ON SPECTRUM OF $\kappa$–RESPLENDENT MODELS

SAHARON SHELAH

Abstract. We prove that some natural “outside” property of counting models up to isomorphism is equivalent (for a first order class) to being stable.

For a model, being resplendent is a strengthening of being $\kappa$-saturated. Restricting ourselves to the case $\kappa > |T|$ for transparency, a model $M$ is $\kappa$-resplendent means:

when we expand $M$ by $< \kappa$ individual constants $\langle c_i : i < \alpha \rangle$, if $(M, c_i)_{i<\alpha}$ has an elementary extension expandable to be a model of $T'$ where $\text{Th}((M, c_i)_{i<\alpha}) \subseteq T', |T'| < \kappa$ then already $(M, c_i)_{i<\alpha}$ can be expanded to a model of $T'$.

Trivially any saturated model of cardinality $\lambda$ is $\lambda$-resplendent. We ask: how many $\kappa$-resplendent models of a (first order complete) theory $T$ of cardinality $\lambda$ are there? Naturally we restrict ourselves to cardinals $\lambda = \lambda^\kappa + 2^{|T|}$. Then we get a complete and satisfying answer: this depend just on $T$ being stable or unstable. In this case proving that for stable $T$ we get few, is not hard; in fact every resplendent model of $T$ is saturated hence determined by its cardinality up to isomorphism. The inverse is more problematic because naturally we have to use Skolem functions with any $\alpha < \kappa$ places. Normally we use relevant partition theorems (Ramsey theorem or Erdős-Rado theorem), but in our case the relevant partitions theorems fails so we have to be careful.

0. Introduction

Our main conclusion speaks on stability of first order theories, but the major (and the interesting) part of the proof has little to do with it and can be read without knowledge of classification theory (only the short proof of 1.8 uses it), except the meaning of $\kappa < \kappa(T)$ which we can take as the property we use, see inside 2.1(1) here (or see [Sh:E59, 1.5(2)] or [Sh:c]). The point is to construct a model in which for some infinite sequences of elements we have appropriate automorphisms, so we need to use “Skolem” functions with infinitely many places. Now having functions with infinite arity make obtaining models generated by indiscernibles harder. More specifically, the theory of the Skolemizing functions witnessing resplendence for $(M, \bar{b})$ is not continuous in $\text{Th}(M, \bar{b})$. So we use a weaker version of indiscernibility hence though having a linear order is usually a very strong requirement (see [Sh:E59, §3]), in our proof we use it as if we only have trees (with $\kappa$ levels).

In [Sh:a] or [Sh:ac, VI 5.3–5.6] we characterized first order $T$ and cardinals $\lambda$ such that for some first order complete $T_1$, $T \subseteq T_1$, $|T_1| = \lambda$ and any $\tau(T)$–reduct of a model of $T_1$ is saturated.

In [Sh:225] we find the spectrum of strongly $\aleph_1$–saturated models, but have nothing comparable for strongly $\aleph_0$–saturated ones (on better computation of the numbers see [Sh:225a], and more in [Sh:331, 3.2]). Our interest was:

Publication 363. This was supposed to be Ch V to the book “Non-structure” and probably will be if it materialize. Was circulated around 1990.
(A): an instance of complete classification for an “outside” question: the question here is the function giving the number up to isomorphism of $\kappa$-resplendent models of a (first order complete) theory $T$ as a function of the cardinality, we concentrate on the case $\lambda = \lambda^\kappa + 2^{|T|}$. 

(B): an “external” definition of stability which happens to be the dividing line.

Earlier we have such an equivalent “external” definition of stability by saturation of ultra-powers, i.e. Keisler order, see [Sh:c]. Baldwin had told me he was writing a paper on resplendent models: for $\aleph_0$-stable one there are few ($\leq 2^{\aleph_0}$) such models in any cardinality; and for $T$ not superstable — there are $2^\lambda$ models of cardinality $\lambda$ (up to isomorphism).

Note that resplendent models are strongly $\aleph_0$-homogeneous and really the non-structure are related. The reader may thank Rami Grossberg for urging me to add more explanation to 1.9.
1. Resplendency

Our aim is to prove 1.2 below (“κ-resplendent” is defined in 1.4).

Convention 1.1. $T$ is a fixed first order complete theory; recall that $\tau(T) = \tau_T$, $\tau(M) = \tau_M$ is the vocabulary of $T$, $M$ respectively and $L$ is first order logic, so $L_\tau \equiv L(\tau)$ is the first order language with vocabulary $\tau$.

We show here

Theorem 1.2. The following are equivalent (see Definition 1.4 below) for a regular uncountable $\kappa$:

(i): $\kappa < \kappa(T)$, see e.g. 2.1(1),
(ii): there is a non-saturated $\kappa$-resplendent model of $T$ (see Definition 1.4 below),
(iii): for every $\lambda = \lambda^\kappa \geq 2^{\vert T \vert}$, $T$ has $\lambda$ non-isomorphic $\kappa$-resplendent models,
(iv): for every $\lambda = \lambda^\kappa \geq 2^{\vert T \vert}$, $T$ has $2^\lambda$ non-isomorphic $\kappa$-resplendent models.

Proof: The implication (i) $\Rightarrow$ (iii) follows from the main Lemma 1.9 below; the implication (iii) $\Rightarrow$ (ii) is trivial, and (ii) $\Rightarrow$ (i) follows from 1.8 below. Lastly, trivially (iv) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (iv) by 3.1+2.22.

Remark 1.3. (1) If we omit condition (iv) we save §3 as well as the dependency on a theorem from [Sh:309] using only an easy relative.
(2) In the proof the main point is (i) $\Rightarrow$ (iii) (and (i) $\Rightarrow$ (iv), i.e., the non-structure part).
(3) Remember: $T$ is unstable if $\kappa(T) = \infty$.
(4) Notice that every saturated model $M$ is $\|M\|$–resplendent (see 1.4(2) below). Actually a little more.

Definition 1.4. (1): A model $M$ is $(\kappa, \ell)$–resplendent (where $\ell = 0, 1, 2, 3$) if:

for every elementary extension $N$ of $M$ and expansion $N_1$ of $N$ satisfying $\vert \tau(N_1) \setminus \tau(N) \vert < \kappa$ and $\alpha < \kappa$, $c_i \in M$ for $i < \alpha$ and $T_1 \subseteq \text{Th}(N_1, c_i)_{i<\alpha}$ satisfying $(\ast)|_{T_1}$ below, there is an expansion $(M_1, c_i)_{i<\alpha}$ of $(M, c_i)_{i<\alpha}$ to a model of $T_1$ when:

$\text{Case 0: } \ell = 0$: $\vert T_1 \vert < \kappa$,
\text{Case 1: } $\ell = 1$: for some $\tau' \subseteq \tau(N_1)$, $\vert \tau' \vert < \kappa$ and $T_1 \subseteq L(\tau' \cup \{c_i : i < \alpha\})$,
\text{Case 2: } $\ell = 2$: $T_1$ is $\kappa$–recursive (see 1.4(4) below),
\text{Case 3: } $\ell = 3$: $T_1 = \text{Th}(N_1, c_i)_{i<\alpha}$ (but remember that $N_1$ has only $\kappa$ relations and functions not of $M$).

(2): $\kappa$-resplendent means $(\kappa, 3)$-resplendent.
(3): Assume $M$ is a model of $T$, $\bar{c} \in ^{\kappa}\vert M \vert$ and $M_\bar{c}$ is an expansion of $(M, \bar{c})$.
We say that $M_\bar{c}$ witnesses $(\kappa, \ell)$–resplendence for $\bar{c}$ in $M$, when:

for every first order $T_1$ such that $\text{Th}(M, \bar{c}) \subseteq T_1$ & $\vert \tau(T_1) \setminus \tau(T) \vert < \kappa$

and $(\ast)|_{T_1}$ holds, we have:

$M_\bar{c}$ is a model of $T_1$ up to renaming the symbols in $\tau(T_1) \setminus \tau(M, \bar{c})$.
(4): For $M, N_1$, $(c_i : i < \alpha)$ and $T_1 \subseteq \text{Th}(N_1, c_i)_{i<\alpha}$ as in part (1), $T_1$ is $\kappa$–recursive when:
(a): \( \kappa = \aleph_0 \) and \( T_1 \) is recursive (assuming the vocabulary of \( T \) is represented in a recursive way or

(b): \( \kappa > \aleph_0 \) and for some \( \tau^* \subseteq \tau(N_1) \), \( |\tau^*| < \kappa \) the following holds:

If \( \forall_\ell \phi_\ell (x_0, \ldots, x_{n-1}) \in L(\tau^\ell) \) for \( \ell = 1, 2 \) and there is an automorphism \( \pi \) of \( \tau^\ell \) (see parts (9)), where \( \tau^\ell \subseteq \tau^* \subseteq \tau(N_1) \) such that \( \pi \) is the identity on \( \tau^* \) and \( \hat{\pi}(\phi_\ell) = \phi_2 \) and \( \beta_0 < \beta_1 < \cdots < \alpha \) then

\[ \phi_1(c_{\beta_0}, c_{\beta_1}, \ldots) \in T_1 \quad \text{iff} \quad \phi_2(c_{\beta_0}, c_{\beta_1}, \ldots) \in T_1. \]

(5): We say \( f \) is an \((M,N)\)-elementary mapping when \( f \) is a partial one-to-one function from \( M \) to \( N \), \( \tau(M) = \tau(N) \) and for every \( \psi(x_0, \ldots, x_{n-1}) \in L(\tau(M)) \) and \( a_0, \ldots, a_{n-1} \in M \) we have:

\[ M \models \psi[a_0, \ldots, a_{n-1}] \quad \text{iff} \quad N \models \psi[f(a_0), \ldots, f(a_{n-1})]. \]

(6): \( f \) is an \( M \)-elementary mapping if it is an \((M,M)\)-elementary mapping.

(7): \( M \) is \( \kappa \)-homogeneous if:

for any \( M \)-elementary mapping \( f \) with \( |\text{Dom}(f)| < \kappa \) and \( a \in M \) there is an \( M \)-elementary mapping \( g \) such that:

\[ f \subseteq g, \quad \text{Dom}(g) = \{a\} \cup \text{Dom}(f). \]

(8): \( M \) is strongly \( \kappa \)-homogeneous if for any \( M \)-elementary mapping \( f \) with \( |\text{Dom}(f)| < \kappa \) there is an automorphism \( g \) of \( M \), such that \( f \subseteq g \).

(9): Let \( \tau_1 \subseteq \tau_2 \) be vocabularies. We say that \( \pi \) is an automorphism of \( \tau_2 \) over \( \tau_1 \) when \( \pi \) is a permutation of \( \tau_2 \), \( \pi \) maps any predicate \( P \in \tau_2 \) to a predicate of \( \tau_2 \) with the same arity, \( \pi \) maps any function symbol of \( F \in \tau_2 \) to a function symbol of \( \tau_2 \) of the same arity and \( \pi|\tau_1 \) is the identity.

(10): For \( \pi, \tau_2 \) as in part (9) let \( \hat{\pi} \) be the permutation of the set of formulas in the vocabulary \( \tau_2 \) which \( \pi \) induce.

Example 1.5. There is, for each regular \( \kappa \), a theory \( T_\kappa \) such that:

(a): \( T_\kappa \) is superstable of cardinality \( \kappa \),

(b): for \( \lambda \geq \kappa \), \( T_\kappa \) has \( 2^\lambda \) non-isomorphic \((\kappa,1)\)-resplendent models.

Note:

Fact 1.6. \( \quad \)

(1) If \( \tau = \tau(M) \), and

\[ |\tau' \subseteq \tau \& |\tau'| < \kappa \quad \Rightarrow \quad M \models \tau' \text{ is saturated} \]

then \( M \) is \((\kappa,1)\)-resplendent.

(2) If \( M \) is saturated of cardinality \( \lambda \) then \( M \) is \( \lambda \)-resplendent.

Proof: Easy, e.g., see [Sh:4] and not used here elsewhere.

Proof of 1.5: Let \( A_0 = \{ \kappa \setminus (i+1) : i < \kappa \} \) and \( A_1 = A_0 \cup \{0\} \). For every linear order \( I \) of cardinality \( \lambda \geq \kappa \) we define a model \( M_I \):

its universe is

\[ I \cup \{ \langle s,t,i,x \rangle : s \in I, t \in I, i < \lambda, x \in A_1 \text{ and } [I \models s < t \Rightarrow x \in A_0] \}, \]

(and of course, without lost of generality, no quadruple \( \langle s,t,i,x \rangle \) as above belongs to \( I \)), its relations are:

\[ P = I, \]
\[ R = \{ \langle s,t,\langle s,t,i,x \rangle \rangle : s \in I, t \in I, \langle s,t,i,x \rangle \in |M_I| \setminus P \}, \]
\[ Q_\alpha = \{ \langle s,t,i,x \rangle : \langle s,t,i,x \rangle \in |M_I| \setminus P, \alpha \in x \} \quad \text{for } \alpha < \kappa. \]
In order to have the elimination of quantifiers we also have two unary functions $F_1$, $F_2$ defined by:

\[
\begin{align*}
s \in I & \Rightarrow F_1(s) = F_2(s) = s, \\
(s, t, i, x) \in |M_I| \setminus I & \Rightarrow F_1((s, t, i, x)) = s&F_2((s, t, i, x)) = t.
\end{align*}
\]

It is easy to see that:

(a): In $M_I$, the formula

\[
P(x) \land P(y) \land (\exists z)(R(x, y, z) \land \bigwedge_{\alpha<\kappa} \neg Q_\alpha(z))
\]

linearly orders $P^M_I$, in fact defines $<_I$;

(b): $\Th(M_I)$ has elimination of quantifiers;

(c): If $\tau \subseteq \tau(M_I)$, $|\tau| < \kappa$ then $M_I \models \tau$ is saturated;

(d): $\Th(M_I)$ does not depend on $I$ (as long as it is infinite) and we call it $T_\kappa$;

(e): $T_\kappa$ is superstable.

Hence: $T_\kappa = \Th(M_I)$ is superstable, does not depend on $I$, and $M_I \cong M_J$ if and only if $I \cong J$.

and by 1.6 $M_I$ is $(\kappa, 1)$-resplendent.

**Fact 1.7.**

1. $M$ is $(\kappa, 3)$-resplendent implies $M$ is $(\kappa, \ell)$-resplendent implies $M$ is $(\kappa, 0)$-resplendent.

2. $M$ is $(\kappa, 0)$-resplendent implies $M$ is $\kappa$-compact.

3. $M$ is $(\kappa, 2)$-resplendent implies $M$ is $\kappa$-homogeneous, even strongly $\kappa$-homogeneous (see Definition 1.4(7),(8)).

4. If $M$ is $(\kappa, 2)$-resplendent $\kappa > \aleph_0$ and $\{a_n : n < \omega\}$ is an indiscernible set in $|M|$, then it can be extended to an indiscernible set of cardinality $|M|$ (similarly for sequences).

5. $M$ is $(\kappa, 3)$-resplendent implies $M$ is $\kappa$-saturated.

6. If $\kappa > |T|$ then the notions of 1.4 “$(\kappa, \ell)$-resplendent” for $\ell = 0, 1, 2, 3$, are equivalent.

Proof: Straightforward, for example

(3) For given $a_i, b_i \in M$ (for $i < \alpha$, where $\alpha < \kappa$) let

\[
T_1 = \{g(a_i) = b_i : i < \alpha\} \cup \\
\{(\forall x, y)(g(x) = g(y) \Rightarrow x = y), (\forall x)(\exists y)(g(y) = x)\} \cup \\
\{(\forall 0, \ldots, x_{n-1})[R(x_0, \ldots, x_{n-1}) = R(g(x_0), \ldots, g(x_{n-1}))]: \\
R \text{ an } n\text{-place predicate of } \tau(M)\} \cup \\
\{(\forall 0, \ldots, x_{n-1})[F(g(x_0), \ldots) = g(F(x_0), \ldots)] : \\
F \text{ an } n\text{-place function symbol of } \tau(M)\}.
\]

(4) For notational simplicity let $a_n = a_n$. Let $T_1$ be, with $P$ a unary predicate, $g$ a unary function symbol,

\[
\{\text{"}g\text{ is a one-to-one function into } P^n\} \cup \{P(a_n) : n < \omega\} \cup \\
\{(\forall 0, \ldots, x_{n-1}) \left[\bigwedge_{l<n} P(x_l) \land \bigwedge_{l<m<n} x_l \neq x_m \land \varphi(a_0, \ldots, a_{n-1}) \land \right. \\
\left. \quad \Rightarrow \varphi(x_0, \ldots, x_{n-1})\right] : \\
\varphi(x_0, \ldots, x_{n-1}) \in \downarrow(\tau(M))\}.
\]
Conclusion 1.8. If $M$ is $\kappa$-resplendent, $\kappa \geq \kappa(T) + \aleph_1$ then $M$ is saturated.

Proof: By 1.7(5) $M$ is $\kappa$-saturated, so without loss of generality $|M| > \kappa$. Hence, by [Sh:a] or [Shcc, III.3.10(1),p.107], it is enough to prove: for $I$ an infinite indiscernible $\subseteq M$, dim($I, M$) = $|M|$. But this follows by 1.7(4).

Main Lemma 1.9. Suppose that $\kappa = \text{cf}(\kappa) < \kappa(T)$ (for example, $T$ unstable, $\kappa$ regular) and $\lambda = \lambda^+ + 2^{|T|}$. Then $T$ has $\lambda$ pairwise non-isomorphic $\kappa$-resplendent models of cardinality $\lambda$.

Before embarking on the proof, we give some explanations.

Remark 1.10. (1) We conjecture that we can weaken in 1.9 the hypothesis $\lambda = \lambda^+ + 2^{|T|}$ to $\lambda = \lambda^{<\kappa} + 2^{|T|}$. This holds for many $\lambda$’s, see [Sh:309, §2]; but we have not looked at this. See §3.

(2) We naturally try to imitate [Sh:a], [Shcc, VII, §2, VIII, §2] or [Sh:E59, §3],[Sh:331].

In the proof of the theorem, the difficulty is that while expanding to take care of resplendency, we naturally will use Skolem functions with infinite arity, and so we cannot use compactness so easily.

If the indiscernibility is not clear, the reader may look again at [Sh:a] or [Shcc, VII, §2], (tree indiscernibility). We get below first a weaker version of indiscernibility, as it is simpler to get it, and is totally harmless if we would like just to get $\lambda$ non-isomorphic models by the old version [Sh:300, III.4.2(3)] or the new [Sh:309, §2].

Explanation 1.11. Note that the problem is having to deal with sequences of $< \kappa$ elements $b =$ $\langle b_i : i < \epsilon \rangle$, $\epsilon$ infinite. The need to deal with such $b$ with all theories of small vocabulary is not serious – there is a “universal one” though possibly of larger cardinality, i.e., if $T \models T$, $b_i \in M$ for $i < \epsilon$, $\epsilon < \kappa$, we can find a f.o. theory $T_2 = T_2(b)$ satisfying $\text{Th}(M, b_i)_{i<\epsilon} \subset T_1$, $|T_1| \leq (2^{|T|}+|\epsilon|)^{<\kappa}$ such that:

- if $\text{Th}(M, b_i)_{i<\epsilon} \subset T'$ and $|\tau(T') \setminus \tau(T) \setminus \{b_i : i < \epsilon\}| < \kappa$
- then renaming the predicates and function symbols outside $T$, we get $T' \subset T_2(b)$

– this is possible by Robinson consistency lemma. Let us give more details.

Claim 1.12. (1) Let $M_0$ be a model, $\tau_0 = \tau(M_0)$, $\bar{b} = \langle b_i : i < \kappa \rangle$ where $b_i \in M_0$ for $i < \epsilon$ and $\theta \geq \aleph_0$ be a cardinal. Let $\tau_1 = \tau_0 \cup \{b_i : i < \epsilon\}$ so $M_1 = \langle M_0, b_i \rangle_{i<\epsilon}$ is a $\tau_1$-model. Then there is a theory $T_2 = T_2[\bar{b}] = T_2[b, M]$, depending only on $\tau_0$, $\tau_1$ and $\text{Th}(M_1)$, i.e., essentially on $\text{tp}(\langle b_i : i < \epsilon\rangle, \emptyset, M_0)$ such that:

(a): $\tau_2 = \tau(T_2) = \tau(\epsilon, \tau_0)$ extends $\tau_1$ and has cardinality $\leq 2^{\tau_1|\theta|+|\epsilon|}$,

(b): for every $M_2$, $T'$, the model $M_2$ is expandable to a model of $T'$, when:

(\alpha): $M_2$ is a $\tau_2$-model,

(\beta): $M_2$ can be expanded to a model of $T_2$,

(\gamma): $\text{Th}(M_2) \subset T'$, equivalently some elementary extension of $M_2$

is expandable to a model of $T'$,

(\delta): $T'$ is f.o. and $|\tau(T') \setminus \tau(M_2)| \leq \theta$,

(\alpha)$^+$: if $\theta > |T| + |\epsilon|$ then $|\tau_2| \leq 2^{<\theta}$ is enough.

(2) If in part (1), clause (\delta) of (b) is weakened to
\[
(d)_2: \ T' \text{ is f.o., and } |\tau(T') \setminus \tau(M_2)| < \theta,
\]
then we can strengthen (a) to

(a)_2: \ \tau_2 = \tau(T_2) \text{ extends } \tau_1 \text{ and has cardinality } \leq \sum_{\mu < \theta} 2^{[\tau_1|+|\mu+\aleph_0+|\varepsilon|]},

(a)_2^*: \text{ if } \theta > (|T| + |\varepsilon|)^+ \text{ then } |\tau_2| \leq \sum_{\mu < \theta} 2^{<\mu} \text{ is enough.}

Proof: 1) We ignore function symbols and individual constants as we can replace them by predicates. Let

\[
\mathbb{T} = \{T': \ T' \text{ f.o. complete theory, } \text{Th}(M_1) \subseteq T' \text{ and } 
\tau(T') \setminus \tau(M_1) \text{ has cardinality } \leq \theta\}.
\]

This is a class; we say that \(T', T'' \in \mathbb{T}\) are isomorphic over \(\text{Th}(M_1)\) (see [Sh:8]) when there is a function \(h\) satisfying:

(a): \(h\) is one-to-one,
(b): \(\text{Dom}(h) = \tau(T')\),
(c): \(\text{Rang}(h) = \tau(T'')\),
(d): \(h\) preserves arity (i.e., the number of places, and of course being predicate/function symbols),
(e): \(h \upharpoonright (\tau(M_1)) = \text{identity},\)
(f): for a f.o. sentence \(\psi = \psi(R_1, \ldots, R_k) \in L[\tau(T')]\), where \(R_1, \ldots, R_k\) are the non-logical symbols occurring in \(\psi\), we have

\[\psi(R_1, \ldots, R_k) \in T' \iff \psi(h(R_1), \ldots, h(R_k)) \in T''.\]

Now note that

\[\exists \mathbb{T}/ \cong \text{ has cardinality } \leq 2^{[\tau_0|+|\varepsilon|+\theta].}\]

Now let \(\{T'_\alpha : \alpha < 2^{[\tau_0|+|\varepsilon|+\theta}\}\) list members of \(\mathbb{T}\) such that every equivalence class of being isomorphic over \(\text{Th}(M_1)\) is represented. \((\tau(T'_\alpha) \setminus \tau_1 : \alpha < 2^{[\tau_0|+|\varepsilon|+\theta}\)}\) are pairwise disjoint.

Note that \(\text{Th}(M_1) \subseteq T'_0\). Let \(T'_2 = \bigcup\{T'_\alpha : \alpha < 2^{[\tau_0|+|\varepsilon|+\theta}\}\) and \(\exists \mathbb{T}_2/ \cong\) is consistent.

Why? By Robinson consistency theorem.

Let \(T_2\) be any completion of \(T'_2\). So condition (a) holds; proving (b) should be easy.

Let us prove \((a)^+\); this is really the proof that a theory \(T, |T| < \theta\), has a model in \(2^{<\theta}\) universal for models of \(T\) of cardinality \(\leq \theta\). We shall define by induction on \(\alpha < \theta\), a theory \(T^2_\alpha\) such that:

(A): \(T^2_0 = \text{Th}(M_1)\),
(B): \(T^2_0\) a f.o. theory,
(C): \(\tau^2_0 = \tau(T^2_0)\) has cardinality \(\leq 2^{[\tau_0|+|\varepsilon|+|\alpha|+\aleph_0},\)
(D): \(T^2_\alpha, \tau^2_\alpha\) are increasing continuous in \(\alpha\),
(E): if \(\tau_1 \subseteq \tau' \subseteq \tau^2_\alpha, |\tau'| \leq |\tau^1| + |\alpha|, \tau' \subseteq \tau'', \tau'' \cap \tau^2_\alpha = \tau, T^2_\alpha \mid L_{\tau'} \subseteq L[\tau''], T'' \text{ complete and } |\tau'' \setminus \tau'| = \{R\}, \)
then we can find \(R' \in \tau^2_{\alpha+1} \setminus \tau^2_\alpha\) such that of the same arity.

\[T'' \text{ replacing } R \text{ by } R' \subseteq T^2_{\alpha+1}.\]

There is no problem to carry out the induction, and \(\bigcup_{\alpha < \theta} T^2_\alpha\) is as required.

2) Similar.
Explanation 1.13. So for $M \models T$, $b \in \kappa^+ M$, we can choose $T_1[b] \supseteq \text{Th}(M, b)$ depending on $\text{Th}(M, b)$ only, such that:

$(\otimes): M \models \text{"} T \text{"}$ is $\kappa$–resplendent  for every $b \in \kappa^+ M$, $(M, b)$ is expandable to a model of $T_2[b]$.

W.l.o.g. $\tau(T_2[b])$ depends on $\ell g(b)$ and $\tau_0$ only, so it is $\tau(\ell g(b), \tau_M)$.

The things look quite finitary but $T_2[b]$ is not continuous in $\text{Th}(M, b)$. I.e.,

$(\ast) \not\Rightarrow (\ast\ast)$, where

$(\ast): b^\alpha \in \kappa^+ M$, for $\alpha \leq \delta$, (\delta a limit ordinal) $\ell g(b^\alpha) = \epsilon$, and for every $n$,

\[ i_1 < \ldots < i_n < \epsilon \text{ and a formula } \varphi(x_{i_1}, \ldots, x_{i_n}) \in L(\tau_M) \text{ for some } \beta < \delta: \]

\[ \beta \leq \alpha \leq \delta \Rightarrow M \models \varphi(b^\beta_{i_1}, \ldots, b^\beta_{i_n}) \equiv \varphi(b^\alpha_{i_1}, \ldots, b^\alpha_{i_n}) \]

$(\ast\ast):$ for any $\varphi \in L(\tau_2)$ for some $\beta < \delta$:

\[ \beta \leq \alpha \leq \delta \Rightarrow [\varphi \in T_2[b^\beta] \iff \varphi \in T_2[b^\beta]] . \]

[You can make $T_1[b]$ somewhat continuous function of the sequence $b$ if we look at sub-sequences as approximations, not the type, but this is not used.]

This explain Why you need "infinitary Skolem functions".

We shall try to construct $M$ such that for every $b \in \kappa^+ M$, $(M, b)$ is expandable to a model of $T_2[b]$, so if $\tau_2 = \tau(T_2[b]) \setminus \tau(M, b)$, this means we have to define finitary relations/functions $R_b$ (for $R \in \tau_2$). We write here $b$ as a sequence of parameters but from another prospective the predicate/function symbol $R_{\bigcirc}(-)$ has $\epsilon + \text{arity}(R)$–places.

Explaning the first construction 1.14. (i.e., 2.19 below)

Eventually we build a generalization of $\text{EM}(\kappa^+ \lambda, \Psi)$, a model with skeleton $\tilde{a}_\eta$ ($\eta \in \kappa^+ \lambda$) witnessing $\kappa < \kappa(T)$, but the functions have any $\alpha < \kappa$ places but not $\kappa$, and the indiscernibility demand is weak. We start as in [Sh:E59, §2], so for some formulas $\langle \varphi_\alpha(x, y, \alpha) : \alpha < \kappa \rangle$ we have (where $\tilde{a}_\eta = \langle a_\eta \rangle$ for $\eta \in \kappa^+ \lambda$):

\[ \eta \in \kappa^+ \lambda \& \nu \in \alpha^+ \lambda \Rightarrow \varphi_\alpha(a_\eta, \tilde{a}_\nu) \equiv \varphi_\nu(a_\eta, a_\nu) \]

Without loss of generality, for any $\alpha < \kappa$ for some sequence $G_\alpha = \langle G_\alpha, \ell : \ell < \ell g(y_\alpha) \rangle$ of unary function symbols for any $\eta \in \kappa^+ \lambda$, $a_{\eta_1} = G_\alpha(a_\eta) := \langle G_\alpha, \ell(a_\eta) : \ell < \ell g(y_\alpha) \rangle$, so we can look at $\{ a_\eta : \eta \in \kappa^+ \lambda \}$ as generators. For $W \in \kappa^+ \lambda$, let $N_W = N[W]$ be the submodel which $\{ a_\eta : \eta \in W \}$ generates. So we would like to have:

$(\alpha): N_W$ has the finitary Skolem function (for $T$), and moreover $N_W$ has the finitary Skolem function for $T_2[b]$ for each $b \in \kappa^+ (N_W)$,

$(\beta):$ monotonicity: $W_1 \subseteq W_2 \Rightarrow N_{W_1} \subseteq N_{W_2}$.

So if $\Psi \subseteq \kappa^+ \lambda$, then $N[\Psi] = \{ N_W : W \in \kappa^+ [\Psi] \}$ is a $\kappa$–resplendent model of cardinality $\lambda$.

$(\gamma)$: Indiscernibility: (We use here very “minimal” requirement (see below)

but still enough for the omitting type in (1) below):

$(1): \eta \in \kappa^+ \lambda \setminus \Psi \Rightarrow N[\Psi]$ omits $p_\eta := \langle \varphi_\alpha(x, a_\eta(\alpha) : \alpha < \kappa) \rangle$ (satisfaction defined in $N[\kappa^+ \lambda]$),

$(2): \eta \in \kappa^+ \lambda \cap \Psi \Rightarrow N[\Psi]$ realizes $p_\eta$.

Now (2) was already guaranteed: $a_\eta$ realizes $p_\eta$.

For (1) it is enough
(1)': if $W \in {}^\kappa[{}^\kappa\lambda]$, $\eta \in {}^\kappa\lambda \setminus W$ then $p_\eta$ is omitted by $N_W$ (satisfaction defined in $N[{}^\kappa\lambda]$).

Fix $W, \eta$ for (1)'. A sufficient condition is

(1)'": for $\alpha < \kappa$ large enough, $(\bar{a}_\eta|\alpha^{-\langle i \rangle} : i < \lambda)$ is indiscernible over $N_W$ in $N[{}^\kappa\lambda]$.

If $\kappa_r(T) < \infty$, immediately suffices; in the general case, and avoiding classification theory, use

$$p'_\eta = \{ \varphi_\alpha(x, \bar{a}_\eta|\langle \alpha + 1 \rangle) \land \neg \varphi(x, \bar{a}_\eta|\langle \eta(\alpha + 1) \rangle) : \alpha < \kappa \}$$

so we use

$$\psi'(x, \bar{y}'_\alpha) = \varphi_\alpha(x, \bar{y}'_\alpha \mid \ell g(\bar{y}_\alpha)) \land \neg \varphi_\alpha(x, \bar{y}'_\alpha \mid (\ell g(\bar{y}_\alpha), 2 \ell g(\bar{y}_\alpha)))$$

in the end.

Note: as $|W| < \kappa$, for every $\eta \in {}^\kappa\lambda$ $W \cap \{ \nu : \eta \mid \alpha(\ast) < \nu \in {}^\kappa\lambda \}$ is a singleton

and $W \in W_{\alpha(\ast)}$ (see below), this will be enough to omit the type. The actual indiscernibility is somewhat stronger.

Further Explanation: On the one hand, we would like to deal with arbitrary sequences of length $< \kappa$, on the other hand, we would like to retain enough freedom to have the weak indiscernibility. What do we do? We define our “$\Phi$” (not as nice as in [Sh:E59, §2], i.e., [Sh:a, Ch.VII §3]) by $\kappa$ approximations indexed by $\alpha \leq \kappa$.

For $\alpha \leq \kappa$, we essentially have $N_W$ for

$$W \in W_{\alpha} : = \{ W : W \subseteq {}^\kappa\lambda, |W| < \kappa \text{ and the function } \eta \mapsto \eta \mid \alpha (\eta \in W) \text{ is one-to-one } \}.$$ 

Now, $W_{\alpha}$ is partially ordered by $\subseteq$ but (for $\alpha < \kappa$) is not directed. For $\alpha < \beta$ we have $W_{\alpha} \subseteq W_{\beta}$ and $W_{\kappa} = \bigcup_{\alpha < \kappa} W_{\alpha}$ is equal to $[{}^\kappa\lambda]^{<\kappa}$.

So if we succeed to carry out the induction for $\alpha < \kappa$, arriving to $\alpha = \kappa$ the direct limit works and no new sequence of length $< \kappa$ arises.
2. Proof of the Main Lemma

In this section we get many models using a weak version of indiscernibility.

**Context 2.1.** (1) $T$ is a fix complete first order theory, $\kappa < \kappa(T)$, $\bar{\varphi} = \langle \varphi_\alpha(x,y) : \alpha < \kappa \rangle$ is a fixed witness for $\kappa < \kappa(T)$, that is

(*) for any $\lambda$, for some model $M$ of $T$ and sequence $\langle a_\eta : \eta \in {}^{\kappa \geq \lambda} \rangle$ with $a_\eta \in M$ we have: if $\varepsilon < \kappa$, $\eta \in {}^\kappa \lambda$, $\alpha < \lambda$ then $M \models \varphi_\varepsilon[a_\alpha,a_{\eta^\varepsilon|-}\eta^\varepsilon)]^{\kappa(\varepsilon=\eta)}$.

(2) Let $\mu$ be infinite large enough cardinal; $\mu = \beth(|T|)$ is O.K.

**Remark:** Why are we allowed in 2.1(1) to use $\varphi_\alpha(x,y)$ instead of $\varphi(x,y)$? We can work in $T^{eq}$, see [Sh:Ch. III] and anyhow this is, in fact, just a notational change.

**Definition 2.2.**

(1) For $\alpha < \kappa$ and $\rho \in {}^{\alpha} \mu$, let $I^o_\rho = I^o_{\rho} = I^o_{\rho} = I^o_{\rho}$ be the model

\[ \{ (\nu \in {}^{(\nu)} \mu) : \nu \models \rho \}, E_i <_i <_{\kappa, \alpha} \]

where

\[ E_i = \{ (\eta, \nu) : \eta \in {}^{\kappa \mu}, \nu \in {}^{\kappa \mu}, \eta \models \sigma \} \]

\[ \leq_i = \{ (\eta, \nu) : \eta E_i \nu \land \eta(i) < \nu(i) \} \]

(2) Let $W_\alpha = W^0_\alpha = \{ W \subseteq {}^{\kappa \mu} : W \ has \ cardinality \ < \kappa \ and \ for \ any \ \eta \neq \nu \ from \ W \ we \ have \ \eta \models \alpha \neq \nu \models \alpha \}, \ and \ W_{\prec \kappa} = \bigcup_{\beta < \kappa} W_\beta.

(3) We say that $W$ is $\alpha$–invariant, or $(\alpha, \mu)$–invariant, when $W \subseteq W_\alpha$ and:

- if $W_1, W_2 \in W_\alpha$ and:

  \[ \eta \models \alpha = b(\eta) \models \alpha \ for \ \eta \in W_1, \]

  then

  \[ W_1 \cap W = W_2 \cap W. \]

(4) We say $W \subseteq W_\alpha$ is hereditary if $W' \subseteq W \Rightarrow W' \subseteq W$.

**Definition 2.3.**

(1) Let $\theta = \Theta_{\kappa, \kappa}$ be the minimal cardinal satisfying:

(a): $\theta < ^{< \kappa} \geq |T|,$

(b): if $M$ is a model of $T$, $b \in {}^{\kappa \geq} M$, then there is a complete (first order) theory $T^*$ with Skolem functions extending $\text{Th}(M, b)$ such that:

- if $T^* \supseteq \text{Th}(M, b)$ and $\tau(T^*) \setminus \tau(M, b)$ has cardinality $\kappa$

  then there is a one-to-one mapping from $\tau(T^*)$ into $\tau(T^*)$ over $\tau(M, b)$ preserving arity and being a predicate / function symbol, and mapping

  $T^*$ into $T^*$.

(2) For $\varepsilon < \kappa$, let $\tau[T, c]$ be a vocabulary consisting of $\tau_T$, the individual constants $b_\xi$ for $\xi \in \varepsilon$, and the $n$–place predicates $R_{T,j,n}$ for $j < \theta$ and $n$–place function symbols $F_{T,j,n}$ for $j < \theta$.

- For $\varepsilon < \kappa$ and a complete theory $T^{eq}$ in the vocabulary $\tau[T, c]$ extending $T$, let $T^{eq}[\tau[T, c]]$ be a complete first order theory in the vocabulary $\tau[T, c]$ such that if $(M, b)$ is a model of $T^{eq}$, then $T^{eq}[\tau[T, c]]$ is as in clause (b) of part (1).

(3) For $M \models T$ and $\varepsilon < \kappa$ and $b \in {}^{\varepsilon} M$, let $T^{eq}[b, M] = T^{eq}[\text{Th}(M, b)].$

**Remark 2.4.** Note that $\theta$ is well defined by 1.12. In fact, $\theta = \Pi\{2^{T+\sigma} : \sigma^+ < \kappa \}$ is O.K.

**Main Definition 2.5.** We say that $m$ is an approximation (or an $\alpha$–approximation, or $(\alpha, \mu)$–approximation) if

(*)$_{\alpha} : \alpha \leq \kappa$ (so $\alpha = \alpha_m = \alpha(m)$),

\[ \langle \varphi_\alpha(x,y) : \alpha < \kappa \rangle \text{ is } \text{ a fix complete first order theory, } \kappa < \kappa(T), \]
\(\ast\))\text{2: } m \text{ consists of the following (so we may give them subscript or superscript m):}

(a): a model \(M = M_m\);

(b): a set \(\mathcal{F} = \mathcal{F}_m\) of symbols of functions, each \(f \in \mathcal{F}\) has an interpretation, a function \(f_m\) with range \(\subseteq M\), but when no confusion arises we may write \(f\) instead of \(f_m\), (or \(f^m\), note that the role of those \(f\)-s is close to that of function symbols in vocabularies, but not equal to);

(c): each \(f \in \mathcal{F}\) has \(\zeta_f < \kappa\) places, to each place \(\zeta\) (i.e., an ordinal \(\zeta < \zeta_f\)) a unique \(\eta \in \alpha\), \(\eta_\zeta = \eta(f, \zeta)\) is attached such that

\[\zeta \neq \xi \Rightarrow \eta_\zeta \neq \eta_\xi,\]

and the \(\zeta\)-th variable of \(f\) varies on \(I_{\eta_\zeta}\), i.e., \(f_m(\ldots, x_\zeta, \ldots)_{\zeta<\zeta_f}\) is well defined iff \(\bigwedge \eta \in \eta_\zeta = I_{\eta_\zeta}\);

we may write \(f_m(\ldots, \nu, \ldots)_{\eta \in w[f]}\) instead \(f_m(\ldots, \nu(f, \zeta), \ldots)_{\zeta<\zeta_f}\), where \(w[f] = [\eta(f, \zeta) : \zeta < \zeta_f]\); and \(f \in \mathcal{F} \Rightarrow (\exists W \in W)[w[f] = [\eta(f, \zeta) : \zeta \in W]}\), see clause (e) below;

(d): for each \(b \in \kappa M\), an expansion \(M_b\) of \((M, b)\) to a model of \(T^\alpha|\alpha|\), (see above in Definition 2.3; so \(M_b\) has Skolem functions and it witnesses \(\kappa\)-resplendency for this sequence in \(M\));

(e): \(W = W_m \subseteq W_\alpha\) which is \(\alpha\)-invariant and hereditary;

(f): for \(W \in W\), \(N_W\) which is the submodel of \(M\) with universe

\[\{f(\ldots, \eta, \ldots)_{\zeta<\zeta_f} : f \in \mathcal{F}, f(\ldots, \eta, \ldots)_{\zeta<\zeta_f} \text{ well defined, and } \eta \in W \text{ for every } \zeta\},\]

such that \(m\) satisfies the following:

(A): \(M\) is a model of \(T\),

(B): \([\text{witness for } \kappa < \kappa(T);]\) for our fixed sequence of first order formulas \(\langle \varphi(x, y) : \zeta < \kappa \rangle\) from \(L(\mathcal{F})\) depending neither on \(\alpha\) nor on \(m\) we have \(f^*_\varphi \in \mathcal{F}\) for \(\zeta < \kappa, \rho \in \alpha\mu\) (we also call them \(f^*_\varphi\)) each \(f^*_\varphi\) such that:

(i): \(f^*_\varphi\) is a one place function \(\zeta_{T, \zeta}\) with \(\zeta_{T, \zeta}\) from clause (e) being \(1, \rho\) respectively,

(ii): \(f^*_\varphi(x_1) = f^*_\varphi(x_2)\) whenever \(x_1 = x_2\) and they are well defined, i.e. \(\rho \in \alpha\mu\) and \(x_1 = x_2\) and they are well defined,

(iii): \(\zeta \in \zeta_{T, \zeta}\) is close to that of function symbols in vocabularies, but not equal to),

we may write \(\varphi(x_1, x_2) = \varphi_{\zeta_{T, \zeta}}(x_1, x_2)\) and they are well defined, i.e. \(\rho \in \alpha\mu\)

\(M \models \varphi_{\zeta_{T, \zeta}}[f^*_\varphi(x_1), f^*_\varphi(x_2)]\)

\(\Rightarrow [x_1 \in (\zeta + 1) = x_2 \in (\zeta + 1)],\)

(C): \(N_W \prec M\), for \(W \in W\),

(D): \(\mathcal{F} = f_m\) witness an amount of resplendency

(\(\alpha\)): the domain of \(f\) is a subset of

\[\mathcal{F}_m = \{f : f = \langle f_\zeta : \zeta < \zeta_f\rangle, \zeta_f < \kappa, f_\zeta \in \mathcal{F}, \text{ and } \zeta_{f_\zeta} \text{ does not depend on } \zeta, \text{ call it } \zeta_f \text{ and } \zeta_{f_\zeta} \text{ does not depend on } \zeta, \text{ call it } \eta(f, \zeta) \text{ does not depend on } \zeta, \text{ call it } \eta_f, \zeta\} = \eta(f, \zeta) \text{ does not depend on } \zeta, \text{ call it } \eta(f, \zeta) \text{ does not depend on } \zeta, \text{ call it } \eta_f, \zeta\} = \eta(f, \zeta) \text{ does not depend on } \zeta, \text{ call it } \eta(f, \zeta) \text{ does not depend on } \zeta, \text{ call it } \eta_f, \zeta\}.\]
(β): for $\bar{f} \in \text{Dom}(f)$, $f(\bar{f})$ is a function with domain
$$\{\sigma(\bar{x}) : \sigma(\bar{x}) \text{ is a } \tau[T,\epsilon_f]-\text{term, and } \bar{x} = \langle x_\xi : \xi \in u \rangle$$
for some finite subset $u = u_\sigma$ of $\epsilon_f$}

and if $\sigma(\bar{x}) \in \text{Dom}(f(\bar{f}))$ then
$$f(\bar{f})(\sigma(\bar{x})) \in \mathcal{F}[\bar{f}] := \{f \in \mathcal{F} : \zeta_f = \zeta_f \& (\forall \zeta < \zeta_f)(\eta(f, \zeta) = \eta(\bar{f}, \zeta))\}.$$

(γ):

- if $\bar{f} \in \text{Dom}(f)$ and $\bar{b} = \langle f_\ell(\ldots, \nu_\ell(\bar{f}, \zeta), \ldots) \rangle_{\zeta < \zeta_f}$, then the universe of $M_\bar{b}$ is $\{f(\ldots, \nu_m(\bar{f}, \zeta), \ldots) \rangle_{\zeta < \zeta_f} : f \in \mathcal{F}[\bar{f}]\}$
- if $(f(\bar{f}))_{(\sigma(\bar{x}))} = f^* \in \mathcal{F}$, $W \in \mathcal{W}$, $\nu_\zeta \in W$ and $\nu_\zeta \mid \alpha = \eta(f, \zeta)$ for $\zeta < \zeta_f$, and $\bar{x} = \langle x_\xi : \xi \in u \rangle$, and $\bar{b} = f(\bar{w}) = \langle f_\ell(\bar{w}) : \epsilon < \epsilon_f \rangle$, then
$$\sigma(\bar{b}) = \sigma(\bar{m})((f_\ell(\ldots, \nu_\ell, \ldots) \rangle_{\zeta < \zeta_f} : \xi \in u)) = f^*(\ldots, \nu_\ell, \ldots) \rangle_{\zeta < \zeta_f}.$$

[explaining (γ): we may consider $\bar{b} \in N_{W_1} \cap N_{W_2}$, and we better have that the witnesses for resplendency demands, specialized to $\bar{b}$, in $N_{W_1}$ and in $N_{W_2}$ are compatible so that in the end resplendency holds].

However, we shall not get far without at least more closure and coherence of the parts of $\mathcal{m}$.

**Definition 2.6.**

1. An approximation $\mathcal{m}$ is called full if $\mathcal{W}_\mathcal{m} = \mathcal{W}_{\alpha(\mathcal{m})}$, and is called semi-full if $\mathcal{W}_{<\alpha(\mathcal{m})} \subseteq \mathcal{W}_\mathcal{m} \subseteq \mathcal{W}_{\alpha(\mathcal{m})}$ and is called almost full if it is semi full when $\alpha$ is limit ordinal and full when $\alpha$ is a non-limit ordinal.
2. An approximation $\mathcal{m}$ is $\beta$-resplendent if $\beta \leq \alpha_\mathcal{m}$ and
$$\text{if } W \in \mathcal{W}_\beta \cap \mathcal{W}_\mathcal{m} \text{ and } f \in \mathcal{F}_\mathcal{m}, \text{ and } \{\eta(f, \zeta) : \zeta < \zeta_f\} \subseteq \{\nu \mid \alpha : \nu \in W\}, \text{ then } f \in \text{Dom}(f_\mathcal{m}).$$
3. In part (2), if we omit $\beta$, we mean $\beta = \alpha_\mathcal{m}$, and “$< \beta^*$” means for every $\beta < \beta^*$.
4. An approximation $\mathcal{m}$ is called term closed if:

- **(\text{E})** Closure under terms of $\tau$:

  Assume that $u \subseteq \alpha_\mathcal{m}$, $|u| < \kappa$, and for some $W \in \mathcal{W}_\mathcal{m}$, $u \subseteq \{\eta \mid \alpha : \alpha \in W\}$, and $\langle \nu_\zeta : \zeta < \zeta^* \rangle$ lists $u$ with no repetitions and $f_\ell \in \mathcal{F}_\mathcal{m}$, $\ell < n$, satisfies $\{\eta(f_\ell, \zeta) : \zeta < \zeta_f\} \subseteq u$, $\sigma$ is an $n$-place $\tau(T)$-term so $\sigma = \sigma(x_0, \ldots, x_{n-1})$. Then for some $f \in \mathcal{F}_\mathcal{m}$ satisfying $\zeta^*$, $\eta(f, \zeta) = \eta_\zeta$, for any choice of $\langle \nu_\zeta : \eta \in u \rangle$ such that $\eta \neq \nu_\zeta \in \alpha_\mathcal{m}$ for $\eta \in u$, and $\{\nu_\zeta : \eta \in u\} \subseteq W \in \mathcal{W}$ for some $W$ we have
$$f_\mathcal{m}(\ldots, \nu_\eta, \ldots)_{\eta \in u[f]} = \sigma(\ldots, f_\mathcal{m}(\ldots, \nu_\eta, \ldots)_{\eta \in u[f]}, \ldots)_{\ell < n}$$

(this clause may be empty, but it helps to understand clause (\text{F}); note that it is not covered by 2.5(D) as the functions are not necessarily with the same domain, hence this says something even for $\sigma$ the identity: so this implies that in clause (f) of Definition 2.5 we can demand $\{\eta_\mathcal{m}(f, \zeta) : \zeta < \zeta_f\} = W$).

- **(\text{F})** Closure under terms of $\tau(M_\mathcal{b})$:

  Assume that $u \subseteq \alpha_\mathcal{m}$, $|u| < \kappa$, and $\langle \nu_\zeta : \zeta < \zeta^* \rangle$ lists $u$ with no repetitions, and for some $W \in \mathcal{W}_\mathcal{m}$, $u \subseteq \{\eta \mid \alpha : \eta \in W\}$. If $\kappa < \omega$
Remark: The reader may concentrate on Definition 2.9.

Observation 2.7. In Definition 2.6(4) in clauses (E),(F) it suffice to restrict ourselves to the case \( n = 1 \) and \( \sigma \) is the identity.

Proof: By 2.5(D)(γ).

Of course some form of indiscernibility will be needed.

Definition 2.8.  

1. Let \( E \) be the family of equivalence relations \( E \) on \( \{ \bar{\nu} \in \kappa \uparrow (\kappa^\mu) : \nu \text{ without repetitions} \} \), or a subset of it, such that \( \bar{\nu}^1 E \bar{\nu}^2 \Rightarrow \ell g(\bar{\nu}^1) = \ell g(\bar{\nu}^2) \).

2. Let \( E_\alpha \) be the family of \( E \in \kappa \) such that \( \bar{\nu} \in \text{Dom}(E) \Rightarrow (\nu_\zeta \mid \alpha : \zeta < \ell g(\bar{\nu})) \) is without repetitions.

3. Let \( E^0_\alpha \) be the following equivalence relation:

\[
\bar{\nu}^1 E^0_\alpha \bar{\nu}^2 \text{ iff for some } \zeta < \kappa \text{ we have (i) } \bar{\nu}^1, \bar{\nu}^2 \in \zeta (\kappa^\mu), (ii) \bar{\nu}^1 \upharpoonright \alpha = \bar{\nu}^2 \upharpoonright \alpha \text{ for } \varepsilon < \zeta, (iii) (\nu_\zeta \mid \alpha : \zeta < \zeta) \text{ is with no repetitions}, (iv) \text{ the set } \{ \varepsilon < \zeta : \nu_\varepsilon^1 \neq \nu_\varepsilon^2 \} \text{ is finite.}
\]

3A. We say that \( (\bar{\nu}^1, \bar{\nu}^2) \) are immediate neighbours if \( \ell g(\bar{\nu}^1) = \ell g(\bar{\nu}^2) \), and for some \( \xi < \ell g(\bar{\nu}^1) \) we have \( (\forall \beta < \xi)(\xi \neq \xi \Rightarrow \nu^1_\xi = \nu^2_\xi) \); so the difference with (3) is that “finite” is replaced by “a singleton”.

4. Let \( E^0_{\alpha, <} \) be defined like \( E^0_\alpha \) strengthening clause (iii) to (iii)’:

\[
\bar{\nu}^1 \upharpoonright \alpha \mid \beta : \varepsilon < \zeta \text{ is with no repetitions.}
\]

5. For \( \alpha < \kappa \) and \( W \subseteq W_\alpha \) let

\[
\text{seq}_W(W) = \{ \bar{\nu} : \bar{\nu} \in \kappa \uparrow (\kappa^\mu) \text{ is with no repetitions, and for some } W \in W \text{ we have (i) } \nu_\xi \mid \xi < \ell g(\bar{\nu}) \subseteq W, \text{ and hence (ii) } \nu_\zeta \mid \alpha : \zeta < \ell g(\bar{\nu}) \text{ is with no repetitions} \}.
\]

6. We define \( E^1_\alpha \) as we define \( E^0\alpha \) in part (3) above, omitting clause (iv). We define \( E^1_{\alpha, <} \) parallelly as in part (4).

Remark: The reader may concentrate on \( E^0_\alpha \), so the “weakly” version below.

Definition 2.9.  

1. An approximation \( m \) is called \( E \)-indiscernible if (a): \( E \in \mathbb{E}_{\alpha(m)} \) refine \( E^1_{\alpha(m)} \).
(b): if $\bar{\nu}^1, \bar{\nu}^2 \in \text{seq}_{\alpha_m}(W_m)$ and $\bar{\nu}^1 E \bar{\nu}^2$, then there is $g$ (in fact, a unique $g = g^m_{\bar{\nu}^1,\bar{\nu}^2}$) such that

(1) $g$ is an $(M_m, M^m)$-elementary mapping,
(2) $\text{Dom}(g) = \{f((\nu^1_m : \zeta < \zeta^f)) : f \in F'_m \text{ and } h \text{ is a one-to-one function from } \zeta_f \text{ into } \epsilon g(\bar{\nu}^f) \text{ such that } \eta(f, \zeta) \in \nu^1_f\}$,
(3) $g(f((\nu^1_m : \zeta < \zeta^f))) = f((\nu^2_m : \zeta < \zeta^f))$ for $f, h$ as above;
(c) Assume $\bar{\nu}^1, \bar{\nu}^2 \in \text{seq}_{\alpha_m}(W_m)$, $f^1, f^2 \in \text{Dom}(F_m)$. $\zeta^* = \zeta_f = \zeta_{f^2}$, and for some one-to-one function $h$ from $\zeta^*$ to $\epsilon g(\bar{\nu}^f)$ we have $h(\bar{\nu}^f, \zeta) = \nu^m_{h(\zeta)}[\alpha \text{ for } \ell, m = 1, 2, \text{ and } \bar{\nu}^1 E \bar{\nu}^2]$. Let

Then there is $g$ such that

(1) $g$ is an $(M_m^m, M^m)$-elementary mapping,
(2) $g = g^m_{\bar{\nu}^1,\bar{\nu}^2}$ from clause (b) above.

(2) An approximation $m$ is strongly indiscernible if it is $E^*_{\alpha(m)}$-indiscernible.

(3) (a) An approximation $m$ is weakly indiscernible when it is $E^0_{\alpha(m)}$-indiscernibility.
(b) An approximation $m$ is weakly/strongly nice if it is term closed and almost indiscernible.
(c) An approximation $m$ weakly/strongly good if it is weakly/strongly nice and is almost full.
(d) An approximation $m$ is weakly/strongly excellent if it is weakly/strongly nice and is resplendent, see Definition 2.6(2),(3).

Discussion 2.10. Why do we have the weak and strong version?

In the proof of the main subclaim 2.19 below the proof for the weak version is easier but we get from it a weaker conclusion: $\leq \lambda^+$ non-isomorphic $\kappa$-resplendent of cardinality $\lambda = \lambda^\kappa$, whereas from the strong version we would get $2^\lambda$. But see §3.

Claim 2.11. Let $m$ be an approximation.

(1) In the definition of $m$ is $E^0_{\alpha(m)}$-indiscernible", it is enough to deal with immediate $E^0_{\alpha(m)}$-neighbors (see Definition 2.8(3)).
(2) If $m$ is weakly/strongly excellent then $m$ is weakly/strongly good.
(3) If $m$ is weakly/strongly good then $m$ is weakly/strongly nice.
(4) If $\alpha_m = 0, E \in E_m$ then $m$ is $E$-indiscernible if and only if $m$ is strongly indiscernible.

Definition 2.12. (1) For approximations $m_1, m_2$ let $m_1 \leq_h m_2"$ or $m_1 \leq m_2$ as witnessed by $h^m$ mean that:
(a) $\alpha(m_1) \leq \alpha(m_2)$,
(b) $W_{m_1} \subseteq W_{m_2}$,
(c) $h$ is a partial function from $F_{m_1}$ into $F_{m_2}$,
(d) if $h(f_2) = f_1$ then they have the same arity (i.e., $\alpha_{f_1} = \alpha_{f_2}$) and $\zeta < \zeta_{f_2}^m \Rightarrow \eta_{m_1}(f_1, \zeta) = \eta_{m_2}(f_2, \zeta)$ for $\alpha(m_1)$,
(e) if $f_1 \in F_{m_1}$ and $W \in W_{m_2}$ and

then there is one and only one $f_2 \in F_{m_2}$ satisfying

$h(f_2) = f_1$ and $\{\eta_m(f_2, \zeta) : \zeta < \zeta_{f_2}^m\} = \{\nu \in \alpha(m_2) : \nu \in W\}$,
Proof: Recall that the sequence $\langle \text{Claim 2.13.} \rangle$
(a): $\Phi_\varepsilon$ is a template proper for $\kappa \geq \mu$ which is nice (see [Sh:E59, 1.7] + [Sh:E59, 1.8(2)]).

(b): $\tau(\Phi_\varepsilon)$ has cardinality $\leq \theta$ (see Definition 2.3).

(c): $\Phi_0 = \Phi$.

(d): the sequence $\langle \Phi_\varepsilon : \varepsilon < \kappa \rangle$ is increasing with $\varepsilon$, that is,

$$\zeta < \varepsilon \implies \tau(\Phi_\zeta) \subseteq \tau(\Phi_\varepsilon) \text{ and } EM^1(\kappa \geq \mu, \Phi_\varepsilon) < EM^1(\kappa \geq \mu, \Phi_\varepsilon).$$

(e): the sequence $\langle \Phi_\varepsilon : \varepsilon < \kappa \rangle$ is continuous, i.e., if $\varepsilon$ is a limit ordinal then

$$\tau(\Phi_\varepsilon) = \bigcup_{\zeta < \varepsilon} \tau(\Phi_\zeta),$$

(f): if $\bar{\sigma} = \langle \sigma_i(x) : i < i^* \rangle$ is a sequence of length $\kappa$ of unary terms in

$$\tau(\Phi_\varepsilon),$$

and $M^{\varepsilon+1} = EM^1(\kappa \geq \mu, \Phi_{\varepsilon+1})$, and for $\nu \in \kappa\mu$ we define $b = b_{\sigma, \nu}$ as

$$\langle \sigma_i^{M^{\varepsilon+1}}(a_\nu) : i < i^* \rangle \in \Gamma^1(EM^1(\nu, \Phi_{\varepsilon+1})),$$

then we can interpret a model $M^{\varepsilon+1}_b$ of $T^* \upharpoonright M^{\varepsilon+1}$ in $M^{\varepsilon+1}$, which means

$$(\alpha): \text{if } \bar{R} \in \mathcal{T}(b, M^{\varepsilon+1}) \upharpoonright \mathcal{T}(T) \setminus \mathcal{T}(\Phi_\varepsilon) \text{ is a } (k+1)\text{-place predicate, then there is a }$$

$$(k+1)\text{-place predicate } R_\varepsilon \in \tau(\Phi_{\varepsilon+1}) \setminus \tau(\Phi_\varepsilon) \text{ such that }$$

$$M^{\varepsilon+1} \models R[c_0, \ldots, c_k] \iff M^{\varepsilon+1} \models R_\varepsilon[c_0, \ldots, c_k, \bar{a}_\nu],$$

$$(\beta): \text{if } \bar{F} \in \mathcal{T}(b, M^{\varepsilon+1}) \upharpoonright \mathcal{T}(T) \setminus \mathcal{T}(\Phi_\varepsilon) \text{ is a } (k+1)\text{-place function symbol, then there is a }$$

$$(k+1)\text{-place function symbol } F_\varepsilon \in \tau(\Phi_{\varepsilon+1}) \setminus \tau(\Phi_\varepsilon) \text{ such that }$$

$$M^{\varepsilon+1} \models F[c_0, \ldots, c_k] = c_\nu \iff M^{\varepsilon+1} \models F_\varepsilon[c_0, \ldots, c_k, \bar{a}_\nu] = c_\nu.$$

Let us carry out the induction; note that there is a redundancy in our contraction: each relevant $b$ is taken care of in the $\varepsilon$-th stage for every $\varepsilon < \kappa$ large enough, independently, for the different $\varepsilon$-s.

For $\varepsilon = 0$:

Let $\Phi_0 = \Phi$.

For a limit $\varepsilon$:

Let $\Phi_\varepsilon$ be the direct limit of $\langle \Phi_\zeta : \zeta < \varepsilon \rangle$.

For $\varepsilon = \zeta + 1$:

Let the family of sequences of the form $\bar{\sigma} = \langle \sigma_i(x) : i < i^* \rangle$, where $\sigma_i(x)$ is a unary term in $\tau(\Phi_\varepsilon)$, $i^* < \kappa$, be listed as $\langle \bar{\sigma}(\gamma)(x) : \gamma < \theta \rangle$, with $\bar{\sigma}(\gamma)(x) = \langle \sigma_i^{\varepsilon}(x) : i < i_\gamma \rangle$.

Let $M_\varepsilon^*$ be a $\theta^+$-resplendent (hence strongly $\theta^+$-homogeneous and $\kappa$-resplendent) elementary extension of $EM^1(\kappa \geq \mu, \Phi_\varepsilon)$, and let $M_\varepsilon = M_\varepsilon^* \upharpoonright \mathcal{T}(T)$, and choose $\nu^* \in \kappa\mu$.

For each $\gamma < \theta$ let $b_{\gamma, \nu^*} := \langle \sigma_i^{\varepsilon}(a_\nu) : i < i_\gamma \rangle$. Now, $(M_\varepsilon, b_{\gamma, \nu^*})$ can be expanded to a model $M^{\varepsilon+1}_{\nu^*}$ of $T^* \upharpoonright M_\varepsilon$, and let

$$\tau(T^* \upharpoonright M_\varepsilon) \upharpoonright \mathcal{T}(T) = \{R^{\varepsilon, \gamma}_{j, n} : j < \theta, n < \omega \} \cup \{F^{\varepsilon, \gamma}_{j, n} : j < \theta, n < \omega \},$$

where $R^{\varepsilon, \gamma}_{j, n}$ is an $n$-place predicate and $F^{\varepsilon, \gamma}_{j, n}$ is an $n$-place function symbol. Next we shall define an expansion $M^{\varepsilon+1}_\nu$ of $M^{\varepsilon+1}_\varepsilon$. Its vocabulary is

$$\tau(\Phi_\varepsilon) \cup \{R^{\varepsilon, \gamma}_{j, n}, F^{\varepsilon, \gamma}_{j, n} : j < \theta, n < \omega \},$$

where $R^{\varepsilon, \gamma}_{j, n}$ is an $(n+1)$-place predicate, $F^{\varepsilon, \gamma}_{j, n}$ is an $(n+1)$-place function symbol, and no one of them is in $\tau(\Phi_\varepsilon)$ (and there are no repetitions in their list).
Almost lastly, for \( \nu \in {}^\kappa \mu \) let \( g_\nu \) be an automorphism of \( M_\kappa \) mapping \( EM^1(\{\nu^*\}, \Phi_\xi) \) onto \( EM^1(\{\xi\}, \Phi_\kappa) \); moreover such that for any \( \tau(\Phi_\xi) \)-term \( \sigma(x) \) we have \( g_\nu(\sigma(a_{\nu^*})) = \sigma(a_\nu) \) (hence \( \xi < \kappa \Rightarrow g_\nu(a_{\nu^*\xi}) = a_{\nu^*} \) using \( \sigma(x) = F_\xi^e(x) \)).

Now we actually define \( M_\kappa^+ \) expanding \( M_\kappa^+ M_\kappa^+: \)

\[
F_{e,\gamma,j,n}^{M_\kappa^+} = \{(g_\nu(a_0), g_\nu(a_1), \ldots, g_\nu(c_{n-1}), g_\nu(a_{\nu^*})) : \\
M_{\kappa^+}^\xi \models F_{j,n}^e(c_0, \ldots, c_{n-1})\}
\]

\( F_{e,\gamma,j,n}^{M_\kappa^+} \) is an \((n + 1)\)-place function such that

\[
M_{\kappa^+}^\xi \models F_{j,n}^e(c_0, \ldots, c_{n-1}) = c \quad \text{implies} \\
F_{e,\gamma,j,n}^{M_\kappa^+}(g_\nu(a_0), \ldots, g_\nu(c_{n-1}), a_{\nu^*}) = g_\nu(c).
\]

We further expand \( M_\kappa^+ \) to \( M_{\kappa^+}^+ \), with vocabulary of cardinality \( \leq \theta \) and with Skolem functions.

Now we apply \( ^{\kappa \geq \mu} \mu \) has the Ramsey property” (see [Sh:E59, 1.14(4)] see “even” there, [Sh:E59, 1.18]) to get \( \Phi_\xi = \Phi_{\xi^+1} \), \( \tau(\Phi_\xi) = \tau(M_{\kappa^+}^+ \), such that for every \( n < \omega, \nu_1, \ldots, \nu_n \in {}^\kappa \mu \), and first order formula \( \varphi(x_1, \ldots, x_n) \in L(\tau(\Phi_\xi)) \), for some \( n_1, \ldots, n_\eta, n_\eta \in {}^\kappa \mu \) we have

\[
(\alpha): M_{\kappa^+}^+ \models \varphi[a_{\nu_1}, \ldots, a_{\nu_n}] \text{ iff } EM^1(\kappa \geq \mu, \Phi_\xi) \models \varphi[a_{\nu_1}, \ldots, a_{\nu_n}], \\
(\beta): (\nu_1, \ldots, \nu_\eta), (\nu_1, \ldots, \nu_n) \text{ are similar in } ^{\kappa \geq \mu}.
\]

It is easy to check that \( \Phi_\xi = \Phi_{\xi^+1} \) is as required.

So we have defined the sequence \( \langle \Phi_\xi : \xi < \kappa \rangle \) satisfying the requirements above, and let \( \Phi_\kappa \) be its limit. It is as required in the claim. \[\blacklozenge\]

**Claim 2.14.** Assume \( \alpha \leq \kappa \) is a limit ordinal and \( \langle m_\gamma, h_\gamma^\alpha : \gamma < \beta < \alpha \rangle \) is an inverse system of approximations.

1. There are \( m_\gamma, h_\gamma^\alpha \) (for \( \gamma < \alpha \)) such that \( \langle m_\gamma, h_\gamma^\alpha : \gamma < \beta < \alpha + 1 \rangle \) is an inverse system of approximations continuous at \( \alpha \).
2. For the following properties, if each \( m_{\gamma+1} \) (for \( \gamma < \alpha \)) satisfies the property, then so does \( m_\gamma \): term closed, semi full, almost full, resplendent, weakly/strongly indiscernible, weakly/strongly nice, \( E \)-indiscernible for any \( E \in E \), weakly/strongly good, weakly/strongly excellent.

**Proof:** Let \( \mathbf{W}_{m_\alpha} = \bigcup_{\beta < \alpha} \mathbf{W}_{m_\beta} \), and let \( M_\beta = M_{m_\beta} \) for \( \beta < \alpha \). We shall define \( \mathcal{F}_\alpha = \mathcal{F}_{m_\alpha}, M_\alpha = M_{m_\alpha} \) and \( N_\alpha^m = N^m_{m_\alpha} \) and \( M^\alpha = M^m_\alpha \) below.

First let \( \mathcal{F}_\alpha \) (formal set, consisting of function symbols not of functions), \( h^\alpha_\beta \) \((\beta < \alpha)\) be the inverse limit of \( \langle \mathcal{F}_\beta, h^\beta_\gamma : \gamma \leq \beta < \alpha \rangle \), i.e.,

\[
\alpha): h^\alpha_\beta \text{ is a partial function from } \mathcal{F}_\alpha \text{ onto } \mathcal{F}_\beta, \text{ and in Definition 2.12,} \\
(\beta): h^\alpha_\beta = h^\beta_\gamma \circ h^\alpha_\beta \text{ for } \gamma < \beta < \alpha, \\
(\gamma): \mathcal{F}_\alpha = \bigcup_{\beta < \alpha} \text{Dom}(h^\alpha_\beta), \\
(\delta): \text{If } \beta_* < \alpha, \text{ then } f_\beta \in \mathcal{F}_\beta, \text{ for } \beta \in [\beta_*, \alpha), \text{ satisfy } h^\beta_\gamma(f_\beta) = f_\gamma \text{ when } \beta_* \leq \gamma < \beta < \alpha, \text{ then for one and only one } f \in \mathcal{F}_\alpha \text{ we have:} \\
\zeta_f = \zeta_{f_\beta} \text{ for } \beta \in [\beta_*, \alpha) \quad \text{and} \quad \eta_{f, \zeta} = \bigcup \{\eta_{f_\beta, \zeta} : \beta_* \leq \beta < \alpha\}, \\
(\varepsilon): \text{every } f \in \mathcal{F}_\alpha \text{ has the form of } f \text{ in } (\delta),
\]
(ζ): $f^*_\rho,\kappa$ are as in (B) of Definition 2.5, i.e., for any $\rho \in {^\kappa} \mu$ and $\zeta < \kappa$ we have $\beta < \alpha \Rightarrow h^*_\rho(f^*_\rho,\beta,\zeta) = f^*_\rho,\kappa$.

Second, we similarly choose $f_{m_\alpha}$.

Thirdly, we choose $M_\alpha$ and interpretation of $f$ (for $f \in \mathcal{F}_\alpha$) and $M^+_\beta$ when

$$\bar{b} \in \{\text{Rang}(f) : f \in \mathcal{F}_\alpha \& (\forall \zeta < \zeta_f)(\exists \nu \in W)(\eta^\nu_\zeta < \nu)\}$$

for some $W \in W_{<\alpha}$. Though we can use the compactness theorem, it seems to me more transparent to use ultraproduct. So let $D$ be an ultrafilter on $\alpha$ containing all co-bounded subsets of $\alpha$. Let $M_\alpha = \prod_{\beta < \alpha} M_\beta / D$. If $f \in \mathcal{F}_\alpha$, let $\beta_f < \alpha$ and $(\langle f_\gamma : \gamma \in [\beta_f, \alpha)\rangle)$ be such that $\beta_f \leq \gamma < \alpha \Rightarrow h^\gamma_\alpha(f) = f_\gamma$, so $\langle \eta^{\nu}_\zeta \mid \beta_f : \zeta < \zeta_f\rangle$ has no repetitions. Now, when $\eta^{\nu}_\zeta < \nu \in {^\kappa} \mu$, let

$$f_m(\ldots, \nu_\zeta, \ldots) = \langle c_\gamma : \gamma < \alpha \rangle / D,$$

where

$$\gamma \in (\beta_f, \alpha) \Rightarrow c_\gamma = (h^\gamma_\alpha(f)_m, \ldots, \nu_\zeta, \ldots) \in M_\gamma,$$

$$\gamma < \beta_f \Rightarrow c_\gamma$$

is any member of $M_\gamma$.

So $M^\alpha_W$ is well defined for $W \in W_{m(\alpha)}$.

Fourth, if $\bar{b} = \langle b_\zeta : \zeta < \zeta(\ast)\rangle \in {^\kappa} > (M^\alpha_W)$, $b_\zeta = f^m_m(\ldots, \nu_\zeta, \ldots)_{\zeta < \zeta(\ast)}$, and $\beta_\zeta < \alpha$ and for $\gamma \in [\beta_\zeta, \alpha)$: $(\gamma, \zeta) : \zeta < \zeta(\ast) \in \text{Dom}(f_{m_\alpha})$, and $h^\gamma_\zeta(f, \gamma, \zeta) = f_\gamma$, then we let $\bar{b}^\beta = (\bar{b}^\beta_\zeta : \zeta < \zeta(\ast))$ where $\bar{b}^\beta_\zeta$ is $f_m^\zeta(\ldots, \nu_\zeta, \ldots)_{\zeta < \zeta(\ast)}$ if $\beta \in [\beta_\zeta, \alpha)$. If $\beta < \beta_\zeta$ and lastly we define $M^\alpha_b = \prod_{\beta \in [\beta_\zeta, \alpha)} M^\beta_{b_\beta} / D$.

We still have to check that if for the same $\bar{b}$ we get two such definitions, then they agree, but this is straightforward.

Fifth, we choose $M^\alpha_b$ for other $\bar{b} \in {^\kappa} > (M_\alpha)$ for which $M^\alpha_b$ is not yet defined to satisfy clause (d) of Definition 2.5; note that by the choice of $W_{m_\alpha}$ those choices do not influence the preservation of weakly/strongly indiscernible. So $m_\alpha$ is well defined and one can easily check that it is as required.  

Claim 2.15. Assume $\alpha = \beta + 1 < \kappa$, and $m_1$ is a $\beta$–approximation.

1. There are $h$, and an $\alpha$–approximation $m_2$ such that $m_1 \leq h$, $m_2$, $M_\alpha = M_m$, $M^\alpha_b = M^m_b$, and $\text{Dom}(h) = \mathcal{F}_m$.

2. If $m_1$ is weakly/strongly nice, then $m_2$ is weakly/strongly nice.

3. If $m_1$ is weakly/strongly indiscernible, then $m_2$ is weakly/strongly indiscernible; simply for $E$–indiscernible, $E \in E_\alpha$.

Proof: (1) Should be clear.

Let $\alpha(m_2) = \alpha$, $W_{m_2} = W_{m_1}$, $M_{m_2} = M_{m_1}$ and $M^m_{m_2} = M^m_{m_1}$ for $\bar{b} \in {^\kappa} > (M_{m_1})$. Then let

$$\mathcal{F}_{m_2} = \{g_f, h : f \in \mathcal{F}_\beta, h$$

is a function with domain $\{\eta_{g, \zeta} : \zeta < \zeta_f\}$

satisfying $h(\eta_{g, \zeta}) \in \text{Suc}(\eta_{g, \zeta}) = \{\eta_{g, \zeta}, \gamma < \mu\}$,

where for $g = g_f$, $h$ we let $\zeta_g = \zeta_f$ and $\eta_{g, \zeta} = h(\eta_{g, \zeta})$, and if $\nu_\zeta \in I_{\eta_{g, \zeta}}$ for $\zeta < \zeta_g$ ($= \zeta_f$), then

$$g^m_{f, h}(\ldots, \nu_\zeta, \ldots) = f^m_m(\ldots, \nu_\zeta, \ldots) \in M_{m_1} < M_{m_2}.$$
We define $h_*$ by:

$$\text{Dom}(h_*) = \mathcal{F}_{m_2}, \quad \text{and} \quad h_*(g_{f,h}) = f.$$ 

Lastly let

$$\text{Dom}(f_{m_2}) = \{ \langle \varepsilon : \varepsilon < \varepsilon(*) \rangle : \text{ for some } f = \langle f_{\varepsilon} : \varepsilon < \varepsilon(*) \rangle \in \text{Dom}(f_{m_1}) \text{ and a function } h \text{ with domain} \}$$

and if $h, f, g = \langle f_{h,f}, \varepsilon : \varepsilon < \varepsilon(*) \rangle \in \text{Dom}(f_{m_2})$ are as above, $\sigma(x)$ is a $\tau[T, \varepsilon(*)]$-term, $x = \langle x_\xi : \xi \in u \rangle$, and $u$ is a finite subset of $\varepsilon(*)$ and $(f_{m_1}(f))(\sigma(x)) = f$, then $(f_{m_1}(\bar{g}))((\bar{\varepsilon}))(\sigma(x)) = g_{f,h}$.

Now check.

2), 3) Easy.

**Definition 2.16.**

1. For approximations $m_1, m_2$, let $m_1 \leq^* m_2$ mean that $\alpha(m_1) = \alpha(m_2)$ and $m_1 \leq_h m_2$ with $h$ being the identity on $\mathcal{F}_{m_1} \subseteq \mathcal{F}_{m_2}$, and $W_{m_1} \subseteq W_{m_2}$ and $f_{m_1} \subseteq f_{m_2}$, the last mean that if $f \in \text{Dom}(f_{m_1})$ then $f \in \text{Dom}(f_{m_2})$ and the function $f_{m_2}(f)$ is equal to the function $f_{m_1}(f)$.

2. Let $m_1 <^* m_2$ mean that

(a): $m_1 \leq^* m_2$, 

(b): if $f \in F_{m_1}$ then $f \in \text{Dom}(f_{m_2})$.

**Observation 2.17.**

1. $\leq^*$ is a partial order, $m_1 \leq^* m_1$, and

$$m_1 <^* m_2 \Rightarrow m_1 \leq m_2, \quad \text{and} \quad m_1 <^* m_2 \leq^* m_3 \Rightarrow m_1 <^* m_3, \quad \text{and} \quad m_1 <^* m_2 \leq m_3 \Rightarrow m_1 <^* m_3.$$ 

2. Each $\leq^*$-increasing chain of length $< \theta^+$ has a lab (essentially its union). If all members of the chain are weakly/strongly indiscernible, then so is the lab.

3. If $m_2 : \varepsilon < \kappa$ is $<^*$-increasing then its lab $m$ is resplendent and $\varepsilon < \kappa \Rightarrow m_* <^* m$. So if each $m_\gamma$ is weakly/strongly good then $m$ is weakly/strongly excellent.

**Proof:** Easy.

As a warm up.

**Claim 2.18.**

1. For any $\alpha$-approximation $m_0$ there is a full, term closed $\alpha$-approximation $m_1$ such that $m_0 \leq^* m_1$.

2. If $m_0$ is an $\alpha$-approximation, then there is an $\alpha$-approximation $m_1$ such that $m_0 <^* m_1$ and $\text{Dom}(f_{m_1}) = F_{m_0}$.

**Proof:**

1. Let $M_{m_1} = M_{m_0}$, and $H_{m_1} = H_{m_0}$ for $\bar{b} \in ^* (M_{m_0})$. Let $W_{m_1} = W_{m_2}$, and let $\langle \bar{\nu}_\gamma : \gamma < \gamma^* \rangle$ list the sequences $\bar{\nu} \in {\kappa}^\gamma(\langle \bar{\mu} \rangle)$ such that $\langle \nu_\zeta : \zeta < \ell(\bar{\nu}) \rangle$ is without repetitions and $\{ \nu_\zeta : \zeta < \ell(\bar{\nu}) \} \notin W_{m_0}$. Let $\bar{\nu}_\gamma = \langle \nu_{\gamma, \zeta} : \zeta < \zeta^* \rangle$ and define $\bar{\rho}_\gamma = \langle \nu_{\gamma, \zeta} \uparrow \alpha : \zeta < \ell(\bar{\nu}) \rangle$, and $W_{\gamma} = \{ \nu_{\gamma, \zeta} : \zeta < \zeta^* \}$ for $\gamma < \gamma^*$. Let $\beta_\gamma = \text{otp}(\gamma \in \gamma^* : (\forall \gamma_2 < \gamma_1)(\bar{\rho}_{\gamma_2} \neq \bar{\rho}_{\gamma_1})$).

For each $W \in W_{m_1} \backslash W_{m_0}$, let $M^m_{W}$ be an elementary submodel of $M_{m_1}$ of cardinality $\theta$ such that

$$W^* \subseteq W \land W^* \in W_{m_0} \Rightarrow M^m_{W^*} \prec M^m_{W^*} \quad \text{and} \quad \bar{b} \in ^* (M_{W^*}) \Rightarrow M^m_{b \in W^*} \prec M^m_{b \in W^*}.$$
Let \( \langle a_{W,i} : i < \theta \rangle \) list the elements of \( M_W^m \). For \( \beta < \beta^* \) and \( i < \theta \) we choose \( f_{\beta,i} \) such that if \( \gamma < \gamma^* \) and \( \beta = \beta \) then \( \zeta_{f_{\beta,i}} = \lg(\nu_\beta) = \lg(\nu_\beta^*) \) and \( \eta(\zeta_{f_{\beta,i}}, \zeta) = \rho_{\gamma,\zeta} \), and we define \( f_{\beta,i} \) by: if \( \nu_\zeta \in \nu_\beta^* \) for \( \zeta < \zeta_{f_{\beta,i}} \), then \( f_{\beta,i} = \nu_\zeta \mu^1 \) and \( f_{\beta,i} = a_{W,i} \).

Next, \( \mathcal{F}_m \) almost is \( \mathcal{F}_{m_0} \cup \{ f_{\beta,i} : \beta < \beta^*, \ i < \theta \} \), just we term-close it. Lastly \( f_{m_1} \) is defined as \( f_{m_0} \) recalling that \( \text{Dom}(f_{m_1}) \) is required just to be a subset of \( F \).

2) Also easy.

Let \( M^* \) be a \( \| M_{m_0} \|^{+} \)-resplendent elementary extension of \( M_{m_0} \). We define an \( \alpha \)-approximation \( m_1 \) as follows:

(a): \( \alpha_{m_1} = \alpha_{m_0} \), \( W_{m_1} = W_{m_0} \), \( M_{m_1} = M^* \),
(b): if \( b \in \kappa^>(M_{m_0}) \), then \( M_{m_0}^b \) is an elementary extension of \( M_{m_0}^b \),
(c): \( f_{m_1} \subseteq f_{m_0} \) and \( \text{Dom}(f_{m_1}) = F_{m_0} \),
(d): \( \mathcal{F}_{m_1} = \mathcal{F}_{m_0} \),
(e): if \( (f_{m_0}(f))\sigma(\xi) = f, \eta(\bar{f}, \zeta) < \nu_{\eta} \in ^{\ast} \mu, \) and

\[
\bar{b} = (f_{m_1}(\ldots, \nu_{\zeta}, \ldots))_{\in \zeta \in \zeta_f} \quad \text{and} \quad \bar{\xi} = (x_{\xi})_{i \in u},
\]

then

\[
f_{m_1} = (\ldots, \nu_{\zeta}, \ldots)_{\in \zeta \in \zeta_f} = \sigma^{\alpha_{m_1}}(\ldots, (x_{\xi})_{i \in u}).
\]

Main Claim 2.19. Assume \( m_0 \) is a weakly nice approximation. Then there is a weakly good approximation \( m_1 \) such that \( m_0 <^* m_1 \) with \( W_{m_1} = W_{m_0} \).

Proof: By 2.18(1)+(2) there is a full term closed \( m_1 \) such that \( m_0 <^* m_1 \) and \( \text{Dom}(f_{m_1}) = F_{m_0} \). We would like to “correct” \( m_1 \) so that it is weakly indiscernible.

Let \( m_2 \) be an \( \alpha_{m_1} \)-approximation as guaranteed in the Claim 2.20 below, so it is good and reflecting we clearly see that \( m_1 \leq^* m_2 \) and even \( m_0 <^* m_2 \).

Main SubClaim 2.20. (1) Assume \( m_0 \) is a weakly nice \( \alpha \)-approximation and \( m_0 <^* m_1 \) and \( \text{Dom}(f_{m_1}) = F_{m_0} \) and \( W_{m_1} \) is an ideal (that is closed under finite union). Then there is a good \( \alpha \)-approximation \( m_2 \) such that:

(a): \( \alpha_{m_2} = \alpha_{m_1} \), \( \mathcal{F}_{m_2} = \mathcal{F}_{m_1} \), \( f_{m_2} = f_{m_1} \), and \( W_{m_2} = W_{m_1} \),
(b): \( m_0 <^* m_2 \),
(c) Assume

(\alpha): \( n < \omega \) and \( f_\ell \in \mathcal{F}_{m_1} \), \( v_\zeta^\ell \in I_{\eta(f_\ell, \zeta)} \) for \( \zeta < \zeta_f \), \( \ell < n \), and \( \Delta \) is a finite set of formulas in \( L(T) \),
(\beta): \( m < \omega \) and for \( k < m \) we have \( f_k = (f_k^\zeta : \zeta < \zeta_k) \in \text{Dom}(f_{m_1}) \) and \( n_k < \omega \) and \( g_{k, \ell} \in \mathcal{F}_{m_2} \) (for \( \ell < n_k \)) satisfying

\[
\{ \eta(g_{k, \ell}, \zeta) : \zeta < \zeta_{g_{k, \ell}} \} = \{ \eta(f_k^\zeta, \zeta) : \zeta < \zeta_{f_k} \},
\]

and \( v_\zeta^{\ell, k} \in I_{\eta(f_k^\zeta, \zeta)} \) for \( \ell < n_k, \zeta < \zeta_{f_k}, \) and \( \Delta_k \) is a finite set of formulas in \( L(\tau(\zeta_f, \tau(T))) \).

Then we can find \( p_{\ell, k}^{\zeta} \) for \( \ell < n_k, \zeta < \zeta_{f_k} \) and \( p_{\ell, k}^{\zeta} \) for \( \ell < n_k, \zeta < \zeta_{f_k} \) for \( \ell < n_k, k < m \) such that

(\i): \( p_{\ell, k}^{\zeta} \in I_{\eta(f_k^\zeta, \zeta)} \) for \( \zeta < \zeta_{f_k} \) and \( p_{\ell, k}^{\zeta} \in I_{\eta(f_k^\zeta, \zeta)} \) for \( \zeta < \zeta_{f_k} \) for \( \ell < n_k, k < m \).
(ii): the sequences \( \langle \rho^\ell : \ell < n, \zeta < \zeta_f \rangle \) and \( \langle \nu^\ell : \ell < n, \zeta < \zeta_f \rangle \) are similar (see Definition ),

(iii): the \( \Delta \)-type realized by the sequence

\[
(f^m_\ell (\ldots, \nu^\ell_{\zeta_f}, \ldots) : \ell < n)
\]

in \( M_{m_2} \) is equal to the \( \Delta \)-type which the sequence

\[
(f^m_\ell (\ldots, \nu^\ell_{\zeta_f}, \ldots) : \ell < n)
\]

realizes in \( M_{m_1} \),

(iv): for \( k < m_1 \), the \( \Delta_k \)-type realized by the sequence

\[
(g^m_k (\ldots, \nu^k_{\zeta_f}, \ldots) : \ell < n_k)
\]

in the model \( M^m_{m_2} \) is equal to the \( \Delta_k \)-type realized by the sequence

\[
(g^m_k (\ldots, \nu^k_{\zeta_f}, \ldots) : \ell < n_k)
\]

in the model \( M^m_{m_1} \),

(v): if \( k_1, k_2 < m \) then

\[
\langle f^k_\ell (\ldots, \nu^k_{\zeta_f}, \ldots) : \ell < n, \zeta < \zeta_{k_1} \rangle = \langle f^k_\ell (\ldots, \nu^k_{\zeta_f}, \ldots) : \ell < n, \zeta < \zeta_{k_2} \rangle
\]

if and only if

\[
\langle f^k_\ell (\ldots, \rho^k_{\zeta_f}, \ldots) : \ell < n, \zeta < \zeta_{k_1} \rangle = \langle f^k_\ell (\ldots, \rho^k_{\zeta_f}, \ldots) : \ell < n, \zeta < \zeta_{k_2} \rangle
\]

(vi): if \( \ell < n_k, k < m, \ell^* < n, \) then

\[
f^m_\ell (\ldots, \nu^{\ell^*}_{\zeta_f}, \ldots) : \ell < n, \zeta < \zeta_f \rangle = f^m_\ell (\ldots, \nu^\ell_{\zeta_f}, \ldots) : \ell < n, \zeta < \zeta_{k_1}
\]

if and only if

\[
f^m_\ell (\ldots, \rho^\ell_{\zeta_f}, \ldots) : \ell < n, \zeta < \zeta_f \rangle = f^m_\ell (\ldots, \rho^\ell_{\zeta_f}, \ldots) : \ell < n, \zeta < \zeta_{k_1}
\]

Discussion 2.21. Now we have to apply the Ramsey theorem to recapture weak indiscernibility. Why we only promise \( m_0 \) excellent? As \( T^* [b, M] \) is not a continuous function of \( (b, M) \) and, more done to earth, as during the proof we need to know the type of \( b \) whenever we consider types in \( M^m_\ell \) in order to know \( T^* [b, M] \).

Usually a partition theorem on what we already have is used at this moment, but partition of infinitary functions tend to contradict ZFC. However, in the set \( \Lambda \) expressing what we need, the formulas are finitary. So using compactness we will reduce our problem to the consistency of the set \( \Lambda \) of first order formulas in the variables

\[
\{ f(\ldots, \eta_{\zeta_f}, \ldots) : f \in \mathcal{F}^{m_1} \text{ and } \zeta < \zeta_f \Rightarrow \eta(f, \zeta) < \eta_{\zeta_f} < n, \mu \}. \]

This can be easily reduced to the consistency of a set \( \Lambda \) of formulas in \( \mathcal{L}(\tau_T) \) (first order).

We can get \( \Lambda \) because for all relevant \( b \) we know \( T^* [b, M] \).
Proof: Let \( Y = \{ y_\zeta(\ldots, y_{\eta-1}, y_\eta) \in w[f] \} : f \in F_{m_1} \) and \( \nu_\eta \in I_\eta \) for \( \eta \in w[f] \) be a set of individual variables with no repetitions, recalling that \( w[f] = [\eta[f, \varepsilon] : \varepsilon < \zeta_f] \).

For each \( f \in \text{Dom}(f_{m_1}) \) and \( \bar{\nu} = (\nu_\eta : \eta \in w[f]) \) such that \( \nu_\eta \in I_\eta \), let \( \tau_{f, \bar{\nu}} \) be \( \tau[T, \bar{g}[f]] \) where \( w[f] = w[f_\bar{\nu}] \) for each \( \varepsilon < \ell_\bar{g}(f) \); pedantically it is a copy of \( \tau_{f, \bar{\nu}} \) for \( (f_1, \nu_1) \neq (f_2, \nu_2) \Rightarrow \tau_{f_1, \nu_1} \cap \tau_{f_2, \nu_2} = \tau_T \). Let \( \tau^* = \bigcup \{ \tau_{f, \bar{\nu}} : f, \bar{\nu} \text{ as above } \} \cup \tau_T \).

Let \( g_{f, \bar{\nu}} \) be a one to one function from \( \tau[T, \bar{g}(f)] \) onto \( \tau_{f, \bar{\nu}} \) which is the identity on \( \tau_T \) preserve the arity and being a predicate function symbol, individual constant.

Let \( g_{f, \bar{\nu}} \) be the mapping from \( L(\tau[T, \bar{g}(f)]) \) onto \( L(\tau_{f, \bar{\nu}}) \) which \( g_{f, \bar{\nu}} \) induce.

We now define a set \( \Lambda \) (the explanations are for the use in the proof of \( \Xi_1 \) below). \( \Xi_0 \Lambda = \Lambda_0 \cup \Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \cup \Lambda_4 \cup \Lambda_5 \cup \Lambda_6 \cup \Lambda_7 \cup \Lambda_8 \) where

(a): \( \Lambda_0 = \{ \varphi(\bar{y}, \bar{z}, \zeta, \zeta, \zeta) : \text{where } \bar{t} = \text{truth if and only if } \}
\]

\[ \varphi_1[\zeta + 1, \mu, \mu, \mu, \xi < \kappa, \mu] \text{ for } \ell = 1 \}
\]

[explanation: to satisfy (iii) in clause (B) of Definition 2.5].

(b): \( \Lambda_1 = \{ y_{f_\zeta, \zeta, \xi} \in y_{f_\zeta, \zeta, \xi} : \zeta < \kappa, \mu \in \alpha, \mu \in I_\mu \text{ for } \ell = 1 \text{ and } \}
\]

\[ \varphi_1[\zeta + 1] \text{ for } \ell = 2 \}
\]

[explanation: to satisfy (ii) in clause (B) of Definition 2.5].

(c): \( \Lambda_2 = \{ f, f' \in L(\eta(f, \zeta)) : f, f' : f, f' : \zeta < \kappa \} \text{ and } \}
\]

\[ \varphi_1[\zeta + 1] \text{ for } \ell = 1 \}
\]

[explanation: this is preservation of the witnesses for closure under terms of \( \tau \), in clause (E) of Definition 2.6(4) for \( m_1 \)].

(d): \( \Lambda_3 = \{ f, f' : f, f' : \zeta < \kappa \} \text{ and } \}
\]

\[ \varphi_1[\zeta + 1] \text{ for } \ell = 1 \}
\]

[explanation: this is for being above \( m_0 \), the \( L(\tau(T)) \)-formulas].

(e): \( \Lambda_4 = \{ \varphi_1[\zeta + 1] \text{ for } \ell = 1 \}
\]

[explanation: this is for being above \( m_0 \), the formulas from the \( M_0^{m_0 naughty} \)].

(f): \( \Lambda_5 \) like \( \Lambda_4 \) for the \( M_0 \)'-s that is

\[ \Lambda_5 = \{ \varphi_1[\zeta + 1] \text{ for } \ell = 1 \}
\]

[explanation: this is for weak indiscernibility, see Definition 2.9(1) clause (b) and Definition 2.9(3)].

(h): \( \Lambda_6 = \{ \varphi_1[\zeta + 1] \text{ for } \ell = 1 \}
\]

[explanation: this has to show the existence of the \( M_0 \); we can avoid this if we change the main definition such that instead \( M_0 \) we have \( M_{f, \bar{\nu}} \)].
\[ L(\tau_{f,\rho}), f_\ell \in \mathcal{F}_m, \zeta_\ell \in \zeta^*, \eta(f_\ell, \zeta) = \eta_{\zeta} \in ^a\mu \text{ for } \zeta < \zeta^*, \text{ and } w(f) \subseteq \{ \eta_{\zeta} : \zeta < \zeta^* \}, \text{ and for } k = 1, 2 \text{ we have } \eta_{\zeta} \cup \nu_{\zeta}^k \in ^a\mu \text{ for } \zeta < \zeta^*, \text{ and } \\
\eta(f, \tau_{f}) = \eta_{\zeta} \Rightarrow \nu_{\zeta} = \nu_{\zeta}^k \}
\]

Clearly (e.g. for the indiscernibility we use term closure)

\((\exists_1)\): \(\Lambda\) is a set of first order formulas in the free variables from \(Y\) and the vocabulary \(\tau^*\) such that an \(\alpha\)-approximation \(\mathfrak{m}\) satisfying (i) below is as required if and only if clause (ii) below holds where

(i): \(\mathcal{F}_m = \mathcal{F}_{m_0}, f_m = f_{m_1}, W_m = W_{m_1}, \)

(ii): interpreting \(y_{f(\ldots, \eta_{\zeta}, \ldots)} \in Y\) as \(f^{\mathfrak{m}(\ldots, \eta_{\zeta}, \ldots)}_{\eta_{\zeta}
}\) and the predicates and relation symbols in each \(\tau_{f,\rho}\) naturally, \(M_m\) is a model of \(\Lambda\) or more exactly not \(M_m\) but the common expansion of the \(M_{m_0}\)'s for \(b \in \{ (f^{\mathfrak{m}(\ldots, \eta_{\zeta}, \ldots)}_{\eta_{\zeta}
} : \epsilon < \epsilon^* ) : f = (f_\ell : \epsilon < \epsilon^*) \in \text{Dom}(f_{m_1}), \)

and \(\lambda_{\zeta} \in I_{\eta(\ell, \zeta)}\).

So it is enough to prove

\((\exists_2)\): \(\Lambda\) has a model.

We use the compactness theorem, so let \(\Lambda^a \subseteq \Lambda\) be finite. We say that \(\nu \in ^a\mu\) appears in \(\Lambda^a\), if for some variable \(y_{f(\ldots, \eta_{\zeta}, \ldots)} \in Y\) appearing as a free variable in some \(\varphi \in \Lambda^a\) we have \(\nu \in \{ \nu_{\eta} : \eta \in w[f]\} \) or some formula in \(\Lambda^a\) belongs to \(L(\tau_{f,\rho}) \setminus L(\tau_{\rho})\). We may also say “\(\nu\) appears in \(\varphi^*\),” and/or “\(\varphi(\ldots, \eta_{\zeta}, \ldots) \eta_{\zeta} \in w[f]\) appears in \(\varphi\)” (or in \(\varphi^*\)).

Let \(n^a_0 = |\Lambda^a|\). Now, for each \(\eta \in ^a\mu\) the set of \(\nu \in I^a_\eta\) appearing in \(\Lambda^a\), which we call \(J^a_\eta\), is finite but on \(\bigcup_{\eta \in ^a\mu} J^a_\eta\) we know only that its cardinality is \(\leq \kappa\). Note that, moreover, \(n^a_1 = \max\{ |J^a_\eta| : \eta \in ^a\mu\}\) is well defined \(< \kappa\) as well as \(m^a_0 = |\Lambda^a \cap \Lambda_0|\).

For each \(\eta \in ^a\mu\) we can find a finite set \(u_\eta \subseteq \kappa\) such that:

\((\otimes)\):

(i): if \(\nu_\ell \neq \nu_2 \in J^a_\eta\), then \(\min \{ \zeta : \nu_\ell(\zeta) \neq \nu_2(\zeta) \} \in u_\eta\)

(ii): if \(\varphi_{\zeta} \in [J^a_{\nu_{\zeta} \in \nu_{\zeta}} \cup \nu_{\zeta}] \subseteq \Lambda^a \cap \Lambda_0\), then 

(iii): \(\alpha \in u_\eta\)

(iv): \(|u_\eta| \leq (n^a_1)^2 + 2m^a_0 + 1\).

Clearly \(n^a_2 = \max\{ |u_\eta| : \eta \in ^a\mu\} \) is well defined \(< \kappa\), so without loss of generality, \(\eta \in ^a\mu\) \(\Rightarrow |u_\eta| = n^a_2\).

Let \(\nu \subseteq ^a\mu\) be finite, in fact of size \(\leq |\Lambda^a| = n^a_0\) such that:

\((\mathcal{I})\):

\(\varphi_{\zeta} \in [y_{f_{\rho_{\zeta} \in \nu_{\zeta}}}, y_{f_{\rho_{\zeta} \in \nu_{\zeta}}}^\ast] \subseteq \Lambda^a \cap \Lambda_0\), so \(\ell \in \{ 1, 2 \} \Rightarrow \nu_\ell \in J^a_\rho_{\ell}\),

then \(\rho_{\ell} \in \nu\) for \(\ell \in \{ 1, 2 \}\).

Now, for all \(\eta \in ^a\mu \setminus v\) we replace in \(\Lambda^a\) all members of \(J^a_\eta\) by one \(\nu_{\eta} \in I^a_\eta\) and we call what we get \(\Lambda^b\), i.e., we identify some variables. It suffices to prove \(\Lambda^b\) is consistent. Now, by the choice of the set \(v\) also \(\Lambda^b\) is of the right kind, i.e., \(\subseteq \Lambda\).

Why? We should check the formulas \(\varphi^*\) in \(\Lambda^a \cap \Lambda_1\) for each \(i \leq 8\); let it be replaced by \(\varphi^* \in \Lambda^b\). If in \(\varphi \in \Lambda_0 \cap \Lambda^a\) by clause (ii) of \(\otimes\) this substitution has no effect on \(\varphi\). If \(\varphi \in \Lambda_1\), either \(\varphi^* = \varphi\) or \(\varphi^*\) is trivially true. If \(\varphi \in \Lambda_3\), clearly \(\varphi^* \in \Lambda_3\). If \(\varphi \in \Lambda_4\) then \(\varphi^* \in \Lambda_4\) as \(m_0\) is nice hence weakly indiscernible, i.e., clause (b) of Definition 2.9(1) (and the demand \(f_\ell \in \mathcal{F}_{m_0}\)). If \(\varphi \in \Lambda_5\), similarly using clause (c) of Definition 2.9(1). Lastly if \(\varphi \in \Lambda_6\) we just note that similarly is preserved and similarly for \(\varphi \in \Lambda_7 \cup \Lambda_8\).
We then transform $\Lambda^b$ to $\Lambda^c$ by replacing each $\varphi$ by $\varphi'$, gotten by replacing, for each $\rho \in v$, every $\nu \in J^b_\rho$ by $\nu^{[k]} \in k_\mu$ where $\nu^{[k]}(\beta) = \nu(\beta)$ if $\beta \in \alpha \cup u_\rho$ and $\nu^{[k]}(\beta) = 0$ otherwise. It suffices to prove the consistency of $\Lambda^c$. Now, the effect is renaming variables and again $\Lambda^c \subseteq \Lambda$. Let $\rho^* = \langle \rho_k^* : k < k^* \rangle$ list the $\rho$ in $\mu$ which appear in $\Lambda^c$ such that $\rho|\alpha \in v$. Let $\eta_k = \rho_k^*|\alpha$ so $\eta_k \in v$, and let $\Upsilon = \langle \rho : \rho = \langle \rho_k : k < k^* \rangle, \eta_k \in \mu^i, (\forall \epsilon)(\alpha \leq \epsilon < \kappa \& \epsilon \notin u_\eta \Rightarrow \rho_k(\epsilon) = 0) \rangle$ and $\rho$ is similar to $\rho^*$ i.e., for $k_1, k_2 < k^*$ and $\epsilon < \kappa$ we have $\rho_{k_1}(\epsilon) < \rho_{k_2}(\epsilon) \Rightarrow \rho_{k_1}^*(\epsilon) < \rho_{k_2}^*(\epsilon)$.

For each $\rho \in \Upsilon$ we can try the following model as a candidate to be a model of $\Lambda^c$. It expand $M_{\alpha,k}$, and if symbols from $T_{f,\rho} \setminus T_{f}$ appear they are interpreted as their $g_{f,\rho}^{k}$ images are interpreted in $M_{\alpha,k}$. Lastly we assign to the variable $y_{f}(\ldots , \nu_\alpha, \ldots )_{\xi<\zeta}$ appearing in $\Lambda^c$ the element $f_m(\ldots , \nu_\alpha, \ldots )_{\xi<\zeta}$ of $M_{\alpha,k}$. Call this the $\rho$-interpretation. Considering the formulas in $\Lambda^c \cap A_\lambda$ for $i \in \{0, \ldots , 5, 7\}$ they always holds. For the formulas in $\Lambda^c \cap A_\lambda$, we can use a partition theorem on trees with $|n_\beta^*| < \aleph_0$ levels (use [Sh:E59, 1.16](4), which is an overkill, but has the same spirit (or [Sh:AP2.6, p.662])).

**Claim 2.22.** There is an increasing continuous inverse system of approximations $(m_\gamma, h_\beta^\gamma : \gamma \leq \kappa, \beta \leq \alpha \leq \kappa)$ such that each $m_\gamma$ is weakly excellent.

*Proof:* By induction on $\alpha \leq \kappa$ we choose $m_\alpha$ and $(h_\beta^\alpha : \beta < \alpha)$ with our inductive hypothesis being

\[ (*) : \quad (a) : (m_\beta, h_\gamma^\beta : \beta_1 \leq \alpha, \gamma < \beta \leq \alpha) \quad \text{is an inverse system of approximations,} \]

\[ (b) : m_\beta \quad \text{is a weakly excellent } \beta\text{-approximation,} \]

For $\alpha = 0$: A weakly excellent good $0$–approximation exists by 2.13.

For $\alpha$ limit:

Clearly $(m_\beta, h_\gamma^\beta : \beta_1 < \alpha, \gamma < \beta < \alpha)$ is an inverse system of good weakly excellent approximations with $\alpha(m_\beta) = \beta$. So by 2.14 we can find $m_\alpha, h_\beta^\alpha (\beta < \alpha)$ as required.

For $\alpha = \beta + 1$:

By 2.15(1+2) there is $m^{*}_{\alpha,0}$ a weakly nice $\alpha$–approximation such that $m_\beta \leq^* m^{*}_{\alpha,0}$. By 2.19 there is a full term closed $\alpha$–approximation $m^{*}_{\alpha,1}$ such that $m^{*}_{\alpha,0} \leq^* m^{*}_{\alpha,1}$ and $m^{*}_{\alpha,1}$ is good. We can choose by induction on $\epsilon \in [1, \kappa]$ good $\alpha$–approximations $m_{\alpha,\epsilon, \kappa}$, $\leq^*$-increasing continuously, $m_{\alpha,\epsilon, \kappa} <^* m_{\alpha,\epsilon+1}$. For $\epsilon = 1$, $m_{\alpha,\epsilon}$ is defined; for $\epsilon$ limit use 2.17(2), for $\epsilon$ successor use 2.19, and $m_{\alpha} = : m_{\alpha,\kappa}$ is good by 2.17(3).

**Claim 2.23.** Assume $m_\alpha, h_\gamma^\alpha$ for $\alpha \leq \kappa, \gamma < \alpha$ as in 2.22 with $\mu = \lambda$ and $\lambda = \lambda^\kappa \geq \theta$ (e.g., $\lambda = \lambda^\kappa \geq 2^{[T]}$). Then there are $> \lambda$ pairwise non-isomorphic $\kappa$–resplendent models of $T$ of cardinality $\lambda$.

*Proof:* Let $m = m_\kappa$ and $I \subseteq \kappa \geq \lambda$, $|I| = \lambda$ and for simplicity $\{ \eta \in \kappa : \eta(\epsilon) = 0$ for every large enough $\epsilon < \kappa \} \cup \kappa \lambda \subseteq I$. Let $M_I$ be the submodel of $M_m$ with universe

\[ \{ f(\ldots , \nu(f, \zeta), \ldots )_{\xi<\zeta} : f \in \mathcal{F}_m \text{ and } \eta(f, \zeta) \in I \cap \kappa \lambda \text{ for every } \xi < \zeta \}. \]
Trivially, \( \|M_I\| \leq \lambda^\kappa = \lambda \) and by clause (B) of Definition 2.5 clearly by 2.1(1) it follows that the sequence \( \langle a_\eta : \eta \in \varepsilon \lambda \rangle \) is with no repetitions for each \( \varepsilon < \lambda \) hence by the indiscernibility the sequence \( \langle a_\eta : \eta \in I \rangle \) is with no repetition, so \( \|M_I\| \geq |I| \geq \lambda \), so \( \|M_I\| = \lambda \).

Now, \( M_I \) is a \( \kappa \)-resplendent model of \( T \) as \( m \) being weakly excellent is full and resplendent.

For \( \zeta < \kappa, \nu \in \varepsilon \lambda \) let \( a_\nu = f^*_{\rho,\zeta}(\eta) (\in M_I) \) for any \( \eta \in I_{\zeta} \cap I \).

The point is:

(\( \otimes \)): For \( \eta \in \varepsilon \lambda, \nu_\gamma \in \gamma^{+1}\lambda, \nu_\gamma \upharpoonright \gamma = \eta \upharpoonright \gamma \), \( \nu_\gamma \neq \eta \upharpoonright (\gamma + 1) \), we have:

\[ \therefore \] the type \( \{ \varphi(x, a_{\eta_\gamma(\gamma + 1)}) \equiv \neg \varphi(x, a_{\nu_\gamma}) : \gamma < \kappa \} \)

is realized in \( M_I \) if and only if \( \eta \in I \).

[Why? The implication "\( \Rightarrow \)" holds by clause (B)(iii) of Definition 2.5. For the other direction, if \( c \in M_I \), then for some \( W \in W_\kappa \), satisfying \( W \subseteq I \), we have \( c \in N^W_\mu \), and as \( \eta \notin I \) and \( |W| < \kappa \) clearly for some \( \alpha < \kappa \) we have

\[ \{ \nu : \eta \upharpoonright \alpha < \nu \in \varepsilon \mu \} \cap W = \emptyset. \]

Let \( c = f_{\alpha \kappa}(\nu_\zeta, \ldots, \nu_{\zeta + 1}), \) where \( f \in F_\kappa \), so \( \nu_\zeta = \eta(f, \zeta) \). By the continuity of the system, for some \( \gamma \in (\alpha, \kappa) \) we have \( f \in \text{Dom}(h^\kappa_\gamma) \), and it suffices to prove that

\[ M_I \models \varphi[c; a_{\eta(\gamma + 1)}] \equiv \varphi[c; a_{\nu_\gamma}] \].

By the definition of a system, \( m \) is full. Choose \( \nu \in I_{\nu_\gamma} \); recalling \( m_{\nu_\gamma} \leq h^\kappa_{\nu_\gamma} m_\kappa \) it suffices to prove that

\[ M_{m_{\nu_\gamma}} \models \varphi[h^\kappa_\gamma(f)(\nu_\zeta, \ldots, \nu_{\zeta + 1})] \equiv \varphi[h^\kappa_\gamma(f)(\nu_\zeta, \ldots, \nu_{\zeta + 1})] \].

But \( m_{\nu_\gamma} \), is weakly excellent, hence it is \( E^0_\gamma \)-indiscernible, and hence the requirement holds.]

Now use [Sh:309, \( \S \)2] to get among those models, \( > \lambda \) non-isomorphic; putting in the eventually zero \( \eta \in \varepsilon \lambda \) does not matter.
Claim 3.1. If there is strongly excellent $\kappa$-approximation $m$ and $\mu \geq \lambda = \lambda^\kappa \geq 2^{|T|}$, then $T$ has $2^\kappa$ non-isomorphic $\kappa$-resplendent models of cardinality $\lambda$.

Proof: This time use Theorem [Sh:331, 2.3]. For any $I \subseteq \kappa^+ \lambda$ which includes $\kappa^+ \lambda$, let $M_I \prec M_m$ be defined as in the proof of 2.23. For $\eta \in \kappa^+ \lambda$ let $a_\eta = f^*_\eta,\kappa(\eta)$, and for $\eta \in \kappa^+ \lambda$ of length $\gamma + 1$ let $\eta' = \eta | _\gamma \gamma \langle \eta(\gamma) + 1 \rangle$, and for any $\nu \in I_\eta$, $\nu' \in I_{\eta'}$ let $a_\eta = \langle f^{\nu,\gamma+1}(\nu), f^{\nu',\gamma+1}(\nu') \rangle$; the choice of $(\nu, \nu')$ is immaterial. Let

$$\varphi(\langle x_\alpha : \alpha < \kappa \rangle) = (\exists y) \left( \bigwedge _{\alpha < \kappa} \varphi(y, x_{\alpha,0}) \equiv \gamma \varphi(y, x_{\alpha,1}) \right).$$

Now we can choose $f_I : M_I \rightarrow \mathcal{M}_{\lambda, \kappa}$ such that

(a): if $f_I(b) = \sigma(\langle t_i : i < i^* \rangle)$ such that $t_i \in I \cap \kappa^+ \lambda$ with no repetitions and $\sigma \in \tau[\mathcal{M}_{\lambda, \tau}]$, then for some $W \in W_m$ and $\gamma < \kappa$ such that $\langle \eta | \gamma : \eta \in W \rangle$ is with no repetition we have $\{ t_i : i < i^* \} = W$ and for some $f \in \mathcal{F}_m$ with $\zeta_f = \zeta$, and $\eta(f, \zeta) = t_\zeta$ for $\zeta < \zeta'$ we have $b = f_m(\ldots, t_\zeta, \ldots)_{\zeta < \zeta'}$ and

(b): $f_I(b) = \eta \in I_\gamma$ if $b = a_\eta$ (see above).

The new point is that we have to prove the statement $(\ast)$ in [Sh:331, 2.3|(v)(\beta)].

So assume that for $\ell = 1, 2$ and $\alpha < \kappa$: $b_\alpha^1 \in 2^{|M_{I_\ell}|}$, $f_{I_\ell}(b_\alpha^1) = \sigma^1_\alpha(b_\alpha^1)$, and $t_\alpha^1 = (t_\alpha^1, \varepsilon < \varepsilon_\alpha)$. Assume furthermore that $\sigma_\alpha = \sigma^2_\alpha$, $t_\alpha^1 = t_\alpha^2$ for $\alpha < \kappa$ (call it then $\sigma^1(t_\alpha)$ though possibly $I_1 \neq I_2$), and the truth value of each statement

$$(\exists \nu \in I_\gamma \cap \kappa^+ \lambda) \left( \bigwedge _{\alpha < \kappa} \nu | _\gamma \varepsilon_\alpha = t_\alpha^\nu, \gamma_1, \gamma_2 \varepsilon_\alpha \right)$$

does not depend on $\ell \in \{ 1, 2 \}$. Assume further that $M_{I_1} \models \varphi(\ldots, b_\alpha^1, \ldots)_{\gamma < \kappa}$, and we shall prove that $M_{I_2} \models \varphi(\ldots, b_\alpha^2, \ldots)_{\gamma < \kappa}$: this suffices.

First note that, as $f_{I_1}, f_{I_2} \subseteq f(\kappa^+ \lambda)$, necessarily $b_\alpha^1 = b_\alpha^2$ (so call it $b_\alpha$). Now, $M_{I_1} \models \varphi(\ldots, b_\gamma^1, \ldots)_{\gamma < \kappa}$ means that for some $c_1 \in M_{I_1}$ we have

$M_{I_1} \models \bigwedge _{\gamma < \kappa} \varphi[c_1, b_\gamma, 0] \equiv \neg \varphi[c_1, b_\gamma, 1],$

and let $c_1 = f_1(\ldots, \eta, \ldots)_{\eta \in w[f_1]}$. Let

$$J = \{ \eta : \eta \leq t_{\alpha, j}, \text{for some } \alpha < \kappa, j < \ell(\sigma_\alpha) \} \quad \text{and} \quad J_\gamma^+ = \{ \eta : \eta \in I_\gamma \text{ or } \ell(\eta) = \kappa \text{ and } (\forall \alpha < \kappa)(\eta | _\alpha \in J) \}.$$

By the assumption, $J$ is $\kappa$-closed, $J \subseteq I_1 \cap I_2$, moreover $J_\gamma^+ = J_\gamma^+$. Let $\gamma < \kappa$ be minimal such that $\eta \in w[f_1] \setminus J_\gamma^+ \Rightarrow \eta | _\gamma \notin J$, and the sequence $\langle \eta(f_1, \zeta) | _\gamma : \zeta < \zeta_f \rangle$ is with no repetitions and $f_1 \in \text{Dom}(h_\kappa)$.

Now we can choose $\nu_\kappa \in I_{\eta(f_1, \zeta)|_\lambda}$, from $I_2$ such that $\eta(f_1, \zeta) \in J_\gamma^+ \Rightarrow \nu_\kappa = \eta(f_1, \zeta)$. Let $f_2 \in \mathcal{F}_m$ be such that $h^s_\zeta(f_2) = h^s_\zeta(f_1)$ and $\eta(f_2, \zeta) = \nu_\kappa$ for $\zeta < \zeta_f = \zeta_{f_2}$. Easily, $c_2 = f_m(\ldots, \nu_\zeta, \ldots)_{\zeta < \zeta_f} \in M_{I_2}$ witness that

$$M_{I_2} \models (\exists y) \left( \bigwedge _{\alpha < \kappa} \varphi(\alpha, b_{\alpha,0}) \equiv \neg \varphi(\alpha, b_{\alpha,1}) \right)$$

(recalling $M_{I_1}, M_{I_2} \prec M(\kappa^+ \lambda)$).

Recall and add
Definition 3.2. 

1. $E_1^\alpha \in E$ (see Definition 2.8) is defined like $E_0^\alpha$ (see Definition 2.8(3)) except that we omit clause (iv) there.

2. For $\alpha < \kappa$ define $E_2^\alpha \in E$ as the following equivalence relation on $\{ \bar{\nu} : \bar{\nu} \in \kappa>(^\kappa \mu), \bar{\nu} \text{ with no repetition} \}$

$$\bar{\nu}^1 E_2^\alpha \bar{\nu}^2 \iff$$

(i) $\bar{\nu}^1, \bar{\nu}^2 \in \kappa>(^\kappa \mu)$ are with no repetition.

(ii) $\bar{\nu}^1, \bar{\nu}^2$ have the same length, all it $\zeta^*$.

(iii) $\nu^1_\zeta | \alpha = \nu^2_\zeta | \alpha$ for $\zeta < \zeta^*$.

(iv) for every $\zeta \in ^\alpha \mu$, the sets $u^\zeta_\eta = \{ \zeta < \zeta^* : \eta \triangleleft \nu^1_\zeta \}$ are finite equal and $\langle \nu^1_\zeta : \zeta \in u^1_\eta \rangle, \langle \nu^2_\zeta : \zeta \in u^2_\eta \rangle$ are similar.

Claim 3.3. 

1. In 2.22 we can demand that every $m_\gamma$ is $E_1^\gamma$-indiscernible i.e. get the strong version.

2. Moreover we can get even $E_2^\alpha$-indiscernibility.

Proof:

1. Very similar to the proof of 2.22. In fact, we need to repeat §2 with minor changes. One point is that defining “good” we use $E_1^\gamma$; the second is that we should not that this indiscernibility demand is preserved in limits, this is 2.14, 2.17. In fact this is the “strongly” version which is carried in §2 the until 2.19. From then on we should replace “weakly” by “strongly” and change the definition of $\Lambda_6, \Lambda_8$ appropriately in the proof of 2.20.

2. Similarly, only we need a stronger partition theorem in the end of the proof of 2.20, but it is there anyhow.

Remark 3.4. Clearly in many cases in 3.1, $\lambda = \lambda^{<\kappa} \geq \theta$ suffices, and it seems to me that with high probability for all. Similarly for getting many $\kappa$-resplendent models no one elementarily embeddable into another.
REFERENCES


