COMPACT LOGICS IN ZFC: CONSTRUCTING COMPLETE EMBEDDINGS OF ATOMLESS BOOLEAN RINGS SH384

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Abstract. Suppose for simplicity that T is first order with Skolem functions in strong enough sense. We prove, in ZFC, that is has a model $\mathfrak{B}$ in which for any two atomless Boolean algebras definable in it, any isomorphism or even complete embedding from one to the other is definable in the model. This has consequences on the compactness of logics gotten by extending first order logic by quantifying over such isomorphisms and even embeddings, this is discussed in reasonable details in the introduction. The we use black boxes given countable models, omitting countable types. But we like to have enough approximations to the models which are quite saturated. We present the frameworks in general terms.

Was supposed to be Ch X of the book "Non-structure" and probably will be if it materialize.

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§ 0. Introduction

Why and how are we looking for logics?

One motive (and way) is in order to find examples in "soft" model theory; there we look for logics with prescribed properties, better with nice definitions, immaterial if they can express really interesting things. Another is looking for logics which have significant expressive power on the one hand while having good model theoretic properties which we can use on the other hand: this is interesting in its own right and is desirable if we hope to find applications. Among "good model theoretic properties" compactness is very natural a priori (and has been prominent so far in applications). Other motive is the naturality of the example and, of course, there are other reasons as well.

Concerning applications, we may have a specific application in mind, but we may first have discussed logics which look natural (or our ability to prove indicate them), they were investigated and later an application was found.

After Mostowski [Mos57] suggestion, cardinality quantifiers were a center of interest, in particular the quantifier $\exists^{\geq \aleph_1}$ received much attention (see on them Kaufman [Kau85] and Schmerl [Sch85]). In particular after works of Furhken, Vaught and Keisler we know that the logic $\mathbb{L}(\exists^{\geq \aleph_1})$ is $\aleph_0$-compact and has a completeness theorem for finitely many (very nice) axiom schemes. Among the extensions considered, many of them $\aleph_0$-compact, we mention $\mathbb{L}(aa)$, stationary logic — introduced in [Sh:43] from general considerations, and then throughly investigated (mainly proved to have the good properties of $\mathbb{L}(\exists^{\geq \aleph_1})$) by Barwise Kaufman Makkai [BKM78]. It was used in [Sh:232] to prove in ZFC the existence of uni-serial rings R which are domains with non-standard uni-serial R-modules (we use completeness theorem of $\mathbb{L}(aa)$, hence absoluteness of consistency). We shall from now on concentrate on fully compact logics.

The logic $\mathbb{L}(\mathfrak{Q}^{\rm cf}_{\leq \lambda})$ is good as an example; the first fully compact logic (stronger than first order, answering a question of Keisler; [Sh:18]) this quantifier says that a given linear order has cofinality $\leq \lambda$. Moreover the logic $[\mathbb{L}(\dot{\mathbf{Q}}^{\rm cf}_{\leq 2^{\aleph_0}})]^{\rm Beth}$ (the Beth closure) may be considered even better — has the Beth property, (see [Sh:199]). Both exemplify the first way but their expressive powers are not impressive. Compact logics stronger than first order even on countable models were found in [Sh:43] — but the proof uses weakly compact cardinal (or diamonds), and it has weak expressive power and not so nicely defined.

$\mathbb{L}(\dot{\mathbf{Q}}^{\rm br})$ (the quantifier: there is a branch in a leveled tree) was introduced in [Sh:73] and proved to be compact and complete (and stronger than first order even on countable models, all properties provable in ZFC, first in this sense), thus fully answering a question of H. Friedman. In Fuchs Shelah [FuSh:316] it was used to disprove a conjecture of Kaplansky (see more Eklof [Ekl92]). Later Osofsky [Oso92], [Oso91] worked quite hard to give "logic free" proofs.

Let us go back to a more direct predecessor. Rubin (see [**?**]) proved various results on the ability to reconstruct the Boolean algebra $\mathbf{B}$ in AUT($\mathbf{B}$) (the automorphism group of a Boolean algebra $\mathbf{B}$), by first order interpretations; he reconstruct not just $\mathbf{B}$ but even higher order logics on it (he has continued to develop this — see his exposition in [Rub89]). This lead to the question whether various restrictions on $\mathbf{B}$ were necessary. This question was largely answered in Rubin Shelah [RuSh:84]; proving mainly, consistency of, Downward Lowenheim-Skolem to $\aleph_1$ and $\aleph_0$-compactness results, i.e.

1) Assume $\diamondsuit_{\aleph_1}$, for any Boolean algebra $\mathbf{B}$ there is a Boolean algebra $\mathbf{B}'$ of cardinality $\aleph_1$ such that $(\mathbf{B}, \mathrm{AUT}(\mathbf{B})) \equiv (\mathbf{B}', \mathrm{AUT}(\mathbf{B}'))$ (i.e. elementarily equivalent in first order logic).
2) Assume $\diamondsuit_{\aleph_1}$ for any first order sentence ψ speaking on models of the form $(\mathbf{B}, \mathrm{AUT}(\mathbf{B}))$ let $\dot{\mathbf{Q}}_\psi$ be the quantifier: $(\mathscr{Q}_\psi x, y)\varphi(x, y)$ say that $\{(x, y) : \varphi(x, y)\}$ is the partial order of a Boolean algebra on its domain which satisfies ψ. The result is: adding to first order logic the countably many quantifiers $\mathscr{Q}_\psi$ gives an $\aleph_0$-compact logic. This gives $\aleph_0$-compact logic stronger than first order logic even on finite structures (partially answering a question of H. Friedman). Those quantifiers are characteristic Lindstrom quantifiers (see definition in 0.1(1) below). {0.1}
3) Assume $\diamondsuit_{\aleph_1}$; for any atomic, non-meagre Boolean algebra $\mathbf{B}$ and sentence $\psi \in \mathbb{L}_{\omega_1,\omega}$ there is an atomic non-meagre Boolean algebra $\mathbf{B}'$ of cardinality $\aleph_1$ with $(\mathbf{B}, \mathrm{AUT}(\mathbf{B})) \models \psi \Rightarrow (\mathbf{B}', \mathrm{AUT}(\mathbf{B}')) \models \psi$. If $\mathbf{B}$ has $\aleph_0$ atoms ($\mathbf{B}$ is possibly in another universe), $\diamondsuit_{\aleph_1}$ can be replaced by CH.

In [Sh:72] this was continued; but

- $(a)'$ we have changed the question to a stronger one: does any first order T has model M in which every, say, automorphism between Boolean Algebras which are definable in M, is definable (so we are not allowed to add Skolem functions!)
- $(b)'$ point the interest in compactness of logics where we extend the syntax by adding a quantifier on automorphism of Boolean algebras (such quantifiers look to me more natural ones). Note that now (i.e. in [Sh:72]) we have a fully compact logic stronger than first order even on finite structures rather than $\aleph_0$-compact; (for example we can say that an atomic Boolean algebra has an automorphism of order two mapping no atom to itself, on finite Boolean algebras this says that it has an even number of atoms). Still also here the proof was not in ZFC; there was a set theoretic assumption needed: λ strongly inaccessible and $\diamondsuit_{\{\delta<\lambda^+:\mathrm{cf}(\delta)=\lambda\}}$ holds, subsequently we have used less — see [Sh:107]
- $(c)'$ deal with higher cardinals
- $(d)'$ point out that for any unstable theory we have such constructions, hence suggest dealing also with other quantifiers and structures. Specifically we have dealt with the strong independence property (prototype is random enough bipartite graph) and (this really appears only in [Sh:107]) ordered fields.

If we look at the proofs, a major problem was how to build a λ-compact models of cardinality λ^+ by λ^+ successive approximations with omitting types (of size λ, which have no "support" of cardinality $< \lambda$), using, when necessary, $\diamondsuit_{\{\delta<\lambda^+:\mathrm{cf}(\delta)=\lambda\}}$ and, of course, $\lambda = \lambda^{<\lambda}$. Parallel difficulties had been encountered in Chang two cardinal theorem. The problem was in limits of cofinality $< \lambda$.

The solution in [Sh:72], [Sh:73] was dealing with special types (we shall return to this in Chapter XI).

In [Sh:82] we succeed with a very reasonable set theoretic assumption on λ to prove that such constructions generally work (the price was $\diamondsuit_\lambda$ which holds for λ successor $\neq \aleph_1$ if GCH holds by works of Gregory and the author. Lately, in [Sh:460], we proved that for $\lambda > \beth_\omega$, for λ successor $\diamondsuit_\lambda$ <u>iff</u> $\lambda^{<\lambda}$, and for λ regular $(D\ell)_\lambda$ <u>iff</u> $\lambda^{<\lambda}$, and really $(D\ell)_\lambda$ suffice — see [Sh:82], [HLSh:162]).

In [Sh:107] we have dealt with those constructions more generally, got the ordered fields case, and got the strong independence property case (which include the Boolean Algebra case). For the ordered field case we use $\diamondsuit_\lambda + \diamondsuit_{\{\delta<\lambda^+:\mathrm{cf}(\delta)=\lambda\}}$; for the strong independence cases (and Boolean algebras) we use just instances of GCH.

The next stage was carried out in Mekler and Shelah [MkSh:375]. The construction of a λ-saturated model in λ^+ using diamonds is replaced by using a black box (introduced in [Sh:172], [Sh:227], extracting general construction principles from [Sh:a, VIII 2.6]), which had been used mainly for constructing abelian groups and modules; see systematic treatment in Chapter IV which include the black boxes needed in [MkSh:375] and here. So the construction is in ZFC, the types omitted are of small cardinality, and diagonalization on λ is replaced by "being definable over a small set"; so non-splitting (see [Sh:a, §2,Ch.I]) or finite satisfiability ([Sh:a, §4,Ch.VII]) are natural to be used (in abelian groups this is automatic as their theory is stable). For inherent reasons we could not "kill" undesirable automorphisms of structures considered finite, and the results on Boolean Algebras were not satisfactory — we have to add a special monadic predicate. However, for automorphisms of ordered fields (or isomorphisms onto) the results were as fine as we can want: compactness in ZFC. We also improve the result on "trees with no undefinable branches" (to all uncountable cardinals rather then just for λ^+, λ regular).

Here we get by a similar construction compactness for $\mathscr{L}^{\mathrm{ceab}} = \mathbb{L}(\mathscr{Q}^{\mathrm{ceab}})$ — quantifying over complete embeddings of one atomless Boolean ring into another. Our motivation was the problem on "can the automorphism groups of a 1-homogeneous[1] Boolean algebra be non-simple[2]"? Much is known on this group and, in particular, that it is "almost" simple — see Rubin and Stepanek [Rv89]. It was known that there may exist such Boolean Algebras as by [Sh:b, Ch.IV] in some generic extension, all automorphisms of $\mathscr{P}(\omega)$/finite are trivial and van Dowen note that the
{3.6} group of trivial automorphisms of $\mathscr{P}(\omega)$/finite is not simple (see the proof of 3.9).

Alternatively, Koppelberg [Kop85] has directly constructed such Boolean Algebras of cardinality $\aleph_1$ assuming (the more natural assumption) CH. So by the completeness theorem here (which is absolute), as the relevant facts are expressible in $\mathbb{L}(\mathscr{Q}^{\mathrm{ceab}})$, the existence is proved in ZFC. Some may want to derive a direct proof. It almost certainly will give more specific desirable information.

* * *

We now explain what are the logics we use; here they are always extensions of first order logic by some generalized quantifiers in the sense explained below.

First recall the classical:

{0.1}

Definition 0.1. 1) A Lindstrom quantifier has the form $\mathscr{Q}_K$, where for some $n = n(K) < \omega$, K is a class of models of the form (A, R) with R an n-place relation, closed under isomorphism; for notational simplicity we identify (A, R) with R when $A = \cup\{a_1, \dots, a_n\} : \langle a_1, \dots, a_n\rangle \in R$ and we can without loss of generality restrict ourselves to clauses where this holds. So actually K is a class of n-place relations.

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[1]A Boolean algebra is 1-homogeneous if it is atomless for every $a,\, b \in B \setminus \{0_{\mathbf{B}}\}$ we have $\mathbf{B} \restriction a \cong B \restriction b$ (equivalently for $a, b \in \mathbf{B} \setminus \{0_B, 1_B\}$ for some automorphism f of B, $f(a) = b$)

[2]That is has no normal subgroup which is not trivial

2) The logic $\mathbb{L}_{\omega,\omega}(\ldots,\mathscr{Q}_{K_i},\ldots)_{i\in I}$ is defined as follows: $\mathbb{L}_{\omega,\omega}(\ldots,\mathscr{Q}_{K_i},\ldots)[\tau]$ for a vocabulary τ is the following set of formulas: it is the closure of the set of atomic formulas by the usual f.o. (= first order) operations and if φ is a formula, so is $\psi = (\mathscr{Q}_{K_i} x_1,\ldots,x_{n(K_i)})\varphi$ and $\mathrm{FVar}(\psi)$, the set of free variable of ψ, is $\mathrm{FVar}(\varphi) \setminus \{x_1,\ldots,x_{n(K)}\}$.

Defining satisfaction $M \models (\mathscr{Q}_{K_i} x_1,\ldots,x_{n(K_i)})\varphi(x_1,\ldots,x_{n_{K_i}},\bar{b})$ <u>iff</u> $\{\langle a_1,\ldots,a_{n(K_i)}\rangle : M \models \varphi(a_1,\ldots,a_{n(K_i)},\bar{b})\} \in K_i$.

This is sufficient to get all logics with finite occurence number, which is natural in our context as the full compactness implies it.

But we prefer to add quantifiers which are restricted second order ones, so the syntax is similar to the one of second order logic, but quantification is restricted. We can reinterpret this as adding many Lindstrom quantifiers but in a way hiding the point; in particular adding a quantifier below entails adding infinitely many Lindstrom quantifiers to the logic, moreover adding two such quantifiers is more than adding Lindstrom quantifier capturing each of them.

{0.2}

Definition 0.2. Consider a first order sentence ψ, $\psi = \psi(P,\bar{Q})$, which means $\bar{Q} = \langle Q_\ell : \ell < \lg(\bar{Q})\rangle$, where Q_ℓ is an $n(Q_\ell)$-place predicate, P is $n(P)$-place predicate (and no more non-logical symbols occurs in ψ), we write $P = P^\psi, \bar{Q} = \bar{Q}^\psi$. We treat function symbols similarly (preferably partial) and we use symbols not occurring in the usual vocabularies, for predicate symbols used as variables $\underset{\sim}{P}, \underset{\sim}{Q}$, writing $\psi(\underset{\sim}{P},\underset{\sim}{\bar{Q}})$. For a vocabulary τ, we define the language $\mathbb{L}_{\omega,\omega}[\ldots,\mathscr{Q}^*_{\psi_i},\ldots]_{i\in I}[\tau]$, as we define first order language but we have also second order variables: we have variables over[3] n-place relations <u>iff</u> $\bigvee\limits_i n(P^{\psi_i}) = n$. Defining the formulas of the language $\mathbb{L}_{\omega,\omega}[\ldots,\mathscr{Q}^*_{\psi_i},\ldots]_{i\in I}$ $[\tau]$, it is the closure of the set of atomic formulas (including $\underset{\sim}{P}(x_1,\ldots,x_n)$, $\underset{\sim}{P}$ a variable on n-place predicates, $x_1,...,x_n$ are variable over elements or are terms if we have function symbols e.g. individual constants), under the usual first order operations, and $(\mathscr{Q}^*_{\psi(\underset{\sim}{P},\underset{\sim}{\bar{Q}})}\underset{\sim}{P},\bar{x}^1,\ldots,\bar{x}^n)[\bar{\vartheta},\varphi]$, where $\bar{\vartheta} = \langle\vartheta_\ell : \ell < \ell g\underset{\sim}{\bar{Q}}\rangle, \vartheta_\ell = \vartheta_\ell(\bar{x}^\ell), \bar{x}^\ell$ is a sequence of individual variables with not repetitions not occurring in $\bar{x}^k$, (for $k \neq \ell$), and not occurring in φ and $\underset{\sim}{P}$ is a variable on $n(\underset{\sim}{P})$-place relations, $\underset{\sim}{P}$ does not occur freely in any ϑ_ℓ and φ, but φ may have other individual variables or predicate variables or members of τ.

Defining satisfaction, let $M \models (\mathscr{Q}^*_{\psi(\underset{\sim}{P},\underset{\sim}{\bar{Q}})}\underset{\sim}{P},\bar{x}^1,\ldots,\bar{x}^m)[\bar{\vartheta},\varphi]$ <u>iff</u> there is an $n(\underset{\sim}{P})$-place relation P^* on $|M|$, for which

- (a) φ is satisfied when we substitute P^* for $\underset{\sim}{P}$ and
- (b) letting $Q_\ell = \{\bar{a}^\ell : M \models \vartheta_\ell(\bar{a}^\ell)\}$ we have $(P^*,Q_0,\ldots) \models \psi$ (again we identify a model with a sequence of relations).

{0.3}

Remark 0.3. We can translate the problem on the logic $\mathbb{L}[\ldots,\mathscr{Q}^*_{\psi(\underset{\sim}{P},\underset{\sim}{\bar{Q}})},\ldots]$ to one on f.o. logic, see 3. More elaborately {3.5A}
1) We can replace in Definition 0.2 M by $M^{[*]}$ as in the proof of 3, so apply f.o. logic to this derived model (there — specific for our quantifier). {0.2A}

[3]We can use different variables for each $\mathscr{Q}^*_{\psi_i}$

2) We can repeat this closure operation ω times (each time every new sort comes from finitely many previous sorts) getting $\langle M^{[\alpha]} : \alpha \leq \omega\rangle$; we can allow our variables each on one of those sorts, so we get a corresponding stronger language $\mathbb{L}^+[\ldots, \mathscr{Q}^*_{\psi_i}, \ldots][\tau]$. In this case we look for $\underset{\sim}{P}$ not only among the $n(\underset{\sim}{P})$-place relations on the elements of the original models but also on the set P''s satisfying some "earlier" formula.

* * *

{0.4}

Concluding Remarks 0.4. 1) Why have we restricted in the quantifier embedings of Boolean algebras to complete embedding? Suppose h is an embedding of an atomic
{2.8A} Boolean ring $\mathbf{B}_1$ (see Definition 2.17) into a Boolean ring $\mathbf{B}_2$ and $a \in \mathbf{B}_2 \setminus \{0_{\mathbf{B}_2}\}$ satisfies $\bigwedge_{x \in \mathbf{B}_1} h(x) \cap a = 0_{\mathbf{B}_2}$; let $\mathscr{J}$ be any maximal ideal (= complement of as ultrafilter) on $\mathbf{B}_1$ and define the embedding the $g = g_{h,\mathscr{J}} : \mathbf{B}_1 \to \mathbf{B}_2$ by

$$g(x) = \{h(x) \text{ if } x \in \mathbf{B}_1, x \in \mathscr{J}, h(x) \cup a \text{ if } x \in \mathbf{B}_1, x \in \mathscr{J}\}.$$

(This is a very reasonable assumption and the use of Boolean ring and not Boolean algebras is just for notational convenience.)

Now if $\mathbf{B}_1$ has an independent subset of cardinality $\mu, 2^\mu > \|M\|$, then there are too many $\mathscr{J}'s$ hence too many g's (i.e. not all of them can be f.o. definable in M). So we can express many second order properties and so can prove the compactness theorem fails.
2) Note that we shall use freely

- $\otimes$ "the following property of $(\mathbf{B}_1, \mathbf{B}_2, f)$ is first order: $\mathbf{B}_1, \mathbf{B}_2$ are Boolean algebras (or just Boolean rings), f an embedding of $\mathbf{B}_1$ into $\mathbf{B}_2$ which is a complete embedding".

(Many people notice that $\otimes$ is expressed in a non-first order way: every maximal antichain is mapped to one hence think that it is a second order property; however it can also be expressed by: "for no $y \in \mathbf{B}_2 \setminus \{0_{\mathbf{B}_2}\}$ is $\{x \in \mathbf{B}_1 : x \neq 0_{\mathbf{B}_1}\}, \{f(x) \cap y = 0_{\mathbf{B}_2}\}$ dense in $\mathbf{B}_1$", this is first order).

{0.5new}

Definition 0.5. A Boolean ring $\mathbf{B}$ is defined like a Boolean algebra but it has no distinguish element 1_B (still a — b is well defined).

* * *

We thank J. Baldwin and A. Siton for helping to clarify this work. See more
{2.2} [Sh:F503]. On f.s. (finitely satisfiable) amalgamation (see 2.4(3)(a) and on non
{2.2} splitting (2.4(3)(b) see here [Sh:c, §4,Ch.VII],[MkSh:375]).

§ 1. The Construction

We first give a quite general model theoretic context (in the next section we make it more specific). Second, we give the set theoretic assumption (those are cases in which a suitable black box from Third, we state the theorem of the section: a construction of a model; in its proof we phrase the relevant black box. We finish discussing some variants.

Let me stress: this section deals with the abstract construction; readers who would like to have something more concrete can start with §2, taking on themselves to believe at least temporarily in 2.8 (which is based on §1). The reader can also read {2.5A} this section with the following interpretation (which is enough for §1) in mind: $\mathscr{K}$ is the class of models of a fixed first order countable theory T^* with Skolem functions, $M \le N$ (i.e. $M \le_{\mathscr{K}} N$) means $M \prec N, M_1 \bigcup\limits_{M_0} M_2^{M_3}$ means $\mathrm{rtp}(M_2, M_1, M_3)$ does not split over M_0 (see Definition 2.4(3)(b) and of course $M_0 \prec M_\ell \prec M_3$) and {2.2} $c\ell_M(A)$ is the Skolem hull of A in M. So it is natural in 2.8 to demand somewhat {2.5A} more.

{1.0}

Explanation 1.1. Concerning the set theory the case we are mainly thinking of (see Theorem 1.11 (and for its context 1.7)) is $\lambda = (2^\mu)^+, \mu$ large enough, $\Theta =$ {1.6} $\{\aleph_0\}, \alpha^* = 1, D_{\aleph_0}$ the filter of compounded subsets of $\omega, \Upsilon_0 = \omega$. E.g. the case $\lambda = (2^\mu)^+, \mu = \aleph_1, \kappa = \aleph_0$ is not really harder — we still have the right black box (with $\Upsilon_0 = \omega^2, \theta_0 = \aleph_0, \alpha^* = 1$; see below) but we have to be more careful in the bookkeeping, in some clauses. Also in the construction of the models "$|\mathscr{K}^{\rm sat}| \le \lambda$" suffice, $|\mathscr{K}^{\rm sat}| < \lambda$ being not necessary but in the application we have in mind the later case is better (if you prefer the first, change $<$ to $\le$ in 1.2(C)(3), 1.7(B)). {1.4} {1.1}

Definition 1.2. 1) We say that $\mathfrak{s}$ is a model theoretical context <u>if</u> $\mathfrak{s} = (\lambda, \mu, \kappa, \mathscr{K}, \le_{\mathscr{K}}, \mathscr{K}^{\rm sat}, c\ell, {\rm gen})$ satisfies:

- (A) $\kappa < \mu < \lambda$ are regular cardinals
- (B) $\mathscr{K}$ almost is an a.e.c. which means that Ķ is a class of models (of, fixed vocabulary $\tau = \tau(\mathscr{K})$) closed under isomorphism, $|\tau(\mathscr{K})| < \mu$.

2) $\le = \le_{\mathscr{K}}$ is a partial order on $\mathscr{K}, M \le N$ implies $M \subseteq N$ and depends just on the isomorphism type of (M, N), (so f is a $\mathscr{K}$-embedding of M into N if f is an isomorphism from M onto M' for some $M' \le_{\mathscr{K}} N$).
3) $M_1 \le N, M_2 \le N, M_1 \subseteq M_2$ implies $M_1 \le M_2$.
4) If $\langle M_i : i < \delta\rangle$ is $\le$-increasing (in $\mathscr{K}$) <u>then</u> $\bigcup\limits_{i<\delta} M_i \in \mathscr{K}$ and $j < \delta \Rightarrow M_j \le \bigcup\limits_{i<\delta} M_i$.
5) If $\langle M_i : i \le \delta\rangle$ is $\le$-increasing (in $\mathscr{K}$) <u>then</u> $\bigcup\limits_{i<\delta} M_i \le M_\delta$.
6) If $A \subseteq M \in \mathscr{K}$ and $|A| < \kappa$ <u>then</u> for some $N \in \mathscr{K}$ we have $A \subseteq N \le M$ and $\|N\| < \mu$.

- (C) $\mathscr{K}^{\rm sat} \subseteq \{(N, M) : N \le M\}$ is closed under isomorphism and $\mathscr{K}^{\rm at} = \{N : (N, M) \in K^{\rm sat}\}$.

7) $(N, M) \in \mathscr{K}^{\rm sat} \Rightarrow \|M\| < \mu$.
8) If $(N, M) \in \mathscr{K}^{\rm sat}$ and $M^* \in \mathscr{K}, \|M^*\| < \lambda$ <u>then</u> $|\{f : f$ is a $\mathscr{K}$-embedding of N into $M^*\}| < \lambda$; this follows from $(D)(7)$ when:

(i) $[\alpha < \lambda \Rightarrow |\alpha|^{<\kappa} < \lambda)]$, and

(ii) if $N = c\ell_N(A)$ (see below) and $|A| < \kappa$ then $f : N \to M^*$ is determined by $f \restriction A$, i.e. if f_ℓ is a $\le_{\mathscr{K}}$-embedding of N into M^* for $\ell = 1,2$ and $f_1 \restriction A = f_2 \cap A$ <u>then</u> $f_1 = f_2$; will be used for M^* of cardinality λ.

(D) For $A \subseteq M \in \mathscr{K}, c\ell_M(A)$ is a subset of M.

9) If $A \subseteq N \le M$ <u>then</u> $c\ell_M(A) = c\ell_N(A)$.
10) If $A \subseteq B \subseteq M \in \mathscr{K}$ <u>then</u> $A \subseteq c\ell_M(A) \subseteq c\ell_M(B)$.
11) If $A \subseteq M$ <u>then</u> $c\ell_M[c\ell_M(A)] = c\ell_M(A)$.
12) For $A \subseteq M, \text{gen}_M A =: \text{Min}\{|B| : B \subseteq A \subseteq c\ell_M(B)\}$.
13) For $A \subseteq M, \text{gen}'_M A =: \text{Min}\{|B| : A \subseteq c\ell_M(B)\}$.
14) If $(N, M) \in \mathscr{K}^{\text{sat}}$ <u>then</u> $\text{gen}_M(N) < \kappa$.
15) The operation $c\ell$ is preserved by isomorphism.

We may omit M in $c\ell_M(A)$ and $\text{gen}_M(A)$, $\text{gen}'_M(A)$ if clear from the context. We may also write $c\ell(A, M), \text{gen}(A, M), \text{gen}'(A, M)$ and let $\text{gen}(M) = \text{gen}(M, M)$.

(E) $\bigcup$ is a 4-place relation on the set of members of $\mathscr{K}$ preserved under isomorphism written $M_1 \underset{M_0}{\bigcup} M_2^{M_3}$ or $\bigcup(M_0, M_1, M_2, M_3)$.

16) $M_1 \underset{M_0}{\bigcup} M_2^{M_3}$ implies $M_0 \le M_\ell \le M_3$ for $\ell = 1, 2$.

17) [Monotonicity] If $M_1 \underset{M_0}{\bigcup} M_2^{M_3}, M_0 \le M'_\ell \le M_\ell, M_\ell \le M'_3 \le M''_3$ and $M_3 \le M''_3$ for $\ell = 1, 2$ <u>then</u> $M'_1 \overset{M'_3}{\underset{M_0}{\bigcup}} M'_2$.

18) [Base enlargement] If $M_1 \underset{M_0}{\bigcup} M_2^{M_3}, M_0 \le M'_0 \le M_1, M'_2 = c\ell(M_2 \cup M'_0)$ <u>then</u> $M'_2 \in \mathscr{K}$ and $M_1 \overset{M_3}{\underset{M'_0}{\bigcup}} M'_2$.

19) [Existence] If $M_0 \le M_1, M_2$ <u>then</u> for some M_3 and g we have $M_1 \le M_3, g$ is a $\le_{\mathscr{K}}$-embedding of M_2 into $M_3, M_1 \overset{M_3}{\underset{M_0}{\bigcup}} g(M_2)$ and $M_3 = c\ell(M_1 \cup g(M_2))$.

20) If $M_1 \overset{M_3}{\underset{M_0}{\bigcup}} M_2$ and $\|M_1\| + \|M_2\| < \mu$ <u>then</u> $\|c\ell_{M_3}(M_1 \cup M_2)\| < \mu$.

21) [Transitivity] If $\langle M_{\ell,\alpha} : \alpha \le \alpha(*)\rangle$ is increasing continuous for $\ell = 0, 1$ and $M_{1,\alpha} \underset{M_{0,\alpha}}{\bigcup} M_{0,\alpha+1}^{M_{1,\alpha+1}}$ for $\alpha < \alpha(*)$ <u>then</u> $M_{1,0} \underset{M_{0,0}}{\bigcup} M_{0,\alpha(*)}^{M_{1,\alpha(*)}}$.

22) [Continuity] If $M_\alpha \overset{N_\alpha}{\underset{M}{\bigcup}} N_0$ for $\alpha < \delta, \langle M_\alpha : \alpha \le \delta\rangle$ is increasing continuous $\langle N_\alpha : \alpha \le \delta\rangle$ is increasing continuous then $M_\delta \overset{N_\delta}{\underset{M}{\bigcup}} N$.

{1.1n}

Remark 1.3. Used only when $\mathrm{gen}_M(M_0)_{<\kappa}, \mathrm{gen}(M_2) < \mu$ so we can add parallel restriction in the other clauses, e.g. in (E)(7) we add $\mathrm{gen}(M_{0,\alpha(*)}) < \mu$ we write
{1.3} $(E)'$ for this version the rephrase ??. Then revise 1.6(2). {1.1A}

Convention 1.4. $\mathfrak{s}$ is a fixed model theoretic context. {1.2}

Definition 1.5. 1) A construction $\mathscr{A}$ (for the model theoretic context $\mathfrak{s}$) is a sequence $\mathscr{A} = \langle M_j, N_i^-, N_i, w_i : i < \alpha, j \leq \alpha \rangle$ such that:

- (a) $\langle M_i : i \leq \alpha \rangle$ is $\leq$-increasing continuous chain in $\mathscr{K}$,
- (b) $M_i \underset{N_i^-}{\overset{M_{i+1}}{\bigcup\!\!\!\!\bigcup}} N_i$ and $M_{i+1} = c\ell(M_i \cup N_i)$
- (c) $w_i \subseteq i$ is closed for $\mathscr{A}$ (see clause (2)(a) below) and $|w_i| < \mu$
- (d) $N_i^- \subseteq M_{w_i}$ (see below)
- (e) $||M_0|| < \mu$,
- (f) (N_i^-, N_i) belongs to $\mathscr{K}^{\mathrm{sat}}$.

2) For a construction $\mathscr{A} = \langle M_j, N_i^-, N_i, w_i : i < \alpha, j \leq \alpha \rangle$:

- (a) u is $\mathscr{A}$-closed (or just closed) <u>if</u>$u \subseteq \alpha$ and $[i \in u \Rightarrow w_i \subseteq u]$
- (b) $M_w = M_w[\mathscr{A}]$ is defined by induction on $\sup(w)$ for any $\mathscr{A}$-closed w as follows:
  - (i) if $w = \emptyset$ <u>then</u> $M_w = M_0$
  - (ii) if $w = u \cup \{j\}$ and $u \subseteq j$ (hence u is $\mathscr{A}$-closed) <u>then</u>

$$M_w = c\ell_{M_\alpha}(M_u \cup N_j)$$

  - (iii) if w has no last element <u>then</u> $M_w = \bigcup\limits_{i \in w} M_{w \cap i}$

(note that $w \cap i$ is $\mathscr{A}$-closed and $\sup(w \cap i) \leq i < \sup(w)$).
3) Let $\alpha = \ell g(\mathscr{A})$ and for $\beta \leq \alpha$ we let $\mathscr{A} \restriction \beta = \langle M_j, N_i^-, N_i, w_i : i < \beta, j \leq \beta \rangle$ and lastly let $M_{\mathscr{A}} = M_\alpha$.

Note: the construction has local character so, e.g. $M_\delta = \bigcup\limits_{\alpha < \delta} M_\alpha$ and not just $M_\delta = c\ell_{M_\delta}(\bigcup\limits_{\alpha<\delta} M_\alpha)$. {1.3}

Fact 1.6. Let $\mathscr{A}$ be a construction.
1) If $u \subseteq \ell g(\mathscr{A})$ is closed for the construction $\mathscr{A}$ <u>then</u> $M_u \leq M_{\mathscr{A}}$ and $M_u \in \mathscr{K}$ of course.
2) If $u_0 \subseteq u_1, u_0 \subseteq u_2$, and u_0, u_1, u_2 are $\mathscr{A}$-closed $\bigwedge\limits_{i \in u_1} \bigwedge\limits_{j \in u_2 \setminus u_0} i < j$ <u>then</u> $M_{u_1} \underset{u_2}{\bigcup\!\!\!\!\bigcup} M^{M_{\mathscr{A}}}_{M_{u_0}}$.
3) If I is a directed partial order and w is $\mathscr{A}$-closed and $u_t \subseteq w$ is $\mathscr{A}$-closed for $t \in I$ and $s <_I t \Rightarrow u_s \subseteq u_t$ <u>then</u> $u = \bigcup\limits_{t \in I} u_t$ is $\mathscr{A}$-closed and

$$M_u = \bigcup_{t \in I} M_{u_t} \leq M_w \leq M_{\alpha^*}$$

4) If u is $\mathscr{A}$-closed and $|u| < \mu$ <u>then</u> $||M_u|| < \mu$.

5) The ordinal $\ell g(\mathscr{A})$ is $\mathscr{A}$-closed, and for $\beta \le \ell g(\mathscr{A})$, also $\mathscr{A} \restriction \beta$ is a construction and β is $\mathscr{A}$-closed.

Proof. By induction on the length of the construction. $\square_{1.6}$

{1.4} **Definition 1.7.** sett is a set theoretic context <u>if</u> sett $= (\lambda, \mu, \kappa, \Theta, S, \bar{S}, \bar{\Upsilon}, \bar{D}^*, \bar{S}^*, S^-, \bar{S}^-, \bar{\tau}, \bar{\mathbf{N}}, \bar{\mathbf{\Upsilon}}, \bar{\rho})$ satisfies:

(A) $\lambda > \mu > \kappa$ are regular, $\Theta \subseteq [\kappa, \mu)$ is a non-empty set of regular cardinals such that

$$\bigwedge_{\theta \in \Theta} \bigwedge_{\alpha < \lambda} |\alpha|^\theta + |\alpha|^{<\kappa} < \lambda$$

(B) $\lambda > 2^\mu$ or at least $\lambda > |\mathscr{K}^{\rm sat}|$.

(C) $S \subseteq \{\delta < \lambda : {\rm cf}(\delta) \in \Theta\}$ is a stationary subset of λ, $\bar{S} = \langle S_\zeta : \zeta < \alpha^* \rangle$ is a sequence of pairwise disjoint stationary subsets of S, $\bar{\Upsilon} = \langle \Upsilon_\zeta : \zeta < \alpha^* \rangle, \Upsilon_\zeta < \mu$, $\bigwedge\limits_{\delta \in S_\zeta} {\rm cf}(\delta) = {\rm cf}(\Upsilon_\zeta), \bar{D}^* = \langle D^*_\zeta : \zeta < \alpha^* \rangle, D^*_\zeta$ a κ-complete filter on Υ_ζ containing all co-bounded subsets of Υ_ζ.

Note that necessarily ${\rm cf}(\Upsilon_\zeta) \in \Theta$, we call it θ_ζ.

(D) S^- a stationary subset of λ, $S \cap S^- = \emptyset, \bar{S}^- = \langle S^-_\zeta : \zeta < \lambda \rangle$ is a sequence of pairwise disjoint stationary subsets of S^- such that $[\delta \in S^- \Rightarrow {\rm cf}(\delta) \ge \kappa]$. Also $\bar{S}^* = \langle S^*_\zeta : \zeta < \alpha^* \rangle$ is a sequence of stationary subsets of λ such that $[\delta \in S^*_\zeta \Rightarrow {\rm cf}(\delta) = \kappa]; S^*_\zeta$ appear in clauses $(I)_{1,\zeta}(\delta), (I)_{2,\zeta}(\delta)$.

(E) For each $\zeta < \alpha^*$, $\bigcup\limits_{\epsilon \ne \zeta} S_\epsilon \cup S^-$ does not reflect in S_ζ.

(F) $\bar{\tau}$ is $\langle \tau_\zeta : \zeta < \alpha^* \rangle, \bar{\mathbf{N}} = \langle \bar{\mathbf{N}}_\zeta : \zeta < \alpha^* \rangle, \bar{\mathbf{\Upsilon}} = \langle \bar{\mathbf{\Upsilon}}_\zeta : \zeta < \alpha^* \rangle$ and for $\zeta < \alpha^*, \tau_\zeta$ is a vocabulary of cardinality $\le \kappa, \bar{\mathbf{N}}_\zeta = \langle \bar{N}^\zeta_\delta : \delta \in S_\zeta \rangle$ and $\bar{\mathbf{\Upsilon}}_\zeta = \langle \mathbf{\Upsilon}^\zeta_\delta : \delta \in S_\zeta \rangle$ for $\zeta < \alpha^*$ are such that $\mathbf{\Upsilon}^\zeta_\delta \subseteq {}^{(\theta_\zeta)}\delta$ and for every $\eta \in \mathbf{\Upsilon}^\zeta_\delta$ we have: η is strictly increasing with limit $\delta, \mathscr{T}^\zeta_\delta = \{\eta \restriction i : i \le \theta_\zeta, \eta \in \mathbf{\Upsilon}^\zeta_\delta\}, \bar{N}^\zeta_\delta = \langle N^\zeta_{\delta,\eta} : \eta \in \mathscr{T}^\zeta_\delta \rangle, N^\zeta_{\delta,\eta}$ is a τ_ζ-model with universe $\in \mathscr{H}_{<\mu}(\lambda), [\eta \triangleleft \nu \in \mathscr{T}^\zeta_\delta \Rightarrow N^\zeta_{\delta,\eta} \subseteq N^\zeta_{\delta,\nu}], i < \ell g(\eta) \Rightarrow \eta \restriction (i+1) \in N^\zeta_{\delta,\eta}$ and $N^\zeta_{\delta,\eta}$ is $\subseteq$-increasing continuous with $\eta, \langle\rangle \triangleleft \eta \in \mathscr{T}^\zeta_\delta \Rightarrow N^\zeta_{\delta,\eta} = \bigcup\{N^\zeta_{\delta,\eta \restriction (i+1)} : i < \ell g(\eta)\}$ and $\ell g(\eta) < \theta_\zeta$ and $\eta \in \mathscr{T}^\zeta_\delta \Rightarrow (\exists \alpha < \delta)(N^\zeta_{\delta,\eta} \in \mathscr{H}_{<\mu}(\alpha)))$.

(G) We have[4] $\bar{\rho} = \langle \rho_\eta : \eta \in \bigcup\{\mathbf{\Upsilon}^\zeta_\delta : delta \in S_\zeta, \zeta < \alpha^*\}\rangle$ such that for $\eta \in \mathbf{\Upsilon}^\zeta_\delta, rho_\eta$ is a strictly increasing sequence of ordinals $< \delta$ with limit $\delta, \ell g(\rho_\eta) = \Upsilon_\zeta$ and $i < \Upsilon_\zeta \Rightarrow \rho_\eta \restriction i \in N^\zeta_{\delta,\eta}$.

(H) For $\eta \ne \nu$ from $\mathbf{\Upsilon}^\zeta_\delta$ we have $\{i < \Upsilon_\zeta : [\rho_\eta(5i+1), \rho_\eta(5i+4)) \cap N_\nu = \emptyset\} \in D^*_\zeta$

(I) For $\zeta < \alpha^*$ clause (α) of $(\mathbf{I})_{1,\zeta}$ or clause (α) of $(I)_{2,\zeta}$ holds and we have $(\mathrm{I})_{1,\zeta} + (I)_{2,\zeta}$ where $I_{1,\zeta}]$ [the guess]

Assume:

- (α) $\zeta < \alpha^*, \theta = \theta_\zeta = \Upsilon_\zeta$
- (β) $\mathscr{T} \subseteq {}^{\theta >}\lambda, \langle\rangle \in \mathscr{T}, \mathscr{T}$ closed under initial segments
- (γ) $\mathscr{T}$ is $(< \theta)$-closed, which means: if $\eta \in {}^j\lambda, j < \theta$ is a limit ordinal, and $\bigwedge\limits_{i<j} \eta \restriction \in \mathscr{T}$ <u>then</u> $\eta \in \mathscr{T}$

[4]note that the $\mathbf{\Upsilon}^\zeta_\delta$-s are pairwise disjoint as from $\eta \in \mathbf{\Upsilon}^\zeta_\delta$ we can reconstruct δ as $\sup {\rm Rang}(\eta)$ and ζ can be reconstructed from δ as $\langle S_\zeta : \zeta < \alpha^* \rangle$ is a sequence of pairwise disjoint sets.

(δ) for every $\eta \in \mathscr{T}$, for some club E of $\lambda, [\alpha \in E \cap S^*_\zeta \Rightarrow \eta\hat{}\langle\alpha\rangle \in \mathscr{T}]$

(ϵ) for $\eta \in \mathscr{T}, N_\eta$ is a model of cardinality $< \mu$ with vocabulary $\tau_\zeta, |\tau_\zeta| \le \kappa$, universe; note:Recall that $\mathscr{H}_{<\mu}(\lambda)$ is the family of sets of hereditary cardinality $< \mu$ considering the ordinals $< \lambda$ as atoms, i.e. $x \in \mathscr{H}_{<\mu}(\lambda)\underline{iff}\,(\exists y)[x \subseteq y$ and $|y| < \mu$ and $(\forall t)[t \in y$ and t not an ordinal $\Rightarrow t \subseteq y] \in \mathscr{H}_{<\mu}(\lambda)$, $\bigwedge\limits_{i<\ell g(\eta)} \eta\restriction(i+1) \in N_\eta$ and $\nu \triangleleft \eta \Rightarrow N_\nu \subseteq N_\eta$ and: if η has limit length then $N_\eta = \bigcup\{N_{\eta\restriction i} : i < \ell g(\eta)\}$.

<u>Then</u> for stationarily many $\delta \in S_\zeta$ for some $\eta \in \mathbf{\Upsilon}^\zeta_\delta$ we have

$$\bigwedge_{i<\theta} \eta\restriction i \in \mathscr{T}$$

$$\bigwedge_{i<\theta} N_{\eta\restriction i} = N^\zeta_{\delta,\eta\restriction i}$$

$$\rho_\eta = \eta$$

moreover, for every club E of λ there are such ζ, δ with $\mathrm{rang}(\eta) \subseteq E \cap S^*_\zeta$.

$(I)_{2,\zeta}$

Assume:

(α) $\theta = \theta_\zeta, \theta^2$ divides $\Upsilon_\zeta, \langle\gamma^\epsilon_\zeta : \epsilon < \theta\rangle$ a strictly increasing sequence of ordinals with limit $\Upsilon_\zeta, a \subseteq \theta = \sup(a)$ such that $\{\delta \in S^*_\zeta : \mathrm{cf}(\delta) = \mathrm{cf}(\gamma^\epsilon_\zeta)\}$ is stationary (for each $\epsilon < \theta$) and $A \subseteq \Upsilon_\zeta$ and $(\forall^*\epsilon \in a)(\forall^*\xi < \gamma^\epsilon_\zeta)(\xi \in A) \Rightarrow A \in D_\zeta$. (Recall that $(\forall^*\epsilon \in a)$ means for every large enough $\epsilon \in a$.

(β) $\mathscr{T} \subseteq {}^{\theta>}\lambda, \langle\rangle \in \mathscr{T}, \mathscr{T}$ closed under initial segments

(γ) $\mathscr{T}$ is $(< \theta)$-closed i.e. for j limit ordinal $< \theta$, if $\eta \in {}^j\lambda$ and $\bigwedge\limits_{i<j} \eta\restriction i \in \mathscr{T}$ then $\eta \in \mathscr{T}$

(δ) for every $\eta \in \mathscr{T}$, for some club E of λ we have $[\alpha \in E \cap S^*_\zeta \Rightarrow \eta\hat{}\langle\alpha\rangle \in \mathscr{T}$

(ϵ) for $\eta \in \mathscr{T}, N_\eta$ is a model of cardinality $< \mu$ with vocabulary τ_ζ, universe $\in \mathscr{H}_{<\mu}(\lambda)$, $\bigwedge\limits_{i<\ell g(\eta)} \eta\restriction(i+1) \in N_\eta$, and $i < \ell g(\eta) \Rightarrow \rho_{\eta\restriction(i+1)} \in N_\eta, [\nu \triangleleft \eta \Rightarrow N_\nu \subseteq N_\eta]$ and if η has limit length then $N_\eta = \bigcup\{N_{\eta\restriction i} : i < \ell g(\eta)\}$

(ζ) for $\eta \in \mathscr{T}, \rho^\eta \in {}^{\Upsilon_\zeta>}\lambda$ is strictly increasing $\ell g(\rho^\eta) = \sup\{\gamma^\epsilon : \epsilon \in a \cap \ell g(\eta)\}$ and $[\epsilon \in a \cap \ell g(\eta) \Rightarrow \eta(\epsilon) = \bigcup\limits_{j<\ell g(\rho^\eta)} \rho^\eta(j)], \rho^\eta$ is $\triangleleft$-increasing and for some $\epsilon < \theta$ we have $\eta \triangleleft \nu \in \mathscr{T} \cap {}^\epsilon\lambda \Rightarrow \rho^\eta \in N_\nu$.

<u>Then</u> for stationarily many $\delta \in \mathscr{T}$ for some $\eta \in \mathbf{\Upsilon}^\zeta_\delta$ we have[5]

$$\bigwedge_{i<\theta} \eta\restriction i \in \mathscr{T}$$

$$\bigwedge_{i<\theta} N_{\eta\restriction i} = N^\zeta_{\delta,\eta\restriction i}$$

$$\rho_\eta = \bigcup_{j<\theta} \rho^{\eta\restriction j}.$$

[5]normally, we also get $\mathrm{rang}(\rho_\eta) \subseteq E$ we thus require choosing the "right" N_η, ρ^η.

Moreover, for a given club E of λ we can demand

$$\mathrm{rang}(\eta) \subseteq S^*_\zeta \cap E.$$

{1.4A} *Remark* 1.8. 1) A natural case is $\alpha^* = 1, \theta_0 = \theta = \Upsilon_0 = \aleph_0$.
2) We can replace τ_ζ by $\langle \tau_{\zeta,\epsilon} : \epsilon \leq \theta\rangle$, increasing continuous sequence of vocabularies and $N^\delta_{\zeta,\eta}$ is a $\tau_{\zeta,\ell g(\chi)}$-model in clause (F) and N_η is a $\tau_{\zeta,\ell g(\eta)}$-model in subclause (ϵ) of $(I)_{1,a_\zeta}, (I)_{2,\zeta}$.

{1.4B} **Definition 1.9.** We say that $(\mathfrak{s}, \mathrm{sett})$ is an $m+s$ context <u>if</u> $\mathfrak{s}$ is a model theoretic context, sett is a set theoretic context and $(\lambda^{\mathfrak{s}}, \mu^{\mathfrak{s}}, \kappa^{\mathfrak{s}}) = (\lambda^{\mathrm{sett}}, \mu^{\mathrm{sett}}, \kappa^{\mathrm{sett}})$.

{1.5} **Claim 1.10.** *1) If $\lambda, \mu, \kappa, \Theta, \langle \Upsilon_\zeta, D^*_\zeta : \zeta < \alpha^*\rangle$ satisfy $\otimes$ below <u>then</u> there is a set theoretic context* sett *with $\lambda^{\mathrm{sett}} = \lambda, \mu^{\mathrm{sett}} = \mu, \kappa^{\mathrm{sett}} = \kappa, \Theta^{\mathrm{sett}} = \Theta, \alpha^{*,\mathrm{sett}} = \alpha^*, (\Upsilon^{\mathrm{sett}}_\zeta, D^{\mathrm{sett}}_\zeta) = (\Upsilon_\zeta, D_\zeta)$ and $\tau^{\mathrm{sett}}_\zeta$, e.g. is $\tau_{\aleph_0,\aleph_0}$*

- $\otimes$ (a) *$\lambda > \mu > \kappa$ are regular cardinals*
  - (b) *$\Theta \subseteq [\kappa, \mu)$ a non-empty set of regular cardinals*
  - (c) *$(\lambda^-)^{<\mu} = \lambda^-$ where λ^- is the predecessor of λ that is $\lambda = (\lambda^-)^+$*
  - (d) *for each $\zeta < \alpha^*$ at least one of the following:*
    - (α) *$\theta_\zeta = \aleph_0, \Upsilon_\zeta < \mu$ is divisible by $\theta_\zeta \times \theta_\zeta$, and $\mathrm{cf}(\Upsilon_\zeta) = \theta_\zeta$,*
    - (β) *$\lambda^- = 2^\chi, \chi$ strong limit singular $> \mu$ and $\theta_\zeta = \Upsilon_\zeta = \mathrm{cf}(\chi)$*
  - (γ) *inducting from (β) (see*

2) We can add the following demand on sett*: if $\zeta < \alpha^*, \theta_\zeta = \aleph_0$ <u>then</u> for $\eta, \nu \in \boldsymbol{\Upsilon}^\zeta_\delta$, $N^\zeta_{\delta,\eta}$ is isomorphic to $N^\zeta_{\delta,\nu}$ by an isomorphism preserving $\in$, (so being an ordinal and their order) mapping $\rho^{\eta\restriction i}$ to $\rho^{\nu\restriction i}$ for $i < \Upsilon^\zeta_\delta$ and map $N^\zeta_{\delta,\eta\restriction i}$ onto $N^\zeta_{\delta,\nu\restriction i}$ for $i < \theta_\zeta$ and $N^\zeta_{\delta,\eta} \cap N^\zeta_{\delta,\nu} = N^\zeta_{\delta,\eta\cap\nu}$ for $\eta, \nu \in \Upsilon^\zeta_\delta$.*

Proof. See [Sh:309, 3.xx]. $\square$

{1.6} **Theorem 1.11.** *Assume that $(\mathfrak{s}, \mathrm{sett})$ is an $m+s$ context, that is $\mathscr{K}, \leq_{\mathscr{K}}, \mathscr{K}^{\mathrm{sat}}, \bigcup\!\!\!\!\bigcup$,*

{1.1} *λ, μ, κ are as in 1.2 and λ, μ, κ, Θ, S, $\langle \theta_\zeta, \Upsilon_\zeta, D^*_\zeta, \bar{\mathbf{N}}^\zeta : \zeta < \alpha^*\rangle$, S, $\langle S_\zeta : \zeta < \alpha^*\rangle$,*
{1.4} *$\langle S^-_\zeta : \zeta < \lambda\rangle \bar{S}^*, \langle \rho_\eta : \eta \in \cup\{\Upsilon^\zeta_\delta : \zeta < \alpha^*, \delta \in S_\zeta\}\rangle$ are as in 1.7.*
<u>Then</u> there are $I, \bar{\mathfrak{B}} = \langle \mathfrak{B}_\alpha : \alpha \leq \lambda\rangle$ (and we let $\mathfrak{B} = \mathfrak{B}_\lambda$) such that:

- (a) *$\bar{\mathfrak{B}}$ is a $\leq_{\mathscr{K}}$-increasing continuous sequence of members of $\mathscr{K}$*
- (b) *for $\alpha < \lambda$, the universe of $\mathfrak{B}_\alpha$ is an ordinal $\gamma_\alpha, \alpha \leq \gamma_\alpha < \lambda$*
- (c) *I is a directed subfamily of $\{N : N \leq_{\mathscr{K}} \mathfrak{B}$ and $\|N\| < \mu\}$ such that $(\forall \bar{a} \in {}^{\kappa>}\mathfrak{B})(\exists N \in I)[\bar{a} \in {}^{\kappa>}N]$ and I is closed under unions of increasing chains of length $< \kappa$, so I is $(<\kappa)$-directed*
- (d) *if $N \in I$, $(N, M) \in \mathscr{K}^{\mathrm{sat}}$ and $\zeta < \lambda$ <u>then</u> for stationarily many $\delta \in S^-_\zeta$, there is a $\leq_{\mathscr{K}}$-embedding g of M into $\mathfrak{B}_{\delta+1}$ satisfying $g\restriction N = \mathrm{id}_N$ and $\mathfrak{B}_\delta \underset{N}{\bigcup\!\!\!\!\bigcup} g(M)$ (you can add $c\ell_{\mathfrak{B}}(\mathfrak{B}_\delta \cup M) = \mathfrak{B}_{\delta+1}$)*
- (e) *if $\alpha \in S^-$ or α is a successor N ordinal and $\bar{a} \in {}^{\kappa>}\mathfrak{B}$ <u>then</u> for some N^-, $N \in I$ we have $N^- \leq_{\mathscr{K}} N \leq_{\mathscr{K}} \mathfrak{B}, N^- \leq_{\mathscr{K}} \mathfrak{B}_\alpha, \bar{a} \in {}^{\kappa>}N$ and $\mathfrak{B}_\alpha \underset{N^-}{\bigcup\!\!\!\!\bigcup} N$*
- (f) *Assume*

(α) $\zeta < \alpha^*, \theta = \theta_\zeta, D = D^*_\zeta, \mathrm{Dom}(D) = \Upsilon = \Upsilon_\zeta$ *so* $\mathrm{cf}(\Upsilon) = \theta$

(β) $\mathscr{T} \leq {}^{\gamma>}\lambda, \langle\rangle \in \mathscr{T}, \mathscr{T}$ *closed under initial segments and for every* $\eta \in \mathscr{T}$ *for some club* E *of* $\lambda, [\alpha \in E \cap S^*_\zeta] \Rightarrow [\eta\hat{\ }\langle\alpha\rangle \in \mathscr{T}]$ *and each* $\eta \in \mathscr{T}$ *strictly increasing*

(γ) $\mathscr{T}^{\lim} = \{\eta : \eta \in {}^\theta\lambda \setminus \mathscr{T}$ *and* $\bigwedge_{\alpha<\theta} \eta{\restriction}\alpha \in \mathscr{T}\}$

(δ) $\langle \bar{N}_\eta, \rho_\eta : \eta \in \mathscr{T} \cup \mathscr{T}^{\lim}\rangle$ *is such that:*

(*i*) $N_\eta \in I$,

(*ii*) *for* $\eta \in \mathscr{T}^{\lim}$ *we have* $\langle N_{\eta\restriction i} : i \leq \ell g(\eta)\rangle$ *is* $\leq_{\mathscr{K}}$*-increasing continuous and* $\langle \rho_{\eta\restriction i} : i \leq \ell g(\eta)\rangle$ *is* $\triangleleft$*-increasing continuous,*

(*iii*) *for* $\eta \in \mathscr{T}^{\lim} \cap {}^\Upsilon\lambda$ *and* $i < \ell g(\eta)$ *we have* $N_\eta \cap \mathfrak{B}_{\eta(5i+4)} \subseteq N_\eta \cap \mathfrak{B}_{\eta(5i+1)}$ *and*

$$\mathfrak{B}_{\eta(5i+4)} \underset{N_\eta \cap \mathfrak{B}_{\eta(5i+1)}}{\bigcup\!\!\!\!\!\!\bigcup} N_\eta$$

(ϵ) *we have* $\mathfrak{B}^+, \mathfrak{B} \leq_{\mathscr{K}} \mathfrak{B}^+$ *and* $\langle N^+_\eta : \eta \in \mathscr{T}^{\lim}\rangle$ *such that:* $N_\eta \leq_{\mathscr{K}} N^+_\eta \leq_{\mathscr{K}} \mathfrak{B}^+$, $\|N^+_\eta\| < \mu$ *and* N^+_η *computable continuously from* $N_{\eta\restriction i}, \mathfrak{B}_{\eta(i)}$ *(for* $i < \ell g(\eta)$*), that is:* $\langle N^+_{\eta,i} : i < \theta\rangle$ *is increasing with union* N^+_η *and for every* $i < \theta$ *for some* $j \in (i, \theta)$, *for every* $\nu \in \mathscr{T}^{\lim}$ *such that* $\eta \restriction j = \nu \restriction j$ *we have: there is an isomorphism* g *from* $c\ell(\mathfrak{B}_{\eta(i)} \cup N^+_{\eta,i})$ *onto* $c\ell(N^+_\eta \cup \mathfrak{B}_\delta) = \bigcup_{i<\theta} c\ell(N^+_{\eta,i} \cup \mathfrak{B}_{\eta(i)})$. *Then for every club* E *of* λ, *for some* $\delta \in S$, *there are* $\eta \in \mathscr{T}^{\lim}$, g *such that:*

(ζ) $i < \ell g(\eta) \Rightarrow \eta(i) \in E$

(η) g *is a* $\leq_{\mathscr{K}}$*-embedding of* N^+_η *into* $\mathfrak{B}_{\delta+1}$ *over* N_η

(θ) $\mathfrak{B}_\delta \underset{N_\eta}{\bigcup\!\!\!\!\!\!\bigcup} g(N^+_\eta)$

(*g*) *If* $\zeta < \alpha^*, \delta \in S_\zeta$ *and* $\eta \in \mathbf{\Upsilon}^\zeta_\delta$ *then for every* $M_1 \in I$ *there is* $M_2 \in I$ *such that* $M_1 \leq M_2$ *and* $g(N^+_\eta) \subseteq M_2$ *and the following set belongs to* D^*_ζ*:*

$$\{i < \theta : \mathfrak{B}_{\eta(5i+4)} \underset{c\ell(M_2 \cup \mathfrak{B}_{\eta(5i+1)}) \cup (N_{\eta\restriction(5i+1)})}{\bigcup\!\!\!\!\!\!\bigcup} M_2\}.$$

Proof. Straightforward. $\square_{1.11}$

{1.7}

Claim 1.12. *1) In 1.11 we can have instead* $\mathscr{K}^{\mathrm{sat}}$ *a family* $\mathscr{F} = \{\mathbf{f}_\alpha : \alpha < \alpha^*\}, \alpha^* < \lambda$, *the domain of each* $\mathbf{f} \in \mathscr{F}$ *is* $\subseteq \{(N, N^+) : N \in \mathscr{K}\}$, *and for every* $M \in \mathscr{K}$ *of cardinality* $< \lambda$ *the number of* $\leq_{\mathscr{K}}$*-embeddings of* N *into* M *is* $< \lambda$, $N \leq N^+, N^+$ *has universe an ordinal* $< \lambda$ *and* $N^+ \leq \mathbf{f}(N, N^+) \in \mathscr{K}_{<\lambda}$. *So the change is in clause (d) of 1.11.* {1.6}
2) We can add in 1.11 {1.6} {1.6}

(*h*) *there is a construction* $\mathscr{A} = \langle M_i, N_i, N^-_i, w_i : i < \lambda\rangle, M_{\mathscr{A}} = \mathfrak{B} = \mathfrak{B}_\lambda, \mathfrak{B}_\alpha = M_{i_\alpha}$, *if* $\delta \in S_\zeta, \zeta < \alpha^*$

(α) $\mathbf{\Upsilon}^\zeta_\delta, \mathscr{T}^\zeta_\delta, \bar{N}^\zeta_\delta, \bar{\rho} = \langle \rho_\eta : \eta \in \mathbf{\Upsilon}^\zeta_\delta\rangle$ *are as in clauses (F), (G) of 1.7,* $\mathscr{T}' = \{\eta \in \mathscr{T}^\zeta_\delta : (N^\zeta_{\delta,\eta}{\restriction}\tau){\restriction}(|N^\zeta_{\delta,\eta}| \cap \lambda)\}$ *has the form* $M_{w[\zeta,\delta,n]}$ *where* $w[\zeta, \eta, \delta]$ *is closed for the construction* $\mathscr{A}$. {1.4}
Either the conclusion of clause (e) holds or

(β) *for some list* $\langle \eta^\zeta_{\delta,i} : i < i^* \rangle$ *of some* $\Upsilon \subseteq (\lim(\mathscr{T})) \cap \mathscr{T}^\zeta_\delta$ *and* $\mathfrak{B}_{\delta,i}(i \leq i^*, i^*$ *zero or limit) we have:* $\mathfrak{B}_{\delta,0} = \mathfrak{B}_\delta, \mathfrak{B}_{\delta,i^*} = \mathfrak{B}_{\delta+1}, \mathfrak{B}_{\delta,i}$ *increasing continuous* $\mathfrak{B}_{\delta,j} = M_{\alpha_\delta+j}$,
$\mathfrak{B}_{\delta,i+1} = c\ell(\mathfrak{B}_{\delta,i} \cup N^+_{\eta^\zeta_{\delta,i}})$
$\mathfrak{B}_{\delta,i} \underset{N_{\eta^\zeta_{\delta,i}}}{\bigcup\!\!\!\!|} N^+_{\eta^\zeta_{\delta,i}}$ *(in fact* $N^-_{\alpha_\delta+j} = N_{\eta^\zeta_{\delta,i}}, N^+_{\alpha_\delta+j} = N_{\eta^\zeta_{\delta,i}}$*)*

(γ) *For every* $M \in I$*, for some* $w \subseteq \ell^*, |w| < \theta_\zeta$ *and* $M', M \leq M' \in I$ *we have:*

(i) $\mathfrak{B}_{\delta+1} \underset{M' \cap \mathfrak{B}_{\delta+1}}{\bigcup\!\!\!\!|} M'$

) (ii) $N^+_{\eta^\zeta_{\delta,i}} \subseteq M$ *for* $i \in w$

(iii) *for every large enough successor* $\alpha < \delta$

$$\mathfrak{B}_{\delta+1} \underset{c\ell(\mathfrak{B}_\alpha \cup \bigcup_{i \in w} N^+_{\eta^\zeta_{\delta,i}})}{\bigcup\!\!\!\!|} M'.$$

{1.7} *Remark* 1.13. 1) Note 1.14(1) will cover the existence of $2^{2^{(N+|T|)}}$-type definition over a model N.

{1.7} 2) In 1.14 we can even waive closure under isomorphism choosing $\mathscr{F}_\alpha$ together with $\mathfrak{B}_\alpha$ by induction on α.

{1.7}

{1.4} **Claim 1.14.** *1) We can weaken the black box in 1.7 by replacing clause* (H) *by:*

$(H)'$ *if* $\eta \neq \nu$ *are from* $\Upsilon^\zeta_\delta, \delta \in S_\zeta$ *<u>then</u>*

$$\{i < \theta : [\eta(5i+1), \eta(5i+4)) \cap \bigcup_{j<\theta} [\nu(5j+1), \nu(5j+4)) = \emptyset\} \in D_\zeta.$$

The results are:

{1.6} (A) *in clause* (f) *of 1.11 we have to strengthen in the assumption* (δ) *the statement* $(*)$ *to*

$(*)^+$ $N_{\eta \restriction i} \leq_{\mathscr{K}} \mathfrak{B}_{\eta(i+1)}$ *and* $\mathfrak{B}_{\eta(5i+1)} \underset{\bigcup_{j<i} N_{\eta \restriction (5j+3)}}{\bigcup\!\!\!\!|} N_\eta$

{1.5} (B) *We have a stronger existence theorem: in 1.10(d) we can add the cases: code by* $\langle h(\eta(i)) : i < \ell g(\eta) \rangle$*, where for* $x \in \mathscr{H}_{<\mu}(\lambda), \{\beta < \lambda : \mathrm{cf}(\beta) = \theta, \beta \in S^-, h(\beta) = x\}$ *is stationary.*

{1.5} *2) For 4) this weakened the version of the set theoretic context (called it 1.7*$^-$*) in 1.10(d)(*β*) the case* $\theta_\zeta = \aleph_0$ *can be omitted [use* [Sh:309, 3.17]*].*

{1.9}

{1.6} **Claim 1.15.** *Assume that we add to the assumptions in 1.11*

$(*)$ *if* $N^- \leq N \in \mathscr{K}_{<\lambda}, N^- \leq M \in \mathscr{K}_{<\lambda}$ *<u>then</u> there are* $< \lambda$ *pairs in* $\{(M^+, g) : M \leq M^+, g$ *a* $\leq_{\mathscr{K}}$*-embedding from* N *into* M^+ *over* N^- *such that* $M \overset{M^+}{\underset{N^-}{\bigcup\!\!\!\!|}} g(N)$ *up isomorphism over* M *(i.e.* $(M^+_1, g_1) \cong (M^+_2, g_2)$ *<u>iff</u>*

there are M'_1, M_2 such that $M^+_1 \leq_{\mathscr{K}} M'_1, M^+_2 \leq_{\mathscr{K}} M'_2, f_1$ an isomorphism from M'_1 onto M'_2 over M with $g_2 = f \circ g$).

Then in 1.11 we can add: {1.6}

(a) *if $N^- \in I, N^- \leq_{\mathscr{K}} N \in \mathscr{K}_{<\mu}$ then for some $\alpha < \lambda$ we have: if for $\ell = 1, 2, g_\ell$ is a $\leq_{\mathscr{K}}$-embedding of N into $\mathfrak{B}$ over $N^-, \mathfrak{B}_\alpha \bigcup\limits_{N^-} g_\ell(N)$ then for some $\beta < \lambda$ and M, f we have $\mathfrak{B}_\beta \leq M \in \mathscr{K}_{<\lambda}, f$ an automorphism of M over $\mathfrak{B}_\alpha$ such that $g_2 = f \circ g_1$.*

(b) *Suppose $\mathbf{g}$ is a function from the set of (N^-, N, α) as above, into λ, depending only on $(N, c)_{c \in N^-}/\cong$ and α then for some club E of $\lambda : \delta \in E \cap S^-$ we have:*
   - (∗) *if $N^- \in I, N^- \leq \mathfrak{B}_\delta$ and $N^- \leq N \in \mathscr{K}_{<\mu}$ then for some $\alpha < \delta$ we have $(N^-, N, \alpha) \in \mathrm{Dom}(\mathbf{g})$ and $\mathbf{g}(N^-, N, \alpha) < \delta$*
   - (∗∗) *for every $\bar{a} \in {}^{\kappa>}\mathfrak{B}$ for some $(N^-, N, \alpha) \in \mathrm{Dom}(\mathbf{g}), \alpha < \delta, \bar{a} \in {}^{\kappa>}N$, so $\mathbf{g}(N^-, N, \alpha) < \delta$.*

Proof: See Stage C of the proof of 2.16. {2.8}

Remark: This is O.K. when $\lambda^- = (\lambda^-)^{<\mu}$; $|\tau_{\mathscr{K}}| < \mu$; when we want otherwise our bookkeeping should be more careful.

{1.10}

Claim 1.16. *We can add in 1.11 the condition (∗) provided we make the other changes listed below:* {1.6}

(∗) *if $\delta \in S^-, \bar{a} \in {}^{\kappa>}\mathfrak{B}$ then for some $N^- \leq N \leq \mathfrak{B}, \mathrm{gen}(N) < \kappa, \mathrm{gen}(N^-) < \kappa, N \leq \mathfrak{B}_\delta$ and $\mathfrak{B}_\delta \bigcup\limits_{N^-} N$ but*
   - (a) *$\mu = \kappa^+, \Theta = \{\kappa\}$*
   - (b) *in clause (f) we add: for if $\eta \in \mathscr{T}^{\lim}, \mathrm{gen}(N^-_\eta) < \kappa$ and $\delta = \cup\{\eta(i)+1 : i < \ell g(\eta)\}$ is a limit ordinal, then for arbitrarily large successor $\alpha < \delta$ we have $\mathrm{gen}(N_\eta \cap \mathfrak{B}) < \kappa$.*

{1.13new}

Remark 1.17. 1) We have sometimes to consider not just $N \leq N'$; say that a free extension $\frac{N'}{N}$ is realized stationarily often but also in the cases that there are quite a few but boundedly many such extensions; we may, for example, consider all triples $N \leq N^1 \leq N^2, N^1 \bigcup\limits_{N} N^2$ and extend $\frac{N^2}{N^1}$.

2) The construction should be such that $(c)(**)$ of 2.7 can be deduced — better, but not necessary here. {2.5}

3) We may reconsider [§4,Ch.VII]Sh:c, we use there almost symmetric cases of $\bigcup\limits_{\mathrm{nsp}}$ (see Definition 2.4). {2.2}

4) We may consider constructions $\mathscr{A}$ where the index set is not in an ordinal and is not well ordered. In particular for the $\mathfrak{B}$ we construct we may consider, for $\delta \in S$ limit, adding $\langle N^+_\eta : \eta \in \mathscr{T}^\zeta_\delta \rangle$ in a natural lexicographic order. It may be useful.

§ 2. Analyzing the Complete Embeddings of Boolean Rings

{25B} In this section we specify our construction (in 2.1–2.9). Then we investigate the properties of the model $\mathfrak{B}$, till we conclude our main result: in $\mathfrak{B}$ all complete embeddings of one atomless Boolean ring to another (both are considered as "sets" by $\mathfrak{B}$) are definable.

{2.1} *Context* 2.1. 1) Let χ^* be a strong limit cardinal, $\mathscr{D}^*$ an ultrafilter on $\mathscr{H}(\chi^*)$ such that for every $x \in \mathscr{H}(\chi^*)$, the set $\{w : w$ a finite subset of $\mathscr{H}(\chi^*)$, (so $w \in \mathscr{H}(\chi^*)$) and $x \in w\}$ belongs to $\mathscr{D}^*$.
2) Let $\mathfrak{C}^*$ be a model with countable vocabulary expanding $(\mathscr{H}(\chi^*), \in, <^*_{\chi^*})$, $<^*_{\chi^*}$ a well ordering of $\mathscr{H}(\chi^*)$, (notice that it follows that $\mathfrak{C}^*$ has definable Skolem functions) such that every definable relation/function is equal to a relation/function of the model; even allowing the quantifier "for the $\mathscr{D}^*$-majority of $x's$"; i.e.

- $(*)_0$ for every formula $\varphi(x, \bar{y})$ for some predicate $R_\varphi(\bar{y})$ for any $\bar{a} \in {}^{\ell g(\bar{y})}(\mathfrak{C}^*)$ we have: $\mathfrak{C}^* \models R_\varphi[\bar{a}]$ <u>iff</u> $\{x \in \mathscr{H}(\chi^*) : \mathfrak{C}^* \models \varphi[x, \bar{a}]\} \in \mathscr{D}^*$.

Let $T^*_0 = \mathrm{Th}(\mathfrak{C}^*)$.

Let T^*_1 denote an expansion of T^*_0 such that $\tau T_1) \setminus \tau(T)$ consist of individual constants only. Let T^* be our fixed T^*_1. For $M \models T^*, A \subseteq M$, let $\mathrm{Sk}_M(A)$ be the

{2.1A} Skolem Hull of A in M. We assume for simplicity (on T_1 see 2.2(2)):

- $(*)_1$ $|\mathrm{Sk}(\emptyset)| = |T^*| \geq \aleph_0$.

3) To avoid confusion, the predicate of T^* corresponding to $\in$ will be $\mathscr{E}$.
4) Let $\mathrm{gen}_M A = \mathrm{Min}\{|B| : B \subseteq A \subseteq \mathrm{Sk}_M(B)\}, \mathrm{gen}'_M A = \min\{|B| : B \subseteq M, A \subseteq \mathrm{Sk}_M(B)\}$. (If M is clear or its choice among the reasonable candidates immaterial, we may omit it).
5) In a model M of $T^*, w \in M$ is pseudo finite if M "thinks" it is finite (so we may say w and mean $\{a \in M : M \models a \mathscr{E} w\}$). We may forget to say "in the sense of M" when clear from the context.

{2.1A}

Fact 2.2. There are such $\mathfrak{C}^*, T^*$ with countable vocabularies.

{2.1B}

Remark 2.3. 1) Of course, we shall use the syntactical properties of T^* only, but thinking on $\mathscr{H}(\chi^*)$ is supposed to clarify.
2) In particular $\mathfrak{C}^*$ (and $\mathscr{H}(\chi^*)$) may be chosen in another universe of set theory (e.g. one obtained by forcing) with the same sets of natural numbers, hence $T^* \in \mathbf{V}$. We can even just use " T^* is any completion of a theory T' such that:

- (a) every finite subset of T, has in some generic extension of $\mathbf{V}$ such a model (but $\varphi(x, \bar{y}) \mapsto R_\varphi(\bar{y})$ is constant)
- (b) T' is "rich" enough i.e. satisfies all what we shall use.

{2.1} 3) Note that because of $(*)_1$ from 2.1(2): if M is a θ-saturated model of $T^*, A \subseteq M, |A| < \theta, |A| \leq |T^*|$ <u>then</u> $\mathrm{gen}'_M(A) < \aleph_0$.

Note: if M is a θ-saturated model of $T^*_0, A \subseteq M, |A| < \theta, |A| \leq |T_1|$ and $|\tau(T^*)| \leq \theta$ <u>then</u> M can be expanded to a model of T^*.

Again for this section

{2.2} **Definition 2.4.** 1) For $x \in \{\text{fs}, \text{nsp}\}$ and regular cardinals $\lambda > \mu > \kappa$ (and T^* as above) we define $\mathfrak{s} = \mathfrak{s}^x_{T^*}(\lambda, \mu, \kappa)$ with the intension that it is a model theoretic context:

(a) $(\lambda^{\mathfrak{s}}, \mu^{\mathfrak{s}}, \kappa^{\mathfrak{s}})$, is (λ, μ, κ)

(b) Let $\mathscr{K} = \mathscr{K}_{T^*}$ be the class of models of T^* and $M \leq_{\mathscr{K}} N$ iff $M \prec N$ (and are in $\mathscr{K}$)

(c) $c\ell^{\mathfrak{s}}(A, M)$ is $\text{Sk}_M(A)$, the Skolem hull of A inside M

(d) $\overset{\mathfrak{s}}{\bigcup\!\!\!\!\bigcup}$ is $\overset{x}{\bigcup\!\!\!\!\bigcup}$ (among model of T^*), as defined in part (3) below

(e) $\mathscr{K}^{\text{at}} = \mathscr{K}^{\text{at}}_\kappa = \mathscr{K}^{\text{at}}_{T^*,\kappa}$ be the class of models of T^* generated by $< \kappa$ elements. Let $\mathscr{K}^{\text{sat}} = \{(N, M) : N \prec M \text{ are both in } \mathscr{K}^{\text{at}}_{T^*,\kappa}\}$.

2) We use only one of the following choices of $\bigcup\!\!\!\!\bigcup$:

[recall that $\text{tp}(A, B, M) = \{\varphi(x_{a_1}, ..., x_{a_n}, \bar{b}) : a_1, ..., a_n \in A, \bar{b} \in B\}$ and $M \models \varphi[a_1, ..., a_n, \bar{b}]$ (so $A, B \subseteq M$)

(a) $\bigcup\!\!\!\!\bigcup = \underset{\text{fs}}{\bigcup\!\!\!\!\bigcup}$ means: $M_1 \underset{M_0}{\bigcup\!\!\!\!\bigcup} M_2^{M_3}$ <u>if and only if</u> $M_0 \prec M_1 \prec M_3$, $M_0 \prec M_2 \prec M_3$, all of them models of T^* and $\text{tp}(M_2, M_1, M_3)$ is finitely satisfiable in M_0

(b) $\bigcup\!\!\!\!\bigcup = \underset{\text{nsp}}{\bigcup\!\!\!\!\bigcup}$ means $M_1 \underset{M_0}{\bigcup\!\!\!\!\bigcup} M_2^{M_3}$ <u>if and only if</u> $M_0 \prec M_1 \prec M_3, M_0 \prec M_2 \prec M_3$, all of them models of T^* and $\text{tp}(M_2, M_1, M_3)$ does not split over M_0 which means: if $\bar{a} \in {}^n(M_2)$ and $\bar{b}_0$, $\bar{b}_1 \in {}^m(M_1)$ and $\text{tp}(\bar{b}_0, M_0, M_1) = \text{tp}(\bar{b}_1, M_0, M_1)$ <u>then</u> for any formula $\varphi = \varphi(x_0, \ldots, x_{n-1}; y_0, \ldots, y_{m_1})$ we have $M_3 \models$ "$\varphi(\bar{a}, \bar{b}_0) \equiv \varphi(\bar{a}, \bar{b}_1)$".

We use $\bigcup\!\!\!\!\bigcup = \underset{\text{nsp}}{\bigcup\!\!\!\!\bigcup}$ except in 2.5. For $x \in \{\text{fs}, \text{nsp}\}$ the extension to $A_1 \overset{M_3}{\underset{M_0}{\bigcup\!\!\!\!\bigcup}} A_2$ {2.3}
$(M_0 \prec M_3, A_1 \subseteq M_3, A_2 \subseteq M_3)$ is natural.

We could have extend those definitions to the case $B \overset{M_4}{\underset{A}{\bigcup\!\!\!\!\bigcup}} C$ but if M_4 has Skolem functions, $M_0 = \text{Sk}(A)$ we abuse our notation by letting $B \overset{M_4}{\underset{A}{\bigcup\!\!\!\!\bigcup}} C$ mean $B \overset{M_4}{\underset{M_0}{\bigcup\!\!\!\!\bigcup}} C$ (note $B \overset{M_4}{\underset{M_0}{\bigcup\!\!\!\!\bigcup}} C$ <u>iff</u> $M_0 \cup B \overset{M_4}{\underset{M_0}{\bigcup\!\!\!\!\bigcup}} M_0 \cup C$).

4) We say $\mathfrak{p}$ is a type-definition over N (speaking on types with the α variables, $\langle x_i : i < \alpha \rangle$), <u>if</u>:

(a) $\bigcup\!\!\!\!\bigcup = \underset{\text{fs}}{\bigcup\!\!\!\!\bigcup}$ and $\mathfrak{p}$ is an ultrafilter on ${}^\alpha N$, and if $N \subseteq A \subseteq M$, or just $N \prec M$ and $A \subseteq M$ <u>then</u> $\mathfrak{p}^A = \{\varphi(\bar{x}, \bar{a}) : \bar{a} \in {}^{\omega >} A\}$ and $\{\bar{b} \in {}^\alpha N : M \models \varphi[\bar{b}, \bar{a}]\} \in \mathfrak{p}$ or (*b*) $\bigcup\!\!\!\!\bigcup = \underset{\rm nsp}{\bigcup\!\!\!\!\bigcup}$ and $\mathfrak{p}$ is a function from $\{\langle \varphi(\bar x,\bar y), q(\bar y)\rangle : \varphi\}$ a formula, $q(\bar y)$ a complete type over N to $\{\text{truth}, \text{false}\}$ and if $A \subseteq M, N \prec M$ <u>then</u> $\mathfrak{p}^A = \{\varphi(\bar x,\bar a) : \bar a \in {}^{\omega>}A\}$ and $\mathfrak{p}(\langle \varphi(\bar x,\bar y), \text{tp}(\bar a, N, M)\rangle) = \text{truth}\}$.

We may write $\mathfrak{p} = \mathscr{P}(\bar x,\bar c)$ if $\bar x = \langle x_i : i < \alpha\rangle$ and $\bar a \subseteq N$ and: in Case (a), for every $u \le \alpha$ the set $\{\bar b \in {}^\alpha N : \bar b{\restriction}u \in {}^u(\text{Rang}(\bar c)\}$ belongs to $\mathfrak{p}$ and in case (b), $\bar a$ list N (or we demand $\langle \varphi(\bar x\bar y), q(\bar y)\rangle \in \text{Dom}(\mathfrak{p}) \Rightarrow q(\bar y) \in \mathscr{S}^{\ell g \bar y}(\text{Rang}(\bar c), N)$.

We say $\mathfrak{p}$ is of kind $\underset{\rm fs}{\bigcup\!\!\!\!\bigcup}$ or of kind $\underset{\rm nsp}{\bigcup\!\!\!\!\bigcup}$ respectively. So we should have written $\mathfrak{p}^{A,M}$ in both cases.

{2.3} **Claim 2.5.** *1)* $\underset{\rm fs}{\bigcup\!\!\!\!\bigcup}, (M_0, M_1, M_2, M_3)$ *implies* $\underset{\rm nsp}{\bigcup\!\!\!\!\bigcup}, (M_0, M_1, M_2, M_3)$.

2) $B \underset{A}{\overset{M_4}{\bigcup\!\!\!\!\bigcup}} C$ *<u>iff</u>* $\text{Sk}(B) \underset{\text{Sk}(A)}{\overset{M_4}{\bigcup\!\!\!\!\bigcup}} \text{Sk}(C)$ *(recalling that we are assuming the existence of Skolem functions).*

3) [finite character] $B \underset{A}{\overset{M_4}{\bigcup\!\!\!\!\bigcup}} C$ *<u>if and only if</u>for every* $\bar b \in {}^{\omega>}B, \bar c \in {}^{\omega>}C$ *we have* $\bar b \underset{A}{\overset{M_4}{\bigcup\!\!\!\!\bigcup}} \bar c$*; this implies continuity, see 1.2(8).*

{1.1} *4) [monotonicity] if* $B \underset{A}{\overset{M_4}{\bigcup\!\!\!\!\bigcup}} C, B' \subseteq B, C' \subseteq C, A \cup B \cup C \subseteq M_4 \prec M_4'$ *<u>then</u>* $B' \underset{A}{\overset{M_4}{\bigcup\!\!\!\!\bigcup}} C'$.

5) [monotonicity] if $B \underset{A}{\overset{M_4}{\bigcup\!\!\!\!\bigcup}} C, A \cup B \cup C \subseteq M_4' \prec M_4$ *<u>then</u>* $B \underset{A}{\overset{M_4'}{\bigcup\!\!\!\!\bigcup}} C$.

6) [transitivity] if $B_0 \underset{A_0}{\overset{M}{\bigcup\!\!\!\!\bigcup}} A_1, B_1 \underset{A_1}{\overset{M}{\bigcup\!\!\!\!\bigcup}} A_2, A_0 \subseteq A_1 \subseteq A_2,\ B_0 \subseteq B_1$ *<u>then</u>* $B_0 \underset{A_0}{\overset{M}{\bigcup\!\!\!\!\bigcup}} A_2$.

7) If $A_\alpha \subseteq M(\alpha < \delta)$ *increases with* $\alpha, \bar a_\alpha \in A_{\alpha+1}, \text{tp}(\bar a_\alpha, A_\alpha, M)$ *does not split over* A_0 *and is increasing with* α *<u>then</u>* $\langle \bar a_\alpha : \alpha < \delta\rangle$ *is an indiscernible sequence over* A_0.

8) [base enlargement] If $M_1 \underset{M_0}{\overset{M_3}{\bigcup\!\!\!\!\bigcup}} M_2$ *and* $M_0 \prec M_0' \prec M_1$ *and* $M_2' = \text{Sk}_{M_3}(M_0' \cup M_2)$ *<u>then</u>* $M_1 \underset{M_0'}{\overset{M_3}{\bigcup\!\!\!\!\bigcup}} M_2'$.

9) [existence] if $M_0 \prec M_\ell$ *for* $\ell = 1, 2$ *<u>then</u> we can find* M_3, f *such that* $M_1 \prec M_3, f$ *is an elementary embedding of* M_2 *into* M_3 *and* $\underset{\rm fs}{\bigcup\!\!\!\!\bigcup}(M_0, M_1, f(M_2), M_3)$.

Proof. Straightforward. $\square_{2.4}$

We now become more specific.

{2.4} **Claim 2.6.** *If* $\lambda > \mu > \kappa$ *are regular and* $(\forall \alpha < \mu)(|\alpha|^{<\kappa} < \lambda)$ *and* $x \in \{\text{fs}, \text{nsp}\}$*,*
{1.1} *<u>then</u>* $\mathfrak{s} = \mathfrak{s}^x_{T^*}(\lambda, \mu, \kappa)$ *is a model theoretic context, see Definition 1.2.*

{2.3} *Proof.* The non-trivial part is clause (E) of 1.2 which follows from 2.5. $\square_{2.6}$

{1.6} So by 1.11

{2.5}

Theorem 2.7. *Assume* $(\mathfrak{s}, \mathrm{sett})$ *is an* $m{+}s$ *context,* $\mathfrak{s} = \mathfrak{s}^{\mathrm{nsp}}_{T^*}(\lambda, \mu, \kappa)$ *and* $\alpha^{*,\mathrm{sett}} = 1$

{1.4} *(and*[6] $\partial < \kappa \Rightarrow 2^{2^{\partial + |T^*|}} \leq \lambda, \lambda, \theta, \kappa, \Upsilon, D^*_\theta, S, S^-$ *are as in 1.7 (and* $\mathrm{cf}(\Upsilon) = \theta$*) and*

{2.1} T^* *as in 2.1.*

{1.6} *Then we can find* $\langle \mathfrak{B}_\alpha : \alpha \leq \lambda \rangle, \mathfrak{B} = \mathfrak{B}_\lambda$ *as in 1.11 for* $(\mathfrak{s}, \mathrm{sett})$.

Actually the properties we shall use are:

- (I) *the set theoretic properties:*
  - (A) $\lambda = \mathrm{cf}(\lambda), \Upsilon$ *an ordinal,* $\mathrm{cf}(\Upsilon) = |\Upsilon| = \kappa = \theta, \lambda > 2^{2^{|T^*|}} + \sum_{\beta < \kappa} 2^{2^{|\beta|}}$ *and* $[\alpha < \lambda \Rightarrow |\alpha|^{<\kappa} < \lambda]$
  - (B) $S \subseteq \{\delta : \delta < \lambda$ *and* $\mathrm{cf}(\delta) = \theta\}$ *is stationary,* $S^- \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) \geq \theta\}$ *is stationary and disjoint to* S
  - (C) T^* *is as in 2.1* {2.1}
  - (D) D^*_θ *is a filter on* Υ *containing the co-bounded subsets, there are* ≥ 2 *disjoint sets of successors* $\neq \emptyset \mod \mathscr{D}^*_\theta$
- (II) *the model theoretic properties:*
  - (a) $\langle \mathfrak{B}_\alpha : \alpha \leq \lambda \rangle$ *is an increasing continuous elementary chain of models of* $T^*, [\alpha < \lambda \Rightarrow \|\mathfrak{B}_\alpha\| < \lambda], \mathfrak{B} = \mathfrak{B}_\lambda$
  - (b) $\mathfrak{B}$ *is* κ*-saturated; moreover if* $\mathfrak{p}$ *is a type-definition over* $N \prec \mathfrak{B}$, $\mathrm{gen}(N) < \kappa$ *then for stationarily many* $\delta \in S^-$*, some* $\bar{a} \in \mathfrak{B}_{\delta+1}$ *realizes* $\mathfrak{p}^{\mathfrak{B}_\delta}$ *and* $\mathfrak{B}_{\delta+1} = \mathrm{Sk}(\mathfrak{B}_\delta \cup \bar{a})$ *and for some* $\bar{b} \in {}^{\kappa>}\mathfrak{B}$ *we have* $\mathfrak{B}_\alpha \underset{\bar{b}}{\overset{\mathfrak{B}}{\bigcup}} \bar{a}$
  - (c) *if* $\bar{a} \in {}^{\kappa>}\mathfrak{B}, \alpha \in S^-$ *then some* $\bar{a}, \bar{b}$ *are as in clause (b)*
  - (d) *This is for the case* $\Upsilon = \theta = \kappa$.

Assume that we have $M, \langle N_\eta, \alpha_\eta, \beta_\eta : \eta \in {}^{\theta \geq}\lambda \rangle, \langle N^+_\eta : \eta \in {}^\theta\lambda \rangle$ *such that:*

- (i) $\eta \in {}^{\theta>}\lambda \Rightarrow N_\eta \prec \mathfrak{B}_{\beta_\eta}$ *and* $\mathrm{gen}(N_\eta) < \theta$ *and* $\eta \in {}^\theta\lambda \Rightarrow \langle N_{\eta \restriction i} : i \leq \theta \rangle$ *is increasing continuous*
- (ii) $\eta \triangleleft \nu \Rightarrow N_\eta \prec N_\nu$*, and if* $\eta \in {}^\theta\lambda$ *then* $N_\eta \prec N^+_\eta \prec M$
- (iii) *for* $\eta \in {}^\theta\lambda, \langle \alpha_{\eta \restriction i} : i < \theta \rangle$ *is increasing continuous and for* i *limit* $\beta_{\eta \restriction i} = \alpha_{\eta \restriction i}$
- (iv) $\eta \triangleleft \nu \Rightarrow \alpha_\eta < \alpha_\nu$ *and* $\alpha_\eta \leq \beta_\eta < \alpha_\nu \leq \beta_\nu$
- (v) $\mathfrak{B} \prec M, \bigwedge_\eta N^+_\eta \prec M, \mathrm{gen}(N^+_\eta) \leq \theta$
- (vi) *for* $\eta \in {}^{(i+1)}\lambda, i \leq \theta$ *we have* $\mathfrak{B}_{\alpha_{\eta \restriction i}} \underset{N_{\eta \restriction i}}{\overset{M}{\bigcup}} N_\eta$
- (vii) *for each* $\eta \in {}^{\theta>}\lambda$ *the sequence* $\langle \alpha_{\eta\hat{\ }\langle\gamma\rangle} : \gamma$ *satisfies* $\sup \mathrm{Rang}(\eta) < \gamma < \lambda \rangle$ *is strictly increasing*
- $(viii)$ *for* $\eta \in {}^\theta\lambda$ *and* $i < \theta$ *we have* $\mathfrak{B}_{\alpha_{\eta \restriction i}} \underset{N_{\eta \restriction i}}{\overset{M}{\bigcup}} N_\eta$

[6] we omit this if in (II)(b) for each N we restrict the family of $\mathfrak{p}$ we use to be of cardinality $\leq \lambda$

(ix) $N^+_\eta = \bigcup_{i<\theta} N^+_{\eta,i}$, $N^+_{\eta,i}$ *increasing continuous and the isomorphism type of* $N^+_{\eta,i}$ *over* N_η *is computed continuously which means: for every* $i < \theta$ *for some* $j \in (i,\theta)$ *we have* $\eta \restriction j \triangleleft \nu \in {}^0\lambda \Rightarrow \mathrm{tp}(N^+_{\eta,i}, \mathfrak{B}_{\alpha_{\eta\restriction i}}, M) = \mathrm{tp}(N^+_{\nu,i}, \mathfrak{B}_{\alpha_{\nu\restriction i}}, M)$ *and* $nonforkin\mathfrak{B}_{\alpha_{\eta\restriction i}} N^{+M}_{\eta i_{N_{\eta i}}}$.

<u>*Then*</u> *for stationarily many* $\delta \in S$, *for some* $\eta \in {}^\theta\lambda$ *such that* $\langle \eta(i) : i < \theta\rangle$ *strictly increasing with limit* δ, *there is an elementary embedding* $g : \mathrm{Sk}_M(\mathfrak{B}_\delta \cup N^+_\eta) \to \mathfrak{B}_{\delta+1}, g\restriction\mathfrak{B}_\delta$ *the identity, and*

(∗) *for every* $\bar a \in {}^{\theta>}\mathfrak{B}$, *for some* $N \prec \mathfrak{B}$ *we have* $\mathrm{gen}(N) \le \theta, N_\eta \cup \bar a \subseteq N$ *and the following set belongs to* D^*_θ

$$\{i < \theta : \text{ we have } (\mathfrak{B}_{\beta_{\eta\restriction i}}) \underset{(N \cap \mathfrak{B}_{\alpha_{\eta\restriction i}}) \cup N_{\eta\restriction i}}{\overset{\mathfrak{B}}{\bigcup\!\!\!\!\bigcup}} \bar a \text{ and } \mathrm{gen}(N \cap \mathfrak{B}_{\alpha_{\eta\restriction i}}) < \theta\} \in$$

(e) *Assume*

(α) $\langle N_i : i \le \theta+1\rangle$ *is an elementary increasing continuous chain of models of* $T^*, \bar a_i \in {}^{\kappa>}(N_{i+1}), N_i = \mathrm{Sk}_{N_{\theta+1}}(\bigcup_{j<i} \bar a_j)$, *for* $i \le j \le \theta+1, \mathfrak{p}_{i,j} = \mathfrak{p}_{i,j}(\bigcup_{\epsilon\in[i,j)} \bar x_\epsilon, \ldots, \bar a_\zeta, \ldots)_{\zeta<i}$ *is a type-definition, and* $\mathrm{tp}(\bigcup_{\epsilon\in[i,j)} \bar a_\epsilon, N_i, N_{i+1}) = \mathfrak{p}^{N_i}_{i,j}$, *and for* $i < \theta$ *the sequence* $\langle \mathfrak{p}_{i,j} : i < j \le \theta+1\rangle$ *commute which means: if* $i_0 < i_1 < i_2$, *and* $N_0 \prec N_1 \prec N_2 \prec N_3$, *and* $\bar a_\epsilon \in N_0$ *for* $\epsilon < i_0$, $\bar a_\epsilon \in N_1$ *for* $\epsilon \in [i_0,i_1), \bar a_\epsilon \in N_2$ *for* $\epsilon \in [i_1,i_2)$ *and* $\langle \bar a_\epsilon : \epsilon \in [i_\ell, i_{\ell+1})\rangle$ *realizes* $(\mathfrak{p}_{i_\ell,i_{\ell+1}}(\bigcup_{\epsilon\in[i_\ell,i_{\ell+1})} \bar x_\epsilon, \ldots, \bar a_\epsilon, \ldots)_{\epsilon<i_\ell})^{N_0}$ *for* $\ell = 0,1$ *then* $\langle \bar a_\epsilon : \epsilon \in [i_0,i_2)\rangle$ *realizes* $(\mathfrak{p}_{i_0,i_2}(\bigcup_{\epsilon\in[i_0,i_2)} \bar x_\epsilon, \ldots, \bar a_\epsilon, \ldots)_{\epsilon<i_0})^{N_0}$

(β) $\mathscr{T} \subseteq {}^{\theta\ge}\lambda$ *is non-empty, closed under initial segments*

(γ) $\langle a_\eta, \beta_\eta : \eta \in \mathscr{T} \cap {}^{\theta>}\lambda\rangle$ *satisfies*

(i) *for* $\eta \triangleleft \nu \in \mathscr{T}, \alpha_\eta < \beta_\eta \le \nu(\ell g(\eta)) < \alpha_\nu < \beta_\nu$

(ii) *for* $\eta \in \mathscr{T} \cap {}^{\theta>}\lambda$ $\langle \alpha_{\eta\restriction i} : i < \ell g(\eta)\rangle$ *is increasing continuous*

(δ) $\langle \bar a_\eta : \eta \in \mathscr{T}\rangle, \mathfrak{B}^+$ *satisfies*

(i) $\mathfrak{B} \prec \mathfrak{B}^+$

(ii) $\bar a_\eta \in \mathfrak{B}_{\beta_\eta}$ *when* $\eta \in \mathscr{T} \cap {}^{\theta>}\lambda$

(iii) *for* $\eta \in \mathscr{T} \cap {}^i\lambda : i < \theta, \bar a_\eta$ *realizes* $\mathfrak{p}_{i,i+1}(\bar x_i, \ldots, \bar a_{\eta\restriction j}, \ldots)_{j<i}$

(iv) *for* $\eta \in \mathscr{T} \cap {}^\theta\lambda, \bar a_\eta \in \mathfrak{B}^+$ *and* $\bigcup_{\epsilon\in[i,\theta+1)} \bar a_{\eta\restriction\epsilon}$ *realizes* $\mathfrak{p}^{\mathfrak{B}_\alpha}_{i,\theta+1}$.

<u>*Then*</u>:

{1.2} (ϵ) *for every club* E *of* λ, *for some strictly increasing continuous sequence* $\langle \alpha_\epsilon : \epsilon \le \theta\rangle$ *of ordinals*[7] *from* E *(in fact listing the set* C_{α_θ} *from 1.5C): if* $\zeta \le \theta$ *is limit and* $\eta \in {}^\zeta\lambda$ *we have:* $\bigwedge_{\epsilon<\zeta} \eta\restriction\epsilon \in \mathscr{T}$, $\bigwedge_{\epsilon<\zeta} \alpha_\epsilon \le \eta(\epsilon) < \alpha_{\epsilon+1}$

[7]if we add $\bar x_\zeta$ is empty for limit $\zeta < 0$ the phrasing is simplified

then there is $\bar{a} \in \mathfrak{B}_{\alpha_\zeta+1}$ *realizing over* $\mathfrak{B}_{\alpha_\zeta}$ *the type required from* $\bar{a}_\eta$ *and* $\zeta < \theta, \eta \notin \mathscr{T}$ *and if* $\zeta = \theta$ *then*

(∗) *for every* $\bar{b} \in {}^{\kappa>}B$ *for some* $A \in {}^{\kappa>}(\mathfrak{B}_\delta)$ *the following set belongs to* D^*_θ*:*

$$\{i < \theta : \text{ we have } \mathfrak{B}_{\alpha_{\eta\restriction i}} \underset{\bigcup_{j<i} a_{\eta\restriction j} \cup (A \cap \mathfrak{B}_{\alpha\restriction i})}{\Cup} (A \cup \bar{b} \cup \bar{a})^{\mathfrak{B}} \text{ and}$$

$$|A \cap \mathfrak{B}_{\alpha_{\eta\restriction i}}| < kappa \text{ (or at least } \mathrm{gen}(\mathrm{Sk}_{\mathfrak{B}}(A \cap \mathfrak{B}_{\alpha_{\eta\restriction i}})) < \kappa).$$

We now further specialize the theorem stating the properties we shall subsequently use. {2.5A}

Theorem 2.8. *Assume*

(α) $\theta = \mathrm{cf}(\theta), \lambda = \mathrm{cf}(\lambda) = \lambda^\theta, \mathrm{cf}(\Upsilon) \equiv \theta, D$ *a filter on* Υ *containing the co-bounded subsets of* Υ *and the black box from IV 3.x exist,* $\lambda > 2^{2^\theta}$ *and* $[\alpha < \lambda \Rightarrow |\alpha|^{<\theta} < \lambda]$

(β) $T^* = \mathrm{Th}(\mathfrak{C}^*)$ *is as in 2.1.* {2.1}

Then there are $\mathfrak{B}, \bar{\mathfrak{B}} = \langle \mathfrak{B}_\alpha : \alpha \leq \lambda \rangle, S$ *such that:*

(a) $\mathfrak{B}$ *is a* θ*-saturated model of* T^*

(b) $\bar{\mathfrak{B}}$ *is an* $\prec$ *-increasing continuous chain of models of* $T^*, [\alpha < \lambda \Rightarrow \|\mathfrak{B}_\alpha\| < \lambda], \mathfrak{B}_\lambda = \mathfrak{B}$

(c) $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$ *is stationary and* $S^- = \{\delta < \lambda : \mathrm{cf}(\delta) = \theta, \delta \notin S\}$ *is stationary disjoint to* S

(d) *if* $\delta \in S^-, \bar{a} \in {}^{\theta>}\mathfrak{B}$ *then for some* $\bar{b} \in {}^{\theta>}(\mathfrak{B}_\delta)$ *the type* $\mathrm{tp}(\bar{a}, \mathfrak{B}_\delta, \mathfrak{B})$ *does not split over* $\bar{b}$

(e) *if* $\bar{a} \in {}^{\theta>}\mathfrak{B}, \mathfrak{p}$ *is a type-definition over* $\mathrm{Sk}_{\mathfrak{B}}(\bar{a})$ *then for stationarily many* $\delta \in S^-, \mathfrak{B}_{\delta+1} = \mathrm{Sk}_{\mathfrak{B}}(\mathfrak{B}_\delta \cup \bar{b}_\delta), \bar{b}_\delta$ *realizes* $\mathfrak{p}^{\mathfrak{B}_\delta}$ *and* $\bar{a} \in {}^{\theta>}(\mathfrak{B}_\delta)$*,*

(f) *if* $\bar{a}, \mathfrak{p}$ *are as in clause (e) and*

(α) $\bar{b}_\alpha \in {}^{\theta>}\mathfrak{B}$ *realizes* $\mathfrak{p}^{\mathfrak{B}_\alpha}$ *for* $\alpha \in W, W$ *a stationary subset of* $\lambda, \bar{b}_\alpha \in {}^{\theta>}(\mathbf{B}_\beta)$ *when* $\beta = \mathrm{Min}(W \setminus (\alpha+1))$ *and* $\bar{a} \in {}^{\theta>}\mathfrak{B}_{\mathrm{Min}(W)}$

(β) $\mathfrak{B}'$ *is as elementary extension of* $\mathfrak{B}, \alpha^0_i \in W$ *for* $i < \theta, \bigwedge_{i<j} \alpha^0_i < \alpha^0_j$ *and* $\bar{c} \in {}^{\theta>}(\mathfrak{B}')$ *are such that for* $i < \theta, \mathrm{tp}(\bar{c}, \mathfrak{B}_{\alpha^0_i})$ *does not split over* $\bar{a} \cup \bigcup_{j<i} \bar{b}_{\alpha^0_j}$

then we can find $\delta \in S, \alpha_i < \alpha'_i < \beta'_i < \beta_i$ *for* $i < \Upsilon$ *from* W *satisfying* $\bigwedge_{i<j} \beta_i < \alpha_j$ *and* $\delta = \cup\{\alpha_i : i < \Upsilon\theta\}$ *and* $\bar{c}' \in {}^{\omega>}(\mathfrak{B})$ *such that:*

(α) $\bar{a}\hat{\ }\ldots\hat{\ }\bar{b}_{\alpha^0_i}\hat{\ }\ldots\hat{\ }\bar{c}$ *and* $\bar{a}\ldots\hat{\ }\bar{b}_{\alpha'_i}\hat{\ }\ldots\hat{\ }\bar{c}'$ *realizes the same type*

(β) *for* $i < \theta, \mathrm{tp}(\bar{a} \cup \bigcup_{j<\Upsilon} \bar{b}_{\alpha'_j} \cup \bar{c}, \mathfrak{B}_{\alpha'_i}, \mathfrak{B})$ *does not split over* $\bar{a} \cup \bigcup_{j\leq i} b_{\alpha_j}$

(γ) *for every* $\bar{d} \in {}^{\theta>}\mathfrak{B}$*, for every* $j < \Upsilon$ *large enough* $\mathrm{tp}(\bar{d}, \mathfrak{B}_{\beta'_j}, \mathfrak{B})$ *does not split over* $\mathfrak{B}_{\alpha'_j} \cup \bar{b}_{\alpha'_j}$*.*

Proof. By 2.7, choose $\mu = \kappa$. $\square_{2.8}$ {2.5}

{2.5B}

Remark 2.9. 1) If we say "in $\mathfrak{B}$, a structure B is definable" we mean defined by a first order formula with parameters in $\mathfrak{B}$, necessarily finitely many. I.e. the set of elements, relations and functions of B are definable. We can assume the set of elements of B is a subset of $\mathfrak{B}$ as using $\mathfrak{C}^{\rm eq}$ (see [Sh:c]) add no generality.
2) We say "b is a structure in $\mathfrak{B}$" or "b representable in $\mathfrak{B}$", if $\mathfrak{B}$ "think" it is such a structure.
{2.5A} 3) In clause (c) of 2.8 only the strictly increasing η's count so we may ignore the rest.

* * *

The rest of this section is dedicated to investigating properties of $\mathfrak{B}$ where

{2.5C}
{2.5A} **Hypothesis 2.10.** For clause (a) of 2.8 holds and $\mathfrak{B}$ is as in the conclusion 2.7
{2.5A} or 2.8 (if it matters we are in the context of 2.7 or 2.8, we shall say; so in 2.8 let $\kappa = \theta$) and for simplicity $\Upsilon = \theta$; so $\lambda, \theta, T^*, \mathfrak{C}^*, \mathscr{D}^*$ and $S, S' \subseteq \{\delta < \lambda : {\rm cf}(\delta) = \theta\}$ are fixed.

{2.6}
Claim 2.11. *For some stationary $S_1 \subseteq S^-$ there is $w_\alpha \in \mathfrak{B}_{\alpha+1}$ for $\alpha \in S_1$ such that:*

- (a) $\mathfrak{B}_{\alpha+1} \models$ "*w_α is finite*"
- (b) $\mathfrak{B}_{\alpha+1} \models$ "*$a \dot\in w_\alpha$*" *for every $a \in \mathfrak{B}_\alpha$*
- (c) $\mathfrak{B}_{\alpha+1} \models$ "*$w \subseteq w_\alpha$*" <u>*iff*</u> $\mathfrak{B}_\alpha \models$ "*w is a finite set*"
- (d) $\mathfrak{B}_{\alpha+1}$ *is the Skolem Hull of* $\mathfrak{B}_\alpha \cup \{w_\alpha\}$
- (e) $\mathfrak{B}_\alpha \underset{{\rm Sk}(\emptyset)}{\overset{\mathfrak{B}_{\alpha+1}}{\Cup}} \{w_\alpha\}$
- (f) *if in $\mathfrak{B}_\alpha, \varphi(x,y,\bar a)$ define a partial order (on $\{x : (\exists y)\varphi(x,y,\bar a)\}$) which is directed (i.e. every pseudo finite subset has an upper bound)* <u>*then*</u> *for some $b \in \mathfrak{B}_{\alpha+1}, b$ is an upper bound of $\{x \in \mathfrak{B}_\alpha : (\exists y)\varphi(x,y,\bar a)\}$ by the partial order $\varphi(x,y,\bar a)$ and $\mathfrak{B}_\alpha \underset{\langle \bar a\rangle}{\overset{\mathfrak{B}_{\alpha+1}}{\Cup}} \langle b\rangle$ (really $b \in \langle \bar a, w_\alpha\rangle$)*
- (g) *there is a type definition $\mathfrak{p}_{\rm uf}$ over* ${\rm Sk}(\emptyset)$, *such that $\alpha \in S \Rightarrow$* ${\rm tp}(w_\alpha, \mathfrak{B}_\alpha, \mathfrak{B}) = \mathfrak{p}_{\rm uf}^{\mathfrak{B}_\alpha}$.

{2.1} *Proof.* Use the ultrafilter $\mathscr{D}^*$ and $(*)_0$ of 2.1(2) to define $\mathfrak{p}_{\rm uf}$. $\square_{2.11}$

{2.6A}
Observation 2.12. *1) If $S_1 \subseteq S^-$ is stationary, for each $\delta \in S_1$ we have $\bar a^\delta \in {}^{\kappa >}\mathfrak{B}$* <u>*then*</u> *for some stationary $S_2 \subseteq S_1$ and $N \prec \mathfrak{B}$, satisfying* ${\rm gen}(N) < \theta$ *and type definition $\mathfrak{p}$ over N, we have for every $\delta \in S_2$,* ${\rm tp}(\bar a^\delta, \mathfrak{B}_\delta, \mathfrak{B}) = \mathfrak{p}^{\mathfrak{B}_\delta}$.
2) If above, a property ϑ is such that for every $\delta \in S_1$ for some $\bar a \in {}^{\kappa>}\mathfrak{B}, \vartheta(\bar a^\delta, \mathfrak{B}_\delta, \mathfrak{B})$ and $\alpha < \beta < \lambda$ and $\vartheta(\bar a^\beta, \mathfrak{B}_\beta, \mathfrak{B}) \Rightarrow \vartheta(\bar a^\beta, \mathfrak{B}_\alpha, \mathfrak{B})$ <u>*then*</u> *for some type definition $\mathfrak{p}$ over some $N \prec \mathfrak{B}$,* ${\rm gen}(N) < \kappa$, *for every $\alpha < \lambda$ such that $N \prec \mathfrak{B}_\alpha$, there is $\bar b^\alpha \in {}^{\vartheta>}\mathfrak{B}$ realizing $\mathfrak{p}^{\mathfrak{B}_\alpha}$ for which $\vartheta(\bar b^\alpha, \mathfrak{B}_\alpha, \mathfrak{B})$ holds.*

Proof. By Fodor's Lemma. $\square_{2.12}$

{2.7}

Claim 2.13. *Suppose M is a model of T^*, $N \prec M$ and I is a dense (infinite) linear order $\langle a_s : s \in I\rangle$ is an indiscernible sequence in M over N such that $i < \delta \Rightarrow \notin N$.*
Then there are M^*, w, u such that:

(a) $M \prec M^*$
(b) $w, u \in M^*$
(c) $M^* \models$ "*w and u are disjoint finite sets*"
(d) $M^* \models$ "$a_s \dot{\epsilon} w$" *for $s \in I$*
(e) $M^* \models$ "$b \subseteq u$" *if $M \models$ "b is a finite set" and $M \models$ "$\neg a_s \dot{\epsilon} b$" for $s \in I$*
(f) *$\langle a_s : s \in I\rangle$ is an indiscernible sequence over $N \cup \{w, u\}$*
(g) $\mathrm{tp}(\langle w\rangle, M, M^*)$ *is* fs *(finitely satisfiable) in* $\mathrm{Sk}_M(\{a_s : s \in I\} \cup N)$
(h) $\mathrm{tp}(\langle w, u\rangle, M, M^*)$ *does not split over* $\mathrm{Sk}_M(\{a_s : s \in I\} \cup N)$.

Proof. We shall show below that by transitivity of "does not split", by 2.11, as $\bigcup_{\mathrm{fs}} \Rightarrow \bigcup_{\mathrm{nsp}}$ and as the existence of difference (among "finite" members of $\mathfrak{B}$)[8] that it suffices to show that {2.6}

⊡ Λ is finitely satisfiable in M, where λ is the union of the following: (the subscript is according to the clause of the claim this set of sentences is concerned with):
$\Lambda_c :=$ {"x is a finite set"}
$\Lambda_d :=$ {"$a_s \dot{\epsilon} x$" : $s \in I$}
$\Lambda_e :=$ {'$b \cap x = \emptyset$" : $b \in M, M \models$ "b a finite set" and $M \models$ "$\neg a_s \dot{\epsilon} b$" for $s \in I$}
$\Lambda_f := \{\varphi(x, a_{s_1}, \dots, a_{s_n}, \bar{d}) \equiv \varphi(x, a_{t_1}, \dots, a_{t_n}, \bar{d}) : \bar{d} \in {}^{\omega >}N, s_1 < \dots < s_n < \delta$ and $t_1 < \dots < t_n < \delta\}$
$\Lambda_g := \{\neg\psi(x, \bar{c}) : \bar{c} \in M, \psi(x, \bar{c})$ not realized by any member of $\mathrm{Sk}_M(\{a_s : s \in I\})\}$.

[Why ⊡ suffice? Let M^* be an elementary extension of M which is $\|M\|^+$-saturated. So some $w \in M^*$ realizes Λ, and some $u^* \in M^*$ realizes $\mathfrak{p}_{\mathrm{uf}}^{\mathrm{Sk}(M+w)}$ where $\mathfrak{p}_{\mathrm{uf}}$ is from clause (g) of 2.11. By the choice of T^* there is $u \in M^*$ such that $M^* \models$ "$u = u^* \backslash w$". {2.6}
Now clause (a) holds by the choice of M^*, clause (b) holds by the choices of w, u^*, u. Also $M^* \models$ "w is finite" as w realizes $\Lambda_c \subseteq \Lambda$, $M^* \models$ "u^* is finite" by the definition of $\mathfrak{p}_{\mathrm{uf}}$, and $M^* \models$ "u is finite disjoint to w" by the choice of u, so clause (c) holds. Clause (d) holds as w realizes $\Lambda_d \subseteq \Lambda$. Clause (e) holds as if $M \models$ "b is finite, $\neg a_s \dot{\epsilon} b$" for $s \in I$ then $M^* \models$ "$b \cap w = \emptyset$ as w realizes $\Lambda_e \subseteq \Lambda$ and $M^* \models$ "$b \subseteq u^*$" by the definition of $\mathfrak{p}_{\mathrm{uf}}$; now as $M^* \models$ "$u = u^* \setminus w$" combining the last two statements we have, we are done. As for clause (f), the sequence $\langle a_s : s \in I\rangle$ is indiscernible over $N \cup \{w\}$ as w realizes $\Lambda_f \subseteq \Lambda$, and as $\mathrm{tp}(u^*, M \cup \{w\}, M^*)$ does not split over $\emptyset$ (by the definition of $\mathfrak{p}_{\mathrm{uf}}$) clearly $\langle a_s : s \in I\rangle$ is indiscernible also over $N \cup \{w\} \cup \{u^*\}$ hence over $\mathrm{Sk}(N \cup \{w, u^*\})$ hence over $N \cup \{w, u\}$ as required in clause (f). Clause (g) holds as w realizes $\Lambda_g \subseteq \Lambda$.

[8] Note: $\mathrm{tp}(w, M, M^*)$ is not fs in N by clauses (c)+(b)+(e) if some $a \in M \setminus N \setminus \{a_i : i < \delta\}$ realizes $\mathrm{tp}(a_0, N, M)$.

Lastly $\mathrm{tp}(w, M, M^*)$ does not split over $\mathrm{Sk}_M(\{a_i : i < \delta\} \cup N)$ because it is finitely satisfiable in it (by clause (g), using 2.5) and $\mathrm{tp}(u^*, M + w, M^*)$ does not split over $\mathrm{Sk}_M(\emptyset)$ hence it does not split over $N \cup \{a_i : i < \delta\}$ so as $u \in \mathrm{Sk}_{M^*}(M + w)$ we can deduce clause (h).] {2.3}

So it is enough to show that $\Lambda =: \Lambda_c \cup \Lambda_d \cup \Lambda_e \cup \Lambda_f \cup \Lambda_g$ is finitely satisfiable in M. Let $\bar{a}^* = \langle a_i : i < \delta \rangle$ and $\Lambda_x = \Lambda_x[M, \bar{a}^*]$ for $x = c, d, e, f, g$.

First note that $\Lambda_c \cup \Lambda_d \cup \Lambda_e$ is finitely satisfiable in M: if Δ is a finite subset of the union we let C_1 be the set of a_s such that "$a_s \dot{\epsilon} x$" $\in \Lambda_d \cap \Delta$.

Let w be the unique $w \in \mathrm{Sk}(N \cup \{a_s : s \in I\})$ satisfying $M \models$ "$w = \{a_s : a_s \in C_1\}$". Clearly w realizes Δ, so $\Lambda_c \cup \Lambda_d \cup \Lambda_e$ is really finitely satisfiable in M.

Secondly, even $\Lambda_c \cup \Lambda_d \cup \Lambda_e \cup \Lambda_g$ is finitely satisfiable in M as the element we have chosen above is in $\mathrm{Sk}(\{a_i : i < \delta\})$.

Thirdly, we shall show that $\Lambda_c \cup \Lambda_d \cup \Lambda_e \cup \Lambda_f \cup \Lambda_g$ is finitely satisfiable with M; let Δ be a finite subset of the union. Let $\Delta_1 = \{\varphi(x, z_1, \ldots, z_n, \bar{d})$: some $\varphi(x, a_{s_1}, \ldots, a_{s_n}, \bar{d}) \equiv \varphi(x, a_{t_1}, \ldots, a_{t_n}, \bar{d})$ belongs to $\Delta \cap \Gamma_f$ for some $s_1 < \ldots < s_n \in I, t_1 < \ldots < t_n \in I$, and $\varphi, \bar{d}\}$.

There is a homogeneous linear order J (i.e. any finite partial order preserving function f from J to J can be extended to an automorphism of J) and an order preserving. Let $\delta = \omega\alpha_*$, so $1 \le \alpha_* \le \delta I \to J$ such that J is of cardinality $|\delta| + \aleph_0$ and $I \cap J = \emptyset$ for notational simplicity. We can find M_1, $M \prec M_1$ and $a_t \in M_1$ for $t \in I$ such that:

- $(*)_1$ $a_t = a_{h(i)}$ for $i < \delta$
- $(*)_2$ $\langle a_t : t \in I \rangle$ indiscernible over N in M_1 (by the order I)
- $(*)_3$ M_1 is strongly $\|M\|^+$-saturated
- $(*)_4$ if $M \models$ "v is a finite set and $\neg(a_i \cdot \dot{\epsilon} v)$" for $i < \delta$ <u>then</u> $M_1 \models$ "$\neg(a_t \cdot \dot{\epsilon} v)$" for $t \in I$.

The Λ's were defined for M and $\langle a_s : s \in I \rangle$, but the previous argument can be applied with M_1 and $\bar{a}^1 := \langle a_t : t \in J \rangle$ replacing M and $\langle a_s : s \in I \rangle$, so Λ_x is replaced by $\Lambda_x[M_1, \bar{a}^1]$ so there is $M^*, M_1 \prec M^*$ and $w^* \in M^*$ which realized $\Lambda_c[M_1, \bar{a}^1] \cup \Lambda_d[M_1, \bar{a}^1] \cup \Lambda_e[M_1, \bar{a}^1] \cup \Lambda_g[M_1, \bar{a}^1]$. Without loss of generality M^* is strongly $\|M_1\|^+$-saturated. Now trivially w^* is O.K. also for $M, \langle a_s : s \in I \rangle$, i.e. w^* realizes $\Lambda_c \cup \Lambda_d \cup \Lambda_e \cup \Lambda_g$. Let $\Delta^+ = \{\varphi(w^*, z_1, \ldots, z_n, \bar{d}) : \varphi(x, z_1, \ldots, z_n, \bar{d}) \in \Delta_1\}$. By Ramsey theorem for some infinite $U \subseteq \delta$, $\langle a_s : s \in U \rangle$ is an Δ^+-indiscernible sequence. As J is homogeneous linear order extending δ there is an automorphism f of J which maps $\{s \in I : a_s$ appear in $\Delta\}$ into U. Because "$\langle a_t : t \in J \rangle$ is an indiscernible sequence over N'', $(*)_3$ and the definitions of $\Lambda_c, \Lambda_d, \Lambda_e, \Lambda_f, \Lambda_g$ we see that some automorphism h of M_1 over N, maps a_t to $a_{f(t)}$ for $t \in I$. Hence h map $\{a_s : a_s$ appear in $\Delta\}$ into a subset of $\{a_t : t \in U\}$ and map $\{a_t : t \in J\}$ onto itself. We can extend h to an automorphism g of M^* as M^* is strongly $\|M_1\|^+$-saturated. Clearly g (and g^{-1}) maps $\Lambda_x[M_1, \bar{a}^1]$ onto itself for $x = c$, d, e, f, g. So $g^{-1}(w^*)$ realizes Δ (i.e. for $x \in \{c, d, e, g\}, \Delta \cap \Lambda_x \subseteq g^{-1}(\Lambda_x[M_1, \bar{a}]) = \Lambda_x[M_1, \bar{a}]$ and for $\Delta \cap \Lambda_f$ we use the choice of h). Hence $\Lambda_c \cup \Lambda_d \cup \Lambda_e \cup \Lambda_f \cup \Lambda_g$ is consistent with M as required. $\square_{2.13}$

{2.7A} **Observation 2.14.** *Assume $M \prec N$ are models of T^*. Assume I is a dense (infinite) linear order, and $\langle \bar{a}_s = (a^1_s, a^2_s) : s \in I \rangle$ is an indiscernible sequence in M over N and $a^\ell_s \notin N$. Assume further that $\mathfrak{p} = \mathfrak{p}(x)$ (or $\mathfrak{p}(\ldots, x, \ldots)$) is a type*

{2.7} *definition over* N. *Then we can have (a)-(c),(f)-(h) from 2.13 (omitting (d),(e)) and then there are* M^*, ω_ℓ, u_ℓ *for* $\ell = 1, 2$ *such that:*

- (a) $M \prec M^*$
- $(b)_\ell$ $\omega_\ell, u_1 \in M^*$
- $(c)_\ell$ $M^* \models$ "ω_ℓ, u_ℓ *are disjoint finite sets*"
- $(d)_\ell$ $M^* \models$ "$a^\ell_s \dot{\epsilon} \omega_\ell$" *for* $s \in I, \ell = 1, 2$
- $(e)_\ell$ $M^* \models$ "$b \subseteq u_\ell$" *if* $M \models''$ b *is a finite set, and* $\neg(a^\ell_s \dot{\epsilon} b)$" *for* $s \in I$
- $(f)_\ell$ $\langle (a^1_s, a^2_s) : s \in I \rangle$ *is an indiscernible sequence in* M *over* $N \cup \{\omega_1, \omega_2, u_1, u_2\}$
- (g) $\mathrm{tp}(\langle \omega_1, \omega_2 \rangle, M, M^*)$ *is* fs *in* $\mathrm{Sk}_m(\{a^1_s, a^2_s : s \in I\} \cup N)$ *in fact is* $\mathrm{Sk}_m(\{a^1_s : s \in I\} \cup N) \times \mathrm{Sk}_M(\{a^2_s : s \in I\} \cup N)$
- (h) $\mathrm{tp}(\langle \omega_1, \omega_2, u_1, u_1 \rangle, M, M^*)$ *does not split over* $\mathrm{Sk}_M(\{a^1_s, a^2_s : s \in I\} \cup N\}$
- $(i)_\ell$ $\mathrm{tp}(\omega_\ell, u_\ell\rangle, M, M^*)$ *does not split over* $\mathrm{Sk}_M(\{a^\ell_s : s \in I\} \cup N)$.

Proof. Similar to the proof of 2.13. {2.7}

Let $b_s = a_{s,0}$, $c_s = a_{s,1}$, so $\langle < c_s, b_s >: s \in I \rangle$ is an indiscernible sequence over N (and $b_s \neq c_t, s \neq t \Rightarrow b_s \neq b_t$ and $s \neq t \Rightarrow c_s \neq c_t$), and repeating the proof of {2.7} 2.13 we can get M^*, u, v, w such that:

- (a) $M \prec M^*$
- (b) u, v, $w \in M^*$
- (c) $M^* \models$ "u, v, w are pairwise disjoint finite sets"
- (d) $M^* \models$ "$b_s \dot{\epsilon} v$ and $c_s \dot{\epsilon} w$" for $s \in I$
- (e) $M^* \models$ "$b \dot{\epsilon} u$" for $b \in M \setminus \{b_s, c_s : s \in I\}$ moreover, if $M \models$ "b is finite" then $\bigwedge\limits_s \neg b_s \dot{\epsilon} b \Rightarrow M^* \models b \subseteq (u \cup w)$" and $\bigwedge\limits_s \neg c_s \dot{\epsilon} b \Rightarrow M^* \models$ "$b \subseteq (u \cup v)$"
- (f) $< \langle b_s, c_s \rangle : s \in I >$ is an indiscernible sequence over $N \cup \{u, v, w\}$
- (g) $\mathrm{tp}(\langle v, w \rangle, M, M^*)$ is finitely satisfiable in $\mathrm{Sk}_M(\{b_s : s \in I\} \cup N)$
- (h) $\mathrm{tp}(\langle u, v, w \rangle, M, M^*)$ does not split over $\mathrm{Sk}_M(\{b_s : s \in I\} \cup N)$.

Clearly "$x \dot{\epsilon} w$" $\in \mathfrak{p}^{M^*}$ or "$\neg(x \dot{\epsilon} w)$" $\in \mathfrak{p}^{M^*}$, and similarly for $x \dot{\epsilon} v$. It is impossible that "$x \dot{\epsilon} v$" and "$x \dot{\epsilon} w$" both belongs to $\mathfrak{p}^{M^*}$ as $M^* \models$ "v, w are disjoint". Now if "$x \dot{\epsilon} v$" $\notin \mathfrak{p}^{M^*}$ we let $u' = u \cup w$ (i.e. M^* satisfies this) and $w' = v$, so for $\ell = 0$ they are as required; and if "$x \dot{\epsilon} w$" $\notin \mathfrak{p}^{M^*}$ then we let $u' = u \cup v, w' = w$, they are as required. $\square_{2.14}$

{2.7B}

Observation 2.15. *As in 2.13 without loss of generality* M *is strongly* $(\|N\| + |\delta|)^+$*-saturated, so in 2.14 we can replace* $(d^-) + (e^*)$ *by* $(d) + (e)$. {2.7} {2.7A}

Proof. Trivial. $\square_{2.15}$

Our main lemma is (see Definition below):

{2.8}

Main Lemma 2.16. *Assume*

- (a) $\mathbf{b}_1$ *is an atomic Boolean ring in* $\mathfrak{B}$*, i.e.* $\mathbf{b}_1 \in \mathfrak{B}, \mathfrak{B} \models$ "$\mathbf{b}_1$ *an atomic Boolean ring" (see 2.17(1) below)* {2.8A}
- (b) $\mathbf{b}_2$ *is a Boolean ring in* $\mathfrak{B}$ *(as above)*
- (c) $\mathbf{f}$ *is a complete embedding (see 2.17(2),(3) below) of* $\mathbf{b}_1$ *into* $\mathbf{b}_2$. {2.8A}

Then *for some* $x \in \mathbf{b}_1[\mathfrak{B}]$*:*

(α) $\mathfrak{B} \models$ "*x is a finite member of* $\mathbf{b}_1$*, i.e. a finite union of atoms*"

(β) $\mathbf{f}{\restriction}\{y \in \mathbf{b}_1^{\rm at} : \mathbf{b}_1 \models$ "$y \cap x = 0$"$\}$ *is definable with parameters in* $\mathfrak{B}$.

{2.8A} **Definition 2.17.** 1) A Boolean ring is just an ideal of a Boolean algebra (so the operations are $x \cap y, x \cup y, x - y$, and we have the individual constant 0 (but not 1)).
2) An *embedding* $\mathbf{f}$ of $\mathbf{b}_1$ into $\mathbf{b}_2$ is an isomorphism from $\mathbf{b}_1$ onto a subalgebra of $\mathbf{b}_2$ (so $h(0_{B_1}) = h(0_{B_2})$).
3) Such an embedding is called complete iff it maps maximal antichains of $\mathbf{b}_1$ to maximal antichains of $\mathbf{b}_2$.
4) For a Boolean ring $\mathbf{B}$, the derived Boolean algebra ba($\mathbf{B}$) is the unique derived (up to isomorphisms) Boolean algebra $\mathbf{B}'$ whose set of elements is $\mathbf{B} \times \{0,1\}$ with the order $(b_1, \ell_1) \le (b_2, \ell_2)$ iff $b_1 \le_{\mathbf{B}_1} b_2$ and $\ell_1 = \ell_2 = 0$ or $b_2 \le_{\mathbf{B}_1} b_1$ and $\ell_1 = \ell_2 = 1$ or $b_1 \cap_{\mathbf{B}_1} b_2 = 0_{\mathbf{B}_1}$ $\ell_1 = 0$ and $\ell_2 = 1$. We identify b and $(b, 0)$ for $b \in \mathbf{B}$.
5) $\mathscr{I}$ is a maximal ideal of B the Boolean ring $\mathbf{B}$ If for some uf D of ba($\mathbf{B}$), $\mathscr{I} = \mathbf{B} \setminus D$.

Proof. We may below write $\mathbf{b}_\ell$ for $\mathbf{b}_\ell[\mathfrak{B}]$.
A Stage: Assume

(a) $N \prec \mathfrak{B}$, gen(N) $< \kappa$, $\mathbf{b}_1$ and $\mathbf{b}_2$ belongs to N,$\mathfrak{p}$ a type-definition over N and "x an atom of $\mathbf{b}_1$" "$\in \mathfrak{p}^{\mathbf{B}}$, we may add[9]" $\neg x \dot{\epsilon} E b$" belongs to $\mathfrak{p}^M$ whenever $N \prec M, M \models$ "b is a finite set".

Then for some $\alpha = \alpha_{\mathfrak{p}} < \lambda$,

($*$) for every $x \in \mathfrak{B}$ realizing $\mathfrak{p}^{\mathfrak{B}_\alpha}$ and $y \in \mathfrak{B}$ we have:
$\text{tp}(y, \mathfrak{B}_\alpha \cup \{x\}, \mathfrak{B}) = \text{tp}(h(x), \mathfrak{B}_\alpha \cup \{x\}, \mathfrak{B}) \Rightarrow y = h(x)$.

Assume the conclusion of stage A fails. Choose $\alpha_0 < \lambda$ such that $N \subseteq \mathfrak{B}_{\alpha_0}$.

As we are assuming that the conclusion of stage A fails, for every $\alpha < \lambda$, $\alpha \ge \alpha_0$ there is $a_\alpha \in \mathfrak{B}$ realizing $\mathfrak{p}^{\mathfrak{B}_\alpha}$ such that $\text{tp}(\mathbf{f}(a_\alpha), \mathfrak{B}_\alpha \cup \{a_\alpha\}, \mathfrak{B})$ is realized not only by $h(a_\alpha)$ but also say by $b_\alpha \in \mathfrak{B}, b_\alpha \ne h(a_\alpha)$.

(Note: if we agree to restrict ourselves to onto isomorphisms we can somewhat simplify the proof).

{2.6} By 2.11 we can for $\delta < \lambda$ choose c_δ a member of $\mathfrak{B}$, such that $\mathfrak{B} \models$ "c_δ is a finite member of $\mathbf{b}_1, x \le_{\mathbf{b}_1} c_\delta$" for any x such that $\mathfrak{B}_\delta \models$ "x is finite member of $\mathbf{b}_1$" and $\mathfrak{B}_\delta \mathop{\bigcup\!\!\!\!\bigcup}_N c_\delta$.

Without loss of generality

$$\mathfrak{B}_\alpha \mathop{\bigcup\!\!\!\!\bigcup}_{\mathfrak{B}_{\alpha_0}} \langle a_\alpha, \mathbf{f}(a_\alpha), b_\alpha, c_\alpha, \mathbf{f}(c_\alpha)\rangle$$

moreover, for some type definition $\mathfrak{p}_1$ over some $N_1 \prec \mathfrak{B}_{\alpha_0}$, gen($N_1$) $< \kappa$ we have
{2.6A} $\text{tp}(\langle a_\alpha, \mathbf{f}(a_\alpha), b_\alpha, c_\alpha, \mathbf{f}(c_\alpha)\rangle, \mathfrak{B}_\alpha, \mathfrak{B}) = {\mathfrak{p}_1}^{\mathfrak{B}_\alpha}$ (see 2.12(2)).

Let $E = \{\delta < \lambda : \delta > \alpha_0$ a limit ordinal and $(\exists \alpha < \delta)[N \cup N_1 \subseteq \mathfrak{B}_\alpha]$ and $[\alpha < \delta \Rightarrow a_\alpha, b_\alpha, c_\alpha \in \mathfrak{B}_\delta]$ and $(\mathfrak{B}_\delta, \mathbf{f}) \prec (\mathfrak{B}, \mathbf{f})$ and [for every $\alpha < \delta, N' \prec \mathfrak{B}_\delta$

[9] why we may add? otherwise the conclusion is trivial

satisfying $\mathrm{gen}(N') < \kappa$ and gr a type definition over N', for some $\beta \in (\alpha, \delta)$ we have $\mathrm{gr}^{\mathfrak{B}_\beta}$ is realized in $\mathfrak{B}_{\beta+1}]\}$.

Let $u \in D^*_\theta$. u a set of successor ordinals with no two successor members. Let the linear order $I = \Sigma\{I_i : i < \theta\}$ be such that: $i \in u \Rightarrow I_i = \{s_i\}, i \in \theta \setminus u \Rightarrow I_i$ is isomorphic to the rationals but may have $I_i = \emptyset$ for i limits and let $I_{<j} = \sum\{I_i : i < \theta\}$ (instead using u we can act as in stage B).

We shall use 2.7 though we could just as well use 2.8. So E is a club of λ by the assumptions and we can define $N_\eta, \alpha_\eta, \beta_\eta$ and $a^*_{\eta,s}(s \in I_{\ell g(\eta)})$ (let $a_{\eta,s} = a_{\eta\restriction i,s}$ if $i \leq \ell g(\eta)$ and $s \in I_i$) for $\eta \in {}^{\theta>}\lambda$ by induction on $\ell g(\eta)$ as in (d) of 2.7 such that $\alpha_\eta, \beta_\eta \in E, \langle N_{\eta\restriction i} : i \leq j\rangle$ is increasing continuous, $N_{<>} = N$ and for η of length $i = j+1, N_\eta = \mathrm{Sk}(N_{\eta\restriction j} \cup \{a_{\eta,s} : s \in I_i\})$ and $j \in u \Rightarrow a_{s_i} = a_{\alpha^*_{\eta\restriction i}}$ and $\alpha_\eta < \alpha^*_\eta < \beta^*_\eta < \beta_\eta$ and also $\alpha^*_\eta, \beta^*_\eta \in E$. {2.5A} {2.5}

such that: if $\ell g(\eta) = i = j+1, t \in I_j$ then for some $\mathfrak{B}, \mathfrak{B} \prec \mathfrak{B}'$ and $\bar{b}$ from $\mathfrak{B}$ the sequence $\langle a_t\rangle \frown \bar{b}$ realizes $(\mathfrak{p}_1)\mathfrak{B}''$ where $\mathfrak{B}'' = \mathrm{Sk}_{\mathfrak{B}}(\mathfrak{B}_{\alpha_{\eta\restriction j}} \cup \{a_{\eta,s} : s <_I t\}$. Note: if $\eta \in {}^\theta\lambda$ then $\langle a_{\eta,s} : s \in I\rangle$ is an indiscernible sequence in $\mathfrak{B}$.

For $\eta \in {}^\theta\lambda$ let $N_\eta = \bigcup_{i<\theta} N_{\eta\restriction i}$, let $\alpha_\eta = \bigcup_{i<\theta} \alpha_{\eta\restriction i}$ and M^* be λ^+-saturated extension of $\mathfrak{B}$, and $N^+_\eta = \mathrm{Sk}_{M^*}(N_\eta, w_\eta, u_\eta)$ be as guaranteed by 2.13 (with $\langle a_{\eta,s} : s \in I\rangle, N, \mathfrak{B}_{\alpha_\eta}$ here standing for $\langle a_i : i < \delta\rangle, N, M$ there). Now w_η can be interpreted as a member of $\mathbf{b}_1$. More exactly there is $w^*_\eta \in \mathfrak{B}$ definable from w_η and the parameters defining $\mathbf{b}_1, (\subseteq N)$ such that $\mathfrak{B} \models$ "$w^*_\eta \in \mathbf{b}_1$, is finite and $x \leq_{\mathbf{b}_1} w^*_\eta \Longrightarrow x \dot{\epsilon} w_\eta$" for every $x \in \mathfrak{B}_\alpha$ such that $\mathfrak{B}_\alpha \models$ "$x \dot{\epsilon} \mathbf{b}_1$ is a finite union of atoms of $\mathbf{b}_1$". Note that for every $i < \theta$ the type $\mathrm{tp}(\langle w_\eta, w^*_\eta\rangle \frown \langle a_{\alpha^*_{\eta\restriction(j+1)}} : j \in [i,\theta), \mathfrak{B}_{\alpha_\eta}, \mathfrak{B}_{\alpha_\eta+1}\rangle$ does not split over $N \cup \{a_{\eta,s} : s \in I\})$ (easy manipulation). {2.7}

By clause II(d) of 2.7 we can find $\delta \in S$, increasing $\eta \in {}^\theta\delta$ and elementary embedding $h : N^+_\eta \to \mathfrak{B}_{\delta+1}$ as guaranteed there. So $v^*_\eta := h(w^*_\eta)$ is a member of $\mathbf{b}_2[\mathfrak{B}]$, so there is $N^* \prec \mathfrak{B}_\delta, \mathrm{gen}(N^*) \leq \theta$ as guaranteed by $(*)$ of II(d) of 2.7 for $\bar{a} := \langle w^*_\eta, v^*_\eta\rangle$, so {2.5} {2.5}

$$W := \{i < \theta : i \text{ is a succcessor ordinal and}$$

$$(\mathfrak{B}_{\beta_{\eta\restriction i}}) \underset{(\mathfrak{B}_{\alpha_{\eta\restriction i}} \cap N^*) \cup N_{\eta\restriction i}}{\Cup} \bar{a}\} \in D^*_\theta$$

hence

$$W' := \{i < \theta : i \text{ is a succcessor ordinal}, i \in u \text{ and}$$

$$(\mathfrak{B}_{\beta^*_{\eta\restriction i}}) \underset{(\mathfrak{B}_{\alpha^*_{\eta\restriction i}} \cap N) \cup N_{\eta\restriction i}}{\Cup} \bar{a}\} \in D^*_\theta.$$

Let $i = j+1 \in W'$. Now as $b_{\alpha^*_{\eta\restriction i}} \in \mathfrak{B}_{\beta^*_{\eta\restriction i}}$ realizes (recalling that the b_α's were chosen in the beginning of the proof)

$$q = \mathrm{tp}(\mathbf{f}(a_{\alpha^*_{\eta\restriction i}}), \mathrm{Sk}(\mathfrak{B}_{\alpha^*_{\eta\restriction i}} + a_{\alpha^*_{\eta resti}}), \mathfrak{B})$$

clearly

$(*)_1$ $\langle b_{\alpha^*_{\eta\restriction i}}, a_{\alpha^*_{\eta\restriction i}}\rangle, \langle \mathbf{f}(a_{\alpha^*_{\eta\restriction i}}), a_{\alpha^*_{\eta\restriction i}}\rangle$ realize the same type over $\mathfrak{B}_{\alpha^*_{\eta\restriction i}}$.

But $c_{\alpha_{\eta\restriction i}}, \mathbf{f}(c_{\alpha_{\eta\restriction i}}) \in \mathfrak{B}_{\alpha^*_{\eta\restriction i}}, \alpha_{\eta\restriction i} < \alpha^*_{\eta\restriction i}$, hence from $(*)_1$ we deduce:

$(*)_2$ $\langle b_{\alpha^*_{\eta\restriction i}}, a_{\alpha^*_{\eta\restriction i}}, c_{\alpha_{\eta\restriction i}}, \mathbf{f}(c_{\alpha_{\eta\restriction i}})\rangle$ and $\langle \mathbf{f}(a_{\alpha^*_{\eta\restriction i}}), a_{\alpha^*_{\eta\restriction i}}, c_{\alpha_{\eta\restriction i}}, \mathbf{f}(c_{\alpha_{\eta\restriction i}})\rangle$ realize the same type over $\mathfrak{B}_{\alpha_{\eta\restriction i}}$.

But $\mathfrak{B}_{\beta^*_{\eta\restriction i}} \bigcup\limits_{\mathfrak{B}_{\alpha_0}} \langle c_{\beta^*_{\eta\restriction i}}, \mathbf{f}(c_{\beta^*_{\eta\restriction i}})\rangle$ and the two sequences in $(*)_2$ are $\subseteq \mathfrak{B}_{\beta^*_{\eta\restriction i}}$, hence:

$(*)_3$ the sequence $\langle b_{\alpha^*_{\eta\restriction i}}, a_{\alpha^*_{\eta\restriction i}}, c_{\alpha_{\eta\restriction i}}, \mathbf{f}(c_{\alpha_{\eta\restriction i}}), c_{\beta^*_{\eta\restriction i}}, \mathbf{f}(c_{\beta*_{\eta\restriction i}})\rangle$, and the sequence $\langle \mathbf{f}(a_{\alpha^*_{\eta\restriction i}}), a_{\alpha^*_{\eta\restriction i}}, c_{\alpha_{\eta\restriction i}}, \mathbf{f}(c_{\alpha_{\eta\restriction i}}), c_{\beta^*_{\eta\restriction i}}, \mathbf{f}(c_{\beta^*_{\eta\restriction i}})\rangle$ realize the same type over $\mathfrak{B}_{\alpha_{\eta\restriction i}}$.

As $i \in W'$, by $(*)_3$ and as the two sequences above are from $\mathfrak{B}_{\beta_{\eta\restriction i}}$:

$(*)_4$ the sequence $\langle b_{\alpha^*_{\eta\restriction i}}, a_{\alpha^*_{\eta\restriction i}}, c_{\alpha_{\eta\restriction i}}, \mathbf{f}(c_{\alpha_{\eta\restriction i}}), c_{\beta^*_{\eta\restriction i}}, \mathbf{f}(c_{\beta^*_{\eta\restriction i}}), w^*_\eta, v^*_\eta\rangle$, and the sequence $\langle \mathbf{f}(a_{\alpha^*_{\eta\restriction i}}), a_{\alpha^*_{\eta\restriction i}}, c_{\alpha_{\eta\restriction i}}, \mathbf{f}(c_{\alpha_{\eta\restriction i}}), c_{\beta^*_{\eta\restriction i}}, \mathbf{f}(c_{\beta^*_{\eta\restriction i}}), w^*_\eta, v^*_\eta\rangle$ realize the same type over $\mathfrak{B}_{\alpha_{\eta\restriction i}}$.

By the various choices, $\mathbf{b}_1[\mathfrak{B}] \models$ "$w^*_\eta \cap c_{\beta^*_{\eta\restriction i}} - c_{\alpha_{\eta\restriction i}} = a_{\alpha^*\eta\restriction i}$", so as h is an embedding of Boolean rings we have $\mathbf{b}_2[\mathfrak{B}] \models$ "$v^*_\eta \cap \mathbf{f}(c_{\beta^*_{\eta\restriction i}}) - \mathbf{f}(c_{\alpha_{\eta\restriction i}}) = \mathbf{f}(a_{\alpha^*_{\eta\restriction i}})$" so by $(*)_4$ above $\mathbf{b}_2[\mathfrak{B}] \models$ "$v^*_\eta \cap \mathbf{f}(c_{\beta^*_{\eta\restriction i}}) - \mathbf{f}(c_{\alpha_{\eta\restriction i}}) = b_{\alpha^*_{\eta\restriction i}}$", contradicting the choice of $b_{\alpha^*_{\eta\restriction i}}$ as $\neq \mathbf{f}(a_{\alpha^*_{\eta\restriction i}})$.

B Stage: Under the assumption of Stage A, there are $\alpha_{\mathfrak{p}} < \lambda, N^* \prec \mathfrak{B}_{\alpha_{\mathfrak{p}}}, \mathrm{gen}(N^*) < \kappa, N \prec N^*$ and $\mathfrak{q}$ a 2-type definition over N^* such that for some club $E_{\mathfrak{p}}$ of λ for every $\alpha \in E_{\mathfrak{p}}$ and $x \in \mathfrak{B}$ realizing $\mathfrak{p}^{\mathfrak{B}_\alpha}$ the pair $\langle x, \mathbf{f}(x)\rangle$ realizes $\mathfrak{q}^{\mathfrak{B}_\alpha}$. $\square_{2.17}$

Proof. Let $W \subseteq \lambda$ be the set of $\alpha < \lambda$ such that: $N \prec \mathfrak{B}_\alpha$ and for some $q_\alpha \in \mathbb{S}^2(\mathfrak{B}_\alpha)$, we have: $x \in \mathfrak{B}$ realizes $\mathfrak{p}^{\mathfrak{B}_\alpha} \Rightarrow \langle x, \mathbf{f}(x)\rangle$ realizes q_α. Clearly if $N \prec \mathfrak{B}_\alpha, \alpha < \beta < \lambda, \alpha \in W$ and $\beta \in W$, then $q_\alpha \subseteq q_\beta$ (as if x realizes $\mathfrak{p}^{\mathfrak{B}_\beta}$ it realizes $\mathfrak{p}^{\mathfrak{B}_\alpha}$ too). Also W is a closed subset of λ. So if $\lambda = \sup(W)$ then W is a club of
{2.6A} λ and by 2.12 we can get $\mathfrak{q}$ such that $\{\alpha \in W : q_\alpha = \mathfrak{q}^{\mathfrak{B}_\alpha}\}$ is stationary, and as q_α (for $\alpha \in W$) increases with α, we get the desired conclusion. So assume $N \prec \mathfrak{B}_{\alpha(*)}, \sup(W) < \alpha(*) < \lambda$. Now for $\alpha \in [\alpha(*), \lambda)$ choose $x_\alpha, y_\alpha \in \mathfrak{B}$ realizing $\mathfrak{p}^{\mathfrak{B}_\alpha}$
{2.6A} such that $\mathrm{tp}(\langle x_\alpha, \mathbf{f}(x_\alpha)\rangle, \mathfrak{B}_\alpha, \mathfrak{B}) \neq \mathrm{tp}(\langle y_\alpha, \mathbf{f}(y_\alpha)\rangle, \mathfrak{B}_\alpha, \mathfrak{B})$. So by 2.12 for some N^* and type definition $\mathfrak{p}_0$ over $N^*, N^* \prec \mathfrak{B}$, we have $\mathrm{gen}(N^*) < \kappa$ and for stationarily many $\alpha \in S^-, \mathrm{cf}(\alpha) \geq \kappa$ and

$$(*)_\alpha \quad \mathrm{tp}(\langle x_\alpha, \mathbf{f}(x_\alpha), y_\alpha, \mathbf{f}(y_\alpha)\rangle, \mathfrak{B}_\alpha, \mathfrak{B}) = \mathfrak{p}_0^{\mathfrak{B}_\alpha}.$$

Note that for limit ordinal $\alpha, \langle x_\alpha, y_\alpha\rangle$ can serve as $\langle x_\beta, y_\beta\rangle$ for every large enough $\beta < \alpha$. So without loss of generality, by Fodor lemma, possibly increasing $\alpha(*)$, for every $[\alpha \in (\alpha(*), \lambda]$, the assertion $(*)_\alpha$ holds.

So without loss of generality for some $N^* \prec \mathfrak{B}_{\alpha(*)}, N \prec N^*, \mathrm{gen}(N^*) < \theta$ and

$$\alpha \in [\alpha(*), \lambda) \Rightarrow \mathrm{tp}(\langle x_\alpha, \mathbf{f}(x_\alpha)\rangle, N^*, \mathfrak{B}) \neq \mathrm{tp}(\langle y_\alpha, \mathbf{f}(y_\alpha)\rangle, N^*, \mathfrak{B}).$$

We can find a 3-type definition $\mathfrak{p}_1$ such that:

⊛ for every $\alpha \in [\alpha(*), \lambda)$ any (equivalently some) triple (a_1, a_2, a_3) realizing $\mathfrak{p}_1$ we have: $\mathrm{tp}(\langle a_1, a_2\rangle, \mathfrak{B}_\alpha, \mathfrak{B}) = \mathrm{tp}(\langle x_\alpha, \mathbf{f}(x_\alpha)\rangle, \mathfrak{B}_\alpha, \mathfrak{B})$ and $\mathrm{tp}(\langle a_1, a_3\rangle, \mathfrak{B}_\alpha, \mathfrak{B}) =$
{2.7p} $\mathrm{tp}(\langle y_\alpha, \mathbf{f}(y_\alpha)\rangle, \mathfrak{B}_\alpha, \mathfrak{B})$. (exists by **??**!!)
[*exists* by the existence of amalgamation of types and of non-splitting.]

Let $\mathfrak{p}_2$ be a type definition over N such that for every α, if c realizes $\mathfrak{p}_2^{\mathfrak{B}_\alpha}$ then $\mathfrak{B} \models$ "$c\dot{\epsilon}\mathbf{b}_1, c$ a finite union of atoms of $\mathbf{b}_1$", and if $\mathfrak{B}_\alpha \models$ "$x\dot{\epsilon}\mathbf{b}_1$ is a finite union
{2.6} of atoms of $\mathbf{b}_1$" then $\mathfrak{B} \models$ "$[\mathbf{b}_1 \models x \leq c]$" (use 2.11, used also in stage (A)). Now possibly increasing N^* we can find a type definition gr over N^* such that for every $\alpha > \alpha(*)$ there is c_α realizing $\mathfrak{p}_2^{\mathfrak{B}_\alpha}$ such that $\langle c_\alpha, \mathbf{f}(c_\alpha)\rangle$ realizes $\mathrm{gr}^{\mathfrak{B}_\alpha}$. Let E be thin enough club of λ. In particular for $\alpha \in [\alpha(*), \lambda)$ we have $c_\alpha, \mathbf{f}(c_\alpha) \in \mathfrak{B}_{\mathrm{Min}[E\setminus(\alpha+1)]}$.

Let $I = \Sigma\{I_i : i < \theta\}$, for non-limit iI_i is a countable dense linear order with a first element $s_{1,i}s_i$, no last element and $s_{1,i} < s \in I_i$. We choose by induction on $i < \theta$ for $\eta \in {}^{\theta>}\lambda, N_\eta, \alpha_\eta, < \alpha^*_\eta < \beta^*_\eta < \beta_\eta$ and β_η and $\otimes 0^1_{\eta,s}0^2_{\eta,s}$ for $s \in I_{\ell g(\eta)}$, but $0^1_{\eta,s} = 0^1_{\eta\restriction i,s}, 0^2_{\eta,s} = 0^2_{\eta\restriction i,s}$, if $\ell g(\eta) \geq i, s \in I_i)$ if $i = 0, \alpha_{<>} = \alpha(*) + 1, N_{<>} = N^*$, if $i = j + 1, \eta \in {}^j\lambda$, we choose $N_{\eta\hat{}<\epsilon>}$,and $\alpha_{\eta\frown\langle\epsilon\rangle} < \alpha^*_{\eta\frown\langle\epsilon\rangle} < \beta^*_{\eta\frown\langle\epsilon\rangle} < \beta_{\eta\frown\langle\epsilon\rangle}$ and $a^\ell_{\eta\frown\langle\epsilon\rangle,t}(t \in I_j)$ by induction on ϵ: arriving to ϵ let $\alpha_{\eta\hat{}<\epsilon>}$ be the minimal ordinal α such that:

$\alpha \in E \setminus \sup[\{\alpha^\ell_{\eta\hat{}<\zeta>} + 1 : \zeta < \epsilon \text{ and } \ell < \omega\} \cup \{\alpha_\eta, \sup\mathrm{rang}(\eta)\}]$.

Next choose $\alpha^*_{\eta\frown\langle\epsilon\rangle} < \alpha^1_{\eta\frown\langle\epsilon\rangle} < \alpha^2_{\eta\frown\langle\epsilon\rangle} < \alpha^3_{\eta\frown\langle\epsilon\rangle} < \alpha^4_{\eta\frown\langle\epsilon\rangle} < \alpha^5_{\eta\frown\langle\epsilon\rangle} < \beta^*_{\eta\frown\langle\epsilon\rangle} < \beta_{\eta\frown\langle\epsilon\rangle}$ from E and $a^1_{\eta\frown\langle\epsilon\rangle,s}a^2_{\eta\frown\langle\epsilon\rangle,s}, a^3_{\eta\frown\langle\epsilon\rangle,s}$ (for $s \in I_j$) be such that:

$(*)$

(a) $\alpha_{\eta\frown\langle\epsilon\rangle} < \alpha^*_{\eta\frown\langle\epsilon\rangle} \in E$

(b) $a^1_{\eta\frown\langle\epsilon\rangle,s_1,j} = x_{\alpha^1_{\eta\frown\langle\epsilon\rangle}}$ and $a^2_{\eta\frown\langle\epsilon\rangle,s_1,j} = h(x_{\alpha^1_{\eta\frown\langle\epsilon\rangle}})$ and $\langle a^1_{\eta\frown\langle\epsilon\rangle,s_1,j}, a^2_{\eta\frown\langle\epsilon\rangle,s_1,j}, a^3_{\eta\frown\langle\epsilon\rangle,s_1}\rangle$ is from $\mathfrak{B}_{\alpha^2_{\eta\frown\langle\epsilon\rangle}}$ and realizes $\mathfrak{p}_2^{\mathfrak{B}_{\alpha^1_{\eta\frown\langle\epsilon\rangle}}}$

(c) for $t \in (s_{1,j}, s_{2,j})_{I_j}$ the triple $\langle a^1_{\eta\frown\langle\epsilon\rangle,t}, a^2_{\eta\frown\langle\epsilon\rangle,t}, a^3_{\eta\frown\langle\epsilon\rangle,t}\rangle$ is from $\mathfrak{B}_{\alpha^3_{\eta\frown\langle\epsilon\rangle}}$ and realizes $\mathfrak{p}_2^{\mathfrak{B}'}$ where $\mathfrak{B}' = \mathrm{Sk}(\mathfrak{B}_{\alpha^2_{\eta\frown\langle\epsilon\rangle}} \cup \{a^\ell_{\eta\frown\langle\epsilon\rangle,s} : s < t, s \in I_j$ and $\ell = 1, 2, 3\})$

(d) $a^1_{\eta\frown\langle\epsilon\rangle,s_2,j} = y_{\alpha^3_{\eta\frown\langle\epsilon\rangle}}, a^2_{\eta\frown\langle\epsilon\rangle,s_2,j} = h(y_{\alpha^3_{\eta\frown\langle\epsilon\rangle}})$ the triple $\langle a^1_{\eta\frown\langle\epsilon\rangle,s_2,j}, a^2_{\eta\frown\langle\epsilon\rangle,s_2,j}, a^3_{\eta\frown\langle\epsilon\rangle,s_2,j}\rangle$ is from $\mathfrak{B}_{\alpha^4_{\eta\frown\langle\epsilon\rangle}}$ and realizes $\mathfrak{p}_2^{\mathfrak{B}_{\alpha^3_{\eta\frown\langle\epsilon\rangle}}}$

(e) for $t \in I_j$, above $s_{2,j}$ as in (c) replaces $(\alpha^2_{\eta\frown\langle\epsilon\rangle}, \alpha^3_{\eta\frown\langle\epsilon\rangle})$ by $(\alpha^4_{\eta\frown\langle\epsilon\rangle}, \alpha^5_{\eta\frown\langle\epsilon\rangle})$

(f) $N_{\eta\frown\langle\epsilon\rangle} = \mathrm{Sk}_{\mathfrak{B}}(N_\eta \cup \{a^\ell_{\eta\frown\langle\epsilon\rangle,s} : s \in I_j)$ and $\ell = 1, 2, 3\}$.

Lastly for i limit, $\eta \in {}^i\lambda, N_\eta := \bigcup_{j<i} N_{\eta\restriction j}, \alpha_\eta = \bigcup_{j<i} \alpha_{\eta\restriction j}$ and $\beta_\eta = \alpha_\eta + 1$.

Let M^* be a λ^+-saturated elementary extension of $\mathfrak{B}$. For any increasing sequence $\eta \in {}^\theta\lambda$ let $\delta_\eta = \cup\{\eta(i) : i < \theta\}$ and let $\bar{\zeta}_s = \langle a^1_{\eta,s}, a^2_{\eta,s}, a^3_{\eta,s}\rangle$ for $t \in I$ (recalling $a^\ell_{\eta,s} = a^\ell_{\eta\restriction j,s}$ if $s \in I_j$).

Now by 2.14 realizing the pair $(a^2_{\eta,s}, a^3_{\eta,s})$ as an element, for some (w_η, u_η) from {2.7A}
M^* we have:

⊛

(a) $M^* \models$ "w_η is finite set of atoms of $\mathbf{b}_1$ and $a^1_{\eta,t}\dot{\epsilon}w_\eta$" for $t \in I$

(b) $M^* \models$ "u_η is finite, $b \subseteq u_\eta$" whenever $\mathfrak{B}_{\delta_\eta} \models$ "b is finite"

(c) $M^* \models$ "$u_\eta \cap w_\eta = \emptyset$"

(d) $\langle (a^1_{\eta,s}, a^2_{\eta,s}, a^3_{\eta,s}) : s \in I\rangle$ is indiscernible in M over $\mathfrak{B}_{\alpha(*)} \cup \{u_\eta, w_\eta\}$.

Let $N^*_\eta = \mathrm{Sk}_{M^*}(N_\eta \cup \{u_\eta, w_\eta\}) \prec M$. So by 2.7II(d) there are $\delta \in S, \eta \in {}^\theta\delta$ increasing, such that $\delta = \sup \mathrm{rang}(\eta)$ and $g : \mathrm{Sk}_{M^*}(\mathfrak{B}_\delta \cup N^+_\eta) \to \mathfrak{B}_{\delta+1}$, an elementary embedding, $g \restriction \mathfrak{B}_\delta =$ the identity. {2.5}

We can consider w_η as a "member" of $\mathbf{b}_1$ so $\mathbf{f}(w_\eta)$ is well defined. Clearly for some every large enough $i < \theta$, the sequence $\langle (a^1_{\eta,s}, a^2_{\eta,s}, a^3_{\eta,s}) : s \in I_i \rangle$ is indiscernible inside $\mathfrak{B}$ over $\mathfrak{B}_{\alpha_{\eta\restriction i}} \cup \{w_\eta, \mathbf{f}(w_\eta), c_{\alpha_{\eta\restriction i}}, \mathbf{f}(c_{\alpha_{\eta\restriction i}}), c_{\beta^*_{\eta\restriction i}}, \mathbf{f}(c_{\beta^*_{\eta\restriction i}})\}$. As in the proof of stage A this gives a contradiction ?? (using [Sh:421, 2.7(e)].

<u>C Stage</u>: For some w^*:

(a) $\mathfrak{B} \models$ "w^* is a finite set"

(b) if $n < \omega, \bar{y} = \langle y_0, \ldots, y_{n-1} \rangle$ then: if $a, b, c \in \mathbf{b}^{\mathrm{at}}_1$ and $\bigwedge_\varphi (b,c) \emptyset^0_\varphi (a, h(a))$ then $c = h(b)$ where letting for all (first order) formula and $\varphi = \varphi(\bar{y}; z_1, ..., z_{n_\varphi})$,

$$\bar{y}_1 \emptyset^n_\varphi \bar{y}_2 := \quad (\forall z_1 \ldots z_{n_\varphi})[\bigwedge_{\ell=1}^{n_\varphi} z_\ell \dot{\in} w^* \to \varphi(\bar{y}_1, z_1, \ldots, z_{n_\varphi}) \equiv \varphi(\bar{y}_2, z_1, \ldots, z_{n_\varphi})]$$

so $\emptyset^n_\varphi$ is $\emptyset^n_\varphi(\bar{y}^*_1, \bar{y}^2, w^*)$.

□

Proof. By stage B if $N^* \prec \mathfrak{B}, \mathrm{gen}(N) < \kappa, \mathfrak{p}$ a type definition over N^*, (so "$[x \dot{\in} \mathbf{b}^{\mathrm{at}}_1]$" $\in \mathfrak{p}^{N^*}$) then there is $\alpha_{\mathfrak{p}}$ as there.

So as

$$[\alpha < \lambda \Rightarrow |\alpha|^{<\kappa} + 2^{2^{|T|^*}} + \sum_{\partial < \kappa} 2^{2^\partial} < \lambda]$$

the set E_0 is a club of λ when we let $E_0 = \{\delta < \lambda : \delta$ is a limit ordinal and if $\alpha < \delta, N^* \prec \mathfrak{B}_\alpha, \mathrm{gen}(N^*) < \kappa, \mathfrak{p}$ 1-type definition over N^* such that "$x \dot{\in} \mathbf{b}^{\mathrm{at}}_1$" $\in \mathfrak{p}^{N^*}$ and the parameters in the definition of $\mathbf{b}^{\mathrm{at}}_1$ are in $\mathfrak{B}_\delta$ then $\delta > \alpha_{\mathfrak{p}}\}$.

Choose $\delta(*) \in E_0 \setminus S$ such that $\mathrm{cf}(\delta(*)) = \kappa$.

Easily: if $a \in \mathbf{b}^{\mathrm{at}}_1[\mathfrak{B}]$ then

$(*)_1$ y realizes in $\mathfrak{B}$ the type $\mathrm{tp}(\mathbf{f}(a), \mathfrak{B}_\delta(*) \cup \{a\}, \mathfrak{B}) \Rightarrow y = \mathbf{f}(a)$

$(*)_2$ $\langle b, c \rangle$ realizes in $\mathfrak{B}$ the type $\mathrm{tp}(\langle a, \mathbf{f}(a) \rangle, \mathfrak{B}_{\delta(*)}, \mathfrak{B}) \Rightarrow c = \mathbf{f}(b)$.

Let $w^* \in \mathfrak{B}$ be a pseudo finite member of $\mathbf{b}_1$ "including" all pseudo finite members
{2.6} of $\mathfrak{B}_{\delta(*)}$ (see 2.11); check it is as required.

noindent <u>D Stage</u>: There is a member $\mathbf{e}_0$ of $\mathfrak{B}$ such that:

(α) $\mathfrak{B} \models$ "$\mathbf{e}_0$ is an equivalence relation over $\mathbf{b}^{\mathrm{at}}_1$, the set of atoms of $\mathbf{b}_1$, with finitely many equivalence classes"
and

(β) for each (first order) formula $\varphi = \varphi(y, \bar{z})$ in $\mathfrak{B}, \mathbf{e}_0$ refines the equivalence relation $\emptyset^1_\varphi$ (for the parameter w^* from stage C) recalling that $\emptyset'_\varphi$ is defined as:

$$a \, \emptyset^1_\varphi b := (\forall \bar{z} \subseteq w^*)[\varphi(a, \bar{z}) \equiv \varphi(b, \bar{z})].$$

□

Proof. By stage C.

As $\mathfrak{B}$ is κ-saturated hence $\aleph_0$-saturated (and as $\mathbf{b}_1$ is represented in $\mathfrak{B}$ not just definable there).

E Stage: There is $s^*, \mathfrak{B} \models$ "$\dot{m}^*$ a natural number" and a member $\dot{f}$ of $\mathfrak{B}$ such that: $\mathfrak{B} \models$ "$\dot{f}$ is a function, $\mathrm{Dom}(\dot{f}) = \mathbf{b}_1^{\mathrm{at}}, \dot{f}(x) = \langle \dot{f}_{\dot{m}}(x) : \dot{m} < \dot{m}^* \rangle$ and $x\mathbf{e}_0 y \to \dot{f}(x) = \dot{f}(y)$", and for each $x\mathbf{e}_0$, (where $x \in \mathbf{b}_1^{\mathrm{at}}[\mathfrak{B}]$, of course) there is an $\dot{m} = \dot{m}_{x/\mathbf{e}_0}$ such that:

⊛ if $\mathfrak{B} \models$ "$x\mathbf{e}_0 y$ and $\dot{f}_{\dot{m}}(y) = z$" then $z = h(y)$.

[Note: we did not say yet that we can find in $\mathfrak{B}$ such a choice function $\langle \dot{m}_{x/\mathbf{e}_0} : x\dot{\epsilon}\mathbf{b}_1^{\mathrm{at}} \rangle$, a pseudo finite sequence]. □

Proof. There is $\mathbf{e}^2 \in \mathfrak{B}$, considered by $\mathfrak{B}$ as an equivalence relation over $\mathbf{b}_1^{\mathrm{at}} \times \mathbf{b}_2$ with finitely many equivalence classes, refining each $\emptyset^2_\varphi$ for $\varphi = \varphi(x, y, \bar{z})$ where:

$$(x_1, y_1)\emptyset^2_\varphi(x_2, y_2) \Leftrightarrow (\forall \bar{z}, \ldots)[\bigwedge_{\ell < \ell g(\bar{z})} z^\ell \dot{\epsilon} w^* \to \varphi(x_1, y_1, z^1, \ldots) \equiv \varphi(x_2, y_2, \bar{z}, \ldots)]$$

(exists as in Stage D). Now each $\mathbf{e}^2$-equivalence class A defines a function $\dot{f}_A$:

$$\dot{f}_A(x) = y : \quad \begin{array}{l} \text{if possible } (x, y) \in A \text{ and } (\forall y')[(x, y)\mathbf{e}^2(x, y') \to y = y'] \\ \text{if not possible } y = 0. \end{array}$$

Now clearly $\dot{f}$ exists in $\mathfrak{B}$ and $\dot{f}$ is as required.

F Stage: First there is $\mathbf{e}_1 \in \mathfrak{B}, \mathfrak{B} \models$ "$\mathbf{e}_1$ an equivalence relation on $\mathbf{b}_1^{\mathrm{at}}$ refining $\mathbf{e}_0$ with finitely many equivalence classes" such that $\mathbf{B} \models$ "$x\mathbf{e}_1 y$" implies x, y realizes same type over $w^* \cup \{\dot{k} : \mathfrak{B} \models \dot{k} < \dot{m}^*\} \cup \{v : \mathfrak{B} \models \text{"}v \subseteq \{\dot{k} : \dot{k} < \dot{m}^*\}\text{"}\} \cup \{\dot{f}\}$ (those sets — in the sense of $\mathfrak{B}$ and are considered by it finite).

Hence if $x, y \in \mathbf{b}_1^{\mathrm{at}}[\mathbf{B}], x\mathbf{e}_1 y$ then

$$\bigwedge_{\dot{m}_1, \dot{m}_2 < \dot{m}^*} [\dot{f}_{\dot{m}_1}(x) = \dot{f}_{\dot{m}_2}(x) \Leftrightarrow \dot{f}_{\dot{m}_1}(y) = \dot{f}_{\dot{m}_2}(y)]$$

and

$$\bigwedge_{\dot{m} < \dot{m}^*} [\mathbf{f}(x) = \dot{d}_{\dot{m}}(x) \Leftrightarrow \mathbf{f}(y) = \dot{f}_{\dot{m}}(y)]$$

(remember the choice of w^* in stage D) and more generally

⊛ for every $w \subseteq \{\dot{m} : \dot{m} < \dot{m}^*\}$, in $\mathfrak{B}$'s sense, we have:

$$x\dot{\epsilon}_1 y \Rightarrow [\bigcap_{\dot{m}\dot{\epsilon}w} \dot{f}_{\dot{m}}(x) - \bigcup_{\dot{m} < \dot{m}^*, \neg \dot{m}\dot{\epsilon}w} \dot{f}_{\dot{m}}(x) = 0_{\mathbf{b}_2} \ \underline{\text{if and only if}} \ \bigcap_{\dot{m}dot\epsilon w} \dot{f}_{\dot{m}}(y) - \bigcup_{\dot{m} < \dot{m}^*, \neg \dot{m}\dot{\epsilon}w} \dot{f}_{\dot{m}}(y) = 0_{\mathbf{b}_2}].$$

Let $A = \{x \in \mathbf{b}_1^{\rm at} : \mathfrak{B} \models$ "$(x/\mathbf{e}_1)$ has $> [\mathbf{b}_1^{\rm at}/\mathbf{e}_1]!!!$ members[10]"$\}$; so A is definable in $\mathfrak{B}$ hence is represented in $\mathfrak{B}$, so we consider it a member of $\mathfrak{B}$, clearly $\mathfrak{B}$ considers $\mathbf{b}_1^{\rm at} \setminus A$ a finite set. Let in $\mathfrak{B}$ the sequence $\langle A_{\dot n} : \dot n < \dot n^* \rangle$ be a list of the $\mathbf{e}_1$-equivalence classes $\subseteq A$.

Let in $\mathfrak{B}, x^*$ be the union of the atoms from $\mathbf{b}_1^{\rm at} \setminus A$, (it is pseudo finite). So it
{2.8} suffices for proving 2.16 to find in $\mathfrak{B}$ a definition of $h \restriction \{x \in \mathbf{b}_1^{\rm at}[\mathfrak{B}] : x \cap x^* = 0\}$, as required. So it suffice to prove that the following sequence is in $\mathfrak{B}$: find in $\mathfrak{B}$ a sequence $\langle \dot m_{\dot n} : \dot n < \dot n^* \rangle$, where $\dot m_{\dot n} < \dot m^*$ and: for every $x \in A_{\dot n}, h(x_{\dot n}) = \dot f_{\dot m_{\dot n}}(x)$. We already know that for each $\dot n < \dot n^*$ there is such $\dot m_{\dot n}$. This is done in stage G.

G Stage: We shall reconstruct in $\mathfrak{B}$ a sequence $\langle \dot m_{\dot n} : \dot n < \dot n^* \rangle$ as above (hence $h \restriction A^{\mathfrak{B}}$ is definable.)

We can find in $\mathfrak{B}, \bar w, \dot k^*$ and a function $\dot g, \mathrm{Dom}(\dot g) = \mathbf{b}_1^{\rm at}[\mathfrak{B}]$ such that

$$\begin{aligned} \mathfrak{B} \models \quad & \text{"for } x \in \mathbf{b}_1^{\rm at} : \dot g(x) \text{ is } \langle a^x_{\dot k} : \dot k < \dot k^* \rangle, \\ & \dot k^* \text{ the natural number } 2^{\dot m^*} - 1, \\ & \bar w = \langle w_{\dot k} : \dot k < \dot k^* \rangle \text{ list the non-empty subsets of } \{\dot m : \dot m < \dot m^*\} \\ & \text{ and } a^x_{\dot k} \dot\in \mathbf{b}_2 \text{ is } \cap \{\dot f_{\dot m}(x) : \dot m \dot\in w_{\dot k}\} \setminus \cup \{\dot f_{\dot m}(x) : \dot m < \dot n^* \\ & \text{ and } \neg(\dot m \dot\in w_{\dot k})\}\text{" in } \mathbf{b}_2\text{'s sense.} \end{aligned}$$

As each $A_{\dot n}$ is large enough (i.e. $\mathfrak{B} \models$ "$|A_{\dot n}| > |\mathbf{b}_1^{\rm at}/\mathbf{e}_1|!!!$") there is in $\mathfrak{B}$ a sequence $\langle x(\dot n, \dot \ell) : \dot n < \dot n^*, \dot \ell < \dot \ell^* \rangle$ such that ($\mathfrak{B}$ satisfies):

- (α) $x(\dot n, \dot \ell) \dot\in A_{\dot n} \subseteq \mathbf{b}_1^{\rm at}, \dot \ell^*$ natural number
- (β) $\dot n < \dot n^*$ and $\dot \ell_1 < \dot \ell_2 < \dot \ell^* \Rightarrow \mathbf{b}_1 \models$ "$x(\dot n, \ell_1) \neq x(\dot n, \ell_2)$" (hence they are disjoint as members of $\mathbf{b}_1$)
- (γ) if $\dot n_1, \dot n_2 < \dot n^*$ and $\dot k_1, \dot k_2 < \dot k^*$ and there are x^1, x^2 satisfying "$x^1 \dot\in A_{\dot n_1} \setminus \{x(\dot n_1, \dot \ell) : \dot \ell < \dot \ell^*\}$ and $x^2 \dot\in A_{\dot n_2} \setminus \{x(\dot n_2, \dot \ell) : \dot \ell < \dot \ell^*\}$", and $\mathbf{b}_2 \models$ "$a^{x^1}_{\dot k_1} \cap a^{x^2}_{\dot k_2} \neq 0_{\mathbf{b}_2}$", <u>then</u> for some "even" $\dot \ell < \dot \ell^*$ (in $\mathfrak{B}$'s sense) $\mathbf{b}_2 \models$ "$a^{x(\dot n_1, \dot \ell)}_{\dot n_1} \cap a^{x(\dot n_2, \dot \ell + 1)}_{\dot k_2} \neq 0_{\mathbf{b}_2}$".

Note: by $(\alpha) + (\beta) : x(\dot n_1, \dot \ell_1) = x(\dot n_2, \dot \ell_2) \Rightarrow \dot n_1 = \dot n_2$ and $\dot \ell_1 = \dot \ell_2$; so the $x(\dot n, \dot \ell)$ are pairwise disjoint in $\mathbf{b}_1$'s sense.

We can find in $\mathfrak{B}$ elements y_0, y_1 such that: $\mathfrak{B} \models$ "$y_0 \dot\in \mathbf{b}_1, y_0 = \cup\{x(\dot n, \dot \ell) : \dot n < \dot n^*$ and $\dot \ell < \dot \ell^*$ even$\}, y_1 \in \mathbf{b}_1, y_1 = \cup\{x(\dot n, \dot \ell) : \dot n < \dot n^*$ and $\dot \ell < \dot \ell^*$ is odd$\}$".

Let $z_m = h(y_m)$ for $m = 0, 1$ (note: h is not assumed to be definable in $\mathfrak{B}$, but we can use z_0, z_1).

Hence

$\otimes_1$ in $\mathfrak{B}$ for $\dot \ell < \dot \ell^*, m < 2$ and $\dot \ell = m \mod 2$ we have $x(\dot n, \dot \ell) \leq y_m$; hence: $\mathbf{f} \restriction A_{\dot n} = \dot f_{\dot m}[\mathfrak{B}] \restriction A_{\dot n} \Rightarrow \mathfrak{B} \models$ "$\dot f_{\dot m}(x(\dot n, \dot \ell)) \leq z_m \Rightarrow \bigwedge\limits_{\dot k < \dot k^*, \dot m \dot\in w_{\dot k}} a^{x(\dot n, \dot \ell)}_{\dot k} \leq z_m$".

But $\mathbf{b}_1[\mathfrak{B}] \models$ "$y_0 \cap y_1 = 0_{\mathbf{b}_1}$" hence $\mathbf{b}_2[\mathfrak{B}] \models$ "$\mathbf{f}(y_0) \cap \mathbf{f}(y_1) = 0_{\mathbf{b}_1}$" hence $\mathbf{b}_2 \models$ "$z_0 \cap z_1 = 0_{\mathbf{b}_2}$" hence

[10]Note: ! is factorial; we could just as well say "finite large enough" and did not bother to make the exact computation.

$\otimes_2$ in $\mathfrak{B}$, for $\dot\ell < \dot\ell^*, m < 2, \dot\ell \neq m \mod 2$ we have $x(\dot n, \dot\ell) \cap y_m = 0_{\mathbf{b}_1}$; hence:

$$\begin{aligned} \mathbf{f}\restriction A_{\dot n} = \dot f_{\dot m}[\mathfrak{B}]\restriction A_{\dot n} \quad &\Rightarrow \mathfrak{B} \models \dot f_{\dot m}(x(\dot n, \dot\ell)) \cap z_m = 0_{\mathbf{b}_1} \\ &\Rightarrow \mathfrak{B} \models \text{ "if } \dot k < \dot k^*, \neg \dot m \dot\epsilon w_{\dot k} \text{ then } a_{\dot k}^{x(\dot n, \dot\ell)} \cap z_m = 0_{\mathbf{b}_2}\text{"}. \end{aligned}$$

Note also

$\otimes_3$ $h\restriction A_{\dot n} = \dot f_{\dot m}\restriction A_{\dot n}$ implies:

(a) for every $x \in A_{\dot n}, \dot f_{\dot m}(x) \neq 0_{\mathbf{b}_2}$

(b) for every $x \neq y \in A_{\dot n}, \mathbf{b}_2[\mathfrak{B}] \models$ "$\dot f_{\dot m}(x) \cap \dot f_{\dot m}(y) = 0$"

(c) for every $x \in A_{\dot n}, \mathbf{b}_2[\mathfrak{B}] \models$ "$\dot f_{\dot m}(x) \cap \mathbf{f}(x^*) = 0_{\mathbf{b}_2}$".

Now the conclusions of $\otimes_1, \otimes_2, \otimes_3$ can be viewed as properties of $(\dot n, \dot m)$ which follows from "$h\restriction A_{\dot n} = \dot f_{\dot m}\restriction A_{\dot n}$", they can be expressed by first order formulas in $\mathfrak{B}$.

Hence

$$Y = \{(\dot n, \dot m) : \ \dot n < \dot n^* \text{ and } \dot m < \dot m^* \text{ and } (\dot n, \dot m) \text{ satisfies the conclusions of } \otimes_1, \otimes_2, \otimes_3\}$$

is a pseudo finite set in $\mathfrak{B}$, and so we can define in $\mathfrak{B}, \dot f' \in \mathfrak{B}$ such that $\mathfrak{B} \models$ "$\dot f'$ is the function $\dot f' : A \to \mathbf{b}_2$ defined by $\dot f'(x) = \cup\{\dot f_{\dot m}(x) : x \dot\epsilon A_{\dot n}$ and $(\dot n, \dot m) \in Y\}$" (union in $\mathfrak{B}$'s sense). If for every $x \in A[\mathfrak{B}]$ ($\subseteq \mathbf{b}_1^{\rm at}[\mathfrak{B}]$), we have $\dot f'(x) \leq \mathbf{f}(x)$ then by $\otimes_1, \otimes_2, \otimes_3$ themselves equality holds and we finish. So it is enough to assume $(\dot n, \dot m)$ satisfies those three properties but $(\exists x \in A_{\dot n})(\bigwedge_{\dot\ell < \dot\ell^*} x \neq x(\dot n, \dot\ell)$ and $\dot f_{\dot m}(x) \nleq_{\mathbf{b}_1} \mathbf{f}(x)$ and eventually get contradiction.

Let $x \in A_{\dot n}[\mathfrak{B}] \setminus \{x(\dot n, \dot\ell) : \dot\ell < e^*\}$ and $\dot f_{\dot m}(x) \nleq_{\mathbf{b}_1} \mathbf{f}(x)$. As $A \cup \{x^*\}$ is a maximal antichain of $\mathbf{b}_1$, clearly $\{\mathbf{f}(x) : \mathfrak{B} \models$ "$x \dot\epsilon A \subseteq \mathbf{b}_1^{\rm at}\} \cup \{\mathbf{f}(x^*)\}$ is a maximal antichain of $\mathbf{b}_2$, and $x \in A[\mathfrak{B}] \Rightarrow \mathbf{f}(x) \neq 0_{\mathbf{b}_2}$. Now for every $x \in A_{\dot n}[\mathfrak{B}], \dot f_{\dot m}(x) \in \mathbf{b}_2[\mathfrak{B}] \setminus \{0_{\mathbf{b}_2}\}$ (by $\otimes_3(a)$) and $\dot f_{\dot m}(x) \cap \mathbf{f}(x^*) = 0_{\mathbf{b}_2}$ (by $\otimes_3(c)$) and as $\mathbf{f}$ is a complete embedding of $\mathbf{b}_1[\mathfrak{B}]$ into $\mathbf{b}_2[\mathfrak{B}]$, by the previous sentence necessarily for some $y \in A[\mathfrak{B}], \mathbf{b}_2[\mathfrak{B}] \models$ "$(\dot f_{\dot m}(x) - \mathbf{f}(x)) \cap \mathbf{f}(y) \neq 0_{\mathbf{b}_2}$" so $y \neq x$.

Lastly, let $\dot n^1 < \dot n^*$ be such that $y \in A_{\dot n^1}[\mathfrak{B}]$. Let $\dot m_1$ be such that $x \in A_{\dot n^1}[\mathfrak{B}] \Rightarrow \mathbf{f}(x) = \dot f_{\dot m_1}(x)$.

By the choice of $\bar w$ and of $\langle a_{\dot k}^z : \dot k < \dot k^* \rangle$ (for $z \in \mathbf{b}_1^{\rm at}[\mathfrak{B}]$) there are $\dot k, \dot k_1 < \dot k^*$ such that $\mathfrak{B} \models$ "$\mathbf{b}_2 \models$ "$\dot f_{\dot m}(x) \cap \mathbf{f}(y) \geq a_{\dot k}^x \cap a_{\dot k_1}^y - \mathbf{f}(x) \neq 0_{\mathbf{b}_2}$". By the choice of $\bar w$ and of $\langle a_{\dot k}^z : \dot k < \dot k^* \rangle$ necessarily $\mathbf{b}_2 \models$ "$\dot f_{\dot m}(x) \geq a_{\dot k}^x > 0_{\mathbf{b}_2}$"; also $\mathbf{f}(y)$ is $\dot f_{\dot m_1}(y)$ and as for x we get $\mathbf{f}(y) = \dot f_{\dot m_1}(y) \geq a_{\dot k_1}^y > 0_{\mathbf{b}_2}$. Clearly $\dot f_{\dot m}(x) \geq a_{\dot k}^x > 0_{\mathbf{b}_2}$ implies $\mathfrak{B} \models \dot m \dot\epsilon w_{\dot k}$; and $\mathbf{f}(y) = \dot f_{\dot m_1}(y) \geq a_{\dot k_1}^y > 0$ implies $\mathfrak{B} \models \dot m_1 \dot\epsilon w_{\dot k_1}$.

By clause (γ) in the choice of the $x(\dot n, \dot\ell)$'s, we know there are $\dot\ell, \dot\ell^1 < \dot\ell^*$ such that $\mathfrak{B} \models$ "$\dot\ell \neq \dot\ell^1 \mod 2$" and $\mathbf{b}_2[\mathfrak{B}] \models$ "$a_{\dot k}^{x(\dot n, \ell)} \cap a_{\dot k_1}^{x(\dot n^1, \dot\ell^1)} \neq 0_{\mathbf{b}_2}$".

Let $m < 2$ be such that $m = \dot\ell \mod 2$, then by the conclusion of $\otimes_1$ (for $\dot\ell, \dot n, \dot m$) as $\mathfrak{B} \models$ "$\dot m \dot\epsilon w_{\dot k}$" we have $\mathbf{b}_2[\mathfrak{B}] \models$ "$a_{\dot k}^{x(\dot m, \dot\ell)} \leq z_m$". By $\otimes_2$ applied to $\dot n^1, \dot\ell^1, \dot m_1$, we have $\oplus_3$ we have $\mathbf{b}_2[\mathfrak{B}] \models$ "$a^{x(\dot n^1,\dot\ell^1)}_{\dot k_1} \cap z_m = 0_{\mathbf{b}_2}$" hence by the last sentence $\mathbf{B}_2 \models a^{x(\dot n,\dot\ell)}_{\dot k} \cap a^{x(\dot n^1,\dot\ell^1)} = 0_{\mathbf{b}_2}$.

This contradicts the previous paragraph so we are done. $\square$

{2.8B} **Observation 2.18.** *Assume $\mathfrak{B} \models$ "$\mathbf{b}_1$ is an atomic Boolean ring and $\mathbf{b}_2$ is a Boolean ring" and $\mathbf{f}$ is a complete embedding of $\mathbf{b}_1[\mathfrak{B}]$ into $\mathbf{b}_2[\mathbf{B}]$.*
1) If $\mathbf{f}{\restriction}\mathbf{b}^{\rm at}_1[\mathfrak{B}]$ is definable in $\mathbf{B}$ then so is $\mathbf{f}$.
2) If $x^*_1 \in \mathbf{b}_1[\mathfrak{B}], x^*_2$ and $\mathbf{f}{\restriction}\{x \in \mathbf{b}^{\rm at}_1[\mathfrak{B}] : x$ disjoint to x^*_1 (in $\mathbf{b}_1[\mathfrak{B}])\}$ is definable in $\mathbf{B}$ <u>then</u> so is $\mathbf{f}{\restriction}\{x \in \mathbf{b}_1[\mathfrak{B}] : x$ disjoint to x^*_1 (in $\mathbf{b}_1[\mathfrak{B}])\}$.

Proof. Easy. $\square_{2.18}$

{2.9} **Claim 2.19.** *Assume*

{2.8} (a) *$\mathbf{b}_1$ is a Boolean ring in $\mathfrak{B}$ (as in 2.16(a)) i.e. $\mathfrak{B} \models$ "$\mathbf{b}_1$ is a Boolean ring"*
(b) *$\mathbf{b}_2$ is a Boolean ring in $\mathfrak{B}$*
(c) *$\mathbf{f}$ is a complete embedding of $\mathbf{b}_1[\mathfrak{B}]$ into $\mathbf{b}_2[\mathfrak{B}]$.*

<u>Then</u> for some $x \in \mathbf{b}_1[\mathfrak{B}]$:

(α) *$\mathfrak{B} \models$ "x is finite i.e. finite union of atoms (of $\mathbf{b}_1$) or is zero" (so if $\mathbf{b}_1$ is atomless then $x = 0_{\mathbf{B}_1}$)*
(β) *$\mathbf{f}{\restriction}(\mathbf{b}^{\mathfrak{B}}_1{\restriction}\{y : y \cap x = 0_{\mathbf{b}_1}\})$ is definable in $\mathfrak{B}$.*

Proof. Without loss of generality $\mathbf{b}_1$ is atomless.
{2.8} [Why? Otherwise apply the proof below to $\mathbf{b}'_1 = \{x \in \mathbf{b}_1 : x$ atomless$\}$ and
{2.8B} 2.16+2.18 to

$$\mathbf{b}''_1 = \{\dot\epsilon\mathbf{b}_1 : \ x \text{ is atomic (i.e. below every non-zero } y \le_{\mathbf{b}_1} x \text{ there is an atom of } \mathbf{b}_1)\}.$$

Now from definition of $\mathbf{f}{\restriction}\mathbf{b}'_1[\mathfrak{B}]$ and of $\mathbf{f}{\restriction}\mathbf{b}''_1[\mathfrak{B}]$ we can define $\mathbf{f}$].

In $\mathfrak{B}$ let $\mathbb{P} = \{\Xi : \Xi$ is a maximal antichain of $\mathbf{b}_1$ (in particular $0_{\mathbf{b}_1} \notin \Xi)\}$. Now $\mathbb{P}$ is partially ordered by: $\Xi_1 \le_{\mathbb{P}} \Xi_2$ iff $(\forall x \in \Xi_2)(\exists y \in \Xi_1)(x \le y)$ (i.e. Ξ_2 is "finer").
{2.6} Clearly $(\mathbb{P}, \le_{\mathbb{P}})$ is a directed set definable in $\mathfrak{B}$ so by 2.11 for every $\alpha < \lambda$ there is $\Xi_\alpha \in \mathbb{P}[\mathfrak{B}]$, which is in $(\mathbb{P}, \le_{\mathbb{P}})[\mathfrak{B}]$ an upper bound of $\mathfrak{B}_\alpha \cap \mathbb{P}$. In $\mathfrak{B}$, for each $\Xi\dot\epsilon\mathbb{P}$ let $\mathbf{b}^1_\Xi = \{x\dot\epsilon\mathbf{b}_1$: for every $y\dot\epsilon\Xi, y \le x \vee y \cap x = 0\}$. So $\mathbf{b}^1_\Xi[\mathfrak{B}]$ is a Boolean subring of $\mathbf{b}_1[\mathfrak{B}]$, even a complete subring, hence $\mathbf{f}{\restriction}\mathbf{b}^1_\Xi[\mathfrak{B}]$ is a complete embedding of $\mathbf{b}^1_\Xi[\mathfrak{B}]$
{2.8} into $\mathbf{b}_2[\mathfrak{B}]$. So by 2.16 there are a_Ξ and $\varphi(x, y, \bar c_\Xi)$ such that $\mathfrak{B} \models$ "a_Ξ is a finite
{2.8B} union of members of Ξ" and $\varphi_\Xi(-,-,\bar c_\Xi)$ define $\mathbf{f}{\restriction}\{x\dot\epsilon\Xi : x \cap a_\Xi = 0\}$. Hence by 2.18 $\mathbf{f}{\restriction}\mathbf{b}^1_\Xi[\mathfrak{B}]$ being a complete embedding is defined too outside a_Ξ so without loss of generality $\varphi(-,-,\bar c_\Xi)$ defined $\mathbf{f}{\restriction}\{x\dot\epsilon\mathbf{b}^1_\Xi : x \cap a_\Xi = 0\}$.
For $\alpha \in S^-$ satisfying ${\rm cf}(\alpha) \ge \kappa$, we have Ξ_α as above (i.e. an upper bound of $\mathfrak{B}_\alpha \cap \mathbb{P}$) hence we have $a_{\Xi_\alpha}, \varphi_{\Xi_\alpha}(-,-,\bar c_{\Xi_\alpha})$ as above, so for some $N_\alpha \prec \mathfrak{B}_\alpha, {\rm gen}(N_\alpha) < \kappa$, we have $\mathfrak{B}_\alpha \biguplus^{\mathfrak{B}}_{N} \langle a_{\Xi_\alpha}\rangle\hat{\ }\bar c_{\Xi_\alpha}$, and so by 2.12 for some model $N^* \prec \mathfrak{B}, {\rm gen}(N^*) < \kappa$,
{2.6A} formula φ and type definition $\mathfrak{p}$ over N^* we have:

$$S' = \{\alpha : \ \alpha < \lambda, \alpha \notin S, {\rm cf}(\alpha) \ge \theta, N_\alpha = N^*, \varphi_{\Xi_\alpha} = \varphi \text{ and } {\rm tp}(a_{\Xi_\alpha}\hat{\ }\bar c_{\Xi_\alpha}, \mathfrak{B}_\alpha, \mathfrak{B}) = \mathfrak{p}^{\mathfrak{B}_\alpha}\}$$

is stationary.

Now if $a \in \mathbf{b}_1[\mathfrak{B}], b = \mathbf{f}(a)$ and $\alpha \in S'$ is large enough then

$(*)^\alpha_{a,b}$ $\mathfrak{B} \models$ "a $\dot{\epsilon}\mathbf{b}^1_{1\Xi_\alpha}\varphi(a - a_{\Xi_\alpha}, b - \mathbf{f}(a_{\Xi_\alpha}), \bar{c}_{\Xi_\alpha})$"

so we can read this from $\mathfrak{p}$; i.e. this can be determined from $\mathrm{tp}(\langle a, b\rangle, N^*, \mathfrak{B})$. On the other hand assume $a \in \mathbf{b}_1[\mathfrak{B}], b \in \mathbf{b}_2[\mathfrak{B}], b \neq \mathbf{f}(a)$ and $(*)^\alpha_{a,b}$ holds for α sufficiently large and we shall get a contradiction.

Note:

(a) any non-zero member x of $\mathbf{b}_1[\mathfrak{B}_\alpha]$ is not included is any finite (even in $\mathfrak{B}$ sense) union of members of Ξ_α in particular in a_{Ξ_α}.

[Why? By our assumptions on T^* and $\mathbf{b}_1$ as $\mathbf{b}_1[\mathfrak{B}]$ is atomless, there is $\Xi \in \mathfrak{B}_\alpha$ such that $\mathfrak{B}_\alpha \models$"$\Xi$ is a maximal antichain of $\mathbf{b}_1$, every member of Ξ is below x or is disjoint to x in $\mathbf{b}_1$, and x is not a finite union of members of Ξ". By the choice of Ξ_α we have $\mathfrak{B} \models$ "$\Xi \leq \Xi_\alpha$", so we are done.]

(b) if $(\mathfrak{B}_\alpha, \mathbf{f}{\restriction}\mathfrak{B}_\alpha) \prec (\mathfrak{B}, \mathbf{f}), d \in (\mathbf{b}_2)^{[\mathfrak{B}_\alpha]} \setminus \{0_{\mathbf{b}_2}\}$ <u>then</u> d is not included in any member of $\Xi'_\alpha := \{\mathbf{f}(c) : c \in \Xi_\alpha$ or is just a pseudo finite union of members of Ξ_α in $\mathfrak{B}$ and $c \cap a_{\Xi_\alpha} = 0_{\mathbf{b}_1}\} \cup \{\mathbf{f}(a_{\Xi_\alpha})\}$.

[Why clause (b)? As $\mathbf{f}$ is complete embeddings of $\mathbf{b}_1[\mathfrak{B}]$ into $\mathbf{b}_2[\mathfrak{B}]$ there is a $c^* \in (\mathbf{b}_1)^{[\mathfrak{B}_\alpha]} \setminus \{0_{\mathbf{b}_2}\}$ such that:

$$0_{\mathbf{b}_1} < c' \leq c^* \Rightarrow \mathbf{b}_1[\mathfrak{B}] \models \mathbf{f}(c') \cap d \neq 0_{\mathbf{b}_2}.$$

Now $\mathfrak{B} \models$ "there are infinitely many $c \in \Xi_\alpha$ which are $\leq c^*$ and $d' \in \Xi_\alpha \Rightarrow d' \cap \mathbf{f}(c^*) \in \{d', 0_{\mathbf{b}_2}\}$", and so $\mathfrak{B} \models$ "$c\dot{\epsilon}\Xi_\alpha, c \cap a_\alpha = 0_{\mathbf{b}_1}, \mathbf{f}(c) \cap d \neq 0_{\mathbf{b}_2}$ for infinitely many c".]

So via $\mathfrak{p}$ we get a definition of $\mathbf{f}$, but not first order, just according to type over N.

The desired conclusion follows by the following claim 2.20. $\square_{2.19}$ {2.10}

Claim 2.20. *Assume $\mathfrak{B}$ is a κ-saturated model of a complete first order theory $T, \mathbf{b}_1, \mathbf{b}_2$ are Boolean rings definable in $\mathfrak{B}$, and $\mathbf{f}$ is a complete embedding of $\mathbf{b}_1[\mathfrak{B}]$ into $\mathbf{b}_2[\mathfrak{B}]$ definable in $\mathfrak{B}$ by an $\mathbb{L}_{\infty,\kappa}$-formula with $< \kappa$ parameters (equivalently, for some $\bar{c} \in {}^{\kappa>}M, \mathrm{tp}(\langle a_1, b_1\rangle, \bar{c}, \mathfrak{B}) = \mathrm{tp}(\langle a_2, b_2\rangle, \bar{c}, \mathfrak{B})\, \mathbf{f}(a_1) = b_2 \Rightarrow \mathbf{f}(a_2) = b_2$). <u>Then</u> $\mathbf{f}$ is definable in $\mathfrak{B}$.* {2.10}

Remark 2.21. 1) Similar to [Sh:72, 1.9.1], [Sh:107, 4.10].
2) For the purposes of this chapter alone we can assume $\mathbf{b}_1$ is atomless, so the reader can read the proof this way, simplifying somewhat. Still our theorem 2.19 {2.9} seemingly is weaken compared to [Sh:72, 107]: the use of T^*, i.e. the expansion of the theory, but see [Sh:F503]; for such generalization see 3.5 we shall need 2.20. {3.40} {2.10}
3) This claim is more than needed in 2.19. {2.9}

Proof. Without loss of generality $\mathfrak{B}$ is μ-saturated, $\mu > 2^{|T|+\kappa}, \kappa > |T|$, and let $\bar{c} \in {}^{\kappa>}\mathfrak{B}$ be as required in the claim. So without loss of generality $\mathrm{rang}(\bar{c})$ is the universe of some $N^* \prec \mathfrak{B}$ and $\mathbf{f}$ maps N^* onto N^*.

If a type q over $C \subseteq \mathfrak{B}, |C| < \kappa$ is realized by a unique element in $\mathfrak{B}$ then some $\varphi(x) \in q$ define the element; hence we have $\langle \varphi_p : p \in \mathscr{P}^*\rangle$, where $\mathscr{P}^* := \{p \in \mathbb{S}^1(\bar{c}) :$ "$x\dot{\epsilon}\mathbf{b}_1$" $\in p\}$ such that: if $a \in \mathbf{b}_1[\mathfrak{B}]$ realizes $p \in \mathscr{P}^*$ then $\varphi_p(a, y)$ defines

$\mathbf{f}(a)$ in $\mathfrak{B}$ [i.e. $\mathfrak{B} \models \varphi[a, \mathbf{f}(a)]$ and $(\forall y)(\varphi[a, y]) \equiv y = \mathbf{f}(a)$]. Let $\varphi_p = \varphi_p(x, y, \bar{c}_p)$ so $\bar{c}_p \subseteq \text{Rang}(\bar{c})$.

Let

$$\mathscr{J} := \{c : \; c \in \mathbf{b}_1[\mathfrak{B}] \text{ is in } \text{Sk}_{\mathfrak{B}}(\bar{c}), c \text{ a real finite union of members of } \mathbf{b}_1^{\text{at}}[\mathfrak{B}]\}.$$

Note that possibly $\mathbf{b}_1^{\text{at}}[\mathfrak{B}] = \emptyset$.

Let

$$\mathscr{P}^- = \{p \in \mathscr{P}^* : \; p \text{ is not realized by any } c \in \mathscr{J} \text{ and if } x \text{ realizes } p, y \in \mathbf{b}_1[\mathfrak{B}] \setminus \mathscr{J} \text{ and } 0 < y \le x \text{ \underline{then} for some } z \in \mathbf{b}_1[\mathfrak{B}], z \text{ realizes } p \text{ and } 0 < z \le y\}.$$

Now $\mathscr{P}^- \subseteq \mathscr{P}^* \subseteq \mathbb{S}^1(\bar{c})$ is closed (hence compact) in the natural topology on $\mathbb{S}^1(\bar{c})$.

[Why? Because if $p \in \mathbb{S}^1(\bar{c}) \setminus \mathscr{P}^-$ then $\varphi'(x)\circledast = \neg(x \dot{\epsilon} \mathbf{b}_1) \in p$ or $\varphi'(x) = (x = 0_{\mathbf{b}_1}) \in p$ or for some $\varphi(x) \in p$ also $\varphi'(x) = (x \dot{\epsilon} \mathbf{b}_1)$ and $(x \neq 0_{\mathbf{b}_1})$ and $\varphi(x)$ and $(\exists y)(0 < y \le x$ and $(\forall z)(0 < z \le x \rightarrow \neg\varphi(x)))$ belong to p; and in cases $\varphi'(x) \in p' \in \mathbb{S}^1(\bar{c}) \Rightarrow p' \notin \mathscr{P}^-$.] Also $\{x \in \mathbf{b}_1[\mathfrak{B}] : \; \text{tp}(x, N, \mathfrak{B}) \in \mathscr{P}^-\}$ is a downward closed dense subset of $\mathbf{b}_1[\mathfrak{B}] \setminus \mathscr{J}$.

[Why? Let $a \le_{\mathbf{b}_1} b$ be from $\mathbf{b}_1[\mathfrak{B}] \setminus \mathscr{J}$ and b realizes $p \in \mathscr{P}^-$, and $c \le_{\mathbf{b}_1[\mathfrak{B}]} a$ be such that $c \in \mathbf{b}_1[\mathfrak{B}] \setminus \mathscr{J}$. Now there is $b' \in \mathbf{b}_1[\mathfrak{B}] \setminus \mathscr{J}$ such that $b' \le_{\mathbf{b}_1[\mathfrak{B}]} c$ and b' realizes p (because $c \le_{\mathbf{b}_1[\mathfrak{B}]} a, c \notin \mathscr{J}$, and $p \in \mathscr{P}^-$), so there is $a' \in \mathfrak{B}$ such that $(a', b'), (a, b)$ realizes the same type over N, so $a' \le_{\mathbf{b}_1} b' \le_{\mathbf{b}_1}, c$ hence $a' \le_{\mathbf{b}_1[\mathfrak{B}]} c$ and a' realizes $\text{tp}(a, N, \mathfrak{B})$ so we are done]. $\square$

We shall later prove:

{2.10A} **Observation 2.22.** *There is $p \in \mathbb{S}^1(\bar{c}) = \mathbb{S}^1(\bar{c}, \mathfrak{B})$ such that $p \in \mathscr{P}$ and:*

$(*)_0$ *$\{x_1 \cap x_2 : x_1, x_2$ realizes $p\} \cup \mathscr{J}$ is a dense subset of $\mathbf{b}_1[\mathfrak{B}]$ (and $c \in \mathfrak{B}$ realizes $p, b \in \mathscr{J} \Rightarrow \mathbf{b}_1[\mathfrak{B}] \models b \cap c = 0_{\mathbf{b}_1}$).*

Now we shall show that this suffices. Letting $\varphi = \varphi_p$, without loss of generality

$$\mathfrak{B} \models (\forall x, y, \bar{z})[\varphi(x, y, \bar{z}) \rightarrow (\exists! t)\varphi(x, t, \bar{z}) \textit{ and } x \dot{\epsilon} \mathbf{b}_1 \textit{ and } y \dot{\epsilon} \mathbf{b}_2)];$$

let $\psi_p^0(x) := (\exists y)\varphi(x, y, \bar{c}_p)$ and so there is in $\mathfrak{B}$ a definition of a function $\dot{h}_p$ from $\psi_p^0[\mathfrak{B}]$ into $\mathbf{b}_2[\mathfrak{B}]$ such that:

$(*)_1$ *d realizes p implies $\mathbf{f}(d) = \dot{h}_p(d)$.*

Note (the meaning of $\cap$ is clear from the context, $\vdash$ is inside $\mathfrak{B}$):

$(*)_2$ $p(x_1) \cup p(x_2) \cup p(x_3) \cup p(x_4) \vdash [(x_1 \cap x_2) \cap (x_3 \cap x_3) = 0_{\mathbf{b}_1}] \equiv [(\dot{h}_p(x_1) \cap \dot{h}_p(x_2)) \cap (\dot{h}_p(x_2) \cap \dot{h}_p(x_4)) = 0_{\mathbf{b}_2}]$

$(*)_3$ $p(x_1) \cup p(x_2) \cup p(x_3) \cup p(x_4) \vdash [x_1 \cap x_2 \le_{\mathbf{b}_1} x_3 \cap x_4] \equiv [\dot{h}_p(x_1) \cap \dot{h}_p(x_2) \le_{\mathbf{b}_2} \dot{h}_p(x_3) \cap \dot{h}_p(x_4)]$.

So by compactness there is $\psi_p(x) \in p(x)$ such that in $(*)_2, (*)_3$ we can replace $p(x_\ell)$ by $\psi_p(x_\ell)$ (and $\psi_p(x) \vdash \psi_p^0(x)$).

Now

$(*)_4$ $\mathbf{f}$ *induces a complete embedding* $\mathbf{f}/\mathscr{J}$ *of* $\mathbf{b}_1/\mathscr{J}$ *into* $\mathbf{b}_2/\mathrm{id}_{\mathbf{f}(\mathscr{J})}$ *where* $\mathrm{id}_{\mathbf{f}(\mathscr{J})} = \{d \in \mathbf{b}_2[\mathfrak{B}] : (\exists a \in \mathscr{J}) d \le \mathbf{f}(a)\}$, *it is an ideal of* $\mathbf{b}_2[\mathfrak{B}]$.

[Why? Being embedding is trivial so let us prove completeness of the embedding. Let $b \in \mathbf{b}_2[\mathfrak{B}] \setminus \mathrm{id}_{\mathbf{f}(\mathscr{J})}$, *as* $\mathrm{id}_{\mathbf{f}(\mathscr{J})}$ *is generated by* $< |T|^+ + \kappa = \kappa$ *elements, there is* $b' \in \mathbf{b}_2[\mathfrak{B}]$ *satisfying* $0_{\mathbf{b}_2} < b' \le b$ *and* $\bigwedge_{a\in\mathscr{J}} b' \cap \mathbf{f}(a) = 0_{\mathbf{b}_2}$. *As* $\mathbf{f}$ *is a complete embedding, there is* $c \in \mathbf{b}_1 \setminus \{0\}$ *such that* $[0_{\mathbf{b}_1} < c' \le c \Rightarrow \mathbf{f}(c') \cap b' \ne 0_{\mathbf{b}_2}]$. *So we are done (see the proof of 0.4(2))]].* {0.4}

Next we claim

$(*)_5$ *if* $c \in \mathbf{b}_1[\mathfrak{B}]$ *(and* $\bigwedge_{a\in\mathscr{J}} a \cap c = 0_{\mathbf{b}_1}$*) and* $\psi_p(c)$ <u>*then*</u> $\mathbf{f}_p(c) \le \mathbf{f}(c) \mod \mathbf{f}(\mathscr{J})$ *in* $\mathbf{b}_2[\mathfrak{B}]$.

[Why? If not, $\dot{h}_p(c) - \mathbf{f}(c) \ne 0_{\mathbf{b}_2}$ *and even* $\notin \mathbf{f}(\mathscr{J})$, *so as* $\mathbf{f}\mathrm{id}_{\mathbf{f}(\mathscr{J})}$ *is a complete embedding, for some* $d \in \mathbf{b}_1[\mathfrak{B}]$ *we have* $\mathbf{f}(d) \cap (\dot{h}_p(c) - \mathbf{f}(c)) \notin \mathscr{J}$, *moreover,* $[0_{\mathbf{b}_1} < d' \le d \Rightarrow \mathbf{f}(d') \cap (\mathbf{f}_p(c) - \mathbf{f}(c)) \notin \mathscr{J}]$. *So* without loss of generality $d \cap c = 0_{\mathbf{b}_1}$ *and* $a \in \mathscr{J} \Rightarrow d \cap a = 0_{\mathbf{b}_1}$, *recalling* $|\mathscr{J}| \le |T| < \kappa$. *By* $(*)_0$ *possibly decreasing* d *(legitimate by the previous sentence), we find* d_1, d_2 *realizing* $p(x)$ *such that* $\mathbf{b}_1[\mathfrak{B}] \models d_1 \cap d_2 = d$. *Let* $d_3 = d_4 = c$ *so* $d_1 \cap d_2 \le d, c = d_1 \cap d_2$, *so* $(d_1 \cap d_2) \cap (d_3 \cap d_4) \le d \cap c = 0_{\mathbf{b}_1}$ *so by the version of* $(*)_2$, *for* ψ_p, *we know* $(\dot{h}_p(d_1) \cap \dot{h}_p(d_2)) \cap (\dot{h}_p(d_3) \cap \dot{h}_p(d_4)) = 0_{\mathbf{b}_2}$; *now the right side is* $\dot{h}_p(c) \cap \dot{h}_p(c) = \dot{h}_p(c)$, *the left side is (as* d_1, d_2 *realizes* p*)* $\mathbf{f}(d_1) \cap \mathbf{f}(d_2) = \mathbf{f}(d_1 \cap d_2) = \mathbf{f}(d)$ *so* $\mathbf{f}(d) \cap \mathbf{f}_p(c) = 0_{\mathbf{b}_2}$; *contradiction].*

$(*)_6$ *if* $c \in \mathbf{b}_1[\mathfrak{B}] \setminus \mathscr{J}$ *and* $\mathfrak{B} \models \psi_p(c)$ <u>*then*</u> $\dot{h}_p(c) \ge \mathbf{f}(c) \mod \mathbf{f}(\mathscr{J})$ *in* $\mathbf{b}_2[\mathfrak{B}]$.

[Why? If not $\mathbf{f}(c) - \dot{h}_p(c) \notin \mathbf{f}(\mathscr{J})$ *so as* $\mathbf{f}/\mathscr{J}$ *is a complete embedding for some* $d \in \mathbf{b}_1 \setminus \mathscr{J}$ *we have* $[d' \notin \mathscr{J}$ *and* $d' \le d \Rightarrow \mathbf{f}(d') \cap (\mathbf{f}(c) - \dot{h}_p(c)) \notin \mathbf{f}(\mathscr{J})]$.

As $\mathbf{f}(d) \cap \mathbf{f}(c) \notin \mathbf{f}(\mathscr{J})$ *necessarily* $d \cap c \notin \mathscr{J}$ *and* without loss of generality *below* d *there is no member of* $\mathscr{J} \setminus \{0_{\mathbf{b}_1}\}$. *By* $(*)_0$ *we can find* d_1, d_2 *realizing* $p(x)$ *such that* $0 < d_1 \cap d_2 \le d \cap c$. *Let* $d_3 = d_4 = c$ *so* $d_1 \cap d_2 \le d_3 \cap d_4$ *hence by the version of* $(*)_3$ *with* ψ_p *we know* $\dot{h}_p(d_1) \cap \dot{h}_p(d_2) \le_{\mathbf{b}_2[\mathfrak{B}]} \dot{h}_p(d_3) \cap \dot{h}_p(d_4)$. *Now the right side is* $\dot{h}_p(c) \cap \dot{h}_p(c) = \dot{h}_p(c)$ *and the left side is (as* d_1, d_2 *realize* p*) just* $\mathbf{f}(d_1) \cap \mathbf{f}(d_2) = \mathbf{f}(d_1 \cap d_2)$; *so* $\dot{h}(d_1 \cap d_2) \le_{\mathbf{b}_2[\mathfrak{B}]} \dot{h}_p(c)$. *But* $0 < d_1 \cap d_2 \le d$, *so by the choice of* d *we have* $\mathbf{f}(d_1 \cap d_2) \cap (\mathbf{f}(c) - \dot{h}_p(c)) \ne 0_{\mathbf{b}_1}$ *contradicting the previous sentence].*

By $(*)_5 + (*)_6$ without loss of generality

$(*)_7$ *if* $c \in \mathbf{b}_1[\mathfrak{B}] \setminus \{0_{\mathbf{b}_1}\}$, $\psi_p(c)$ *and* $\bigwedge_{d\in\mathscr{J}} c \cap d = 0_{\mathbf{b}_1}$ <u>*then*</u> $\mathbf{f}(c) = \mathbf{f}_p(c)$.

[Why? If not, this fail even if we strengthen ψ_p, *(note the* without loss of generality*],) so by* $(*)_5 + (*)_6$ *for every* $\psi(x) \in p$ *there is* $c_\psi \in \mathbf{b}_1[\mathfrak{B}]$ *satisfying* $\psi(c_\psi)$ *and* $\psi_p(c_\psi)$ *and* $\bigwedge_{d\in\mathscr{J}} d \cap c_\psi = 0_{\mathbf{b}_1}$ *and* $\dot{h}_p(c_\psi) \ne \mathbf{f}(c_\psi)$. *Now by* $(*)_5 + (*)_6$ *for some* $d_\psi \in \mathscr{J}$, $\mathbf{f}_p(c_\psi) - \mathbf{f}(d_\psi) = \mathbf{f}(c_\psi) - \mathbf{f}(d_\psi)$; *however* $\mathbf{f}(c_\psi) \cap \mathbf{f}(d_\psi) = 0_{\mathbf{b}_2}$ *as* $c_\psi \cap d_\psi = 0_{\mathbf{b}_2}$ *by the definition of* $\mathscr{J}$ *hence* $\dot{h}_p(c_\psi) - \dot{h}(d_\psi) = \dot{h}(c_\psi) - \mathbf{f}(d_\psi) = \mathbf{f}(c_\psi) \ne \dot{h}_p(c_\psi)$. *So as* $\dot{h}_p(c_\psi) \ne \mathbf{f}(c_\psi)$ *necessarily there is* $d'_\psi \in \mathscr{J} \cap \mathbf{b}_1^{\mathrm{at}}[\mathfrak{B}]$ *such that* $\mathbf{f}(d'_\psi) \cap \dot{h}_p(c_\psi) \ne 0_{\mathbf{b}_2}$. *By saturation we can find* $c \in \mathbf{b}_1[\mathfrak{B}]$, *satisfying* $\bigwedge_{\psi\in p} c \cap c_\psi = 0_{\mathbf{b}_1}$ *but* $\bigwedge_{d\in\mathscr{J}} d \le_{\mathbf{b}_1} c$

(the finite satisfaction is exemplified by members of $\mathscr{J}$ *). So* $\bigwedge_{\psi \in p} \dot{h}_p(c_\psi) \cap \mathbf{f}(c) \neq 0_{\mathbf{b}_1}$*; by saturation for some* $c' \in \mathbf{b}_1[\mathfrak{B}]$ *realizing* p, $c' \cap c = 0_{\mathbf{b}_1}$ *and* $\dot{h}_p(c') \cap \mathbf{f}(c) \neq 0_{\mathbf{b}_2}$ *(the finite satisfaction is exemplified by the* c_ψ*'s) but* $\dot{h}_p(c') = \mathbf{f}(c')$, $h_p c) = h(c)$ *and* $\mathbf{f}$ *an embedding, contradiction.]*

Now we can, in $\mathfrak{B}$*, define a partial function* $\dot{h}^+_p : \mathbf{b}_1[\mathfrak{B}] \to \mathbf{b}_2[\mathfrak{B}]$ *as follows:* $h^+_p(x)$ *is the* $\leq_{\mathbf{b}_2}$*-minimal element* y *of* $\mathbf{b}_2[\mathfrak{B}]$ *such that:*

$$\psi_p(z_1) \text{ and } \psi_p(z_2) \text{ and } z_1 \cap z_2 \leq_{\mathbf{b}_1} x \Rightarrow \dot{h}_p(z_1) \cap \dot{h}_p(z_2) \leq_{\mathbf{b}_2} y.$$

Let $\mathscr{J}^* = \{a \in \mathbf{b}_1[\mathfrak{B}] : \mathbf{f}^+_p \restriction \{b \in \mathbf{b}_1[\mathfrak{B}] : b \leq a\}$ *is a complete embedding into*
{0.4} $\mathbf{b}_2[\mathfrak{B}]\}$*. This set is definable into* $\mathfrak{B}$ *by 0.4(2) and by* $(*)_7$ *(and* $(*)_0$*) it include every* $a \in \mathbf{b}_1[\mathfrak{B}]$ *such that* $\bigwedge_{b \in \mathscr{J}} b \cap a = 0$*, so by saturation there is* $b^* \in \mathscr{J}$ *such that* $a^* \in \mathbf{b}_1[\mathfrak{B}]$ *and* $a^* \cap b^* = 0_{\mathbf{b}_1} \Rightarrow a^* \in \mathscr{J}^*$*.*

Now

$(*)_8$ $\mathscr{J}' = \{a \in \mathscr{J} : a \in \mathbf{b}^{\rm at}_1, a \cap b^* = 0_{\mathbf{b}_1}, \mathbf{f}(a) \neq \dot{h}_p(a)\}$ *is really finite.*

[Why? If a_n $(n < \omega)$ *are distinct members; thinning the sequence* $\langle a_n : n < \omega\rangle$ *by Ramsey theorem* without loss of generality *if for some* $n, \mathbf{f}_p(a_n) - \mathbf{f}(a_n) \neq 0_{\mathbf{b}_2}$ *then* $\dot{h}_p(a_n) \not\leq_{\mathbf{b}_2[\mathfrak{B}]} \bigcup_{\ell < k} \dot{h}_p(a_\ell)$ *for any* n, k *hence there are non-zero* $d^1_n \leq_{\mathbf{b}_2[\mathfrak{B}]} \mathbf{f}_p(a_n)$ *disjoint in* $\mathbf{b}_2[\mathfrak{B}]$ *to* $\mathbf{f}(a_k)$ *for any* n, k *otherwise* $d^1_{\mathbf{f}} = 0_{\mathbf{b}_1}$*, Similarly if* $\mathbf{f}(a_m) - ??h_p(a_m) \neq 0_{\mathbf{b}_2}$ *for some* m*, then there are non-zero* $d^2_m \leq \mathbf{f}(a_n)$ *disjoint to* $h_p(a_k)$ *for* $n, k < \omega$*. Now we get a contradiction by saturation so* $(*)_8$ *hold].*

So without loss of generality (possibly increasing b^**)* $\mathscr{J}' = \emptyset$*, so we can finish as* $\mathbf{f}(x) = y \Leftrightarrow (x \in \mathbf{b}_1$ *and* $y \in \mathbf{b}_2$ *and* $\mathbf{f}^*_p(x - b^*) = y - (\mathbf{f}(B^*)$ *and* $\bigwedge_{b \leq_{\mathbf{b}_1} b^*} (x \cap b^* = b \to y \cap \mathbf{f}(b^*) = \mathbf{f}(b))$*.*

<u>Proof of the Observation 2.22</u>: For each $p \in \mathscr{P}^-$ let $\mathscr{I}_p$ be the ideal of $\mathbf{b}_1[\mathfrak{B}]$ which $\{d \in \mathbf{b}_1[\mathfrak{B}] : d \text{ realizes } p\}$ generates. Let $\{p_i : i < i^*\} \subseteq \mathscr{P}^-$ be maximal such that $i \neq j \Rightarrow \mathscr{I}_{p_i} \neq \mathscr{I}_{p_j}$ (equivalently $i \neq j \Rightarrow I_{p_i} \cap I_{p_j} = \{0_{\mathbf{b}_1}\}$ and also equivalently, there are $c_i \leq_{\mathbf{b}_1} c_j$ with c_i realizing p_i, c_j realizing p_j). We can find, for $i < i^*$ and $n < \omega$ an element $d_{i,n}$ of $\mathfrak{B}$ realizing p_i such that $[(i,n) \neq (j,m) \Rightarrow d_{i,n} \cap d_{j,n} = 0_{\mathbf{b}_1}]$. By partition theorems and compactness (i.e. $\mathfrak{B}$ quite saturated) without loss of generality: for each i, $\langle d_{i,n} : n < \omega\rangle$ is an indiscernible sequence over $\bar{c} \cup \bigcup_{j \neq i} \{d_{j,m} : m < \omega, j < i^* \text{ and } j \neq i\}$.

Let us define Λ

$$\begin{aligned} \Lambda \;&= \{\varphi(x_{i,\ell}) \equiv \varphi(x_{j,m}) : i,j < i^* \text{ and } \ell, m < 2 \text{ and } \varphi \text{ a formula over } \bar{c}\} \\ &\cup \{x_{i,\ell} \dot{\in} \mathbf{b}_1 : i < i^*\} \cup \{x_{i,0} \cap x_{i,1} = d_{i,0} : i < i^*\}. \end{aligned}$$

If Λ is realized in $\mathfrak{B}$ by an assignment $x_{i,\ell} \mapsto c_{i,\ell}$, then $p := \text{tp}(c_{i,\ell}, \bar{c}, \mathfrak{B})$ (which does not depend on i, ℓ) is a type as required. So it suffices to show any finite $\Lambda' \subseteq \Lambda$ is realized in $\mathfrak{B}$, so let w be the set of $i < i^*$ such that $x_{i,\ell}$ appears in Λ' (clearly it is finite).

We choose the following assignment $x_{i,\ell} \mapsto c_{i,\ell}$ where:

$$c_{i,\ell} = \bigcup\{d_{j,m} : j = i \text{ and } j \in w \text{ and } m = 0 \text{ or } j \neq \text{ and } j \in w \text{ and } m = \ell\}$$

(it is a really finite union in $\mathbf{b}_1$).

* * *

§ 3. Compactness for the Boolean Ring Complete Embedding Quantifiers

We present here the compactness results (we can add more quantifiers; see §4).

{3.1} **Theorem 3.1.** *1) Adding to first order logic quantification on complete embeddings*
{3.2} *of an atomless Boolean ring to a Boolean ring getting $\mathscr{L}^{\rm ceab}$ (see 3.3(1) below) result in a compact logic.*
{3.2} *2) If every finite subset of $T \subseteq \mathscr{L}^{\rm ceab}$ (see 3.3(1) below) has a model in some generic extension of* $\mathbf{V}$ *(or any model of set theory with the same natural numbers)*
{2.5} *and λ, θ, κ are as in 2.7, <u>then</u> T has a κ-saturated model of cardinality λ.*

{3.1A} *Remark* 3.2. Note that by 3.7(1) we have a completeness theorem for $\mathscr{L}(\dot{\mathbf{Q}}^{\rm ceab})$:
{3.5} a sentence ψ has model iff $\psi \cup \{$the reasonable axioms for $\mathfrak{C}^*$ (of 2.1) in suitably
{2.1} expanded vocabulary$\}$ is consistent (first order) — see 3.7(1). If we would like to
{3.5} have a nice set of axiom schemes, we need 3.5, below see 3.7(2).

{3.5} {3.2} **Definition 3.3.** The logic $\mathscr{L}^{\rm ceab}$ is gotten from first order logic by allowing free variables on partial unary functions, and allowing the quantifier $(\dot{\mathbf{Q}}^{\rm ceab} f, x, y)[\psi^1(x,y), \psi^2(x,y), \varphi(f)]$ where ψ^1, ψ^2 are formulas (with parameters) in which f does not appear freely and is a variable on partial unary functions, the individual variable x, y does not appear freely in φ, the free variables of this formula are those of ψ^1, ψ^2, except x, y and those of φ except f.

Lastly, $M \models (\dot{\mathbf{Q}}^{\rm ceab} f, x, y)[\psi^1, \psi^2, \varphi]$ if $\psi^1(x,y)$, $\psi^2(x,y)$ define Boolean rings, the first atomless (i.e. the ψ^ℓ define the corresponding partial orders) and there is a complete embedding $\mathbf{f}$ from the Boolean ring which ψ^1 defines (in M) to the Boolean ring which ψ^2 defines (in M) which satisfies φ when we substitute $\mathbf{f}$ for $\underset{\sim}{f}$.

{3.2A} *Remark* 3.4. Note the "$\mathbf{f}$ is a complete embedding of the Boolean ring $\mathbf{B}_1$ into the
{0.4} Boolean ring $\mathbf{B}_2$" is a first order property (see 0.4(2)).

Proof. 1) Let T be a theory in $\mathscr{L}^{\rm ceab}$, let M_Δ be a model of Δ for any finite subset Δ of T. Let χ^* be strong limit cardinal to which $\langle M_\Delta : \Delta \subseteq T \text{ finite}\rangle$ belongs.
{2.1} Expand $\mathscr{H}(\chi)$ as in 2.1 and get $\mathfrak{C}^*, T^*$ such that each M_Δ is an individual constant
{2.5} as well as T. Now get $\mathfrak{B}$ by 2.7. For some $M, \mathfrak{B} \models$ "M is a model of a finite set of
{2.9} sentences t", such that for every $\psi \in T$ we have $M \models$ "$\psi \dot{\in} t$". By 2.19 the quantifier $\dot{\mathbf{Q}}^{\rm ceab}$ has the same interpretation in the universe and in $\mathfrak{B}$, so we are done.
2) Same proof. $\square_{3.1}$

The following is not in our main line, but helps in axiomatization.

{3.4} **Lemma 3.5.** *1) In the theorems* [Sh:107, 4.9] *and* [Sh:107, 5.2(2)] *(for every P, Q, R definable with parameter) we can add complete embeddings of Boolean rings. I.e. let T be a complete first order theory. Assume $(D\ell)_\lambda, |T| < \lambda^+$ and $\diamondsuit_{\{\delta<\lambda^+:{\rm cf}(\delta)=\lambda\}}$. <u>Then</u> if an λ-saturated model M of T of cardinality λ^+ is constructed in the game (*[Sh:107, 2.8, 2.12]*) the second player can guarantee in addition that:*

- (*A*) *If $\mathbf{b}_1$, $\mathbf{b}_2$ are Boolean rings definable M (no atomlessness assumed), $\mathbf{f}$ a complete embedding of $\mathbf{b}_1$ into $\mathbf{b}_2$ <u>then</u> $\mathbf{f}$ is first order definable with parameters from M.*
- (*B*) *If $\psi_\ell(x, y, \bar{c}_\ell)$ define in M a dense linear order I_ℓ for $\ell = 1, 2$ and $\mathbf{f}$ is an isomorphism from I_1 onto I_2 <u>then</u> every interval I'_1 of I_1 contains a subinterval I''_1 such that $\mathbf{f} \restriction I''_1$ is definable in M with parameters.*

(*C*) *Similarly for the strong independence property.*

(*D*) *In clause (B) we can weaken the demand on* $\mathbf{f}$ *to "*$\mathbf{f}: I_1 \to I_2$ *is order preserving with dense range (in* I_2*)".*

Proof. Follows by [Sh:107] and 2.20 (and degenerate version of the proof of 2.16, 2.20). $\square_{3.5}$ {2.10}

Remark 3.6. 1) Player II has time enough for all such assignments.
2) We can weaken the set theoretic demands (but no real need here).

{3.5}

Lemma 3.7. *1)* $\mathbb{L}(\dot{\mathbf{Q}}^{\text{ceab}})$ *is complete, i.e. the set of sentences have no models is recursively enumerable and is absolute.*
2) In fact the natural axioms schemes suffice (in addition to the first order ones).
3) If you have a definition $\varphi(x,y)$ *of a complete embedding from the atomless Boolean ring defined by* $\psi^1(x,y)$ *to the atomless Boolean ring defined by* ψ^2*, then* $(\dot{\mathbf{Q}}^{\text{ceab}} f,x,y)[\psi^1(x,y),\psi^1(x,y)(\forall x,y)[f(x)=y \leftrightarrow \varphi(x,y)]]$.
4) $(\dot{\mathbf{Q}}^{\text{ceab}} f,x,y)[\psi^1(x,y),\psi^2(x,y),\varphi(f)] \to [\psi^1(x,y)$ *defines an atomless Boolean Ring and* $\psi^2(x,y)$ *defines a Boolean ring and* $(\dot{\mathbf{Q}}^{\text{ceab}} f,x,y)[\psi^1(x,y),\psi^2(x,y),\varphi'(f)]$ *where* $\varphi'(f)=\varphi(f)$ *and "(f is a complete embedding from the Boolean ring which* $\psi_1(-,-)$ *defines into the one which* $\psi_2(-,-)$ *defines)].*
5) $(\dot{\mathbf{Q}}^{\text{ceab}} x,y,f)[\psi_1(x,y),\psi_2(x,y),\varphi(f)] \to (\dot{\mathbf{Q}}^{\text{ceab}} x_1,y_1,f_1)[\psi'_1(x_1,y_1),\psi'_2(x_1,y_1),\varphi'(f_1)]$ *when:*
"f is a complete embedding of the atomless Boolean ring which $\psi^1(x,y)$ *defines to the atomless Boolean ring which* $\psi^2(x,y)$ *defines" and* $\varphi(f)$ *and* $\psi'_1(-,-)$ *define an atomless Boolean ring and* $\psi'_2(-,-)$ *defines a Boolean ring* $\vdash (\exists f')[\varphi'(f')$ *and* f' *is a complete embedding from the Boolean ring which* $\psi'_1(-,-)$ *defines into the Boolean ring which* $\psi'_2(-,-)$ *defines"].*

Proof. 1) By the proof of 3.1. {3.1}
2) For a sentence ψ in our logic $\mathbb{L}(\dot{\mathbf{Q}}^{\text{ceab}})$ for applying 2.11 we need essentially Skolem function. How this relate to our axioms? Proof theoretists has done such things but we prefer other ways. So we have to use another proof. {2.6}

We need to pass to the theory $T^{\otimes}$ which is $T+$ the axiom used from 2.1 i.e. with "Skolem functions" for $(\dot{\mathbf{Q}}^{\text{ceab}} f)$ (which is described below). So we shall prove the consistency of this relevant theory by supplying a model of ψ by 3.5. However, in our universe $\mathbf{V}$ there may be no cardinal to which it applies. Choose $\lambda > |T^{\otimes}|$ suitable to 3.5. Choose $A \subseteq \beth_2(\lambda)^+$ such that $T \in \mathbf{L}[A]$ and even $\mathscr{H}(\lambda^+) \in \mathbf{L}[A]$. Now 3.5 apply in $\mathbf{L}[A]$ as for $\mu > (\beth_2(\lambda)^2)^{\mathbf{V}}$, μ a regular cardinal of $\mathbf{L}[A]$ we have $2^\mu = \mu^+, \mu = \mu^{<\mu}, \diamondsuit_\mu$ and $\diamondsuit^*_{\{\delta<\mu^+:\text{cf}(\delta)=\mu\}}$; so T has a model by 3.5. Apply 2.1 in $\mathbf{L}[A]$ to get the consistency of the first order $T^{\otimes} \in \mathbf{L}[A]$ and apply 2.7, 2.19 to get a model in λ, so it belongs to $\mathbf{V}$. $\square_{3.7}$ {2.1} {3.4} {3.4} {3.4} {3.4} {2.9}

Discussion 3.8. However we may like to allow applying the new quantifier not just to Boolean rings defined on a set of elements but also to ones defined on sets of triples of elements, or even to triples consisting of elements, complete embedding of one definable atomless Boolean rings etc. The definition of the logic should be clear, but we can translate the model.

Formal Description of the Translation Let the vocabulary of ψ be $\subseteq \tau$; for every τ-model M we construct a model $M^{[*]}$. We define by induction on $n, M^{[n]}, M^{[0]} =$ {3.5A}

M. For each n, let $\{(\varphi^n_k(x,y,\bar z_1), \psi^n_k(x,y,\bar z_2)) : k < \omega\}$ is a list of the pairs of the first order formulas for the vocabulary of $M^{[n]}$, of the indicated form in Definition 3.3. {3.2}

Let $S^n_k (k < \omega)$ be new sorts, the universes of $M^{[n+1]}$ are: for sorts of $M^{[n]}$ the same as before, for $S^n_k, \{(\mathbf{f}, \bar c^1, \bar c^2) : \varphi^n_k(x,y,\bar c^1)$ define in $M^{[n]}$ an atomless Boolean ring, $\psi^n_k(x,y,\bar c^2)$ define in $M^{[n]}$ an atomless Boolean ring, $\mathbf{f}$ is a complete embedding of the first Boolean ring into the second$\}$.

Relations:

- (a) the old one:
- (b) for each k unary functions mapping $(\mathbf{f}, \bar c^1, \bar c^2)$ to $\bar c^1 \hat{\ } \bar c^2$
- (c) binary function mapping $((\mathbf{f}, \bar c^1, \bar c^2), b)$ to $\mathbf{f} f(b)$.

Let T^*_τ be the set of first order sentences describing the construction of $M^{[*]}$ from M, except that we replace the definition of the universe S^n_k, by demanding the scheme saying: if a formula with parameters in $M^{[*]}$ defining a complete embedding of the atomless Boolean ring defined by $\varphi^n_k(x,y,\bar c^1)$ into the Boolean ring defined by $\psi^n_k(x,y,\bar c^2)$, then for some $(\mathbf{f}, \bar c^1, \bar c^2)$ in the interpretation of $S^n_k, \mathbf{f}$ is the complete embedding mentioned above.

Note that models M^* of T^*_τ gives in general non-standard interpretation of $\dot{\mathbf{Q}}^{\rm ceab}$. Now if ψ is consistent according to the axiom scheme described above then there is a model M^* of T^*_τ (in the vocabulary of T^*_τ) such that in the interpretation of $\dot{\mathbf{Q}}^{\rm ceab}$ in this model M^* (which in general is non-standard), ψ holds. Now we would like
{3.4} to use 3.5 (multisortness does not matter) to get there is a standard model of the first a model $M^\otimes$ of the first order theory $T^\otimes$ of M^*, which is standard for $\dot{\mathbf{Q}}^{\rm ceab}$.

{3.6}

Conclusion 3.9. *There is a 1-homogeneous atomless Boolean Algebra* $\mathbf{B}$*, such that* $\mathrm{Aut}(\mathbf{B})$ *is not simple, where*

{3.6A}

Definition 3.10. 1) A Boolean Algebra $\mathbf{B}$ is 1-homogeneous if whenever in $\mathbf{B}, 0 < x < 1, 0 < y < 1$, then there is an automorphism $\mathbf{f}$ of $\mathbf{B}$ such that $\mathbf{f}(x) = y$.
2) A group $\mathbb{G}$ is not simple if it has a normal subgroup $\neq \mathbb{G}$, with at least two elements.

Proof. Clearly there is $\psi \in \mathbb{L}(\dot{\mathbf{Q}}^{\rm ceab})$ which has a model iff there is a 1-homogeneous
{3.5} atomless Boolean Algebra. So by 3.7 (or directly by 3.1(2)) it suffices to have some generic entension of $\mathbf{V}$ (or just a model of set theory with the same natural numbers) in which ψ has a model.

Now from this we can just quote one of the following:

It is proved in [Sh:b, Ch.IV], that in some generic extension any automorphism $\mathbf{f}$ of $(\mathscr{P}(\omega)/\text{finite})$ is trivial; this means that some one to one function f with domain and range cofinite subsets of ω, induced $\mathbf{f} = [f]$, i.e. $\mathbf{f}\ (A/\text{finite}) = \{f(n) : n \in A \cap \mathrm{Dom}(f)\}/\text{finite}$ (this involves detailed analysis that this Boolean Algebra has no automorphism which are "simply defined"). Van Dowen notes that this group is not simple[11]

[11] The subgroup is $\mathbb{G} := \{\mathbf{f} : \mathbf{f}$ is induced by a permutation of $\omega\}$. More fully if f, g are as above and induce the same $\mathbf{f}$, i.e. $[f] = [g]$ then $|\omega \setminus \mathrm{Dom}(f)| - |\omega \setminus \mathrm{Rang}(f)| = |\omega \setminus \mathrm{Dom}(g)| - |\omega \setminus \mathrm{Rang}(g)|$ (those numbers are integers) so $n([f]) =: |\omega \setminus \mathrm{Dom}(f)| - |\omega \setminus \mathrm{Rang}(f)|$ is well defined and $\mathbf{f} \mapsto n(\mathbf{f})$ is a homomorphism from $\mathrm{Aut}(\mathscr{P}(\omega)/\text{finite})$ into $\mathbb{Z}$; the kernel is a normal group as required.

By [Sh:107] we can replace mere consistency by GCH; the proof is very indirect. However, S. Koppelberg [Kop85] proved the existence of such Boolean Algebra A if CH holds (this involves detailed analysis of the specifics of the case).

The sentence ψ can be:

(*i*) $(P, \leq)$ is an atomless Boolean Algebra

(*ii*) $(Q, \circ, e)$ is a group

(*iii*) $F(-,-)$ is such that for $x \in Q, F(x,-)$ is a permutation of P which is an automorphism of $(P, <)$

(*iv*) $x \mapsto F(x,-)$ is an embedding of $(Q, \circ, e)$ into the group of permutations of P

(*v*) Q' is a proper normal subgroup of $(Q, \circ, e)$, $|Q'| > 1$

(*vi*) if $x, y \in P$ are neither the maximal element of P nor the minimal element of P then for some $z \in Q$, $F(z, x) = y$

(*vii*) every automorphism of $(P, \leq)$ has the form $F(x,-)$ for some $x \in Q$.

$\square_{3.9}$

§ 4. Adding More Quantifiers

We show here for completeness that we can ncorporate the results of [MkSh:375] for trees and for ordered fields but we somewhat strengthen the statement: we can quantify over embedding of one order field $\dot{\mathbb{F}}_1$ onto some dense subfield of another $\dot{\mathbb{F}}_2$ (rather than onto isomorphism).

{4.1}
{2.5} **Claim 4.1.** *Assume ($T^*, \mathfrak{B}$ as in 2.7 and)*

- (a) *$(I, \mathbb{P}, \leq_I, \leq_{\mathbb{P}}, h)$ is a leveled partial order; i.e.*
  - (i) *$(I, \leq_I)$ is a partial order*
  - (ii) *$(\mathbb{P}, \leq_{\mathbb{P}})$ is a directed partial order (with no maximal element)*
  - (iii) *$h : I \to \mathbb{P}$ satisfies: $x <_I y \Rightarrow h(x) <_{\mathbb{P}} h(y)$*
  - (iv) *if $x \in I, t <_{\mathbb{P}} h(x)$ then for some unique $y \in I, y < x, h(y) = t$*
- (b) *$(I, \mathbb{P}, \leq_I, \leq_{\mathbb{P}}, h)$ is first order definable in $\mathfrak{B}$ with (finitely many) parameters so without loss of generality $I, \mathbb{P}$ are subsets of $\mathfrak{B}$*
- (c) *$B \subseteq I$ is a full branch, i.e. $(B, <_I \restriction B)$ is directed and $h \restriction B$ is one to one onto $\mathbb{P}$.*

Then B is first order definable in $\mathfrak{B}$ (with parameters).

Remark 4.2. Clearly h is a homomorphism from $(B, \leq_I \restriction B)$ onto $(\mathbb{P}, <_{\mathbb{P}})$.

Proof. For each $\alpha \in S^-$ let $b_\alpha \in \mathbb{P}^{\mathfrak{B}}$ be such that $[b \in \mathbb{P}[\mathfrak{B}_\alpha] \Rightarrow b <_{\mathbb{P}} b_\alpha]$ (exists
{2.6} by 2.11(g)), and let $t_\alpha \in B (\subseteq I)$ be such that $h(t_\alpha) = b_\alpha$. For some $N_\alpha \prec \mathfrak{B}_\alpha, \mathrm{gen}(N_\alpha) < \kappa$ and $\mathfrak{B}_\alpha \biguplus_{N_\alpha}^{\mathfrak{B}} \langle N_\alpha, t_\alpha \rangle$. As $\{\delta < \lambda : \mathrm{cf}(\delta) = \theta, \delta \notin S\}$ is stationary, λ regular and $(\forall \alpha < \lambda)[\alpha^{<\kappa} < \lambda]$, clearly for some N,

$$S_N := \{\delta \in S^- : N_\delta = N\} \text{ is stationary and even for some } p \in S(N)$$

$$S_N^p := \{\delta \in S_N : \mathrm{tp}(t_\delta, N, \mathfrak{B}) = p\} \text{ is stationary.}$$

Without loss of generality the parameters defining $(I, \mathbb{P}, \leq_I, \leq_{\mathbb{P}}, h)$ are in N. Now if $\alpha < \beta$ are in S_N^p, then $t_\alpha <_I t_\beta$ (as h is an isomorphism from $(B, \leq_I, \restriction B)$ onto $(\mathbb{P}, <_{\mathbb{P}})$), but $\mathrm{tp}(t_\beta, \mathfrak{B}_\beta, \mathfrak{B})$ does not split over N, hence if $t \in I \cap \mathfrak{B}_\beta$ realizes p then $t <_I t_\beta$. As this holds for any $\beta \in S_N^p \setminus \{\mathrm{Min}(S_N^p)\}$, clearly $p(x) \cup p(y) \cup \{\neg(\exists z)[x \leq_I z \text{ and } y \leq_I z]\}$ is not realized in $\mathfrak{B}$.

But N is generated by $< \kappa$ elements and $\mathfrak{B}$ is κ-saturated so the type is not finitely satisfiable in $\mathfrak{B}$, so for some $\psi(x) \in p$

$$B \models (\forall x, y)\ [\psi(x)\ \&\ \psi(y) \Rightarrow (\exists z)\ [x \leq_I z\ \&\ y \leq_I z]\}$$

m and without loss of generality $\psi(\mathfrak{B}) \subseteq I$. So $\psi'(x) = (\exists y)(x \leq_I y \& \psi(y))$ defines the branch B. $\square_{\text{ref4.1}}$

{4.2}
{3.4} **Conclusion 4.3.** *In 3.1, 3.5 we can add to the logic quantification over full branches*
{4.5} *of leveled partial orders, $\dot{\mathbf{Q}}^{\mathrm{br}}$ (see 4.6(2)). Also adding the reasonable axiom schema,*
{3.5} *we can add this quantifier in 3.7.*

{4.3}
{2.5} **Claim 4.4.** *Assume $\mathfrak{B}$ as in 2.7, $\mathbf{b}_1$, $\mathbf{b}_2$ are Boolean rings representable (or just first order definable with parameters) in $\mathfrak{B}$, $h : \mathbf{b}_1 \to \mathbf{b}_2$ is a homomorphism and $a_i \in \mathbf{b}_1[\mathfrak{B}]$ for $i < \lambda$ are pairwise disjoint.*

Then for some $\alpha < \lambda$ for every $i \in [\alpha, \lambda)$, $\mathbf{f}(a_i)$ is the unique member of $\mathfrak{B}$ realizing $\mathrm{tp}(\mathbf{f}(a_\alpha), \mathfrak{B}_\alpha \cup \{a_i\}, \mathfrak{B})$.

Proof. Similar to 2.16 (but we have to replace the use of the c_α's by poorer means: {2.8}
which $x \in \mathfrak{B}_{\beta_i} \setminus \mathfrak{B}_{\alpha_i}$ is below $\mathbf{f}(a_{\delta,\eta,\theta})$). $\square_{4.4}$

{4.4}
Claim 4.5. *In 3.1, 3.5 (and 3.7) we can add to the logic the quantifiers $\dot{\mathbf{Q}}^{\mathrm{ofde}}$ and $\dot{\mathbf{Q}}^{\mathrm{br}}$.* {3.5}

where

{4.5}
Definition 4.6. 1) $(\dot{\mathbf{Q}}^{\mathrm{ofde}} f, \bar{x}, \bar{y})[\psi^1(\bar{x}), \psi^2(\bar{y}), \varphi(f)]$ (defined syntactically as in 3.3) means: $M \models (\dot{\mathbf{Q}}^{\mathrm{ofde}} f, \bar{x}, \bar{y})[\psi^1(\bar{x}), \psi^2(\bar{y}), \varphi(f)]$ iff $\psi^1(\bar{x}), \psi^2(\bar{y})$ defines ordered field $\mathbb{F}^1, \mathbb{F}^2$ respectively such that there is an embedding $\mathbf{f}$ of $\mathbb{F}^1$ into $\mathbb{F}^2$ with dense range satisfying $\varphi[h]$. {3.2}

2) $[\dot{\mathbf{Q}}^{\mathrm{br}} y, x][\psi(x), \varphi(y)]$ (defined syntactically as in 3.3) means: $M \models (\dot{\mathbf{Q}}^{\mathrm{br}} y, \bar{x})[\psi(\bar{x}), \varphi(y)]$ {3.2}
iff $\psi(\bar{x})$ defines a leveled partial order (as in 4.1(a)) and there is a full branch of it {4.1}
y (see 4.1(c)) such that $M \models \varphi[y]$. {4.1}

Proof. By 3.5. $\square_{4.5}$ {3.4}

Concluding Remarks 4.7. For ordered field we cannot prove the theorem for embedding, but we can for dense embedding; i.e. the range is dense.

Why not? Suppose we have an ordered field $\mathbb{F}_1$, an ordered real closed field $\dot{f}_2$ and an embedding $\mathbf{f} : \mathbb{F}_1 \to \mathbb{F}_2$ such that the interval $(-\epsilon, +\epsilon)_{\mathbb{F}_2}$ is disjoint to $\mathrm{Rang}(\mathbf{f})(\epsilon \in \mathbb{F}_2, \epsilon > 0)$. Let $\{a_i : i < i^*\} \subseteq \mathbb{F}_1$ be a maximal family of algebraically independent elements, $\langle b_i : i < a^* \rangle$ be such that $b_i \in \mathbb{F}_2, -\epsilon < b_i < \epsilon$. We "correct" $\mathbf{f}$ by letting $\mathbf{f}'(a_i) = \mathbf{f}(a_i) + b_i$, and completing $\mathbf{f}(a)(a \in \mathbb{F})$ by algebraicity.

§ 5. Continuing [Sh:924]

In [Sh:924] assuming $\diamondsuit_{\aleph_1}$, for any countable model M of PA we find N such that:

$(*)_{M,N}$ (a) $M \prec N$
(b) for $a, b \in N$ the linear orders $N_{<a}, N_{<b}$ are isomorphic iff $aE^3_N b$.

Discussion 5.1. We wonder: (see [Sh:924, xxx]): can we deal with larger cardinal? i.e.

Question 5.2. : 1) Assume μ strong limit of cofinality $\aleph_1, \lambda = \mu^+ = 2^\mu, \square_\mu$ holds, $\diamondsuit^*_\lambda$ or so.

Can we find N as above:

$(*)_1$ (a) so let $\mu = \sum\limits_n \mu_n, 2^{\mu_n} < \mu_{n+1}$ when $\mu^-_n = \sum\limits_{\ell<n} \mu_\ell + \aleph_0$.

We consider

$(*)_2$ K is the set of $(\bar{M}, \Gamma)$ such that:
- (a) $\bar{M} = \langle M_n : n < \omega \rangle$
- (b) $M_n \prec M_{n+1}$
- (c) M_n is a μ^+_n-saturated model of T of cardinality 2^{μ_n}
- (d) Γ is a set of countable type omitting by $\cup M_n$
  - $p = \{a_{p_n} < x < a_{q,n} : n < \omega\}$ where
  - $M_{n+1} \models a_{p,n} < a_{p,n+1} < a_{p(n)}$
  - $[a_{p,n}, b_{p,n}]_{M_{n+1}} \cap M_n = \emptyset$.

So our problem is (T countable for transparency)

$(*)_3$ a problem $\mathbf{x}$ consists of:
- (a) a term $\sigma(x, y)$
- (b) $n(*)$ and $a_*, b_* \in M_{n(*)}, a_* > b_*, b_*/E^3_\mu/E^3_\mu$
- (c) $\bar{\varphi}' = (\varphi_1(x), \varphi_2(x)) \in \Phi_{M_{n(*)}}$ is as in?

$(*)_4$ a solution is $\bar{\varphi}', a_1, a_2$
- (a) $\bar{\varphi} \le_{\text{AP}} \bar{\varphi}' \in \Phi_{M_{n(*)+1}}$
- (b) if $d \in M_{n(*)}$, then $\varphi' \Vdash$ "$a_1 < x < a_2$"
  - if $b \in M_{n(*)}$ then $\varphi'(x) \vdash \sigma(x, b) \notin [F(a_1), F(a_2))$.

Naturally

$(*)_5$ we can find $w \in M_{n(*)+1}$ such that:
- (a) $M_{n(*)+1} \models$ "w is a very small set"
- (b) $d \in M_{n(*)} = M_{n(*)+1} \models$ "$d \in w$".

So in $(*)_4$ we shall try to replace "$b \in M_{n(*)+1}$" by $M_{n(*)+1} \models$ "$b \in w$". So now we can hopefully repeat the proof of [Sh:924, 3.7=Ld31=pg.4.7].

Second Way: discussion

1) Question: The first way assume "$\mathbb{R}_M = \mathbb{R}$" is compatible with 3-rigidity. Is this so?

2) Try to force a model of T generated by a perfect set $\langle \lambda_\eta : \eta \in {}^\omega 2 \rangle$, with the set of AP of approximations being $\varphi(\bar{x}_n), \bar{x}_n = \langle x_\eta : \eta \in {}^n 2 \rangle$. Even find $a_* > b_*$ (so a weaker result, just fixing the complete 2-type), e.g.

(∗) $\varphi(\bar{x}_n) \vdash \bigwedge_\eta x_\eta < a_*$

- $\mathfrak{C}_T \models$ "$\varphi(\mathfrak{C}_T)/a_*^{2(n)}$ is $> 1_m$ for some m <u>or</u> is $\geq \frac{1}{d}, d \in \mathfrak{C}_T$ standard small enough

(∗) we have $(\bar{\varphi}^1, \bar{\varphi}^2)$ parallel to [Sh:924, §3].

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