1 Summary of Results

In their fundamental work Paul Erdős and Alfred Renyi [1] considered the evolution of the random graph \( G(n, p) \) as \( p \) “evolved” from 0 to 1. At \( p = 1/n \) a sudden and dramatic change takes place in \( G \). When \( p = c/n \) with \( c < 1 \) the random \( G \) consists of small components, the largest of size \( \Theta(\log n) \). But by \( p = c/n \) with \( c > 1 \) many of the components have “congealed” into a “giant component” of size \( \Theta(n) \). Erdős and Renyi called this the double jump, the terms phase transition (from the analogy to percolation) and Big Bang have also been proffered.

Now imagine an observer who can only see \( G \) through a logical fog. He may refer to graph theoretic properties \( A \) within a limited logical language. Will he be able to detect the double jump? The answer depends on the strength of the language. Our rough answer to this rough question is: the double jump is not detectible in the First Order Theory of Graphs but it is detectible in the Second Order Monadic Theory of Graphs. These theories will be described below. We use the abbreviations fotog and somtag for these two theories respectively.

For any property \( A \) and any \( c > 0 \) we define

\[
f(c) = f_A(c) = \lim_{n \to \infty} Pr[G(n, c/n) \models A]
\]

Here \( G \models A \) means that \( G \) satisfies property \( A \). Beware, however, that we cannot presuppose the existence of \( f(c) \) as the limit might not exist.

**Theorem 1.** Let \( A \) be a fotog sentence. The \( f_A(c) \) exists for all \( c \) and \( f_A \) is an infinitely differentiable function. Moreover \( f_A \) belongs to the minimal family of functions \( \mathcal{F} \) which contain the functions 0, 1 and \( c \), are closed under
addition, subtraction, and multiplication by rationals and are closed under base $e$ exponentiation so that $f \in \mathcal{F} \Rightarrow e^f \in \mathcal{F}$.

Examples. Let $A$ be “there exists a triangle”. Then $f(c) = e^{-c^3/6}$. Let $B$ be “there exists an isolated triangle”, i.e., a triangle with none of the three vertices adjacent to any other vertices besides themselves. Then

$$f_B(c) = e^{-c^3 e^{-3c/6}}$$

Finally, call a triangle $x, y, z$ unspiked if there is no point $w$ which is adjacent to exactly one of $x, y, z$ and no other point. Let $C$ be the property that there is no unspiked triangle. Then

$$f_C(c) = e^{-c^3 e^{-3c^3 e^{-c^3}}/6}$$

Since 1 is not a special value for these $f$ we say, roughly, that the double jump is not detectible in fotog. Our remaining results all concern somtog.

Theorem 2. There is a somtog $A$ with

$$f_A(c) = \begin{cases} 0 & \text{if } c < 1 \\ 1 & \text{if } c > 1 \end{cases}$$

Theorem 3. For all somtog $A$ and $c < 1$ the value $f_A(c)$ is well defined.

Theorem 4. For all $c_0 > 1$ there is a somtog $A$ with $f_A(c)$ not defined for $c \geq c_0$.

Theorem 5. For any $c < 1$ and $\epsilon > 0$ there is a decision procedure that will determine $f_A(c)$ within $\epsilon$ for any somtog $A$.

Theorem 6. Let $c > 1$. Then there is no decision procedure that separates the somtog $A$ with $f_A(c) = 0$ from those with $f_A(c) = 1$.

Certainly the situation with $c=1$ is most interesting but we do not discuss it in this paper.

Description of Theories: The First Order Theory of Graphs (fotog) consists of an infinite number of variable symbols $(x, y, z, \ldots)$, equality $(x = y)$ and adjacency (denoted $x \sim y$) symbols, the usual Boolean connectives $(\land, \lor, \neg, \ldots)$ and universal and existential quantification $(\forall x, \exists y, \ldots)$ over the variables which represent vertices of a graph. Second Order Monadic Theory of Graphs (somtog) also includes an infinite number of set symbols $(S, T, U, \ldots)$ which represent subsets of the vertices and membership $(\in)$ between vertices and sets $(x \in S)$. The set symbols may be quantified over $(\forall S, \exists T, \ldots)$ as well as the variables. As an example, in fotog we may write

$$\forall x \forall y \exists z \exists w [x \sim z \land z \sim w \land w \sim y]$$
which means that all pairs of vertices are joined by a path of length three. However, one cannot say in fotog that the graph is connected. In somtog we define \( \text{path}(x, y, S) \) to be that \( x, y \in S \) and every \( z \in S \) is adjacent to precisely two other \( w \in S \) except for \( x \) and \( y \) which are each adjacent to precisely one point of \( S \). This has the interpretation that \( S \) gives an induced path between \( x \) and \( y \). The statement \( \exists S \text{path}(x, y, S) \) holds if and only if \( x \) and \( y \) lie in the same component since if they do a minimal path \( S \) would be an induced path. We write \( \text{conn}(x, y, R) \) for \( \exists S \subseteq R \land \text{path}(x, y, S) \) which means that in the restriction to \( R \), \( x \) and \( y \) lie in the same component. The property \( \text{G is connected} \) is represented by the somtog sentence \( \forall_x \forall_y \exists_S \text{path}(x, y, S) \). This ability to express \( x \) and \( y \) being joined by some path of arbitrary size seems to give the essential strength of somtog over fotog. Now we can prove Theorem 2. Let \( \text{circ}(S) \) be the sentence that \( S \) is connected and that every \( v \in S \) is adjacent to precisely two \( w \in S \). Consider the sentence

\[
A : \exists S, T, R \text{circ}(S) \land \text{circ}(T) \land S \cap T = \emptyset \land S \subseteq R \land T \subseteq R \land \forall_{x, y \in R} \text{conn}(x, y, R)
\]

This has the interpretation that the graph contains a component \( (R) \) with two disjoint circuits. For this \( A \) it is well known that \( f_A(c) = 0 \) when \( c < 1 \) and \( f_A(c) = 1 \) when \( c > 1 \).

\section{The First Order World}

The results of this section were done independently and in more complete detail in Lynch[1]. Here we attempt to give a more impressionistic picture of \( G(n, c/n) \) through First Order glasses.

What does \( G(n, p) \) look like? To begin with, there are lots of trees. More precisely, for any tree \( T \) and any \( r \) there are almost surely more than \( r \) copies of \( T \) as components of the graph. (This includes the trivial case where \( T \) is a single vertex.) What about more complicated structures. Let \( B(x, R) \) denote the set of vertices within distance \( R \) of \( x \), where we use shortest path as the metric. The veracity of a fotog \( A \) depends only on the values \( B(x, R) \) for a fixed (dependent on \( A \)) \( R \). (This is most certainly not the case for somtog.) For any fixed \( c \) and \( R \) a.a. all \( B(x, R) \) will be either trees or unicylic graphs. (For \( c > 1, G(n, c/n) \) will have many cycles in the giant component but they will be far apart.) To make things a bit bigger let’s define \( \mathcal{H} = \mathcal{H}_R \) to be the set of graphs \( H \) consisting of a cycle of size at most \( R \) and trees of depth at most \( R \) rooted at each vertex of the
cycle. For $C \subseteq G$ let $B(C, R)$ denote the set of vertices within distance $R$ of some $x \in C$. For any $H \in \mathcal{H}$ let $X_H$ denote the number of cycles $C$ with $B(C, R) \cong H$. When $H$ has $v$ vertices, and hence $v$ edges, with $w$ vertices not at depth $R$

$$E[X_H] = \binom{n}{v} p^v (1-p)^{w_n} / \left| \text{Aut}(H) \right| \sim \lambda_H$$

where

$$\lambda_H = c^v e^{-wc}/v! \left| \text{Aut}(H) \right|$$

Moreover, the $X_H$ act as independent Poisson distributions in that for any $H_1, \ldots, H_s \in \mathcal{H}$

$$\lim_{n \to \infty} P\left[ X_{H_i} = c_i, 1 \leq i \leq s \right] = \prod_{i=1}^{s} \lambda_{H_i}^e e^{-\lambda_{H_i}}$$

The values $B(C, R)$ can be generated as follows. For each $3 \leq i \leq R$ the number of $i$-cycles is Poisson with mean $c^i/2i$. We generate these cycles and then from each vertex generate a pure birth process with Poisson $c$ births.

To check the veracity of a fotog $A$ one needs only examine $B(x, R)$ and further one doesn’t need to be able to count higher than $R$. Let’s say two numbers are $R$-same if they are either equal or both at least $R$. Say two rooted trees of depth 1 are $R$-equivalent if the degrees of their roots are $R$-same. Clearly there are $R + 1$ equivalence classes. Suppose by induction $R$-equivalent has been defined on rooted trees of depth $i$. A rooted tree of depth $i+1$ may be identified naturally with a set of rooted trees of depth $i$. We call two rooted trees of depth $i+1$ $R$-equivalent if every $R$-equivalence class of rooted trees of depth $i$ appears the $R$-same number of times as a subtree. We say $H, H' \in \mathcal{H}$ are $R$-equivalent if their cycles are the same size and the vertices can be ordered about the cycles so that the rooted trees emanating from the corresponding vertices are $R$-equivalent. We say graphs $G, G'$ are $R$-equivalent if every equivalence class of $H \in \mathcal{H}$ appears the $R$-same number of times as a $B(C, R)$.

We use the following result about fotog. For every fotog $A$ there is an $R$ with the following property. Let $G, G'$ be $R$-equivalent, both with all $B(x, R)$ either trees or unicyclic. Suppose further that every $R$-equivalence class of rooted trees of depth $R$ appears at least $R$ times as a $B(x, R)$ in both graphs. Then

$$G \models A \iff G' \models A$$
With $R$ fixed and $c$ fixed the rooted trees above certainly appear. By induction on the depth we may show that every $R$-equivalence class of $H \in \mathcal{H}$ appears Poisson $\lambda(c)$ times in $G(n, c)$ where $\lambda \in \mathcal{F}$. Each $R$-equivalence class of $G$ occurs with probability $f(c) \in \mathcal{F}$. Then $Pr[A]$ is the finite sum of such $f(c)$ and is also in $\mathcal{F}$.

3 The Second Order Monadic World Before the Double Jump

Here we prove Theorem 5 and hence the weaker Theorem 3. Fix $c > 0$ and $\epsilon > 0$. There are $K, L$ so that with probability at least $1 - \epsilon$ all components of $G(n, c/n)$ are either trees or unicyclic components of size at most $K$, and there are less than $L$ of the latter. Each possible family of unicyclic components holds with a calculatable limit probability. Knowing the precise unicyclic components and that $G(n, c/n)$ has “all” trees and no other components determines the veracity of $A$. Hence $Pr[A]$ is determined within $\epsilon$.

4 The Second Order Monadic World After the Double Jump

We prove Theorem 6 by a reduction to the Trakhtenbrot-Vought Theorem, which states that there is no decision procedure which separates those fotog $A$ which hold for some finite graph from those which do not. By a clean topological $T_k(CTK_k)$ in a graph $G$ we mean an induced subgraph consisting of $k$ vertices, one path between every pair of points, and nothing else. In §5 we show for all $c > 1$ and all integers $k$ that $G(n, c/n)$ almost surely contains a $CTK_k$. For any fotog $A$ we define a somtog $A^+$ of the form

$$A^+ : \exists_{S, T, U\text{clean}(S, T)} A^*$$

Here $\text{clean}(S, T)$ represents that $S$ is the vertex set of a $CTK_k$ on $T$. That is

(i) $S \subset T$
(ii) Every $x, y \in S$ have a unique $T_{x, y} \subset T$ with $\text{part}(x, y, T_{x, y})$ and $T_{x, y} \cap S = \{x, y\}$.
(iii) There is no edge between any $T_{x, y}$ and $T_{x', y'}$ except at the endpoints.

Now we transform $A$ to $A^*$ by

(i) replacing $\forall_x$ and $\exists_y$ by $\forall_{x \in S}$ and $\exists_{y \in S}$
(ii) replacing $x \sim y$ by $T_{x, y} \cap U \neq \emptyset$, with $T_{x, y}$ defined above.
If $A$ holds for no finite graph then $A^+$ holds for no finite graph. Suppose $A$ holds for a finite graph $H$ on, say, $k$ vertices $1 \ldots k$. Almost surely $G$ contains a $CTK_k$ on vertices $T$ with endpoints $S$. Label $S$ by $x_1 \ldots x_k$ arbitrarily. Let $U$ consist of one vertex from $T_{x_i,x_j}$ (not an endpoint) for each pair $\{x_i,x_j\}$ with $\{i,j\} \in H$. Then $A^+$ holds. That is, $A^+$ holds a.a.

A decision procedure that could separate somtog $B$ with $f_B(c) = 1$ from those with $f_B(c) = 0$ could, when applied to $B = A^+$, be used to determine if $A$ held for some finite graph, and this would contradict the Trakhtenbrot-Vought Theorem.

**Nonconvergence.** To prove Theorem 4 we use a somewhat complicated graph. Let $k_1$ be a positive real and $K$ a positive integer. ($k_1 = 5, K = 100$ is a good example.) We define, for all sufficiently large $n$, a graph $H = H(k_1,K,n)$. Let $w$ be the nearest integer to $k_1 \log n$ divisible by $K$ (a technical convenience) so that $w \sim k_1 \log n$. (Asymptotics are in $n$ for fixed $k_1,K$. Begin with two points $S_0,S_1$ and three vertex disjoint paths, each of length $w$, between them. Call this graph $H^-$. Let $AR$ (which stands for arithmetizable) consist of every $K$-th vertex on each of the paths, excluding the endpoints. Thus $AR$ will have $\frac{w}{K} - 1$ points from each path, a total of $w_1 = 3\lceil\frac{w}{K}\rceil - 1$ points. Order the three paths arbitrarily and order the points of $AR$ on a path from $S_0$ to $S_1$ so that the points of $AR$ are labelled $1, \ldots , w_1$. Now, using this labelling, between every pair $i, 2i$ add a path of length $w$. (These paths all use new vertices with no additional adjacencies.)

Now between every pair $i, 2^i$ add a path of length $w$. Now between every pair $i, tower(i)$ add a path of length $w$. (The function $tower(i)$ is defined inductively by $tower(1) = 2, tower(i+1) = 2tower(i)$.). Finally between every pair $i, wow(i)$ add a path of length $w$. (The function $wow(i)$ is defined inductively by $wow(i) = 2, wow(i+1) = tower(wow(i))$.) This completes the description of the graph $H = H(k_1,K,n)$.

In §5 we prove that for every $c > 1$ there exist $k_1,K$ so that $G(n,c/n)$ almost surely contains an induced copy of $H = H(k_1,K,n)$. We assume that here, and with $H$ in mind construct a somtog sentence $A = A_K$ which shows nonconvergence.

The sentence $A = A_K$ will be built up in stages. First we say there exist vertices $S_0,S_1$ and sets $P_1, P_2, P_3$ so that each $P_i$ gives a path from $S_0$ to $S_1$, the $P_i$ overlap only at $S_0, S_1$, and there are no edges between $P_i$ and $P_j$ except at the endpoints. Second we say there exists a set $AR \subset P_1 \cup P_2 \cup P_3 = \{S_0,S_1\}$ so that for any path $x_1 \ldots x_K$ in any $P_i$ that $AR$ contains exactly one of the $x_1, \ldots, x_K$. (Here the sentence depends on the choice of the fixed integer $K$.) We define an auxiliary binary relation $<$ on
ARG. If \( x, y \in AR \cap P_i \) we define \( x < y \) by the existence of a subset of \( P_i \) which is a path from \( S0 \) to \( x \) which does not contain \( y \). If \( x \in AR \cap P_i \) and \( y \in AR \cap P_j \) with \( i \neq j \) we define \( x < y \) to be \( i = 1, j = 2 \) or \( i = 1, j = 3 \) or \( i = 2, j = 3 \). On \( AR \) we define the auxilliary binary predicate \( \text{next}(i, j) \) by \( i < j \) and there does not exist \( k \in AR \) with \( i < k \) and \( k < j \). We define the unary predicate \( \text{ONE}(i) \) by \( i \in AR \) and there is no \( j < i \) and \( \text{TWO}(i) \) by \( i \in AR \) and \( j < i \Rightarrow \text{ONE}(j) \). We say there are unique \( i, j \) with \( \text{ONE}(i) \), \( \text{TWO}(j) \). For convenience we write 1, 2 for these elements henceforth.

Now to arithmetize \( AR \). We say there exists vertex sets \( \text{DOUBLE} \), \( \text{EXP} \), \( \text{TOWER} \) and \( \text{WOW} \). We define auxilliary binary predicate \( \text{double} \) on \( AR \) by \( \text{double}(x, y) \) if \( x < y \) and there is a path from \( x \) to \( y \) in \( \text{DOUBLE} \); and we similarly define binary predicates \( \text{exp} \), \( \text{tower} \) and \( \text{wow} \). We say \( \text{double}(1, 2) \) and \( \text{double}(x, y) \cap \text{double}(x, z) \rightarrow y = z \) and \( \text{double}(x, y) \cap \text{next}(x, x_1) \cap \text{next}(y, y_1) \cap \text{next}(y_1, y_2) \rightarrow \text{double}(x_1, y_2) \) and if \( \text{double}(x, y) \) and \( \text{next}(x, x_1) \) and there do not exist \( y_1, y_2 \) with \( \text{next}(y, y_1) \cap \text{next}(y_1, y_2) \) then there does not exist \( z \) with \( \text{double}(x_1, z) \) and finally if \( \text{double}(x, y) \) and \( x' < x \) then there exists \( y' \) with \( \text{double}(x', y') \). We say \( \text{exp}(1, 2) \) and \( \text{exp}(x, y) \cap \text{exp}(x, z) \rightarrow y = z \) and \( \text{exp}(x, y) \cap \text{next}(x, x_1) \cap \text{double}(y, y_1) \rightarrow \text{exp}(x_1, y_1) \) and if \( \text{exp}(x, y) \) and \( \text{next}(x, x_1) \) and there does not exist \( y_1 \) with \( \text{double}(y, y_1) \) then there does not exist \( z \) with \( \text{exp}(x_1, z) \) and finally if \( \text{exp}(x, y) \) and \( x' < x \) then there exists \( y' \) with \( \text{exp}(x', y') \). The properties for \( \text{tower} \) are in terms of \( \text{exp} \) exactly as the properties for \( \text{exp} \) were in terms of \( \text{double} \) and the properties for \( \text{wow} \) are in terms of \( \text{tower} \) in the same way.

On \( AR \) we define unary predicates \( \text{even}(x) \) by \( \exists y \text{double}(y, x) \) and \( \text{invwow}(x) \) by there existing \( y \) with \( \text{wow}(x, y) \) but for all \( x' > x \) there do not exist \( y' \) with \( \text{wow}(x', y') \). The sentence \( A = A_K \) concludes by saying there exists \( x \) with \( \text{even}(x) \cap \text{invwow}(x) \).

Now we show that \( \lim Pr[G(n, p) \models A_K] \) does not exist, moreover that the limit sup is one and the limit inf is zero. On the integers define \( \text{wow}^{-1}(y) \) to be the biggest integer \( x \) with \( \text{wow}(x, y) \leq y \). First let \( n \rightarrow \infty \) through that subsequence for which \( \text{wow}^{-1}(w_1) \) is even. (Recall \( w_1 = \Theta (\log n) \) was the size of \( AR \).) Suppose \( G(n, p) \) contains an induced copy of \( H \). (\( k_1, K \) depend only on \( c \) and so are already fixed.) On \( H \) there do exist the vertices \( S0, S1 \), the sets \( P_1, P_2, P_3, AR, \text{DOUBLE}, \text{EXP}, \text{TOWER} \) with all the properties of \( A_K \). (Indeed, \( A_K \) was created with that in mind.) Under the labelling \( 1, \ldots, w_1 \) the predicates \( \text{double}, \ldots \) correspond to the actual numbertheoretic predicates and the \( x = \text{wow}^{-1}(w_1) \) has \( \text{invwow}(x) \) and \( \text{even}(x) \) so \( A_K \) holds. But \( G(n, p) \) contains an induced copy of \( H \) almost surely so the limiting probability on this subsequence is one.
In the other direction, let \( n \) go to infinity through a subsequence with the property that for all \( m \) with (leaving some room) \( \log \log n < m < n \) the value \( \text{wow}^{-1}(m) \) is odd. (Such \( n \) exist since \( \text{wow} - 1 \) is constant for such a long time.) Here is the crucial random graph fact: There is a \( \delta = \delta(c) \) so that in \( G(n, c/n) \) almost surely all subconfigurations consisting of two vertices and three paths between them have size at least \( \delta \log n \). (This uses a simple expectation argument. The number of configurations of \( t \) vertices and \( t + 1 \) edges giving the above graph is \( O(n^t \mathcal{P}^{t+1}) = O(c^{t+1}/n) = o(1) \) when \( t < \delta \log n \).) Thus almost surely any \( AR \) that satisfies the conditions of \( A_K \) will have \( |AR| = m > \delta' \log n > \log \log n \). The conditions on double, \ldots force \( AR \) to be arithmeticized so that \( \exists x \text{invwow}(x) \cap \text{even}(x) \) will not occur when \( \text{wow}^{-1}(m) \) is odd. Thus almost surely \( A_K \) will not be satisfied.

5 A Variance Calculation

We fix \( c > 1 \), set \( p = c/n \) and let \( G \sim G(n,p) \). We consider a graph \( H = H(k_1, K, n) \) as defined in \( \S 4 \). We give a description of \( H \) suitable for our purposes. Set \( w = [k_1 \log n] \). Take two vertices and draw three vertex disjoint paths each of length \( w \). This gives a graph \( H^- \). On \( H^- \) a set of pairs of vertices \( \{a, a'\} \) are specified, no \( a \) lying in more than eight such pairs. We let \( l \) denote the precise number of such pairs so that \( l \sim \epsilon \log n \). By making \( K \) large we can make \( \epsilon \) as small as desired. Between each such pair a path of length \( w \) is placed with new vertices. This gives the graph \( H \). It has \( v = \Theta(\ln^2 n) \) vertices and \( e \) edges where \( e = 3w + lw \sim k_1 \log^2 n \) and \( e - v = l + 1 \sim \epsilon \log n \). Let us denote the vertices of \( H \) by \( 1, \ldots, v \).

5.1 The Second Moment Method

Our object in this section is to show that, for appropriate \( k_1, \epsilon \), the random \( G(n,p) \) almost surely contains an induced copy of \( H \). Let \( X \) be the number of \( v \)-tuples \( (a_1, \ldots, a_v) \) of distinct vertices of \( G \) so when \( \{i, j\} \in E(H) \) then \( \{a_i, a_j\} \in E(H) \). That is, \( X \) is a count of copies of \( H \) in \( G \) though these copies may have extra edges and a given copy may be multiply counted if \( H \) has automorphisms. Clearly

\[
E[X] = (n)_v p^e \sim n^v p^e = c^e / n^{e-v}
\]

which is

\[
\eta(\epsilon \log n)(k_1 \log c^{-1} + o(1))
\]
from the estimates above. We first require that

\[ k_1 \log c > 1 \]

which assures that \( E[X] \) is a positive power of \( n^{\log n} \). The crucial calculation will be to show

\[ \text{Var}[X] = o(E[X]^2) \quad (V1) \]

From this, by Chebyshev’s Inequality \( X > .99 E[X] \) (say) almost surely. True, \( X \) counts noninduced copies of \( H \). But let \( X^+ \) be a count of all copies of any \( H^+ \) consisting of \( H \) with one additional edge added. There are \( \Theta(\log^4 n) \) choices of that edge and for a given choice the expected number of such copies is \( pE[X] \) so that \( E[X^+] = O(\log^4 n/n)E[X] = o(E[X]) \) and so by Markov’s Inequality almost surely \( X^+ < E[X]/2 \), say. So almost surely there are more than \( .99 E[X] \) copies of \( H \) and fewer than \( .5 E[X] \) total copies of graphs containing \( H \) and one more edge so therefore there is at least one copy of \( H \) with no additional edge, i.e., the desired induced copy.

Hence it suffices to show \( (V1) \).

Remark. To illustrate the complexities suppose we condition \( G(n, p) \) on a fixed copy of \( H^- \) and let \( Z \) be the expected number of extensions to \( H \). The expectation argument above gives that \( E[Z] \sim (n^{w-1}p^w)^l = (c^w/n)^l \) which is \( n^{\Theta(\log n)} \). However for there to be any extensions each of the at least \( l/8 \) vertices of \( H^- \) that is supposed to have a path coming out of it must have at least one edge besides those of \( H^- \). Any particular vertex fails this condition with probability \( e^{-c} \) and these events are independent so that the probability that \( Z \neq 0 \) is bounded from above by \( (1 - e^{-c})^{l/8} \) which is polynomially small. This illustrates that the expected number of thingees being large does not a priori guarantee that almost surely there is a thingee.

Of course, \( (V1) \) is equivalent to showing \( E[X^2] \sim E[X]^2 \). By the symmetry of copies, \( E[X^2] \) is \( E[X] \) times the expected number of copies of \( H \) given the existence of a particular copy of \( H \). Let

\[ V = \{1, \ldots, m\} \]

and let us specify a particular copy of \( H \) on vertex set \( V \) with \( 1, \ldots, 3w - 1 \) being the vertices of \( H^- \). Let \( G^* = G^*(n, p) \) be the random graph on vertex set \( 1, \ldots, n \) where for \( i, j \in V \) and \( \{i, j\} \in E(H) \) we specify that \( \{i, j\} \in E(G) \) but all other pairs \( i, j \) are adjacent in \( G \) with independent probabilities \( p \). (Note that even those \( i, j \) with \( i, j \in V \) but \( \{i, j\} \notin E(H) \)

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have probability $p$ of being in $G^*$. Let $E^*[X]$ denote the expectation of $X$ in $G^*$. Then it suffices to show

$$E^*[X] \sim E[X] \quad (V2)$$

5.2 The Core Calculation: Expectation for Paths

We shall work up to $E^*[X]$ in stages. Let $P^s(a,b)$ denote the expected number of paths of length $s$ between vertices $a,b$, with the graph distribution $G^*(n,p)$. (As a benchmark note that in $G(n,p)$ this expectation would be $(n-2)^s p^s \sim n^s p^s$.) $P^s(a,b)$ is simply the sum over all tuples $(a_0, \ldots, a_s)$ with $a_0 = a, a_s = b$ of distinct vertices of $G$ of $p^\alpha$ where $\alpha$ is the number of edges of the path $a_0 \cdots a_s$ which are not in $H$. Let $P^s_-(a,b)$ denote the expected number of such paths where we further require that $a$ is not adjacent to $a_1$ in $H$. (When $a \notin V$ these are the same.) We shall define inductively $x_s, x_s^-$ which provide upper bounds to $P^s(a,b)$ and $P^s_-(a,b)$ respectively under the further assumption that $b \notin V$. (We shall see that $P^s_-(a,b)$ is dominated by paths which do not overlap $H$ but that for $P^s(a,b)$ there is a contribution from those paths which are paths in $H$ for their initial segment.) Clearly we may set $x_1 = x_1^- = p$. Let $x_s, x_s^-$ satisfy the following:

$$x_s^- = px_{s-1}^- + px_{s-1}$$

$$x_s = x_s^- + \sum_{k=1}^{s-1} 50kx_{s-k}^-$$

We claim such $x_s, x_s^-$ provide the desired upper bounds. To bound $P^s_-(a,b)$ split paths $a_1 \cdots a_{s-1}b$ according to $a_1 \in V$ ($\leq m$ possibilities) and $a_1 \notin V$ ($\leq n$ possibilities). Note we are excluding the case where $a, a_1$ are adjacent in $H$. For a given $a_1$ the expected number of paths is $pP^s_{s-1}(a_1,b)$ (as we must have $a, a_1$ adjacent). When $a_1 \notin V$ this is by induction at most $px_{s-1}^-$ and when $a_1 \in V$ this is by induction at most $px_{s-1}$ so $P^s_-(a,b) \leq x_s^-$ by induction. Bounding $P^s(a,b)$ is a bit more complex. Those paths for which $a, a_1$ are not adjacent in $H$ contribute at most $x_s^-$ by induction. Otherwise, let $k$ be the least integer for which $a_k, a_{k+1}$ are not adjacent in $H$. (As $b \notin V$ this is well defined and $1 \leq k < s$.) We pause for a technical calculation.

We claim that in $H$ for any $k \leq w$ there are at most $50k$ paths of length $k$ beginning at any particular vertex $v$. Suppose $a \in H^-$. There are at most four such paths staying in $H^-$. Once leaving $H^-$ the path is determined (since critically $k \leq w$, the path length) and there are at most $8k$ ways of
determining when and how to leave \( H^- \). The argument with \( a \notin H^- \) is similar, we omit the details. Of course \( 50k \) is a gross overestimate but we only use that it is a \( O(k) \) bound.

Back to bounding \( P_s(a, b) \). For a given \( k \) there are at most \( 50k \) choices for \( a_1 \cdots a_k \) and fixing those there is a contribution of \( P_{s-k}^- (a_k, b) \leq x_{s-k}^- \) to \( P_s(a, b) \). Thus \( P_s(a, b) \leq x_s \) by induction.

Now to bound the values \( x_s, x_{s}^- \) given by the inductive formulae. Let \( L \) be fixed (dependent only on \( c \)) so that

\[
L > 1 + \sum_{k=1}^{\infty} 50kc^{-k}
\]

and set

\[
X_s = Lp^sn^{s-1}
\]

\[
X_s^- = p^sn^{s-1}(1 + L\frac{ms}{n})
\]

We claim \( x_s \leq X_s \) and \( x_s^- \leq X_s^- \). For this we merely check (recall \( pm = c \))

\[
X_s^- + \sum_{k=1}^{s-1} 50kX_{s-k}^- \leq (1 + L\frac{ms}{n})p^sn^{s-1}(1 + \sum_{k=1}^{s-1} 50kc^{-k}) \leq Lp^sn^{s-1} = X_s
\]

as \( Lms/n = o(1) \) and that

\[
 pnX_{s-1}^- + pmX_{s-1} = X_s^-
\]

Thus we have shown

\[
P_w(a, b) \leq Ln^{w-1}p^w
\]

when \( b \notin V \) and further

\[
P_w(a, b) \leq n^{w-1}p^w[1 + O(\frac{\log^3 n}{n})]
\]

when \( a, b \notin V \).

Now (thinking of \( a, b \in V \)) we seek a general bound \( y_s \) for \( P_s(a, b) \). We set \( y_1 = 1 \) (as perhaps \( a, b \) are adjacent in \( H \)) and define inductively

\[
y_s = 3 + pmx_{s-1} + pmy_{s-1} + 50sp + \sum_{k=1}^{s-2} 50k[pmx_{s-k-1} + pmy_{s-k-1}]
\]

We claim \( P_s(a, b) \leq y_s \) for \( 1 \leq s \leq w \). Of the potential paths \( aa_1 \cdots a_{s-1}b \) there are at most three which are paths in \( H \) and they contribute at most
three. There are less than 50 cases where $aa_1 \cdots a_{s-1}$ is a path in $H$ but $a_{s-1}, b$ are not adjacent in $H$ and they each contribute $p$. The cases with $a_1 \notin V$ contribute at most $pmx_{s-1}$. The cases with $a_1 \in V$ but not adjacent to $a$ in $H$ contribute at most $pmy_{s-1}$. Otherwise let $1 \leq k \leq s-2$ be the least $k$ so that $a_k, a_{k+1}$ are not adjacent in $H$. There are at most $50k$ choices of $a_1 \cdots a_k$. Then there are at most $n$ choices of $a_{k+1} \notin V$ and each contributes $px_{s-k-1}$ and at most $m$ choices of $a_{k+1} \in V$ and each contributes $py_{s-k-1}$.

Now fix a constant $M$ satisfying

$$M > M_1 = L[1 + \sum_{k=1}^{\infty} 50kc^{-k}]$$

We claim that for $1 \leq s \leq w$

$$P_s(a, b) \leq 4 + Mp^s n^{s-1}$$

By the previous bounds on $x_s$ we bound

$$pmx_{s-1} + \sum_{k=1}^{s-2} 50kpmx_{s-k-1} < M_1 p^s n^{s-1}$$

We bound $3 + 50sp < 3.01$. By induction we bound

$$pmy_{s-1} + \sum_{k=1}^{s-2} 50kpmy_{s-k-1} < 50s^2 pm[4 + Mp^s n^{s-1}] < .01 + (M - M_1)p^s n^{s-1}$$

since $50s^2 pm = O(\log^4 n/n) = o(1)$, completing the claim. We are really interested in the case $s = w$. Note $p^w n^{w-1} = c^w / n$ is asymptotically a positive power of $n$ by the choice made of $k_1$ earlier. Thus the +4 may be absorbed in $M$ and we have that

$$P_w(a, b) < Mp^w n^{w-1}$$

for all $a, b$ while if $a, b \notin V$ then we have the better bound

$$P_w(a, b) < p^w n^{w-1}[1 + O(\ln^3 n / n)]$$
5.3 Expectation of Copies of $H$

Now we turn to the full problem of bounding $E^*[X]$. Recall we have labelled $H$ so that $1, \ldots, 3w-1$ are the vertices of $H^-$. Recall $l$ denotes the number of $w$-paths in going from $H^-$ to $H$ and recall $l \sim \epsilon \log n$. $E^*[X]$ is the sum over all $m$-tuples $(a_1, \ldots, a_m)$ of distinct vertices of the probability $(G^*(n, p))$ that these as (in this order) give a copy of $H$. For each $a_1, \ldots, a_{3w-1}$ the contribution of $m$-tuples with this start is bounded from above by

$$p^\alpha \left[ Mp^{w} n^{w-1} \right]^{l-A} \left[ p^{w} n^{w-1} [1 + O(\frac{\ln^3 n}{n})] \right]^{A}$$

Here $\alpha$ is the number of adjacencies $i, j$ in $H^-$ with $a_i, a_j$ not adjacent in $H$. $A$ is the number of pairs $i, j$ in $H^-$ which are joined in $H$ by a $w$-path and for which neither $a_i$ nor $a_j$ is in $V$. $l - A$ is then the remaining number of pairs $i, j$ in $H^-$ joined in $H$ by a $w$-path.

To see this note that for fixed $a_1, \ldots, a_{3w-1}$ and any choice of $w$-paths $P_1, \ldots, P_l$ that are vertex disjoint the probability that they are all paths in $G$ is simply the product of the probabilities for each path. Adding over all $P_1, \ldots, P_l$ is then at most the product over $j$ of adding the probabilities for each $P_j$, and these are precisely what the bracketed terms bound. The $p^\alpha$, of course, is the probability that the $a_i$ have the proper edges of $H^-$. Now we split the contribution to $E[X^+]$ into two classes. First consider all those $a_1, \ldots, a_{3w-1}$ with no $a_i \in V$. There are at most $n^{3w-1}$ such tuples and each gives $p^{3w}$ with $3w$ being the number of edges in $H^-$. For each $A = l$ so this gives a

$$\left[ p^{w} n^{w-1} [1 + O(\frac{\ln^3 n}{n})] \right]^{l}$$

factor. As $l = O(\ln n)$

$$[1 + O(\frac{\ln^3 n}{n})]^l = 1 + o(1)$$

so this entire contribution is asymptotic to $n^{3w-1+l(w-1)} p^{3w+lw}$ which is asymptotic to $E[X]$, the expectation in $G(n, p)$. That is, the main contribution (among the $a_1 \cdots a_{3w-1}$ that don’t overlap $V$) to $E^*[X]$ is by those copies of $H$ that don’t overlap $H$ at all. To show (V2) it now suffices to show that the remaining contributions to $E^*[X]$ are $o(E[X])$.

There are $O(\ln^3 n)$ choices of a pair $i \leq 3w - 1$ and $a_i \in V$. Fix such a pair and consider the contribution to $E^*[X]$ with $a_i$ this fixed value. In
fix a cycle $C$ of length $2w$ that $i$ lies on. The expected number of cycles of length $2w$ in $G$ through $a_i$ bounded from above by $P_{2w}(a_i, a_i)$ which we’ve shown is at most $Mn^{2w-1}p^{2w}$. (The analysis done for $P_s$ for $1 \leq s \leq w$ extends with no change to $s = 2w$.) Let $Q, Q'$ be the points of $H^-$ of degree three. Fixing such a cycle the points $a_Q, a_{Q'}$ are now fixed and the expected number of paths of length $w$ between them is at most $P_w(a_Q; a_{Q'}) \cdot M^{n-w+1}p^w$. Together the expected number of extensions of $a_i$ to a copy of $H^-$ is less than $O(n^{3/2}w^{-1/2}p^{3/2}w)$, o® from the expected number of copies of $H^-$ by a factor of $n^{-1+o(1)}$. The $O(ln^2 n)$ factor of the choices of $i, a_i$ can be absorbed into to $o(1)$ so that the expected number of copies of $H^-$ overlaps $V$ is still only $n^{1+o(1)}M^{l}E[X]$.

Recall $l \sim \epsilon \log n$. Up to now all constants $k_1, L, M$ have depended only on $c$ and not $\epsilon$. Now (and formally this is at the very start of the proof, in the definition of $H$) we fix $K$ so large that $\epsilon$ is so small so that

$$\epsilon(\log M) < 1$$

This assures that $n^{-1+o(1)}M^{l}$ is $n$ to a negative power. Thus this contribution to $E^*[X]$ is only $o(E[X])$. Hence $E^*[X] \sim E[X]$ which concludes the argument.

5.4 Clean Topological $k$-Cliques

For the proof of Theorem 6 we require that for every $c > 1$ and every integer $k$ that $G(n, c/n)$ almost surely contains a $CTK_k$. Fix $c, k$. We fix a real $k_1$ with

$$k_1 \log c > 1$$

Set $w = [k_1 \ln n]$. We define $H = H(k, k_1, n)$ to consist of $k$ “special” points and between each pair of special points a path of length $w$. Set $v = k + \binom{k}{2}(w-1)$, the number of vertices and $t = \binom{k}{2} - k$, $e = v + t$ so that $e$ is the number of edges. Note $e, v = (k_1 \binom{k}{2} + o(1))\log n$. We show that almost surely $G(n, c/n)$ contains a copy of $H$. As the argument is very
simpler (and simpler) than that just given, we shall give the argument in outline form. Letting $X$ denote the number of copies of $H$ we have

$$E[X] = (n)_{c}p^e \sim n^v p^e = e^c n^{-t} = n^{k1(\frac{k}{2})\log e - t + o(1)}$$

which is a positive power of $n$. Now we need show $E^*[X] \sim E[X]$ where $E^*[X]$ is the expected number of copies of $H$ conditioning on a fixed copy of $H$. Let us specify the fixed copy to be on vertex set $V = \{1, \ldots, v\}$ with $1, \ldots, k$ being the special vertices and let $G^*(n, p)$ be $G(n, p)$ conditioned on this copy. As before we let $P_w(a, b)$ be the expected number of paths of length $w$ between $a, b$ in $G^*$. Then, as before, there is a constant $M$ so that

$$P_w(a, b) < Mn^{w-1}p^w$$

for all $a, b$ while

$$P_w(a, b) = n^{w-1}p^w(1 + o(1))$$

if either $a$ or $b$ is not in $V$. (We will not need the more precise error bound for this problem.)

We split the contribution to $E^*[X]$ into two groups. The $a_1, \ldots, a_k$ which do not overlap $V$ contribute

$$n^k \left[ n^{w-1}p^w(1 + o(1)) \right]^{(k)}$$

to $E^*[X]$. Since $k$ is fixed this is asymptotically $n^v p^e$ which is asymptotically $E[X]$. There are only $n^{k-1+o(1)}$ different $a_1, \ldots, a_k$ which do overlap $V$. For each the contribution to $E^*[X]$ is at most

$$\left[ Mn^{w-1}p^e \right]^{(k)}$$

Since $M$ and $k$ are constants this only a constant times the contribution to $E[X]$. Thus the total contribution from these intersecting $a_1, \ldots, a_k$ is $n^{-1+o(1)}E[X]$ and thus $E^*[X] \sim E[X]$ as required.

References.