FACTOR = QUOTIENT, UNCOUNTABLE BOOLEAN ALGEBRAS, NUMBER OF ENDOMORPHISM AND WIDTH

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Abstract. We prove that assuming suitable cardinal arithmetic, if B is a Boolean algebra every homomorphic image of which is isomorphic to a factor, then B has locally small density. We also prove that for an (infinite) Boolean algebra B, the number of subalgebras is not smaller than the number of endomorphisms, and other related inequalities. Lastly we deal with the obtainment of the supremum of the cardinalities of sets of pairwise incomparable elements of a Boolean algebra.

We show in the first section:

0.1 Conclusion. *It is consistent, that for every $Q = F$ Boolean algebra B^*, for some $n < \omega, \{x : B^*|x$ has a density $\leq \aleph_n\}$ is dense (so B^* has no independent subset of power $\aleph_n$).*

Where:

0.2 Definition. *A B.A. is $Q = F$ (quotient equal factor) if: every homomorphic image of B is isomorphic to some factor of B i.e $B|a$ for some $a \in B$.*

The "consistent" is really a derivation of the conclusion from a mild hyphothesis on cardinal arithmetic (1.2). The background of this paper is a problem of Bonnet whether every $Q = F$ Boolean algebra is superatomic.

Noting that : "$B|x$ has density $\leq \aleph_n$" is a weakening of "x is as atom of B'", we see that **0.1** is relevant.

The existence of non trivial example is proved in R. Bonnet, S. Shelah [2].

M. Bekkali, R. Bonnet and M. Rubin [1] characterized all interval Boolean algebras with this property.

In the second section we give a more abstract version. In a paper in preparation , Bekkali and the author use theorem **2.1** to show that every $Q = F$ tree Boolean algebras are superatomic.

In the third section we deal with the number of endomorphism (e.g. $aut(B)^{\aleph_0} \leq end(B)$) and in the fourth with the width of a Boolean algebra.

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Notation

B denote a Boolean algebra.

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B^+ is the set of non zero members of B.
$B|x$ where $x \in B^+$ is $B|\{y : y \leq x\}$.
$\mathrm{comp}\, B$ is the completion of B, so it is an extension of B.
Remember : if B_1 is a subalgebra of B_2, B is complete, h a homormorphism from B_1 to B, then h can be extended to a homomorphism from B_2 to B.
id_A is the identity function on A.
${}^{\gamma}\alpha = \{\eta : \eta$ is a sequence of lenghts γ of ordinals $< \alpha\}$.
${}^{\gamma>}\alpha = \bigcup_{\beta<\alpha} {}^{\beta}\alpha$.

§1 Maybe every "quotient equal factor" BA has locally small density .

1.1 Hypothesis.

(1) *B^* is a $Q = F$ Boolean algebra.*
(2) *for each $n < \omega$, B^* has a factor B_n s.t.: $0 < x \in B_n \Rightarrow density(B_n|x) \geq \aleph_n$.*

1.2 Hypothesis. *For $\alpha \geq \omega$ we have $2^{|\alpha|} > \aleph_\alpha$.*

1.3 Desired Conclusion. *Contradiction.*
We shall use **1.1** *all the times, but* **1.2** *only in* **1.17**.

1.4 Definition. *K^*_λ is the class of Boolean algebra such that:*

$$(\forall x \in B^+)[density(B|x) = \lambda].$$

1.5 Claim. *If B is atomless, $x \in B^+$ then for some y, $0 < y \leq x$,and infinite cardinal λ we have $B|y \in K^*_{\mathrm{density}(B|x)}$.*

1.6 Claim. *If $B \in K^*_\lambda$, $B \subseteq B' \subseteq comp(B)$ then,*

$$B' \in K^*_\lambda.$$

1.7 Claim. *If $B \in K^*_\lambda$,$\aleph_0 \leq \mu < \lambda$, μ regular then some subalgebra B' of B is in K^*_μ. If in addition B is a $Q = F$ algebra, then some homomorphic image B'' of B is in K^*_μ.*

Proof. Choose by induction on $i < \mu$, $B_i \subseteq B$, $|B_i| \leq \mu$, $[i < j \Rightarrow B_i \subseteq B_j]$ such that:
if $x \in B^+_i$ then there is $y(x, i) \in B_{(i+1)}$ satisfying:

(1) $0 < y(x, i) \leq x$,
(2) for no $z \in B_i$, $0 < z \leq y(x, i)$.

(possible as for each i and $x \in B^+_i$ $density(B_i|x) = \lambda$). Let now $B' = \cup_{i<\mu} B_i$; it is a subalgebra of B. Now $x \in (B')^+ \Rightarrow density(B'|x) = \mu$ as on the one hand $|B'| \leq \sum_{i<\mu} |B_i| \leq \mu \times \mu = \mu$ implies $density(B'|x) \leq \mu$ for every $x \in B'$ and on the other hand if $x \in (B')^+, A \subseteq B'|x, |A| < \mu$ then for some $i < \mu, A \subseteq B_i$, hence $y(x, i) \in (B')^+$ wittness A is not dense in $B'|x$, Now, if B is a $Q = F$ Boolean Algebra, then $id_{B'}$ can be extended to a homomorphism h' from B into $comp(B')$, so $h'(B)$ is as required.

1.8 Conclusion. *$\{\lambda : \lambda$ regular and B^* has a factor in $K^*_\lambda\}$ is an initial segment of $\{\aleph_\alpha : \aleph_\alpha$ regular$\}$.*

1.9 Definition.

(1) *$\alpha(*)$ is minimal such that for no $\lambda > \aleph_{\alpha(*)}$ does B^* has a factor in K^*_λ. Let $\kappa^* = \kappa(*) =: |\alpha(*)|$.*
(2) *Let for $\alpha < \alpha^*$, $b_\alpha \in B^*$ be such that $B^*|b_\alpha \in K^*_{\aleph_{\alpha+1}}$.*
(3) *Let J_α be the ideal $\{b \in B^* : B^*|b \in K^*_{\aleph_{\alpha+1}}\}$.*

1.10 Definition. *$B[{}^{\omega>}\lambda]$ is the Boolean algebra generated freely by $\{x_\eta : \eta \in {}^{\omega>}\lambda\}$ except $x_\eta \le x_{\eta|m}$ $(m \le lg(\eta), \eta \in {}^{\omega>}\lambda)$.*

1.11 Claim.

(1) *For $\alpha < \beta < \alpha^*$, $J_\alpha \cap J_\beta = \{0\}$ and J_α is an ideal.*
(2) *For no $B' \subseteq B^*$ and proper ideal I of B', $[x \in B' \backslash I \Rightarrow density(B'|x/I) > \aleph_{\alpha(*)}]$.*

Proof.
(1) Trivial.
(2) By **1.5** for some $\beta > \alpha(*)$ and proper ideal J of B' (of the form $\{x \in B' : x - b \in I\}$) we have $B'/I \in K^*_{\aleph_\beta}$ so there is a homomorphism h from B^* into $comp(B'/I)$ extending $x \mapsto x/I\,(x \in B')$. So B^* has a factor isomorphic to $Rang\, h$, but this Boolean algebra is in $K^*_{\aleph_\beta}, \aleph_\beta > \aleph_{\alpha(*)}$. So by **1.7** we get contradiction to the choice of $\alpha(*)$.

1.12 Definition. *1) $I^* =: \{x \in B^* : \bigcup_{(\alpha<\alpha(*))} J_\alpha$ is dense below $x\}$.*
2) For $A \subseteq \alpha(*) : I^*_A =: \{x \in B^* : \bigcup\{J_\alpha : \alpha \in A\}$ is dense below $x\}$.

1.13 Claim.

(1) *I^*, I^*_A are ideals of B.*
(2) *$A \subseteq B \Rightarrow I^*_A \subseteq I^*_B \subseteq I^*$.*

1.14 Claim. *For every $A \subseteq \alpha(*)$, there are c_A, h_A such that:*

(1) *$c_A \in B^*$ and I^*_A is dense below c_A,*
(2) *h_A is a homomorphism from B^* onto $B^*|c_A$,*
(3) *$h_A|I^*_A$ is one to one,*
(4) *If $B|x \cap I^*_A = \{0\}$ then $h_A(x) = 0$,*
(5) *$B^*|c_A$ is a subalgebra of the completion of the subalgebra $\{h_A(x) : x \in I^*_A\}$.*

Proof. Let $ba(A)$ be

$$I^*_A \cup \{1 - x : x \in I^*_A\},$$

this is a subalgebra of B^*.

Let h_1 be a homomorphism from B^* to $\text{comp}\big(b\,a(A)\big)$ extending $id_{ba(A)}$ and $h_1(x) = 0$ if $B^*|x \cap I^*_A = \{0\}$.

Now $h_1(B^*)$ is a quotient of B^* hence there is an isomorphism h_2 from $h_1(B*)$ onto some $B^*|c_A$.

Let

$$h_A = h_2 \circ h_1$$

so (1), (2), (3), (4), (5) holds.

1.15 Claim. *Let* $A \subseteq \alpha(*)$

(1) *If* $x \in \bigcup_{\alpha \in A} J_\alpha$ *then* $h_A(x) \in \bigcup_{\alpha \in A} J_\alpha$.
(2) *If* $x \in I^*_A \setminus \bigcup_{\alpha \in A} J_\alpha$ *then* $h_A(x) \notin \bigcup_{\alpha \in A} J_\alpha$.
(3) $\bigcup_{\alpha \in A} h_A(J_\alpha)$ *is dense and downwared closed in* $B^* | c_A$.
(4) $c_A \in I^*_A$.
(5) *If* $x \in J_\alpha$ *and* $\alpha \in A$ *then* $h_A(x) \in J_\alpha$.

Proof.

(1) By (5) of **1.14**.
(2) is easy too.
(3) is easy too.
(4) is easy too.
(5) is easy too.

1.16 Claim. *We can find* $x_\eta \in B^*$ *for* $\eta \in {}^{\omega >}\lambda$ *where* $\lambda = 2^{\kappa(*)}$ *such that:*

(1) $m < lg(\eta) \Rightarrow 0 < x_\eta \leq x_{(\eta | m)}$.
(2) *If* $\bigwedge_{e=1}^{k} \nu_e \ntrianglelefteq \eta$, $\eta \in {}^{\omega >}\lambda$, $\bigwedge_e \nu_e \in {}^{\omega}\lambda$ *then*

$$B^* \models x_\eta \nleq \bigcup_{e=1}^{k} x_{\nu_e}.$$

Proof. Now we can choose $\langle A^0_\eta : \eta \in {}^{\omega >}\lambda \rangle$ which is a family of subsets of $\kappa(*)$ such that any non trivial Boolean combination of then has cardinality $\kappa(*)$.

Let for $\eta \in {}^{\omega >}\lambda$ $\quad A_\eta =^{def} \bigcap_{e \leq lg(\eta)} A_{\eta | e}$. Let

(a)
$$x_{<>} = c_{A_{<>}} = h_{A_{<>}}(1_{B^*}).$$

(b)
$$x_{<i>} = h_{A_{<>}}(c_{A_{<i>}}) = h_{A_{<>}} h_{A_{<i>}}(1_{B^*})$$

and generally,

$$x_{<i_0, i_1, \ldots, i_{n-1}>} = h_{A_{<>}} h_{A_{<i_0>}} h_{A_{<i_0, i_1>}} \ldots h_{A_{<i_0, i_1, \ldots, i_{n-1}>}}(1_{B^*}).$$

We prove **(a)** by induction of $lg\eta$.

The reader may check

1.17 Final Contradiction. $\{x_\eta : \eta \in {}^{\omega >}\lambda\}$ *from* **Claim 1.16** *contradict by* **1.11(2)** *and the choice of* $\aleph_{\alpha(*)}$, *because* $\lambda = 2^{\kappa(*)} = 2^{|\alpha(*)|} > \aleph_{\alpha(*)}$.
[of course $\aleph_{\alpha(*)} \leq |B^*|$*]*
Actually, we have prove more.

1.18 Remark. (1) So we have in **1.17** prove that if set theory is as in **Hypothesis 1.2**, then there is no Boolean algebra as in **1.1**, hence proving **Conclusion 0.1**

(2) Note: if **1.2**, any $Q = F$ Boolean algebra has no factor $\mathcal{P}(\omega)$.

§2 $Q = F$ Boolean algebras: a general theorem .

2.1 Theorem. *Suppose:*

1. *B^* is a $(Q = F)$ Boolean algebra.*
2. *N is a family of (non zero) members of B^* (the "nice" elements).*
3. *κ a cardinal $(\geq \aleph_0)$, $\langle K_\alpha : \alpha < \kappa \rangle$ a sequence.*
4. *K_α is family of Boolean algebras closed under isomorphism and for $\alpha \neq \beta$ we have $K_\alpha \cap K_\beta = \emptyset$.*
5. *for every α some factor of B^* is in K_α.*
6. *if $x \in (B^*)^+$, $(B^*|x) \in K_\alpha$ then for some $y \leq x$, $B^*|y \in K_\alpha$, $y \in N$.*
7. *if $x_1, x_2 \in N$, $B^*|x_1 \in K_{\alpha_1}$, $B^*|x_2 \in K_{\alpha_2}$, $\alpha_1 \neq \alpha_2$ then $x_1 \cap x_2 = 0$.*
8. *if $x \in B^*, N' \subseteq N$ then*
   (α) for some $y_1, \cdots, y_n \in N'$, for every $z \in N'$ we have $B^* \models x \cap z \subseteq y_1 \cup \cdots \cup y_n$
   or
   (β) for some $y \in N' : y \leq x$.
   Actually we use (8) only for N' of the form $\{y \in N : (\exists \alpha)[\alpha \in A \,\&\, B^*|y \in K_\alpha\}$.
9. *if $x < y \in B^*, B^*|x \in K_\alpha$ and $B^*|y \in K_\beta$, then $\alpha = \beta$.*

Then the Boolean algebra $B\,[{}^{\omega>}(2^\kappa)]$ can be embedded into B^*, remember ${}^{\omega>}(2^\lambda)$ is
the tree $\{\eta : \eta$ a finite sequence of ordinals $< 2^\kappa\}$ and Def 1.10.

Proof of theorem 2.1.
Let for $\alpha < \kappa$, $Y_\alpha =: \{y \in N : B^*|y \in K_\alpha\}$.
For $A \subseteq \kappa$ let us define
I_A = the ideal generated by $Y_A = \bigcup_{\alpha \in A} Y_\alpha = \{y \in N : B^*|y \in K_\alpha$ for some $\alpha \in A\}$ and
$J_A =: \{z \in B_\alpha :$ for every $y \in Y_A$ we have $z \cap y = 0\}$.
Clearly J_A is an ideal.
Now for each $A \subseteq \kappa$ B^*/J_A is a quotient of B^*. Hence by condition (1) there are $y^*_A \in B^*$ and an isomorphism $h_A : B^*/J_A \to B^*|y^*_A$ onto.
Let $g_A : B^* \to B^*/J_A$ be canonical, so $h_A \circ g_A(1_{B^*}) = y^*_A$. Let $f_A = h_A \circ g_A$.
Define for $y \in B^*$ the following: $\text{cont}(y) =: \{\alpha : (\exists y' \leq y)[B^*|y' \in K_\alpha]\}$.
(i.e. the content of y). We next prove
$(*)_1$ cont $(y^*_A) \supseteq A$.

Proof. By conditon (5) for each $\alpha \in A$, there is $x_\alpha \in B^*$, such that: $B^*|x_\alpha \in K_\alpha$. By condition (6) wlog $x_\alpha \in N$, hence $x_\alpha \in I_A$, hence $g_A|(B^*|x_\alpha)$, is one to one, hence $B^*|x_\alpha \cong B^*|f_A(x_\alpha)$, hence by condition (4) $B^*|f_A(x_\alpha) \in K_\alpha$; now $Rang f_A = B^*|y^*_A$, so $\alpha \in \text{cont}(y^*_A)$. So we have prove $(*)_1$.

$(*)_2$ $\text{cont}(y^*_A) \subseteq A$

Proof. Suppose $\alpha \in cont(y^*_A)$, so there is $z \leq y^*_A$, such that $B^*|z \in K_\alpha$. As f_A is a homomorphism from B^* onto $B^*|y^*_A$, there is $x \in B^*$ such that $f_A(x) = z$.

Now the kernel of f_A is J_A, and $B^*|0 \notin K_\alpha$ so $x \notin J_A$; and clearly $(B^*|x)/J_A$ is in K_α.

Hence by $(*)_3$ below $\alpha \in A$.

$(*)_3$ if $x \in B^* \setminus J_A$ and $(B^*/J_A)|f_A(x) \in K_\alpha$ then $\alpha \in A$

Proof. We apply condition (8) to x and $N' = Y_A$. So one of the following two cases ocurrs:

Case α: There are $n < \omega$, $y_1, \cdots, y_n \in N'$ such that:
$(\forall z \in N')\, x \cap z \subseteq y_1 \cup \ldots \cup y_n$.
So $x - (y_1 \cup \cdots \cup y_n) \in J_A$ (by definition of J_A).
Let $x_1 = x \cap (y_1 \cup \cdots \cup y_n)$ hence $B^*|f_A(x) = (B^*|x)/J_A \cong (B^*|x_1)/J_A \cong B^*|x_1$,(last isomorphism as $\wedge_e y_e \in Y_A$ hence $y_1 \cup \cdots \cup y_n \in I_A$ hence $f_A|(B^*|x_1)$ is one to one). So $B^*|x_1 \in K_\alpha$, hence by condition (6) for some $x_2 \leq x_1$,we have $x_2 \in N \& B^*|x_2 \in K_\alpha$.
Let $B^*|y_e \in K_{\alpha_e}$ where $\alpha_e \in A$(by definition of Y_A). Clearly $x_2 \leq x_1 \leq y_1 \cup \cdots \cup y_n$, so for some $e, y_e \cap x_2 \neq 0$ hence $\alpha = \alpha_e \in A$ (by condition (7)) so we get the trivial desired conclusion.

Case β: There is $y \in N'$, $y \leq x$.
As $y \in N' = Y_A = \cup_{\beta \in A} Y_\beta$ for some $\beta \in A$ we have $y \in K_\beta$, also $y \in N' \subseteq N$ so $J_A \cap (B^*|y) = \{0\}$ so $(B^*/J_A)|g_A(y) \in K_\beta$. Remembering $(B^*/J_A)|g_A(x) \in K_\alpha$, as $B^*/J_A \cong B^*|y^*_A$ we get by condition (9) that $\alpha = \beta$. So $(*)_3$ hence $(*)_2$ is proved.

Next we prove
$(*)_4$ if $B \subseteq A \subseteq \kappa$, and $\text{cont}(y) = B$, *then* $cont[f_A(y)] = B$.

Proof.

inclusion $\supseteq$
First let $\alpha \in B$, then for some $x \leq y$,$B^*|x \in K_\alpha$, and by condition (6) *wlog* $x \in N$, hence $x \in I_A$ (as $\alpha \in B \subseteq A$) hence $f_A|(B^*|x)$ is one to one and onto $B^*|f_A(x)$ so $f_A(x) \leq f_A(y)$, $B^*|f_A(x) \in K_\alpha$, so $\alpha \in \text{cont}(f_A(y))$.

inclusion $\subseteq$
Second let us assume $\alpha \in cont[f_A(y)]$. So (as f_A is onto $B^*|y^*_A$, and if $f_A(x) \leq f_A(y)$ then $f_A(x \cap y) = f_A(y)$, $x \cap y \leq y$) there is $x \leq y$ such that $B^*|f_A(x) \in K_\alpha$. Now apply condition (8) to x and Y_A. So case (α) or case (β) below holds.

Case α: There are $n < \omega$, $y_1, \cdots, y_n \in Y_A$ such that for every $z \in Y_A$ we have $x \cap z \subseteq y_1 \cup \cdots \cup y_n$.
Hence $x - (y_1 \cup \cdots \cup y_n) \in J_A$ let $x_1 = x \cap (y_1 \cup ... \cup y_n)$ so $f_A(x) = f_A(x_1)$ so $B^*|f_A(x_1) \in K_\alpha$ and of course $x_1 \leq y_1 \cup \cdots \cup y_n$, (and $x_1 \leq y$) so $f_A|(B^*|x_1)$ is one to one.
Now f_A is one to one on $B^*|x_1$ hence $B^*|x_1 \cong B^*|f_A(x_1) \in K_\alpha$. Now $x_1 \leq x \leq y$, so x_1 wittness $\alpha \in cont(y)$, which is B.

Case β: There is $t \leq x$, $t \in Y_A$.
Now $t \leq x \leq y$, $B^*|t \in K_\beta$ for some $\beta \in A$ as $t \in Y_A$. Now f_A is one to one on $B^*|f_A(t) \in K_\beta$.
Also $f_A(t) \leq f_A(x)$ hence by assumption (9) we have $\beta = \alpha$. Also $t \leq x \leq y$ so t wittness $\beta \in \text{cont}(y)$, so $\alpha = \beta \in \text{cont}(y) = B$ as required.
So we have proved $(*)_4$

end of proof of theorem 2.1: Let $\lambda = 2^\kappa$.

Let $\langle \mathcal{U}_\eta : \eta \in {}^{\omega >}\lambda \rangle$ be a family of subsets of κ, any finite Boolean combination of them has power κ (or just $\neq \emptyset$).

Let $\mathcal{U}^*_\eta = \cap_{e \leq lg\eta} \mathcal{U}_{\eta|e}$. Now define for every $\eta \in {}^{\omega >}\kappa$ and $e \leq lg(\eta)$ an element y^e_η of B^*:

$y^e_\eta =^{def} f_{\mathcal{U}^*_{\eta|e}} f^*_{\mathcal{U}_{\eta|(e+1)}} \cdots f_{\mathcal{U}^*_{\eta|(n-1)}} f_{\mathcal{U}^*_\eta}(1_{B^*})$ and $y^\otimes_\eta =^{def} y^\circ_\eta$

Now: (a) prove for each $\eta \in {}^n\kappa$ by downward induction on $e \in \{0, 1, ..., n\}$ that $cont(y^e_\eta) = \mathcal{U}^*_\eta$; for $e = n$ this is $(*)_1 + (*)_2$ as $y^n_\eta = y^*_{\mathcal{U}_\eta}$;

for $e < n$ (assuming for $e + 1$) this is by $(*)_4$.

Next note: (b) if $\nu = \eta\hat{\ } < \alpha >$ *then* $y^\otimes_\nu \leq y^\otimes_\eta$

[prove by domnward induction for $e \in \{0, 1, ..., lg\eta\}$ we have : $y^{e+1}_v \leq y^e_\eta$; remember $f_\mathcal{U}$ is order preserving].

Lastly note (c) if $\eta \in {}^{\omega >}\lambda, n < \omega$, and $\nu_e \in {}^{\omega >}\lambda$ is not initial segment of η for $e = 1, \ldots, n$ then $y^\circ_\eta - \cup^n_{e=1} y^\circ_{\nu_e} \neq$; this follows by (a) and the definition of $cont(y^\circ_\eta)$.

Now by (a), (b), (c) there is an embedding g from the subalgebra of B^* which $\{y^\circ_\eta : \eta \in {}^{\omega >}\lambda\}$ generates mapping y°_η to x_η .

§3 The Number of Subalgebras .

3.1 Definition. For a BA A

1. $Sub(A)$ is the set of subalgebras of A.
2. $Id(A)$ is the set of ideals of A.
3. $End(A)$ is the set of endomorphisms of A.
4. $Pend(A)$ is the set of partial endomorphisms (i.e. homomorphisms from a subalgebra of A into A).
5. $Psub(A)$ is the family of subsets of A closed under union ,intersection and substruction but 1 may be not in it though 0 is [so not neccessarily closed under complementation].
6. We let $sub(A)$, $id(A)$, $end(A)$, $aut(A)$, $pend(A)$, $psub(A)$ be the cardinality of $Sub(A)$, $Id(A)$, $End(A)$, $Aut(A)$, $Pend(A)$ and $Psub(A)$ respectively

In D. Monk [4] list of open problems appear:

PROBLEM 63. Is there a BA A such that $aut(A) > sub(A)$?
See [4] page 125 for backgraound.

3.2 Theorem. *For a BA A we have: $aut(A)$ is not bigger than $sub(A)$.*

3.3 Conclusion.

1. *For a BA A we have $end(A)$ is not bigger than $sub(A)$.*
2. *For a BA A we have $pend(A)$ is exactly $sub(A)$.*
3. *For a BA A and a in A, $0 < a < 1$ we have $sub(A) = Max\{sub(A|a), sub(A|-a)\}$.*

We shall prove it in 3.5

Remark. Of course - A is infinite- we many times forget to say so.

3.4 Proof of the theorem. Let μ be $sub(A)$.

3.4A Observation: $Psub(A)$ has cardinality $Sub(A)$ [why? for the less trivial inequality, $\leq$, for every X in $Psub(A)$ which is not a subset of $\{0, 1\}$ choose a member $a \neq 0, 1$ in X and let $Y_a[X]$ be the subalgebra generated by $\{x : x \in$

$X, x \leq a\}$ let $Z_a[X]$ be the subalgebra generated by $\{x : x \in X, x \cap a = 0\}$; now X can be reconstructed from $\langle a, Y, Z\rangle$ as $\{x \in A : x \cup a \in Y$ and $x - a \in Z\}$. So $|Psub(A)| \leq |A| \times |Sub(A)|^2 + 4 = |Sub(A)| + \aleph_0 = |Sub(A)|$. (remember that $|A| \geq \aleph_0$, and that $|A| \leq |Sub(A)|$, as the number of non-atoms of A is $\leq |\{Z_a[A] : a \in A\}|$).

For any automorphism f of A we shall choose a finite sequence of members of $Psub(A)$ (in particular of ideals of A), and this mapping is one to one, thus we shall finish.

Let $J =^{def} \{x : x \in A,$ and for every $y \in A$ below $x,$ we have $f(y) = y\}$; let I^* be the ideal of A generated by I, the set of elements x for which $f(x) \cap x = 0$.

Observe : $x \in I$ implies $f(x) \in I$ [why? as $y = f(x), x \in I$ implies $f(x) \cap x = 0$ hence $f(y) \cap y = f(f(x)) \cap f(x) = f(f(x) \cap x) = f(0) = 0$].

Observe: $J \cup I$ is a dense subset of A [if $x \in B^+$ and there is no $y \in J, 0 < y \leq x$ *then wlog* $x \neq f(x)$. If $x \not\leq f(x)$ *then* $z =: x - f(x)$ satisfies $0 < z \leq x, f(z) \cap z \leq f(x) \cap z = f(x) \cap (x - f(x)) = 0$; so $z \in I$. If $x \leq f(x)$ *then* $x < f(x)$; as f is an automorphism of B^*, for some z in B^* we have $f(z) = x$, so $x < f(x)$ means $f(z) < f(f(z))$ hence $z < f(z)$, and $z' = x - z = f(z) - z$ is in I, is > 0 but $< x$, as required.]

Next let $\{x_i : i < \alpha\}$ be a maximal sequence of distinct members of I satisfying : for any $i, j < \alpha$ we have $x_i \cap f(x_j) = 0$, let I_1 be the ideal generated by $\{x_i : i < \alpha\}$ and let I_2 be the ideal generated by $\{f(x_i) : i < \alpha\}$.

Clearly $I_1 \cap I_2 = \{0\}$ let $I_0 = \{y : y \in I^*,$ and for every $x \in I_1, y \cap x = 0$ and $f(y) \in I_1\}$ and let I_3 be the ideal of A generated by $\{f(x) : x \in I_2\}$

Observe : each I_t $(t = 0, 1, 2, 3)$ is an ideal of A, contained in I^* [why? for $t = 1$ by choice each x_i is in I, $t = 0$ by it's definition , I_2, I_3 by their definition and an observation above, i.e., $x \in I \Rightarrow f(x) \in I$]

Observe: for $t = 0, 1, 2$ we have: $I_t \cap I_{t+1} = \{0\}$ [why ? for $t = 1$, by the choice of $\{x_i : i < \alpha\}$, for $t = 0$ by definition of I_0, lastly for $t = 2$ applies its definition + f being an automorphism.]

Observe: $I_0 \cup I_1 \cup I_2$ is a dense subset of I^* [why? assume x in I^* but below it there is no non zero member of this union , so we can replace it by any non zero element below it; as $x \in I^*$, there is below it a non zero element y with $y \cap f(y) = 0$ so wlog $x \cap f(x) = 0$; why have we not choose $x_\alpha = x$? there are two case:

Case 1: For some $j < \alpha$, $x \cap f(x_j)$ is not zero
so there is a non zero element below x in I_2.

Case 2: for some $j < \alpha, f(x) \cap x_j$ is not zero
so there is a non zero y in I_1 below $f(x)$ hence (as f is an automorphism) there is x' below x such that $f(x') = y$ so wolg $f(x)$ is in I_1, but then by its definition, either below x there is a member of I_1 or x is in I_0 so we have finished proving the observation.]

Observe: for $t = 0, 1, 2$, $x \in I_t \Rightarrow f(x) \in I_{t+1}$ [check].

Now we define $C_t = C_t^f$, a member of $Psub(A)$ for $t = 0, 1, 2$ as follows: C_t is the set $\{x \cup f(x) : x \in I_t\}$. The closure under the relevant operations follows as I_t is closed under them and f is an isomorphism and for x, y in I_t, $x \cap f(y) = 0$, this is needed for substraction.

Also for every automorphism f of A we assign the sequence $\langle J, I, I_0, I_1, I_2, I_3, C_0, C_1, C_2\rangle$ (some reddundancy). Suppose for $f_1, f_2 \in Aut(A)$ we get the same tuple; it is enough to show that their restriction to J and to I are equal -as the union is dense. Concerning J this is trivial - they are the identity on it so we discuss I, by an observation above it is enough to choeck it for $I_t, t = 0, 1, 2$ but for each t, from C_t and I_t, I_{t+1} we can [or see below] reconstruct $f|I_t$.

So we have finished the proof.

3.4B Remark. We can phrased this argument as a claim: let I, J, be ideals of A with intersection $\{0\}$; for every f, a one to one homomorphism from I to J let X_f be the set $\{x \cup f(x) : x \in I\}$ *then* "f mapped to X_f" is a one to one mapping from $HOM(I, J)$ to $Psub(A)$ (the former include $\{g|I : g \in Aut(A), g$ maps I onto $J\}$, for which we use this). For subalgebra relative to $\cup, \cap$ only f needs not be one to one.

3.5 Proof of the conclusion from the theorem.

(1) For a given Boolean Algebra B assume $\mu =: sub(A)$ is $< end(A)$. For any endomorphism f of A we attach the pair $(Kernel(f), Range(f))$. The number of possible such pairs is at most $id(A) \times sub(A)$, which is at most μ (we are dealing with infinite BAs and $id(A) \le psub(A) \le sub(A)$) so as we assume $\mu < end(A)$, there are distinct f_i , endomorphisms of A for $i < \mu^+$, an ideal I and a subalgebra R of B such that for every i we have $Kernel(f_i) = I$ and $Range(f_i) = R$.

We now define a homomorphism g_i from B/I to R by : $g_i(x/I) = f_i(x)$. Easily the definition does not depend on the representative, so g_i is as required and it is one to one and onto. So $\{g_i \circ (g_0)^{-1} : i < \mu^+\}$ is a set of μ^+ distinct automorphisms of R.

So

$$(*)_1 \qquad \mu < aut(R)$$

but, by the theorem

$$(*)_2 \qquad aut(R) \le sub(R)$$

obviously

$$(*)_3 \qquad sub(R) \le sub(A)$$

remember

$$(*)_4 \qquad \mu = sub(A)$$

together a contradiction

(2) As $R \mapsto$ (identity map on R) is one to one from $sub(A)$ into $Pend(A)$, obviously $sub(A) \le pend(A)$, so we are left with proving the other inequality. Same proof, only the domain is a subalgebra too and it has an ideal. So for every such partial endomorphism h of A we attach two subalgebras $D_h = Domain(h)$ and $R_h = Range(h)$ an ideal I_h of D_h $\{x : x \in D_h$ and

$f(x) = 0\}$. They are all in $Psub(A)$, so their number is at most $sub(A)$ and if we fixed them the amount of freedom we have left is : an automorphism of R (and $aut(R) \leq sub(R) \leq sub(A)$).

(3) Let B be a subalgebra of A. We shall attached to it several ideals and subalgebras of $B|a, B|(-a)$ such that B can be reconstructed from them; this clearly suffice. Let C be the subalgebra $\langle B, a\rangle, C_o = \{x \in C : x \leq a\}, C_1 = \{x \in C : x \leq 1 - a\}$. The number of possible C is clearly the number of pairs $\langle C_o, C_1\rangle$ which is clearly $sub(A|a) + sub(A|-a)$; fix C. Let $I =: \{x : x \in B, x \leq a\}$, it is an ideal of $C|a$; so the number of such I is at most

$id(C|a) \leq pend(C|A) \leq sub(C|a) \leq sub(A|a)$.

So we can fix it . Similarly we can fix $J = \{y : y \in B, y \leq (1 - a)\}$, now I and J are subsets of B , now check : the amount of freedom we have left is an isomorphism g from $(C|a)/I$ onto $(C| - a)/J$ such that $B = \{$ the subalgebra of C generated by $I \cup J \cup \{x \cup y : x \in (C|a), y \in (C| - a)$ and $g(x/I) = (y/J)\}$.

So we can finish easily.

We originally want to prove $Aut(A)^{\aleph_0} \leq$ sub A and even $|\{f \in End(A) :\ f$ is onto $\}|^{\aleph_0} \leq subA$. But we get more: intermediate invariants with reasonable connections.

3.6 Definition: For a Boolean algebra A

(0) A partial function f from A to A is everywhere onto if:

$x \in Dom(f) \& y \in Rang(f) \& y \leq f(x) \Rightarrow (\exists z)[z \leq x \& f(z) = y]$.

(1) $End_0(A) = End(A)$.

$End_1(A) = \{f \in End(A) :\ Rang(f)$ include a dense ideal$\}$.

$End_2(A)$ is the set of endomorphism f of A which are onto.

$End_3(A) = \{f :$ for some dense ideal I, f is an onto endomorphism of $I\}$.

$End_4(A) = \{f :$ for some ideal I of A, f is an onto endomorphism of $I\}$.

$End_5(A) = \{f :$ for some dense ideals $I, J,\ f$ is an homorphism from I onto $J\}$.

$End_6(A) = \langle f :$ for some ideals I, J of A, f is an endomorphism from I onto $J\}$.

Note $I = A$ is allowed. All kinds of endomorphism, commute with $\cap, \cup$, preserve 0 but not necessarily $-$.

(2) For $l = 1 \cdots, 6$ we let $Aut_l(A) = \{f \in End_l(A) :\ f(x) = 0 \Leftrightarrow x = 0$ for $x \in Dom\, f\}$.

(3) For function f, g whose domain is $\subseteq A$ let: $f \sim g$ if $Kerf = Kerg$ and $\{x : f(x) = g(x)$ or both are defined not $\}$ include a dense ideal of $B/Kerf$.

(4) Let $end_e(A) = |End_e(A)|$, $aut_e(A) = |Aut_e(A)|$.

Let $end^{\sim}_e(A) = |\{f/\sim :\ f \in End_e(A)\}|$ and $aut^{\sim}_e(A) = |\{f/\sim : f \in Aut_e(A)\}|$.

(5) We allow to replace A by an ideal I with the natural changes.

(6) We define $Endv_e(A), endv_e(A)$ similarly replacing "onto" by "everywhere onto" and define $Endl_e(A), endl_e(A)$ similarly omitting "onto". We define $Endu_e(A)$ as the set of $f \in End_e(A)$ such that for every $x \in Dom\, f, f(x) \neq 0$ and ideal I of $A|x$ which is dense, we have $f(x) = sup_A\{f(y) : y \in I\}$. We defined naturally $Autv_e(A), Autl_e(A), Autv_e(A)$ etc.

Note: In $Endv_1(A)$ we mean: for some dense ideal I of $A, y \leq f(x) \in I \Rightarrow (\exists z \leq x)(f(z) = y)$ and $Endl_1(A) = End(A)$.

3.7 Claim.

(1) $End_6(A) \supseteq End_5(A) \cup End_4(A) \supseteq End_5(A) \cap End_4(A) \supseteq End_3(A) \supseteq End_2(A)$ *and* $End_2(A) \subseteq End_1(A) \subseteq End_0(A) = EndA$.

(2) $end_6(A) \geq max\{end_5(A), end_4(A)\} \geq min\{end_5(A), end_4(A)\} \geq end_3(A) \geq end_2(A)$, *and* $end_2(A) \leq end_1(A) \leq end(A)$.

(3) *In (1) we can replace* End *by* Aut *or* $Endv$ *or a*$Endl$ *or* $Autv$ *or* $Autl$, *or* $Endu$ *or* $Autu$ *and in (2)* end *by* aut *etc. We can in (2) replace* end_l *(or* aut_l*) by* $end_l^\sim$ *(or* $aut_l^\sim$*) etc.*

(4) $Aut_l(A) \subseteq Endl_l(A)$ *hence* $aut_l(A) \leq endl_l(A)$*; and* $Endv_e(A) \subseteq End_e(A) \subseteq Endl_e(A)$ *hence* $endv_e(A) \leq end_e(A) \leq endl_e(A); Endv_e(A) = End_e(A)$ *if* $e = 0$. *Also* $Aut(A) = Aut_2^\sim(A)$ *hence* $aut(A) = aut_2^\sim(A)$.

(5) *If* $f, g, \in Endv_6(A)$, $f \sim g$ *then* f, g, *are compatible functions,i.e.* $x \in Domf \cap Domg \Rightarrow f(x) = g(x)$.

(6) $sub(A) \geq aut_6(A)$ *and* $aut(A) \leq aut_3^\sim(A)$.

(7) $Aut_e(A) = Autv_e(A)$.

(8) $end_e(A) \leq id(A) + end_e^\sim(A)$ *etc.*

Proof. E.g.

(5) Suppose $x \in Domf \cap Domg$, $f(x) \neq g(x)$, so $wlog\, f(x) \not\leq g(x)$ so let $z =: f(x) - g(x) \neq 0$. As $z \leq f(x)$ for some $t \in Domf$ we have $0 < f(t) \leq z$, and $wlog\, f(t) = g(t)$, and we get contradiction.

(6) As in proof of 3.2 or see proof of 3.12 (noting $|\{I : I$ a dense ideal of $A\}| \leq sub(A)$ by 3.4A).

3.8 Definition. For a Boolean Algebra A

(1) $Idc(A) = \{I : I$ an ideal of A, and $I^c = I\}$ where $I^c = \{x \in A : I$ is dense below $x\}$.

(2) $idc(A) = |Idc(A)|$.

(3) $Did(A) = \{I : I$ a dense ideal of $A\}$.

(4) For ideals I, J $I + J$ is the minimal ideal I of A which include $I \cup J$ i.e. $\{x \cup y : x \in I$ and $y \in J\}$. Similarly $\sum_{\zeta < \xi_0} I_\zeta$.

(5) If we replace A by an ideal I (in 3.8 (1),(2),(3), 3.1 (2)) means we restrict ourselves to subideals of it.

3.9 Claim. *For a Boolean algebra A:*

(1) $id(A) \geq idc(A) = |comp(A)|$.

(2) $|A| \leq idc(A) = idc(A)^{\aleph_0}$ *(when A is infinite, of course).*

(3) *If* $f \in Endu_5(A)$, *then* f *has a unique extension to an endomorphism* f^+ *of* $comp(A)$ *where* $f^+(x) = sup_A\{f(y) : y \leq x, y \in Domf\}$. *If* f *is everywhere onto it is the unique extension of* f *in* $End(compA)$.

(4) *For* $g \in End(A)$, $(\exists f \in End_5(A))f^+ \supseteq g$ *iff* $g \in Endu(A)$.

(5) *For* $f, g \in End_5(A)$ *we have* $f \sim g \Leftrightarrow f^+ = g^+$.

(6) *For* $f \in End\, u_6(A)$, *letting* $a = sup_{comp(A)}\{x : B|x \subseteq Dom\, f\} \in comp(A)$ *and* $b = sup_{comp(A)}\{x : A|x \subseteq Rangf\} \in comp(A)$, *we define* $f^+ \in HOM(comp(A|a), comp(A|b))$ *extending* f *by* $f(x) = sup\{f(y) : y \leq x, y \in Domf\}$, *also* f^+ *is onto.*

3.10 Claim.

(1) $id(A) \leq endl_3(A)$.
(2) $idc(A) \leq endl^{\sim}_3(A)$.
(3) $endl_e(A) = id(A) + endl^{\sim}_e(A)$ *for* $e = 3, 4, 5, 6$.
(4) *If* $f \in End_5(A)$ *then* $f^+ \in End_1(compA)$.
(5) $aut_e(A) \leq id(A) + aut^{\sim}_m(A)$ *when* e, $m \in \{3, 5\}$ *or* e, $m \in \{4, 6\}$.

Proof.

(1) For $I \in Id(A)$ let $J_I = \{x \in B : (B|x) \cap I = \{0\}\}$, so J_I is an ideal, $J_I \cap I = \{0\}$, $J_I \cup I$ is dense. Let F_I be the following map: $Dom f_I = I + J_I$, $f_I|I = id_I$, $f_I|J_I = 0_{J_I}$. Now $I \mapsto f_I$ is a one to one mapping from $Id(A)$ to $Endl_3(A)$.
(2) The mapping above works.
(3) Note that $id(A) \leq end_3(A)$ by part(1), and $endl^{\sim}_3(A) \leq endl_3(A)$ trivially. A is infinite hence all those cardinals are infinite so $\chi =: id(A) + endl^{\sim}_3(A) \leq endl_3(A)$. The inverse inequality is easy too.

(4), (5) Left to the reader.

3.11 Claim.

(1) *For* $x \in A$:
$\big(\exists f \in Aut(A)\big)[x \neq f(x)]$ *iff*
$\big(\exists f \in Aut_6(A)\big)\big(x \in Dom f \& x \neq f(x)\big)$ *iff* $(\exists y, z)[0 < y \leq x \& 0 < z \leq 1 - x \& A|y \cong A|z]$.
(2) *If* $\langle I_\zeta : \zeta < \alpha \rangle$ *is a sequence of ideals of* A ,$[\zeta \neq \xi \Rightarrow I_\zeta \cap I_\xi = \{0\}]$ *and* $I = \sum_{i<\alpha} I_i$ *then:*
$id(I) \geq \pi_{\zeta<\alpha}\, id(I_\zeta)$.
$idc(I) \geq \pi_{\zeta<\alpha}\, idc(I_\zeta)$.
$aut(I) \geq \pi_{\zeta<\alpha}\, aut(I_\zeta)$.
$end(I) \geq \pi_{\zeta<\alpha}\, end(J_\zeta)$.
Similarly for $end_l, aut_l, end^{\sim}_l, aut^{\sim}_l$ *etc.*

There are many more easy relations, but for our aim the main point is

3.12 Main Lemma. *For an infinite Boolean Algebra A:*

(1) $aut^{\sim}_e(A)$ *for* $e = 3, 5$ *are equal or both finite (and we can restrict ourselves to automorphisms of order 2).*
(2) *If for some* $e \in \{3, 4, 5, 6\}$ *we have* $aut^{\sim}_e(A) > idc(A)$ *then* $aut^{\sim}_e(A)$ *for* $e = 3, 4, 5, 6$ *are all equall.*
(3) $aut^{\sim}_3(A) = aut^{\sim}_3(A)^{\aleph_0}$ *or* $aut_3(A)$ *is finite.*
(4) $autv^{\sim}_e(A) = autv_3(A) + idc(A)$ *for* $e = 4, 6$.

Proof. Let $J =: \{x \in A :$ for every $f \in Aut(A)$, $f|(A|x) = id_{A|x}\}$.

The function $F_{1,\dots}, F_5$ satisfying $y \leq x \Rightarrow F_e(y) \leq F_e(x)$ are functions from A to ord defined below; and we let:

$K =: \{x \in A :$
(i) for some $y, x \cap y = 0$ and $A|x \cong A|y$.

(ii) for $e = 1, \cdots, 5$ and $0 < y < x$ we have $F_e(y) = F_e(y)\}$.

Where:

$F_1(x) =$ the cardinality of $A|x$.

$F_2(x) = idc(A|x)$.

$F_3(x) = aut(A|x)$.

$F_4(x) = aut_3(A|x)$.

$F_5(x) = aut_3^{\sim}(A|x)$.

Now

$(*)_1$ J is an ideal of B.

$(*)_2$ K is downward closed.

$(*)_3$ $K \cup J$ is dense.

Choose $\{x_i : i < \alpha\}$ maximal such that:

(a) $x_i \in K$, $x_i > 0$,

(b) if $i \neq j$, $0 < y' \leq x_i$, $0 < y'' \leq x_j$ *then* $A|y' \not\cong A|y''$.

Let $K_i = \{y : \text{for some } y' \leq x_i,\ A|y \cong A|y'\}$, and K_i^+ : the ideal K_i generate. Now

$(*)_4$ $\bigcup_{i<\alpha} K_i$ is dense in K [hence $J \cup \bigcup_{i<\alpha} K_i$ is dense in A].

$(*)_5$ For $i \neq j$ we have $K_i \cap K_j = \{0\}$, $K_i \cap J = \{0\}$.

Clearly,

$(*)_6$ for $f \in Aut_6(A)$ we have:

(a) f is the identity on $J \cap Dom f$.

(b) for $x \in Dom f$ we have $x \in K_i \Leftrightarrow f(x) \in K_i$.

So

$(*)_7$ $aut_e^{\sim}(A) = \pi_{i<\alpha}\, aut_e^{\sim}(K_i^+)$ for $e = 3, 5$ and $aut_e^{\sim}(A) = idc(J) \times \pi_{i<\alpha} aut_e^{\sim}(K_i^+)$ for $e = 4, 6$

We shall prove:

$(*)_8$ For each i, one of the following ocurrs:

(a) $aut_e^{\sim}(K_i^+)$ for $e = 3, 4, 5, 6$ are all finite > 1,

(b) For some infinite κ we have $aut_e^{\sim}(K_i^+) = autv_e^{\sim}(K_i^+)^{\kappa} \geq idc(K_i)$ for $e = 3, 4, 5, 6$

(really we can use $F_6(i) = sup\{\kappa^+ : B|x$ has κ pairwise disjoint non zero members$\}$ and any such κ is OK for (b)).

Case 1. x_i is an atom.

This is clear: let $\lambda_i =: |K_i|$, if it is infinite, $aut_e^{\sim}(K_i^+) = 2^{\lambda_i}$ for $e = 3, 4, 5, 6$ so case (b) in $(*)_8$ ocurrs.

If λ_i is finite, $1 < aut_e^{\sim}(K_i^+) < \aleph_0$ (we can compute exactly), so case (a) in $(*)_8$. In fact in all cases we can use just automorphism of order 2.

So we can assume

Case 2 . not Case 1, so $B|x_i$ is atomless, hence $idc(B|x_i) \geq 2^{\aleph_0}$

Let $\langle J_{i,\zeta}, J'_{i,\zeta} : \zeta < \zeta_i\rangle$ be a sequence such that:

(α) $J_{i,\zeta}$, $J'_{i,\zeta}$ are ideals $\subseteq K_i^+$ and $\neq \{0\}$,

(β) $\{0\} \neq J_{i,\zeta} \subseteq A|x_i$,

(γ) $J_{i,\zeta} \cong J'_{i,\zeta}$, an $h_{i,\zeta}$ an isomorphism from $J_{i,\zeta}$ onto $J'_{i,\zeta}$,

(δ) $\wedge_{\zeta<\xi<\zeta_i} J'_{i,\zeta} \cap J'_{i,\xi} = \{0\}$,

(ϵ) if $y \in K^+_\zeta$ is disjoint to all members $\cup_{\xi \leq \zeta} J'_{i,\xi}$ of then for some $y' \leq y$ and $z \in J_{i,\zeta}, B|z' \cong B|y'$ hence

(ζ) if $\zeta < \xi < \zeta_i$ then $J_{i,\xi} \cap J_{i,\xi}$ is a dense subset of $J_{i,\xi}$

Now

$(*)_9$ $\bigcup_{\zeta < \zeta_i} J'_{i,\zeta}$ is a dense subset of K^+_i.

$(*)_{10}$ $\zeta_i \geq 2$ (as there is $y, x_i \cap y = 0$ and $A|y \cong A|x_i$ as $x_i \in K$).

$(*)_{11}$ $idc(K^+_i) = \pi_{\zeta < \zeta_i} idc(J'_{i,\zeta})$.

By the definition of K (see choice of F_2), we have $0 < y \leq x_i \Rightarrow idc(A|y) = (A|x)$ hence $idc(J'_{i,\zeta}) = idc(A|x_i)$ so

$(*)_{12}$ $idc(K^+_i) = [idc(A|x_i)]^{|\zeta_i|}$ hence $[idc(K^+_i)]^{|\zeta_i|} = idc(K^+_i)$.

Easily

$(*)_{13}$ $aut^{\sim}_6(K^+_i) \geq aut^{\sim}_3(K^+_i) \geq idc(A|x_i)$.

(and even by automorphism of order 2).

Last inequality: for each $z \in Idc(A|x_i)$ there is $z', z' \cap x_i = 0$, and $A|z \cong A|z'$ and let g be such isomorphism, let I_z be the ideal of A generated by $\{x \in K^+_i : x \leq z$ or $x \leq z'$ or $x \cap z = x \cap z' = 0\}$ $g_z \in Aut(I_z) \subseteq Aut_3(K^+_3), g_z(y) =: (y - z \cup z') \cup g(y \cap z) \cup g^{-1}(y \cap z')$.

Case 2A. $\zeta_i < \aleph_0$ (and not Case 1)

Let for $n < \omega$ $x_{i,n} \leq x_i, x_{i,n} \neq 0$, and $[n \neq m \Rightarrow x_{i,n} \cap x_{i,m} = 0]$, and $I_{i,\omega} = \{y \in K^+_i : \wedge_n y \cap x_{i,n} = 0\}$ and $I_{i,n} = \{z : z \leq x_{i,n}\}$: So $aut^{\sim}_3(K^+_i) \geq aut^{\sim}_3(\sum_{\alpha \leq \omega} I_{i,\alpha}) \geq \pi_{n<\omega} aut^{\sim}_3(B|x_{i,n}) = aut^{\sim}_3(B|x_i)^{\aleph_0}$.

Similarly to the proof of Case 2B below (but easier) we can show that Case (b) of $(*)_8$ ocurrs. From now on we assume

Case 2B. Not Case 2A, so $\zeta_i \geq \aleph_0$

For every $f \in Aut_6(K^+_i)$ let, for $\zeta, \xi < \zeta_i$:

$L^f_{\zeta,\xi} = \{x \in J'_{i,\zeta} : f(x) \in J'_{i,\xi}\}$.

$M^f_{\zeta,\xi} = \{f(x) : x \in J'_{i,\zeta}, f(x) \in J'_{i,\xi}\}$.

So the number of possible $\bar{L}^f = \langle sup L^f_{\zeta,\xi} : \zeta \neq \xi < \zeta_i \rangle$ and $\bar{M}^f = \langle sup(M^f_{\zeta,\xi}) : \zeta \neq \xi < \zeta \rangle$ is $\leq [idc(A|x_i)]^{|\zeta_i|}$ and for fixed $\bar{L}, \bar{M}$ the number of $f \in aut^{\sim}_6(K^+_i)$ for which $\bar{L}^f = \bar{L}, \bar{M}^f = \bar{M}$ is $\leq \pi_{\zeta,\xi} |\{f/\sim : f$ an isomorphism from a dense subset of $L_{\zeta,\xi}$ onto a dense subset of $M_{\zeta,\xi}\}| \leq \pi_{\zeta,\xi} aut^{\sim}_3(L_{\zeta,\xi}) \leq \pi_{\zeta,\xi}\, aut^{\sim}_3(A|x_i) = [aut^{\sim}_3(A|x_i)]^{|\zeta_i|}$).

(In the last equality we use F_5 in the definition of K: for the last $\leq$, note we can replace $L_{\zeta,\xi}$ by isomosrphic ideal $L^*_{\zeta,\xi}$ of $A|x_i$, and letting $L^-_{\zeta,\xi} = \{y\ :: y \in A|x_i,\ (\forall z \in L^*_{\zeta,\xi})[y \cap z = 0]\}$ we can extend every $f \in Aut_3(L^*(\zeta,\xi)$ to $f' \in Aut_3(A|x)$ by letting $f'(x \cup y) = f(x) \cup y$ for $x \in Dom(f)$, $y \in L^-_{\zeta,\xi}$)

So

$(*)_{14}$ $aut^{\sim}_6(K^+_i) \leq [aut^{\sim}_3(A|x_i) + idc(A|x_i)]^{|\zeta_i|}$.

Now for each ξ such that $2\xi + 1 < \zeta_i$ we can choose $y_{i,2\xi} \in J_{i,2\xi}, y_{i,2\xi+1} \in J_{i,2\xi+1}$ such that $A|y_{i,2\xi} \cong A|y_{i,2\xi+1}$ and $\langle g_{i,\xi,\alpha} : \alpha < aut^{\sim}_3(A|x_i)\rangle$ such that $g_{i,\xi,\alpha}$ is an isomorphism from a dense subset of $A|y_{i,\xi,\alpha}$ onto a dense subset of $A|y_{i,2\xi+1}$, $\langle g^+_{i,\xi,\alpha} : \alpha < aut^{\sim}_3(A|x_i)\rangle$ pairwise distinct. Let ζ^*_i be minimal such that $2\zeta^*_i \leq \zeta_i$.

Now for every sequence $\bar{\alpha} = \langle \alpha_\xi : \xi < \zeta^*_i \rangle$, $\alpha_\xi < aut^{\sim}_3(A|x_i)$, we define $g_{i,\bar{\alpha}} \in Aut_3(K^+_i)$ of order two (see condition (γ)):

$$g_{i,\bar{\alpha}}|(y_{i,2\xi} \cup y_{i,2\xi+1}) = h_{2\xi+1} \circ g_{i,\xi,\alpha} \circ h^{-1}_{2\xi} \cup h_{2\xi} \circ g^{-1}_{i\xi,\alpha} \circ h^{-1}_{2\xi+1}$$

and if $y \in K_i^+ \setminus \{0\}$ and $\wedge_{\xi < \zeta_i^*}(y \cap y_{i,\xi} = 0\rangle$ *then* $g_{i,\bar\alpha}(y) = 0$.

Lastly if y is the disjoint union of $y_0, \cdots, y_n$ and each $g_{\bar\alpha}(y_e)$ was defined then we define $g_{\bar\alpha}(y_0 \cup \cdots \cup y_n) = g_{\bar\alpha}(y_0) \cup \cdots \cup g_{\bar\alpha}(y_n)$. The reader may check that $g_{\bar\alpha} \in Aut_3(K_i^+)$.

The mapping $\bar\alpha \mapsto g_{i,\bar\alpha}$ show:

$(*)_{15}$ $aut_3^\sim(K_i^+) \geq aut_3^\sim(A|x_i)^{|\zeta_i|}$.

Easily, choosing $y_i \in J_{i,1}$ (possible as $\zeta_i \geq 2$)

$(*)_{16}$ $aut_3^\sim(K_i^+) \geq aut_3^\sim(A|(x_i \cup y_i)) \geq idc(A|y) = idc(A|x_i) \geq \aleph_0$.

Together by $(*)_{13}, (*)_{14}, (*)_{15}, (*)_{16}$ for $e = 3, 6$:

$$aut_e^\sim(K_i^+) = aut_3^\sim(A|x_i)^{|\zeta_i|}.$$

By 3.7(1) this holds for $e = 4, 5$, hence $(*)_8$ has been proved.

Now by $(*)_7 + (*)_8$ the four parts of 3.12 follows.

3.13 Conclusion : $aut(A)^{\aleph_0} \leq sub(A)$, also $aut_e(A)^{\aleph_0} \leq sub(A)$ for $e = 3, 4, 5, 6$.

Note: even if $aut(A)$ is finite, A infinite, still $2^{\aleph_0} \leq sub(A)$ (for A infinite).

Proof. By 3.7(3) + 3.7(6) $aut(A) = aut_0^\sim(A) \leq id(A) + aut_3^\sim(A) \leq id(A) + Pend(A) \leq sub(A)$ but by S. Shelah [6] $id(A)^{\aleph_0} = id(A)$ and by 3.12(3) $aut_3^\sim(A)^{\aleph_0} = aut_3^\sim(A)$, together we can finish the first inequality, the second is similar using 3.10(5).

3.14 Claim : *For an (inifinite) Boolean algebra A we have:*

(1) $end_e(A) = id(A) + aut_3(A)$ *for* $e = 3, 4, 5, 6$.
(2) $end_6(A)^{\aleph_0} \leq sub(A)$.
(3) $end_e(A)^{\aleph_0} \leq sub(A)$ *for* $e = 2, 3, 4, 5, 6$.

Proof.

(1) Clearly $end_3(A) \geq aut_3(a)$ and $end_3(A) \geq id(A)$ (as the mapping in the proof of 3.10(1) examplify).

On the other hand we can attach to every $f \in End_6(A)$ three ideals $I_1(f) = Dom\, f$, $I_2(f) = Rang(f)$ and $I_3(f) = Ker(f)$. Now the number of triples $\bar I$ of ideal of A has cardinality $id(A)$ and for each such $\bar I$:

$\{f \in End_6(A) : I_e(f) = I_e$ for $e = 1, 2, 3\}$ has cardinality $|Aut(I_2)|$ which is $\leq aut_3(A)$. By 3.7(2) we can finish.

(2) Remember also $id(A)^{\aleph_0} = id(A)$ by [6] and part (1) and 3.13.
(3) By part (2) and 3.7.

§4 The width of the Boolean algebra .

4.1 Definition. For a Boolean algebra B let: (1) $A \subseteq B$ is an antichain if $x \in A \& y \in A \& x \neq y \Rightarrow x \not\leq y$ (i.e. A is a set of pairwise incomparable elements).

(2) Width of B, $w(B)$ is $sup\{|A| : A \subseteq B$ antichain$\}$, $w^+(B) = \cup\{|A|^+ : A \subseteq B$ antichain$\}$

E. C. Milner and M. Poizat [3], answering a question of E. K. van Dowen, D. Monk and M. Rubin [7] proved $cf(w^+(B)) \neq \aleph_0$.

In S. Shelah [5] we claim: if $\lambda > cf\lambda > \aleph_0$, for some generic extension of the universe preserving cardinalities and cofinalities, for some $B, w^+(B) = \lambda$. We retract this and replace it by the theorem 4.2 below.

For weakly inaccessibles we still have the consistency. Moreover, if λ is a limit uncountable regular cardinal, $S \subseteq \lambda$ stationary not reflecting and $\diamond_S$ it then we have such an example for λ.

4.2 Theorem. *For an infinite Boolean algebra* $B, w^+(B)$ *is an uncountable regular cardinal.*

Proof. As B is infinite it has an antichain $A, |A| = \aleph_0$, [if B has finitely many atoms clear, if not it has a subalgebra which is atomless, without loss of generality countable and check]. So $\lambda =: w^+(B) > \aleph_0$. Assume $\kappa =: cf\lambda < \lambda$; let $\lambda = \sum_{i<\kappa} \lambda_i$, $cf\lambda + \sum_{j<i} \lambda_j < \lambda_i < \lambda$ and let $A_i \subseteq B$ be an antichain of cardinality λ_i^+ (exist by the choice of λ). Let $A = \bigcup_{i<\kappa} A_i$, so $|A| = \lambda$. Choose such $\langle A_i : i < \kappa \rangle$ such that, if possible

$(*) i < j < \kappa,\ x \in A_i,\ y \in A_j \Rightarrow y \not\leq x.$

For $x \in B$ let $A[>, x] = \{y \in A : y > x\}, A[<, x] = \{y \in A : y < x\}$,

$A[>, \mu] = \{x \in A : |A[>, x]| < \mu\}, A[<, \mu] = \{x \in A : |A[<, x]| < \mu\}$.

Case 1. For some $\mu < \lambda$, $A[>, \mu]$ has cardinality λ.

By Hajnal free subset theorem, there is a set $E \subseteq A[>, \mu]$ of cardinality λ such that:

$x \neq y \,\&\, x \in E \,\&\, y \in E \Rightarrow x \notin A[>, y] \,\&\, y \notin A[>, x]$. So E witness $w^+(B) > \lambda$.

Case 2. For some $\mu < \lambda$, $A[<, \mu]$ has cardinality λ.

Same proof.

Case 3. For every $i < \kappa$ there is $x \in A$ such that $\lambda_i < |A[>, x]| < \lambda$.

Let for $i < \kappa$, $x_i \in A$ be such that $\lambda_i < |A[>, x_i]| < \lambda$. Let $u \subseteq \kappa$ be such that: $\kappa = sup u$ and for $i \in u$, $\lambda_i > \sum_{j \in u \cap i} |A[>, x_i]|$ (choose the members of u inductively). By renaming without loss of generality $u = \kappa$. Clearly $A[>, x_i] \setminus \cup_{j<i} A[>, x_j]$ has cardinality $> \lambda_i$.

As $\lambda_i > \kappa$ (by its choice) and $A = \cup_{j<\kappa} A_j$, clearly for each i there is $\alpha(i) < \kappa$ such that $(A[>, x_i] \setminus \cup_{j<i} A[>, x_j]) \cap A_{\alpha(i)}$ has cardinality $> \lambda_i$; necessarily $\alpha(i) \geq i$.

For some unbounded $u \subseteq \kappa$ we have $[i \in u \& j \in u \& i < j \Rightarrow \alpha(i) < j]$; without loss of generality $u = \kappa$, $\alpha_i = i$. Let A_i^* be a subset of $(A[>, x_i] \setminus \cup_{j<i} A[<, x_j]) \cap A_{\alpha(i)}$ of cardinality λ_i^+. Now $\langle A_i^* : i < \kappa \rangle$ satisfies: $A_i^* \subseteq B$ is an antichain of cardinality λ_i^+ and

$(*)'$ $i < j, x \in A_i^*, y \in A_j^* \Rightarrow x \not\leq y$ (otherwise $x_i \leq x \leq y \notin A[>, x_i]$, contradiction).

So $\langle A_i' =: \{1_B - x : x \in A_i^*\} : i < \kappa \rangle$ satisfies $A_i' \subseteq B$ is an antichain of B of cardinality λ_i^+ and also (*) above (check). So by the choice of $\langle A_i : i < \kappa \rangle$, it satisfies (*). By $(*) + (*)'$, $A^* = \cup_{i<\kappa} A_i^*$ is an antichain of B of cardinality λ, so $w^+(B) > \lambda$.

Case 4. For every $i < x$ there is $x \in A$ such that $\lambda_i < |A[<, x]| < \lambda$.

Similar to Case 3.

Case 5. None of the previous cases.

By "not Case 3" for some $i(*) < \kappa$, for no $x \in A$ is $\lambda_{i(*)} < |A[>, x]| < \lambda$. By not Case 2, $A[<, \lambda_{i(*)}^+]$ has cardinality $< \lambda$. By not Case 1 $A[>, \lambda_{i(*)}^+]$ has cardinality $< \lambda$.

Choose $x^* \in A \setminus A[<, \lambda_{i(*)}^+] \setminus A[>, \lambda_{i(*)}^+]$ so $A[>, x^*]$ has cardinality $\geq \lambda_{i(*)}^+ > \lambda_{i(*)}$, hence by the choice of $i(*)$ we have $A[>, x^*]$ has cardinality λ.

As $\lambda_{i(*)} > \kappa$, for some $j(*), A[<, x^*] \cap A_{j(*)}$ has cardinality $> \lambda_{i(*)}$, so choose distinct $y_i \in A[<, x^*] \cap A_{j(*)}$ for $i < \kappa$. Now $y_i < x^*$ (as $y_i \in A[<, x^*]$), and $[i \neq j \Rightarrow y_i \not\leq y_j]$ (as they are distinct and in $A_{j(*)}$).

Let $A'_i = A_i \cap A[>, x^*]$, so $A'_i \subseteq A_i$ hence is an antichain of B, and

$$\cup_i A'_i = (\cup A_i) \cap A[>, x^*] = A \cap A[>, x^*] = A[>, x^*]$$

So each A'_i is an antichain, its member are $> x^*$ and $| \cup_i A'_i|$ is λ as $|A[>, x^*]|$ is.

Now

$$A' = \cup_{i<\kappa}\{y_i \cup (x - x^*) : x \in A'_i\}$$

is an antichain of B of cardinality λ, so $w^+(B) > \lambda$, as required.

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