \beta \land i \in A_\delta f_\alpha(i) < f_\beta(i)\}$. is good for $(\mu, \theta, \chi(*)$).

Lemma 3.44. Let $\chi(1) = \chi(*) + (\leq \chi(*)\theta)$.

In 3.41, if $\lambda^0 = \lambda^{(1)}$, we can strengthen (b1) to (b1) +(of 3.20).


Lemma 3.45. $\frac{3.17}{3.11} \times 3.29$ and $\frac{3.19}{3.11} \times 3.37$ hold (we need also the parallel to 3.33).

Proof. Left to the reader.

* * *

Now we draw some conclusions.

The first, 3.46, gives what we need in 2.7 (so 2.3).

Conclusion 3.46. Suppose $\lambda^0 = \lambda^{<\chi(*)}$, $\text{cf}(\lambda) \geq \chi(*) + \theta^+$, $\theta = \text{cf}(\theta) < \chi(*) = \text{cf}(\chi(*)$). Then we can find

$$W = \{(\bar{M}_\alpha, \eta^\alpha) : \alpha < \alpha(*)\}, M_1^\alpha = (N_1^\alpha, A_1^\alpha, B_1^\alpha),$$

where

$$A_1^\alpha \subseteq \lambda \cap |N_1^\alpha|, B_1^\alpha \subseteq \lambda \cap |N_1^\alpha|, A_1^\alpha \neq B_1^\alpha,$$

and functions $\xi, h$ such that:

(a0), (a1) as in 3.12;
(a2) as in 3.12 except that in the game, player I can choose $M_1$, only as above;
First assume 

Proof. First assume \( \lambda \) is regular, and \( \mathbf{W} = \{ (\bar{M}^\alpha, \eta^\alpha) : \alpha < \alpha(\ast) \}, \bar{\zeta}, h \) be as in the conclusion of 3.12. Let \( w = \{ \text{cd}(\alpha, \beta) : \alpha, \beta < \lambda \} \), and \( G_1, G_2 : w \rightarrow \lambda \) be such that for \( \alpha \in E, \alpha = \text{cd}(G_1(\alpha), G_2(\alpha)) \).

Let

\[
Y = \{ \alpha < \alpha(\ast) : \bar{M}^\alpha \text{ has the form } (N^\alpha, A^\alpha, B^\alpha),
\]

\[
A^\alpha, B^\alpha \text{ distinct subsets of } \lambda \in \text{card}(\lambda) \text{ (equivalently, monadic relations), } h(\alpha) \in E, \text{ and }
\]

\[
G_2(h(\alpha)) = \min \{ \gamma : \gamma \in A^\alpha \setminus B^\alpha \text{ or } \gamma \in B^\alpha \setminus A^\alpha \}.
\]

Now we let

\[
\mathbf{W^*} = \{ (\bar{M}^\alpha, \eta^\alpha) : \alpha \in Y, \bar{\zeta}^* = \bar{\zeta}|Y, h^* = G_1 \circ h \}.
\]

They exemplify that 3.46 holds.

What if \( \lambda \) is singular? Still \( \text{cf}(\lambda) \geq \chi(\ast) + \theta^* \) and we can just use 3.18 instead of 3.12. \( \square \) 3.46

\[
\text{Claim 3.47. 1) In 3.12, if } \lambda = \lambda^{< \chi(\ast)}, \text{ we can let } h : S \rightarrow \mathcal{H}^\chi(\ast) \text{ be onto; generally we can still make } \text{Rang}(h) \in S, \text{ whenever } |A| = \lambda.
\]

\[
2) \text{In 3.12, by its proof, whenever } S' \subseteq S \text{ is stationary, and } \bigwedge (h^{-1}(\zeta) \cap S') \text{ stationary (by } \zeta \text{)} \text{ then } \{ (\bar{M}^\alpha, \eta^\alpha) : \alpha < \alpha(\ast), \bar{\zeta}(\alpha) \in S' \} \text{ satisfies the same conclusion.}
\]

\[
3) \text{For any unbounded } a \subseteq \theta \text{ we can let player I choose also } \eta(i) \text{ for } i \in \theta \setminus a, \text{ without changing our conclusions.}
\]

\[
4) \text{Similar statements hold for the parallel claims.}
\]

\[
5) \text{It is natural to have } \chi(\ast) = \chi^+.
\]

Proof. Straightforward. \( \square 3.47 \)

\[
\text{Fact 3.48. We can make the following changes in (a1), (a2) of 3.12 (and in all similar lemmas here) getting equivalent statements:}
\]

\[
(*) M^\alpha \in \mathcal{H}^{\prec \chi}(\lambda + \lambda); \text{ in the game, for some arbitrary } \lambda^* \geq \lambda \text{ (but fix during the game) player I chooses the } M^\alpha \in \mathcal{H}(\lambda^*) \text{ (of cardinality } \chi(\ast)), \text{ and in the end instead } \bigwedge M^\alpha = M^\alpha \text{ we have “there is an isomorphism from } M_\theta \text{ onto } M^\alpha \text{ taking } M_\theta \text{ onto } M^\alpha \text{ and is the identity on } M_\theta \cap \mathcal{H}^{\prec \chi}(\lambda) \text{ and maps } |M_\theta| \setminus \mathcal{H}(\lambda) \text{ into } \mathcal{H}^{\prec \chi}(\lambda + \lambda) \setminus \mathcal{H}^{\prec \chi}(\lambda) \text{ and preserves } \in \text{ and } \notin \text{ and “being an ordinal” and “not being an ordinal”.}
\]

Exercise 3.49. If \( D \) is a normal fine filter on \( \mathcal{P}(\mu), \lambda \) is regular, \( \lambda \leq \mu, S \subseteq \delta < \lambda : \text{cf}(\delta) = \theta \) is stationary, moreover:

\[
(*)_{D, S} \{ a \in \mathcal{P}(\mu) : \sup(a \cap \lambda) \in S \} \neq \emptyset \mod D
\]

then we can partition \( S \) to \( \lambda \) stationary disjoint subsets each satisfying (*).

[Hint: like the proof of 3.3.]
Notation 3.50. 1) Let $\kappa$ be an uncountable regular cardinal. We let $\text{seq}_c^\kappa(\mathcal{A})$, where $\mathcal{A}$ is an expansion of a submodel of some $\mathcal{H}_\kappa(\lambda)$ with $|\tau(\mathcal{A})| \leq \chi$, be the set of sequences $(M_i : i < \alpha)$, which are increasing continuous, $M_i \subseteq \mathcal{A}$, $|M_i| < \kappa$, $M_i \cap \kappa \in \mathcal{K}$, $\kappa = \kappa_1^+ \Rightarrow \kappa_1 + 1 \subseteq M_i$, $(M_j : j \leq i) \in M_{i+1}$. (If $\alpha = \delta$ is limit, $M_\delta = \bigcup_{i < \delta} M_i$).

2) If $\kappa = \kappa_1^+$, we may write $\leq \kappa_1$ instead of $< \kappa$.

We repeat the definition of filters introduced in [Sh:52, Definition 3.2].

Definition 3.51. 1) $E_\kappa^\theta(A)$ is the following filter on $[A]^{<\kappa}$: $Y \in E_\kappa^\theta(A)$ if and only if for (every) $\chi$ large enough, for some $x \in \mathcal{H}(\chi)$ the set $\{(\bigcup M_i) \cap A : (M_i : i \leq \theta) \in E_\kappa^\theta(\mathcal{H}(\chi), x)\}$ is included in $Y$.

Exercise 3.52. Let $\lambda$, $\kappa$, $\theta$, and $Y \subseteq [\lambda]^{<\kappa}$ be given. Then

$$(a) \Rightarrow (b) \Rightarrow (c),$$

where

$$Y = \{M_\alpha : \alpha < \alpha(\ast)\}, \zeta, h$$

satisfy 3.12,

and

$$(\ast) \alpha \neq \beta \land \bigwedge_{i < \theta} \eta_\alpha \in M_\beta \Rightarrow \alpha < \beta.$$

(b) $\Diamond_{E_\kappa^\theta(\lambda)}$ holds.

(c) Like (a) without $(\ast)$.

Exercise 3.53. If $\lambda\kappa^+ = \lambda$, $\theta \leq \kappa$ then $\Diamond_{E_\kappa^\theta(\lambda)}$ (main case: $\kappa = \theta$).

Exercise 3.54. If $\lambda = \mu^+$, $\lambda^\kappa = \lambda$, $\theta = \aleph_0$, $\kappa = \kappa^\theta$, then there is a coding set with diamond (see [Sh:247]).

Exercise 3.55. Suppose that $\text{cf}(\lambda) > \aleph_0$, $2^\lambda = \chi^{\text{cf}(\lambda)}$, $\chi(\ast) > \text{cf}(\chi)$, $(\forall \alpha < \lambda)[\alpha]^{\chi(\ast)} < \lambda$, $\mathfrak{C}$ is a model expanding $(\mathcal{H}_{\leq \chi(\ast)}(\lambda), \in), |\tau(\mathfrak{C})| \leq \aleph_0$. Then we can find $\{M_\alpha : \alpha < \alpha(\ast)\}$ such that:

(i) $\tilde{M}_\alpha = (M^\alpha : i < \sigma)$, $M^\alpha \subseteq \mathcal{H}_{\leq \chi(\ast)}(\lambda)$, $M^\alpha \cap \chi(\ast)$ is an ordinal, $M^\sigma | \tau(\mathfrak{C}) \times E$, $[i < j \Rightarrow M_i^\alpha < M_j^\alpha]$, $(M_j^\alpha : j \leq i) \in M_{i+1}^\alpha$,

(ii) if $f_\alpha$ is a $\kappa_\alpha^+$-place function from $\lambda$ to $\mathcal{H}_{\leq \chi(\ast)}(\lambda)$ then for some $\alpha$, $M_\alpha^\alpha \times (\mathfrak{C}, f_\alpha)|_{\alpha < \alpha}$.

Exercise 3.56. Suppose $\theta = \text{cf}(\mu) < \mu$, $(\forall \alpha < \mu)[\alpha]^{\theta} < \mu$, $2^\mu = \mu^\theta$, and $\lambda = (2^\mu)^+$, $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \theta\}$. Let $\mu = \sum_{i < \theta} \mu_i$, $\mu_i$ regular strictly increasing, and $\text{cf}(\prod_{i < \theta} \mu_i / E) = 2^\alpha$. Then we can find

$W = \{(\tilde{M}_\alpha, \eta^\alpha) : \alpha < \alpha(\ast)\}, \tilde{\zeta} : \alpha(\ast) \rightarrow S$, $h : \alpha(\ast) \rightarrow \lambda$

such that:
for $\delta \in S$ there is a club $C_\delta$ of $\delta$ of order type $\theta$ such that

$$\alpha \in C_\delta \land \operatorname{otp}(\alpha \cap C_\delta) = \gamma + 1 \implies \operatorname{cf}(\alpha) = \mu_\gamma.$$  \hfill{6.26F}

Remark 3.57. We do not know if the existence of a Black Box for $\lambda^+$ with $h$ one-to-one follows from ZFC (of course it is a consequence of $\Diamond$). On the other hand, it is difficult to get rid of such a Black Box (i.e., prove the consistency of non-existence).

If $\lambda = \lambda^{<\lambda}$ then we have $h : S \rightarrow \lambda$, $S \subseteq \{\delta < \lambda^+ : \operatorname{cf}(\delta) < \lambda\}$ such that $C_\delta$ is a club of $\delta$, $\operatorname{otp}(C_\delta) = \operatorname{cf}(\delta)$ and

$$\forall \text{club } (C \subseteq \alpha \in C_\delta)$$

$$[\operatorname{cf}(\alpha) > \aleph_0 \land \min_{C' \text{ club of } C_\alpha'} \sup(h|_{C'}) = \operatorname{otp}(C \cap \alpha)].$$

This is hard to get rid of, (i.e., hard to find a forcing notion making it no longer a black box, without collapsing too many cardinals); compare with Mekler-Shelah [MkSh:274].
\section*{4. On Partitions to Stationary Sets}

We present some results on the club filter on $[\kappa]^{\aleph_0}$ and some relatives, and \Diamond (see Definition [Sh:E62, 4.6=Ld12] or 4.4(2) below). There are overlaps of the claims hence redundant parts which still have some interest.

\begin{claim}
Assume $\kappa$ is a cardinal $> \aleph_1$, then $[\kappa]^{\aleph_0}$ can be partitioned to $\kappa^{\aleph_0}$ (pairwise disjoint) stationary sets.
\end{claim}

\begin{proof}
Follows by 4.2 below, [in details, let $\tau$ be the vocabulary $\{c_n : n < \omega\}$ where each $c_n$ is an individual constant. By 4.2 below there is a sequence $M = (M_n : u \in [\kappa]^{\aleph_0})$ of $\tau$-models, with $M_n$ having universe $u$ such that $M$ is a diamond sequence.

For each $\eta \in \Diamond \lambda$ let $\mathcal{N}_\eta$ be the set $u \in [\kappa]^{\aleph_0}$ such that for every $n < \omega$ we have $c_n^{M_u} = \eta(n)$.

By the choice of $\tilde{M}$ necessarily each set $\mathcal{N}_\eta$ is a stationary subset of $[\kappa]^{\aleph_0}$, and trivially those sets are pairwise disjoint. \hfill \Box_{4.1}
\end{proof}

\begin{claim}
Let $\kappa > \aleph_1$. Then we have diamond on $[\kappa]^{\aleph_0}$ (modulo the filter of clubs on it, see 4.4(2) or [Sh:E62, 4.6=Ld12]), and we can find $A_\alpha \subseteq [\kappa]^{\aleph_0}$ for $\alpha < \lambda := 2^{\aleph_0}$ such that each is stationary but the intersection of any two is not.
\end{claim}

\begin{proof}
The existence of the $A_\alpha$'s for $\alpha < \lambda$ follows from the other result. Let $\tau$ be a countable vocabulary, $\tau_1 = \tau \cup \{<\}$; First we prove it when $\kappa = \aleph_2 \in [\aleph_2, 2^{\aleph_0})$.

Let $\omega \setminus \{0\}$ be the disjoint union of $s_n$ for $n < \omega$, each $s_n$ is infinite with the first element $> n + 3$ when $n > 0$. Let $\langle C_\delta : \delta \in S_0^0 \rangle$ be club guessing, where $S_0^0 = \{\delta < \omega_2 : \text{cf}(\delta) = \aleph_0\}$, such that $C_\delta \subseteq \delta = \sup(C_\delta)$ has order type $\omega$.

Let $\langle (\mathfrak{A}_k, \alpha_k^\xi) : \xi < 2^{\aleph_0} \rangle$ list without repetitions the pairs $(\mathfrak{A}_k, \alpha_k)$, $\mathfrak{A}_k$ a model with vocabulary $\tau_1$ and universe a limit countable ordinal and $\alpha_0 = (\alpha_n : n < \omega)$ an increasing sequence of ordinals with limit sup(\mathfrak{A}) and $\mathfrak{A}|\alpha_0 < \mathfrak{A}$. Let $E_n$ be the following equivalence relation relation on $2^{\aleph_0} : \xi E_n \zeta$ iff $(\mathfrak{A}_n|\alpha_n^\xi, \alpha_n^\zeta|n)$ is isomorphic to $(\mathfrak{A}_n^\xi|\alpha_n^\xi, \alpha_n^\zeta|n)$ which means: there is an isomorphism $f$ from $\mathfrak{A}_n|\alpha_n^\xi$ onto $\mathfrak{A}_n|\alpha_n^\zeta$ which maps $\mathfrak{A}_n^\xi|\alpha_k^\xi$ onto $\mathfrak{A}_n^\zeta|\alpha_k^\zeta$ for $k < n$ and is an order preserving function (for the ordinals, alternatively we restrict ourselves to the case $< \text{interpreted as a well ordering}$).

We can find subsets $t^\xi_n$ of $\omega$ such that:

\begin{itemize}
  \item[$\ast$] for $\zeta, \epsilon < 2^{\aleph_0}$ we have $t^\xi_n \cap s_n = t^\epsilon_n \cap s_n$ iff $\mathfrak{A}_n^\xi|\alpha_n^\xi = \mathfrak{A}_n^\epsilon|\alpha_n^\epsilon$ and $\alpha_k^\zeta = \alpha_k^\epsilon$ for $k \leq n$. Also $t^\xi_n \cap s_n$ is infinite and $\epsilon \neq \zeta \Rightarrow \aleph_0 > |t^\epsilon \cap t^\xi| \cap s_n$ for simplicity (so $t^\xi_n \cap s_n$ depend just on $\xi/E_n$, in fact code it).
\end{itemize}

For $\zeta < 2^{\aleph_0}$ let

$$\mathcal{N}_\zeta := \{a \in [\kappa]^{\aleph_0} : t_\zeta = |C_\sup(a) \cap \beta| : \beta \in a\},$$

and let

$$\mathcal{N}_\zeta^\prime = \{a \in \mathcal{N} : \text{otp}(a) = \text{otp}(\mathfrak{A}_n^\zeta)\},$$

and for $a \in \mathcal{N}_\zeta^\prime$ let $N_a$ be the model isomorphic to $\mathfrak{A}_n^\zeta$ by the function $f_a$, where $\text{Dom}(f_a) = a$, $f_a(\gamma) = \text{otp}(\gamma \cap a)$.

Let $\mathcal{N}$ be the union of $\mathcal{N}_\zeta^\prime$ for $\zeta < 2^{\aleph_0}$. Clearly $\zeta \neq \xi \Rightarrow \mathcal{N}_\zeta \cap \mathcal{N}_\xi = \emptyset$, and so $\mathcal{N}_\zeta \cap \mathcal{N}_\xi^\prime = \emptyset$. Hence $N_a$ is well defined for $a \in \mathcal{N}$.\hfill \Box_{4.2}
Let $K_n$ be the set of pairs $(\mathcal{A}, \alpha)$ such that $\mathcal{A}$ is a $\tau_1$-model with universe a countable subset of $\kappa$ with no last member, and $\alpha$ is an increasing sequence of ordinals $< \kappa$ of length $n$ such that $\alpha_k < \sup(\mathcal{A})$ and $[\alpha_k, \alpha_{k+1}) \cap \mathcal{A} \neq \emptyset$ and $\mathcal{A}|\alpha_k \prec \mathcal{A}$. So clearly there is a function $cd_n : K_n \to \mathcal{P}(s_n)$ such that: if $\zeta < 2^{\aleph_0}$ then $cd_n(\mathcal{A}, \alpha) = \ell^\kappa \cap s_n$ iff the pairs $(\mathcal{A}, \alpha), ((\mathcal{A}^\kappa, \alpha^\kappa)|n)$ are isomorphic.

Let $M$ be a $\tau_1$-model with universe $\kappa$. Now (see [Sh:E62, 1.16=L1.15], or history in the introduction of §3, and the proof of 3.24) we can find a full subtree $\mathcal{T}$ of $\omega^\kappa(\aleph_2)$ (i.e., it is non-empty, closed under initial segments and each member has $\aleph_2$ immediate successors) and elementary submodels $N_\eta$ of $M$ for $\eta \in \mathcal{T}$ such that:

1. $\text{Rang}(\eta) \subseteq N_\eta$,
2. if $\eta$ is an initial segment of $\rho$ then $N_\eta$ is a submodel $N_\rho$, moreover $N_\eta \cap \aleph_2$ is an initial segment of $N_\rho \cap \aleph_2$.

Now let $E$ be the set of $\delta < \aleph_2$ satisfying: if $\rho \in \mathcal{T}$ and $\rho \in \omega^\kappa \delta$ then $N_\rho \cap \aleph_2$ is a bounded subset of $\delta$, and $\delta$ is a limit ordinal. Let $E_1$ be the set of $\delta \in E$ such that if $\rho \in \mathcal{T} \cap \omega^\kappa \delta$ then for every $\beta < \delta$ there is $\gamma$ such that $\beta < \gamma < \delta$ and $\eta^\kappa(\gamma) \in \mathcal{T}$. So by the choice of $(C_\delta : \delta \in S)$ for some $\delta \in S$ we have $C_\delta \subseteq E_1$.

Let $(\alpha_\delta, k : k < \omega)$ list $C_\delta$ in increasing order.

Now we choose by induction on $n$ a triple $(\eta_n, s_n^\kappa, \alpha_n, k_n)$ such that:

1. $\eta_n \in \mathcal{T}$ has length $n$ (so $\eta_0$ is necessarily $()$)
2. if $n = m + 1$ then $\eta_n$ is a successor of $\eta_m$
3. $s_n^\kappa$ is $cd_n((N_{\eta_m}, \langle \alpha_\ell : \ell < n \rangle))$ if the pair $(N_{\eta_m}, \langle \alpha_\ell : \ell < n \rangle)$ belongs to $K_n$ and is $s_n$ otherwise; actually it is so,
4. $\alpha_n = \sup(N_{\eta_m}) + 1$
5. $k_n = \min\{k : N_{\eta_n} \subseteq \alpha_\delta, k \}$ and $k_0 = 0$ and $\bar{n}[0, k_n] \subseteq \bigcup_{\ell < n} s_\ell \cup \{0\}$
6. if $n = m + 1$ and $k_m < k_n$ then
   1. $\min(N_{\eta_n} \setminus N_{\eta_m}) > \alpha_\delta, k_m - 1$
   2. $(k_m, k_n)$ is disjoint to $\bigcup_{\ell < n} s_\ell^\kappa$
   3. $k_n \in \bigcup\{s_\ell^\kappa : \ell < n\}$
   4. $k_n$ is minimal under those restrictions.
7. if $n = m + 1$ and $k_m = k_n$ then we cannot find $k \in (k_m, \omega)$ satisfying $(\beta), (\gamma)$ of clause (f).

There is no problem to carry the induction. In the end let $\eta = \bigcup_n \eta_n \in \text{lim}(i, \mathcal{T})$, so we get a $\tau_1$-model $N_{\eta} =: \bigcup\{N_{\eta_n} : n < \omega\}$, and an increasing sequence $(\alpha_n : n < \omega)$ of ordinals with limit $\sup(\mathcal{A})$. Now by the choice of $((\mathcal{A}^\kappa, \alpha^\kappa) : \zeta < 2^{\aleph_0})$ clearly for some $\zeta$ we have $(N_{\eta}, \alpha)$, $(\mathcal{A}^\kappa, \alpha^\kappa)$ are isomorphic, so necessarily $(N_{\eta}|\alpha_n, \alpha^\kappa|n)$ belongs to $K_n$ and necessarily $cd_n(N_{\eta}, \langle \alpha_\ell : \ell < n \rangle) = s_n^\kappa$.

Also clearly $\sup(N_{\eta}) = \delta$ and $\{k_n : n < \omega\} = \{\text{Rang}(\beta) : \beta \in N_{\eta}\} = \{\alpha_\delta, k_n : n < \omega\}$.

Letting $a$ be the universe of $N_{\eta}$ it follows that $a \in \mathcal{T}$ so $N_a$ is well defined and isomorphic to $\mathcal{A}^\kappa$ hence to $N_{\eta}$ using $\delta^M$ we get $N_a = N_{\eta}$. But $N_{\eta} \prec M$. So $\langle N_a : a \in \mathcal{T} \rangle$ is really a diamond sequence, well for $\tau_1$-models rather than $\tau$-models, but this does no harm and will help for $\kappa > \aleph_2$.
Second, we consider the case $\kappa > \aleph_2$. For each $c \in [\kappa]^{\aleph_0}$, if $\text{otp}(c) = \text{otp}(c \cap \kappa_2, <, N_{\kappa_2})$, let $g_c$ be the unique isomorphism from $(c \cap \kappa_2, <, N_{\kappa_2})$ onto $(c, <)$, the usual order, and let $M_c$ be the $\tau$-model with universe $c$ such that $g$ is an isomorphism from $N_{\kappa_2}$ onto $M_c$. Clearly it is an isomorphism and the $M_c$'s form a diamond sequence.

Why? For notational simplicity $\tau$ has predicates only. Let $M_0 = M$ be a $\tau$-model with universe $\kappa$, let $M_1$ be an elementary submodel of $M$ of cardinality $\aleph_2$ such that $\omega_2 \subseteq M_1$, let $h$ be a one-to-one function from $M_1$ onto $\omega_2$ let $M_2$ be a $\tau$-model with universe $\omega_2$ such that $h$ is an isomorphism from $M_1$ onto $M_2$, and let $M_3$ be the $\tau_1$-model expanding $M_2$ such that $h$ is an isomorphism from $M_1$ onto $M_2$, and let $M_3$ be the $\tau_1$-model expanding $M_2$ such that $<_{M_3} = \{(h(a), h(\beta)) : \alpha < \beta \text{ are from } M_1\}$. So for some $a \in \mathcal{X} \subseteq [\kappa]^{\aleph_0}$ we have $N_a \prec M_3$ and $h(a) = \beta \in N_a \wedge \alpha < \omega_2 \Rightarrow \alpha \in a$ (the set of $a$-s satisfying this contains a club of $[\aleph_2]^{\aleph_0}$). Let $c = \{\alpha : h(a) \in a\}$, so clearly $c \cap \omega_2 = a$ and $M_c \prec M_1$ hence $M_4 \prec M$, so we are done. $\square$

Discussion 4.3. Some concluding remark are:
1) We can use other cardinals, but it is natural if we deal with $D_{\kappa, <, \theta, \aleph_0}$ (see below).
2) The context is very near to §3, but the stress is different.

Definition 4.4. Let $\kappa \geq \theta \geq \sigma$, $\theta$ uncountable regular. If $\theta = \mu^+$ we may write $\mu$ instead of $< \theta$.
1) Let $D = D_1 = D_{\kappa, <, \theta, \aleph_0}^1$ be the filter $[\kappa]^{< \theta}$ generated by $\{A_x^1 : x \in \mathcal{X}(\chi)\}$ where
\[
A_x^1 = \{N \cap \kappa : N \text{ is an elementary submodel of } (\mathcal{X}(\chi), \in) \text{ and } N \in \bigcup_{n < \omega} N_n, \text{ } N_n \in N_{n+1} \\
\text{and } \|N_n\| < \theta \text{ and } N_n \cap \theta \in \theta\}.
\]
2) Let $D = D_2 = D_{\kappa, <, \theta, \sigma}^2$ be the filter on $[\kappa]^{< \theta}$ generated by $\{A_x^2 : x \in \mathcal{X}(\chi)\}$ where
\[
A_x^2 = \{N \cap \kappa : N \text{ is an elementary submodel of } (\mathcal{X}(\chi), \in) \text{ and } N \in \bigcup_{n \leq \xi} N_\tau, \text{ } N_\tau \text{ increasing and } N_\tau \cap \theta \in \theta\}
\]
3) For a filter $D$ on $[\kappa]^{< \theta}$ let $\mathcal{D}$ mean: fixing any countable vocabulary $\tau$ there are $S \in D$ and $N = (N_a : a \in S)$, each $N_a$ a $\tau$-model with universe $a$, such that for every $\tau$-model $M$ with universe $\lambda$ we have $\{a \in S : N_a \subseteq M\} \neq \emptyset$ mod $D$.
4) Instead $\leq \theta$ we may write $\theta$. 

Claim 4.5. Assume $\theta \leq \sigma$ and $\kappa > \sigma^+$ and let $D = D_{\kappa, \theta, \aleph_0}$.
1) $[\kappa]^\theta$ can be partitioned to $\sigma^{\aleph_0}$ (pairwise disjoint) $D$-positive sets.
2) Assume in addition that $\sigma^{\aleph_0} \geq 2^\theta$. Then
   (a) we can find $A_\alpha \subseteq [\kappa]^{\theta}$ for $\alpha < \lambda := 2^{\kappa^\theta}$ such that each is $D$-positive but they are pairwise disjoint mod $D$,
   (b) if $\lambda = \kappa^\theta$ and $\tau$ is a countable vocabulary then $\mathcal{D}_{\lambda, \theta, \aleph_0}$, moreover there are $S^* \subseteq [\lambda]^{\theta}$ and function $N^*$ with domain $S^*$ such that
      (a) for distinct $a, b$ from $S^*$ we have $a \cap \kappa \neq b \cap \kappa$,
      (b) for $a \in S^*$ we have $N^*(a) = N_a^*$ is a $\tau$-model with universe $a$, 

(c) for a \( \tau \)-model \( M \) with universe \( \lambda \), the set \( \{ a : N^a_0 = M \upharpoonright a \} \) is stationary.

Proof. Similar to earlier ones: part (1) like Claim 4.1 case (a), part (2) like the proof of Claim 4.2. \( \square \)

Claim 4.6. 1) If \( \theta \leq \kappa_0 \leq \kappa_1 \) and \( \diamondsuit S_0 \) i.e. \( \diamondsuit D_{\kappa_0,0,0} \), where \( S_0 \) is a subset of \( [\kappa_0]^\theta \) which is \( D_{\kappa_0,\theta,0} \)-positive and \( S_1 := \{ a \in [\kappa_1]^\theta : a \cap \kappa_0 \in S_0 \} \), then \( \diamondsuit S_1 \), i.e. \( \diamondsuit D_{\kappa_1,0,0} \).

2) In part (1), if in addition \( \kappa_0 = (\kappa_0)^\theta \) and \( \kappa_2 = (\kappa_1)^\theta \) then we can find \( S_2 \subseteq [\kappa_2]^\theta \) such that:

   (a) \( a \in S_2 \) implies \( a \cap \kappa_0 \in S_0 \),
   (b) if \( b, c \) are distinct members of \( S_2 \) then \( b \cap \kappa_1, c \cap \kappa_1 \) are distinct, and
   (c) \( \diamondsuit S_2 \).

3) If \( \kappa = \kappa^\theta \) then \( \diamondsuit D_{\kappa,0,0} \).

Remark 4.7. This works for other uniform definition of normal filters.

Above, \( \kappa^\theta = \kappa \) can be replaced by: every tree with \( \leq \theta \) nodes has at most \( \theta^* \)-branches and \( \kappa^\theta = \kappa \).

Proof. 1) Easy.
2) Implicit in earlier proof, 4.2.
3) See [Sh:212], [Sh:247] \( \square \)

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