

A PARTITION THEOREM FOR PAIRS OF FINITE SETS

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ABSTRACT. Every partition of $[[\omega_1]^{<\omega}]^2$ into finitely many pieces has a cofinal homogeneous set. Furthermore, it is consistent that every directed partially ordered set satisfies the partition property if and only if it has finite character.

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1. Introduction.

A branch of combinatorics called *Ramsey theory* deals with phenomena of the following kind: If a sufficiently large set of objects is endowed with a certain structure then a large subset can be found whose all elements are “alike”.

A simple instance is the pigeon-hole principle: If there are more pigeons than pigeon-holes then some pigeon-hole is occupied by more than one pigeon. Another is this (the finite Ramsey theorem): For every integer $k > 2$ there is an integer n with the property that if A is a set of at least n elements and if the set of all (unordered) pairs $\{a, b\} \subset A$ is divided into two classes then there is a subset $H \subset A$ of size k such that all pairs $\{a, b\} \subset H$ belong to the same class.

Many such principles have been formulated and proved, with applications in various branches of mathematics; most are variants of Ramsey’s Theorem [2].

Ramsey’s Theorem states (in particular) that every partition of the set $[\mathbf{N}]^2$ (into finitely many pieces) has an infinite homogeneous set, i.e. a set $H \subseteq \mathbf{N}$ cofinal in $(\mathbf{N}, <)$ such that $[H]^2$ is included in one piece of the partition. The following generalization of Ramsey’s Theorem was suggested in [3]:

Let A be an infinite set, and let $[A]^{<\omega}$ denote the set of all finite subsets of A . A set $H \subseteq [A]^{<\omega}$ is *cofinal* in $[A]^{<\omega}$ if for every $x \in [A]^{<\omega}$ there exists a $y \in H$ such that $x \subseteq y$. Note that if a cofinal set H is partitioned into two pieces, $H = H_1 \cup H_2$, then at least one of the two sets H_1, H_2 is cofinal.

Let $F : [[A]^{<\omega}]^2 \rightarrow \{1, \dots, k\}$ be a partition of pairs of finite subsets of A ; a set $H \subseteq [A]^{<\omega}$ is *homogeneous* for F if all pairs $(a, b) \in [H]^2$ with the property that $a \subset b$ belong to the same piece of the partition, i.e.

$$F(x_1, x_2) = F(y_1, y_2)$$

whenever $x_1, x_2, y_1, y_2 \in H$ and $x_1 \subset x_2, y_1 \subset y_2$.

The question raised in [3] asked whether for every infinite A , every partition of $[[A]^{<\omega}]^2$ has a cofinal homogeneous set.

It is not difficult to see that if A is countable, then $[A]^{<\omega}$ has a cofinal subset of order type ω and so $[A]^{<\omega}$ satisfies the partition property as a consequence of Ramsey’s Theorem. For an arbitrary A , the partition property in question is a generalization of Ramsey’s Theorem for pairs.

We answer the question in the affirmative in the case when $|A| = \aleph_1$:

Theorem 1. *If $|A| = \aleph_1$, then every partition of $[[A]^{<\omega}]^2$ into finitely many pieces has a cofinal homogeneous set.*

The question remains open for sets of size greater than \aleph_1 . By an unpublished theorem of Galvin, Martin’s Axiom implies the partition property for all sets A of cardinality less than 2^{\aleph_0} .

More generally, let S be a partially ordered set, and assume that S is directed and does not have a maximal element. A set $H \subseteq S$ is *cofinal* in S if for every $x \in S$ there exists a $y \in H$ such that $x \leq y$. Let $r \geq 2$ and $k \geq 2$, and let $F : [S]^r \rightarrow \{1, \dots, k\}$ be a partition

of r -tuples in S . A set $H \subseteq S$ is *homogeneous* for F if for all x_1, \dots, x_r and y_1, \dots, y_r such that $x_1 < \dots < x_r$ and $y_1 < \dots < y_r$ we have

$$F(x_1, \dots, x_r) = F(y_1, \dots, y_r).$$

Using the standard arrow notation, the formula

$$S \rightarrow (\text{cofinal subset})_k^r$$

states that for every partition $F : [S]^r \rightarrow \{1, \dots, k\}$ there exists a cofinal subset H of S homogeneous for F .

The following is an unpublished result of Galvin [1]:

Theorem 2. (*F. Galvin*) Assume $MA(\kappa)$. Let S be a partially ordered set of power κ , which is directed, and suppose for all $a \in S$, $\{b \in S : b < a\}$ is finite. Let $f : \{(x, y) \in S \times S : x < y\} \rightarrow \{\text{red}, \text{blue}\}$. Then there is a cofinal $H \subseteq S$ such that f is constant on $\{(x, y) \in H \times H : x < y\}$.

Galvin's method admits a generalization to partitions of r -tuples, for any $r \geq 2$ (see the proof of Theorem 4 below). Thus assuming Martin's Axiom the following holds:

Theorem 2'. Let S be a directed partially ordered set of cardinality less than 2^{\aleph_0} , without maximal element and such that for every $a \in S$ the set $\{x \in S : x < a\}$ is finite. Then

$$S \rightarrow (\text{cofinal subset})_k^r \quad \text{for all } r, k < \omega.$$

Note that every partially ordered set S with the properties stated above is isomorphic to a cofinal subset of $[S]^{<\omega}$.

The statement that for every cofinal $S \subseteq [\omega_1]^{<\omega}$,

$$S \rightarrow (\text{cofinal subset})_2^2$$

is not a theorem of ZFC, as by an unpublished result of Laver [4] a counterexample exists under the assumption of the continuum hypothesis:

Theorem 3. (*R. Laver*) Let κ be a cardinal such that $\kappa^{\aleph_0} = \kappa$. Then there exist a cofinal set $S \subset [\kappa]^{<\omega}$ and a partition $F : [S]^2 \rightarrow \{1, 2\}$ such that no cofinal subset of S is homogeneous for F .

With Laver's permission we include the proof of Theorem 3 below.

We say that a partially ordered set S has *finite character* if S has a cofinal set S' such that every $x \in S'$ has only finitely many predecessors in S' . Thus Galvin's theorem implies that

$$S \rightarrow (\text{cofinal subset})_k^r$$

holds for every set S of size \aleph_1 that has finite character, if Martin's Axiom holds together with $2^{\aleph_0} > \aleph_1$, and Laver's theorem implies that if $2^{\aleph_0} = \aleph_1$ then a partial order S exists that has size \aleph_1 and finite character but

$$S \rightarrow (\text{cofinal subset})_2^2$$

fails.

Theorem 4. *In the Cohen model for $2^{\aleph_0} = \aleph_2$ the following statements are equivalent for every directed set of cardinality \aleph_1 :*

- (1) $S \rightarrow (\text{cofinal subset})_2^2$
- (2) $S \rightarrow (\text{cofinal subset})_k^r$ for all $r, k < \omega$
- (3) S has finite character.

The consistency proof of “(3) implies (2)” is essentially the Galvin’s result; we will show that (1) implies (3) in the Cohen model.

2. Proof of Theorem 1.

Throughout this section we consider a fixed partition $F : [[\omega_1]^{<\omega}]^2 \rightarrow \{1, \dots, k\}$. The pairs $\{x, y\}$ such that $x \subset y$ are divided into two classes; we shall refer to these two classes as *colors*.

We reserve lower case letters such as a, b, c for finite subsets of ω_1 , and capital letters such as A, B, C for at most countable subsets of ω_1 .

A *partial coloring* of a finite set a is a function f whose domain is a set of subsets of a , and whose values are in the set $\{1, \dots, k\}$. A *total coloring* of a is a partial coloring whose domain is the set of all subsets of a .

If $a \subset b$ and if f is a partial coloring of a then b is *f-correct* if for every $x \in \text{dom}(f)$, the pair (x, b) has the color $f(x)$ (i.e. $F(x, b) = f(x)$).

If $a \subseteq A$ then b is an *A-extension* of a , $a \leq_A b$, if $a \subseteq b$ and $b \cap A = a$. An *A-extension* b of a is *proper* if $a \subset b$.

We shall consider pairs (a, A) where a is finite, A is at most countable and $a \subseteq A$. If $a \subseteq A$ and $b \subseteq B$ then

$$(a, A) \leq (b, B)$$

means that $A \subseteq B$ and b is an A -extension of a . Note that \leq is transitive.

Definition 2.1. Let $a \subseteq A$, and let f be a partial coloring of a . We say that the pair (a, A) is *good for f* if for every $(b, B) \geq (a, A)$ there exists a proper B -extension c of b that is f -correct.

Remark. If (a, A) is good for f and if $f' \subseteq f$ and $(a', A') \geq (a, A)$ then (a', A') is good for f' .

Lemma 2.2. *For every (b, B) there exist a total coloring g of b and some $(c, C) \geq (b, B)$ such that (c, C) is good for g .*

Moreover, we may require that c is a proper B -extension of b and is g -correct.

Proof. First assume that g and $(c, C) \geq (b, B)$ are as claimed in the first part of the lemma. Then there is some $d >_C c$ that is g -correct, and $(d, C \cup d)$ is good for g . Hence it suffices to find for each (b, B) a total coloring g of b and some $(c, C) \geq (b, B)$ good for g .

Thus assume that the lemma fails and let (b, B) be such that for every total coloring g of b , no $(c, C) \geq (b, B)$ is good for g .

There are finitely many total colorings g_1, \dots, g_m of b . We construct a sequence (b_i, B_i) , $i = 1, \dots, m$ so that

$$(b, B) \leq (b_1, B_1) \leq \dots \leq (b_m, B_m)$$

as follows:

As (b, B) is not good for g_1 , there exists some $(b_1, B_1) \geq (b, B)$ such that no proper B_1 -extension of b_1 is g_1 -correct.

Next, as (b_1, B_1) is not good for g_2 , there exists some $(b_2, B_2) \geq (b_1, B_1)$ such that no proper B_2 -extension of b_2 is g_2 -correct.

And so on. For each $i = 1, \dots, m$, no proper B_i -extension of b_i is g_i -correct.

Now let c be an arbitrary proper B_m -extension of b_m . Let us consider the following total coloring g of b :

$$g(x) = F(x, c) \quad (\text{the color of } (x, c)).$$

We have $g = g_i$ for some $i \leq m$. It is now clear that c is a g_i -correct proper B_i -extension of b_i , a contradiction. \square

Lemma 2.3. *If (a, A) is good for f , then for every $(b, B) \geq (a, A)$ there exists a total coloring g of b extending f , and some $(c, C) \geq (b, B)$ such that c is a g -correct proper B -extension of b and (c, C) is good for g .*

Proof. The proof proceeds as in Lemma 2.2, the difference being that we consider only the total colorings g_1, \dots, g_m of b that extend f . After having constructed $(b_1, B_1) \leq \dots \leq (b_m, B_m)$, we find (because (a, A) is good for f and $(a, A) \leq (b_m, B_m)$) a proper B_m -extension c of b_m that is f -correct. Then g (defined as above) extends f and so $g = g_i$ for some $i \leq m$. The rest of the proof is as before. \square

We shall use Lemma 2.2 and Lemma 2.3 to construct an end-homogeneous cofinal set $H \subseteq [\omega_1]^{<\omega}$.

Definition 2.4. A set H is *end-homogeneous* if for all $x, y, z \in H$, if $x \subset y$ and $x \subset z$, then (x, y) and (x, z) have the same color.

Note that if H is a cofinal end-homogeneous set, then one of the sets

$$H_i = \{a \in H : F(a, x) = i \text{ for all } x \in H \text{ such that } a \subset x\} \quad (i = 1, \dots, k)$$

is cofinal, and is homogeneous. It follows that it suffices to construct a cofinal end-homogeneous set.

Definition 2.5. An *approximation* is a triple (A, G, H) where A is an infinite countable subset of ω_1 , G and H are disjoint cofinal subsets of $[A]^{<\omega}$, H is end-homogeneous, and for every $a \in G$, (a, A) is good for f_a^H , where f_a^H is the partial coloring of a defined on $\{x \subset a : x \in H\}$ by

$$f_a^H(x) = \text{the color of } (x, y), \text{ where } y \text{ is any } y \in H \text{ such that } x \subset y.$$

Let

$$(A, G, H) \leq (A', G', H')$$

mean that $A \subseteq A'$, $G \subseteq G'$ and $H \subseteq H'$. We want to construct an increasing sequence of approximations $(A_\alpha, G_\alpha, H_\alpha)$, such that $\bigcup_\alpha A_\alpha = \omega_1$. Then $H = \bigcup_\alpha H_\alpha$ is an end-homogeneous set, cofinal in $[\omega_1]^{<\omega}$.

It is easy to verify that if λ is a countable limit ordinal, and if $(A_\alpha, G_\alpha, H_\alpha)$, $\alpha < \lambda$, is an increasing sequence of approximations, then $(\bigcup_\alpha A_\alpha, \bigcup_\alpha G_\alpha, \bigcup_\alpha H_\alpha)$ is an approximation. Thus to complete the proof, it suffices to prove the following two lemmas:

Lemma 2.6. *There exists an approximation.*

Lemma 2.7. *Let (A, G, H) be an approximation and let $\xi \in \omega_1 - A$ be arbitrary. Then there is an approximation $(\bar{A}, \bar{G}, \bar{H}) \geq (A, G, H)$ such that $\xi \in \bar{A}$.*

Proof of Lemma 2.6. We construct A as the union of a sequence $c_0 \subset c_1 \subset \dots \subset c_n \subset \dots$ of finite sets, as follows. Let b_0 be an arbitrary finite subset of ω_1 . By Lemma 2.2. there exist a total coloring g_0 of b_0 and some (c_0, C_0) such that (c_0, C_0) is good for g_0 and $c_0 \supset b_0$ is g_0 -correct.

Now let $n \geq 0$; we have constructed $(c_0, C_0), \dots, (c_n, C_n)$ such that $c_0 \subset \dots \subset c_n$. Fix for each $i \leq n$ an enumeration of C_i of order-type ω . Let $b_{n+1} \supseteq c_n$ be a finite set such that for each $i \leq n$, b_{n+1} contains the first n elements of C_i . This will guarantee that $\bigcup_{n=0}^\infty c_n = \bigcup_{n=0}^\infty C_n$.

By Lemma 2.2. there exist a total coloring g_{n+1} of b_{n+1} and some (c_{n+1}, C_{n+1}) such that (c_{n+1}, C_{n+1}) is good for g_{n+1} and $c_{n+1} \supset b_{n+1}$ is g_{n+1} -correct.

We let $A = \bigcup_{n=0}^\infty C_n$. To construct G and H , consider the partition F restricted to the set $[\{c_n\}_{n=0}^\infty]^2$. By Ramsey's Theorem, $\{c_n\}_{n=0}^\infty$ has an infinite homogeneous (let us say green) subsequence. Let us denote this subsequence

$$d_0 \subset e_0 \subset d_1 \subset e_1 \subset \dots \subset d_i \subset e_i \subset \dots$$

and let $G = \{d_i\}_{i=0}^\infty$, $H = \{e_i\}_{i=0}^\infty$. Clearly, G and H are disjoint cofinal subsets of $[A]^{<\omega}$. Moreover, H is homogeneous, and we claim that for every $a \in G$, (a, A) is good for f_a^H . Since $a = c_n$ for some n , (c_n, C_n) is good for g_n and $(c_n, C_n) \leq (c_n, A)$, it suffices to show that $f_a^H \subseteq g_n$. If $x \in \text{dom } f_a^H$ then because g_n is a total coloring of b_n and $x = c_m$ for some $m < n$, and because c_n is g_n -correct, we have $g_n(x) = \text{the color of } (x, a)$, which is green, because both x and a are in the homogeneous sequence. But $f_a^H(x)$ is also green. Hence $f_a^H \subseteq g_n$, and (A, G, H) is an approximation. \square

Proof of Lemma 2.7. We construct \bar{A} as the union of a sequence $c_0 \subset c_1 \subset \dots \subset c_n \subset \dots$ of finite sets, as follows. First, we choose an increasing cofinal sequence $a_0 \subset a_1 \subset \dots \subset a_n \subset \dots$ in G . Let b_0 be some A -extension of a_0 such that $\xi \in b_0$. As (a_0, A) is good for $f_{a_0}^H$, there exist a total coloring g_0 of b_0 extending $f_{a_0}^H$, an A -extension c_0 of a_0 such that $c_0 \supset b_0$ and some $C_0 \supseteq A \cup c_0$ such that c_0 is g_0 -correct and (c_0, C_0) is good for g_0 .

Now assume that (c_n, C_n) has been constructed and c_n is an A -extension of a_n . Let b_{n+1} be some A -extension of a_{n+1} such that $b_{n+1} \supseteq c_n$. Moreover, we choose b_{n+1} large enough to contain the first n elements of each $C_i - A$, $i = 0, \dots, n$ (in some fixed enumeration). This will guarantee that $\bigcup_{n=0}^\infty c_n = \bigcup_{n=0}^\infty C_n$.

As (a_{n+1}, A) is good for $f_{a_{n+1}}^H$, there exist a total coloring g_{n+1} of b_{n+1} extending $f_{a_{n+1}}^H$, an A -extension c_{n+1} of a_{n+1} such that $c_{n+1} \supset b_{n+1}$ and some $C_{n+1} \supseteq A \cup c_{n+1}$ such that c_{n+1} is g_{n+1} -correct and (c_{n+1}, C_{n+1}) is good for g_{n+1} .

We let $\bar{A} = \bigcup_{n=0}^{\infty} C_n$. To construct \bar{G} and \bar{H} , consider the partition restricted to the set $[\{c_n\}_{n=0}^{\infty}]^2$. By Ramsey's Theorem, $\{c_n\}_{n=0}^{\infty}$ has an infinite homogeneous (let us say green) subsequence. Let us denote this subsequence

$$d_0 \subset e_0 \subset d_1 \subset e_1 \subset \dots \subset d_i \subset e_i \subset \dots$$

and let $\bar{G} = G \cup \{d_i\}_{i=0}^{\infty}$, $\bar{H} = H \cup \{e_i\}_{i=0}^{\infty}$. Clearly, \bar{G} and \bar{H} are disjoint cofinal subsets of $[\bar{A}]^{<\omega}$, and $G = \bar{G} \cap [A]^{<\omega}$, $H = \bar{H} \cap [A]^{<\omega}$. It remains to show that \bar{H} is end-homogeneous, and that for every $a \in \bar{G}$, (a, \bar{A}) is good for $f_a^{\bar{H}}$.

To prove that \bar{H} is end-homogeneous, we have to show that the color of (x, y) for $x, y \in \bar{H}$ does not depend on y . If $x \in \bar{H} - H$, say $x = e_i$, then every $y \supset x$ in \bar{H} is some e_m , and (e_i, e_m) is green. If $x \in H$, then the color of (x, y) is determined by H , and should be equal to $f_a^H(x)$, for any $a \supset x$ in G . We have to show that (x, e_i) has this color, for all $e_i \supset x$. So let i be such that $e_i \supset x$; we have $e_i = c_n$ for some n . As c_n is an A -extension of a_n , it follows that $x \subset a_n$. Since c_n is g_n -correct and $g_n \supseteq f_{a_n}^H$, c_n is $f_{a_n}^H$ -correct. Therefore (x, c_n) has color $f_{a_n}^H(x)$.

Finally, we prove that for every $a \in \bar{G}$, (a, \bar{A}) is good for $f_a^{\bar{H}}$. If $a \in G$, then $f_a^{\bar{H}}$ is just f_a^H because $\{x \subset a : x \in \bar{H}\} = \{x \subset a : x \in H\}$. Because (a, A) is good for f_a^H , and $A \subseteq \bar{A}$, (a, \bar{A}) is good for $f_a^{\bar{H}}$. So let $a \in \bar{G} - G$, say $a = d_i = c_n$. We know that (c_n, C_n) is good for g_n , and $C_n \subseteq \bar{A}$, so it suffices to show that $f_a^{\bar{H}} \subseteq g_n$, and then it follows that (a, \bar{A}) is good for $f_a^{\bar{H}}$.

So let $x \subset a$ be an element of \bar{H} . If $x \in H$ then, because $a = c_n$ is an A -extension of a_n , $x \subset a_n$ and so $x \in \text{dom}(f_{a_n}^H)$. We already know that \bar{H} is end-homogeneous, so $f_a^{\bar{H}}(x) = f_{a_n}^H(x) =$ the color of (x, y) for any $y \supset x$ in \bar{H} . Because g_n is an extension of $f_{a_n}^H$, we have $f_a^{\bar{H}}(x) = g_n(x)$.

If $x \in \bar{H} - H$ then $x = c_m$ for some $m < n$, and $f_a^{\bar{H}}(x) =$ green (because $x = e_i$ for some i). Now g_n is a total coloring of b_n , and $b_n \supseteq c_m$, so $x \in \text{dom}(g_n)$ and it remains to show that $g_n(x) =$ green. But c_n is g_n -correct, and so $g_n(x) =$ the color of $(c_m, c_n) =$ green. \square

3. Proof of Theorem 4.

We shall prove that the equivalence between the partition property and finite character, for directed partial orders of size \aleph_1 , holds in the model $V[G]$ obtained by adding \aleph_2 Cohen reals to a ground model for ZFC.

We shall first prove that (3) implies (1) in the Cohen model, and then outline how the proof can be modified to show that (3) implies (2). Assume that S is a directed partially ordered set of size \aleph_1 in the model $V[G]$, and assume that each $a \in S$ has finitely many predecessors. Let F be a partition of $[S]^2$. As $|S| = |F| = \aleph_1$, $V[G]$ is a generic extension of $V[S, F]$ by the Cohen forcing, and we may assume that S and F are in the ground model. Thus it suffices to prove that adding \aleph_1 Cohen reals produces a cofinal homogeneous set for F .

In fact, we define a forcing notion P that produces a generic cofinal homogeneous set for F , and then show that P is equivalent to adding \aleph_1 Cohen reals. The forcing notion P

is essentially the one used by Galvin in his proof of the partition property for $[\omega_1]^{<\omega}$ from Martin's Axiom.

Let D be an ultrafilter on S with the property that for every $a \in S$, $\{x \in S : a \leq x\} \in D$. We say that $a \in S$ is *red*, if for D -almost all $x > a$, (a, x) is red; otherwise a is *green*. Either almost all $a \in S$ are red, or almost all are green; let us assume that almost all $a \in S$ are red. A forcing condition in P is a finite red-homogeneous set of red points. A condition p is stronger than q if $p \supseteq q$ and if for no $x \in p - q$ and no $y \in q$ we have $x < y$.

Using the ultrafilter D one can easily verify that for every $p \in P$ and every $a \in S$ there exists some $x \geq a$ such that $p \cup \{x\}$ is a condition stronger than p . Therefore a generic set is a cofinal homogeneous set.

We shall finish the proof by showing that the forcing P is equivalent to adding \aleph_1 Cohen reals. Let S_α , $\alpha < \omega_1$, be an elementary chain of countable submodels of $(S, <, \text{red})$, with limit S . For each α , let $P_\alpha = \{p \in P : p \subset S_\alpha\}$. Each P_α is a countable forcing notion, therefore equivalent to adding a Cohen real. It suffices to prove that every maximal antichain in P_α is a maximal antichain in P . This will follow from this claim: For every $p \in P$ there is a $\bar{p} \in P_\alpha$ such that every $q \in P_\alpha$ stronger than \bar{p} is compatible with p . Note that conditions p and q are compatible if and only if no element of $p - q$ is less than any element of q and no element of $q - p$ is less than any element of p .

Let $p \in P$. Let Z be the (finite) set $\{x \in S_\alpha : x \leq a \text{ for some } a \in p\}$, and let $u \in S_\alpha$ be a red point such that $u > x$ for all $x \in Z$, and that (x, u) is red for all $x \in p \cap S_\alpha$. Such a u exists as S_α is an elementary submodel. Now let $\bar{p} = (p \cap S_\alpha) \cup \{u\}$.

Clearly, \bar{p} is a condition in P_α . Let $q \in P_\alpha$ be stronger than \bar{p} and let us show that q and p are compatible. First, let $x \in q - p$ and $y \in p$. We claim that x is not less than y : since q is stronger than \bar{p} , x is not less than u , hence $x \notin Z$ and because $x \in S_\alpha$, the claim follows.

Second, let $x \in q$ and $y \in p - q$. We claim that y is not less than x : this is because $x \in S_\alpha$, $y \notin S_\alpha$, and since S_α is an elementary submodel and x has finitely many predecessors, all $z < x$ are in S_α .

Hence p and q are compatible.

We shall now outline how the above proof is modified to show that (3) implies (2) in the Cohen model. For instance, let $k = 2$ and $r = 3$. The above proof produces in fact a homogeneous cofinal set H such that $D \cup \{H\}$ has the finite intersection property. (For every condition and every $A \in D$ there exists a stronger condition q such that $q \cap A \neq \emptyset$.)

Let F be a partition of $[S]^3$ into {red, green}. For each $a \in S$, let F_a be the partition of $[S]^2$ given by $F_a(x, y) = F(a, x, y)$. Let D be an ultrafilter on S as before, and let P_a denote the forcing that produces a homogeneous cofinal set for F_a . The product of $\{P_a : a \in S\}$ is isomorphic to adding \aleph_1 Cohen reals and if $\{H_a : a \in S\}$ are the generic homogeneous cofinal sets then $D \cup \{H_a : a \in S\}$ has the finite intersection property.

We may therefore assume that the sets H_a are in the ground model, and $H_a \in D$ for each $a \in S$. We say that $a \in S$ is *red*, if H_a is red-homogeneous; otherwise a is *green*. Assuming that almost all $a \in S$ are red, a forcing condition is a finite red-homogeneous set of red points. This forcing produces a cofinal homogeneous set for the partition F , and is equivalent to adding \aleph_1 Cohen reals. \square

We shall now prove that (1) implies (3) in the Cohen model $V[G]$. So let $S \in V[G]$ be

a directed partially ordered set of size \aleph_1 and assume that S has the partition property. Consider the forcing notion P that adds, with finite conditions, a generic partition of $[S]^2$:

The forcing conditions in P are functions whose domain is a finite subset of $[S]^2$, with values $\{\text{red, green}\}$, and let \dot{F} be the canonical name for a P -generic set. Clearly, P is equivalent to adding \aleph_1 Cohen reals, and if Q is the forcing that adds \aleph_2 Cohen reals, we have $Q \times P \simeq Q$. We shall prove:

Lemma. *P forces that if \dot{F} has a cofinal homogeneous set, then S has finite character.*

Granted the lemma, we complete the proof of Theorem 2 as follows: Let \dot{S} be a Q -name for $S \in V[G]$, and let R be the forcing such that $V^Q = V[\dot{S}]^R$. We have $R \simeq Q$ and so $R \simeq R \times P$. The assumption is that R (and therefore $R \times P$) forces that every partition of S has a cofinal homogeneous set. Hence $R \times P$ forces that \dot{F} has a cofinal homogeneous set, and it follows from the lemma that $R \times P$ forces that S has finite character. Hence in $V[G]$, S has finite character.

Proof of Lemma. Let \dot{H} be a P -name for a cofinal homogeneous set for \dot{F} , and assume that P forces that $[\dot{H}]^2$ is green. Let S_α , $\alpha < \omega_1$, be an elementary chain of countable submodels of $(S, <, P, \Vdash, \dot{F}, \dot{Q})$. First we claim that every condition forces the following: For every α , if $a \in \dot{H} - S_\alpha$ then the set $\{x \in \dot{H} \cap S_\alpha : x < a\}$ is finite.

So let us assume otherwise, and let $a \notin S_\alpha$ and $p \in P$ be such that $p \Vdash a \in \dot{H}$ and that $p \Vdash \{x \in \dot{H} \cap S_\alpha : x < a\}$ is infinite. There is therefore some $x < a$, $x \in S_\alpha$ such that $(x, a) \notin \text{dom } p$ and that some q stronger than p forces $x \in \dot{H}$. Since S_α is an elementary submodel, there is some q stronger than the restriction of p to $[S_\alpha]^2$ such that $\text{dom}(q) \subset [S_\alpha]^2$ and that q forces $x \in \dot{H}$. Now q and p are compatible conditions, and moreover, (x, a) is not in the domain of $q \cup p$, so let r be the extension of $p \cup q$ that forces that (x, a) is red. Then $r \Vdash (x \in \dot{H} \text{ and } a \in \dot{H} \text{ and } (x, a) \text{ is red})$ which is a contradiction since $x < a$ and $[\dot{H}]^2$ is forced to be green.

Now we shall construct, in V^P , a cofinal subset C of H such that each $a \in C$ has only finitely many predecessors in C . For each α , let $a_{\alpha 0} \in S_{\alpha+1} - S_\alpha$ be, if it exists, an element of H that is not below any $x \in H \cap S_\alpha$. Then let $a_{\alpha n}$, $n < \omega$, be an increasing sequence starting with $a_{\alpha 0}$, cofinal in $H \cap S_{\alpha+1}$. Finally, let $C = \{a_{\alpha n} : \alpha < \omega_1, n < \omega\}$.

The set C is cofinal in H . If $a_{\alpha n} \in C$, then by the claim proved above, $a_{\alpha n}$ has only finitely many predecessors in $C \cap S_\alpha$, and because $a_{\beta 0}$ is not less than $a_{\alpha n}$ for any $\beta > \alpha$, $a_{\alpha n}$ has only finitely many predecessors in C . \square

4. Proof of Laver's Theorem.

Let a_α and (M_α, H_α) , $\alpha < \kappa$, enumerate, respectively, the set $[\kappa]^{<\omega}$ and the set of all pairs (M, H) where $M \in [\kappa]^{\leq \aleph_0}$ and $H \subseteq [M]^{<\omega}$ is cofinal in $[M]^{<\omega}$. Furthermore, assume that $a_\alpha \subseteq \alpha$ and $M_\alpha \subseteq \alpha$ for all α .

We construct a cofinal set $S = \{s_\alpha : \alpha < \kappa\}$ and a partition $F : [S]^2 \rightarrow \{1, 2\}$ as follows: Let $\alpha < \kappa$. Let $b_0 = a_\alpha \cup \{\alpha\}$; α is the largest element of b_0 . Choose, if possible, two distinct elements c_0 and d_0 of H_α , and let $b_1 = b_0 \cup c_0 \cup d_0$. Note that α is the largest element of b_1 . Let α_1 be the largest element of b_1 below α , and choose, if possible, c_1 and

d_1 in H_{α_1} , distinct from c_0 and d_0 and from each other, and let $b_2 = b_1 \cup c_1 \cup d_1$. Let α_2 be largest in b_2 below α_1 , and choose c_2, d_2 in H_{α_2} distinct from c_0, d_0, c_1, d_1 . This procedure terminates after finitely many, say k , steps, and we let $s_\alpha = b_k$.

For each $i \leq k$, let $F(c_i, s_\alpha) = 1$ and $F(d_i, s_\alpha) = 2$, provided c_i and d_i are defined. Note that $\max s_\alpha = \alpha$, and that if β is the i th largest element of s_α and if M_β is infinite then c_i and d_i are defined; hence there exist c and d in H_β such that $F(c, s_\alpha) = 1$ and $F(d, s_\alpha) = 2$.

Let $S = \{s_\alpha : \alpha < \kappa\}$, and let F be a partition of $[S]^2$ that satisfies the conditions specified above. We claim that no cofinal subset of S is homogeneous for F .

Thus let H be a cofinal subset of S . There exists an infinite countable set $M \subset \kappa$ such that $H \cap [M]^{<\omega}$ is cofinal in $[M]^{<\omega}$; let $\beta < \kappa$ be such that $M_\beta = M$ and $H_\beta = H \cap [M]^{<\omega}$. As H is cofinal, there is an $x \in H$ such that $\beta \in x$; as $H \subseteq S$, there is some α such that $x = s_\alpha$.

Since M_β is infinite, there exist $c, d \in H_\beta$ such that $F(c, s_\alpha) = 1$ and $F(d, s_\alpha) = 2$. Hence H is not homogeneous for F . \square

5. Open problems.

- (1) $[\aleph_2]^{<\omega} \rightarrow (\text{cofinal subset})_2^2$ (in ZFC)
- (2) $[\aleph_1]^{<\omega} \rightarrow (\text{cofinal subset})_2^3$ (in ZFC)
- (3) Is it consistent that there exists a directed partial ordering of size \aleph_1 that does not have finite character but has the partition property?
- (4) $[A]^{<\omega} \rightarrow (\text{cofinal subset})_k^r$
for all infinite sets A and all integers $r, k \geq 2$.

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