

On closed P -sets with ccc in the space ω^*

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Abstract. It is proved that – consistently – there can be no ccc closed P -sets in the remainder space ω^* .

In this paper we show how to construct a model of set theory in which there are no P -sets satisfying ccc (countable antichain condition) in the ultrafilter space $\omega^* = \beta\omega \setminus \omega$. The problem of the existence of such sets (which are generalizations of P -points) was known since some time and occurred explicitly in $\mathfrak{c}vM-R$ —. In the proof we follow the construction from $\mathfrak{c}S$ — of a model in which there are no P -points. A particular case of P -sets, which are supports of approximative measures has been settled in $\mathfrak{c}M$ —, where the author shows that there can be no such measures on $P(\omega)/fin$. (Under CH , e.g. the Gleason space $\mathbf{G}(2^\omega)$ of the Cantor space is a ccc P -set in ω^* which carries no approximative measure).

Sec.1. Closed P -sets in the space ω^* can be identified with P -filters F on ω . Thus, the dual ideal $I = \{\omega \setminus A : A \in F\}$ has the property:

$$(1.1) \quad \begin{array}{l} \text{If } A_n \in I, \text{ for } n \in \omega, \text{ then there is an } A \in I \\ \text{such that } A_n \subseteq_* A, \text{ for each } n \in \omega. \end{array}$$

Further, the countable chain condition imposed upon F implies that I is fat in the following sense (see $\mathfrak{c}F-Z$ —):

$$(1.2) \quad \begin{array}{l} \text{if } A_n \in I, \text{ for } n \in \omega \text{ and } \lim_n \min A_n = \infty, \\ \text{then there is an infinite } Z \subseteq \omega \text{ such that } \bigcup_{n \in Z} A_n \in I. \end{array}$$

Indeed, let $e_n = A_n \setminus A$, where $A \in I$ is as in (1.1). Since $\min A_n$ are arbitrarily large, we can find an infinite set $Y \subseteq \omega$ such that the family $\{e_n : n \in Y\}$ is disjoint. If $\{Y_\alpha : \alpha < c\}$ is an almost disjoint family of subsets of Y , then the unions

$$S_\alpha = \bigcup \{e_n : n \in Y_\alpha\}, \quad \alpha < c$$

are almost disjoint and hence the closures S_α^* in the space ω^* are disjoint. By ccc we have

$$S_\alpha^* \cap \bigcap \{B^* : B \in F\} = \emptyset,$$

for some α and consequently $S_\alpha \in I$. It follows that the union

$$\bigcup_{n \in Y_\alpha} A_n = \bigcup_{n \in Y_\alpha} (A_n \cap A) \cup \bigcup_{n \in Y_\alpha} (A_n \setminus A)$$

is in I as a subset of $S_\alpha \cup A$.

Let us fix a given ccc P -filter F and its dual I . We shall define a forcing $\mathbf{P} = \mathbf{P}(F)$.

A partial ordering (T, \leq_T) , where $T \subseteq \omega$, will be called a tree, if for each $i \in T$ the set of predecessors $\{j \in T : j \leq_T i\}$ is linearly ordered and

$$i \leq_T j \text{ implies } i \leq j, \text{ for all } i, j \in T.$$

We define a partial ordering for trees

$$T \leq_t S \text{ iff } (S, \leq_S) \text{ is a subordering of } (T, \leq_T) \\ \text{and each branch of } T \text{ contains cofinally a (unique) branch of } S.$$

There is a tree T_0 such that $T_0 \in I$ and T_0 is order isomorphic to the full binary tree of height ω .

Deleting the numbers $\leq n$ from T_0 we obtain a subtree denoted by $T_0^{(n)}$ (we have $T_0^{(n)} \leq_t T_0^{(m)}$, for $n \leq m$). Let \mathcal{T} consist of all the trees $T \in I$ such that

$$T \leq_t T_0^{(n)}, \text{ for some } n \in \omega.$$

Note that each tree $T \in \mathcal{T}$ has finitely many roots.

DEFINITION. *Elements of the forcing \mathbf{P} are of the form $p = \langle T_p, f_p \rangle$, where $T_p \in \mathcal{T}$ and $f_p : T_p \rightarrow \{0, 1\}$. The ordering of \mathbf{P} is defined thus*

$$p \leq q \text{ iff } T_p \leq_t T_q \text{ and } f_p \supseteq f_q.$$

Let $\{b_\alpha : \alpha < c\}$ be a fixed enumeration of all the branches of T_0 in V . For a generic $G \subseteq \mathbf{P}$ let $T_G = \bigcup_{p \in G} T_p$ and $f_G = \bigcup_{p \in G} f_p$.

Each branch B of T_G contains cofinally a unique b_α . Let us write $B = B_\alpha$ and define

$$X_\alpha = \{i \in \omega : i \in B_\alpha \text{ and } f_G(i) = 1\}$$

Since $T_p \in I$, for any $p \in \mathbf{P}$, hence $\omega \setminus T_p \cap A$ is infinite, for each $A \in F$. It follows that the sets

$$D_{n\varepsilon}^{A\alpha} = \{p \in \mathbf{P} : \exists i > n \diamond i \in b_\alpha^p \text{ and } f_p(i) = \varepsilon\}$$

are dense, for each $A \in F$, $n \in \omega$, $\alpha < c$ and $\varepsilon = 0, 1$ (here b_α^p denotes the branch of T_p extending b_α).

Thus, \mathbf{P} adds uncountably many almost disjoint Gregorieff-like sets.

Sec.2. Let $\mathbf{Q} = \mathbf{Q}(F)$ be a countable product of $\mathbf{P} = \mathbf{P}(F)$. Thus the elements $q \in \mathbf{Q}$ can be written in the form

$$q = \langle f_0^q, f_1^q, \dots \rangle, \text{ where } \langle \text{dm}(f_i^q), f_i^q \rangle \in \mathbf{P}, \text{ for each } i < \omega.$$

By $q^{(n)}$ we denote the condition $\langle g_i : i < \omega \rangle$ where

$$g_i = \begin{cases} f_i^q \upharpoonright \text{dm}(f_i^q)^{(n)}, & \text{for } i < n \\ f_i^q & \text{for } i \geq n \end{cases}$$

Here $T^{(n)}$ is a tree obtained from T by deleting the numbers $\leq n$.

LEMMA 2.1. *For each decreasing sequence $p_0 \geq p_1 \geq \dots$ there is a q and an infinite $Z \subseteq \omega$ such that*

$$q \leq p_n^{(n)}, \text{ for each } n \in Z.$$

PROOF. Let $T_{ni} = \text{dm}(f_i^{p_n})$, where $p_n = \langle f_i^{p_n} : i < \omega \rangle$. Since $\min T_{ni}^{(n)} \geq n$, we may use (1.2) to define by induction a descending sequence $Z_0 \supseteq Z_1 \supseteq \dots$ of infinite subsets of ω such that

$$\bigcup_{n \in Z_i} T_{ni}^{(n)} \text{ is in } I, \text{ for each } i < \omega.$$

There is an infinite $Z \subseteq \omega$, such that $Z \subseteq_* Z_i$, for each $i < \omega$. Define

$$T_i = T_{ii} \cup \bigcup_{n \in Z} T_{ni}^{(n)}$$

and

$$f_i^q = f_i^{p_i} \cup \bigcup_{n \in Z} f_i^{p_n} \upharpoonright T_{ni}^{(n)}.$$

Then, $\text{dm}(f_i^q) = T_i$ and $q = \langle f_i^q : i < \omega \rangle$ is as required. *QED*

For $q \in \mathbf{Q}$ and $n \in \omega$ let $S(q, n)$ be the set of all sequences $s = \langle s_0, \dots, s_{n-1} \rangle$ satisfying the following properties

1. s_0, \dots, s_{n-1} are finite zero-one functions.
2. The domains $t_0 = \text{dm}(s_0), \dots, t_{n-1} = \text{dm}(s_{n-1})$ are finite trees such that

$$t_0 \cap T_0^{(n)} = \dots = t_{n-1} \cap T_{n-1}^{(n)} = \emptyset,$$

where $T_0 = \text{dm}(f_0^q), \dots, T_{n-1} = \text{dm}(f_{n-1}^q)$

3. Ordered sums $t_0 \oplus T_0^{(n)}, \dots, t_{n-1} \oplus T_{n-1}^{(n)}$ are trees belonging to \mathcal{T} .

Note that from the definition of \mathcal{T} it follows that $S(q, n)$ is always finite. Let us denote

$$s * q^{(n)} = \langle s_0 \cup f_0^q, \dots, s_{n-1} \cup f_{n-1}^q, f_n^q, \dots \rangle$$

for q, n, s as above. Obviously, we have

$$(2.2) \quad \text{the set } \{s * q^{(n)} : s \in S(q, n)\} \text{ is predense below } q^{(n)}$$

(i.e. the boolean sum $\sum_{s \in S(q, n)} s * q^{(n)} = q^{(n)}$).

Now, we obtain easily an analogue of VI, 4.5 in œS —.

$$(2.3) \quad \begin{array}{l} \text{For arbitrary } p \in \mathbf{Q}, n < \omega \text{ and } \tau \in V^{(\mathbf{Q})} \\ \text{such that } \mathbf{Q} \Vdash \text{“}\tau \text{ is an ordinal” there is} \\ \text{a } q \leq p \text{ and ordinals } \{\alpha(s) : s \in S(p, n)\} \\ \text{so that} \\ q^{(n)} \Vdash \text{“}\bigvee_s \tau = \alpha(s)\text{”} \end{array}$$

Indeed, if $S(q, n) = \{s^0, \dots, s^{m-1}\}$, then we define inductively conditions p_0, \dots, p_m so that $p_0 = p$ and $p_{k+1} \leq s^k * p_k^{(n)}$ is such that

$$p_{k+1} \Vdash \text{“}\tau = \alpha\text{”}, \text{ for some ordinal } \alpha = \alpha(s^k)$$

Now, $q = s * p_m^{(n)}$, where s is such that $p = s * p^{(k)}$ (we may assume $s \in S(p, n)$), satisfies (2.3).

2.4. THEOREM. \mathbf{Q} is α -proper, for every $\alpha < \omega_1$, and has the strong PP-property.

PROOF. Let countable $N \prec H(\kappa)$ for sufficiently large κ , be such that $\mathbf{Q} \in N$ and suppose that $p \in \mathbf{Q} \cap N$. To prove that \mathbf{Q} is proper we have to find a $q \leq p$, which is N -generic. Let $\{\tau_n : n < \omega\}$ be an enumeration of all the \mathbf{Q} -names for ordinals, such that $\tau_n \in N$, for $n < \omega$. Using (2.3) we define inductively a sequence $p_0 = p \geq p_1 \geq \dots$ and ordinals $\alpha(n, s)$ so that

$$p_n^{(n)} \Vdash \text{“}\bigwedge_{i \leq n} \bigvee_s \tau_i = \alpha(n, s)\text{” for each } n < \omega$$

(i.e. in the n -th step we apply (2.3) for all names τ_0, \dots, τ_n). Note that the p 's and α 's can be found in N , since $N \prec H(\kappa)$. By Lemma 2.1 there is a q and an infinite $Z \subseteq \omega$ such that

$$q \leq p_n^{(n)}, \text{ for each } n \in Z.$$

Hence also $q^{(m)} \leq p_n^{(n)}$ holds for arbitrarily large n and all $m < \omega$ and thus

$$q^{(m)} \Vdash \text{“}\tau_n \in N\text{”},$$

for all $n, m < \omega$.

By III, 2.6 of œS —, each $q^{(m)}$ is N -generic.

To see that \mathbf{Q} is α -proper let $\langle N_\xi : \xi \leq \alpha \rangle$ be a continuous sequence of elementary countable submodels of $H(\kappa)$ such that $\mathbf{Q} \in N_0$ and

$$\langle N_\xi : \xi \leq \eta \rangle \in N_{\eta+1}, \text{ for each } \eta < \alpha.$$

Assume that \mathbf{Q} is β -proper, for each $\beta < \alpha$ and let $q_0 \in \mathbf{Q} \cap N_0$. If $\alpha = \beta + 1$, we have a $q \leq q_0$ which is N_ξ -generic, for each $\xi \leq \beta$ and we may assume that the $q^{(n)}$ have the same property, for all $n < \omega$. since $N_\alpha \prec H(\kappa)$ and all the parameters are in N_α , such a q can be found in N_α and as above we construct a $q_\alpha \leq q$ which is N_α -generic and so are the $q_\alpha^{(n)}$, for $n < \omega$. Thus, q_α and all the $q_\alpha^{(n)}$ are N_ξ -generic for all $\xi \leq \alpha$. If α is a limit ordinal, we fix an increasing sequence $\langle \xi_n : n < \omega \rangle$ such that $\alpha = \sup_{n < \omega} \xi_n$ and by the inductive hypothesis there is a sequence $q_0 \geq q_{\xi_0} \geq q_{\xi_1} \geq \dots$ such that, for each $n < \omega$, q_{ξ_n} is N_ξ -generic, for each $\xi \leq \xi_n$ and $q_{\xi_n} \in N_{\xi_{n+1}}$ and that $q_{\xi_n}^{(m)}$ have the same property for each $m < \omega$. By Lemma 2.1 there is a $q \in \mathbf{Q}$ such that $q \leq q_{\xi_n}^{(n)}$, for infinitely many $n < \omega$. Thus, $q \leq q_0$ and q is N_ξ -generic for each $\xi < \alpha$ and hence also for each $\xi \leq \alpha$.

Finally, to prove the *PP*-property let $h : \omega \rightarrow \omega$ diverge to infinity and suppose that $p \Vdash "f : \omega \rightarrow \omega"$. Define

$$k_n = \min\{i : h(i) > 2^n \cdot \text{card}S(p, n)\}, \text{ for } n < \omega$$

and, using (2.3), define inductively the sequence $p = p_0 \geq p_1 \geq \dots$ such that

$$p_n^{(n)} \Vdash " \bigwedge_{i < k_n} \bigvee_{s \in S(p_i, i)} f(i) = \alpha(s, i) "$$
 for each $n < \omega$ and some integers $\alpha(s, i) < \omega$.

Let T be the tree built up of integers

$$\{\alpha(s, i) : i < \omega \text{ and } s \in S(p_i, i)\}$$

If $q \leq p_n^{(n)}$, for infinitely many n , then we have $q \Vdash "f \in \text{Lim } T"$ and $T \cap \omega^{k_n}$ has less elements than $h(k_n)$, for all $n < \omega$, which finishes the proof. *QED*

The last point to be discussed is how does $\mathbf{Q} = \mathbf{Q}(F)$ act in the course of iteration.

2.5.LEMMA. *If \mathbf{R} is ω^ω -bounding (i.e. the set of old functions $: \omega \rightarrow \omega$ dominates) and $\mathbf{Q}(F)$ is a complete subforcing of \mathbf{R} , then in $V(\mathbf{R})$ the filter F cannot be extended to a ccc P -filter.*

PROOF. Let X_α^n be the α -th set added by n -th factor of the product $\mathbf{Q} = \mathbf{P}^\omega$. Suppose that for some $r \in \mathbf{R}$ and a ccc P -filter $E \in V(\mathbf{R})$ we have

$$r \Vdash "F \subseteq E"$$

Note that for each $n < \omega$, the relation $X_\alpha^n \in E$ hold for at most countably many α 's, since E is ccc. Hence, there is an α such that for all $n < \omega$ we have $\omega \setminus X_\alpha^n \in E$ and, since E is a P -filter, there is an $A \in E$ and a function g , so that $A \subseteq \bigcap_{n < \omega} (\omega \setminus X_\alpha^n) \cup \diamond 0, g(n)$ i.e. for some $r_1 \leq r$ we have

$$(2.6) \quad r_1 \Vdash " \bigcap_{n < \omega} (\omega \setminus X_\alpha^n) \cup \diamond 0, g(n) \in E "$$

Since \mathbf{R} is ω^ω -bounding we may assume that $g \in V$. By the assumption, \mathbf{Q} is a complete subforcing of \mathbf{R} and hence there is a $q \in \mathbf{Q}$ such that r is compatible with each $q' \leq q$.

On the other hand, since $T_n = \text{dm}(f_n^q) \in \mathcal{T}$, there is a set $B \in I$ and an increasing sequence $a_0 < a_1 < \dots$ such that $T_n \setminus \diamond 0, a_n \subseteq B$, $g(n) < a_n$ and $\diamond a_n, a_{n+1} \setminus B \neq \emptyset$, for each $n < \omega$. Define $q' \leq q$ as follows. For a given n extend T_n by adding elements of $\diamond a_n, a_{n+1} \setminus B$ on the α -th branch b_α^q and put $f_n^{q'}(i) = 1$, for each $i \in \diamond a_n, a_{n+1} \setminus B$. Obviously, we have

$$q' \Vdash “(\omega \setminus X_\alpha^n) \cup \diamond 0, g(n) \cap \diamond a_n, a_{n+1} \setminus B = \emptyset”, \text{ for each } n$$

and hence

$$q' \Vdash “ \bigcap_{n < \omega} (\omega \setminus X_\alpha^n) \cup \diamond 0, g(n) \setminus B \cap \bigcup_{n < \omega} \diamond a_n, a_{n+1} = \emptyset ”$$

Consequently $q' \Vdash “ \bigcap_{n < \omega} (\omega \setminus X_\alpha^n) \cup \diamond 0, g(n) \subseteq_* B ”$, which contradicts (2.6). *QED*

The rest of the proof is routine. Beginning with a model V of $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$ we iterate with countable supports the forcings $\mathbf{Q}(F)$, for all *ccc* P -filters F booked at each stage $\alpha < \omega_2$ of the iteration. From œS—, V.4 we know that the resulting forcing \mathbf{R} (obtained after ω_2 stages) is proper and ω^ω -bounding. Hence, in $V \diamond G$ there are no *ccc* P -sets.

References

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