# CHARACTERIZING AN $\aleph_\epsilon$-SATURATED MODEL OF SUPERSTABLE NDOP THEORIES BY ITS $\mathbb{L}_{\infty,\aleph_\epsilon}$-THEORY SH401

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Abstract. Assume a complete countable first order theory is superstable with NDOP. We knew that any $\aleph_\varepsilon$-saturated model of the theory is $\aleph_\varepsilon$-prime over a non-forking tree of "small" models and its isomorphism type can be characterized by it $\mathbb{L}_{\infty,\kappa}$(dimension quantifiers)-theory; or if you prefer - appropriate cardinal invariants. We go here one step further providing cardinal invariants which are as finitary as seems reasonable.

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## Introduction

After the main gap theorem was proved (see [Sh:c]), in discussion, Harrington expressed a desire for a finer structure - of finitary character (when we have a structure theorem at all). I point out that the logic $\mathbb{L}_{\infty,\aleph_0}$(d.q.) (where d.q. stands for dimension quantifier) does not suffice: suppose; e.g. for $T =$ Th$(\lambda \times {}^\omega 2, E_n)_{n<\omega}$ where $(\alpha,\eta)E_n(\beta,\nu) =: \eta \restriction n = \nu \restriction n$ and for $S \subseteq {}^\omega 2$ define $M_S = M \restriction \{(\alpha,\eta) : [\eta \in S \Rightarrow \alpha < \omega_1]$ and $[\eta \in {}^\omega 2 \backslash S \Rightarrow \alpha < \omega]\}$. Hence, it seems to me we should try $\mathbb{L}_{\infty,\aleph_\epsilon}$(d.q.) (essentially, in $\mathfrak{C}$ we can quantify over sets which are included in the algebraic closure of finite sets, see below 1.1, 1.3), and Harrington accepts this interpretation. Here the conjecture is proved for $\aleph_\epsilon$-saturated models.

I.e., the main theorem is $M \equiv_{\mathbb{L}_{\infty,\aleph_\epsilon}(\text{d.q.})} N \Leftrightarrow M \cong N$ for $\aleph_\epsilon$-saturated models of a superstable countable (first order) theory $T$ without dop. For this we analyze further regular types, define a kind of infinitary logic (more exactly, a kind of type of $\bar a$ in $M$), "looking only up" in the definition (when thinking of the decomposition theorem). Recall that for as $\aleph_\varepsilon$-saturated model $M$ of a superstable DNOP theory an $\aleph_\varepsilon$-decomposition is $\langle M_\eta, a_\eta : \eta \in \mathscr{T}\rangle$ where

(a) $I \subseteq {}^{\omega>}$ord is nonempty closed under initial segments
(b) $M_\eta \prec M$ is $\aleph_\varepsilon$-saturated
(c) $\nu \triangleleft \eta \in I \Rightarrow M_\nu \prec M_\eta$
(d) if $\nu = \eta\hat{\ }\langle\alpha\rangle \in I$ then $M_\nu$ is $\aleph_\varepsilon$-prime over $M_\eta \cup \{a_\nu\}$ and tp$(a_\eta, M_\eta)$ is orthogonal to $M_\rho$ for $\rho \triangleleft \nu$ and (the last is not essential but clarifies)
(e) $\langle M_\eta : \eta \in I\rangle$ is non-forking enough: for every $\nu \in I$ the set $\{a_\eta : \eta \in \text{Suc}_I(\nu)\} \subseteq M$ is independent over $M_\nu$.

The point is that if $\eta = \nu\hat{\ }\langle\alpha\rangle, M_{\eta_\nu}, a_\eta$ are chosen then to a large extent $\langle M_\rho, a_\rho : \eta \triangleleft \rho \in I\rangle$ is determined. But the amount of "to a large extent" which suffices in [Sh:c], is not sufficient here, we need to find a finer understanding. In particular, we certainly do not like to "know" $(M_\nu, a_\eta)$. So we consider a pair $(A, B)$ where $A \subseteq M_\nu, A \cup \{a_\eta\} \subseteq B \subseteq M_\eta$, stp$_*(B, A) \vdash$ stp$_*(B, M_\nu)$ and we try to define the type of such pairs in a way satisfying:

(a) it can be impressed in our logic $\mathbb{L}_{\infty,\aleph_\varepsilon}$
(b) it expresses the essential information in $\langle M_\rho, a_\rho : \eta \triangleleft \rho \in I\rangle$.

To carry out the isomorphism proof we need: (1.27) the type of the sum is the sum of types (infinitary types) assuming first order independence. The main point of the proof is to construct an isomorphism between $M_1$ and $M_2$ when $M_1 \equiv_{\mathbb{L}_{\infty,\aleph_\epsilon}(d.q.)} M_2, Th(M_\ell) = T$ where $T$ and $\equiv_{\mathbb{L}_{\infty,\aleph_\varepsilon}(\text{q.d.})}$ are as above. So by [Sh:c, X] it is enough to construct isomorphic decompositions. The construction of isomorphic decompositions is by $\omega$ approximations, in stage $n$, $\sim n$ levels of the decomposition tree are approximated, i.e. we have $I^\ell_n \subseteq {}^{n\geq}\text{Ord}$ and $\bar a^{n,\ell}_\eta \in M_\ell$ for $\eta \in I_n, \ell = 1,2$ such that $\text{tp}(\bar a^{n,1}_{\eta\restriction 0}{}^\frown \bar a^{n,1}_{\eta\restriction 1}{}^\frown \ldots {}^\frown \bar a^{n,1}_\eta, \emptyset, M) = \text{tp}(\bar a^{n,2}_{\eta\restriction 0}{}^\frown \bar a^{n,2}_{\eta\restriction 1}{}^\frown \ldots {}^\frown \bar a^{n,2}_\eta, \emptyset, M_2)$ with $\bar a^{\eta,\ell}_\eta$ being $\varepsilon$-finite, so in stage $n+1$, choosing $\bar a^{n+1,\ell}_{\langle\rangle}$ we cannot take care of all types $\bar a^{n+1,\ell}_{\langle\rangle}{}^\frown \bar a^{\eta,\ell}_{\langle\alpha\rangle}$ so addition theorem takes care. So though we are thinking on $\aleph_\varepsilon$-decompositions (i.e., the $M_\eta$'s are $\aleph_\varepsilon$-saturated), we get just a decomposition.

In the end of §1 (in 1.37) we point up that the addition theorem holds in fuller generalization. In the second section we deal with a finer type needed for shallow $T$, in the appendix we discuss how absolute is the isomorphism type.

Of course, we may consider replacing "$\aleph_\varepsilon$-saturated models of an NDOP superstable countable $T$" by "models of an NDOP $\aleph_0$-stable countable $T$". But the use of $\varepsilon$-finite sets seems considerably less justifiable in this context, it seems more reasonable to use finite sets, i.e., $\mathbb{L}_{\infty,\aleph_0}(\text{d.q.})$. But subsequently Hrushovski and Bouscaren proved that even if $T$ is $\aleph_0$-stable, $\mathbb{L}_{\infty,\aleph_0}(\text{d.q.})$ is not sufficient to characterize models of $T$ up to isomorphism. This is not sufficient even if one considers the class of all $\aleph_\varepsilon$-saturated models rather than all models. The first example is $\aleph_0$-stable shallow of depth 3, and the second one is superstable (non $\aleph_0$-stable), NOTOP, non-multi-dimensional.

If we deal with $\aleph_\epsilon$-saturated models of shallow (superstable NDOP) theories $T$, we can bound the depth of the quantification $\gamma = DP(T)$; i.e. $\mathbb{L}^\gamma_{\infty,\aleph_\epsilon}$ suffice.

We assume the reader has a reasonable knowledge of [Sh:c, V,§1,§2] and mainly [Sh:c, V,§3] and of [Sh:c, X].

Here is a slightly more detailed guide to the paper. In 1.1 we define the logic $\mathbb{L}_{\infty,\aleph_\epsilon}$ and in 1.3 give a back and forth characterization of equivalence in this logic which is the operative definition for this paper.

The major tools are defined in 1.7, 1.11. In particular, the notion of $\text{tp}_\alpha$ defined in 1.5 is a kind of a depth $\alpha$ look-ahead type which is actually used in the final construction. In 1.28 we point out that equivalence in the logic $\mathbb{L}_{\infty,\aleph_\epsilon}$ implies equivalence with respect to $\text{tp}_\alpha$ for all $\alpha$. Proposition 1.14 contains a number of important concrete assertions which are established by means of Facts 1.16-1.23. In general these explain the properties of decompositions over a pair $\binom{B}{A}$. Claim 1.27 (which follows from 1.26) is a key step in the final induction. Definition 1.30 establishes the framework for the proof that two $\aleph_\epsilon$-saturated structures that have the same $\text{tp}_\infty$ are isomorphic. The induction step is carried out in 1.35.

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-1.1 Notation: The notation is of [Sh:c], with the following additions (or reminders).

If $\eta = \nu{}^\frown\langle\alpha\rangle$ then we let $\eta^- = \nu$; for $I$ a set of sequences ordinals we let

$\mathrm{Suc}_T(\eta) = \{\nu : \text{for some } \alpha, \nu = \eta\hat{\ }\langle\alpha\rangle \in I\}$.
We work in $\mathfrak{C}^{\rm eq}$ and for simplicity every first order formula is equivalent to a relation.

(1) $\perp$ means orthogonal (so $q$ is $\perp p$ means $q$ is orthogonal to $p$), remember $p \perp A$ means $p$ orthogonal to $A$; i.e. $p \perp q$ for every
$q \in S(ac\ell(A))$ (in $\mathfrak{C}^{\rm eq}$)

(2) $\perp_{\rm a}$ means almost orthogonal

(3) $\perp_{\rm w}$ means weakly orthogonal

(4) $\frac{\bar a}{B}$ and $\bar a/B$ means $\mathrm{tp}(\bar a, B)$

(5) $\frac{A}{B}$ or $A/B$ means $\mathrm{tp}_*(A, B)$

(6) $A + B$ means $A \cup B$

(7) $\bigcup\limits_{A}\{B_i : i < \alpha\}$ means $\{B_i : i < \alpha\}$ is independent over $A$

(8) $A \bigcup\limits_{B} C$ means $\{A, C\}$ is independent over $B$

(9) $\{C_i : i < \alpha\}$ is independent over $(B, A)$ means that[1]
$$j < \alpha \Rightarrow \ \mathrm{tp}_*(C_j, \cup\{C_i \cup B : i \neq j\}) \text{ does not fork over } A$$

(10) regular type means stationary regular type $p \in S(A)$ for some $A$

(11) for $p \in S(A)$ regular and $C$ a set of elements realizing $p$, $\dim(C, p)$ is
$$\mathrm{Max}\{|\mathbf{I}| : \mathbf{I} \subseteq C \text{ is independent over } A\}$$

(12) $ac\ell(A) = \{c : \mathrm{tp}(c, A) \text{ is algebraic}\}$

(13) $dc\ell(A) = \{c : \mathrm{tp}(c, A) \text{ is realized by one and only one element}\}$

(14) $\mathrm{Dp}(p)$ is depth (of a stationary type, see [Sh:c, X,Definition 4.3,p.528,Definition 4.4,p.529]

(15) $\mathrm{Cb}(p)$ is the canonical base of a stationary type $p$ (see [Sh:c, III,6.10,p.134])

(16) $B$ is $\aleph_\varepsilon$-atomic over $A$ <u>if</u> for every finite sequence $\bar b$ from $A$, for some find $A_0 \subseteq A$ we have $\mathrm{stp}(\bar b, A_0)) \vdash \ \mathrm{stp}(\bar b, A)$, equivalently for some $\varepsilon$-finite $A_0 \subseteq acl(A)$ we have $\mathrm{tp}(\bar b, A_0)) \vdash \ \mathrm{tp}(\bar b, acl(A))$.

[1] Actually by the non-forking calculus this is equivalent to $\{C_i : i \leq \alpha\}$ is independent over $A$ where we let $C_\alpha = B$.

## §1 $\aleph_\epsilon$-SATURATED MODELS

We first define our logic, but as said in §0, we shall only use the condition from 1.4. $T$ is always superstable complete first order theory.

**1.1 Definition.** 1) The logic $\mathbb{L}_{\infty,\aleph_\epsilon}$ is slightly stronger than $\mathbb{L}_{\infty,\aleph_0}$, it consists of the set of formulas in $\mathbb{L}_{\infty,|T|^+}$ such that any subformula of $\psi$ of the form $(\exists \bar{x})\varphi$ is actually the form

$$(\exists \bar{x}^0, \bar{x}^1)\left[\varphi_1(\bar{x}^1, \bar{y}) \;\&\; \bigwedge_{i<\ell g \bar{x}^1} \big(\theta_i(x^1_i, \bar{x}^0) \;\&\; (\exists^{<\aleph_0} z)\theta_i(z, \bar{x}^0)\big)\right],$$

with $\bar{x}^0$ finite, $\bar{x}^1$ not necessarily finite but of length $< |T|^+$; so $\varphi$ "says" $\bar{x}^1 \subseteq ac\ell(\bar{x}^0)$; note that always our final proof of the theorem uses $|T| \geq \aleph_0$.
2) $\mathbb{L}_{\infty,\aleph_\epsilon}(d.q.)$ is like $\mathbb{L}_{\infty,\aleph_\epsilon}$ but we have cardinality quantifiers and moreover dimensional quantifiers (as in [Sh:c, XIII,1.2,p.624]), see below.
3) The logic $\mathbb{L}^\gamma_{\infty,\aleph_\epsilon}$ consist of the formulas of $\mathbb{L}_{\infty,\aleph_\epsilon}$ such that $\varphi$ has quantifier depth $< \gamma$ (but we start the inductive definition by defining the quantifier depth of all first order as zero).
4) $\mathbb{L}^\gamma_{\infty,\aleph_\epsilon}(d.q.)$ is like $\mathbb{L}^\gamma_{\infty,\aleph_\epsilon}$ but we have cardinality quantifiers and moreover dimensional quantifiers.

*1.2 Remark.* 1) In fact the dimension quantifier is used in a very restricted way (see Definition 1.9 and Claim 1.28 + Claim 1.30).
2) The reader may ignore this logic altogether and use just the characterization of equivalence in claim 1.4.

*1.3 Convention.* 1) $T$ is a fixed first order complete theory, $\mathfrak{C}$ is the "monster" model, as in [Sh:c], so is $\bar{\kappa}$-saturated; $\mathfrak{C}^{\rm eq}$ is as in [Sh:c, III,6.2,p.131]. We work in $\mathfrak{C}^{\rm eq}$ so $M, N$ vary on elementary submodels of $\mathfrak{C}^{\rm eq}$ of cardinality $< \bar{\kappa}$. We assume $T$ is superstable with NDOP (countability is used only in the Proof of 1.5 for bookkeeping, i.e., in the proof of 1.30 (and 1.29).
Remember $a, b, c, d$ denote members of $\mathfrak{C}^{\rm eq}$, $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ denote finite sequences of members of $\mathfrak{C}^{\rm eq}$, $A, B, C$ denote subsets of $\mathfrak{C}^{\rm eq}$ of cardinality $< \bar{\kappa}$.
Remember $ac\ell(A)$ is the algebraic closure of $A$, i.e.
$\{b :$ for some first order and $n < \omega, \varphi(x, \bar{y})$ and $\bar{a} \subseteq A$ we have $\mathfrak{C}^{\rm eq} \models \varphi[b, \bar{a}]$ & $(\exists^{\leq n} y)\varphi(y, \bar{a})\}$
and $\bar{a}$ denotes $\mathrm{Rang}(\bar{a})$ in places where it stands for a set (as in $ac\ell(\bar{a})$. We write

$\bar a \in A$ instead of $\bar a \in {}^{\omega>}(A)$.
2) $A$ is $\epsilon$-finite, if for some $\bar a \in {}^{\omega>}A, A = ac\ell(\bar a)$. (So for stable theories a subset of an $\epsilon$-finite set is not necessarily $\epsilon$-finite but as $T$ is superstable, a subset of an $\epsilon$-finite set is $\epsilon$-finite as if $B \subseteq ac\ell(\bar a), \bar b \in B$ is such that $\mathrm{tp}(\bar a, B)$ does not fork over $\bar b$, then trivially $ac\ell(\bar b) \subseteq A$ and if $ac\ell(\bar b) \neq B,\ \mathrm{tp}_*(B, \bar a\hat{\ }\bar b)$ forks over $B$, hence ([Sh:c, III,0.1]) $\mathrm{tp}(\bar a, B)$ forks over $\bar b$, a contradiction.
So if $ac\ell(A) = ac\ell(B)$, then $A$ is $\epsilon$-finite iff $B$ is $\epsilon$-finite).
3) When $T$ is superstable by [Sh:c, IV,Table 1,p.169] for $\mathbf{F} = \mathbf{F}^a_{\aleph_0}$, all the axioms there hold and we write $\aleph_\varepsilon$ instead of $\mathbf{F}$ and may use implicitly the consequences in [Sh:c, IV,§3].

We may instead Definition 1.1 use directly the standard characterization from 1.4; as actually less is used we state the condition we shall actually use:

**1.4 Claim.** *For models $M_1, M_2$ of $T$ we have $M_1 \equiv_{\mathbb{L}_{\infty,\aleph_\epsilon}(\text{d.q.})} M_2$ iff*

$\otimes$ *there is a non-empty family $\mathscr{F}$ such that:*
- (a) *each $f \in \mathscr{F}$ is an $(M_1, M_2)$-elementary mapping, (so $\mathrm{Dom}(f) \subseteq M_1$, $Rang(f) \subseteq M_2$)*
- (b) *for $f \in \mathscr{F}, \mathrm{Dom}(f)$ is $\epsilon$-finite (see 1.3(2)) above)*
- (c) *if $f \in \mathscr{F}, \bar a_\ell \in M_\ell\ (\ell = 1, 2)$ then for some $g \in \mathscr{F}$ we have: $f \subseteq g$ and $ac\ell(\bar a_1) \subseteq \mathrm{Dom}(f)$ and $ac\ell(\bar a_2) \subseteq Rang(f)$*
- (d) *if $f \cup \{\langle a_1, a_2\rangle\} \in \mathscr{F}$ and $\mathrm{tp}(a_1, \mathrm{Dom}(f))$ is stationary and regular then $\dim(\{a^1_1 \in M_1 : f \cup \{\langle a^1_1, a_2\rangle\} \in \mathscr{F}\}, M_1)$ $= \dim(\{a^1_2 \in M_2 : f \cup \{\langle a_1, a^1_2\rangle\} \in \mathscr{F}\}, M_2)$.*

Our main theorem is

**1.5 Theorem.** *Suppose $T$ is countable (superstable complete first order theory) with NDOP.*
*Then*

1. *the $\mathbb{L}_{\infty,\aleph_\epsilon}(\text{d.q.})$ theory of an $\aleph_\epsilon$-saturated model characterizes it up to isomorphism.*
2. *Moreover, if $M_1, M_2$ are $\aleph_\epsilon$-saturated models of $T$ (so $M_\ell \prec \mathfrak{C}^{\text{eq}}$) and $\otimes_{M_0,M_1}$ of 1.4 holds, then $M_1, M_2$ are isomorphic.*

* * *

By 1.4, it suffices to prove part (2).

The proof is broken into a series of claims (some of them do not use NDOP, almost all do not use countability; but we assume $T$ is superstable complete all the time (1.7(1)).

<u>1.6 Discussion</u>: Let us motivate the notation and Definition below.

Recall from the introduction that we are thinking of a triple $(M, N, a)$ which may appear in $\aleph_\varepsilon$-decomposition $\langle M_\eta, a_\eta : \eta \in I\rangle$ of $N$, in the sense that for some $\eta \in I \backslash \{<>\}$ we have $(M, M', a) = (M_{\eta^-}, M_\eta, a_\eta)$ so $M, M'$ are $\aleph_\varepsilon$-saturated, $a_\eta \in M' \backslash M'$, $M'$ is $\aleph_\varepsilon$-prime over $M + a$ and $\mathrm{tp}(a, M)$ is regular. But this is "too large for us" hence we consider an approximation $(A, B)$ where $A \subseteq M (= M_{\eta^-}), A \subseteq B \subseteq M' (= M_\eta)), a = a_\eta \in B$ and $B/M (= B/M_{\eta^-})$ does not fork over $A$. We would like to define the $\alpha$-type of $(A, B)$ in $N$, which tries to say something on the decomposition above $(M, M', a) = (M_{\eta^-}, M_\eta, a_\eta)$, i.e., on $\langle M_\rho, a_\rho : \eta \triangleleft \rho \in I\rangle$. There are two natural "successor" of $(A, B)$ we may choose in this context: the first 1.7 below replaces $(A, B)$ to $(A', B')$ such that $A \subseteq A' \subseteq M (= M_{\eta^-}), B \subseteq B' \subseteq M' (= M_\eta)$ and (as $M'$ is $\aleph_\varepsilon$-prime over $M + a$) we have $\mathrm{stp}_*(B', A' \cup B) \vdash \mathrm{stp}(B', M)$, so $\mathrm{tp}(B', A' \cup B)$ is almost orthogonal to $A'$; we can think of this as "advancing in the same model"; in other words as $A, B$ are $\varepsilon$-finite, we have to increase them in order to capture even $(M, M')$. This is formalized by $\leq_{\mathrm{a}}$ in Definition 1.7 below.

The second is to pass from $(M_{\eta^-}, M_\eta, a)$ to $(M_\eta, M_\nu, a_\nu)$ for some $\nu$ an immediate successor (in $I$) of $\eta \in I$. So the old $B$ is included in the new $A'$ and $B' = A' \cup \{a\}$ where $\mathrm{tp}(a, A')$ is regular and is orthogonal to $A$ (as in the decomposition we require $\mathrm{tp}(a_\eta, M_{\eta^-})(M_\nu$ when $\nu \triangleleft \eta^-)$. This is formalized by $\leq_{\mathrm{b}}$ in Definition 1.7 below.

**1.7 Definition.** 1) $\Gamma = \{(A, B) : A \subseteq B$ are $\epsilon$-finite$\}$. Let
$\Gamma(M) = \{(A, B) \in \Gamma : A \subseteq B \subseteq M\}$.
2) For members $(A, B)$ of $\Gamma$ we may also write $\binom{B}{A}$; <u>if</u> $A \nsubseteq B$ we mean $\binom{B \cup A}{A}$.
3) $\binom{B_1}{A_1} \leq_{\mathrm{a}} \binom{B_2}{A_2}$ (usually we omit a) <u>if</u> (both are in $\Gamma$ and)
$A_1 \subseteq A_2, B_1 \subseteq B_2, B_1 \underset{A_1}{\biguplus} A_2$ and $\frac{B_2}{B_1 + A_2} \perp_{\mathrm{a}} A_2$.
4) $\binom{B_1}{A_1} \leq_b \binom{B_2}{A_2}$ <u>if</u> $A_2 = B_1, B_2 \backslash A_2 = \bar{b}$ and $\frac{\bar{b}}{A_2}$ is regular orthogonal to $A_1$.
5) $\leq^*$ is the transitive closure of $\leq_{\mathrm{a}} \cup \leq_{\mathrm{b}}$. (So it is a partial order, whereas in general $\leq_{\mathrm{a}} \cup \leq_{\mathrm{b}}$ and $\leq_{\mathrm{b}}$ are not).
6) We can replace $A, B$ by sequences listing them (we do not always strictly distinguish).

*Remark.* The following observation may clarify.

*1.8 Observation.* If $\binom{B_1}{A_1} \leq^* \binom{B_2}{A_2}$ <u>then</u> we can find $\langle B'_\ell : \ell \leq n\rangle$ and

$\langle c_\ell : 1 \le \ell < n\rangle$ for some $n \ge 1$, satisfying $\binom{B_1}{A_1} \le_{\rm b} \binom{B'_1}{B'_0}, c_\ell \in B'_{\ell+1}, \frac{c_\ell}{B'_\ell}$ regular, $\frac{B'_{\ell+1}}{c_\ell + B'_\ell} \perp_{\rm a} B'_\ell, A_2 = B'_{n-1}, B_2 = B'_n$.

*Remark.* 1) Note that actually $\le_{\rm a}$ is transitive. This means that in a sense $\le_{\rm b}$ is enough, $\le_{\rm a}$ inessential.
2) We may in 1.7(4) use $\bar b = \langle c\rangle$, does not matter.

*Proof.* By the definition of $\le^*$ there are $k < \omega$ and $\binom{B^\ell}{A^\ell}$ for $\ell \le k$ such that: $\binom{B^\ell}{A^\ell} \le_{x(\ell)} \binom{B^{\ell+1}}{A^{\ell+1}}$ for $\ell \le k$ and $x(\ell) \in \{\rm a,b\}$ and $\binom{B^0}{A^0} = \binom{B_1}{A_1}, \binom{B^k}{A^k} = \binom{B_2}{A_2}$ and without loss of generality $\mathrm{x}(2\ell) = \mathrm{a}, \mathrm{x}(2\ell+1) = \mathrm{b}$. Let $N_0 \prec \mathfrak{C}$ be $\aleph_\epsilon$-prime over $\emptyset$ such that $A^0 \subseteq N_0, B_0 \bigcup\limits_{A^0} N_0$ and $f_0 = \mathrm{id}_{A_0}$. We choose by induction on $\ell \le k, N_{\ell+1}, f_{\ell+1}$ such that:

- $(a)$ $\mathrm{Dom}(f_{\ell+1}) = B^\ell$
- $(b)$ $N_\ell \prec N_{\ell+1}$
- $(c)$ if $\mathrm{x}(\ell) = b$ then $f_{\ell+1}$ is an extension of $f_\ell$ (which necessarily has domain $A_\ell$, check) with domain $B^\ell$ such that $f_\ell(B^\ell) \bigcup\limits_{f_\ell(A^\ell)} N_\ell$ and $N_{\ell+1}$ is $\aleph_\epsilon$-prime over $N_\ell \cup f_\ell(B^\ell)$
- $(d)$ if $\mathrm{x}(\ell) =$ a, then $f_{\ell+1}$ maps $A^\ell$ into $N_{\ell-1}, B^\ell$ into $N_\ell$ and $N_{\ell+1} = N_\ell$.

This is straightforward. Now on $\langle N_\ell : \ell \le k+1\rangle$ we repeat the argument (of choosing $\langle B_\ell : \ell \le n\rangle$) in the proof of 1.14(6) above, i.e., choose $B^\ell \subseteq N_\ell$ by downward induction on $\ell$ large enough as required. $\square_{1.8}$

**1.9 Definition.** 1) We define $\mathrm{tp}_\alpha[\binom{B}{A}, M]$ (for $A \subseteq B \subseteq M, A$ and $B$ are $\epsilon$-finite and $\alpha$ is an ordinal) and $\mathscr{S}_\alpha(\binom{B}{A}, M), \mathscr{S}_\alpha(A, M)$ and $\mathscr{S}^r_\alpha(\binom{B}{A}, M), \mathscr{S}^r_\alpha(A, M)$ by induction on $\alpha$ (we mean simultaneously; of course, we use appropriate variables):

- $(a)$ $\mathrm{tp}_0[\binom{B}{A}, M]$ is the first order type of $A \cup B$
- $(b)$ $\mathrm{tp}_{\alpha+1}[\binom{B}{A}, M] =$ the triple $\langle Y^{1,\alpha}_{A,B,M}, Y^{2,\alpha}_{A,B,M}, \mathrm{tp}_\alpha(\binom{B}{A}, M)\rangle$ where:
  $Y^{1,\alpha}_{A,B,M} =: \left\{\mathrm{tp}_\alpha[\binom{B'}{A'}, M] : \text{for some } A', B' \text{ we have } \binom{B}{A} \le_{\rm a} \binom{B'}{A'} \in \Gamma(M)\right\}$,

and $Y^{2,\alpha}_{A,B,M} =: \Big\{\langle \Upsilon, \lambda^{\Upsilon}_{M,B}\rangle : \Upsilon \in \mathscr{S}^r_\alpha(B,M)\Big\}$
where $\lambda^{\Upsilon}_{M,B} = \dim\Big[\Big\{d : \text{tp}_\alpha[\binom{B+d}{B}, M] = \Upsilon\Big\}, B\Big]$:

(c) for $\delta$ a limit ordinal, $\text{tp}_\delta[\binom{B}{A}, M] = \langle \text{tp}_\alpha[\binom{B}{A}, M] : \alpha < \delta\rangle$ (this includes $\delta = \infty$, really $\|M\|^+$ suffice).

(d) $\mathscr{S}_\alpha(A, M) = \Big\{\text{tp}_\alpha[\binom{B}{A}, M] : \text{for some } B \text{ such that } B \subseteq M, \text{ and } \binom{B}{A} \in \Gamma(M)\Big\}$

(e) $\mathscr{S}^r_\alpha(\binom{B}{A}, M) = \Big\{\text{tp}_\alpha[\binom{B+c}{B}, M] : \text{for some } c \in M \text{ we have } \frac{c}{B} \perp A \text{ and } \frac{c}{B} \text{ is regular}\Big\}$

(f) $\mathscr{S}^r_\alpha(A, M) = \Big\{\text{tp}_\alpha[\binom{A+c}{A}, M] : c \in M \text{ and } \frac{c}{A} \text{ regular}\Big\}$.

2) We define also $\text{tp}_\alpha[A, M]$, for $A$ an $\epsilon$-finite subset of $M$:

(a) $\text{tp}_0[A, M] =$ first order type of $A$

(b) $\text{tp}_{\alpha+1}[A, M]$ is the triple $\langle Y^{1,\alpha}_{A,M}, Y^{2,\alpha}_{A,M}, \text{tp}_\alpha[A, M]\rangle$ where
$Y^{1,\alpha}_{A,M} =: \mathscr{S}_\alpha(A; M)$ and
$Y^{2,\alpha}_{A,M} =: \Big\{\langle \Upsilon, \dim\{d \in M : \text{tp}_\alpha[\binom{A+d}{A}, M] = \Upsilon\}\rangle : \Upsilon \in \mathscr{S}^r_\alpha(A, M)\Big\}$

(c) $\text{tp}_\delta[A, M] = \langle \text{tp}_\alpha(A, M) : \alpha < \delta\rangle$

3) $\text{tp}_\alpha[M] = \text{tp}_\alpha[\emptyset, M]$.





1.10 Discussion: Clearly $\text{tp}[\binom{B}{A}, M]$ is intended, on the one hand, to be expressible by our logic and, on the other hand, to express the isomorphism type of $M$ "in the direction of $\binom{B}{A}$". To really say it we need to go back to the $\aleph_\varepsilon$-decompositions of $M$, a central notion of [Sh:c, Ch.X].

For the reader's benefit, by the referee request, let us review informally the proof in [Sh:c, Ch.X]. Let $M$ be an $\aleph_\varepsilon$-saturated model, and we choose $\langle M_\eta : \eta \in I \cap {}^n\text{Ord}\rangle, \langle a_\eta : \eta \in I \cap {}^{n+1}\lambda\rangle$ by induction on $n$. For $n = 0$, of course, $I \cap {}^0\text{Ord} = \{<>\}$, we let $N_{<>} \prec M$ be $\aleph_\varepsilon$-prime over $\emptyset$ and let $\mathbf{I}_{<>}$ a maximal subset of $\{c \in M : \text{tp}(c, N_{<>}) \text{ regular}\}$ which is independent over $N_{<>}$, let $\langle a_{<\alpha>} : \alpha < |\mathbf{I}_{<>}|\rangle$ list $\mathbf{I}_{<>}$. Similarly for $n+1, \eta \in I \cap {}^{n+1}\text{Ord}$ let $N_\eta \prec M$ be $\aleph_\varepsilon$-prime over $M_{\eta^-} + a_\eta$, let $\mathbf{I}_\eta$ be a maximal subset of $\{c \in M : \text{tp}(c, M_\eta)$ is regular orthogonal to $M_{\eta^-}\}$ independent over $N_\eta$. Lastly, let $\langle c_{\eta\hat{\ }<\alpha>} : \alpha < |\mathbf{I}_\eta|\rangle$ list $\mathbf{I}_\eta$ and let $I \cap {}^{n+1}\mathrm{Ord} = \{\eta\hat{\ } < \alpha >: \eta \in I \cap {}^n\mathrm{Ord} \text{ and } \alpha < |\mathbf{I}_\eta|\}$.

To carry this we use the existence of $\aleph_\varepsilon$-prime models (and the local character of indpendent). Also looking at the set $\cup\{M_\eta : \eta \in I\}$, its first order type is determined by the non-forking calculus. In fact, for any $\eta \in I\backslash\{<>\}$, the set $\cup\{N_\nu : \eta\triangleleft\nu \in I\}, \cup\{N_\eta : \neg(\eta \le \nu) \text{ and } \nu \in I\}$ are independent over $N_\eta$. Let $N \prec M$ be $\aleph_\varepsilon$-prime over $\cup\{N_\eta : \eta \in I\}$, now if $M = N$ we are done decomposing $M$, if not some $c \in M\backslash N$ realize a regular type (we use density of regular types). By NDOP, the $\mathrm{tp}(c, N)$ is not orthogonal to some $N_\eta$. Choose $\eta$ of minimal length hence $\nu\triangleleft\eta \Rightarrow \mathrm{tp}(c, M_\eta) \perp N_\nu$. By properties of regular types, without loss of generality $\mathrm{tp}(c, N)$ does not fork over $N_\eta$, so we get a contradiction to the maximality of $\{a_\nu : \nu \in \mathrm{Suc}_I(\eta)\}$ (this explains the role of $\mathscr{P}$ in Definition 1.11(5) below).
We are interested in the possible trees $\langle N_\nu : \eta \triangleleft \nu \in I\rangle$.

Now the tree determines $M$ up to isomorphism, but there are "incidental" choices, so two trees may give isomorphic models (for investigating the number of non isomorphic models it is enough to find sufficiently pairwise far trees $I$).

Here we like to get exact information and in as finitary way as we can. So we replace $(M_{\eta^-}, M_\eta, a_\eta)$ by $\binom{B}{A}$, where $A \subseteq M_{\eta^-}, A + a_\eta \subseteq B \subseteq M_\eta$, $\mathrm{tp}(B, M_{\eta^-})$ does not fork over $A$.

Now for $\eta \in I\backslash\{<>\}$ we are interested in the possible trees $\langle N_\nu : \eta \triangleleft \nu \in I\rangle$, over $(N_{\eta^-_*}, N_\eta, a_\eta)$. But not only different trees may be equivalent (giving isomorphic $\aleph_\varepsilon$-prime models) <u>but</u> the other part of the tree, $\langle N_\nu : \nu \in I \text{ but } \neg(\eta \triangleleft \nu)\rangle$ may apriori cause non equivalent trees to contribute the same toward understanding $M$. This is done in [Sh:c, Ch.XII], <u>but</u> here we have to deal with $\varepsilon$-finite $A, B$.

The following claim 1.11 really does not add to [Sh:c, Ch.X], it just collects the relevant information which is proved there, or which follows immediately (paricularly using the parameter $(A, B)$). We allow here $a_\eta/M_\eta$ - to be not regular, but this is not serious: we can here deal exclusively with this case and we can omit this requirement in [Sh:c, Ch.X]; however, this does not eliminate the use of regular types (in the proof that $M$ is $\aleph_\varepsilon$-prime over every $\aleph_\varepsilon$-decomposition of it).

**1.11 Definition.** 1) $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition inside $M$ above (or for over) the pair $\binom{B}{A}$ (but we may omit the "$\aleph_\epsilon -$") if:

- $(a)$ $I$ a set of finite sequences of ordinals closed under initial segments
- $(b)$ $\langle\rangle, \langle 0\rangle \in I, \eta \in I\backslash\{\langle\rangle\} \Rightarrow \langle 0\rangle \trianglelefteq \eta$, let $I^- = I\backslash\{\langle\rangle\}$, really $a_{\langle\rangle}$ is meaningless
- $(c)$ $A \subseteq N_{\langle\rangle}, B \subseteq N_{\langle 0\rangle}, N_{\langle\rangle} \underset{A}{\bigcup} B$ and $dc\ell(a_{\langle 0\rangle}) \subseteq dc\ell(B)$,
- $(d)$ if $\nu = \eta\hat{\ }\langle\alpha\rangle \in I$ <u>then</u> $N_\nu$ is $\aleph_\epsilon$-primary over $N_\eta \cup \bar{a}_\nu, N_{\langle\rangle}$ is $\aleph_\epsilon$-prime over $A$

(e) for $\eta \in I$ such that $k = \ell g(\eta) > 1$ the type $a_\eta / N_{\eta \restriction (k-1)}$ is orthogonal to $N_{\eta \restriction (k-2)}$

(f) $\eta \triangleleft \nu \Rightarrow N_\eta \prec N_\nu$

(g) $M$ is $\aleph_\epsilon$-saturated and $N_\eta \prec M$ for $\eta \in I$

(h) if $\eta \in I \backslash \{\langle\rangle\}$, then $\{a_\nu : \nu \in \mathrm{Suc}_I(\eta)\}$ is (a set of elements realizing over $N_\eta$ types orthogonal to $N_{\eta^-}$ and is) an independent set over $N_\eta$.

2) We replace "inside $M$" by "of $M$" if in addition

(i) in clause (h) the set is maximal.

3) $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition inside $M$ <u>if</u> (a), (d), (e), (f), (g), (h) of part (1) holds and in clause (h) we allow $\eta = \langle\rangle$ (call this $(h)^+$). We add "over $A$" if $A \subseteq M_{<>}$.
4) $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition of $M$ <u>if</u> in addition to 1.11(3) and we have the stronger version of clause (i) of 1.11(2) by including $\eta = \langle\rangle$, i.e. we have:

$(i)^+$ for $\nu \in I$, the set $\{a_\eta : \eta \in \mathrm{Suc}_I(\nu)\}$ is a maximal subset of $M$ independent over $N_\nu$.

We may add "over $A$" <u>if</u> $A \subseteq M$.
5) If $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition inside $M$ we let

$$\mathscr{P}(\langle N_\eta, a_\eta : \eta \in I\rangle, M) = \Big\{p \in S(M) : p \text{ regular and for some } \eta \in I\backslash\{\langle\rangle\} \text{ we have } p \text{ is orthogonal to } N_{\eta^-} \text{ but not to } N_\eta\Big\}.$$

As said earlier it is natural to use regular types.

**1.12 Definition.** 1) We say that $\langle N_\eta, a_\eta : \eta \in I\rangle$, an $\aleph_\epsilon$-decomposition inside $M$, is $J$-regular <u>if</u> $J \subseteq I$ and:

$(*)$ for each $\eta \in I \backslash J$ there[2] is $c_\eta$ such that $a_\eta \in ac\ell(N^-_\eta + c_\eta)$ $\frac{c_\eta}{N_\eta}$ is regular and if $\eta \neq \langle\rangle$ then $\frac{a_\eta}{N_\eta + c_\eta} \perp_{\mathrm{a}} N_{(\eta^-)}$.

2) We say "$\langle N_\eta, a_\eta : \eta \in I\rangle$ is a regular $\aleph_\epsilon$-decomposition inside $M$ [of $M$]" <u>if</u> it is an $\aleph_\epsilon$-decomposition inside $M$ [of $M$] which is $\emptyset$-regular.
3) We say "$\langle N_\eta, a_\eta : \eta \in I\rangle$ is a regular $\aleph_\epsilon$-decomposition inside $M$ [of $M$] over $\binom{B}{A}$" <u>if</u> it is an $\aleph_\epsilon$-decomposition inside $M$ [of $M$] over $\binom{B}{A}$ which is $\{\langle\rangle\}$-regular.

[2] without loss of generality $c_\eta = a_\eta$

**1.13 Claim.** *1) Every $\aleph_\epsilon$-saturated model has an $\aleph_\epsilon$-decomposition (i.e. of it).*
*2) If $M$ is $\aleph_\epsilon$-saturated, $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition inside $M$, <u>then</u> for some $J$, and $N_\eta, a_\eta$ for $\eta \in J\backslash I$ we have: $I \subseteq J$ and $\langle N_\eta, a_\eta : \eta \in J\rangle$ is an $\aleph_\varepsilon$-decomposition of $M$ (even a $(J\backslash I)$-regular one).*
*3) If $M$ is $\aleph_\epsilon$-saturated, $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition of $M$ <u>then</u> $M$ is $\aleph_\epsilon$-prime and $\aleph_\epsilon$-minimal[3] over $\bigcup\limits_{\eta\in I} N_\eta$; if in addition $\langle N_\eta, a_\eta : \eta \in \{\langle\rangle, \langle 0\rangle\}\rangle$ is an $\aleph_\epsilon$-decomposition inside $M$ above $\binom{B}{A}$, <u>then</u> $\langle N_\eta, a_\eta : \eta \in I \ \& \ (\eta \neq \langle\rangle \rightarrow \langle 0\rangle \trianglelefteq \eta)\rangle$ is an $\aleph_\epsilon$- decomposition of $M$ above $\binom{B}{A}$.*
*4) If $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition inside $M$ above $\binom{B}{A}$, <u>then</u> it is an $\aleph_\epsilon$-decomposition inside $M$.*
*5) If $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition inside $M$ [above $\binom{B}{A}$], $\eta \in I$, $[\eta \in I\backslash\{\langle\rangle\}], \alpha = \mathrm{Min}\{\beta : \eta\hat{\ }\langle\beta\rangle \notin I\}, \nu =: \eta\hat{\ }\langle\alpha\rangle, a_\nu \in M\backslash N_\eta, \frac{a_\nu}{N_\eta}$ is orthogonal to $M_{\eta^-}$ if $\eta^-$ if $\neq \langle\rangle, N_\nu \prec M$ is $\aleph_\epsilon$-primary over $N_\eta + a_\nu$ and $a_\nu \underset{N_\eta}{\bigcup\!\!\!\!\!\cup} \left(\bigcup\limits_{\rho\in I} N_\rho\right)$ (enough to demand $\{a_\rho : \rho^- = \eta$ and $\rho \in I\}$ is independent over $a_\nu/N_\eta$) <u>then</u> $\langle N_\rho, a_\rho : \rho \in I \cup \{\nu\}\rangle$ is an $\aleph_\epsilon$-decomposition inside $M$ [over $\binom{B}{A}$].*
*6) Assume $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition of $M$, <u>if</u> $p$ is regular (stationary) and is not orthogonal to $M$ (e.g. $p \in S(M)$) <u>then</u> for one and only one $\eta \in I$, there is a regular (stationary) $q \in S(N_\eta)$ not orthogonal to $p$ such that: if $\eta^-$ is well defined (i.e. $\eta \neq \langle\rangle$), then $p \perp N_{\eta^-}$.*
*7) Assume $I = \bigcup\limits_{\alpha<\alpha(*)} I_\alpha$, for each $\alpha$ we have $\langle N_\eta, a_\eta : \eta \in I_\alpha\rangle$ is an $\aleph_\epsilon$-decomposition inside $M$ [above $\binom{B}{A}$] and for each $\eta \in I$ for every $n < \omega$ and $\nu_\ell = \eta\hat{\ }\langle\beta_\ell\rangle \in I$ for $\ell < n$, for some $\alpha$ we have: $\{\nu_\ell : \ell < n\} \subseteq I_\alpha$ (e.g. $I_\alpha$ increasing). <u>Then</u> $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition inside $M$ [above $\binom{B}{A}$].*
*8) In (7), if $\eta \neq \langle\rangle$ and some $\nu_\ell$ is not $\triangleleft$-maximal in $I$ and $\frac{a_{\nu_\ell}}{N_\eta}$ is regular, it is enough:*

$$\ell_1 < \ell_2 < n \Rightarrow \bigvee_{\alpha<\alpha(*)} [\{\nu_{\ell_1}, \nu_{\ell_2}\} \subseteq I_\alpha].$$

*9) If $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition inside $M, I_1, I_2 \subseteq I$ are closed under initial segments and $I_0 = I_1 \cap I_2$ <u>then</u> $\left(\bigcup\limits_{\eta\in I_1} N_\eta\right) \underset{\bigcup\limits_{\eta\in I_0} N_\eta}{\bigcup\!\!\!\!\!\cup} \left(\bigcup\limits_{\eta\in I_2} N_\eta\right)$.*

[3]here we use NDOP

*10) Assume that for $\ell = 1,2$ that $\langle N^\ell_\eta, a^\ell_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition inside $M_\ell$, and for $\eta \in I$ the function $f_\eta$ is an isomorphism from $N^1_\eta$ onto $N^2_\eta$ and $\eta \triangleleft \nu \Rightarrow f_\eta \subseteq f_\nu$. Then $\bigcup\limits_{\eta \in I} f_\eta$ is an elementary mapping; if in addition $\langle N^\ell_\eta, a^\ell_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition of $M_\ell$ (for $\ell = 1,2$) then $\bigcup\limits_{\eta \in I} f_\eta$ can be extended to an isomorphism from $M_1$ onto $M_2$.*
*11) If $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$- decomposition inside $M$ (above $\binom{B}{A}$) and $M^- \prec M$ is $\aleph_\epsilon$-prime over $\bigcup\limits_{\eta \in I} N_\eta$ then $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition of $M$ (above $\binom{B}{A}$).*
*12) If $\langle N_\eta, a_\eta; \eta \in I\rangle$ is an $\aleph_\varepsilon$-decomposition inside $M$/ of $M$ [above $\binom{M}{A}$] and $a'_\eta \in N_\eta$ and $N_\eta$ is $\aleph_\varepsilon$-prime over $N_{\eta^-} + a'_\eta$ for $\eta \in I \backslash \{<>\}$ [and $a'_{<0>} = a_{<0>}$ or at least $dcl(a'_{<0>}) \subseteq dcl(B)$] then $\langle N_\eta, a'_\eta : \eta \in I\rangle$ is an $\aleph_\varepsilon$-decomposition inside $M$/ of $M$ [above $\binom{B}{A}$].*

*Proof.* 1), 2), 3), 5), 6), 9)-12). Repeat the proofs of [Sh:c, X]. (Note that here $a_\eta/N_\eta$ is not necessarily regular, a minor change).
4), 7) Check.
8) As $\mathrm{Dp}(p) > 0 \Rightarrow p$ is trivial, by [Sh:c, ChX,7.2,p.551] and [Sh:c, ChX,7.3]. $\square_{1.13}$

We shall prove:

**1.14 Claim.** *1) If $M$ is $\aleph_\epsilon$-saturated, $\binom{B}{A} \in \Gamma(M)$, then there is $\langle N_\eta, a_\eta : \eta \in I\rangle$, an $\aleph_\epsilon$-decomposition of $M$ above $\binom{B}{A}$.*
*2) Moreover if $\langle N_\eta, a_\eta : \eta \in I\rangle$ satisfies clauses $(a)-(h)$ of Definition 1.11(1), we can extend it to satisfy clause $(i)$ of 1.11(2), too.*
*3) If $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition of $M$ above $\binom{B}{A}$, $M^- \prec M$ is $\aleph_\epsilon$-prime over $\bigcup\limits_{\eta \in I} N_\eta$ then:*

- $(a)$ *$\langle N_\eta : \eta \in I\rangle$ is a $\aleph_\epsilon$-decomposition of $M^-$*
- $(b)$ *we can find an $\aleph_\epsilon$-decomposition $\langle N_\eta, a_\eta : \eta \in J\rangle$ of $M$ such that $J \supseteq I$ and $[\eta \in J \backslash I \Leftrightarrow (\eta \neq \langle\rangle$ and $\neg \langle 0\rangle \triangleleft \eta)]$, moreover the last phrase follows from the previous ones.*

*4) If in 3)(b) the set $J \backslash I$ is countable (finite is enough for our applications), then necessarily $M, M^-$ are isomorphic, even adding all members of an $\epsilon$-finite subset of $M^-$ as individual constants.*
*5) If $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition of $M$ above $\binom{B}{A}, I \subseteq J$ and*

*$\langle N_\eta, a_\eta : \eta \in J\rangle$ is an $\aleph_\epsilon$-decomposition of $M$, $M^- \prec M$ is $\aleph_\epsilon$-prime over $\bigcup_{\eta \in I} N_\eta$ and $\binom{B}{A} \le^* \binom{B_1}{A_1}$ and $B_1 \subseteq M$ and $c \in M$ and $\frac{c}{B_1} \perp A_1$ and $\frac{c}{B_1}$ is (stationary and) regular <u>then</u>*

($\alpha$) $\frac{c}{B_1} \perp \frac{\cup\{N_\eta : \eta \in J \backslash I\}}{N_{\langle\rangle}}$

($\beta$) *$\frac{c}{B_1}$ is not orthogonal to some $p \in \mathscr{P}(\langle N_\eta, a_\eta : \eta \in I\rangle, M^-)$.*

*6) If $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition of $M$ above $\binom{B}{A}$, <u>then</u> the set $\mathscr{P} = \mathscr{P}(\langle N_\eta : \eta \in I\rangle, M)$ depends on $\binom{B}{A}$ and $M$ only (and not on $\langle N_\eta : \eta \in I\rangle$ or $M^-$ when $M^- \prec M$ is $\aleph_\varepsilon$-prime over $\cup\{N_\eta : \eta \in I\}$), recalling:*

$$\mathscr{P} = \mathscr{P}(\langle N_\eta : \eta \in I\rangle, M) = \Big\{p \in S(M) : p \text{ regular and for some } \eta \in I \backslash \{<>\}, \text{ we have}: p \text{ is orthogonal to } N_{\eta^-} \text{ but not to } N_\eta\Big\}.$$

*So let $\mathscr{P}(\binom{B}{A}, M) =: \mathscr{P}(\langle N_\eta : \eta \in I\rangle, M)$.*

*(7) If $\frac{B}{A}$ is regular of depth zero or just $\frac{b}{A} \le_{\rm a} \frac{B}{A}$, $\frac{b}{A}$ regular of depth zero and $M$ is $\aleph_\epsilon$-saturated and $B \subseteq M$ <u>then</u>*

(a) *for any $\alpha$, we have $tp_\alpha(\binom{B}{A}, M)$ depend just on $tp_0(\binom{B}{A}, M)$*

(b) *if $\binom{B}{A} \le^* \binom{B'}{A} \in \Gamma(M)$ <u>then</u> $\mathrm{tp}_\alpha(\binom{B'}{A}, M)$ depends just on $\mathrm{tp}_0(\binom{B}{A}, M)$ (and $(A, B, A', B)$ but not on $M$).*

*8) For $\alpha < \beta$, from $\mathrm{tp}_\beta(\binom{B}{A}, M)$ we can compute $\mathrm{tp}_\alpha(\binom{B}{A}, M)$.*

*9) If $f$ is an isomorphism from $M_1$ onto $M_2$, $A_1 \subseteq B_1$ are $\varepsilon$-finite subsets of $M_1$ and $f(A_1) = A_2$, $f(B_1) = B_2$ <u>then</u>*

$$\mathrm{tp}_\alpha(\begin{pmatrix} B_1 \\ A_1 \end{pmatrix}, M_1) = \mathrm{tp}_\alpha(\begin{pmatrix} B_2 \\ A_2 \end{pmatrix}, M_2)$$

*(more pedantically $\mathrm{tp}_\alpha(\binom{B_2}{A_2}, M_2) = f[\mathrm{tp}_\alpha(\binom{B_1}{A_1}, M_1)]$ or considered the $A_\ell, B_\ell$ as indexed sets).*

We delay the proof (parts (1), (2), (3) are proved after 1.22, part (4), (6) after 1.23, and after it parts (5), (7), (8). Part (9) is obvious.

*1.15 Definition.* 1) If $\binom{B}{A} \in \Gamma(M), M$ is $\aleph_\epsilon$-saturated let $\mathscr{P}^M_{\binom{B}{A}}$ be the set $\mathscr{P}$ from Claim 1.14(6) above (by 1.14(6) this is well defined as we shall prove below).
2) Let $\mathscr{P}_{\binom{B}{A}} = \left\{ p : p \text{ is (stationary regular and) parallel to some } p' \in \mathscr{P}^{\mathfrak{C}^{\text{eq}}}_{\binom{B}{A}} \right\}$.

**1.16 Definition.** If $\langle N^\ell_\eta, a_\eta : \eta \in J\rangle$ is a decomposition inside $\mathfrak{C}$ for $\ell = 1,2$ we say that $\langle N^1_\eta, a_\eta : \eta \in J\rangle \le^*_{\text{direct}} \langle N^2_\eta, a_\eta : \eta \in J\rangle$ <u>if</u>:

- $(a)$ $N^1_{\langle\rangle} \prec N^2_{\langle\rangle}$
- $(b)$ $N^2_{\langle\rangle} \underset{N^1_{\langle\rangle}}{\bigcup\!\!\!\!\bigcup} \{a_{\langle\alpha\rangle} : \langle\alpha\rangle \in J\}$
- $(c)$ for $\eta \in J \backslash \{\langle\rangle\}, N^2_\eta$ is $\aleph_\epsilon$-prime over $N^1_\eta \cup N^2_{\eta^-}$.

**1.17 Claim.** *1) $M$ is $\aleph_\epsilon$-prime over $A$ <u>iff</u> $M$ is $\aleph_\epsilon$-primary over $A$ <u>iff</u> $M$ is $\aleph_\epsilon$-saturated, $A \subseteq M, M$ is $\aleph_\varepsilon$-atomic over $A$ (see -1.1(16)) for every $\mathbf{I} \subseteq M$ indiscernible over $A$ we have: dim$(\mathbf{I}, M) \le \aleph_0$ <u>iff</u> $M$ is $\aleph_\epsilon$-saturated, $A \subseteq M, M$ is $\aleph_\varepsilon$-atomic over $A$ and for every finite $B \subseteq M$ and regular (stationary) $p \in S(A \cup B)$, we have dim$(p, M) \le \aleph_0$.*
*2) If $N_1, N_2$ are $\aleph_\epsilon$-prime over $A$, <u>then</u> they are isomorphic over $A$.*

*Proof.* By [Sh:c, IV,4.18] (see Definition [Sh:c, IV,4.16], noting that we replace $\mathbf{F}^a_{\aleph_0}$ by $\aleph_\varepsilon$ and that part (4) there disappears when we are speaking on $\mathbf{F}^a_{\aleph_0}$). $\square_{1.16}$

However, we need more specific information saying that "minor changes" preserve being $\aleph_\varepsilon$-prime; this is done in 1.18 below, parts of it are essentially done in [Sh 225] but we give full proof.

*1.18 Fact.* 0) If $A$ is countable, $N$ is $\aleph_\epsilon$-primary over $A$ <u>then</u> $N$ is $\aleph_\epsilon$-primary over $\emptyset$.

1) If $N$ is $\aleph_\epsilon$-prime over $\emptyset, A$ countable, $N^+$ is $\aleph_\epsilon$-prime over $N \cup A$ <u>then</u> $N^+$ is $\aleph_\epsilon$-prime over $\emptyset$.

2) If $\langle N_n : n < \omega\rangle$ is increasing, each $N_n$ is $\aleph_\epsilon$-prime over $\emptyset$ or just $\aleph_\varepsilon$-constructible over $\emptyset$ and $N_\omega$ is $\aleph_\epsilon$-prime over $\bigcup\limits_{n<\omega} N_n$ <u>then</u> $N_\omega$ is $\aleph_\epsilon$-prime over $\emptyset$, (note that if each $N_n$ is $\aleph_\varepsilon$-saturated then $N_\omega = \bigcup\limits_{n<\omega} N_n$).

2A) If $N$ is $\aleph_\varepsilon$-prime over $C, \bar a{}^\frown \bar b \subseteq N$, tp$(\bar b, \bar a)$ is regular (stationary) and orthogonal to $C$ <u>then</u> dim(tp$(\bar b, \bar a), N) \le \aleph_0$; also if $q \in S(C \cup \bar a)$ is a non-forking extension

of $\text{tp}(\bar b, \bar a)$ then $\dim(q, C \cup \bar a) = \dim(\text{tp}(\bar b, \bar a), N) = \aleph_0$.
2B) If $C \cup \bar a\hat{\ }\bar b \subseteq N$ and $\bar a/\bar b$ is a regular type orthogonal to $C$ and $q \in S^{\ell g(\bar a)}(N)$ is a non-forking extension of $\bar a/\bar b$ <u>then</u> $\dim(p \restriction (C + \bar b), N) \le \dim(\bar a/\bar b, N) \le \dim(p \restriction (C+\bar b), N) + \aleph_0$; moreover, $\dim(p \restriction (C+\bar b), N) \le \dim(\bar a/\bar b, N) < \dim(p \restriction (C + \bar b), N)^+ + \aleph_0$.
3) If $N_2 \bigcup\limits_{N_0} N_1$, each $N_\ell$ is $\aleph_\epsilon$ -saturated, $N_2$ is $\aleph_\epsilon$-prime over $N_0 \cup \bar a$, and $N_3$ is $\aleph_\epsilon$-prime over $N_2 \cup N_1$ <u>then</u> $N_3$ is $\aleph_\epsilon$-prime over $N_1 \cup \bar a$.
4) If $N_1 \prec N_2$ are $\aleph_\epsilon$-primary over $\emptyset$ <u>then</u> for some $\aleph_\epsilon$-saturated $N_0 \prec N_1$ (necessarily $\aleph_\epsilon$-primary over $\emptyset$) we have: $N_1, N_2$ are isomorphic over $N_0$.
5) In part (4), if $A \subseteq N_1$ is $\epsilon$-finite <u>then</u> we can demand $A \subseteq N_0$.
6) If $M_0$ is $\aleph_\epsilon$-saturated, $A \bigcup\limits_{M_0} B, M_1$ is $\aleph_\epsilon$-primary over $M_0 \cup A$ <u>then</u> $M_1 \bigcup\limits_{M_0} B$.
7) Assume $N_0 \prec N_1 \prec N_2$ are $\aleph_\epsilon$-saturated, $N_2$ is $\aleph_\epsilon$-primary over $N_1 + a$ and $\frac{a}{N_1} \perp N_0$ (and $a \notin N_1$). <u>If</u> $N_0' \prec N_0, N_0' \prec N_1' \prec N_1, N_1' \bigcup\limits_{N_0'} N_0$ and $N_1$ is $\aleph_\epsilon$-primary over $N_0 \cup N_1', A_1^* \subseteq N_1', A_2^* \subseteq N_2$ are $\varepsilon$-finite and $\text{tp}_*(A_2^*, N_1)$ does not fork over $A_1^*$, <u>then</u> we can find $a', N_2'$ such that: $N_2'$ is $\aleph_\epsilon$-saturated, $\aleph_\epsilon$-primary over $N_1' + a', N_1' \prec N_2' \prec N_2, N_1 \bigcup\limits_{N_1'} N_2'$ and $N_2$ is $\aleph_\epsilon$-primary over $N_1 \cup N_2'$ and $A_2^* \subseteq N_2'$.
8) Assume $N_0' \prec N_0 \prec N_1$ and $a \in N_1$ and $N_1$ is $\aleph_\epsilon$-prime over $N_0 + a$ and $\frac{a}{N_0} \pm N_0'$ and $A_0^* \subseteq N_0', A_1^* \subseteq N_1$ are $\varepsilon$-finite and $\text{tp}_*(A_1^*, N_0)$ does not fork over $A_0^*$ <u>then</u> we can find $a', N_1'$ such that $a' \in N', N_0' \prec N_1' \prec N_1, N_1' \bigcup\limits_{N_0'} N_0, N_1'$ is $\aleph_\epsilon$-prime over $N_0' + a$ and $N_1$ is $\aleph_\epsilon$-prime over $N_0 + N_1'$ and $A_1^* \subseteq N_1'$.
9) If $N_1$ is $\aleph_\epsilon$-prime over $\emptyset$ and $A \subseteq B \subseteq N_1$ and $A, B$ are $\epsilon$-finite, <u>then</u> we can find $N_0$ such that: $A \subseteq N_0 \prec N_1, N_0$ is $\aleph_\epsilon$-prime over $\emptyset, A \subseteq N_0, B \bigcup\limits_{A} N_0$, and $N_1$ is $\aleph_\epsilon$-prime over $N_0 \cup B$.
10) If $N_0$ is $\aleph_\epsilon$-prime over $A$ and $B \subseteq N_0$ is $\epsilon$-finite, <u>then</u> $N_0$ is $\aleph_\epsilon$-prime over $A \cup B$ (and also over $A'$ if $A \subseteq A' \subseteq \text{acl}(A)$).

*1.19 Remark.* In the proof of 1.18(1)-(6),(10) we do not use "$T$ has NDOP".

*Proof.* 0) There is $\{a_\alpha : \alpha < \alpha^*\}$, a list of members of $N$ in which every member of $N \backslash A$ appears such that for $\alpha < \alpha(*)$ we have: $\text{tp}(a_\alpha, A \cup \{a_\beta : \beta < \alpha\})$ is $\aleph_\varepsilon$-isolated (which means just $\mathbf{F}^a_{\aleph_0}$-isolated).
[Why? by the definition of "$N$ is $\aleph_\epsilon$-primary over $A$"). Let $\{b_n : n < \omega\}$ list $A$ (if $A = \emptyset$ the conclusion is trivial so without loss of generality $A \neq \emptyset$, hence we can find such a sequence $\langle b_n : n < \omega \rangle$). Now define $\beta^* = \omega + \beta$ and $b_{\omega+\alpha} = a_\alpha$

for $\alpha < \alpha^*$. So $\{b_\beta : \beta < \beta^*\}$ lists the elements of $N$ (possibly with repetitions, remember $A \subseteq N$ and check). We claim that $\mathrm{tp}(b_\beta, \{b_\gamma : \gamma < \beta\})$ is $\mathbf{F}^a_{\aleph_0}$-isolated for $\beta < \beta^*$.

(Why? if $\beta \geq \omega$, let $\beta' = \beta - \omega$ (so $\beta < \alpha^*$), now the statement above means $\mathrm{tp}(a_{\beta'}, A \cup \{a_\gamma : \gamma < \beta'\})$ is $\mathbf{F}^a_{\aleph_0}$-isolated which we know; if $\beta < \omega$ this statement is trivial)]. By the definition of "$\mathbf{F}^a_{\aleph_0}$-primary", clearly $\langle b_\beta : \beta < \omega + \alpha\rangle$ exemplify that $N$ is $\mathbf{F}^a_{\aleph_0}$- primary over $\emptyset$.
1) Note

$(*)_1$ if $N$ is $\aleph_\epsilon$-primary over $\emptyset$ and $A \subseteq N$ is finite then $N$ is $\aleph_\epsilon$-primary over $A$ [why? see [Sh:c, IV,3.12](3),p.180 (of course, using [Sh:c, IV, Table 1,p.169] for $\mathbf{F}^a_{\aleph_0}$]

$(*)_2$ if $N$ is $\aleph_\epsilon$-primary over $\emptyset, A \subseteq N$ is finite and $p \in S^m(N)$ does not fork over $A$ and $p \restriction A$ is stationary then for some $\{\bar a_\ell : \ell < \omega\}$ we have: $\bar a_\ell \in N$ realize $p, \{\bar a_\ell : \ell < \omega\}$ is independent over $A$ and $p \restriction (A \cup \bigcup\limits_{\ell<\omega} \bar a_\ell) \vdash p$

[why? [Sh:c, IV,proof of 4.18] (i.e. by it and [Sh:c, 4.9](3),4.11) or let $N'$ be $\aleph_\epsilon$-primary over $A \cup \bigcup\limits_{\ell<\omega} \bar a_\ell$ and note: $N'$ is $\aleph_\epsilon$-primary over $A$ (proof like the one of 1.18(0)) but also $N$ is $\aleph_\epsilon$-primary over $A$ so by uniqueness of the $\aleph_\epsilon$-primary model $N'$ is isomorphic to $N$ over $A$, so without loss of generality $N' = N$; and easily $N'$ is as required].

Now we can prove 1.18(1), for any $\bar c \in {}^{\omega >}A$, we can find a finite $B^1_{\bar a} \subseteq N$ such that $\mathrm{tp}(\bar c, N)$ does not fork over $B^1_{\bar c}$, let $\bar b_{\bar c} \in {}^{\omega >}N$ realize $\mathrm{stp}(\bar a, B^1_{\bar a})$ and let $B_{\bar c} = B^1_{\bar c} \cup \bar b_{\bar c}$, so $\mathrm{tp}(\bar c, N)$ does not fork over $B_{\bar c}$ and $\mathrm{tp}(\bar c, B_{\bar c})$ is stationary, hence we can find $\langle \bar a^{\bar c}_\ell : \ell < \omega\}$ as in $(*)_2$ (for $\mathrm{tp}(\bar c, B_{\bar c})$). Let
$A' = \cup\{B_{\bar c} : \bar c \in {}^{\omega >}A\} \cup \{\bar a^{\bar c}_\ell : \bar c \in {}^{\omega >}A$ and $\ell < \omega\}$, so $A'$ is a countable subset of $N$ and $\mathrm{tp}_*(A, A') \vdash \ \mathrm{tp}(A, N) = \ \mathrm{stp}(A, N)$. As $N$ is $\aleph_\epsilon$-primary over $\emptyset$ we can find a sequence $\langle d_\alpha : \alpha < \alpha^*\rangle$ and $\langle w_\alpha : \alpha < \alpha^*\rangle$ such that $N = \{d_\alpha : \alpha < \alpha^*\}$ and $w_\alpha \subseteq \alpha$ is finite and $\mathrm{stp}(d_\alpha, \{d_\beta : \beta \in w_\alpha\}) \vdash \ \mathrm{stp}(d_\alpha, \{d_\beta : \beta < \alpha\})$ and $\beta < \alpha \Rightarrow d_\beta \neq d_\alpha$.

We can find a countable set $W \subseteq \alpha^*$ such that $A' \subseteq \{d_\alpha : \alpha \in W\}$ and $\alpha \in W \Rightarrow w_\alpha \subseteq W$. Let $A'' = \{a_\alpha : \alpha \in W\}$. By [Sh:c, IV,§2,§3] without loss of generality $W$ is an initial segment of $\alpha^*$.
Easily

$$\alpha < \alpha^* \ \& \ \alpha \notin W \Rightarrow \ \mathrm{stp}(d_\alpha, \{d_\beta : \beta \in w_\alpha) \vdash \ \mathrm{stp}(d_\alpha, A \cup \{d_\beta : \beta < \alpha\}).$$

As $N^+$ is $\aleph_\epsilon$-primary over $N \cup A$ we can find a list $\{d_\alpha : \alpha \in [\alpha^*, \alpha^{**})\}$ of $N^+ \backslash (N \cup A)$ such that $\mathrm{tp}(d_\alpha, N \cup A \cup \{d_\beta : \beta \in [\alpha^*, \alpha^{**})\})$ is $\aleph_\epsilon$-isolated. So $\langle d_\alpha : \alpha \notin W, \alpha < \alpha^{**} \rangle$ exemplifies that $N^+$ is $\aleph_\epsilon$-primary over $A \cup A''$, hence by 1.18(0) we know that $N^+$ is $\aleph_\epsilon$-primary over $\emptyset$.
2) We shall use the characterization of "$N$ is $\mathbf{F}^a_{\aleph_0}$-prime over $A$" in 1.17, more exactly we use the last condition in 1.17(1) for $A = \emptyset, M = N_\omega$. Clearly $N_\omega$ is $\aleph_\epsilon$-saturated (as it is $\aleph_\epsilon$-prime over $\bigcup\limits_{n<\omega} N_n$). Suppose $B \subseteq N_\omega$ is finite and $p \in S(B)$ is (stationary and) regular.

Case 1: $p$ not orthogonal to $\bigcup\limits_{n<\omega} N_n$.

So for some $n < \omega, p$ is not orthogonal to $N_n$, hence there is a regular $p_1 \in S(N_n)$ such that $p, p_1$ are not orthogonal. Let $A_1 \subseteq N_n$ be finite such that $p_1$ does not fork over $A$ and $p_1 \restriction A_1$ is stationary. So by [Sh:c, V,§2] we know $\dim(p, N_\omega) = \dim(p_1 \restriction A_1, N_\omega)$, hence it suffices to prove that the latter is $\aleph_0$. Now this holds by [Sh:c, V,1.16](3),p.237 or immitate the proof of $(*)_2$ above.

Case 2: $p$ is orthogonal to $\bigcup\limits_{n<\omega} N_n$.

Note that if each $N_n$ is $\aleph_\varepsilon$-prime then $\bigcup\limits_{n<\omega} N_n$ is $\aleph_\varepsilon$-saturated hence $N = \bigcup\limits_{n<\omega} N_n$ hence this case does not arise. Let $A = \bigcup\limits_{n<\omega} N_n$, so $\dim(p, N) \leq \aleph_0$ follows from 2A) below.
Alternatively (and work even if we replace $N_\eta$ by a set $A_n, \mathbf{F}^a_{\aleph_0}$-constructible over $\emptyset$, see below).
2A) By 2B).
2B) The first inequality as immediate (as $T$ is superstable and $\bar{a}, \bar{b}$ are finite), so let us concentrate on the second. Let $B \subseteq C$ be a finite set such that $\mathrm{tp}_*(\bar{a}\hat{\ }\bar{b}, C)$ does not fork over $B$ and $\mathrm{stp}_*(\bar{a}\hat{\ }\bar{b}, B) \vdash \mathrm{stp}_*(\bar{a}\hat{\ }\bar{b}, C)$. Recall $q \in S(N)$ extend $\bar{a}/\bar{b}$ and do not fork over $\bar{b}$, let $b^* \in \mathfrak{C}$ realize $q$ and let $q_1 = \mathrm{stp}(\bar{b}^*, B \cup \bar{b})$ and $q_2 = \mathrm{stp}(\bar{b}^*, C \cup \bar{b})$. Now by the assumption of our case $q_1$ is orthogonal to $\mathrm{tp}_*(C, B)$ hence (see [Sh:c, V,§3]) $q_1 \vdash q_2$ and let $\{a_\alpha : \alpha < \alpha^*\} \subseteq (q_1 \restriction (\bar{b} \cup B))(N)$ be a maximal set independent over $C + \bar{b}$, so $|\alpha^*| \leq \dim(\bar{a}/(C + \bar{b}), N)$ and $q \restriction (C \cup \bar{b} \cup \{a_\alpha : \alpha < \alpha^*\}) \vdash q$. Also clearly $\mathrm{stp}_*(\{a_\alpha : \alpha < \alpha^*\}, \bar{b} \cup B) \vdash \mathrm{stp}_*(\{a_\alpha : \alpha < \alpha^*\}, \bar{b} \cup C)$. Together $\dim(q_1, N) \leq |\alpha^*|$ and as $|B| < \aleph_0 = \kappa_r(T)$ clearly $\dim(\bar{a}/\bar{b}, N) < \aleph_0 + \dim(q_1, N)^+$, so we are done.

We can use a different proof for part (2), note:

$\otimes_1$ if $\kappa = \mathrm{cf}(\kappa) \geq \kappa_r(T)$ and $B_\alpha$ is $\mathbf{F}^a_\kappa$-constructible over $A$ for $\alpha < \delta, \delta \leq \kappa$

and $\alpha < \beta < \delta \Rightarrow B_\alpha \subseteq B_\beta$ then $\bigcup\limits_{\alpha<\delta} B_\alpha$ is $\mathbf{F}^a_\kappa$-constructible over $A$

[why? see [Sh:c, IV,§3], [Sh:c, IV,5.6,p.207] for such arguments, assume $\mathscr{A}_\alpha = \langle A, \langle a^\alpha_i : i < i_\alpha\rangle, \langle B^\alpha_i : i < i_\alpha\rangle\rangle$ is an $\mathbf{F}^a_\kappa$-construction of $B_\alpha$ over $A$. Without loss of generality $i < j < i_\alpha \Rightarrow a^\alpha_i \neq a^\alpha_i$, and choose by induction on $\zeta, \langle u^\alpha_\zeta : \alpha < \delta\rangle$ such that: $u^\alpha_\zeta \subseteq i_\alpha, u^\alpha_\zeta$ increasing continuous in $i, u^\alpha_0 = \emptyset, |u^\alpha_{\zeta+1} \backslash u^\alpha_\zeta| \leq \kappa, u^\alpha_\zeta$ is $\mathscr{A}_\alpha$-closed and $\alpha < \beta < \delta$ implies $\{a^\alpha_j : j \in u^\alpha_\zeta\} \subseteq \{a^\beta_j : j \in u^\beta_\zeta\}$ and $\mathrm{tp}_*(\{a^\beta_i : i \in u^\beta_\zeta\}, A \cup \{a^\alpha_i : i < i_\alpha\})$ does not fork over $A \cup \{a^\alpha_i : i \in u^\alpha_\zeta\}$. Now find a list $\langle a_j : j < j^*\rangle$ such that for each $\zeta, \{j : a_j \in a^\alpha_i : i \in u^\alpha_\varepsilon$ for some $\alpha < \delta, \varepsilon < \zeta\}$ is an initial segment $\beta_\zeta$ of $j^*$ and $\beta_{\zeta+1} \leq \beta_\zeta + \kappa$.]

We use $\otimes_1$ for $\kappa = \aleph_0$.
So each $N_n$ is $\aleph_\epsilon$-constructible over $\emptyset$ hence $\bigcup\limits_{n<\omega} N_n$ is $\aleph_\epsilon$-constructible over $\emptyset$ and also $N_\omega$ is $\aleph_\epsilon$-constructible over $\bigcup\limits_{n<\omega} N_n$ hence $N_\omega$ is $\aleph_\epsilon$-constructible over $\emptyset$. But $N_\omega$ is $\aleph_\epsilon$-saturated hence $N_\omega$ is $\aleph_\epsilon$-primary over $\emptyset$. Alternatively use: if $B$ is $\mathbf{F}^a_\kappa$-constructible over $A, \kappa \geq \kappa_r(T)$ and $\mathbf{I}$ is indiscernible over $A, |\mathbf{I}| > \kappa$ <u>then</u> for some $\mathbf{J} \subseteq \mathbf{I}$ of cardinality $\leq \kappa, \mathbf{I} \backslash \mathbf{J}$ is an indiscernible set over $B$.
3) Suppose $N'_3$ is $\aleph_\epsilon$-saturated and $N_1 + \bar{a} \subseteq N'_3$. As $N_2$ is $\aleph_\epsilon$-prime over $N_0 + \bar{a}$ and $N_0 + \bar{a} \subseteq N_1 + \bar{a} \subseteq N'_3$ we can find an elementary embedding $f_0$ of $N_2$ into $N'_3$ extending $\mathrm{id}_{N_0+\bar{a}}$. By [Sh:c, V,3.3], the function $f_1 = f_0 \cup \mathrm{id}_{N_1}$ is an elementary mapping and clearly $\mathrm{Dom}(f_1) = N_1 \cup N_2$. As $N_3$ is $\aleph_\epsilon$-prime over $N_1 \cup N_2$ and $f_1$ is an elementary mapping from $N_1 \cup N_2$ into $N'_3$ which is an $\aleph_\varepsilon$-saturated model there is an elementary embedding $f_3$ of $N_3$ into $N'_3$ extending $f_2$. So as for any such $N'_3$ there is such $f_3$, clearly $N_3$ is $\aleph_\epsilon$-prime over $N_1 + \bar{a}$, as required.
4) Let $N_0$ be $\aleph_0$-prime over $\emptyset$ and let $\{p_i : i < \alpha\} \subseteq S(N_0)$ be a maximal family of pairwise orthogonal regular types. Let $\mathbf{I}_i = \{\bar{a}^i_n : n < \omega\} \subseteq \mathfrak{C}$ be a set of elements realizing $p_i$ independent over $N_0$ and let $\mathbf{I} = \bigcup\limits_{i<\alpha} \mathbf{I}_i$ and $N'_1$ be $\mathbf{F}^a_{\aleph_0}$-prime over $N_0 \cup \mathbf{I}$.
Now

$(*)$ if $\bar{a}, \bar{b} \subseteq N'_1$ and $\bar{a}/\bar{b}$ is regular (hence stationary), then $\dim(\bar{a}/\bar{b}, N'_1) \leq \aleph_0$.

[Why? If $\bar{a}/\bar{b} \perp N_0$ then $\dim(\bar{a}/\bar{b}, N'_1) \leq \aleph_0$ by part (2A) and the choice of the $p_i$ and $\mathbf{I}_i$ for $i < \alpha$. If $\bar{a}/\bar{b} \pm N_0$, then for some $\bar{b}'\hat{\ }\bar{a}' \subseteq N_0$ realizing $\mathrm{stp}(\bar{b}\hat{\ }\bar{a}, \emptyset)$, we have $\bar{a}'/\bar{b}' \pm \bar{a}/b$ hence $\dim(\bar{a}/b, N'_1) = \dim(\bar{a}'/\bar{b}', N'_1)$, so without loss of generality $\bar{b}\hat{\ }\bar{a} \subseteq N_0$, similarly without loss of generality there is $i(*) < \alpha$ such that $\bar{a}/\bar{b} \subseteq p_{i(*)}$ and $p_{i(*)}$ does not fork over $\bar{b}$ now easily $\dim(\bar{a}/\bar{b}, N'_1) = \dim(\bar{a}/\bar{b}, N_0) + \dim(p_{i(*)}, N_0) \leq \aleph_0 + \aleph_0 = \aleph_0$ (see [Sh:c, V,1.6](3)). So we have proved $(*)$].

Now use 1.17(1) to deduce: $N_1'$ is $\mathbf{F}^a_{\aleph_\epsilon}$-prime over $\emptyset$ hence (by uniqueness of $\aleph_\epsilon$-prime model, 1.17(2)) $N_1' \cong N_1$.

By renaming without a loss of generality $N_1' = N_1$. Now

$(**)(\alpha)$ $(N_1, c)_{c\in N_0}, (N_2, c)_{c\in N_0}$ are $\aleph_\epsilon$-saturated and

$(\beta)$ if $\bar a \in \mathfrak{C}, \bar b \in N_\ell, \bar a/\bar b$ a regular type and $\bar a \bigcup\limits_{\bar b}(N_0 + \bar b)$ (for $\ell = 1$ or $\ell = 2$), then
$\dim(\bar a/(\bar b \cup N_0), N_\ell) = \aleph_0$.

[Why? Remember that we work in $(\mathfrak{C}^{\rm eq}, c)_{c\in N_0}$. The "$\aleph_\epsilon$-saturated" follows from the second statement.
Note: $\dim(\bar a/(\bar b \cup N_0), N_\ell) \le \dim(\bar a/\bar b, N_\ell) \le \aleph_0$ (first inequality by monotonicity, second inequality by 1.17(1) and the assumption "$N_\ell$ is $\aleph_\epsilon$-prime over $\emptyset$"). If $\bar a/\bar b$ is not orthogonal to $N_0$ then for some $i < \alpha$ we have $p_i \pm (\bar a/\bar b)$ so easily (using "$N_\ell$ is $\aleph_\varepsilon$-saturated") we have $\dim(\bar a/(\bar b \cup N_0), N_\ell) = \dim(p_i, N_\ell) \ge \|\mathbf{I}_i\| = \aleph_0$; so together with the previous sentence we get equality. Lastly, if $\bar a/\bar b \perp N_0$ by part (2B) of 1.18, we have $\dim(\bar a/(\bar b \cup N_0), N_\ell) < \aleph_0 \Rightarrow \dim(\bar a/\bar b, N_\ell) < \aleph_0$ which contradicts the assumption "$N_\ell$ is $\aleph_\epsilon$-saturated".] So we have proved $(**)$ hence by 1.17(1) we get "$N_1, N_2$ are isomorphic over $N_0''$ as required.
5) This is proved similarly as if $N$ is $\aleph_\varepsilon$-prime over $A$ and $B \subseteq N$ is $\varepsilon$-finite then $N$ is $\aleph_\varepsilon$-prime over $A + B$ and also over $A'$ if $A + B \subseteq A' \subseteq \operatorname{acl}(A + B)$, see part (10).
6) By [Sh:c, V,3.2].
7) First assume that $A_2^* \subseteq N_1$ and $a/N_1$ is regular. As $N_1$ is $\aleph_\epsilon$-prime over $N_0 \cup N_1'$ and as $T$ has NDOP (i.e. does not have DOP) we know (by [Sh:c, X,2.1,2.2,p.512]) that $N_1$ is $\aleph_\epsilon$-minimal over $N_0 \cup N_1'$ and $\frac{a}{N_1}$ is not orthogonal to $N_0$ or to $N_1'$. But $a/N_1 \perp N_0$ by an assumption, so $a/N_1$ is not orthogonal to $N_1'$ hence there is a regular $p' \in S(N_1')$ not orthogonal to $\frac{a}{N_1}$ hence (by [Sh:c, V,1.12,p.236]) $p'$ is realized say by $a' \in N_2$. By [Sh:c, V,3.3], we know that $N_2$ is $\aleph_\epsilon$-prime over $N_1 + a'$. We can find $N_2'$ which is $\aleph_\epsilon$-prime over $N_1' + a'$ and $N_2''$ which is $\aleph_\epsilon$-prime over $N_1 \cup N_2'$ hence by part (3) of 1.18 we know that $N_2''$ is $\aleph_\varepsilon$-prime over $N_1 + a'$ so by uniqueness, i.e. 1.17(1), without loss of generality $N_2'' = N_2$ hence we are done.
In general by induction on $\alpha$ choose $N_{2,\alpha}'$ such that $N_{2,0}'$ is $\aleph_\varepsilon$-prime over $N_1' \cup A_2^*, N_{2,\alpha}'$ is increasing with $\alpha$ and $N_1 \bigcup\limits_{N_1'} N_{2,\alpha}'$. Easily for some $\alpha, N_{2,\alpha}'$ is defined but not $N_{2,\alpha+1}'$. Necessarily $N_2$ is $\aleph_\varepsilon$-prime over $N_1' \cup N_{2,\alpha}'$. Lastly let $a' \in N_{2,\alpha}'$ be such that $\operatorname{tp}(a, N_1 \cup N_{2,\alpha}')$ does not fork over $N_1 + a'$. Easily $N_{2,\alpha}'$ is $\aleph_\varepsilon$-prime over $N_1' + a'$ by (1.17(1)).
8) Similar easier proof.
9) Let $N_0'$ be $\aleph_\epsilon$-prime over $A$ such that $B \bigcup\limits_A N_0'$, and let $N_1'$ be $\aleph_\epsilon$-prime over

$N'_0 \cup B$. By 1.18(1), we know that $N'_1$ is $\aleph_\epsilon$-prime over $\emptyset$, and by 1.18(10) below $N'_1$ is $\aleph_\epsilon$-prime over $A \cup B$, hence by 1.17(2) we know that $N'_1, N_1$ are isomorphic over $A \cup B$ hence without loss of generality $N'_1 = N_1$ and so $N_0 = N'_0$ is as required.
10) By [Sh:c, IV,3.12](3),p.180. $\square_{1.18}$

*1.20 Fact.* Assume $\langle N^1_\eta, a_\eta : \eta \in I\rangle \le^*_{\text{direct}} \langle N^2_\eta, a_\eta : \eta \in I\rangle$ (see Definition 1.16) and $A \subseteq B \subseteq N^1_{<0>}$ and $\bigwedge\limits_{\eta \in I} N^2_\eta \prec M$.

(1) If $\nu = \eta\hat{\ }\langle\alpha\rangle \in I$, then $N^2_\eta \underset{N^1_\eta}{\bigcup\!\!\!\!\bigcup} N^1_\nu$ and even $N^2_\eta \underset{N^1_\eta}{\bigcup\!\!\!\!\bigcup} \left(\bigcup\limits_{\substack{\rho \in I \\ \neg\eta\triangleleft\rho}} N^1_\rho\right)$; and

$\eta \triangleleft \nu \in I$ implies $N^2_\nu \underset{N^1_\eta}{\bigcup\!\!\!\!\bigcup} \left(\bigcup\limits_{\substack{\rho \in T \\ \neg\eta\triangleleft\rho}} N^1_\rho\right)$.

(2) $\langle N^2_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition inside $M$ above $\binom{B}{A}$ <u>iff</u> $\langle N^1_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition inside $M$ above $\binom{B}{A}$.

(3) Similarly replacing "$\aleph_\epsilon$-decomposition inside $M$ above $\binom{B}{A}$" by "$\aleph_\epsilon$-decomposition of $M$ above $\binom{B}{A}$".

*Proof.* 1) We prove the first statement by induction on $\ell g(\eta)$. If $\eta = <>$ this is clause (b) by the Definition 1.16 and clause (d) of Definition 1.11(1) (and [Sh:c, V,3.2]). If $\eta \ne <>$, then $\frac{a_\nu}{N_\eta} \perp N^1_{(\eta^-)}$ (by condition (e) of Definition 1.11(1)). By the induction hypothesis $N^2_{(\eta^-)} \underset{N^1_{(\eta^-)}}{\bigcup\!\!\!\!\bigcup} N^1_\eta$ and we know $N^2_\eta$ is $\aleph_\epsilon$-primary over $N^2_{(\eta^-)} \cup N^1_\eta$; we know this implies that no $p \in S(N^1_\eta)$ orthogonal to $N^1_{\eta^-}$ is realized in $N^2_\eta$ hence $\frac{a_\nu}{N^1_\eta} \perp \frac{N^2_\eta}{N^1_\eta}$, so $\frac{a_\nu}{N^1_\eta} \vdash \frac{a_\nu}{N^2_\eta}$ hence $\frac{N^1_\nu}{N^1_\eta} \perp \frac{N^2_\eta}{N^1_\eta}$ hence $N^1_\nu \underset{N^1_\eta}{\bigcup\!\!\!\!\bigcup} N^2_\eta$ as required.
The other statements hold by the non-forking calculus (remember if $\eta = \nu\hat{\ }\langle\alpha\rangle \in I$ then use $\mathrm{tp}(\cup\{N^1_\rho : \eta \trianglelefteq \rho \in I\}, N^1_\eta)$ is orthogonal to $N^1_\nu$ or see details in the proof of 1.21(1)($\alpha$)).
2) By Definition 1.16, for $\ell = 1,2$ we have: $\langle N^\ell_\eta, a_\eta : \eta \in I\rangle$ is a decomposition inside $\mathfrak{C}$ and by assumption $\bigwedge\limits_{\eta \in I} N^1_\eta \prec N^2_\eta \prec M$. So for $\ell = 1,2$ we have to prove "$\langle N^\ell_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition inside $M$ for $\binom{B}{A}$" assuming this holds for $1-\ell$. We have to check Definition 1.11(1).

Clauses 1.5(1)(a),(b) for $\ell$ holds because they hold for $1-\ell$.
Clause 1.5(1)(c) holds as by the assumptions $A \subseteq B \subseteq N^1_{<0>} \prec N^2_{<0>}, A \subseteq N^1_{<>}$ and $N^1_{<0>} \underset{N^1_{<>}}{\bigcup\!\!\!\!\bigcup} N^2_{<>}$.
Clauses 1.5(1)(d),(e),(f),(h) holds as $\langle N^\ell_\eta, a_\eta : \eta \in I\rangle$ is a decomposition inside $\mathfrak{C}$ (for $\ell = 1$ given, for $\ell = 2$ easily checked).
Clause 1.5(1)(g) holds as $\bigwedge\limits_\eta N^1_\eta \prec N^2_\eta \prec M$ is given and $M$ is $\aleph_\epsilon$-saturated.

3) First we do the "only if" direction; i.e. prove the maximality of $\langle N^1_\eta, a_\eta : \eta \in I\rangle$ as an $\aleph_\epsilon$-decomposition inside $M$ for $\binom{B}{A}$ (i.e. condition (i) from 1.11(2)), assuming it holds for $\langle N^2_\eta, a_\eta : \eta \in I\rangle$. If this fails, then for some $\eta \in I\backslash\{<>\}$ and $a \in M, \{a_{\eta\char`^<\alpha>} : \eta\char`^ < \alpha > \in I\} \cup \{a\}$ is independent over $N^1_\eta$ and $a \notin \{a_{\eta\char`^\langle\alpha\rangle} : \eta\char`^\langle\alpha\rangle \in I\}$ and $\frac{a}{N^1_\eta} \perp N^1_{\eta^-}$. Hence, if $\eta\char`^\langle\alpha_\ell\rangle \in I$ for $\ell < k$ then $\bar a = \langle a\rangle\char`^\langle a_{\eta\char`^<\alpha_\ell>} : \ell < k\rangle$ realizes over $N^1_\eta$ a type orthogonal to $N^1_{\eta^-}$, but $N^1_{\eta^-} \prec N^1_\eta, N^1_{\eta^-} \prec N^2_{\eta^-}$ and $N^1_\eta \underset{N^1_{\eta^-}}{\bigcup\!\!\!\!\bigcup} N^2_\eta$ (see 1.20(1), hence (by [Sh:c, V,2.8]) $\text{tp}(\bar a, N^2_\eta) \perp N^2_{\eta^-}$ hence $\{a\}\cup\{a_{\eta\char`^\langle\ell\rangle} : \ell < k\}$ is independent over $N^2_\eta$ but $k, \eta\char`^\langle\alpha_\ell\rangle \in I$ for $\ell < k$ were arbitrary so $\{a\} \cup \{a_{\eta\char`^\langle\alpha\rangle} : \eta\char`^\langle\alpha\rangle \in I\}$ is independent over $N^2_\eta$ contradicting condition (i) from Definition 1.11(2) for $\langle N^2_\eta, a_\eta : \eta \in I\rangle$.

For the other direction use: if the conclusion fails, then for some $\eta \in I\backslash\{<>\}$ and
$a \in M\backslash N^2_\eta\backslash\{a_{\eta\char`^<\alpha>} : \eta\char`^ < \alpha > \in I\}$ the set $\{a_{\eta\char`^\langle\alpha\rangle} : \eta\char`^\langle\alpha\rangle \in I\} \cup \{a\}$ is independent of $N^2_\eta$ and $\text{tp}(a, N^2_\eta)$ is orthogonal to $N^2_{\eta^-}$; let $N' \prec M$ be $\aleph_\varepsilon$-prime over $N^2_\eta + a$. But $N^2_\eta$ is $\aleph_\varepsilon$-prime over $N^1_\eta \cup N^2_{\eta^-}$ (by the definition of $\le$-direct) so by NDOP $\text{tp}(a, N^2_\eta) \pm N^1_\eta$ hence there is a regular $q \in S(N^1_\eta)$ such that $q \pm \text{ tp}(a, N^2_\eta)$. Hence some $a' \in N'$ realizes $q$, clearly $\{a_{\eta\char`^<\alpha>} : \eta\char`^ < \alpha > \in I\}\cup\{a'\}$ is independent over $N^2_\eta$ (and $a' \notin \{a^1_{\eta\char`^<\alpha>} : \eta\char`^ < \alpha > \in I\}$) hence over $(N^2_\eta, N^1_\eta)$ and easily we get contradiction. $\square_{1.20}$

*1.21 Fact.* Assume $\langle N^1_\eta, a^1_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition inside $M$.
1) If $N^1_{<>} \prec N^2_{<>} \prec M, N^2_\eta$ is $\aleph_\epsilon$-prime over $\emptyset$ and $N^2_{<>} \underset{N^1_{<>}}{\bigcup\!\!\!\!\bigcup} \{a^1_{<\alpha>} : <\alpha> \in I\}$ <u>then</u>

$$(\alpha) \quad \left[N^2_{<>} \underset{N^1_{<>}}{\bigcup\!\!\!\!\bigcup} \bigcup_{\eta \in I} N^1_\eta\right] \text{ and}$$

$(\beta)$ we can find $N^2_\eta$ $(\eta \in I \backslash \{<>\})$ such that $N^2_\eta \prec M$, and $\langle N^1_\eta, a^1_\eta : \eta \in I\rangle \le^*_{\text{direct}} \langle N^2_\eta, a^1_\eta : \eta \in I\rangle$.

2) If Cb $\frac{a^1_{<\alpha>}}{N^1_{<>}} \subseteq N^0_{<>} \prec N^1_{<>}$ or at least $N^0_{<>} \prec N^1_{<>}$ and $\frac{a^1_{<\alpha>}}{N^1_{<>}} \pm N^0_{<>}$ whenever $<\alpha>\in I$ <u>then</u> we can find $N^0_\eta \prec M$ and $a^0_\eta \in N_\eta$ (for $\eta \in I\backslash\{<>\}$) such that $\langle N^0_\eta, a^0_\eta : \eta \in I\rangle \le^*_{\text{direct}} \langle N^1_\eta, a^0_\eta : \eta \in I\rangle$.
3) In part (2), if in addition we are given $\langle B^*_\eta : \eta \in I\rangle$ such that $B^*_\eta$ is an $\varepsilon$-finite subset of $N_\eta$, $\text{tp}_*(B^*_\eta, N_\eta)$ does not fork over $B^*_{\eta^-}$ and $B^*_{<>} \subseteq N^0_{<>}$ <u>then</u> we can demand in the conclusion that $\eta \in I \Rightarrow B^*_\eta \subseteq N^0_\eta$.

*Proof.* 1) For proving $(\alpha)$ let $\{\eta_i : i < i^*\}$ list the set $I$ such that $\eta_i \triangleleft \eta_j \Rightarrow i < j$, so $\eta_0 =<>$ and without loss of generality for some $\alpha^*$ we have $\eta_i \in \{<\alpha>:<\alpha>\in I\} \Leftrightarrow i \in [1,\alpha^*)$. Now we prove by induction on $\beta \in [1, i^*)$ that $N^2_{<>} \underset{N^1_{<>}}{\bigcup\!\!\!\!\cup} \cup\{N^1_{\eta_i} : i < \beta\}$. For $\beta = 1$ this is assumed. For $\beta$ limit use the local character of non-forking.

If $\beta = \gamma + 1 \in [1, \alpha^*)$, then by repeated use of [Sh:c, V,3.2] (as $\{a_{\eta_0} : j \in [1,\beta)\}$ is independent over $(N^1_{<>}, N^2_{<>})$ and $N^1_{<>}$ is $\aleph_\epsilon$-saturated and $N^1_{\eta_j}(j \in [1,\gamma))$ is $\aleph_\epsilon$-prime over $N^1_{<>} + a_{\eta_j})$ we know that $\text{tp}(a_{\eta_\gamma}, N^2_{<>} \cup \bigcup\limits_{i<\gamma} N^1_{\eta_i})$ does not fork over $N^1_{<>}$. Again by [Sh:c, V,3.2], the type $\text{tp}_*(N^1_{\eta_\gamma}, N^2_{<>} \cup \bigcup\limits_{i<\gamma} N^1_{\eta_i})$ does not fork over $N^1_{<>}$ hence $\bigcup\limits_{i<\beta} N^1_{\eta_i} \underset{N^1_{<>}}{\bigcup\!\!\!\!\cup} N^2_{<>}$ and use symmetry.

Lastly, if $\beta \in \gamma + 1 \in [\alpha^*, i^*), \text{tp}(a_{\eta^-_\gamma}, N_{\eta_\gamma})$ is orthogonal to $N^1_{<>}$ and even to $N^1_{(\eta^-_\gamma)^-}$ so again by non-forking and [Sh:c, V,3.2] we can do it, so clause $(\alpha)$ holds.

For clause $(\beta)$, we choose $N^2_{\eta_i}$ for $i \in [1, i^*)$ by induction on $i < i^*$ such that $N^2_{\eta_i} \prec M$ is $\aleph_\epsilon$-prime over $N^2_{\eta^-_i} \cup N^1_{\eta_i}$. By the non-forking calculus we can check Definition 1.7.
2) We let $\{\eta_i : i < i^*\}$ be as above, now we choose $N^0_{\eta_i}, a^0_{\eta_i}$ by induction on $i \in [1, i^*)$ such that:

$(*)$ $N^0_{\eta_i} \prec N^1_{\eta_i}$ and $N^1_{\eta^-_i} \underset{N^0_{\eta^-_i}}{\bigcup\!\!\!\!\cup} N^0_{\eta_i}$ and $N^1_{\eta_i}$ is $\aleph_\epsilon$-prime over $N^0_{\eta_i} \cup N^1_{\eta^-_i}$

$(**)$ $a^0_{\eta_i} \in N^0_{\eta_i}$ and $N^0_{\eta_i}$ is $\aleph_\epsilon$-prime over $N^0_{\eta^-_i} + a^0_{\eta_i}$.

The induction step has already been done: if $\ell g(\eta_i) > 1$ by 1.18(7) and if $\ell g(\eta_i) = 1$

by 1.18(8).
3) Similar. $\square_{1.21}$

*1.22 Fact.* 1) <u>If</u> $\langle N^1_\eta, a_\eta : \eta \in I\rangle \le^*_{\rm direct} \langle N^2_\eta, a_\eta : \eta \in I\rangle$ and both are $\aleph_\epsilon$-decompositions of $M$ above $\binom{B}{A}$, <u>then</u>

$$\mathscr{P}(\langle N^1_\eta, a^1_\eta : \eta \in I\rangle, M) = \mathscr{P}(\langle N^2_\eta, a^2_\eta : \eta \in I\rangle, M).$$

*Proof.* By Defintion 1.11(5) it suffices to prove, for each $\eta \in I \backslash \{<>\}$ that

(*) for regular $p \in S(M)$ we have
$p \perp N^1_{\eta^-}$ & $p \pm N^1_\eta \Leftrightarrow p \perp N^2_{\eta^-}$ & $p \pm N^2_\eta$.

Now consider any regular $p \in S(M)$: first assume $p \perp N^1_{\eta^-}$ & $p \pm N^1_\eta$ where $\eta \in I\backslash\{<>\}$ so $p \pm N^2_\eta$ (as $N^1_\eta \prec N^2_\eta$ and $p \pm N^1_\eta$) and we can find a regular $q \in S(N^1_\eta)$ such that $q \pm p$; so as $p \perp N^1_{\eta^-}$ also $q \perp N^1_{\eta^-}$, now $q \perp N^2_{\eta^-}$ (as $N^1_\eta \underset{N^1_{\eta^-}}{\bigcup} N^2_{\eta^-}$ and $q \perp N^1_\eta$ see [Sh:c, V,2.8]) hence $p \perp N^2_{\eta^-}$.

Second, assume $p \perp N^2_{\eta^-}$ & $p \pm N^2_\eta$ where $\eta \in I\backslash\{<>\}$; remember $N^1_{\eta^-}, N^1_\eta, N^2_\eta, N^3_\eta$ are
$\aleph_\epsilon$-saturated, $N^1_\eta \underset{N^1_{\eta^-}}{\bigcup} N^2_{\eta^-}$ and $N^2_\eta$ is $\aleph_\epsilon$-prime over $N^1_\eta \cup N^2_{\eta^-}$ and $T$ does not have DOP. Hence $N^2_\eta$ is $\aleph_\epsilon$-minimal over $N^1_\eta \cup N^2_{\eta^-}$ and every regular $q \in S(N^2_\eta)$ is not orthogonal to $N^1_\eta$ or to $N^2_{\eta^-}$. Also as $p \pm N^2_\eta$ there is a regular $q \in S(N^2_\eta)$ not orthogonal to $p$, so as $p \perp N^2_{\eta^-}$ also $q \perp N^2_{\eta^-}$; hence by the previous sentence $q \pm N^1_\eta$ hence $p \pm N^1_\eta$. Lastly, as $p \perp N^2_{\eta^-}$ and $N^1_{\eta^-} \prec N^2_{\eta^-}$ clearly $p \perp N^1_{\eta^-}$, as required. $\square_{1.22}$

At last we start proving 1.14.

*Proof of 1.14.* 1) Let $N^0 \prec \mathfrak{C}$ be $\aleph_\epsilon$-primary over $A$, without loss of generality $N^0 \underset{A}{\bigcup} B$ (but not necessarily $N^0 \prec M$), and let $N^1$ be $\aleph_\epsilon$-primary over $N^0 \cup B$. Now by 1.18(0) the model $N^0$ is $\aleph_\epsilon$-primary over $\emptyset$ and by 1.18(1) the model $N^1$ is $\aleph_\epsilon$-primary over $\emptyset$ hence (by 1.18(10)) is $\aleph_\epsilon$-primary over $B$, hence without loss of generality $N^1 \prec M$. Let $N_{<>} =: N^0, N_{<0>} = N^1, I = \{<>, <0>\}$ and $a_{<0>} = B$. More exactly $a_\eta$ is such that $\mathrm{dcl}(\{a_\eta\}) = \mathrm{dcl}(B)$. Clearly $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition inside $M$ above $\binom{B}{A}$. Now apply part (2) of 1.14 proved below.

2) By 1.13(4) we know $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition inside $M$, by 1.18(2) we then find $J \supseteq I$ and $N_\eta, a_\eta$ for $\eta \in J\backslash I$ such that $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition of $M$. By 1.18(3), $\langle N_\eta, a_\eta : \eta \in J'\rangle$ is an $\aleph_\epsilon$-decomposition of $M$ above $\binom{B}{A}$ where $J' =: \{\eta \in J : \eta = <> \text{ or } \langle 0\rangle \trianglelefteq \eta \in J\}$.
3) Part (a) holds by 1.13(2),(3). As for part (b) by 1.13(2) there is $\langle N_\eta, a_\eta : \eta \in J\rangle$, an $\aleph_\epsilon$-decomposition of $M$ with $I \subseteq J$; easily $[\langle 0\rangle \trianglelefteq \eta \in J \Rightarrow \eta \in I]$.

$\square_{1.14(1),(2),(3)}$

*1.23 Fact.* If $\langle N^\ell_\eta, a^\ell_\eta : \eta \in I^\ell\rangle$ are $\aleph_\epsilon$-decompositions of $M$ above $\binom{B}{A}$, for $\ell = 1, 2$ and $N^1_{<>} = N^2_{<>}$ then $\mathscr{P}(\langle N^1_\eta, a^1_\eta : \eta \in I^1\rangle, M) = \mathscr{P}(\langle N^2_\eta, a^2_\eta : \eta \in I^2\rangle, M)$.

*Proof.* By 1.14(3)(b) we can find $J^1 \supseteq I^1$ and $N^1_\eta, a^1_\eta$ for $\eta \in J^1\backslash I^1$ such that $\langle N^1_\eta, a^1_\eta : \eta \in J^1\rangle$ is an $\aleph_\epsilon$-decomposition of $M$ and moreover we have $\eta \in J^1\backslash I^1 \Leftrightarrow \eta \neq \langle\rangle \;\;\&\;\; \neg(\langle 0\rangle \triangleleft \eta)$. Let $J^2 = I^2 \cup (J^1\backslash I^1)$ and for $\eta \in J^2\backslash I^2$ let $a^2_\eta =: a^1_\eta$, $N^2_\eta =: N^1_\eta$. Easily $\langle N^2_\eta, a^2_\eta : \eta \in J^2\rangle$ is an $\aleph_\epsilon$-decomposition of $M$. By 1.13(6) we know that for every regular $p \in S(M)$ there is (for $\ell = 1, 2$) a unique $\eta(p,\ell) \in J^\ell$ such that $p \pm N_{\eta(p,\ell)} \;\;\&\;\; p \perp N_{\eta(p,\ell)^-}$ (note $\langle\rangle^-$ – meaningless). By the uniqueness of $\eta(p,\ell)$, if $\eta(p,1) \in J^1\backslash I^1$ then as it can serve as $\eta(p,2)$ clearly it is $\eta(p,2)$ so $\eta(p,2) = \eta(p,1) \in J^1\backslash I^1 = J^2\backslash I^2$; similarly $\eta(p,2) \in J^2\backslash I^2 \Rightarrow \eta(p,1) \in J^1\backslash I^1$ and $\eta(p,1) = \langle\rangle \Leftrightarrow \eta(p,2) = \langle\rangle$. So

$(*)$ $\eta(p,1) \in I^1\backslash\{\langle\rangle\} \Leftrightarrow \eta(p,2) \in I^2\backslash\{\langle\rangle\}$.

But

$(**)$ $\eta(p,\ell) \in I^\ell\backslash\{\langle\rangle\} \Leftrightarrow p \in \mathscr{P}(\langle N^\ell_\eta, a^\ell_\eta : \eta \in I^\ell\rangle, M)$.

Together we finish. $\square_{1.23}$

We continue proving 1.14.

*Proof of 1.14(4).* Let $A^* \subseteq M^-$ be $\varepsilon$-finite, so we can find an $\varepsilon$-finite $B^* \subseteq \cup\{N_\eta : \eta \in I\}$ such that $\text{stp}(A^*, B^*) \vdash \text{stp}(A^*, \cup\{N_\eta : \eta \in I\})$. Hence, there is a finite non empty $I^* \subseteq I$ such thatd $<> \in I^*$, $I^*$ is closed under initial segments and $B^* \subseteq \cup\{N_\eta : \eta \in I^*\}$, so of course $\text{stp}_*(A^*, \cup\{N_\eta : \eta \in I^*\}) \vdash \text{stp}(A^*, \cup\{N_\eta : \eta \in I\})$.

We can also find $\langle B^*_\eta : \eta \in I^*\rangle$ such that $B^*_\eta$ is an $\varepsilon$-finite subset of $N_\eta, B^*_\eta = \text{acl}(B^*_\eta)$ and $B^* \subseteq \cup\{B^*_\eta : \eta \in I^*\}, \eta \neq <> \Rightarrow a_\eta \in B^*_\eta$ and if $\eta \triangleleft \nu \in I^*$ then $B^*_\eta \subseteq B^*_\nu$ and $\text{tp}_*(B^*_\nu, N_\nu)$ does not fork over $B^*_\eta$. Without loss of generality $B \subseteq B^*_{<0>}$. For $\eta \in I\backslash I^*$ let $B^*_\eta = B^*_{\eta\restriction\ell}$ where $\ell < \ell g(\eta)$ is maximal such that $\eta \restriction \ell \in I^*$, such $\ell$ exists as $\ell g(\eta)$ is finite and $<> \in I^*$.

Let $N^1_\eta = N_\eta$ and $a^1_\eta = a_\eta$ for $\eta \in I$ and without loss of generality $J \ne I$ hence $J \backslash I \ne \emptyset$.

Let $N^2_{<>} \prec M$ be $\aleph_\epsilon$-prime over $\bigcup\limits_{\nu \in J \backslash I} N_\nu$; letting $J \backslash I = \{\eta_i : i < i^*\}$ be such that $[\eta_i \triangleleft \eta_j \Rightarrow i < j]$ we can find $N^2_{<>,i}$ (for $i \le i^*$) increasing continuous, $N^2_{<>,0} = N_{<>}$ and $N^2_{<>,i+1}$ is $\aleph_\epsilon$-prime over $N^2_{<>,i} \cup N_{\eta_i}$ hence over $N^2_{<>,i} + a_{\eta_i}$. Lastly, without loss of generality $N^2_{<>,i^*} = N^2_{<>}$.

By 1.18(1),(2) we know $N^2_{<>}$ is $\aleph_\epsilon$-primary over $\emptyset$ and (using repeatedly 1.18(6) + finite character of forking) we have $N^2_{<>} \underset{N^1_{<>}}{\bigcup\!\!\!\!\bigcup} a_{<0>}$. By 1.18(4) (with $N^1_{<>}, N^2_{<>}, B^*_{<>} \supseteq \mathrm{Cb}(a_{<>}/N^1_{<>})$ here standing for $N_1, N_2, A$ there) we can find a model $N^0_{<>}$ such that $a_{<0>} \underset{N^0_{<>}}{\bigcup\!\!\!\!\bigcup} N^1_{<>}$ and $\mathrm{Cb}(a_{<>}/N^1_{<>}) \subseteq B^*_{<>} \subseteq N^0_{<>}, N^0_{<>} \prec N^1_{<>}, N^0_{<>}$ is $\aleph_\epsilon$-primary over $\emptyset$ and $N^1_{<>}, N^2_{<>}$ are isomorphic over $N^0_{<>}$. By 1.21(1) we can for $\eta \in I$ choose $N^2_\eta \prec M$ with $N^1_\eta \prec N^2_\eta$ and $\langle N^1_\eta, a^1_\eta : \eta \in I\rangle \le^*_{\text{direct}} \langle N^2_\eta, a^0_\eta : \eta \in I\rangle$. Similarly, by 1.21(2) (here $\mathrm{Suc}_I(<>) = \{\langle 0\rangle\}$) we can choose an $\aleph_\epsilon$-decomposition $\langle N^0_\eta, a^0_\eta : \eta \in I\rangle$ with $\langle N^0_\eta, a^0_\eta : \eta \in I\rangle \le^*_{\text{direct}} \langle N^1_\eta, a^1_\eta : \eta \in I\rangle$. Moreover, we can demand $\eta \in I^* \Rightarrow B^*_\eta \subseteq N^0_\eta$ using 1.21(3). By 1.13(12) + 1.14(3) we know that $\langle N^1_\eta, a^0_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition of $M^-$ and easily $\langle N^2_\eta, a^0_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition of $M$. Now choose by induction on $\eta \in I$ an isomorphism $f_\eta$ from $N^1_\eta$ onto $N^2_\eta$ over $N^0_\eta$ such that $\nu \triangleleft \eta \Rightarrow f_\nu \subseteq f_\eta$ and $\eta \in I^* \Rightarrow f_\eta \restriction B^*_\eta = \mathrm{id}_{B^*_\eta}$. For $\eta =<>$ we have chosen $N^0_\eta$ such that $N^1_\eta, N^2_\eta$ are isomorphic over $N^0_\eta$. For the induction step note that $f_{(\eta^-)} \cup \mathrm{id}_{N^0_\eta}$ is an elementary mapping as $N^2_{(\eta^-)} \underset{N^0_{(\eta^-)}}{\bigcup\!\!\!\!\bigcup} N^0_\eta$ and $f_{(\eta^-)} \cup \mathrm{id}_{N^0_\eta}$ can be extended to an isomorphism $f_\eta$ from $N^1_\eta$ onto $N^2_\eta$ as $N^\ell_\eta$ is $\aleph_\epsilon$-primary (in fact even $\aleph_\epsilon$-minimal) over $N^\ell_{(\eta^-)} \cup N^0_\eta$ for $\ell = 1, 2$ (which holds easily). If $\eta \in I^*$ there is no problem to add $f_\eta \restriction B^*_\eta = \mathrm{id}_{B^*_\eta}$. Now by 1.13(3) the model $M^-$ is $\aleph_\epsilon$-saturated and $\aleph_\epsilon$-primary and $\aleph_\epsilon$-minimal over $\bigcup\limits_{\eta \in J} N_\eta = \bigcup\limits_{\eta \in I} N^1_\eta$; similarly $M$ is $\aleph_\epsilon$ -primary over $\bigcup\limits_{\eta \in I} N^2_\eta$. Now $\bigcup\limits_\eta f_\eta$ is an elementary mapping from $\bigcup\limits_{\eta \in I} N^1_\eta$ onto $\bigcup\limits_{\eta \in I} N^2_\eta$ hence can be extended to an isomorphism $f$ from $M^-$ into $M$. Moreover as $\mathrm{stp}_*(A^*, \cup\{B^*_\eta : \eta \in I^*\}) \vdash \mathrm{stp}(A^*, \{N^1_\eta : \eta \in I\})$, by [Sh:c, CH.XII,§4] we have $\mathrm{tp}_*(A^*, \cup\{B^*_\eta : \eta \in I^*\} \vdash \mathrm{tp}(A^*, \cup\{N^1_\eta : \eta \in I\}$ hence $\mathrm{tp}_*(A^*, \cup\{B^*_\eta : \eta \in I^*\})$ has a unique extension as a complete type over $\cup\{N^1_\eta : \eta \in I\}$ hence over $\cup\{N^2_\eta : \eta \in I\}$

so without loss of generality $f \restriction A^* = \mathrm{id}_{A^*}$. By the $\aleph_\epsilon$-minimality of $M$ over $\bigcup_{\eta \in I} N_\eta$ (see 1.13(3)), $f$ is onto $M$, so $f$ is as required. $\square_{1.14(4)}$

We delay the proof of 1.14(5).

*Proof of 1.14(6).* Let $\langle N^\ell_\eta, a^\ell_\eta : \eta \in I^\ell \rangle$ for $\ell = 1, 2$, be $\aleph_\epsilon$-decompositions of $M$ above $\binom{B}{A}$, so $\mathrm{dcl}(a^\ell_{<>}) = \mathrm{dcl}(B)$. Let $p \in S(M)$, and assume that $p \in \mathscr{P}(\langle N^1_\eta, a^1_\eta : \eta \in I^1 \rangle, M)$, i.e. for some $\eta \in I^1 \backslash \{<>\}$, $(p_\eta \perp N_{\eta^-})$ and $p_\eta \pm N_\eta$. We shall prove that the situation is similar for $\ell = 2$; i.e. $p \in \mathscr{P}(\langle N^2_\eta, a^2_\eta : \eta \in I^2 \rangle, M)$; by symmetry this suffices.

Let $n = \ell g(\eta)$, choose $\langle B_\ell : \ell \leq n \rangle$ and $d$ such that:

($\alpha$) $A \subseteq B_0$,

($\beta$) $B \subseteq B_1$,

($\gamma$) $a_{\eta \restriction \ell} \subseteq B_\ell \subseteq N^1_{\eta \restriction \ell}$, for $\ell \leq n$

($\delta$) $B_{\ell+1} \underset{B_\ell}{\bigcup\!\!\!\!\cup} N^1_{\eta \restriction \ell}$,

($\epsilon$) $\frac{B_{\ell+1}}{B_\ell + a^1_{\eta \restriction (\ell+1)}} \vdash \frac{B_{\ell+1}}{N^1_{\eta \restriction \ell} + a^1_{\eta \restriction (\ell+1)}}$,

($\zeta$) $d \in B_n$, $\frac{d}{B_n \backslash \{d\}}$ is regular $\pm p$, (hence $\perp B_{n-1}$)

($\eta$) $B_\ell$ is $\epsilon$-finite.

[Why such $\langle B_\ell : \ell \leq n \rangle$ exists? We prove by induction on $n$ that for any $\eta \in I$ of length $n$ and $\epsilon$-finite $B' \subseteq N_\eta$, there is $\langle B_\ell : \ell \leq n \rangle$ satisfying $(\alpha) - (\epsilon)$, $(\eta)$ such that $B' \subseteq B_n$. Now there is $p' \in S(N^1_\eta)$ regular not orthogonal to $p$, let $B^1 \subseteq N^1_\eta$ be an $\epsilon$-finite set extending $\mathrm{Cb}(p')$. Applying the previous sentence to $\eta$, $B^1$ we get $\langle B_\ell : \ell \leq n \rangle$, let $d \in N_\eta$ realize $p' \restriction B_n$.

Now as $n > 0$, $\mathrm{tp}(d, B_n) \perp N_{\eta^-}$ hence $\mathrm{tp}(d, B_n) \perp B_{n-1}$, hence $\mathrm{tp}(d, B_n) \perp tp_*(N_{\eta^-}, B_n)$, hence as $\mathrm{tp}(d, B_n)$ is stationary, by [Sh:c, V,1.2](2),p.231, the types $\mathrm{tp}(d, B_n)$,$\mathrm{tp}_*(N_{\eta^-}, B_n)$ are weakly orthogonal so $\mathrm{tp}(d, B_n) \vdash \mathrm{tp}(d, N_{\eta^-} \cup B_n)$ hence $\frac{B_n + d}{B_{n-1} + a^1_\eta} \vdash \frac{B_n + d}{N^1_{\eta^-} + a^1_\eta}$.

Now replace $B_n$ by $B_n \cup \{d\}$ and we finish].

Note that necessarily

$(\delta)^+$ $B_n \underset{B_m}{\bigcup\!\!\!\!\cup} N^1_{\eta \restriction m}$ for $m \leq n$.

[Why? By the non-forking calculus].

$(\epsilon)^+$ $\frac{B_n}{B_m + a^1_{\eta \restriction (m+1)}} \perp_a B_m$ for $m < n$.

[Why? As $N^1_{\eta \restriction m}$ is $\aleph_\epsilon$-saturated].

Choose $D^* \subseteq N^2_{<>}$ finite such that $\frac{B_n}{N^2_{<>}+B}$ does not fork over $D^* + B$.
[Note: we really mean $D^* \subseteq N^2_{<>}$, not $D^* \subseteq N^1_{<>}$].

We can find $N^3_{<>}, \aleph_\varepsilon$-prime over $\emptyset$ such that $A \subseteq N^3_{<>} \prec N^2_{<>}$ and $D^* \bigcup\limits_A N^3_{<>}$ and $N^2_{<>}$ is $\aleph_\varepsilon$-prime over $N^3_{<>} \cup D^*$ (by 1.18(9)). Hence $B_n \bigcup\limits_A N^3_{<>}$ and $B_n \bigcup\limits_B N^3_{<>}$ (by the non-forking calculus). As $\mathrm{tp}_*(B, N^2_{<>})$ does not fork over $A \subseteq N^3_{<>} \subseteq N^2_{<>}$ by 1.21(2) we can find $N^3_\eta, a^3_\eta$ (for $\eta \in I^2 \backslash \{<>\}$), such that $\langle N^3_\eta, a^3_\eta : \eta \in I\rangle$ is an $\aleph_\varepsilon$-decomposition inside $M$ above $\binom{B}{A}$ and $\langle N^3_\eta, a^3_\eta : \eta \in I^2\rangle \leq^*_{\text{direct}} \langle N^2_\eta, a^3_\eta : \eta \in I^2\rangle$ and $a^3_{<0>} = a^2_{<0>}$ (remember $\mathrm{dcl}(a^2_{<0>}) = \mathrm{dcl}(B)$). By 1.20(2) we know $\langle N^3_\eta, a^3_\eta : \eta \in I^2\rangle$ is an $\aleph_\epsilon$-decomposition of $M$ above $\binom{B}{A}$.
By 1.22 it is enough to show $p \in \mathscr{P}(\langle N^3_\eta, a^3_\eta : \eta \in I^2\rangle, M)$. Let $N^4_{<>} \prec N^2_{<>} \prec M$ be $\aleph_\epsilon$-prime over $N^3_{<>} \cup B_0$. Now by the non-forking calculus $B \bigcup\limits_A (N^3_{<>} \cup B_0)$ [why? because

(a) as said above $B_n \bigcup\limits_B N^3_{<>}$ but $B_0 \subseteq B_n$ so $B_0 \bigcup\limits_B N^3_{<>}$, and

(b) as $B \bigcup\limits_A N^1_{<>}$ and $B_0 \subseteq N^1_{<>}$ we have $B \bigcup\limits_A B_0$ so $B_0 \bigcup\limits_A B$

hence (by (a) + (b) as $A \subseteq B$)

(c) $\frac{B_0}{N^3_{<>}+B}$ does not fork over $A$,

also

(d) $B \bigcup\limits_A N^3_{<>}$ (as $A \subseteq N^3_{<>} \subseteq N^2_{<>}$ and $\mathrm{tp}(B, N^2_{<>})$ does not fork over $A$)

putting (c) and (d) together we get

(e) $\bigcup\limits_A \{B_0, B, N^3_{<>}\}$

hence the conclusion].

Hence $B \bigcup\limits_{N^3_{<>}} B_0$ so $B \bigcup\limits_{N^3_{<>}} N^4_{<>}$ (by 1.18(6)) and so (as $N^3_{<0>}$ is $\aleph_\varepsilon$-prime over $N^3_{<>} + dcl(a^3_{<>}) = N^3_{<>} + dcl(B)$) we have $N^4_{<>} \bigcup\limits_{N^3_{<>}} N^3_{<0>}$ and by 1.21(1) we can choose $N^4_\eta \prec M$ (for $\eta \in I^2 \backslash \{<>\}$), such that $\langle N^4_\eta, a^3_\eta : \eta \in I^2\rangle \geq_{\text{direct}} \langle N^3_\eta, a^3_\eta : \eta \in I^2\rangle$. So by 1.20(1) $\langle N^4_\eta, a^3_\eta : \eta \in I^2\rangle$ is an $\aleph_\epsilon$-decomposition of $M$ above $\binom{B}{A}$ hence $a^3_{<0>}/N^4_\eta$ does not fork over $A$ but $A \subseteq B_0 \subseteq N^4_{<>}$ so $a^3_{<>}/N^4_\eta$ does not fork over $B_0$

and by 1.22 it is enough to prove $p \in \mathscr{P}\big(\langle N^4_\eta, a^3_\eta : \eta \in I^2\rangle, M\big)$. Now as said above $B \underset{N^3_{<>}}{\bigcup} N^4_{<>}$ and $B \underset{A}{\bigcup} N^3_{<>}$ so together $B \underset{A}{\bigcup} N^4_{<>}$, also we have $A \subseteq B_0 \subseteq N^4_{<>}$, hence $B \underset{B_0}{\bigcup} N^4_{<>}$ and

$\frac{B_n}{B_0+B} \equiv \frac{B_n}{B_0+a^3_{<0>}} \perp_a B_0$ (by $(\epsilon)^+$ above) but $a^3_{<>} \underset{B_0}{\bigcup} N^4_{<>}$ hence $\frac{B_n}{N^4_{<>}+a^3_{<0>}}$ is $\aleph_\epsilon$-isolated. Also letting $B'_n = B_n \backslash \{d\}$ we have $\frac{B'_n}{N^4_{<>}+a^3_{<>}}$ is $\aleph_\varepsilon$-isolated and $\frac{d}{B'_n} \perp B_0$ (by clause $(\zeta)$), and clearly $d \underset{B'_n}{\bigcup} (N^4_{<>} \cup B'_n)$ so $\frac{d}{B'_n} \perp N^4_{<>}$. Hence we can find $\langle N^5_\eta, a^5_\eta : \eta \in I^5\rangle$ an $\aleph_\epsilon$-decomposition of $M$ above $\binom{B}{A}$ such that $N^5_{<>} = N^4_{<>}$, $\mathrm{dcl}(B) = \ \mathrm{dcl}(a^5_{<0>}), B_n \backslash \{d\} \subseteq N^5_{<0>}$ and $d = a^5_{<0,0>}$ (on $d$ see clause $(\zeta)$ above) so $d \underset{B_n}{\bigcup} N^5_{<0>}$.

By 1.23 it is enough to show $p \in \mathscr{P}(\langle N^5_\eta, a^5_\eta : \eta \in I^5\rangle, M)$ which holds trivially as $\mathrm{tp}(d, B_n \backslash \{d\})$ witness. $\square_{1.14(6)}$

*Proof of 1.14(5).* By 1.8, with $A, B, A_1, B_1$ here standing for $A_1, B_1, A_2, B_2$ there, we know that there are $\langle B'_\ell : \ell \leq n\rangle, \langle c_\ell : 1 \leq \ell < n\rangle$ as there. By 1.18(9) we can choose $N^1_{<>}$ such that $B_0 \subseteq N^1_{<>}, N^1_{<1>} \underset{B_0}{\bigcup} B_n, N^1_{<>}$ is $\aleph_\epsilon$-primary over $\emptyset$. Then we choose $\langle N^1_\eta, a^1_\eta : \eta \in \Big\{<>, <0>, <0,0>, \dots, \underbrace{<0,\dots,0>}_{n}\Big\}\rangle$, (where $N^1_{\underbrace{<0,\dots,0>}_{n}} \prec M$ and $\ell > 0 \Rightarrow a^1_{\underbrace{<0,\dots,0>}_{\ell}} = c_\ell, B'_{\ell g(\eta)} \subseteq N^1_\eta$ and we choose $N^1_\eta$ by induction on $\ell g(\eta)$ being $\aleph_\varepsilon$-prime over $N^1_{\eta^-} \cup a^1_\eta$ hence $a^1_\eta / N^1_{\eta^-}$ does not fork over $B'_{\ell g(\eta^-)}$ hence $N^1_\eta$ is $\aleph_\varepsilon$-prime also over $N^1_{\eta^-} + B'_{\ell g(\eta)}$. So $\langle N^1_\eta, a^1_\eta : \eta \in \{<>, \dots\}\rangle$ is an $\aleph_\epsilon$-decomposition inside $M$ for $\binom{B_1}{A_1}$. Now apply first 1.14(2) and then 1.14(6).

*Proof of 1.14(7).* Should be easy. Note that

$(*)_1$ for no $\binom{B'}{A'}$ do we have $\binom{B}{A} \leq_b \binom{B'}{A'}$
Why? By the definition of Depth zero.

$(*)_2$ if $\binom{B}{A} <_a \binom{B'}{A'}$, then also $\binom{B'}{A'}$ satisfies the assumption.

Hence

$(**)$ for no $\binom{B_1}{A_1}, \binom{B_2}{A_2}$ do we have

$$\binom{B}{A} \leq_a \binom{B_1}{A_1} <_b \binom{B_2}{A_2}.$$

[Why? As also $\binom{B_1}{A_1}$ satisfies the assumption].

Now we can prove the statement by induction on $\alpha$ for all pairs $\binom{B}{A}$ satisfying the assumption. For $\alpha = 0$ the statement is a tautology. For $\alpha$ limit ordinal reread clause (c) of Definition 1.9(1). For $\alpha = \beta + 1$, reread clause (b) of Definition 1.9(1): on $\text{tp}_\beta(\binom{B}{A}, M)$ use the induction hypothesis also for computing $Y^{1,B}_{A,B,M}$ (and reread the definition of $\text{tp}_0$, in Definition 1.9(1), clause (a)). Lastly $Y^{2,\beta}_{A,B,M}$ is empty by $(*)$ above.

*Proof of 1.14(8),(9).* Read Definition 1.9. $\square_{1.14(5),(7),(8),(9)}$

*Discussion.* In particular, the following Claim 1.26 implies that if $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition of $M$ above $\binom{B}{A}$ and $M^-$ is $\aleph_\epsilon$-prime over $\cup\{N_\eta : \eta \in I\}$ then $\binom{B}{A}$ has the same $\text{tp}_\alpha$ in $M$ and $M^-$.

**1.24 Claim.** *1) Assume that $M_1 \prec M_2$ are $\aleph_\varepsilon$-saturated, $\binom{B}{A} \in \Gamma(M_1)$. <u>Then</u> the following are equivalent:*

- *(a) if $p \in \mathscr{P}\big(\binom{B}{A}, M_1\big)$ (see 1.14(6) for definition; so $p \in S(M_1)$ is regular) then $p$ is not realized in $M_2$*
- *(b) there is an $\aleph_\varepsilon$-decomposition of $M_1$ above $\binom{B}{A}$, which is also an $\aleph_\varepsilon$-decomposition of $M_2$ above $\binom{B}{A}$*
- *(c) every $\aleph_\varepsilon$-decomposition of $M_1$ above $\binom{B}{A}$ is also an $\aleph_\varepsilon$-decomposition of $M_2$ above $\binom{B}{A}$.*

*2) If $M$ is $\aleph_\varepsilon$-saturated, $\binom{B_1}{A_1} \leq^* \binom{B_2}{A_2}$ are both in $\Gamma(M)$ <u>then</u> $\mathscr{P}\big(\binom{B_2}{A_2}, M\big) \subseteq \mathscr{P}\big(\binom{B_1}{A_1}, M\big)$.*
*3) The conditions in 1.24(1) above implies*

- *(d) $p \in \mathscr{P}\big(\binom{B}{A}, M_2\big) \Rightarrow p \pm M_1$.*

*Proof.* 1) <u>(c) ⇒ (b)</u>.

By 1.14(1) there is an $\aleph_\varepsilon$-decomposition of $M_1$ above $\binom{B}{A}$. By clause (c) it is also an $\aleph_\varepsilon$-decomposition of $M_2$ above $\binom{B}{A}$, just as needed for clause (b).

<u>(b) ⇒ (a)</u>

Let $\langle N_\eta, a_\eta : \eta \in I\rangle$ be as said in clause (b). By 1.14(3)(b) we can find $J_1, I \subseteq J_1$ and $N_\eta, a_\eta$ (for $\eta \in J_1 \backslash I$) such that $\langle N_\eta, a_\eta : \eta \in J_1\rangle$ is an $\aleph_\varepsilon$-decomposition of $M_1$ and $\nu \in J_1 \backslash I \Rightarrow \nu(0) > 0$. Then we can find $J_2, J_1 \subseteq J_2$ and $N_\eta, a_\eta$ (for $\eta \in J_2 \backslash J_1$) such that $\langle N_\eta : \eta \in J_2\rangle$ is an $\aleph_\varepsilon$-decomposition of $M_2$ (by 1.14(2)). By 1.14(3)(b), $\nu \in J_2 \backslash I \Rightarrow \nu(0) > 0$. So $\eta \in I \backslash \{\langle\rangle\} \Rightarrow \operatorname{Suc}_{J_2}(\eta) = \operatorname{Suc}_I(\eta)$, hence

(∗) if $\eta \in I \backslash \{\langle\rangle\}$ and $q \in S(N_\eta)$ is regular orthogonal to $N_{\eta^-}$ <u>then</u> the stationarization of $q$ in $S(M_1)$ is not realized in $M_2$.

Now if $p \in \mathscr{P}\big(\binom{B}{A}, M_1\big)$ then $p \in S(M_1)$ is regular and (see 1.14(1), 1.11(5)) for some $\eta \in I \backslash \{\langle\rangle\}, p \perp N_{\eta^-}, p \pm N_\eta$, so there is a regular $q \in S(N_\eta)$ not orthogonal to $p$. Now no $c \in M_2$ realizes the stationarization of $q$ over $M_1$ (by (∗) above), hence this applies to $p$, too.

<u>(a) ⇒ (c)</u>

Let $\langle N_\eta, a_\eta : \eta \in I\rangle$ be an $\aleph_\varepsilon$-decomposition of $M_1$ above $\binom{B}{A}$. We can find $\langle N_\eta, a_\eta : \eta \in J\rangle$ an $\aleph_\varepsilon$-decomposition of $M_1$ such that $I \subseteq J$ and $\nu \in J \backslash I \Rightarrow \nu(0) > 0$ (by 1.14(3)(b)), so $M$ is $\aleph_\varepsilon$-prime over $\cup\{N_\eta : \eta \in J\}$. We should check that $\langle N_\eta : a_\eta : \eta \in I\rangle$ it is also an $\aleph_\varepsilon$-decomposition of $M_2$ above $\binom{B}{A}$, i.e. Definition 1.11(1),(2). Now in 1.11(1), clauses (a)-(h) are immediate, so let us check clause (i) (in 1.11(2)). Let $\eta \in I \backslash \{\langle\rangle\}$, now is $\{a_{\eta \hat{} \langle\alpha\rangle} : \eta \hat{} \langle\alpha\rangle \in I\}$ really maximal (among independent over $N_\eta$ sets of elements of $M_2$ realizing a type from $\mathscr{P}_\eta = \{p \in S(N_\eta) : p \text{ orthogonal to } N_{\eta^-}\}$)?. This should be clear from clause (a) (and basic properties of dependencies and regular types).

2) By 1.14(5).

3) Left to the reader. $\square_{1.24}$

<u>1.25 Conclusion</u>: Assume $M_1 \prec M_2$ are $\aleph_\varepsilon$-saturated and $\binom{B_1}{A_1} \leq^* \binom{B_2}{A_2}$ both in $\Gamma(M_1)$. If clause $(a)$ (equivalently $(b)$ or $(c)$) of 1.24 holds for $\binom{B_1}{A_1}, M_1, M_2$ then they hold for $\binom{B_2}{A_2}, M_1, M_2$.

*Proof.* By 1.24(2), clause (a) for $\binom{B_1}{A_1}, M_1, M_2$ implies clause (a) for $\binom{B_2}{A_2}, M_1, M_2$. $\square_{1.25}$

**1.26 Claim.** *If* $\binom{B_1}{A_1} \in \Gamma(M)$ *and* $\langle N_\eta, a_\eta : \eta \in I\rangle$ *is an* $\aleph_\epsilon$*-decomposition of* $M$ *above* $\binom{B_1}{A_1}$ *and* $M^- \subseteq M$ *is* $\aleph_\epsilon$*-saturated and* $\bigcup_{\eta \in I} N_\eta \subseteq M^-$ *and* $\alpha$ *is an ordinal* *then*

$$\mathrm{tp}_\alpha\left[\binom{B_1}{A_1}, M\right] = \mathrm{tp}_\alpha\left[\binom{B_1}{A_1}, M^-\right]$$

*Proof.* We prove this by induction on $\alpha$ (for all $B, A, \langle N_\eta, a_\eta : \eta \in I\rangle, I, M$ and $M^-$ as above). We can find an $\aleph_\epsilon$-decomposition $\langle N_\eta, a_\eta : \eta \in J\rangle$ of $M$ with $I \subseteq J$ (by 1.13(4) + 1.13(2)) such that $\eta \in J \backslash I \Leftrightarrow \eta \neq \langle\rangle$ and $\neg\langle 0\rangle \trianglelefteq \eta$ and so $M$ is $\aleph_\epsilon$-prime over $\bigcup_{\eta \in J} N_\eta$ and also over $M^- \cup \{N_\eta : \eta \in J \backslash I\}$.

Case 0: $\alpha = 0$.
Trivial.

Case 1: $\alpha$ is a limit ordinal.
Trivial by induction hypothesis (and the definition of $\mathrm{tp}_\alpha$).

Case 2: $\alpha = \beta + 1$.
We can find $M^* \prec M^-$ which is $\aleph_\epsilon$-prime over $\bigcup_{\eta \in I} N_\eta$, so as equality is transitive it is enough to prove

$$\mathrm{tp}_\alpha\left(\binom{B_1}{A_1}, M^*\right) = \mathrm{tp}_\alpha\left(\binom{B_1}{A_1}, M^-\right)$$

and

$$\mathrm{tp}_\alpha\left(\binom{B_1}{A_1}, M^*\right) = \mathrm{tp}_\alpha\left(\binom{B_1}{A_1}, M\right).$$

By symmetry, this means that it is enough to prove the statement when $M^-$ is $\aleph_\epsilon$-prime over $\bigcup_{\eta \in I} N_\eta$.

Looking at the definition of $\mathrm{tp}_{\beta+1}$ and remembering the induction hypothesis our problems are as follows:
First component of $\mathrm{tp}_\alpha$:

given $\binom{B_1}{A_1} \leq_{\mathrm{a}} \binom{B_2}{A_2}, B_2 \subseteq M$, <u>it suffices to find</u> $\binom{B_3}{A_3}$ such that:

(*) there is $f \in \mathrm{AUT}(\mathfrak{C})$ such that: $f \restriction B_1 = \mathrm{id}_{B_1}, f(A_2) = A_3$, $f(B_2) = B_3$ and $B_3 \subseteq M^-$ and $\mathrm{tp}_\beta[\binom{B_2}{A_2}, M] = \mathrm{tp}_\beta[\binom{B_3}{A_3}, M^-]$ (pedantically we should replace $B_\ell, A_\ell$ by indexed sets).

We can find $J', M'$ such that:

(i) $I \subseteq J' \subseteq J, |J' \backslash I| < \aleph_0, J'$ closed under initial segments,
(ii) $M' \prec M$ is $\aleph_\epsilon$-prime over $M^- \cup \cup\{N_\eta : \eta \in J' \backslash I\}$
(iii) $B_2 \subseteq M'$.

The induction hypothesis for $\beta$ applies, and gives

$$\mathrm{tp}_\beta[\binom{B_2}{A_2}, M] = \mathrm{tp}_\beta[\binom{B_2}{A_2}, M'].$$

By 1.14(4) there is $g$, an isomorphism from $M'$ onto $M^-$ such that $g \restriction B_1 = \mathrm{id}$. So clearly $g(B_2) \subseteq M^-$ hence

$$\mathrm{tp}_\beta[\binom{B_2}{A_2}, M'] = \mathrm{tp}_\beta[\binom{g(B_2)}{g(A_2)}, M^-].$$

So $\binom{B_3}{A_3} =: g\binom{A_2}{B_2}$ is as required.

<u>Second component for $\mathrm{tp}_\alpha$</u>:

So we are given $\Upsilon$, a $\mathrm{tp}_\beta$ type, (and we assign the lower part as $B$) and we have to prove that the dimension in $M$ and in $M^-$ are the same, i.e. $\dim(\mathbf{I}, M) = \dim(\mathbf{I}^-, M)$, where: $\mathbf{I} = \{c \in M : \Upsilon = \mathrm{tp}_\beta(\binom{c}{B_1}, M)\}$ and $\mathbf{I}^- = \{c \in M^- : \Upsilon = \mathrm{tp}_\beta(\binom{c}{B_1}, M^-)\}$.
Let $p$ be such that: $\mathrm{tp}_\beta\left(\binom{c}{B_1}, M\right) = \Upsilon \Rightarrow p = \frac{c}{B_1}$. Necessarily $p \perp A_1$ and $p$ is regular (and stationary).

Clearly $\mathbf{I}^- \subseteq \mathbf{I}$, so without loss of generality $\mathbf{I} \neq \emptyset$ hence $p$ is really well defined, now

(*) for every $c \in \mathbf{I}$ for some $k < \omega, c'_\ell \in M^-$ realizing $p$ for $\ell < k$ we have $c$ depends on $\{c'_0, c'_1, \ldots, c'_{k-1}\}$ over $B_1$.
[Why? Clearly $p \perp N_{<>}$ (as $B_1 \bigcup\limits_{A_1} N_{<>}$ and $p \perp A_1$) hence

$\mathrm{tp}_*(\bigcup\limits_{\eta \in J \backslash I} N_\eta, N_{<>}) \perp p$ hence

$\mathrm{tp}_*(\bigcup_{\eta \in J\setminus I} N_\eta, M^-) \perp p$, but $M$ is $\aleph_\epsilon$-prime over $M^- \cup \bigcup_{\eta \in J\setminus I} N_\eta$ hence by [Sh:c, V,3.2,p.250] for no $c \in M\setminus M^-$ is $\mathrm{tp}(c, M^-)$ a stationarization of $p$ hence by [Sh:c, V,1.16](3) clearly $(*)$ follows].

If the type $p$ has depth zero, then (by 1.14(7)):

$$\mathbf{I} = \{c \in M : \mathrm{tp}(c, B) = p\} \text{ and}$$
$$\mathbf{I}^- = \{c \in M^- : \mathrm{tp}(c, B) = p\}.$$

Now we have to prove $\dim(\mathbf{I}, A) = \dim(\mathbf{I}^-, A)$, as $A$ is $\varepsilon$-finite and $M, M^-$ are $\aleph_\epsilon$-saturated and $\mathbf{I}^- \subseteq \mathbf{I}$ clearly $\aleph_0 \leq \dim(\mathbf{I}^-, A) \leq \dim(\mathbf{I}, A)$. Now the equality follows by $(*)$ above.

So we can assume "$p$ has depth $>$ zero", hence (by [Sh:c, X,7.2]) that the type $p$ is trivial; hence, see [Sh:c, X7.3], in $(*)$ without loss of generality $k = 1$ and dependency is an equivalence relation, so for "same dimension" it suffices to prove that every equivalence class (in $M$ i.e. in $\mathbf{I}$) is representable in $M^-$ i.e. in $\mathbf{I}^-$. By the remark on $(*)$ in the previous sentence $(\forall d_1 \in \mathbf{I})(\exists d_2 \in \mathbf{I}^-)\big[\neg d_1 \bigcup_{B_1} d_2\big]$. So it is enough to prove that:

$\bigotimes$ if $d_1, d_2 \in M$ realize same type over $B_1$, which is (stationary and) regular, and are dependent over $B_1$ and $d_1 \in M^-$ <u>then</u> there is $d_2' \in M^-$ such that $\frac{d_2'}{B_1+d_1} = \frac{d_2}{B_1+d_1}$ and $\mathrm{tp}_\beta\big[\binom{B_1+d_2}{B_1}, M\big] = \mathrm{tp}_\beta\big[\binom{B_1+d_2'}{B_1}, M^-\big]$.

Let $M_0 = N_{\langle\rangle}$. There are $J', M_1, M_1^+$ such that

$(*)_1(i)$ $J' \subseteq J$ is finite (and of course closed under initial segments)

$(ii)$ $\langle\rangle \in J', \langle 0\rangle \notin J'$

$(iii)$ $M_1 \prec M$ is $\aleph_\varepsilon$-prime over $\cup\{N_\eta : \eta \in J'\}$

$(iv)$ $M_1^+ \prec M$ is $\aleph_\varepsilon$-prime over $M_1 \cup M^-$ (and $M_1 \bigcup_{M_0} M^-$)

$(v)$ $d_2 \in M_1^+$.

Now the triple $\binom{B_1+d_2}{B_1}, M_1, M$ satisfies the demand on $\binom{B_1}{A_1}, M^-, M$ (because $\binom{B_1}{A_1} \leq^* \binom{B_1+d_2}{B_1}$,by 1.25. Hence by the induction hypothesis we know that

$$\mathrm{tp}\big[\binom{B_1+d_2}{B_1}, M\big] = \mathrm{tp}_\beta\big[\binom{B_1+d_2}{B_1}, M_1^+\big].$$

By 1.29(4) there is an isomorphism $f$ from $M_1^+$ onto $M^-$ which is the identity on $B_1 + d_1$; let $d_2' = f(d_2)$ so:

$$\mathrm{tp}_\beta\big[\binom{B_1+d_2}{B_1}, M_1^+\big] = \mathrm{tp}_\beta\big[\binom{B_1+d_2'}{B_1}, M^-\big].$$

Together

$$\mathrm{tp}_\beta\big[\binom{B_1+d_2}{B_1}, M\big] = \mathrm{tp}_\beta\big[\binom{B_1+d_2'}{B_1}, M^-\big].$$

As $\{d_1, d_2\}$ is not independent over $B_1$, also $\{f(d_1), f(d_2)\} = \{d_1, f(d_2)\}$ is not independent over $B_1$, hence, as $p$ is regular

(∗) $\{d_2, f(d_2)\}$ is not independent over $B_2$.

Together we have proved $\bigoplus$, hence finished proving the equality of the second component.

Third component: Trivial.

So we have finished the induction step, hence the proof. $\square_{1.26}$



**1.27 Claim.** *1) Suppose $M$ is $\aleph_\epsilon$-saturated, $A \subseteq B \subseteq M$, $\binom{B}{A} \in \Gamma$, $\bigwedge_{\ell=1}^{2}[A \subseteq A_\ell \subseteq M]$,*
*$A = ac\ell(A)$, $A_\ell$ are $\epsilon$-finite, $\frac{A_1}{A} = \frac{A_2}{A}$, $B \bigcup_A A_1$ and $B \bigcup_A A_2$.*

*Then $\mathrm{tp}_\alpha\big[\binom{A_1 \cup B}{A_1}, M\big] = \mathrm{tp}_\alpha\big[\binom{A_2 \cup B}{A_2}, M\big]$ for any ordinal $\alpha$.*

*2) Suppose $M$ is $\aleph_\epsilon$-saturated, $B \subseteq M$, $\binom{B}{A} \in \Gamma$, $\bigwedge_{\ell=1}^{2}[A \subseteq A_\ell \subseteq M]$, $A = ac\ell(A)$, $B = ac\ell(B)$, $A_\ell = ac\ell(A_\ell)$, $A_\ell$ is $\epsilon$-finite, $\frac{A_1}{A} = \frac{A_2}{A}$, $B \bigcup_A A_1$, $B \bigcup_A A_2$, $f : A_1 \xrightarrow{\text{onto}} A_2$ an elementary mapping, $f \restriction A = \mathrm{id}_A$, $g \supseteq f \cup \mathrm{id}_B$, $g$ elementary mapping from $B_1 = ac\ell(B \cup A_1)$ onto $B_2 = ac\ell(B \cup A_2)$.*

*Then $g\Big(\mathrm{tp}_\alpha\big[\binom{B_1}{A_1}, M\big]\Big) = \mathrm{tp}_\alpha\big[\binom{B_2}{A_2}, M\big]$ for any ordinal $\alpha$.*

*3) Assume that*

(a) *$A_\ell = acl(A_\ell) \subseteq B_\ell = acl(B_\ell) \subseteq M^\ell$ for $\ell = 1, 2$*

(b) *$A_\ell \subseteq A_\ell^+ \subseteq acl(A_\ell^+) \subseteq M^\ell$ for $\ell = 1, 2$*

(c) $B_\ell \bigcup_{A_\ell} A^+_\ell$ *for* $\ell = 1, 2$

(d) $f$ *is an elementary mapping from* $A_1$ *onto* $A_2$

(e) $g$ *is an elementary mapping from* $A^+_1$ *onto* $A^+_2$

(f) $f \restriction A_1 = g \restriction A_1$

(g) $h$ *is an elementary mapping from* $B^+_1 = acl(B_1 \cup A^+_1)$ *onto* $B^+_2 = acl(B_2 \cup A^+_2)$ *extending* $f$ *and* $g$

(h) $f(\mathrm{tp}_\alpha[\binom{B_1}{A_1}, M_1]) = \mathrm{tp}_\alpha[\binom{B_2}{A_2}, M_2]$.

<u>Then</u> $h(\mathrm{tp}_\alpha[\binom{B^+_1}{A^+_1}, M_1]) = \mathrm{tp}_\alpha[\binom{B^+_2}{A^+_2}, M_2]$.

*Proof.* 1) Follows from part (2).
2) We can find $A_3 \subseteq M$ such that:

(i) $\frac{A_3}{A} = \frac{A_1}{A}$

(ii) $A_3 \bigcup_A (B \cup A_1 \cup A_2)$.

Hence without loss of generality $A_1 \bigcup_B A_2$ and even $\bigcup_A \{B, A_1, A_2\}$. Now we can find $N_{<>}$, an $\aleph_\epsilon$-prime model over $\emptyset, N_{<>} \prec M, A \subseteq N_{<>}$ and $(B \cup A_1 \cup A_2) \bigcup_A N_{<>}$ (e.g. choose $\{A^\alpha_1 \cup A^\alpha_i \cup B^\alpha : \alpha \leq \omega\} \subseteq M$ indiscernible over $A, A^\omega_1 = A_1, A^\omega_2 = A_2, B^\omega = B$ and let $N_{<>} \prec M$ be $\aleph_\epsilon$-primary over $\bigcup_{n<\omega} (A^n_1 \cup A^n_2 \cup B^n \cup A)$).

Now find $\langle N_\eta, a_\eta : \eta \in J\rangle$ an $\aleph_\epsilon$-decomposition of $M$ with $\mathrm{dcl}(a_{<0>}) = \mathrm{dcl}(B), \mathrm{dcl}(a_{<1>}) = \mathrm{dcl}(A_1), \mathrm{dcl}(a_{<2>}) = \mathrm{dcl}(A_2)$.

Let $I = \{\eta \in J : \eta = <> \text{ or } <0> \trianglelefteq \eta\}$ and $J' = I \cup \{<1>, <2>\}$. Let $N^2_{<>} \prec M^*$ be $\aleph_\epsilon$-prime over $N_{<1>} \cup N_{<2>}$. By 1.21 there is $\langle N^2_\eta, a_\eta : \eta \in I\rangle$ an $\aleph_\epsilon$-decomposition of $M$ above $\binom{B}{A}$ such that $\langle N_\eta, a_\eta : \eta \in I\rangle \leq_{\mathrm{direct}} \langle N^2_\eta, a_\eta : \eta \in I\rangle$. Let $M' \prec M$ be $\aleph_\epsilon$-prime over $\bigcup_{\eta \in I} N^2_\eta$ and $M^- \prec M'$ be $\aleph_\epsilon$-prime over $\bigcup_{\eta \in I} N_\eta$. So $M^- \prec M' \prec M$ and $M'$ is $\aleph_\epsilon$-prime over $M^- \cup N_{<1>} \cup N_{<2>}$.

Now by 1.26 we have $\mathrm{tp}_\alpha\big[\binom{B}{A_\ell}, M\big] = \mathrm{tp}_\alpha\big[\binom{B}{A_\ell}, M'\big]$ for $\ell = 1, 2$ hence it suffices to find an automorphism of $M'$ extending $g$. Let $B^+ = ac\ell(N_{<>} \cup B), A^*_\ell = ac\ell(B \cup A_\ell)$, let $\bar{\mathbf{a}}_\ell$ list $A^*_\ell$ be such that $\bar{\mathbf{a}}_2 = g(\bar{\mathbf{a}}_1)$. Clearly $\mathrm{tp}(\bar{\mathbf{a}}_\ell, B^+)$ does not fork over $A \subseteq B$ and $ac\ell(B) = B$ and so $\mathrm{stp}(\bar{\mathbf{a}}_1, B^+) = \mathrm{stp}(\bar{\mathbf{a}}_2, B^+)$. Also $\mathrm{tp}_*(A_2, B^+ \cup A_1)$ does not fork over $A$ hence $\mathrm{tp}(\bar{\mathbf{a}}_2, B^+ \cup \bar{\mathbf{a}}_1)$ does not fork over $A \subseteq B^+$ hence $\{\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2\}$ is independent over $B^+$ hence there is an elementary mapping $g^+$ from

$ac\ell(B^+ \cup \bar{\mathbf{a}}_1)$ onto $ac\ell(B^+ \cup \bar{\mathbf{a}}_2), g^+ \supseteq \mathrm{id}_{B^+} \cup g$ and even $g' = g^+ \cup (g^+)^{-1}$ is an elementary embedding.

Let $\bar{\mathbf{a}}'_1$ lists $ac\ell(N_{<>} \cup A_1)$ so clearly $\bar{\mathbf{a}}'_2 =: g^+(\bar{\mathbf{a}}'_1)$ list $ac\ell(N_{<>} \cup A_2)$. Clearly $g' \restriction (\bar{\mathbf{a}}'_1 \cup \mathbf{a}'_2)$ is an elementary mapping from $\bar{\mathbf{a}}'_1 \cup \bar{\mathbf{a}}'_2$ onto itself. Now $N^2_{<>}$ is $\aleph_\epsilon$-primary over $N_{<>} \cup A_1 \cup A_2$ and $N_{<>} \cup A_1 \cup A_2 \subseteq \bar{\mathbf{a}}'_1 \cup \bar{\mathbf{a}}'_2 \subseteq ac\ell(N_{<>} \cup A_1 \cup A_2)$ so by 1.18(10) $N^2_{<>}$ is $\aleph_\epsilon$-primary over $N_{<>} \cup \bar{\mathbf{a}}'_1 \cup \bar{\mathbf{a}}'_2$ hence we can extend $g' \restriction (\bar{\mathbf{a}}'_1 \cup \bar{\mathbf{a}}'_2)$ to an automorphism $h_{<>}$ of $N^2_{<>}$ so clearly $h_{<>} \restriction N_{<>} = \mathrm{id}_{N_{<>}}$. Let $\bar{\mathbf{a}}^+_1$ list $ac\ell(B^+ \cup A_1)$ and $\bar{\mathbf{a}}^+_2 = g^+(\bar{\mathbf{a}}^+_1)$. So $\mathrm{tp}(\bar{\mathbf{a}}^+_\ell, N^2_{<>})$ does not fork over $\bar{\mathbf{a}}'_1 (\subseteq N^2_{<>})$ and $ac\ell(\bar{\mathbf{a}}'_1) = \mathrm{Rang}(\bar{\mathbf{a}}'_1)\ (= ac\ell(N_{<>} \cup A_1))$ and $h_{<>} \restriction \bar{\mathbf{a}}'_1 = g^+ \restriction \bar{\mathbf{a}}'_1$ hence $h_{<>} \cup g^+$ is an elementary embedding. Remember $g^+$ is the identity on $B^+ = ac\ell(N_{<>} \cup B)$, and $\mathrm{tp}_*(N_{<0>}, N^2_{<>})$ does not fork over $N_{<>}$ hence $\mathrm{tp}_*(N_{<0>}, B^+ \cup N^2_{<>})$ does not fork over $B^+$, so as $ac\ell(B^+) = B^+$ necessarily $(h_{<>} \cup g^+) \cup \mathrm{id}_{N_{<0>}}$ is an elementary embedding. But this mapping has domain and range including $N_{<0>} \cup N^2_{<>}$ and included in $N^2_{<0>}$, but the latter is $\aleph_\epsilon$-primary and $\aleph_\epsilon$-minimal over the former. Hence $(h_{<>} \cup g^+) \cup \mathrm{id}_{N_{<>}}$ can be extended to an automorphism of $N^2_{<0>}$ which we call $h_{<0>}$.

Now we define by induction on $n \in [2, \omega)$ for every $\eta \in I$ of length $n$, an automorphism $h_\eta$ of $N^2_\eta$ extending $h_{\eta^-} \cup \mathrm{id}_{N_\eta}$, which exists as $N^2_\eta$ is $\aleph_\epsilon$-primary over $N^2_{\eta^-} \cup N_\eta$ (and $N^2_{\eta^-} \underset{N_{\eta^-}}{\biguplus} N_\eta$). Now $\bigcup\limits_{\eta \in I} h_\eta$ is an elementary mapping (as $\langle N^2_\eta : \eta \in I \rangle$ is a non-forking tree; i.e. 1.13(10)), with domain and range $\bigcup\limits_{\eta \in I} N^2_\eta$ hence can be extended to an automorphism $h^*$ of $M'$, (we can demand $h^* \restriction M^- = \mathrm{id}_{M^-}$ but not necessarily). So as $h^*$ extends $g$, the conclusion follows.

3) Similarly to (2). $\square_{1.27}$

**1.28 Claim.** *1) For every $\Upsilon = \mathrm{tp}_\delta\left[\binom{B}{A}, M\right]$, and $\bar{\mathbf{a}}, \bar{\mathbf{b}}$ listing $A, B$ respectively there is $\psi = \psi(\bar{x}_A, \bar{x}_B) \in \mathbb{L}_{\infty, \aleph_\epsilon}(\mathrm{q.d.})$ of depth $\delta$ such that:*

$$\mathrm{tp}_\delta\left[\binom{B}{A}, M\right] = \Upsilon \Leftrightarrow M \models \psi[\bar{\mathbf{a}}, \bar{\mathbf{b}}].$$

*2) Assume $\bigotimes_{M_1, M_2}$ of 1.4 holds as exemplified by the family $\mathscr{F}$ and $\binom{B}{A} \in \Gamma(M_1)$ and $g \in \mathscr{F}, \mathrm{Dom}(g) = B$; and $\alpha$ an ordinal <u>then</u>*

$$\mathrm{tp}_\alpha\left(\binom{B}{A}, M\right) = \mathrm{tp}_\alpha\left(\binom{g(B)}{g(A)}, M_2\right).$$

*3) Similarly for $\mathrm{tp}_\alpha([A], M), \mathrm{tp}_\alpha[M]$.*

*Proof.* Straightforward (remember we assume that every first order formula is equivalent to a predicate). $\square_{1.28}$

*1.29 Proof of Theorem 1.2.* [The proof does not require that the $M^\ell$ are $\aleph_\epsilon$-saturated, but only that 1.27, 1.28 hold except in constructing $g_{\alpha(*)}$ (see $\otimes_{14}, \otimes_{15}$ in 1.30(E), we could instead use NOTOP].

So suppose

$(*)_0$ $M^1 \equiv_{\mathbb{L}_{\infty,\aleph_\epsilon}(\text{d.q.})} M^2$ or (at least) $\bigotimes_{M^1,M^2}$ from 1.4 holds.

We shall prove $M^1 \cong M^2$. By 1.28 (i.e. by 1.28(1) if the first possibility in $(*)_0$ holds and by 1.28(2) if the second possibility in $(*)_0$ holds)

$(*)_1$ $\text{tp}_\infty[M^1] = \text{tp}_\infty[M^2]$.

So it suffices to prove:

**1.30 Claim.** *Assume that $T$ is countable. If $M^1, M^2$ are $\aleph_\epsilon$-saturated models (of $T, T$ as in 1.5), then:*

$(*)_1 \Rightarrow M^1 \cong M^2$.

*Proof.* Let $\langle W_k, W'_k : k < \omega\rangle$ be a partition of $\omega$ to infinite sets (so pairwise disjoint).

1.31 Explanation: (If seems opaque, the reader may return to it after reading parts of the proof).

We shall now define an approximation to a decomposition. That is we are approximating a non-forking tree $\langle N^\ell_\eta, a^\ell_\eta : \eta \in I^*\rangle$ of countable elementary submodels of $M^\ell$ for $\ell = 1,2$ and $\langle f^*_\eta : \eta \in I^*\rangle$ such that $f^*_\eta$ an isomorphism from $N^1_\eta$ onto $N^2_\eta$ increasing with $\eta$ such that $M^\ell$ is $\aleph_\epsilon$-prime over $\bigcup\limits_{\eta \in I^*} N^\ell_\eta$.

In the approximation $Y$ we have:

- $(\alpha)$ $I$ approximating $I^*$,
  [it will not be $I^* \cap {}^{n\geq}\text{Ord}$ but we may "discover" more immediate successors to each $\eta \in I$; as the approximation to $N_\eta$ improves we have more regular types, but some member of $I$ will be later will drop]
- $(\beta)$ $A^\ell_\eta$ approximates $N^\ell_\eta$ and is $\epsilon$-finite
- $(\gamma)$ $a^\ell_\eta$ is the $a^\ell_\eta$ (if $\eta$ survives, i.e. will not be dropped)

($\delta$) $B^\ell_\eta, b^\ell_{\eta,m}$ expresses commitments on constructing $A^\ell_\eta$: we "promise" $B^\ell_\eta \subseteq N^\ell_\eta$ and $B^\ell_\eta$ is countable; $b^\ell_{\eta,m}$ for $m < \omega$ list $B^\ell_\eta$ (so in the choice $B^\ell_\eta \subseteq M^\ell$ there is some arbitrariness)

($\varepsilon$) $f_\eta$ approximate $f^*_\eta$

($\zeta$) $p^\ell_{\eta,m}$ also expresses commitments on the construction.

Since there are infinitely many commitments that we must meet in a construction of length $\omega$ and we would like many chances to meet each of them, the sets $W_k, W'_k$ are introduced as a further bookkeeping device. At stage $n$ in the construction we will deal e.g. with the $b^\ell_{\eta,m}$ for $\eta$ that are appropriate and for $m \in W_k$ for some $k < n$ and analogously for $p^\ell_{\eta,m}$ and the $W'_k$.

Note that while the $A^\ell_\eta$ satisfy the independence properties of a decomposition, the $B^\ell_\eta$ do not and may well intersect non-trivially. Nevertheless, a conflict arises if an $a^\ell_{\eta\hat{}<i>}$ falls into $B^\ell_\eta$ since the $a^\ell_{\eta\hat{}<i>}$ are supposed to represent independent elements realizing regular types over the model approximated by $A^\ell_\eta$ but now $a^\ell_{\eta\hat{}<i>}$ is in that model. This problem is addressed by pruning $\eta\hat{} < i >$ from the tree $I$.

**1.32 Definition.** An approximation $Y$ to an isomorphism consist of:

($a$) <u>natural numbers $n, k^*$ and index set</u>: $I \subseteq {}^{n\geq}\text{Ord}$ (and $n$ minimal)

($b$) $\langle A^\ell_\eta, B^\ell_\eta, a^\ell_\eta, b^\ell_{\eta,m} : \eta \in I \text{ and } m \in \bigcup\limits_{k<k^*} W_k\rangle$ for $\ell = 1, 2$ (this is an approximated decomposition)

($c$) $\langle f_\eta : \eta \in I\rangle$

($d$) $\langle p^\ell_{\eta,m} : \eta \in I \text{ and } m \in \bigcup\limits_{k<k^*} W'_k\rangle$

such that:

(1) $I$ closed under initial segments

(2) $<>\in I$

(3) $A^\ell_\eta \subseteq B^\ell_\eta \subseteq M^\ell, A^\ell_\eta$ is $\epsilon$-finite, $ac\ell(A^\ell_\eta) = A^\ell_\eta, B^\ell_\eta$ is countable, $B^\ell_\eta = \{b^\ell_{\eta,m} : m \in \bigcup\limits_{k<k^*} W_k\}$

(4) $A^\ell_\nu \subseteq A^\ell_\eta$ if $\nu \triangleleft \eta \in I$

(5) if $\eta \in I \backslash \{<>\}$, then $\frac{a^\ell_\eta}{A^\ell_{(\eta^-)}}$ is a (stationary) regular type and $a^\ell_\eta \in A^\ell_\eta$; if in addition $\ell g(\eta) > 1$ then $\frac{a^\ell_\eta}{A^\ell_{(\eta^-)}} \perp A^\ell_{(\eta^{--})}$ (note that we may decide $a^\ell_{<>}$ be not defined or $\in A^\ell_{<>}$)

(6) $\frac{A^\ell_\eta}{A^\ell_{\eta^-} + a_\eta} \perp_a A^\ell_{\eta^-}$ if $\eta \in I, \ell g(\eta) > 0$

(7) if $\eta \in I$, not $\triangleleft$-maximal in $I$, <u>then</u> the set $\{a^\ell_\nu : \nu \in I \text{ and } \nu^- = \eta\}$ is a maximal family of elements realizing over $A^\ell_\eta$ regular types $\perp A^\ell_{(\eta^-)}$ (when $\eta^-$ is defined), independent over $(A^\ell_\eta, B^\ell_\eta)$, (and we can add: if $\nu^-_1 = \nu^-_2 = \eta$ and $\frac{a^\ell_{\nu_1}}{A_\eta} \pm \frac{a^\ell_{\nu_2}}{A_\eta}$ then $a^\ell_{\nu_1}/A_\eta = a^\ell_{\nu_2}/A_\eta$)

(8) $f_\eta$ is an elementary map from $A^1_\eta$ onto $A^2_\eta$

(9) $f_{(\eta^-)} \subseteq f_\eta$ when $\eta \in I, \ell g(\eta) > 0$

(10) $f_\eta(a^1_\eta) = a^2_\eta$

(11) $(\alpha)$ $f_\eta\left(\mathrm{tp}_\infty\left[\left(\frac{A^1_\eta}{A^1_{(\eta^-)}}\right), M^1\right]\right) = \mathrm{tp}_\infty\left[\left(\frac{A^2_\eta}{A^2_{(\eta^-)}}\right), M^2\right]$ when $\eta \in I \backslash \{<>\}$

$(\beta)$ $f_{<>}\left(\mathrm{tp}_\infty\left[A^1_{<>}, M^1\right]\right) = \mathrm{tp}_\infty\left[A^2_{<>}, M^2\right]$

(12) $B^\ell_\eta \prec M^\ell$ moreover, $B^\ell_\eta \subseteq_{\mathrm{na}} M^\ell$, i.e., if $\bar a \subseteq N^\ell_\eta, b \in M^\ell \backslash B^\ell_\eta$ and $M^\ell \models \varphi(b, \bar a)$ there for some $b' \in B^\ell_\eta, \models \varphi(b', \bar a)$ and $b \notin acl(\bar a) \Rightarrow b' \notin acl(A)$

(13) $\langle p^\ell_{\eta,m} : m \in \bigcup_{k<k^*} W'_k \rangle$ is a sequence of types over $A^\ell_\eta$ (so $\mathrm{Dom}(p^\ell_{\eta,m})$ may be a proper subset of $A^\ell_\eta$).

*1.33 Notation.* We write $n = n_Y = n[Y], I = I_Y = I[Y], A^\ell_\eta = A^\ell_\eta[Y]$, $B^\ell_\eta = B^\ell_\eta[Y], f_\eta = f^Y_\eta = f_\eta[Y], a^\ell_\eta = a^\ell_\eta[Y], b^\ell_\eta = b^\ell_\eta[Y], k^* = k^*_Y = k^*[Y]$ and $p^\ell_{\eta,m} = p^\ell_{\eta,k}[Y]$.

*Remark.* We may decide to demand: each $\frac{a^\ell_{\eta\hat{}<i>}}{A_\eta}$ is strongly regular; also: if two such types are not orthogonal then they are equal (or at least have same witness $\varphi$ for $(\varphi, \frac{a_{\eta\hat{}<i>}}{A_\eta})$ regular). This is easy here as the models are $\aleph_\epsilon$-saturated (so take $p' \pm p$, $\mathrm{rk}(p')$ minimal).

*1.34 Observation.* $(*)_1$ implies that there is an approximation, (see 1.29).

*Proof.* Let $I = \{<>\}, A^\ell_{<>} = ac\ell(\emptyset), k^* = 1$ and then choose countable $B^\ell_{<>}$ to satisfy condition (12) and then choose $f_\eta, p^\ell_k, b^\ell_{\eta,m}$ (for $k \in W'_0$ and $m \in W_0$) as required.

*1.35 Main Fact.* For any approximation $Y, i \in \bigcup_{k<k^*_Y} (W_k \cup W'_k)$ and $m \le n_Y$ and $\ell(*) \in \{1,2\}$ we can find an approximation $Z$ such that:

$(\bigotimes)(\alpha)$ $n_Z = \text{Max}\{m+1, n_Y\}, I_Z \cap {}^{m\ge}\text{Ord} = I_Y \cap {}^{m\ge}\text{Ord}$, (we mean $m$ not $n_Y$) and $k^*_Z = k^*_Y + 1$

$(\beta)$ $(a)$ if $\eta \in I_Y, \ell g(\eta) < m$ then

$$A^\ell_\eta[Z] = A^\ell_\eta[Z],$$

$$a^\ell_\eta[Z] = a^\ell_\eta[Z]$$

$$B^\ell_\eta[Z] = B^\ell_\eta[Y]$$

$(b)$ if $\eta \in I_Y \cap I_Z, k < k^*_Y$ and $j \in W'_k$ then $p^\ell_{\eta,j}[Z] = p^\ell_{\eta,j}[Y]$

$(c)$ if $\eta \in I_Y \cap I_Z, k < k^*_Y$ and $j \in W_k$ then $b^\ell_{\eta,j}[Z] = b^\ell_{\eta,j}[Y]$

$(\gamma)^1$ <u>if</u> $\eta \in I_Y, \ell g(\eta) = m, k < k^*_Y$ and[4] $i \in W_k$ and the element $b \in M^{\ell(*)}$ satisfies clauses $(a), (b)$ below <u>then</u> for some such $b$ we have: $A^{\ell(*)}_\eta[Z] = ac\ell(A^{\ell(*)}_\eta[Y] \cup \{b\})$; where

$(a)$ $b^{\ell(*)}_{\eta,i}[Y] \notin A^{\ell(*)}_\eta[Y]$ and $\ell g(\eta) > 0 \Rightarrow \frac{b}{A^{\ell(*)}_\eta[Y]} \perp_a A^{\ell(*)}_{\eta^-}[Y]$

$(b)$ one of the conditions (i),(ii) listed below holds for $b$

$(i)$ $b = b^{\ell(*)}_{\eta,i}[Y]$ and $\ell g(\eta) > 0 \Rightarrow \frac{b}{A^{\ell(*)}_\eta[Y]} \perp_a A^{\ell(*)}_{\eta^-}[Y]$ <u>or</u>

$(ii)$ for no $b$ is $(i)$ satisfied (so $\ell g(\eta) > 0$) and $b \in M^{\ell(*)}$, $b^\ell_{\eta,i} \underset{A^{\ell(*)}_\eta[Y]}{\uplus} b$ and $\ell g(\eta) > 0 \Rightarrow \frac{b}{A^{\ell(*)}_\eta[Y]} \perp_a A^{\ell(*)}_{\eta^-}[Y]$

[4] recall that $i$ is part of the information given in the main fact, and of course, $k$ is uniquely determined by $i$

$(\gamma)^2$ assume $\eta \in I_Y, \ell g(\eta) = m, k < k^*_Y$ and $i \in W'_k$ then we have:

(a) $\underline{\text{if}}$ $p^{\ell(*)}_{\eta,i}$ is realized by some $b \in M^{\ell(*)}$ such that

$$\text{Rk}\left(\frac{b}{A^{\ell(*)}_\eta[Y]}, L, \infty\right) = \text{R}\left(p^{\ell(*)}_{\eta,i}, L, \infty\right) \text{ and}$$

$$\left[\ell g(\eta) > 0 \Rightarrow \frac{b}{A^{\ell(*)}_\eta[Y]} \perp_a A^{\ell(*)}_{\eta^-}[Y]\right]$$

$\underline{\text{then}}$ for some such $b$ we have

$$A^{\ell(*)}_\eta[Z] = ac\ell\left(A^{\ell(*)}_\eta[Y] \cup \{b\}\right)$$

(b) $\underline{\text{if}}$ the assumption of clause (a) fails but $p^{\ell(*)}_{\eta,i}$ is realized

by some $b \in M^{\ell(*)} \backslash A^{\ell(*)}_\eta$ such that

$$\left[\ell g(\eta) > 0 \Rightarrow \frac{b}{A^{\ell(*)}_\eta[Y]} \perp_a A^{\ell(*)}_{\eta^-}[Y]\right]$$

$\underline{\text{then}}$ for some such $b$ we have

$$A^{\ell(*)}_\eta[Z] = ac\ell\left[A^{\ell(*)}_\eta[Y] \cup \{b\}\right]$$

$(\delta)$ If $\eta \in I_Y$ and $\ell g(\eta) = m$, $\underline{\text{then}}$ $B^\ell_\eta[Z] = \{b^\ell_{\eta,j}[Y] : j \in \cup\{W_k : k < k^*_Z\}\}$ is a countable subset of $M^\ell$, containing $\{B^\ell_\nu[Z] : \nu \trianglelefteq \eta \text{ and } \nu \in Y\} \cup B^\ell_\eta[Y]$, with $B^\ell_\eta[Z] \prec M^\ell$ moreover $B^\ell_\eta[Z] \subseteq_{na} M^\ell$ i.e. if $\bar{a} \subseteq B^\ell_\eta[Z], \varphi(x,\bar{y})$ is first order and $(\exists x \in M^\ell \backslash ac\ell(\bar{a}))\, \varphi(x,\bar{a})$ then $(\exists x \in B^\ell_\eta[Z] \backslash ac\ell(\bar{a}))\varphi(\bar{x},\bar{a}))$ and $\{a^\ell_{\eta\hat{\ }<\alpha>}[Y] : \eta\hat{\ }\langle\alpha\rangle \in I_Y \text{ and } a^\ell_{\eta\hat{\ }<\alpha>}[Y] \notin B^\ell_\eta[Z]\}$ is independent over $(B^\ell_\eta[Z], A^\ell_\eta[Y])$

$(\epsilon)$ if $\eta \in I_Y, \ell g(\eta) > m$, then $\eta \in I_Z \Leftrightarrow a^\ell_{\eta\restriction(m+1)}[Y] \notin B^\ell_{\eta\restriction m}[Z]$

$(\zeta)$ if $\eta \in I_Y \cap I_Z, \ell g(\eta) > m$ then $A^\ell_\eta[Z] = ac\ell(A^\ell_\eta[Y] \cup A^\ell_{\eta\restriction m}[Z])$ and $B^\ell_\eta[Z] = B^\ell_\eta[Y]$

$(\eta)$ if $\eta \in I_Z \backslash I_Y$ then $\eta^- \in I_Y$ and $\ell g(\eta) = m+1$

$(\theta)$ $\{p^\ell_{\eta,i}[Z] : i \in W'_{k^*_Z-1}\}$ is "rich enough", e.g. include all finite types over $A^\ell_\eta$

$(\iota)$ $\{b^\ell_{\eta,i} : i \in W_{k^*_z-1}\}$ list $B^\ell_\eta[Z]$, each appearing infinitely often.

*Proof.* First we choose $A^{\ell(*)}_\eta[Z]$ for $\eta \in I$ of length $m$ according to condition $(\gamma) = (\gamma)^1 + (\gamma)^2$. (Note: one of the clauses $(\gamma)^1, (\gamma)^2$ necessarily holds trivially as $\bigcup\limits_k W_k \cap \bigcup\limits_k W'_k = \emptyset$).

Second, we choose (for such $\eta$) an elementary mapping $f^Z_\eta$ extending $f^Y_\eta$ and a set $A^{3-\ell(*)}_\eta[Z] \subseteq M^{3-\ell(*)}$ satisfying "$f^Z_\eta$ is from $A^1_\eta[Z]$ onto $A^{3-\ell(*)}_\eta[Z]$" such that

$(*)_2$ if $m > 0$, then
$$f^Z_\eta\big(\mathrm{tp}_\infty\big(\big(\tfrac{A^1_\eta[Z]}{A^1_{\eta^-}[Y]}\big), M_1\big)\big) = \mathrm{tp}_\infty\big(\big(\tfrac{A^2_\eta[Z]}{A^2_{\eta^-}[Y]}\big), M_2\big)$$

$(*)_3$ if $m = 0$, then $f^Z_\eta\left(\mathrm{tp}_\infty\left(A^1_\eta[Z], M_1\right)\right) = \mathrm{tp}_\infty\left(A^2_\eta[Z], M_2\right)$.

[Why possible? If we ask just the equality of $\mathrm{tp}_\alpha$ for an ordinal $\alpha$, this follows by the first component of $\mathrm{tp}_{\alpha+1}$. But (overshooting) for $\alpha \geq \left[(\|M_1\| + \|M_2\|)^{|T|}\right]^+$, equality of $\mathrm{tp}_\alpha$ implies equality of $\mathrm{tp}_\infty$].

Third, we choose $B^\ell_\eta[Z]$ for $\eta \in I_Y, \ell g(\eta) = m$ according to condition $(\delta)$ (here we use the countability of the language, you can do it by extending it $\omega$ times) in both sides, i.e. for $\ell = 1, 2$.

Fourth, let $I' = \{\eta \in I :$ if $\ell g(\eta) > m$ then $a^\ell_{\eta\restriction(m+1)}[Y] \notin B^\ell_{\eta\restriction m}[Z]\}$ (this will be $I_Y \cap I_Z$).

Fifth, we choose $A^\ell_\eta[Z]$ for $\eta \in I'$: if $\ell g(\eta) < m$, let $A^\ell_\eta[Z] = A^\ell_\eta[Y]$, if $\ell g(\eta) = m$ this was done, lastly if $\ell g(\eta) > m$, let $A^\ell_\eta[Z] = ac\ell\left(A^\ell_\eta[Y] \cup A^\ell_{\eta\restriction m}[Z]\right)$.

Sixth, by induction on $k \leq n_Y$ we choose $f^Z_\eta$ for $\eta \in I'$ of length $k$: if $\ell g(\eta) < m$, let $f^Z_\eta = f^Y_\eta$, if $\ell g(\eta) = m$ this was done, lastly if $\ell g(\eta) > m$ choose an elementary mapping from $A^1_\eta$ onto $A^2_\eta$ extending $f^Y_\eta \cup f^Z_{\eta^-}$ (possible as $f^Y_\eta \cup f^2_{\eta^-}$ is an elementary mapping and $\mathrm{Dom}(f^Y_\eta) \cap \mathrm{Dom}(f^Z_{\eta^-}) = A^{\ell(*)}_{\eta^-}$, $\mathrm{Dom}(f^Y_\eta) \underset{A^{\ell(*)}_{\eta^-}}{\bigcup\!\!\!\!\cup} \mathrm{Dom}(f^Z_{\eta^-})$ and $A^{\ell(*)}_{\eta^-} = ac\ell(A^{\ell(*)}_{\eta^-})$.) Now $f^Z_\eta$ satisfies clause (11) of Definition 1.32 when $\ell g(\eta) > m$ by applying 1.27(3).

Seventh, for $\eta \in I'$, of length $m < n_Z$, let $v_\eta =: \{\alpha : \eta\hat{\ }\langle\alpha\rangle \in I\}$, and we choose $\{a^1_{\eta\hat{\ }<\alpha>}[Z] : \alpha \in u_\eta\}, [\alpha \in u_\eta \Rightarrow \eta\hat{\ }\langle\alpha\rangle \notin I]$, a set of elements of $M^1$ realizing (stationary) regular types over $A^1_\eta[Z]$, orthogonal to $A_{\eta^-}[Y]$ when $\ell g(\eta) > 0$, such that it is independent over $\Big(\cup\{a^1_{\eta\hat{\ }<\alpha>}[Y] : \eta\hat{\ }\langle\alpha\rangle \in I'\} \cup B^1_\eta[Z], A^1_\eta[Z]\Big)$ and maximal under those restrictions. Without loss of generality $\sup(v_\eta) < \min(u_\eta)$ and for $\alpha_1 \in v_\eta \cup u_\eta$ and $\alpha_2 \in u_\eta$ we have:

$(*)_1$ if (for the given $\alpha_2$ and $\eta$) $\alpha_1$ is minimal such that $\frac{a^1_{\eta\hat{}<\alpha_1>}[Z]}{A^1_\eta[Z]} \pm \frac{a^1_{\eta\hat{}<\alpha_2>}[Z]}{A^1_\eta[Z]}$ then $\frac{a^1_{\eta\hat{}<\alpha_1>}[Z]}{A^1_\eta[Z]} = \frac{a^1_{\eta\hat{}<\alpha_2>}[Z]}{A^1_\eta[Z]}$

$(*)_2$ <u>if</u> $\alpha_1 < \alpha_{2_1}$ and $a^1_{\eta\hat{}\langle\alpha_1\rangle}[Z]/A^1_\eta[Z] = a^1_{\eta\hat{}\langle\alpha_2\rangle}[Z]/A^1_\eta[Z]$ and for some $b \in M^1$ realizing $\frac{a^1_{\eta\hat{}<\alpha_2>}[Z]}{A^1_\eta[Z]}$ we have

$b \underset{A^1_{\eta^\ell}[Z]}{\uplus} a^1_{\eta\hat{}<\alpha_2>}$ and

$\mathrm{tp}_\infty\left[\left(\frac{b}{A^1_{\eta\hat{}<\alpha_2>}}\right), M\right] = \mathrm{tp}_\infty\left[\left(\frac{a^1_{\eta\hat{}<\alpha_1>}}{A^1_\eta\hat{}<\alpha_2>}\right), M\right]$

and $\alpha_1$ is minimal (for the given $\alpha_2$ and $\eta$) <u>then</u>

$\mathrm{tp}_\infty\left[\left(\frac{a^1_{\eta\hat{}<\alpha_2>}}{A^1_{\eta\hat{}<\alpha_2>}}\right), M\right] = \mathrm{tp}_\infty\left[\left(\frac{a^1_{\eta\hat{}<\alpha_1>}}{A^1_{\eta\hat{}<\alpha_1>}}\right), M\right]$.

Easily (as in [Sh:c, X]) if $\alpha \in u_\eta$ and $\eta\hat{}\langle\beta\rangle \in I'$ then $\frac{a^1_{\eta\hat{}<\alpha>}[Z]}{A^1_\eta[Z]} \perp \frac{a^1_{\eta\hat{}<\beta>}[Y]}{A^1_\eta[Y]}$.

For $\alpha \in u_\eta$ let $A^1_{\eta\hat{}<\alpha>}[Z] = ac\ell\left(A^1_\eta[Y] \cup \{a^1_{\eta\hat{}<\alpha>}[Z]\}\right)$.

<u>Eighth</u>, by the second component in the definition of $\mathrm{tp}_{\alpha+1}$ (see Definition 1.9) we can choose (for $\alpha \in u_\eta$) $a^2_{\eta\hat{}<\alpha>}[Z], A^2_{\eta\hat{}<\alpha>}[Z]$ and then $f^Z_{\eta\hat{}<\alpha>}$ as required (see (7) of Definition 1.32).

<u>Ninth</u> and lastly, we let $I_Z = I' \cup \{\eta\hat{}<\alpha>: \eta \in I', \ell g(\eta) = m < n_Z$ and $\alpha \in u_\eta\}$ and we choose $B^\ell_\eta$ for $\eta \in I_Z \backslash I_Y$ and the $p^\ell_{\eta,i}, b^\ell_{\eta,j}$ as required (also in other case left).

$\square_{1.35}$

*1.36 Finishing the Proof of 1.11.* We define by induction on $n < \omega$ an approximation $Y_n = Y(n)$. Let $Y_0$ be the trivial one (as in observation 1.30(C)).

$Y_{n+1}$ is gotten from $Y_n$ as in 1.35 for $m_n, i_n \leq n, \ell_n(*) \in \{1,2\}$ defined by reasonable bookkeeping (so $i_n \in \bigcup_{k<k^*_{Y(n)}} (W_k \cup W'_k)$) such that any triples appear infinitely often; without loss of generality: if $n_1 < n_2$ & $\eta \in I^\ell_{n_1} \cap I^\ell_{n_2}$ then $\eta \in \bigcap_{n=n_1}^{n_2} I_n$.

Let $I^* = I[*] = \lim(I^{Y(n)}_\ell) =: \{\eta :$ for every large enough $n, \eta \in I_n\}$;

for $\eta \in I^*$ let: $A^\ell_\eta[*] = \bigcup_{n<\omega} A^\ell_\eta[Y_n], f^\ell_\eta[*] = \bigcup_{n<\omega} f^{Y(n)}_\eta$ and

$B^\ell_\eta[*] = \bigcup_{n<\omega} B^\ell_\eta[Y_n]$.

Easily

$\bigoplus_0$ $<>\in I^*$ and $I^* \subseteq {}^{\omega>}\mathrm{Ord}$ is closed under initial segments

$\bigoplus_1$ for $\eta \in I^*, \langle B^\ell_\eta[Y_n] : n < \omega$ and $\eta \in I[Y_n]\rangle$ is an increasing sequence of $\subseteq_{\mathrm{na}}$-elementary submodels of $M^\ell$
[Why? By clause (12) of Definition 1.32, Main Fact 1.35, clauses $(\beta)(a), (\delta), (\zeta)$.]

hence

$\bigoplus_2$ for $\eta \in I^*, B^\ell_\eta[*] \subseteq_{na} M^\ell$.

Also

$\bigoplus_3$ $\nu \triangleleft \eta \in I^* \Rightarrow B^\ell_\nu[*] \subseteq B^\ell_\eta[*]$.

[Why? Because for infinitely many $n, m_n = \ell g(\eta)$ and clause $(\delta)$ of Main Fact 1.35].

$\bigoplus_4$ if $\eta \in I[Y_{n_1}] \cap I^*, \eta^- = \nu$ and $n_1 \leq n_2$ <u>then</u>

$$A^\ell_\eta[Y_{n_1}] \underset{A^\ell_\nu[Y_{n_1}]}{\biguplus} A^\ell_\nu[Y_{n_2}].$$

[Why? Prove by induction on $n_2$ (using the non-forking calculus), for $n_2 = n_1$ this is trivial, so assume $n_2 > n_1$. If $m_{(n_2-1)} > \ell g(\nu)$ we have $A^\ell_\nu[Y_{n_2}] = A^\ell_\nu[Y_{n_2-1}]$ (see 1.35, clause $(\beta)(a)$ and we have nothing to prove. If $m_{(n_2-1)} < \ell g(\nu)$ then we note that $A^\ell_\nu[Y_{n_2}] = \mathrm{acl}(A^\ell_\nu[Y_{n_2-1}] \cup A^\ell_{\nu\restriction m_{(n_2-1)}}[Y_{n_2}])$ and $A^\ell_\nu[Y_{n_2-1}] \underset{A^\ell_{\nu\restriction m_{(n_2-1)}}}{\biguplus} A^\ell_{\nu\restriction m_{(n_2-1)}}[Y_{n_2}]$ (as $\nu \in I[Y_{n_2}]$, by 1.35 clause $(\delta)$ last phrase) and now use clauses (5), (6) of Definition 1.35. Lastly if $m_{(n_2-1)} = \ell g(\nu)$ again use $\nu \in I[Y_{n_2}]$ by 1.35, clause $(\delta)$, last phrase].

$\bigoplus_5$ if $\eta \in I[Y_{n_1}] \cap I^*, \eta^- = \nu$ and $n_1 \leq n_2$ <u>then</u>

$$\frac{A^\ell_\eta[*]}{A^\ell_\nu[*] + a^\ell_\eta[*]} \perp_a A^\ell_\nu.$$

[Why? By clause (6) of Definition 1.32, and orthogonality calculus].

$\bigoplus_6$ if $\eta \in I^*$, then $A^\ell_\eta[*] \subseteq B^\ell_\eta[*] \prec M^\ell$ moreover
$\bigotimes_7$ $A^\ell_{\eta(*)}[*] \subseteq_{\mathrm{na}} B^\ell_\eta[*] \subseteq_{\mathrm{na}} M^\ell$.

[Why? The second relation holds by $\bigotimes_2$. The first relation we prove by induction on $\ell g(\eta)$; clearly $A^\ell_\eta[*] = ac\ell(A^\ell_\eta[*])$ because $A^\ell_\eta[Y_n]$ increases with $n$ by 1.35 and $A^\ell_\eta[Y_n] = acl(A^\ell_\eta[Y_n])$ by clause (3) of Definition 1.32. We prove "$A^\ell_{\eta(*)}[*] \subseteq_{\rm na} B^\ell_\eta[*]$" by induction on $m = \ell g(\eta)$, so suppose this is true for every $m' < m, m = \ell g(\eta), \eta \in I^*$, let $\varphi(x)$ be a formula with parameters in $A^\ell_\eta[*]$ realized in $M^\ell$ as above say by $b \in M^\ell$. As $\langle A^\ell_\eta[Y_n] : n < \omega, \eta \in Y_n\rangle$ is increasing with union $A^\ell_\eta[*]$, clearly for some $n$ we have $b \underset{A^\ell_\eta[Y_n]}{\bigcup\!\!\!\!\!\!\bigcup} A^\ell_\eta[*]$.

So $\{\varphi(x)\} = p^\ell_{\eta,i}$ for some $i$ and for some $n' > n$ defining $Y_{n'+1}$ we have used 1.35 with $(\ell(*), i, m)$ there being $(\ell, i, \ell g(\eta))$ here, hence we consider clause $(\gamma)^2$ of 1.35. So the case left is when the assumption of both clauses (a) and (b) of $(\gamma)^2$ fail, so we have $\ell g(\eta) > 0$ and

$$b' \notin A^\ell_\eta[Y_{n'}], b' \in M^\ell \models \varphi[b'] \Rightarrow \frac{b'}{A^\ell_\eta[Y_{n'}]} \pm A^\ell_{\eta^-}[Y_{n'}].$$

We can now use the induction hypothesis (and [BeSh 307, 5.3,p.292]).]

$\bigoplus_8$ if $\eta \in I^*$ and $\ell = 1, 2$, <u>then</u>
$\{a^\ell_{\eta\char`^<\alpha>}[*] : \eta\char`^\langle\alpha\rangle \in I^*\}$ is a maximal subset of

$$\{c \in M_\ell : \frac{c}{A^\ell_\eta[*]} \text{ regular, } c \underset{A^\ell_\eta[*]}{\bigcup\!\!\!\!\!\!\bigcup} B^\ell_\eta[*] \text{ and } \ell g(\eta) > 0 \Rightarrow \frac{c}{A^\ell_\eta[*]} \perp A^\ell_{\eta^-}[*]\}$$

independent over $(A^\ell_\eta[*], B^\ell_\eta[*])$.

[Why? Note clause (7) of Definition 1.32 and clause $(\delta)$ of Main Fact 1.35].

$\bigotimes_9$ $A^\ell_{<>}[*] = B^\ell_{<>}[*]$

[Why? By the bookkeeping every $b \in B^\ell_{<>}[*]$ is considered for addition to $A^\ell_{<>}[*]$ see 1.35, clause $(\gamma)^1$, subclause (b)(i) and for $\langle\rangle$ there is nothing to stop us].

$\bigotimes_{10}$ if $\eta \in I^* \backslash <>$ and $p \in S(A^\ell_\eta[*])$ is regular orthogonal to $A^\ell_{\eta^-}[*]$ then $\frac{B^\ell_\eta[*]}{A^\ell_\eta[*]} \perp p$.

[Why? If not, as $A^\ell_\eta[*] \subseteq_{\rm na} B^\ell_\eta[*]$ by [BeSh 307, Th.B,p.277] there is $c \in B^\ell_\eta[*] \backslash A^\ell_\eta[*]$ such that: $\frac{c}{A^\ell_\eta[*]}$ is $p$. As $c \in B^\ell_\eta[*] = \bigcup\limits_{n<\omega} B^\ell_\eta[Y_n]$, for every $n < \omega$ large enough $c \in B^\ell_\eta[Y_n]$, and $p$ does not fork over $A^\ell_\eta[Y_n]$. So for some such $n$

the triple $(i_n, \ell_n, m_n)$ is such that $\ell_n = \ell, m_n = \ell g(\eta)$ and $b^\ell_{\eta, i_n} = c$, so by clause $(\gamma)^1$(b)(ii) of 1.35 we have $c \in A^\ell_\eta[Y_n] \subseteq A^\ell_\eta[*]$.]

$\bigotimes_{11}$ if $\eta \in I^*, \ell \in \{1,2\}$ <u>then</u> $\{a_{\eta\hat{}<\alpha>} : \eta\hat{}\langle\alpha\rangle \in I^*\}$ is a maximal subset of $\{c \in M^\ell : \frac{c}{A^\ell_\gamma[*]}$ regular, $\perp A^\ell_{\eta^-}[*]$ when meaningful$\}$ independent over $A^\ell_\eta[*]$.

[Why? If not, then for some $c \in M, \{a^\ell_{\eta\hat{}\langle\alpha\rangle} : \eta\hat{}\langle\alpha\rangle \in I^*\} \cup \{c\}$ is independent over $A^\ell_\eta[*]$ and $\text{tp}(c, A^\ell_\eta[*])$ is regular (and stationary). Hence by $\otimes_{10}$ we have $\{a^\ell_\eta[Y_n] : \eta\hat{}\langle\alpha\rangle \in I^*\} \cup \{c\}$ is independent over $(A^\ell_\eta[*], B^\ell_\eta[*])$. Now for large enough $n$ we have $c \underset{A^\ell_\eta[Y_n]}{\bigcup\!\!\!\!\!\!\smile} A^\ell_\eta[*]$ and by $\otimes_{10}$ we have $c \underset{A^\ell_\eta[Y_n]}{\bigcup\!\!\!\!\!\!\smile} B^\ell_\eta[*]$, hence $c \underset{A^\ell_\eta[*]}{\bigcup\!\!\!\!\!\!\smile} B^\ell_\eta[Y_n]$, and so by Definition 1.32(7) $\{c\} \cup \{a^\ell_{\eta\hat{}\langle\alpha\rangle}[Y_n] : \eta\hat{}\langle\alpha\rangle \in I[Y_n]\}$ is not independent over $(A^\ell_\eta[Y_n], B^\ell_\eta[Y_n])$, but $\{a^\ell_{\eta\hat{}\langle\alpha\rangle}[Y_n] : \eta\hat{}\langle\alpha\rangle \in I[Y_n]\}$ is independent over $(A^\ell_\eta[Y_n], B^\ell_\eta[Y_n])$. So there is a finite set $w$ of ordinals such that $\alpha \in w \Rightarrow \eta\hat{}\langle\alpha\rangle \in I[Y_n]$ and $\{c\} \cup \{a^\ell_{\eta\hat{}\langle\alpha\rangle}[Y_n] : \alpha \in w\}$ is not independent over $(A^\ell_\eta[Y_n], B^\ell_\eta[Y_n])$, and without loss of generality $w$ is minimal. Let $n_1 \in [n, \omega)$ be such that $\alpha \in w \;\;\&\;\; a^\ell_{\eta\hat{}<\alpha>} \in B^\ell_\eta[*] \Rightarrow a^\ell_\eta \in B^\ell_\eta[Y_{n_1}]$; clearly exist as $w$ is finite and let $u = \{\alpha \in w : a^\ell_{\eta\hat{}<\alpha>} \notin B^\ell_\eta[*]$; clearly $\alpha \in u \Rightarrow \eta\hat{} < \alpha > \in I^*$. Now $\{a^\ell_{\eta\hat{}\langle\alpha\rangle}[*] : \eta\hat{}\langle\alpha\rangle \in I^*\} \cup B^\ell_\eta[*]$ includes $\{a^\ell_{\eta\hat{}\langle\alpha\rangle}[Y_n] : \alpha \in w\}$, easy contradiction to the second sentence above.]

$\bigoplus_{12}$ $f^*_\eta = \bigcup\limits_{m<\omega} f_\eta[Y_n]$ (for $\eta \in I^*$) is an elementary map from $A^1_\eta[*]$ onto $A^2_\eta[*]$.

[Easy].

$\bigoplus_{13}$ $f^* =: \bigcup\limits_{\eta \in I^*} f^*_\eta$ is an elementary mapping from $\bigcup\limits_{\eta \in I^*} A^1_\eta[*]$ onto $\bigcup\limits_{\eta \in I^*} A^2_\eta[*]$.

[Clear using by $\otimes_5 + \otimes_6 + \otimes_{12}$ and non-forking calculus].

$\bigoplus_{14}$ We can find $\langle d^\ell_\alpha : \alpha < \alpha(*)\rangle$ such that:

$(a)$ $d^\ell_\alpha \in M^\ell, \beta < \alpha \Rightarrow d^\ell_\beta \neq d^\ell_\alpha$
$\text{tp}\big(d^\ell_\alpha, \bigcup\limits_{\eta \in I[*]} A^\ell_\eta[*] \cup \{d^\ell_\beta : \beta < \alpha\}\big)$ is $\aleph_\epsilon$-isolated and $\mathbf{F}^\ell_{\aleph_0}$-isolated, and

$(b)$ $g_\alpha = \bigcup\limits_{\eta \in I^*} f^*_\eta \cup \{\langle (d^1_\alpha, d^2_\alpha) : \alpha < \alpha(*)\rangle\}$ is an elementary mapping,

(c) $\alpha(*)$ is maximal, i.e., we cannot find $d^1_{\alpha(*)}$ such that the demand in (a) holds for $\alpha(*)+1$.

[Why? We can try to choose by induction on $\alpha$, a member $d^1_\alpha$ of $M^1 \backslash \bigcup\limits_{\eta \in I[*]} \cup \{d^1_\beta : \beta < \alpha\}$ such that $\text{tp}(d^1_\alpha, \bigcup\limits_{\eta \in I[*]} A^\ell_\eta[*] \cup \{d^1_\beta : \beta < \alpha\})$ is $\aleph_\varepsilon$-isolated and $\mathbf{F}^\ell_{\aleph_0}$-isolated. So for some $\alpha(*)$, $d^1_\alpha$ is well defined iff $\alpha < \alpha(*)$ (as $\beta < \alpha \Rightarrow d^1_\beta \neq d^1_\alpha \in M^1$). Now choose by induction on $\alpha < \alpha(*), d^2_\alpha \in M^2$ as required above, possible by "$M^2_i$ being $\aleph_\varepsilon$-saturated (see [Sh:c, XII,2.1,p.591], [Sh:c, IV,3.10,p.179].]

$\bigotimes_{15}$ $\text{Dom}(g_{\alpha(*)})$, $\text{Rang}(g_{\alpha(*)})$ are universes of elementary submodels of $M^1, M^2$ respectively called $M'_1, M'_2$ respectively.

[Why? See [Sh:c, XII,1.2](2),p.591 and the proof of $\otimes_{14}$.
Alternatively, choose a formula $\psi(x, \bar{a})$ such that:

(a) $\bar{a} \subseteq \text{Dom}(g_{\alpha(*)})$ and $\models \exists x \psi(x, \bar{a})$ but no $b \in \text{Dom}(g_{\alpha(*)})$ satisfy $\varphi(x, \bar{a})$

(b) under clause (a), $\text{Rk}(\psi(x, \bar{a}), \mathbb{L}_{\tau|T|}, \infty)$ is minimal
(or just has no extension in $S(\text{Dom}(g_{\alpha(*)}))$ forking over $\bar{a}$).

Let $\{\varphi_\ell(x, \bar{y}_\ell) : \ell < \omega\}$ list that $\mathbb{L}_{\tau(T)}$-formulas and we choose by induction on $\ell$ as formula $\psi_n(x, \bar{a}_n)$ such that:

(i) $\bar{a} \subseteq \text{Dom}(g_{\alpha(*)})$

(ii) $\models (\exists x)\psi_n(x, \bar{a}_n)$

(iii) $\psi_{n+1}(x, \bar{a}_{n+1}) \vdash \psi_n(x, \bar{a}_n)$

(iv) $\psi_0(x, \bar{a}_0) = \psi(x, \bar{a})$

(v) for any formula $\psi'(x, \bar{a}')$ satisfying the demands on $\psi_{n+1}(x, \bar{a}_{n+1})$ we have $\text{Rk}(\psi_{n+1}(x, \bar{a}_{n+1}), \{\varphi_n(x, \bar{y}_n)\}, 2) < \text{Rk}(\psi'(x, \bar{a}), \{\varphi_n(x, \bar{y})\}, 2)$
(on this rank see [Sh:c, II,§2]).

So $p = \{\psi_n(x, \bar{a}_n) : n < \omega\}$ has an extension in $S(\text{Dom}(g_{\alpha(*)}))$ call it $q$. Now $q$ is $\aleph_\varepsilon$-isolated because $\psi(x, \bar{a}) \in q \in S(\text{Dom}(g_{\alpha(*)})$. For every $n, \psi_{n+1}(x, \bar{a}_n) \vdash q \restriction \{\varphi_n(x, \bar{y}_n)\}$ by clause (v) above so as $\psi_{n+1}(x, \bar{a}_n) \in q$ and this holds for every $n$ clearly $q$ is $\mathbf{F}^\ell_{\aleph_0}$-isolated.

$\bigotimes_{16}$ If $M^\ell \neq M'_\ell$ then for some $d \in M_\ell \backslash M'_\ell$, $\frac{d}{M'_\ell}$ is regular.

[Why? By [BeSh 307, Th.5.9,p.298] as $N^\ell_\eta \subseteq_{\rm na} M^\ell$ by $\bigotimes_7$].

$\bigotimes_{17}$ if $M^\ell \ne M'_\ell$ then for some $\eta \in I^*$, there is $d \in M^\ell \backslash M'_\ell$ such that $\frac{d}{A^\ell_\eta[*]}$ is regular, $d \underset{A^\ell_\eta[*]}{\bigcup} M'_\ell$ and $\left[\ell g(\eta) > 0 \Rightarrow \frac{d}{A^\ell_\eta[*]} \perp A^\ell_{\eta^-}[*]\right]$.

[Why? By [Sh:c, XII,1.4,p.529] every non-algebraic $p \in S(M'_\ell)$ is not orthogonal to some $A^\ell_\eta[*]$ so by $\otimes_{16}$ we can choose $\eta \in I^*$ and $d \in M^\ell \backslash M'_\ell$ such that $\frac{d}{M'_\ell}$ is regular $\pm A^\ell_\eta[*]$; without loss of generality $\ell g(\eta)$ is minimal, now $A^\ell_\eta[*] \subseteq_{\rm na} M^\ell$ and by [BeSh 307, 4.5,p.290] without loss of generality $d \underset{A^\ell_\eta[*]}{\bigcup} M'_\ell$; the last clause is by "$\ell g(\eta)$ minimal"].

$\bigoplus_{18}$ $M_\ell = M'_\ell$.

[Why? By $\bigoplus_{11} + \bigoplus_{17}$].

$\bigoplus_{19}$ there is an isomorphism from $M_1$ onto $M_2$ extending $\bigcup\limits_{\eta \in I^*} f^*_\eta$.

[Why? By $\bigoplus_{14} + \bigotimes_{15}$ we have $M'_1 \cong M'_2$, so by $\otimes_{18}$ we are done]. $\square_{1.36}$ $\square_{1.30}$

**1.37 Lemma.** *Assume* $B \underset{A}{\bigcup} C, A = ac\ell(A) = B \cap C$ *and* $A, B, C$ *are* $\epsilon$*-finite,* $A \cup B \cup C \subseteq M, M$ *an* $\aleph_\epsilon$*-saturated model of* $T$*. For notational simplicity make* $A$ *a set of individual constants.*
*Then* $\text{tp}_{\mathbb{L}_{\infty,\aleph_\epsilon}(\text{d.q.})}(B + C; M) = \text{tp}_{\mathbb{L}_{\infty,\aleph_\epsilon}(d.q.)}(B; M) + \text{tp}_{\mathbb{L}_{\infty,\aleph_\epsilon}(d.q.)}[C; M]$
*where*

**1.38 Definition.** 1) For any logic $\mathbb{L}$ and $\bar b$ a sequence from a model $M$, let

$$\text{tp}_{\mathscr{L}}(\bar b; M) = \Big\{\varphi(\bar x) : M \models \varphi[B], \varphi \text{ a formula in the vocabulary of } M, \text{ from the logic } \mathscr{L} \text{ (with free variables from } \bar x, \text{ where } \bar x = \langle x_i : i < \ell g(\bar b)\rangle)\Big\}.$$

2) Replacing $\bar b$ by a set $B$ means we use the variables $\langle x_b : b \in B\rangle$.
3) Saying $p_1 = p_2 + p_3$ in 1.37 means that we can compute $p_1$ from $p_2$ and $p_3$ (and the knowledge how the variables fit and the knowledge of $T$, of course).

*Proof of the Lemma 1.37.*
It is enough to prove:

**1.39 Claim.** *Assume*

- $(a)$ $M^1, M^2$ *are* $\aleph_\epsilon$*-saturated and*
- $(b)$ $A^i_1 \underset{A^i_0}{\biguplus} A^i_2$ *for* $i = 1, 2$
- $(c)$ $A^i_0 = ac\ell(A^i_0)$ *and* $A^i_m$ *is* $\epsilon$*-finite for* $i = 1, 2$ *and* $m < 3$
- $(d)$ *for* $m = 0, 1, 2$ *we have* $f_m : A^1_m \xrightarrow{\text{onto}} A^2_m$ *is an elementary mapping preserving* $\text{tp}_\infty$ *(in* $M^1, M^2$ *respectively) and*
- $(e)$ $f_0 \subseteq f_1, f_2$.

Then *there is an isomorphism from* $M^1$ *onto* $M^2$ *extending* $f_1 \cup f_2$.

*Proof of 1.39.* Repeat the proof of 1.5, but starting with $Y_0$ such that $A^\ell_{<>}[Y_0] = A^\ell_0, A^\ell_{<>}[Y_0] = A^\ell_1, A^\ell_{<1>}[Y_0] = A^\ell_2, f^{Y_0}_{<>} = f_0, f^{Y_0}_{<0>} = f_1, f^{Y_0}_{<1>} = f_2$ and that $\langle\rangle, \langle 0\rangle, \langle 1\rangle$ belongs to all $I[Y_0]$. During the construction we preserve $\langle 0\rangle, \langle 1\rangle \in I[Y_n]$ and for helping to preserve this we add also the demand

$$\circledast_{2,m} \quad B^\ell_{<>}[Y_n] \underset{A^\ell_0}{\biguplus} A^\ell_1 \cup A^\ell_2.$$

During the proof, when we have to increase $B^\ell_{<>}$, we use 1.18(1) + 1.16(1). $\square_{1.39}$

Discussion: A natural version of 1.39 to say is the conclusion only that $\text{tp}_\alpha[\binom{A^1_a \cup A^1_2}{A^1_0}, M^1] = \text{tp}_\alpha[\binom{A^2_1 \cup A^2_2}{A^2_0}, M^2]$ and to prove this by induction on $\alpha$. The case $\alpha = 0$ and $\alpha$ limit are obvious. If $\alpha = \beta + 1$, for the condition of $\leq_{\rm a}$, we use the induction hypothesis and claim 1.27(1). The condition involving $\leq_{\rm b}$ is similar but harder. $\square_{1.39}$

## §2 Finer Types

We shall use here alternative types showing us probably a finer way to manipulate tp.

*2.1 Convention.* $T$ is superstable, NDOP
$M, N$ are $\aleph_\epsilon$-saturated $\prec \mathfrak{C}^{\rm eq}$.

**2.2 Definition.** $\Gamma_3 = \left\{ \binom{\bar b}{\bar a} : \bar a \subseteq \bar b \text{ are } \epsilon\text{-finite} \right\}$

$$\Gamma_1 = \left\{ \begin{pmatrix} p \\ \bar a \end{pmatrix} : \bar a \text{ is } \epsilon\text{-finite, } p \in S(\bar a) \text{ is regular (so stationary)} \right\}$$

$$\Gamma_2 = \left\{ \begin{pmatrix} p, r \\ \bar a \end{pmatrix} : \begin{array}{l} \bar a \text{ is } \epsilon\text{-finite, } p \text{ is a regular type of depth } > 0, \\ p \pm \bar a \text{ (really only the equivalence class } p/\pm \text{ matters),} \\ r = r(x, \bar y) \in S(\bar a) \text{ is such that for } (c, \bar b) \text{ realizing } r, \\ c/(\bar a + \bar b) \text{ is regular } \pm p, \text{ and } \dfrac{\bar b}{\bar a} = (r \restriction \bar y) \perp p \end{array} \right\}.$$

We may add (to $\Gamma_x$) superscripts:

($\alpha$) $f$ if $\bar a$ (or $\bar a \char`^ \bar b$) is finite

($\beta$) $s$ : for $\Gamma_3$ if $\frac{\bar b}{\bar a}$ is stationary, for $\Gamma_1$ if $p$ is stationary which holds always and for $\Gamma_2$ if $r$ is stationary and every automorphism of $\mathfrak{C}$ over $\bar a$ fix $p/\pm$

($\gamma$) $c$ if $\bar a$ (or $\bar a, \bar b$) are algebraically closed.

**2.3 Claim.** *If $p$ is regular of depth $> 0$ and $p \pm \bar a$ and $\bar a$ is $\epsilon$-finite <u>then</u> for some $\bar a', \bar a \subseteq \bar a' \subseteq ac\ell(\bar a)$ and for some $q$ <u>we have</u> $\binom{p,q}{\bar a'} \in \Gamma_2^s$.*

*Proof.* Use, e.g., [Sh:c, V,4.11,p.272], assume $\frac{\bar b}{\bar a} \pm p$; we can define inductively equivalence relations $E_n$, with parameters from $ac\ell(\bar a^\ell), \bar a^\ell = \bar a \char`^ (\bar b/E_0) \char`^ \cdots \char`^ (\bar b/E_{n-1})$, such that $\text{tp}(\bar b/E_n, ac\ell(\bar a^n))$ is semi-regular. By superstability this stop for some $n$ hence $\bar b \subseteq ac\ell(\bar a^n)$. For some first $m$ $\text{tp}(\bar b/E_m, ac\ell(\bar a^n))$ is $\pm p$, by [Sh:c, X,7.3](5),p.552 the type is regular (as because $p$ is trivial having depth $> 0$; see [Sh:c, X,7.2,p.551]). $\square_{2.3}$

**2.4 Definition.** We define by induction on an ordinal $\alpha$ the following (simultaneously): note — if a definition of something depends on another which is not well defined, neither is the something)

$$\mathrm{tp}^1_\alpha\big[\binom{p}{\bar a}, M\big] \qquad \text{for} \qquad \binom{p}{\bar a} \in \Gamma_1, \bar a \subseteq M$$

$$\mathrm{tp}^2_\alpha\big[\binom{p, r}{\bar a}, M\big] \qquad \text{for} \qquad \binom{p, r}{\bar a} \in \Gamma_1, \bar a \subseteq M$$

$$\mathrm{tp}^3_\alpha\big[\binom{\bar b}{\bar a}, M\big] \qquad \text{for} \qquad \binom{\bar b}{\bar a} \in \Gamma^c_3, \bar a \subseteq \bar b \subseteq M$$

<u>Case A</u> <u>$\alpha = 0$</u>: $\mathrm{tp}^1_\alpha\left[\binom{p}{\bar a}, M\right]$ is $\mathrm{tp}((c, \bar a), \emptyset)$ for any $c$ realizing $p$.

$$tp^2_\alpha\big[\binom{p, r}{\bar a}, M\big] \text{ is } \mathrm{tp}((c, \bar b, \bar a), \emptyset) \text{ for any } (c, \bar b) \text{ realizing } r$$

$$\mathrm{tp}^3_\alpha\big[\binom{\bar b}{\bar a}, M\big] \text{ is } \mathrm{tp}((\bar b, \bar a), \emptyset)$$

(i.e., the type and the division of the variables between the sequences).

<u>Case B</u> <u>$\alpha = \beta + 1$</u>:

(a) $\mathrm{tp}^1_\alpha\left[\binom{p}{\bar a}, M\right]$ is:

<u>Subcase a1</u>: <u>if</u> $p$ has depth zero, it is $w_p(M/\bar a)$ (the $p$-weight, equivalently, the dimension)

<u>Subcase a2</u>: <u>if</u> $p$ has depth $> 0$ (hence is trivial), then it is $\{\langle y, \lambda^y_{\bar a, p}\rangle : y\}$ where

$$\begin{aligned} \lambda^y_{\bar a, p} = \dim(\mathbf{I}^y_{\bar a, p}[M], a) \text{ where } \mathbf{I}^y_{\bar a, p}[M] = \\ \Big\{ c \in M : c \text{ realize } p \text{ and } y = \ \mathrm{tp}^3_\beta \left[ \binom{ac\ell(\bar a + c)}{ac\ell(\bar a)}, M \right] \\ \text{where } \bar a^* \text{ list } acl(\bar a) \text{ and } \bar c^* \text{ list } acl(\bar a + c) \Big\} \end{aligned}$$

an alternative probably more transparent and simpler in use is:

$$\lambda^y_{\bar a,p} = \ \dim\Big\{c \in M : c \text{ realizes } p \text{ and}$$

$$y = \{\mathrm{tp}^3_\beta\left[\binom{ac\ell(\bar a + c')}{ac\ell(\bar a)}, M\right] : c' \in p(M) \text{ and } c' \underset{\bar a}{\bigcup} c\}$$

pedantically $y = \{\mathrm{tp}^3_\beta\binom{< c' > \hat{}\, \bar a^* \hat{}\, \bar c^*}{\bar a \hat{}\, \bar a^*}, M]$; where

$\bar a^*$ list $acl(\bar a)$ and

$\bar c^*$ list $acl(\bar a + c'), c' \in p(M)$ and $c' \underset{\bar a}{\bigcup} c\}\Big\}$.

(b) $\mathrm{tp}^2_\alpha\left[\binom{p,r}{\bar a}, M\right]$ is:

$\mathrm{tp}^1_\alpha\left[\binom{c/\bar b^+}{\bar b^+}, M\right]$ for any $(c,\bar b)$ realizing $r, \bar b^+ = ac\ell(\bar a + \bar b)$, i.e., $\bar b^+$ lists $acl(\bar a + \bar b)$ (so not well defined if we get at least two different cases; so remember $c/b^+ \in S(\bar b^+)$).

(c) $\mathrm{tp}^3_\alpha\left[\binom{\bar b}{\bar a}, M\right]$ is $\left\{\langle p, \mathrm{tp}^2_\alpha\left[\binom{p,r}{\bar b}, M\right]\rangle : \binom{p,r}{\bar b} \in \Gamma^s_2 \text{ and } p \perp \bar a\right\}$.

Case C: $\alpha$ limit: For any $\ell \in \{1,2,3\}$ and suitable object OB:

$$\mathrm{tp}^\ell_\alpha[OB, M] = \langle \mathrm{tp}^\ell_\beta[OB, M] : \beta < \alpha\rangle.$$

**2.5 Definition.** 1) For $\binom{p}{\bar a} \in \Gamma_1$ where $\bar a \in M$, let (remembering 1.14(8)):

$$\mathscr{P}^M_{\binom{p}{\bar a}} = \Big\{q \in S(M) : q \text{ regular and} : q \pm p \text{ or for some}$$

$$c \in p(M) \text{ we have } q \in \mathscr{P}^M_{\binom{c}{\bar a}}\Big\}.$$

2) For $\binom{p,r}{\bar a} \in \Gamma_2$ let

$$\mathscr{P}^M_{\binom{p,r}{\bar a}} = \Big\{q \in S(M) : q \text{ regular and} : q \pm p \text{ or for some}$$

$$(c,\bar b) \in r(M), q \in \mathscr{P}^M_{\binom{c}{\bar a+\bar b}}\Big\}.$$

3) For a set $\mathscr{P}$ of (stationary) regular types not orthogonal to $M_1$, let $M_1 \leq_{\mathscr{P}} M_2$ means $M_1 \prec M_2$ and for every $p \in \mathscr{P}$ and $\bar{c} \in M_2$, $\frac{\bar{c}}{M_1} \perp p$.
4) If (in (3)), $\mathscr{P} = \mathscr{P}^{M_1}_{\binom{p}{\bar{a}}}$ we may write $\binom{p}{\bar{a}}$ instead $\mathscr{P}$, similarly if $\mathscr{P} = \mathscr{P}^{M_1}_{\binom{p,r}{\bar{a}}}$ we may write $\binom{p,r}{\bar{a}}$.

**2.6 Claim.** *:*
*1) From* $\mathrm{tp}^1_\alpha\big[\binom{p}{\bar{a}}, M\big]$ *we can compute* $\mathrm{tp}^1_\infty\big[\binom{p}{\bar{a}}, M\big]$ *if* $\mathrm{Dp}(p) < \alpha$.

*2) From* $\mathrm{tp}^2_\alpha\big[\binom{p,q}{\bar{a}}, M\big]$ *we can compute* $\mathrm{tp}^2_\infty\big[\binom{p,q}{\bar{a}}, M\big]$ *if* $\mathrm{Dp}(p) < \alpha$.

*3) From* $\mathrm{tp}^3_\alpha\big[\binom{\bar{b}}{\bar{a}}, M\big]$ *we can compute* $\mathrm{tp}^3_\infty\big[\binom{b}{\bar{a}}, M\big]$ *if* $\mathrm{Dp}(\bar{b}/\bar{a}) < \alpha$.

*4) If Definition 2.5(2) we can replace "some* $(c,\bar{b}) \in r(M)$*" by "every* $(c,\bar{b}) \in r(M)$*".*

*Proof.* 1),2),3) We prove this by induction on $\alpha$. By the definition.
4) Left to the reader.

*2.7 Observation.* From $\mathrm{tp}^\ell_\alpha(OB, M)$ we can compute $\mathrm{tp}^\ell_\beta[OB, M]$ and $\mathrm{tp}^\ell_\beta[OB, M]$ is well defined if $\beta \leq \alpha$ and the former is well defined.

**2.8 Lemma.** *For every ordinal* $\alpha$ *the following holds:*

(1) $\mathrm{tp}^1_\alpha$ *is well defined*[5]
(2) $\mathrm{tp}^2_\alpha$ *is well defined.*
(3) $\mathrm{tp}^3_\alpha$ *is well defined.*
(4) *If* $\bar{a} \in M_1$, $\binom{p}{\bar{a}} \in \Gamma_1$, $M_1 \leq_{\binom{p}{\bar{a}}} M_2$ *then*
$\mathrm{tp}^1_\alpha\big[\binom{p}{\bar{a}}, M_1\big] = \mathrm{tp}^1_\alpha\big[\binom{p}{\bar{a}}, M_2\big]$.
(5) *If* $\bar{a} \in M_1$, $\binom{p,r}{\bar{a}} \in \Gamma^s_2$, $M_1 \leq_{\binom{p}{\bar{a}}} M_2$ *then*
$\mathrm{tp}^2_\alpha\big[\binom{p,r}{\bar{a}}, M_1\big] = \mathrm{tp}^2_\alpha\big[\binom{p,r}{\bar{a}}, M_2\big]$.
(6) *If* $\bar{a} \subseteq \bar{b} \subseteq M_1$, $\binom{\bar{b}}{\bar{a}} \in \Gamma^c_3$, $M_1 \leq_{\binom{\bar{b}}{\bar{a}}} M_2$ *then*
$\mathrm{tp}^3_\alpha\Big[\binom{\bar{b}}{\bar{a}}, M_1\Big] = \mathrm{tp}^3_\alpha\Big[\binom{\bar{b}}{\bar{a}}, M_2\Big]$.

[5] i.e. in all the cases we have tried to define it in Definition 2.9

*Proof.* We prove it, by induction on $\alpha$, simultaneously (for all clauses and parameters).

If $\alpha$ is zero, they hold trivially by the definition.

If $\alpha$ is limit, they hold trivially by the definition and induction hypothesis. So for the rest of the proof let $\alpha = \beta + 1$.

*Proof of* $(1)_\alpha$. If $p$ has depth zero — check directly.

If $p$ has depth $> 0$ - by $(3)_\beta$

(i.e. induction hypothesis) no problem.

*Proof of* $(2)_\alpha$. Like 1.27 (and $(4)_\alpha$).

*Proof of* $(3)_\alpha$. Like $(2)_\alpha$.

*Proof of* $(4)_\alpha$. Like 1.26 (and $(3)_\beta, (6)_\beta$).

*Proof of* $(5)_\alpha$. By $(2)_\alpha$ we can look only at $(c, \bar{b}^+)$ in $M_1$, then use $(4)_\alpha$.

*Proof of* $(6)_\alpha$. By $(5)_\alpha$. $\square_{2.8}$



**2.9 Lemma.** *For an ordinal $\alpha$ restricting ourselves to the cases (the types $p, p_1$ being) of depth $< \alpha$:*

- $(A1)$ *Assume $\binom{p}{\bar{a}} \in \Gamma_1, \bar{a} \subseteq \bar{a}_1 \subseteq M, \bar{a}_1$ is $\epsilon$-finite, $\frac{\bar{a}_1}{\bar{a}} \perp p$ and $p_1$ is the stationarization of $p$ over $\bar{a}_1$.*
  *Then from $\mathrm{tp}^1_\alpha\left[\binom{p}{\bar{a}}, M\right]$ we can compute $\mathrm{tp}^1_\alpha\left[\binom{p_1}{\bar{a}_1}, M\right]$.*
- $(A2)$ *Under the assumption of $(A1)$ also the inverse computations are O.K.*
- $(A3)$ *Assume $\binom{p_\ell}{\bar{a}} \in \Gamma_1$ for $\ell = 1, 2, \bar{a} \subseteq M$ and $p_1 \pm p_2$.*
  *Then from $\mathrm{tp}^1_\alpha\left[\binom{p_1}{\bar{a}}, M\right]$ (and $\mathrm{tp}((\bar{a}, c_1, c_2), \emptyset)$ where $c_1, c_2$ realizes $p_1, p_2$ respectively, of course) we can compute $\mathrm{tp}^1_\alpha\left[\binom{p_2}{\bar{a}}, M\right]$.*
- $(B1)$ *Assume $\binom{p_\ell, r_\ell}{\bar{a}} \in \Gamma_2^{sc}$ for $\ell = 1, 2, \bar{a} \in M$ and $p_1 \pm p_2$.*
  *Then (from the first order information on $\bar{a}, p_1, p_2, r_1, r_2$, of course, and) $\mathrm{tp}^2_\alpha\left[\binom{p_1, r_2}{\bar{a}_1}, M\right]$ we can compute $\mathrm{tp}^2_\alpha\left[\binom{p_2, r_2}{\bar{a}}, M\right]$.*
- $(B2)$ *Assume $\bar{a} \subseteq \bar{a}_1 \subseteq M, \frac{\bar{a}_1}{\bar{a}} \perp p, \binom{p, r}{\bar{a}} \in \Gamma_2^{cs}, r \subseteq r_1 \in S(\bar{a}_1), r_1$ does not fork over $\bar{a}$, ( so $\binom{p, r_1}{\bar{a}_1} \in \Gamma_2$).*
  *Then from $\mathrm{tp}^2_\alpha\left[\binom{p, r_1}{\bar{a}}, M\right]$ we can compute $\mathrm{tp}^2_\alpha\left[\binom{p, r_2}{\bar{a}}, M\right]$.*

(*B*3) *Under the assumption of* (*B*2)*, the inverse computation are O.K.*

(*C*1) *Assume* $\binom{\bar b}{\bar a} \in \Gamma_3^c, \bar a \subseteq \bar b \subseteq M, \bar a \subseteq \bar a_1, \bar b \bigcup_{\bar a} \bar a_1, \bar b_1 = ac\ell(\bar a_1 + \bar b)$.

<u>*Then*</u> *from* $\mathrm{tp}^3_\alpha\big[\binom{\bar b}{\bar a}, M\big]$ *we can compute* $\mathrm{tp}^3_\alpha\big[\binom{\bar b_1}{\bar a_1}, M\big]$.

(*C*2) *Under the assumptions of* (*C*1) *the inverse computation is 0.K.*

(*C*3) *Assume* $\binom{\bar b}{\bar a} \in \Gamma_3, \bar b \subseteq b^*, \frac{\bar b^*}{\bar b} \perp_a \bar a, \bar b^* = ac\ell(\bar b^*)$. <u>*Then*</u> *from* $\mathrm{tp}^3_\alpha\big[\binom{\bar b}{\bar a}, M\big]$ *we can compute*

$\left\{ \mathrm{tp}^3_\alpha \left[\binom{\bar b'}{\bar a}, M\right] : \bar b \subseteq b' \subseteq M \text{ and } \frac{\bar b'}{\bar b} = \frac{\bar b^*}{\bar b} \right\}$.

*Proof.* We prove it, simultaneously, for all clauses and parameters, by induction on $\alpha$ and the order of the clauses.

For $\alpha = 0$: easy.

For $\alpha$ limit: very easy.

So assume $\alpha = \beta + 1$.

*Proof of* $(A1)_\alpha$. As $p$ is stationary $\perp \frac{\bar a_1}{\bar a}$, for every $c \in p(M), \frac{c}{\bar a} \vdash \frac{c}{\bar a_1}$, which necessarily is $p_1$, hence $p(M) = p_1(M)$. Also the dependency relation on $p(M)$ is the same over $\bar a_1$, hence dimension. So it suffices to show:

(∗) for $c \in p(M)$, from $\mathrm{tp}^3_\beta\big[\binom{ac\ell(\bar a + c)}{ac\ell\, \bar a}, M\big]$ we can compute $\mathrm{tp}^3_\beta\big[\binom{ac\ell(\bar a_1 + c)}{ac\ell\, \bar a_1}, M\big]$.

But this holds by $(C1)_\beta$.

*Proof of* $(A2)_\alpha$. Similar using $(C2)_\beta$.

*Proof of* $(A3)_\alpha$. If $p_1$ (equivalently $p_2$) has depth zero — the dimensions are equal. Assume they have depth $> 0$ hence are trivial and dependency over $\bar a$ is an equivalence relation on $p_1(M) \cup p_2(M)$.

Now for $c_1 \in p_1(M)$, from $\mathrm{tp}^3_\beta\big[\binom{ac\ell(a+c_1)}{ac\ell(\bar a)}, M\big]$ we can compute for every complete type over $ac\ell(\bar a + c_1)$ not forking over $\bar a$, and $\bar d$ realizing $r, \mathrm{tp}^3_\beta\big[\binom{ac\ell(\bar a + \bar d + c_1)}{ac\ell(\bar a + \bar d)}, M\big]$ - by $(C1)_\beta$, then we can compute for each such $r, \bar d$,

$$\Big\{\mathrm{tp}^3_\beta\left[\begin{pmatrix} ac\ell(\bar a+\bar d+c_2) \\ ac\ell(\bar a+\bar d)\end{pmatrix}, M\right] : c_2\in p_2(M) \text{ and } \frac{c_2}{ac\ell(\bar a+\bar d+c_1)}\perp_a(\bar a+\bar d)$$
$$\text{(necessarily } c_2 \bigcup_{\bar a}\bar d)\Big\}$$

(this by $(C3)_\beta$).

*Proof of* $(B1)_\alpha$. As in earlier cases we can restrict ourselves to the case $\mathrm{Dp}(p_\ell)>0$. We can find $(c_\ell,\bar b_\ell)\in r_\ell(M), \bar b_1\bigcup_{\bar a}\bar b_2, c_1\bar b_1\bigcup_{\bar a}\bar b_2$ (by [Sh:c, X,7.3](6)]. By 2.8(2) (and the definition) from $\mathrm{tp}^2_\alpha\big[\binom{p_1,r_1}{\bar a},M\big]$ we can compute that it is equal to $\mathrm{tp}^1_\alpha\big[\binom{c_1/ac\ell(\bar a+\bar b_1)}{ac\ell(\bar a+b_1)},M\big]$.

By $(A1)_\alpha$ we can compute $\mathrm{tp}^1_\alpha\big[\binom{c_1/ac\ell(\bar a+\bar b_1+\bar b_2)}{ac\ell(\bar a+b_1+\bar b_2)},M\big]$ hence by $(A3)_\alpha$ we can compute $\mathrm{tp}^1_\alpha\big[\binom{c_2/ac\ell(\bar a+\bar b_1+\bar b_2)}{ac\ell(\bar a+\bar b_1+\bar b_2)},M\big]$.

Now use $(A2)_\alpha$ to compute $\mathrm{tp}^1_\alpha\big[\binom{c_2/ac\ell(\bar a+\bar b_2)}{ac\ell(\bar a+\bar b_2)},M\big]$ and by 2.8(2), 2.4(2) it is equal to $\mathrm{tp}^2_\alpha\big[\binom{p,r}{\bar a},M\big]$.

*Proof of* $(B2)_\alpha$. Choose $(c,\bar b)\in r(M)$ such that $c\bar b\bigcup_{\bar a}\bar a_1$.

From $\mathrm{tp}^2_\alpha\big[\binom{p,r_1}{\bar a},M\big]$ we can compute $\mathrm{tp}^1_\alpha\big[\binom{c/(\bar a+\bar b)}{\bar a+\bar b},M\big]$ (just — see 2.8(2) and Definition 2.4), from it we can compute $\mathrm{tp}^1_\alpha\big[\binom{c/(\bar a+\bar b+\bar a_1)}{(\bar a+\bar b+\bar a_1)},M\big]$ (by $(A1)_\alpha$), from it we can compute $\mathrm{tp}^2_\alpha\big[\binom{p,r_2}{\bar a_2},M\big]$ (see 2.8(2) and Definition 2.4).

*Proof of* $(B3)_\alpha$. Let $\binom{p,r}{\bar b_1}\in\Gamma^s_r, p\perp\bar a_1$ be given. So necessarily $\frac{\bar a_1}{\bar a}\pm p$ (this to enable us to use $(B2,3)$. It suffices to compute $\mathrm{tp}^2_\alpha\big[\binom{p,r}{\bar b_1},M\big]$ and we can discard the case $\mathrm{Dp}(p)=0$.

So $p$ is regular $\pm\bar b_1,\perp\bar a_1$, hence $p\pm\bar b, p\perp\bar a$, and as $\bar a\subseteq\bar b, \bar b=ac\ell(\bar b)$ we can find $r$, $\binom{p,r_1}{\bar b}\in\Gamma_2$, (see 2.3) and we know $\mathrm{tp}^2_\alpha\big[\binom{p,r_1}{\bar b},M\big]$, and we can find $r_2$, a complete type over $\bar b_1$ extending $r_1$ which does not fork over $\bar b_1$. From $\mathrm{tp}^2_\alpha\big[\binom{p,r_1}{\bar b},M\big]$ we can compute $\mathrm{tp}^2_\alpha\big[\binom{p,r_2}{\bar b_1},M\big]$ by $(B2)_\alpha$, and from it $\mathrm{tp}^2_\alpha\big[\binom{p,r}{\bar b_1},M\big]$ by $(B1)_\alpha$.

*Proof of* $(C2)_\alpha$. Similar, use $(B3)_\alpha$ instead of $(B2)_\alpha$.

*Proof of* $(C3)_\alpha$. Without loss of generality $\frac{\bar{b}^*}{\bar{b}}$ is semi regular, let $p^*$ be a regular type not orthogonal to it and without loss of generality $\mathrm{Dp}(p^*) > 0 \Rightarrow \frac{\bar{b}^*}{\bar{b}}$ regular (as in 2.3).

If $p^*$ has depth zero, then the only $p$ appearing in the definition $\mathrm{tp}^3_\alpha\big(\big[\frac{\bar{b}}{\bar{a}}\big], M\big)$ is $p^*$ (up to $\pm$) and this is easy. Then $\mathrm{tp}^2_\alpha$ is just the dimension and we have no problem.

So assume $p^*$ has depth $> 0$. We can by $(B1)_\alpha, (B2)_\alpha$ compute $\mathrm{tp}^2_\alpha\big[\big(\substack{p',q'\\ \bar{b}^*}\big), M\big]$ when $p' \pm \bar{b}, p' \pm p^*$ (regardless of the choice of $\bar{b}^*$). Next assume $p' \pm p^*$; by $(B1)_\alpha$ without loss of generality $q'$ does not fork over $\bar{b}$. As $\mathrm{Dp}(p^*) > 0$, it is trivial (and we assume $w_p(\bar{b}^*, \bar{b}) = 1$) hence $\bar{b}^*/\bar{b}$ is regular so in $\mathrm{tp}^2_\alpha\big[\big(\substack{p,q'\\ \bar{b}^*}\big), M\big]$ we just lose a weight 1 for one specific $\mathrm{tp}^3_\beta$ type: the one $\bar{b}^*$ realizes concerning which we have a free choice. We are left with the cases $p' \pm \bar{b}, p' \pm p^*$; well we know $\mathrm{tp}^3_\beta$ but we have to add $\mathrm{tp}^3_\alpha$? Use Claim 2.6(3) (and $(A1)_\alpha$ as we add a parameter). $\square_{2.9}$

**2.10 Claim.** $\mathrm{tp}^3_\gamma\big[\binom{\bar{b}}{\bar{a}}, M\big], \mathrm{tp}^3_\gamma[\bar{a}, M], \mathrm{tp}^3_\gamma[M]$ *are expressible by formulas in* $\mathbb{L}^\gamma_{\infty,\aleph_\epsilon}$(d.q.).

By 2.9 we have

*2.11 Conclusion.* If $\mathrm{Dp}(T) < \infty$ then:
1) From $\mathrm{tp}^3_\infty\big[\binom{B}{A}, M\big]$ we can compute $\mathrm{tp}_\infty\big[\binom{B}{A}, M\big]$ (the type from §1).
2) Similarly from $\mathrm{tp}^3_\infty[A, M]$ we can compute $\mathrm{tp}_\infty[(A), M]$.

From 2.6, 2.10, 2.11 and 1.30 we get

**2.12 Corollary.** *If* $\gamma = \mathrm{Dp}(T)$ *and* $M, N$ *are* $\aleph_\epsilon$*-saturated,* <u>*then*</u>

$$M \cong N \Leftrightarrow \mathrm{tp}^3_\gamma[M] = \mathrm{tp}^3_\gamma[N] \Leftrightarrow M \cong_{\mathbb{L}^\gamma_{\infty,\aleph_\epsilon}(d.q.)} N.$$

## Appendix

The following clarifies several issues raised by Baldwin. A consequence of

$\otimes$ the existence of nice invariants for characterization up to isomorphism (or characterization of the models up to isomorphism by their $\mathscr{L}$-theory for suitable logic $\mathscr{L}$)

naturally give absoluteness, e.g. extending the universe say by nice forcing preserve non-isomorphism. So negative results for

$(*)$ is non-isomorphism (of models of $T$) preserved by forcing by "nice forcing notions"?

implies that we cannot characterize models up to isomorphism by their $\mathscr{L}$-theory when the logic $\mathscr{L}$ is "nice", i.e. when $Th_{\mathscr{L}}(M)$ preserved by nice forcing notions. So coding a stationary set by the isomorphism type can be interpreted as strong evidence of "no nice invariants", see [Sh 220]. Baldwin, Laskowski, Shelah [BLSh 464] show that not only for every unsuperstable; but also for some quite trivial superstable (with NDOP, NOTOP) countable $T$, there are non-isomorphic models which can be made isomorphic by some ccc (even $\sigma$-centered) forcing notion. This shows that the lack of a really finite characterization is serious.
Can we still get from the characterization in this paper an absoluteness result? Note that for preserving $\aleph_\epsilon$-saturation (for simplicity for models of countable $T$) we need to add no reals[6], and in order not to erase distinction of dimensions we want not to collapse cardinals, so the following questions is natural, for a first order (countable) complete $T$:

$(*)_T$ assume $\mathbf{V}_1 \subseteq \mathbf{V}_2$ are transitive models of ZFC with the same cardinals and reals, the theory $T \in \mathbf{V}_1$. If the models $M_1, M_2$ are from $\mathbf{V}_1$ and they are models of $T$ not isomorphic in $\mathbf{V}_1$; must they still be not isomorphic in $V_2$[7]

$(*)_{T,\kappa}$ like $(*)^1_T$ we assume in addition $\mathscr{P}(\kappa)^{\mathbf{V}_1} = \mathscr{P}(\kappa)^{\mathbf{V}_2}$.

**2.13 Theorem.** *1) For countable first order complete $T$ the answer to $(*)_T$ and $(*)_{T,\kappa}$ for any $\kappa$ is negative except when possibly $T$ is superstable, NDOP, NOTOP.*
*2) For any first order complete $T$ for the class of $\aleph_\varepsilon$-saturated models, the answer to $(*)_{T,|T|}$ is negative except possible when $T$ is superstable with NDOP.*

[6](the set of $\{ac\ell(\bar a) : \bar a \in {}^{\omega>}M\}$ is absolute but the set of their enumeration and of the $\{f \restriction (ac\ell(\bar a)) : f \in \text{AUT}(\mathfrak{C}), f(\bar a) = \bar a\}$ is not).

[7]Note we did not say they have the same $\omega$-sequences of ordinals, e.g. if $V_2 = V_1^P$, $P$ Prikry forcing, then the assumption of $(*)_T$ holds though a new $\omega$-sequence of ordinals was added. So for $V_1 \subseteq V_2$ as in $(*)_T$, the $\mathscr{L}_{\infty,\aleph_1}$-theory is not necessarily preserved.

*Proof.* By quoting.

So we restrict ourselves to these. It should be quite transparent that $\mathbb{L}_{\infty,\aleph_\epsilon}(q.d.)$-theory is preserved from $\mathbf{V}_1$ to $\mathbf{V}_2$ (as well as the set of sentences in the logic).

Hence

**2.14 Theorem.** *For the class of $\aleph_\epsilon$-saturated models of superstable NDOP, NOTOP theory $T$ the answer to $(*)_{T,|T|}$ is yes.*

*Proof.* In $\mathbf{V}_1$ for $\ell = 1,2$ let $\mathscr{A}_\ell = \{A \subseteq M_\ell^{\rm eq} : A$ is $\varepsilon$-finite and $\mathrm{acl}(A) = A\}$. Clearly the same definition gives $\mathscr{A}_\ell$ in $\mathbf{V}_2$ and $M_1, M_2$ are $\aleph_\varepsilon$-saturated also in $\mathbf{V}_2$.

$A \in \mathscr{A}_\ell \Rightarrow |A| \le |T|$ and let $\mathscr{F}^* = \{f$: for some $A_1 \in \mathscr{A}_2$ and $A_2, \mathscr{A}_2, f$ is a one-to-one function from $A_1$ onto $A_2$ which is an $(M_1, M_2)$-elementary mapping$\}$. Again this definition gives the same set. We can define a rank function rk:$\mathscr{F}^* \to$ Ord $\cup$ $\{\infty\}$ such that $\mathrm{rk}(f) = \infty$ iff $(M_1, a)_{a \in \mathrm{Dom}(f)} \equiv_{\mathbb{L}_{\infty,\aleph_\varepsilon},(d.q.)} (M_2, f(a))_{a \in \mathrm{Dom}(f)}$ and it too is absolute.

Easily in both universes

(*a*) $M_1 \cong M_2$ iff $M_1 \equiv_{\mathbb{L}_{\infty,\aleph_\varepsilon}(d.q)} M_2$.

[Why? By Theorem 1.5.]

(*b*) $M_2 \equiv_{\mathbb{L}_{\infty,\aleph_0}(d.q)} M_2$ <u>iff</u> there is $\mathscr{F} \subseteq \mathscr{F}^*$ as in $\otimes_{M_1,M_2}$ from 1.4.

[Why? By 1.4.]

(*c*) there is $\mathscr{F} \subseteq \mathscr{F}^*$ as in $\circledast_{M_1,M_2}$ from 1.4 <u>iff</u> $\infty$ belongs to the range of the rank function.

[Why? Well known.]

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