CHARACTERIZING AN $\aleph_\epsilon$-Saturated
MODEL OF SUPERSTABLE NDOP
THEORIES BY ITS $L_{\infty,\aleph_\epsilon}$-THEORY
SH401

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Abstract. Assume a complete countable first order theory is superstable with
NDOP. We knew that any $\aleph_\epsilon$-saturated model of the theory is $\aleph_\epsilon$-prime over a
non-forking tree of “small” models and its isomorphism type can be characterized by
its $L_{\infty,\kappa}$-(dimension quantifiers)-theory; or if you prefer - appropriate cardinal invari-
ants. We go here one step further providing cardinal invariants which are as finitary
as seems reasonable.

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After the main gap theorem was proved (see [Sh:c]), in discussion, Harrington expressed a desire for a finer structure - of finitary character (when we have a structure theorem at all). I point out that the logic $L_{\infty, \aleph_0}$ (d.q.) (where d.q. stands for dimension quantifier) does not suffice: suppose; e.g. for $T = \text{Th}(\lambda \times \omega^2, E_n)_{n<\omega}$ where $(\alpha, \eta) \in E_n(\beta, \nu) := \eta \upharpoonright n = \nu \upharpoonright n$ and for $S \subseteq \omega^2$ define $M_S = M \upharpoonright \{((\alpha, \eta)) : [\eta \in S \Rightarrow \alpha < \omega_1] \text{ and } [\eta \in S \setminus S \Rightarrow \alpha < \omega]\}$. Hence, it seems to me we should try $L_{\infty, \aleph_0}$ (d.q.) (essentially, in $\mathcal{E}$ we can quantify over sets which are included in the algebraic closure of finite sets, see below 1.1, 1.3), and Harrington accepts this interpretation. Here the conjecture is proved for $\aleph_0$-saturated models. I.e., the main theorem is $M \equiv L_{\infty, \aleph_0}$ (d.q.) $\iff M \cong N$ for $\aleph_0$-saturated models of a superstable countable (first order) theory $T$ without dop. For this we analyze further regular types, define a kind of infinitary logic (more exactly, a kind of type of $\bar{a}$ in $M$), “looking only up” in the definition (when thinking of the decomposition theorem). Recall that for as $\aleph_0$-saturated model $M$ of a superstable DNOP theory an $\aleph_0$-decomposition is $\langle M_\eta, a_\eta : \eta \in \mathcal{F} \rangle$ where

(a) $I \subseteq \omega^>\text{ord}$ is nonempty closed under initial segments
(b) $M_\eta \prec M$ is $\aleph_0$-saturated
(c) $\nu \prec \eta \in I \Rightarrow M_\nu \prec M_\eta$
(d) if $\nu = \eta \smallfrown (\bar{\alpha}) \in I$ then $M_\nu$ is $\aleph_0$-prime over $M_\eta \cup \{a_\nu\}$ and $\text{tp}(a_\eta, M_\eta)$ is orthogonal to $M_\rho$ for $\rho \prec \nu$ and (the last is not essential but clarifies)
(e) $\langle M_\eta : \eta \in I \rangle$ is non-forking enough: for every $\nu \in I$ the set $\{a_\eta : \eta \in \text{Suc}(I(\nu))\} \subseteq M$ is independent over $M_\nu$.

The point is that if $\eta = \nu \smallfrown (\bar{\alpha})$, $M_\eta$, $a_\eta$ are chosen then to a large extent $\langle M_\rho, a_\rho : \eta \prec \rho \in I \rangle$ is determined. But the amount of “to a large extent” which suffices in [Sh:c], is not sufficient here, we need to find a finer understanding. In particular, we certainly do not like to “know” $(M_\nu, a_\eta)$. So we consider a pair $(\mathcal{A}, B)$ where $A \subseteq M_\nu$, $A \cup \{a_\eta\} \subseteq B \subseteq M_\eta$, $\text{stp}_s(B, A) \vdash \text{stp}_s(B, M_\nu)$ and we try to define the type of such pairs in a way satisfying:

(a) it can be impressed in our logic $L_{\infty, \aleph_0}$
(b) it expresses the essential information in $\langle M_\rho, a_\rho : \eta \prec \rho \in I \rangle$.

To carry out the isomorphism proof we need: (1.27) the type of the sum is the sum of types (infinitary types) assuming first order independence. The main point of the proof is to construct an isomorphism between $M_1$ and $M_2$ when $M_1 \equiv L_{\infty, \aleph_0}$ (d.q.) $M_2, \text{Th}(M_\ell) = T$ where $T$ and $\equiv L_{\infty, \aleph_0}$ (q.d.) are as above. So by [Sh:c, X] it is
enough to construct isomorphic decompositions. The construction of isomorphic decompositions is by $\omega$ approximations, in stage $n$, $\sim n$ levels of the decomposition tree are approximated, i.e. we have $I_n^\ell \subseteq n^2 $ Ord and $\bar{a}_n^{n,\ell} \in M_\ell$ for $\eta \in I_n, \ell = 1, 2$ such that $\text{tp}(\bar{a}_n^{n,1} \cdot \bar{a}_n^{n,1} \cdots \bar{a}_n^{n,1}, \emptyset, M) = \text{tp}(\bar{a}_n^{n,2} \cdot \bar{a}_n^{n,2} \cdots \bar{a}_n^{n,2}, \emptyset, M)$ with $\bar{a}_n^{n,\ell}$ being $\varepsilon$-finite, so in stage $n + 1$, choosing $\bar{a}_{n+1}^{n+1,\ell}$ we cannot take care of all types $\bar{a}_{n+1\ell}^{n+1,\ell}$ so addition theorem takes care. So though we are thinking on $\aleph_\varepsilon$ decompositions (i.e., the $M_\eta$’s are $\aleph_\varepsilon$-saturated), we get just a decomposition.

In the end of §1 (in 1.37) we point up that the addition theorem holds in fuller generalization. In the second section we deal with a finer type needed for shallow $T$, in the appendix we discuss how absolute is the isomorphism type.

Of course, we may consider replacing “$\aleph_\varepsilon$-saturated models of an NDOP superstable countable $T$” by “models of an NDOP $\aleph_0$-stable countable $T$”. But the use of $\varepsilon$-finite sets seems considerably less justifiable in this context, it seems more reasonable to use finite sets, i.e., $L^\infty,\aleph_0 (d.q.)$. But subsequently Hrushovski and Bouscaren proved that even if $T$ is $\aleph_0$-stable, $L^\infty,\aleph_0 (d.q.)$ is not sufficient to characterize models of $T$ up to isomorphism. This is not sufficient even if one considers the class of all $\aleph_\varepsilon$-saturated models rather than all models. The first example is $\aleph_0$-stable shallow of depth 3, and the second one is superstable (non $\aleph_0$-stable), NOTOP, non-multi-dimensional.

If we deal with $\aleph_\varepsilon$-saturated models of shallow (superstable NDOP) theories $T$, we can bound the depth of the quantification $\gamma = DP(T)$; i.e. $L^\infty,\aleph_\varepsilon$ suffice.

We assume the reader has a reasonable knowledge of [Sh:c, V, §1, §2] and mainly [Sh:c, V, §3] and of [Sh:c, X].

Here is a slightly more detailed guide to the paper. In 1.1 we define the logic $L^\infty,\aleph_\varepsilon$ and in 1.3 give a back and forth characterization of equivalence in this logic which is the operative definition for this paper.

The major tools are defined in 1.7, 1.11. In particular, the notion of $\text{tp}_\alpha$ defined in 1.5 is a kind of a depth $\alpha$ look-ahead type which is actually used in the final construction. In 1.28 we point out that equivalence in the logic $L^\infty,\aleph_\varepsilon$ implies equivalence with respect to $\text{tp}_\alpha$ for all $\alpha$. Proposition 1.14 contains a number of important concrete assertions which are established by means of Facts 1.16-1.23. In general these explain the properties of decompositions over a pair $(B, A)$. Claim 1.27 (which follows from 1.26) is a key step in the final induction. Definition 1.30 establishes the framework for the proof that two $\aleph_\varepsilon$-saturated structures that have the same $\text{tp}_\infty$ are isomorphic. The induction step is carried out in 1.35.

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-1.1 Notation: The notation is of [Sh:c], with the following additions (or reminders).

If $\eta = \nu^\ast (\alpha)$ then we let $\eta^- = \nu$; for $I$ a set of sequences ordinals we let
$\text{Suc}_T(\eta) = \{\nu : \text{for some } \alpha, \nu = \eta^*(\alpha) \in I\}$.
We work in $\mathcal{C}^{eq}$ and for simplicity every first order formula is equivalent to a relation.

(1) $\perp$ means orthogonal (so $q$ is $\perp p$ means $q$ is orthogonal to $p$),
remember $p \perp A$ means $p$ orthogonal to $A$; i.e. $p \perp q$ for every
$q \in S(\text{acl}(A))$ (in $\mathcal{C}^{eq}$)
(2) $\perp_a$ means almost orthogonal
(3) $\perp_w$ means weakly orthogonal
(4) $\frac{\bar{a}}{B}$ and $\bar{a}/B$ means $tp(\bar{a}, B)$
(5) $\frac{A}{B}$ or $A/B$ means $tp_*(A, B)$
(6) $A + B$ means $A \cup B$
(7) $\bigcup\{B_i : i < \alpha\}$ means $\{B_i : i < \alpha\}$ is independent over $A$
(8) $A \bigcup C$ means $\{A, C\}$ is independent over $B$
(9) $\{C_i : i < \alpha\}$ is independent over $(B, A)$ means that$^1$
\[ j < \alpha \Rightarrow tp_*(C_j, \bigcup\{C_i \cup B : i \neq j\}) \text{ does not fork over } A \]
(10) regular type means stationary regular type $p \in S(A)$ for some $A$
(11) for $p \in S(A)$ regular and $C$ a set of elements realizing $p$, $\dim(C, p)$ is
\[ \text{Max}\{|I| : I \subseteq C \text{ is independent over } A\} \]
(12) $\text{acl}(A) = \{c : \text{tp}(c, A) \text{ is algebraic}\}$
(13) $\text{dc}(A) = \{c : \text{tp}(c, A) \text{ is realized by one and only one element}\}$
(14) $Dp(p)$ is depth (of a stationary type, see [Sh:c, X, Definition 4.3, p.528, Definition
4.4, p.529])
(15) $\text{Cb}(p)$ is the canonical base of a stationary type $p$ (see [Sh:c, III.6.10, p.134])
(16) $B$ is $\aleph_\varepsilon$-atomic over $A$ if for every finite sequence $\bar{b}$ from $A$, for some find
$A_0 \subseteq A$ we have $stp(\bar{b}, A_0) \vdash stp(\bar{b}, A)$, equivalently for some $\varepsilon$-finite
$A_0 \subseteq \text{acl}(A)$ we have $tp(\bar{b}, A_0) \vdash tp(\bar{b}, \text{acl}(A))$.

$^1$Actually by the non-forking calculus this is equivalent to $\{C_i : i \leq \alpha\}$ is independent over $A$
where we let $C_\alpha = B$. 

§1 \( \aleph_\gamma \)-saturated models

We first define our logic, but as said in §0, we shall only use the condition from 1.4. \( T \) is always superstable complete first order theory.

1.1 Definition. 1) The logic \( L_{\infty, \aleph_\gamma} \) is slightly stronger than \( L_{\infty, \aleph_0} \), it consists of the set of formulas in \( L_{\infty, |T|^+} \) such that any subformula of \( \psi \) of the form \( (\exists \bar{x}) \phi \) is actually the form

\[
(\exists \bar{x}^0, \bar{x}^1) \left[ \varphi_1(\bar{x}^1, \bar{y}) \land \bigwedge_{i<\ell_\bar{x}^1} (\theta_i(\bar{x}^1_i, \bar{x}^0) \land (\exists^{<\aleph_0}z)\theta_i(z, \bar{x}^0)) \right],
\]

with \( \bar{x}^0 \) finite, \( \bar{x}^1 \) not necessarily finite but of length \( < |T|^+ \); so \( \varphi \) “says” \( \bar{x}^1 \subset acl(\bar{x}^0) \); note that always our final proof of the theorem uses \( |T| \geq \aleph_0 \).

2) \( L_{\infty, \aleph_\gamma}(d.q.) \) is like \( L_{\infty, \aleph_\gamma} \) but we have cardinality quantifiers and moreover dimensional quantifiers (as in [Sh:c, XIII,1.2,p.624]), see below.

3) The logic \( L_{\infty, \aleph_\gamma}^\gamma \) consist of the formulas of \( L_{\infty, \aleph_\gamma} \) such that \( \varphi \) has quantifier depth \( < \gamma \) (but we start the inductive definition by defining the quantifier depth of all first order as zero).

4) \( L_{\infty, \aleph_\gamma}(d.q.) \) is like \( L_{\infty, \aleph_\gamma}^\gamma \) but we have cardinality quantifiers and moreover dimensional quantifiers.

1.2 Remark. 1) In fact the dimension quantifier is used in a very restricted way (see Definition 1.9 and Claim 1.28 + Claim 1.30).

2) The reader may ignore this logic altogether and use just the characterization of equivalence in claim 1.4.

1.3 Convention. 1) \( T \) is a fixed first order complete theory, \( \mathfrak{C} \) is the “monster” model, as in [Sh:c], so is \( \kappa \)-saturated; \( \mathfrak{C}^{eq} \) is as in [Sh:c, III,6.2,p.131]. We work in \( \mathfrak{C}^{eq} \) so \( M, N \) vary on elementary submodels of \( \mathfrak{C}^{eq} \) of cardinality \( < \kappa \). We assume \( T \) is superstable with NDOP (countability is used only in the Proof of 1.5 for bookkeeping, i.e., in the proof of 1.30 and 1.29).

Remember \( a, b, c, d \) denote members of \( \mathfrak{C}^{eq} \), \( \bar{a}, \bar{b}, \bar{c}, \bar{d} \) denote finite sequences of members of \( \mathfrak{C}^{eq} \), \( A, B, C \) denote subsets of \( \mathfrak{C}^{eq} \) of cardinality \( < \kappa \).

Remember \( acl(A) \) is the algebraic closure of \( A \), i.e.

\[
\{ b : \text{for some first order and } n < \omega, \varphi(x, \bar{y}) \text{ and } \bar{a} \subseteq A \text{ we have } \mathfrak{C}^{eq} \models \varphi[b, \bar{a}] \land (\exists^{\leq n}y)\varphi(y, \bar{a}) \}
\]

and \( \bar{a} \) denotes \( Rang(\bar{a}) \) in places where it stands for a set (as in \( acl(\bar{a}) \)). We write
$\bar{a} \in A$ instead of $\bar{a} \in \omega^>(A)$.

2) $A$ is $\epsilon$-finite, if for some $\bar{a} \in \omega^> A, A = acl(\bar{a})$. (So for stable theories a subset of an $\epsilon$-finite set is not necessarily $\epsilon$-finite but as $T$ is superstable, a subset of an $\epsilon$-finite set is $\epsilon$-finite as if $B \subseteq acl(\bar{a}), \bar{b} \in B$ is such that $tp(\bar{a}, B)$ does not fork over $\bar{b}$, then trivially $acl(\bar{b}) \subseteq A$ and if $acl(\bar{b}) \neq B,$ $tp_*(B, \bar{a}^* \bar{b})$ forks over $B$, hence $([Sh:c, III,0.1])$ $tp(\bar{a}, B)$ forks over $\bar{b}$, a contradiction.
So if $acl(A) = acl(B)$, then $A$ is $\epsilon$-finite iff $B$ is $\epsilon$-finite).

3) When $T$ is superstable by $[Sh:c, IV,Table 1,p.169]$ for $F = F^\kappa_{\aleph_0}$, all the axioms there hold and we write $\aleph_\epsilon$ instead of $F$ and may use implicitly the consequences in $[Sh:c, IV,\S3]$.

We may instead Definition 1.1 use directly the standard characterization from 1.4; as actually less is used we state the condition we shall actually use:

**1.4 Claim.** For models $M_1, M_2$ of $T$ we have $M_1 \equiv_{L_\infty, \aleph_\epsilon(d.q.)} M_2$ iff

- there is a non-empty family $\mathcal{F}$ such that:
  
  (a) each $f \in \mathcal{F}$ is an $(M_1, M_2)$-elementary mapping, (so Dom($f$) $\subseteq M_1$, Rang($f$) $\subseteq M_2$)
  
  (b) for $f \in \mathcal{F}$, Dom($f$) is $\epsilon$-finite (see 1.3(2)) above
  
  (c) if $f \in \mathcal{F}, \bar{a}_\ell \in M_\ell (\ell = 1, 2)$ then for some $g \in \mathcal{F}$ we have:
     
     $f \subseteq g$ and $acl(\bar{a}_1) \subseteq$ Dom($f$) and $acl(\bar{a}_2) \subseteq$ Rang($f$)
     
  (d) if $f \cup \{\langle a_1, a_2 \rangle\} \in \mathcal{F}$ and $tp(a_1, Dom(f))$ is stationary and regular then $dim(\{a_1^1 \in M_1 : f \cup \{\langle a_1^1, a_2 \rangle\} \in \mathcal{F}\}, M_1)$
     
     $= dim(\{a_2^2 \in M_2 : f \cup \{\langle a_1, a_2^2 \rangle\} \in \mathcal{F}\}, M_2)$.

Our main theorem is

**1.5 Theorem.** Suppose $T$ is countable (superstable complete first order theory) with NDOP.

Then

(1) the $L_{\infty, \aleph_\epsilon}(d.q.)$ theory of an $\aleph_\epsilon$-saturated model characterizes it up to isomorphism.

(2) Moreover, if $M_1, M_2$ are $\aleph_\epsilon$-saturated models of $T$ (so $M_\ell < C^{eq}$) and $\boxtimes_{M_0, M_1}$ of 1.4 holds, then $M_1, M_2$ are isomorphic.

By 1.4, it suffices to prove part (2).
The proof is broken into a series of claims (some of them do not use NDOP, almost all do not use countability; but we assume $T$ is superstable complete all the time (1.7(1)).

1.6 Discussion: Let us motivate the notation and Definition below.

Recall from the introduction that we are thinking of a triple $(M, N, a)$ which may appear in $\mathcal{K}_\varepsilon$-decomposition $(\langle M_\eta, a_\eta : \eta \in I \rangle)$ of $N$, in the sense that for some $\eta \in I \setminus \{<\}$ we have $(M, M', a) = (M_\eta, M_\eta, a_\eta)$ so $M, M'$ are $\mathcal{K}_\varepsilon$-saturated, $a_\eta \in M' \setminus M$, $M'$ is $\mathcal{K}_\varepsilon$-prime over $M + a$ and $\text{tp}(a, M)$ is regular. But this is “too large for us” hence we consider an approximation $(A, B)$ where $A \subseteq M(= M_{\eta^-})$, $A \subseteq B \subseteq M'(= M_\eta)$, $a = a_\eta \in B$ and $B/M(= B/M_{\eta^-})$ does not fork over $A$. We would like to define the $\alpha$-type of $(A, B)$ in $N$, which tries to say something on the decomposition above $(M, M', a) = (M_{\eta^-}, M_\eta, a_\eta)$, i.e., on $(M_\rho, a_\rho : \eta < \rho \in I)$. There are two natural “successor” of $(A, B)$ we may choose in this context: the first 1.7 below replaces $(A, B)$ to $(A', B')$ such that $A \subseteq A' \subseteq M(= M_{\eta^-})$, $B \subseteq B' \subseteq M'(= M_{\eta})$ and (as $M'$ is $\mathcal{K}_\varepsilon$-prime over $M + a$) we have $\text{stp}_{\ast}(B', A' \cup B) \vdash \text{stp}(B', M)$, so $\text{tp}(B', A' \cup B)$ is almost orthogonal to $A'$; we can think of this as “advancing in the same model”; in other words as $A, B$ are $\varepsilon$-finite, we have to increase them in order to capture even $(M, M')$. This is formalized by $\leq_a$ in Definition 1.7 below.

The second is to pass from $(M_{\eta^-}, M_\eta, a_\eta)$ to $(M_\eta, M_\nu, a_\nu)$ for some $\nu$ an immediate successor of $\eta$ in $I$. So the old $B$ is included in the new $A'$ and $B' = A' \cup \{a\}$ where $\text{tp}(a, A')$ is regular and is orthogonal to $A$ (as in the decomposition we require $\text{tp}(a_\eta, M_{\eta^-})(M_\nu$ when $\nu < \eta^-)$. This is formalized by $\leq_b$ in Definition 1.7 below.

1.7 Definition. 1) $\Gamma = \{(A, B) : A \subseteq B \text{ are } \varepsilon\text{-finite}\}$. Let $\Gamma(M) = \{(A, B) \in \Gamma : A \subseteq B \subseteq M\}$.

2) For members $(A, B)$ of $\Gamma$ we may also write $(B/A)$; if $A \not\subseteq B$ we mean $(B \cup A/A)$.

3) $(B_1/A_1) \leq_a (B_2/A_2)$ (usually we omit $a$) if both are in $\Gamma$ and

- $A_1 \subseteq A_2$, $B_1 \subseteq B_2$, $B_1 \cup A_2$ and $B_2/A_1 + A_2 \perp_a A_2$.

4) $(B_1/A_1) \leq_b (B_2/A_2)$ if $A_2 = B_1, B_2 \setminus A_2 = \emptyset$ and $B_2/A_2$ is regular orthogonal to $A_1$.

5) $\leq^*$ is the transitive closure of $\leq_a \cup \leq_b$. (So it is a partial order, whereas in general $\leq_a \cup \leq_b$ and $\leq_b$ are not).

6) We can replace $A, B$ by sequences listing them (we do not always strictly distinguish).

Remark. The following observation may clarify.

1.8 Observation. If $(B_1/A_1) \leq^* (B_2/A_2)$ then we can find $(B'_\ell : \ell \leq n)$ and
\[ \langle c_\ell : 1 \leq \ell < n \rangle \text{ for some } n \geq 1, \text{ satisfying } \langle \frac{B_\ell}{A_\ell} \rangle \leq_b \langle \frac{B'_\ell}{A'_\ell} \rangle, c_\ell \in B'_{\ell+1}, \text{ regular,} \]
\[ \frac{B'_{\ell+1}}{c_\ell+B'_{\ell}} \perp_a B'_{\ell}, A_2 = B'_{n-1}, B_2 = B'_n. \]

**Remark.** 1) Note that actually \( \leq_a \) is transitive. This means that in a sense \( \leq_b \) is enough, \( \leq_a \) inessential.
2) We may in 1.7(4) use \( \bar{b} = \langle c \rangle \), does not matter.

**Proof.** By the definition of \( \leq^* \) there are \( k < \omega \) and \( \langle \frac{B'_\ell}{A'_\ell} \rangle \) for \( \ell \leq k \) such that:
\[ \langle \frac{B'_\ell}{A'_\ell} \rangle \leq_{x(\ell)} \langle \frac{B'_{\ell+1}}{A'_{\ell+1}} \rangle \text{ for } \ell \leq k \text{ and } x(\ell) \in \{a,b\} \text{ and } \langle \frac{B'_0}{A'_0} \rangle = \langle \frac{B_1}{A_1} \rangle, \langle \frac{B'_k}{A'_k} \rangle = \langle \frac{B_2}{A_2} \rangle \text{ and} \]
without loss of generality \( x(2\ell) = a,x(2\ell+1) = b \). Let \( N_0 < \mathcal{C} \) be \( \aleph_\varepsilon \)-prime over \( \emptyset \) such that \( A^0 \subseteq N_0, B_0 \subseteq N_0 \) and \( f_0 = \text{id}_{A_0} \). We choose by induction on \( \ell \leq k, N_{\ell+1}, f_{\ell+1} \) such that:
\[ \begin{align*}
(1) & \text{ Dom}(f_{\ell+1}) = B^\ell \\
(2) & N_\ell < N_{\ell+1} \\
(3) & \text{if } x(\ell) = b \text{ then } f_{\ell+1} \text{ is an extension of } f_\ell \text{ which necessarily has domain } A_\ell, \text{ check) with domain } B^\ell \text{ such that } f_\ell(B^\ell) \bigcup N_\ell \text{ and } N_{\ell+1} \text{ is } \aleph_\varepsilon\text{-prime} \\
& \text{ over } N_\ell \cup f_\ell(B^\ell) \\
(4) & \text{if } x(\ell) = a \text{, then } f_{\ell+1} \text{ maps } A^\ell \text{ into } N_{\ell-1}, B^\ell \text{ into } N_\ell \text{ and } N_{\ell+1} = N_\ell.
\end{align*} \]
This is straightforward. Now on \( \langle N_\ell : \ell \leq k+1 \rangle \) we repeat the argument (of choosing \( \langle B_\ell : \ell \leq n \rangle \)) in the proof of 1.14(6) above, i.e., choose \( B^\ell \subseteq A_\ell \) by downward induction on \( \ell \) large enough as required. \( \square_{1.8} \)

**1.9 Definition.** 1) We define \( \text{tp}_\alpha[\langle B^\alpha_A, M \rangle \text{ for } A \subseteq B \subseteq M, A \text{ and } B \text{ are} \varepsilon\text{-finite and } \alpha \text{ is an ordinal} \) and \( \mathcal{S}_\alpha(\langle B^\alpha_A, M \rangle, \mathcal{S}_\alpha(A, M) \text{ and } \mathcal{S}'_\alpha(\langle B^\alpha_A, M \rangle, \mathcal{S}'_\alpha(A, M) \text{ by induction on } \alpha \text{ (we mean simultaneously; of course, we use appropriate variables):} \]
\[ \begin{align*}
(1) & \text{tp}_0[\langle B^A_A, M \rangle \text{ is the first order type of } A \cup B \\
(2) & \text{tp}_{\alpha+1}[\langle B^A_A, M \rangle = \text{the triple } \langle Y_{A,B,M}^{1,\alpha}, Y_{A,B,M}^{2,\alpha}, \text{tp}_\alpha(\langle B^A_A, M \rangle) \rangle \text{ where} \\
& \text{where:} \\
& Y_{A,B,M}^{1,\alpha} = \text{tp}_\alpha[\langle B^A_A, M \rangle : \text{for some } A', B' \text{ we have } \langle B^A_A \rangle \leq_a \langle B'^A_A \rangle \in \Gamma(M) \rangle,}
\end{align*} \]
and \( Y_{A,B,M}^{2,\alpha} =: \{ (\Upsilon, \lambda_{\Upsilon,M,B}) : \Upsilon \in \mathcal{I}_\alpha(A,B,M) \} \)

where

\[
\lambda_{\Upsilon,M,B} = \dim \left[ \{ d : \text{tp}_\alpha([B+d],M) = \Upsilon \} \right], \quad B:
\]

(\( c \)) for \( \delta \) a limit ordinal, \( \text{tp}_\delta([B],M) = \langle \text{tp}_\alpha([B],M) : \alpha < \delta \rangle \)
(this includes \( \delta = \infty \), really \( \|M\| + \) suffice).

(\( d \)) \( \mathcal{I}_\alpha(A,M) = \left\{ \text{tp}_\alpha([B],M) : \text{for some } B \text{ such that } B \subseteq M, \right. \)

\( (\begin{Bmatrix} B \\ \end{Bmatrix}) \in \Gamma(M) \}

(\( e \)) \( \mathcal{I}_\alpha(A,M) = \left\{ \text{tp}_\alpha([B],M) : \text{for some } c \in M \text{ we have} \right. \)

\( c \perp A \) and \( c \) is regular

(\( f \)) \( \mathcal{I}_\alpha(A,M) = \left\{ \text{tp}_\alpha([A],M) : c \in M \text{ and } c \) regular \}

2) We define also \( \text{tp}_\alpha[A,M] \), for \( A \) an \( \epsilon \)-finite subset of \( M \):

(\( a \)) \( \text{tp}_0[A,M] = \) first order type of \( A \)

(\( b \)) \( \text{tp}_{\alpha+1}[A,M] = \) the triple \( \langle Y_{A,M}^{1,\alpha}, Y_{A,M}^{2,\alpha}, \text{tp}_\alpha[A,M] \rangle \) where

\( Y_{A,M}^{1,\alpha} =: \mathcal{I}_\alpha(A,M) \) and

\( Y_{A,M}^{2,\alpha} =: \left\{ (\Upsilon, \dim\{ d \in M : \text{tp}_\alpha([A+d],M) = \Upsilon \}) : \Upsilon \in \mathcal{I}_\alpha(A,M) \right\} \)

(\( c \)) \( \text{tp}_\delta[A,M] = \langle \text{tp}_\alpha(A,M) : \alpha < \delta \rangle \)

3) \( \text{tp}_\alpha[M] = \text{tp}_\alpha[\emptyset,M] \).

1.10 Discussion: Clearly \( \text{tp}([B],M) \) is intended, on the one hand, to be expressible by our logic and, on the other hand, to express the isomorphism type of \( M \) “in the direction of \( \langle B \rangle^n \)”. To really say it we need to go back to the \( \aleph_\epsilon \)-decompositions of \( M \), a central notion of [Sh:c, Ch.X].

For the reader’s benefit, by the referee request, let us review informally the proof in [Sh:c, Ch.X]. Let \( M \) be an \( \aleph_\epsilon \)-saturated model, and we choose \( \langle M_\eta : \eta \in I \cap ^n \text{Ord}, (a_\eta : \eta \in I \cap ^{n+1} \text{Ord}) \rangle \) by induction on \( n \). For \( n = 0 \), of course, \( I \cap ^0 \text{Ord} = \{ \langle \rangle \} \), we let \( N_{\langle \rangle} < M \) be \( \aleph_\epsilon \)-prime over \( \emptyset \) and let \( I_{\langle \rangle} \) a maximal subset of \( \{ c \in M : \text{tp}(c, N_{\langle \rangle}) \text{ regular} \} \) which is independent over \( N_{\langle \rangle} \), let \( \langle a_{\langle \alpha \rangle} : \alpha < |I_{\langle \rangle}| \rangle \) list \( I_{\langle \rangle} \). Similarly for \( n+1, \eta \in I \cap ^{n+1} \text{Ord} \), let \( N_\eta < M \) be \( \aleph_\epsilon \)-prime over \( M_{\eta} + a_\eta \), let \( I_\eta \) be a maximal subset of \( \{ c \in M : \text{tp}(c, M_\eta) \text{ is regular} \)
orthogonal to \( M_{\eta^-} \) independent over \( N_\eta \). Lastly, let \( \langle c_\eta^{<\alpha} : \alpha < |I_\eta| \rangle \) list \( I_\eta \) and let \( I \cap \\name{Ord}^{n+1} = \{\eta^- < \alpha : \eta \in I \cap |I| \\text{ and } \alpha < |I_\eta| \} \).

To carry this we use the existence of \( \aleph_c \)-prime models (and the local character of indpendent). Also looking at the set \( \bigcup\{M_\eta : \eta \in I\} \), its first order type is determined by the non-forking calculus. In fact, for any \( \eta \in I\{<>\} \), the set \( \bigcup\{\eta_\nu : \eta_\nu \in I\}, \bigcup\{N_\eta : \neg(\eta \leq \nu) \text{ and } \nu \in I\} \) are independent over \( N_\eta \). Let \( N < M \) be \( \aleph_c \)-prime over \( \bigcup\{\eta_\nu : \eta \in I\} \), now if \( M = N \) we are done decomposing \( M \), if not some \( c \in M \setminus N \) realize a regular type (we use density of regular types). By NDOP, the \( \text{tp}(c, N) \) is not orthogonal to some \( N_\eta \). Choose \( \eta \) of minimal length hence \( \nu < \eta \Rightarrow \text{tp}(c, M_\eta) \perp N_\nu \). By properties of regular types, without loss of generality \( \text{tp}(c, N) \) does not fork over \( N_\eta \), so we get a contradiction to the maximality of \( \{a_\nu : \nu \in \text{Suc}_I(\eta)\} \) (this explains the role of \( \mathcal{P} \) in Definition 1.11(5) below).

We are interested in the possible trees \( \langle N_\nu : \eta \cup \nu \in I \rangle \).

Now the tree determines \( M \) up to isomorphism, but there are “incidental” choices, so two trees may give isomorphic models (for investigating the number of non isomorphic models it is enough to find sufficiently pairwise far trees \( I \)).

We like to get exact information and in as finitary way as we can. So we replace \( (M_{\eta^-}, M_\eta, a_\eta) \) by \( (B_A^{\langle} \rangle \), where \( A \subseteq M_{\eta^-}, A + a_\eta \subseteq B \subseteq M_\eta, \text{tp}(B, M_{\eta^-}) \) does not fork over \( A \).

Now for \( \eta \in I\{<>\} \) we are interested in the possible trees \( \langle N_\nu : \eta \cup \nu \in I \rangle \), over \( (N_{\eta^-}, N_\eta, a_\eta) \). But not only different trees may be equivalent (giving isomorphic \( \aleph_c \)-prime models) but the other part of the tree, \( \langle N_\nu : \nu \in I \text{ but } \neg(\eta \cup \nu) \rangle \) may apriori cause non equivalent trees to contribute the same toward understanding \( M \).

This is done in [Sh:c, Ch.XII], but here we have to deal with \( \varepsilon \)-finite \( A, B \).

The following claim 1.11 really does not add to [Sh:c, Ch.X], it just collects the relevant information which is proved there, or which follows immediately (particularly using the parameter \( (A, B) \)). We allow here \( a_\eta/M_{\eta^-} \) - to be not regular, but this is not serious: we can here deal exclusively with this case and we can omit this requirement in [Sh:c, Ch.X]; however, this does not eliminate the use of regular types (in the proof that \( M \) is \( \aleph_c \)-prime over every \( \aleph_c \)-decomposition of it).

1.11 Definition. 1) \( \langle N_\eta, a_\eta : \eta \in I \rangle \) is an \( \aleph_c \)-decomposition inside \( M \) above (or for over) the pair \( (B_A^{\langle} \rangle \) (but we may omit the “\( \aleph_c - \)”) if:

\( (a) \) \( I \) a set of finite sequences of ordinals closed under initial segments

\( (b) \) \( \langle \rangle, \langle 0 \rangle \in I, \eta \in I \setminus \{\langle \rangle \} \Rightarrow \langle 0 \rangle \not\subseteq \eta, \) let \( I^- = I \setminus \{\langle \rangle \} \), really \( a_\langle \rangle \) is meaningless

\( (c) \) \( A \subseteq N_\langle \rangle, B \subseteq N_{\langle 0 \rangle}, N_\langle \rangle \setminus B \) and \( \text{dc} \langle a_\langle 0 \rangle \rangle \subseteq \text{dc} \langle B \rangle \),

\( (d) \) if \( \nu = \eta^\langle \alpha \rangle \in I \) then \( N_\nu \) is \( \aleph_c \)-primary over \( N_\eta \cup \bar{a}_\nu, N_\langle \rangle \) is \( \aleph_c \)-prime over \( A \).
(e) for $\eta \in I$ such that $k = \ell g(\eta) > 1$ the type $a_\eta/N_{\eta}(k-1)$ is orthogonal to $N_{\eta}(k-2)$.

(f) $\eta \prec \nu \Rightarrow N_\eta \prec N_\nu$.

(g) $M$ is $\aleph_\nu$-saturated and $N_\eta \prec M$ for $\eta \in I$.

(h) if $\eta \in I \setminus \{\langle \rangle \}$, then $\{a_\nu : \nu \in \text{Suc}_I(\eta)\}$ is (a set of elements realizing over $N_\eta$ types orthogonal to $N_\eta$ and is) an independent set over $N_\eta$.

2) We replace “inside $M$” by “of $M$” if in addition

(i) in clause (h) the set is maximal.

3) $\langle N_\eta, a_\eta : \eta \in I \rangle$ is an $\aleph_\nu$-decomposition inside $M$ if $(a), (d), (e), (f), (g), (h)$ of part (1) holds and in clause (h) we allow $\eta = \langle \rangle$ (call this (h)$^+$). We add “over $A$” if $A \subseteq M_{\prec}$.

4) $\langle N_\eta, a_\eta : \eta \in I \rangle$ is an $\aleph_\nu$-decomposition of $M$ if in addition to 1.11(3) and we have the stronger version of clause (i) of 1.11(2) by including $\eta = \langle \rangle$, i.e. we have:

\[(i)^+ \text{ for } \nu \in I, \text{ the set } \{a_\eta : \eta \in \text{Suc}_I(\nu)\} \text{ is a maximal subset of } M \text{ independent over } N_\nu.\]

We may add “over $A$” if $A \subseteq M$.

5) If $\langle N_\eta, a_\eta : \eta \in I \rangle$ is an $\aleph_\nu$-decomposition inside $M$ we let

$\mathcal{P}(\langle N_\eta, a_\eta : \eta \in I \rangle, M) = \left\{ p \in S(M) : p \text{ regular and for some } \eta \in I \setminus \{\langle \rangle \} \text{ we have} \right.$

\[p \text{ is orthogonal to } N_\eta - \text{ but not to } N_\eta \right\}.

As said earlier it is natural to use regular types.

1.12 Definition. 1) We say that $\langle N_\eta, a_\eta : \eta \in I \rangle$, an $\aleph_\nu$-decomposition inside $M$, is $J$-regular if $J \subseteq I$ and:

\[(*) \text{ for each } \eta \in I \setminus J \text{ there are } c_\eta \text{ such that } a_\eta \in a\ell(N_\eta^- + c_\eta)

\[
\frac{c_\eta}{N_\eta^-} \text{ is regular and if } \eta \neq \langle \rangle \text{ then } \frac{a_\eta}{N_\eta^+ + c_\eta} \perp_{N_\eta^-}.
\]

2) We say “$\langle N_\eta, a_\eta : \eta \in I \rangle$ is a regular $\aleph_\nu$-decomposition inside $M$ [of $M$]” if it is an $\aleph_\nu$-decomposition inside $M$ [of $M$] which is $\emptyset$-regular.

3) We say “$\langle N_\eta, a_\eta : \eta \in I \rangle$ is a regular $\aleph_\nu$-decomposition inside $M$ [of $M$] over $\langle \rangle$” if it is an $\aleph_\nu$-decomposition inside $M$ [of $M$] over $\langle \rangle$ which is $\{\langle \rangle\}$-regular.

\[\text{without loss of generality } c_\eta = a_\eta\]
1.13 Claim. 1) Every $\aleph_\varepsilon$-saturated model has an $\aleph_\varepsilon$-decomposition (i.e. of it).
2) If $M$ is $\aleph_\varepsilon$-saturated, $\langle N_\eta, a_\eta : \eta \in I \rangle$ is an $\aleph_\varepsilon$-decomposition inside $M$, then for some $J$, and $N_\eta, a_\eta$ for $\eta \in J\setminus I$ we have: $I \subseteq J$ and $\langle N_\eta, a_\eta : \eta \in J \rangle$ is an $\aleph_\varepsilon$-decomposition of $M$ (even a $(J\setminus I)$-regular one).
3) If $M$ is $\aleph_\varepsilon$-saturated, $\langle N_\eta, a_\eta : \eta \in I \rangle$ is an $\aleph_\varepsilon$-decomposition of $M$ then $M$ is $\aleph_\varepsilon$-primary and $\aleph_\varepsilon$-minimal$^3$ over $\bigcup_{\eta \in I} N_\eta$; if in addition $\langle N_\eta, a_\eta : \eta \in \{\langle\rangle, (0)\} \rangle$ is an $\aleph_\varepsilon$-decomposition inside $M$ above $(\bigcup_{\eta \in I} N_\eta)^B$, then $\langle N_\eta, a_\eta : \eta \in I \& (\eta \neq \langle\rangle \rightarrow (0) \leq \eta) \rangle$ is an $\aleph_\varepsilon$-decomposition of $M$ above $(\bigcup_{\eta \in I} N_\eta)^B$.
4) If $\langle N_\eta, a_\eta : \eta \in I \rangle$ is an $\aleph_\varepsilon$-decomposition inside $M$ above $(\bigcup_{\eta \in I} N_\eta)^B$, then it is an $\aleph_\varepsilon$-decomposition inside $M$.
5) If $\langle N_\eta, a_\eta : \eta \in I \rangle$ is an $\aleph_\varepsilon$-decomposition inside $M$ [above $(\bigcup_{\eta \in I} N_\eta)^B$], $\eta \in I$, $[\eta \in I \setminus \{\langle\rangle\}], \alpha = \min\{\beta : \eta^\varepsilon(\beta) \notin I\}, \nu = \eta^\varepsilon(\alpha), a_\nu \in M\setminus N_\eta, a_\eta$ is orthogonal to $M_\eta$ if $\eta^\varepsilon(\nu) \neq \langle\rangle$, $N_\nu < M$ is $\aleph_\varepsilon$-primary over $N_\eta + a_\nu$ and $a_\nu \bigcup_{\eta \in I} N_\eta\bigcup_{\rho \in I} N_\rho$ (enough to demand $\{a_\rho : \rho^\varepsilon = \eta$ and $\rho \in I\}$ is independent over $a_\nu/N_\eta$) then $\langle N_\rho, a_\rho : \rho \in I \cup \{\nu\} \rangle$ is an $\aleph_\varepsilon$-decomposition inside $M$ over $(\bigcup_{\eta \in I} N_\eta)^B$.
6) Assume $\langle N_\eta, a_\eta : \eta \in I \rangle$ is an $\aleph_\varepsilon$-decomposition of $M$, if $p$ is regular (stationary) and not orthogonal to $M$ (e.g. $p \in S(M)$) then for one and only one $\eta \in I$, there is a regular (stationary) $q \in S(N_\eta)$ not orthogonal to $p$ such that: if $\eta^\varepsilon$ is well defined (i.e. $\eta \neq \langle\rangle$), then $p \perp N_\eta$.
7) Assume $I = \bigcup_{\alpha < \alpha(\star)} I_\alpha$, for each $\alpha$ we have $\langle N_\eta, a_\eta : \eta \in I_\alpha \rangle$ is an $\aleph_\varepsilon$-decomposition inside $M$ [above $(\bigcup_{\eta \in I} N_\eta)^B$] and for each $\eta \in I$ for every $n < \omega$ and $\nu_\ell = \eta^\varepsilon(\beta_\ell) \in I$ for $\ell < n$, for some $\alpha$ we have: $\{\nu_\ell : \ell < n\} \subseteq I_\alpha$ (e.g. $I_\alpha$ increasing). Then $\langle N_\eta, a_\eta : \eta \in I \rangle$ is an $\aleph_\varepsilon$-decomposition inside $M$ [above $(\bigcup_{\eta \in I} N_\eta)^B$].
8) In (7), if $\eta \neq \langle\rangle$ and some $\nu_\ell$ is not $\alpha$-maximal in $I$ and $a_\nu/N_\eta$ is regular, it is enough:
\[\ell_1 < \ell_2 < n \Rightarrow \bigvee_{\alpha < \alpha(\star)} \{\nu_\ell_1, \nu_\ell_2\} \subseteq I_\alpha\].
9) If $\langle N_\eta, a_\eta : \eta \in I \rangle$ is an $\aleph_\varepsilon$-decomposition inside $M, I_1, I_2 \subseteq I$ are closed under initial segments and $I_0 = I_1 \cap I_2$ then $\bigcup_{\eta \in I_1} N_\eta \bigcup_{\eta \in I_0} N_\eta \bigcup_{\eta \in I_2} N_\eta$.

$^3$here we use NDOP
10) Assume that for $\ell = 1,2$ that $\langle N_\eta^\ell, a_\eta^\ell : \eta \in I \rangle$ is an $\mathcal{K}_\epsilon$-decomposition inside $M_\ell$, and for $\eta \in I$ the function $f_\eta$ is an isomorphism from $N_\eta^1$ onto $N_\eta^2$ and $\eta \smallsetminus \nu \Rightarrow f_\eta \subseteq f_\nu$. Then $\bigcup_{\eta \in I} f_\eta$ is an elementary mapping; if in addition $\langle N_\eta^\ell, a_\eta^\ell : \eta \in I \rangle$ is an $\mathcal{K}_\epsilon$-decomposition of $M_\ell$ (for $\ell = 1,2$) then $\bigcup_{\eta \in I} f_\eta$ can be extended to an isomorphism from $M_1$ onto $M_2$.

11) If $\langle N_\eta, a_\eta : \eta \in I \rangle$ is an $\mathcal{K}_\epsilon$-decomposition inside $M$ (above $\langle B_A \rangle$) and $M^- \prec M$ is $\mathcal{K}_\epsilon$-prime over $\bigcup_{\eta \in I} N_\eta$ then $\langle N_\eta, a_\eta : \eta \in I \rangle$ is an $\mathcal{K}_\epsilon$-decomposition of $M$ (above $\langle B_A \rangle$).

12) If $\langle N_\eta, a_\eta : \eta \in I \rangle$ is an $\mathcal{K}_\epsilon$-decomposition inside $M/\mathcal{A}$ of $M$ (above $\langle M_A \rangle$) and $a_\eta' \in N_\eta$ and $N_\eta$ is $\mathcal{K}_\epsilon$-prime over $N_\eta^- + a_\eta'$ for $\eta \in I \\setminus \{<\} \{a_{<0}\}$ or at least $\text{dcl}(a_{<0}) \subseteq \text{dcl}(B)$ then $\langle N_\eta, a_\eta' : \eta \in I \rangle$ is an $\mathcal{K}_\epsilon$-decomposition inside $M/\mathcal{A}$ of $M$ (above $\langle B_A \rangle$).

Proof. 1), 2), 3), 5), 6), 9)-12). Repeat the proofs of [Sh:c, X]. (Note that here $a_\eta/N_\eta$ is not necessarily regular, a minor change).

4), 7) Check.

8) As $\text{Dp}(p) > 0 \Rightarrow p$ is trivial, by [Sh:c, ChX,7.2,p.551] and [Sh:c, ChX,7.3]. $\square_{1.13}$

We shall prove:

1.14 Claim. 1) If $M$ is $\mathcal{K}_\epsilon$-saturated, $\langle B_A \rangle \in \Gamma(M)$, then there is $\langle N_\eta, a_\eta : \eta \in I \rangle$, an $\mathcal{K}_\epsilon$-decomposition of $M$ above $\langle B_A \rangle$.

2) Moreover if $\langle N_\eta, a_\eta : \eta \in I \rangle$ satisfies clauses (a) -- (h) of Definition 1.11(1), we can extend it to satisfy clause (i) of 1.11(2), too.

3) If $\langle N_\eta, a_\eta : \eta \in I \rangle$ is an $\mathcal{K}_\epsilon$-decomposition of $M$ above $\langle B_A \rangle$, $M^- \prec M$ is $\mathcal{K}_\epsilon$-prime over $\bigcup_{\eta \in I} N_\eta$ then:

(a) $\langle N_\eta : \eta \in I \rangle$ is a $\mathcal{K}_\epsilon$-decomposition of $M^-$

(b) we can find an $\mathcal{K}_\epsilon$-decomposition $\langle N_\eta, a_\eta : \eta \in J \rangle$ of $M$ such that $J \supseteq I$ and $[\eta \in J \setminus I \iff (\eta \neq \langle \rangle \text{ and } \neg \langle \rangle \smallsetminus \eta)]$, moreover the last phrase follows from the previous ones.

4) If in 3)(b) the set $J \setminus I$ is countable (finite is enough for our applications), then necessarily $M, M^-$ are isomorphic, even adding all members of an $\epsilon$-finite subset of $M^-$ as individual constants.

5) If $\langle N_\eta, a_\eta : \eta \in I \rangle$ is an $\mathcal{K}_\epsilon$-decomposition of $M$ above $\langle B_A \rangle, I \subseteq J$ and
\(<N_\eta, a_\eta : \eta \in J>\) is an \(\aleph_\epsilon\)-decomposition of \(M, M^- \prec M\) is \(\aleph_\epsilon\)-prime over \(\bigcup_{\eta \in I} N_\eta\)
and \((B^l_A) \leq^* (B^1_A)\) and \(B_1 \subseteq M\) and \(c \in M\) and \(c \upharpoonright B_1 \perp A_1\) and \(c \upharpoonright B_1\) is (stationary and) regular.

\([\alpha]\) \(c \upharpoonright B_1 \perp \bigcup_{\eta \in J \setminus \{\rangle\}} N_\eta\)
\([\beta]\) \(c \upharpoonright B_1\) is not orthogonal to some \(p \in \mathcal{P}(\langle N_\eta, a_\eta : \eta \in I>, M\)).

6) If \(\langle N_\eta, a_\eta : \eta \in I\rangle\) is an \(\aleph_\epsilon\)-decomposition of \(M\) above \((B^l_A)\), \(\text{then the set } \mathcal{P} = \mathcal{P}(\langle N_\eta : \eta \in I\rangle, M)\) depends on \((B^l_A)\) and \(M\) only (and not on \(\langle N_\eta : \eta \in I\rangle\) or \(M^-\) when \(M^- \prec M\) is \(\aleph_\epsilon\)-prime over \(\bigcup\{N_\eta : \eta \in I\}\)), recalling:

\[\mathcal{P} = \mathcal{P}(\langle N_\eta : \eta \in I\rangle, M) = \left\{ p \in S(M) : p \text{ regular and for some } \eta \in I \setminus \{\rangle\}, \text{ we have :} \right\}
\[p \text{ is orthogonal to } N_\eta^- \text{ but not to } N_\eta \}\]

So let \(\mathcal{P}(\langle B^l_A\rangle, M) =: \mathcal{P}(\langle N_\eta : \eta \in I\rangle, M)\).

7) If \(B^l_A\) is regular of depth zero or just \(B^l_A \leq^* B^{'l}_A\) regular of depth zero and \(M\) is \(\aleph_\epsilon\)-saturated and \(B \subseteq M\) \(\text{then}\)

\(\text{(a) for any } \alpha, \text{ we have } tp_{\alpha}(\langle B^l_A\rangle, M) \text{ depend just on } tp_{0}(\langle B^l_A\rangle, M)\)
\(\text{(b) if } (B^l_A) \leq^* (B^{'l}_A) \in \Gamma(M) \text{ then } tp_{\alpha}(\langle B^{'l}_A\rangle, M) \text{ depends just on } tp_{0}(\langle B^l_A\rangle, M) \text{ (and} \langle A, B, A', B \rangle \text{ but not on } M)\).

8) For \(\alpha < \beta\), from \(tp_{\beta}(\langle B^l_A\rangle, M)\) we can compute \(tp_{\alpha}(\langle B^l_A\rangle, M)\).

9) If \(f\) is an isomorphism from \(M_1\) onto \(M_2, A_1 \subseteq B_1\) are \(\epsilon\)-finite subsets of \(M_1\) and \(f(A_1) = A_2, f(B_1) = B_2\) \(\text{then}\)

\[tp_{\alpha}(\langle B^l_A_1\rangle, M_1) = tp_{\alpha}(\langle B^l_A_2\rangle, M_2)\]

(more pedantically \(tp_{\alpha}(\langle B^l_A_2\rangle, M_2) = f[tp_{\alpha}(\langle B^l_A_1\rangle, M_1)]\) or considered the \(A_\ell, B_\ell\) as indexed sets).

We delay the proof (parts (1), (2), (3) are proved after 1.22, part (4), (6) after 1.23, and after it parts (5), (7), (8). Part (9) is obvious.

\[\text{(401) revisi...} \]
1.15 Definition. 1) If \((B, A) \in \Gamma(M)\), \(M\) is \(\aleph_\varepsilon\)-saturated let \(\mathcal{P}\) from Claim 1.14(6) above (by 1.14(6) this is well defined as we shall prove below). 2) Let \(\mathcal{P} = \left\{ p : p \text{ is (stationary regular and) parallel to some } p' \in \mathcal{P} \right\}\).

1.16 Definition. If \(\langle N_\eta, a_\eta : \eta \in J \rangle\) is a decomposition inside \(\mathcal{C}\) for \(\ell = 1, 2\) we say that \(\langle N_1, a_\eta : \eta \in J \rangle \leq_{\text{direct}} \langle N_2, a_\eta : \eta \in J \rangle\) if:

(a) \(N^1_0 \prec N^2_0\)
(b) \(N^2_0 \uplus \left\{ a_\alpha : \langle \alpha \rangle \in J \right\} \cap N^1_0\)
(c) for \(\eta \in J \setminus \{ \langle \rangle \}, N_\eta \) is \(\aleph_\varepsilon\)-prime over \(N^1_\eta \cup N^2_\eta\).

1.17 Claim. 1) \(M\) is \(\aleph_\varepsilon\)-prime over \(A\) iff \(M\) is \(\aleph_\varepsilon\)-primary over \(A\) iff \(M\) is \(\aleph_\varepsilon\)-saturated, \(A \subseteq M, M\) is \(\aleph_\varepsilon\)-atomic over \(A\) (see 1.1(18)) for every \(I \subseteq M\) indiscernible over \(A\) we have: dim \(I, M\) \(\leq \aleph_0\) iff \(M\) is \(\aleph_\varepsilon\)-saturated, \(A \subseteq M, M\) is \(\aleph_\varepsilon\)-atomic over \(A\) and for every finite \(B \subseteq M\) and regular (stationary) \(p \in S(A \cup B)\), we have dim \(p, M\) \(\leq \aleph_0\).

2) If \(N_1, N_2\) are \(\aleph_\varepsilon\)-prime over \(A\), then they are isomorphic over \(A\).

**Proof.** By [Sh:c, IV,4.18] (see Definition [Sh:c, IV,4.16], noting that we replace \(F_{\aleph_0}\) by \(\aleph_\varepsilon\) and that part (4) there disappears when we are speaking on \(F_{\aleph_0}\)).

However, we need more specific information saying that “minor changes” preserve being \(\aleph_\varepsilon\)-prime; this is done in 1.18 below, parts of it are essentially done in [Sh 225] but we give full proof.

1.18 Fact. 0) If \(A\) is countable, \(N\) is \(\aleph_\varepsilon\)-primary over \(A\) then \(N\) is \(\aleph_\varepsilon\)-primary over \(\emptyset\).

1) If \(N\) is \(\aleph_\varepsilon\)-prime over \(\emptyset\), \(A\) countable, \(N^+\) is \(\aleph_\varepsilon\)-prime over \(N \cup A\) then \(N^+\) is \(\aleph_\varepsilon\)-prime over \(\emptyset\).

2) If \(\langle N_n : n < \omega \rangle\) is increasing, each \(N_n\) is \(\aleph_\varepsilon\)-prime over \(\emptyset\) or just \(\aleph_\varepsilon\)-constructible over \(\emptyset\) and \(N_\omega\) is \(\aleph_\varepsilon\)-prime over \(\emptyset\), (note that if each \(N_n\) is \(\aleph_\varepsilon\)-saturated then \(N_\omega = \bigcup_{n<\omega} N_n\)).

2A) If \(N\) is \(\aleph_\varepsilon\)-prime over \(C, \bar{a}, \bar{b} \subseteq N, \text{tp}(\bar{b}, \bar{a})\) is regular (stationary) and orthogonal to \(C\) then dim \(\text{tp}(\bar{b}, \bar{a}), N\) \(\leq \aleph_0\); also if \(q \in S(C \cup \bar{a})\) is a non-forking extension
Proof. In the proof of 1.18(1)-(6),(10) we do not use “T has NDOP”.

1.19 Remark. In the proof of 1.18(1)-(6),(10) we do not use “T has NDOP”.

Proof. 0) There is \( \{a_\alpha : \alpha < \alpha^*\} \), a list of members of \( N \) in which every member of \( N\setminus A \) appears such that for \( \alpha < \alpha^* \) we have: \( tp(a_\alpha, A \cup \{a_\beta : \beta < \alpha\}) \) is \( \aleph_\varepsilon \)-isolated (which means just \( F^\varepsilon_{\aleph_0} \)-isolated).

[Why? by the definition of “\( N \) is \( \aleph_\varepsilon \)-primary over \( A^* \)”]. Let \( \{b_n : n < \omega\} \) list \( A \) (if \( A = \emptyset \) the conclusion is trivial so without loss of generality \( A \neq \emptyset \)), hence we can find such a sequence \( \langle b_n : n < \omega \rangle \). Now define \( \beta^* = \omega + \beta \) and \( b_{\omega+\alpha} = a_\alpha \).
for $\alpha < \alpha^*$. So $\{b_\beta : \beta < \beta^*\}$ lists the elements of $N$ (possibly with repetitions, remember $A \subseteq N$ and check). We claim that $\text{tp}(b_\beta, \{b_\gamma : \gamma < \beta\})$ is $F_{\aleph_0}^*$-isolated for $\beta < \beta^*$.

(Why? if $\beta \geq \omega$, let $\beta' = \beta - \omega$ (so $\beta < \alpha^*$), now the statement above means $\text{tp}(a_{\beta'}, A \cup \{a_\gamma : \gamma < \beta\})$ is $F_{\aleph_0}^*$-isolated which we know; if $\beta < \omega$ this statement is trivial]. By the definition of \( \text{"F}_{\aleph_0}^*\text{-primary"} \), clearly $\langle b_\beta : \beta < \omega + \alpha \rangle$ exemplify that $N$ is $F_{\aleph_0}^*\text{-primary over } \emptyset$.

1) Note

\( (\ast)_1 \) if $N$ is $\aleph_c$-primary over $\emptyset$ and $A \subseteq N$ is finite then $N$ is $\aleph_c$-primary over $A$ [why? see [Sh: c, IV, 3.12](3), p.180 (of course, using [Sh: c, IV, Table 1, p.169] for $\text{F}_{\aleph_0}^*$)]

\( (\ast)_2 \) if $N$ is $\aleph_c$-primary over $\emptyset$, $A \subseteq N$ is finite and $p \in S^m(N)$ does not fork over $A$ and $p \upharpoonright A$ is stationary then for some $\{\bar{a}_\ell : \ell < \omega\}$ we have: $\bar{a}_\ell \in N$ realize $p$, $\{\bar{a}_\ell : \ell < \omega\}$ is independent over $A$ and $p \upharpoonright (A \cup \bigcup_{\ell < \omega} \bar{a}_\ell) \vdash p$ [why? [Sh: c, IV, proof of 4.18] (i.e. by it and [Sh: c, 4.9](3), 4.11) or let $N'$ be $\aleph_c$-primary over $A \cup \bigcup_{\ell < \omega} \bar{a}_\ell$ and note: $N'$ is $\aleph_c$-primary over $A$ (proof like the one of 1.18(0)) but also $N$ is $\aleph_c$-primary over $A$ so by uniqueness of the $\aleph_c$-primary model $N'$ is isomorphic to $N$ over $A$, so without loss of generality $N' = N$; and easily $N'$ is as required].

Now we can prove 1.18(1), for any $\bar{c} \in \omega^+A$, we can find a finite $B_1^\bar{c} \subseteq N$ such that $\text{tp}(\bar{c}, N)$ does not fork over $B_1^\bar{c}$, let $\bar{b}_\ell \in \omega^+N$ realize $\text{stp}(\bar{a}, B_1^\bar{a})$ and let $B_c = B_1^\bar{c} \cup \bar{b}_c$, so $\text{tp}(\bar{c}, N)$ does not fork over $B_c$ and $\text{tp}(\bar{c}, B_c)$ is stationary, hence we can find $\langle \bar{a}_\ell^\bar{c} : \ell < \omega\rangle$ as in $(\ast)_2$ (for $\text{tp}(\bar{c}, B_c)$). Let $A' = \cup\{B_c : \bar{c} \in \omega^+A\} \cup \{ \bar{a}_\ell^\bar{c} : \bar{c} \in \omega^+A \text{ and } \ell < \omega\}$, so $A'$ is a countable subset of $N$ and $\text{tp}(A, A') \vdash \text{tp}(A, N) = \text{stp}(A, N)$. As $N$ is $\aleph_c$-primary over $\emptyset$ we can find a sequence $\langle d_\alpha : \alpha < \alpha^* \rangle$ and $\langle w_\alpha : \alpha < \alpha^* \rangle$ such that $N = \{d_\alpha : \alpha < \alpha^*\}$ and $w_\alpha \subseteq \alpha$ is finite and $\text{stp}(d_\alpha, \{d_\beta : \beta \in w_\alpha\}) \vdash \text{stp}(d_\alpha, \{d_\beta : \beta < \alpha\})$ and $\beta < \alpha \Rightarrow d_\beta \neq d_\alpha$.

We can find a countable set $W \subseteq \alpha^*$ such that $A' \subseteq \{d_\alpha : \alpha \in W\}$ and $\alpha \in W \Rightarrow w_\alpha \subseteq W$. Let $A'' = \{a_\alpha : \alpha \in W\}$. By [Sh: c, IV, §2, §3] without loss of generality $W$ is an initial segment of $\alpha^*$.

Easily

\[
\alpha < \alpha^* \land \alpha \notin W \Rightarrow \text{stp}(d_\alpha, \{d_\beta : \beta \in w_\alpha\}) \vdash \text{stp}(d_\alpha, A \cup \{d_\beta : \beta < \alpha\}).
\]
As $N^+$ is $\aleph_\epsilon$-primary over $N \cup A$ we can find a list $\{d_\alpha : \alpha \in [\alpha^*, \alpha^{**})\}$ of $N^+ \setminus (N \cup A)$ such that $tp(d_\alpha, N \cup A \cup \{d_\beta : \beta \in [\alpha^*, \alpha^{**})\})$ is $\aleph_\epsilon$-isolated. So $\langle d_\alpha : \alpha \notin W, \alpha < \alpha^{**}\rangle$ exemplifies that $N^+$ is $\aleph_\epsilon$-primary over $A \cup A''$, hence by 1.18(0) we know that $N^+$ is $\aleph_\epsilon$-primary over $\emptyset$.

2) We shall use the characterization of “$N$ is $F^{\alpha}_{\aleph_0}$-prime over $A$” in 1.17, more exactly we use the last condition in 1.17(1) for $A = \emptyset, M = N_\omega$. Clearly $N_\omega$ is $\aleph_\epsilon$-saturated (as it is $\aleph_\epsilon$-prime over $\bigcup_{n<\omega} N_n$). Suppose $B \subseteq N_\omega$ is finite and $p \in S(B)$ is (stationary and) regular.

**Case 1:** $p$ not orthogonal to $\bigcup_{n<\omega} N_n$.

So for some $n < \omega, p$ is not orthogonal to $N_n$, hence there is a regular $p_1 \in S(N_n)$ such that $p, p_1$ are not orthogonal. Let $A_1 \subseteq N_n$ be finite such that $p_1$ does not fork over $A$ and $p_1 \upharpoonright A_1$ is stationary. So by [Sh:c, V, 2] we know $\dim(p, N_\omega) = \dim(p_1 \upharpoonright A_1, N_\omega)$, hence it suffices to prove that the latter is $\aleph_0$. Now this holds by [Sh:c, V, 1.16](3), p. 237 or imitate the proof of (*)$_2$ above.

**Case 2:** $p$ is orthogonal to $\bigcup_{n<\omega} N_n$.

Note that if each $N_n$ is $\aleph_\epsilon$-prime then $\bigcup_{n<\omega} N_n$ is $\aleph_\epsilon$-saturated hence $N = \bigcup_{n<\omega} N_n$ hence this case does not arise. Let $A = \bigcup_{n<\omega} N_n$, so $\dim(p, N) \leq \aleph_0$ follows from 2A) below.

Alternatively (and work even if we replace $N_\eta$ by a set $A_n, F^{\alpha}_{\aleph_0}$-constructible over $\emptyset$, see below).

2A) By 2B).

2B) The first inequality as immediate (as $T$ is superstable and $\bar{a}, \bar{b}$ are finite), so let us concentrate on the second. Let $B \subseteq C$ be a finite set such that $tp_*(\bar{a} \bar{b}, C)$ does not fork over $B$ and $stp_*(\bar{a} \bar{b}, B) \models stp_*(\bar{a} \bar{b}, C)$. Recall $q \in S(N)$ extend $\bar{a} \bar{b}$ and do not fork over $\bar{b}$, let $b^* \in C$ realize $q$ and let $q_1 = stp(\bar{a} b^*, B \cup \bar{b})$ and $q_2 = stp(\bar{b}, C \cup \bar{b})$. Now by the assumption of our case $q_1$ is orthogonal to $tp_*(C, B)$ hence (see [Sh:c, V, 3]) $q_1 \vdash q_2$ and let $\{a_\alpha : \alpha < \alpha^*\} \subseteq (q_1 \upharpoonright (\bar{b} \cup B))(N)$ be a maximal set independent over $C + \bar{b}$, so $|\alpha^*| \leq \dim(\bar{a}/(C + \bar{b}), N)$ and $q \upharpoonright (C \cup \bar{b} \cup \{a_\alpha : \alpha < \alpha^*\}) \models q$. Also clearly $stp_*(\{a_\alpha : \alpha < \alpha^*\}, \bar{b} \cup B) \models stp_*(\{a_\alpha : \alpha < \alpha^*\}, \bar{b} \cup C)$. Together $\dim(q_1, N) \leq |\alpha^*|$ and as $|B| < \aleph_0 = \kappa_r(T) \leq \kappa_s(B)$, clearly $\dim(\bar{a}/\bar{b}, N) < \aleph_0 + \dim(q_1, N)^+$, so we are done.

We can use a different proof for part (2), note:

$\otimes_1$ if $\kappa = cf(\kappa) \geq \kappa_r(T)$ and $B_\alpha$ is $F^\alpha_\kappa$-constructible over $A$ for $\alpha < \delta, \delta \leq \kappa$
and \( \alpha < \beta < \delta \Rightarrow B_\alpha \subseteq B_\beta \) then \( \bigcup_{\alpha<\delta} B_\alpha \) is \( \mathbf{F}_\kappa^a \)-constructible over \( A \)

[why? see [Sh:c, IV,§3], [Sh:c, IV,5.6,p.207] for such arguments, assume \( \mathcal{A}_\alpha = \langle A, \{a_i^\alpha : i < \iota_\alpha\}, \{B_i^\alpha : i < \iota_\alpha\} \rangle \) is an \( \mathbf{F}_\kappa^a \)-construction of \( B_\alpha \) over \( A \).

Without loss of generality \( i < j < i_\alpha \Rightarrow a_i^\alpha \neq a_i^\alpha \), and choose by induction on \( \zeta, \langle u_\zeta^\alpha : \alpha < \delta \rangle \) such that: \( u_\zeta^\alpha \subseteq i_\alpha, u_\zeta^\alpha \) increasing continuous in \( i, u_0^\alpha = \emptyset, |u_{\zeta+1}^\alpha| \leq \kappa, u_\zeta^\alpha \) is \( \mathcal{A}_\alpha \)-closed and \( \alpha < \beta < \delta \) implies \( \{a_j^\beta : j \in u_\zeta^\alpha\} \subseteq \{a_j^\beta : j \in u_\zeta^\alpha\} \) and \( \text{tp}_*(\{a_j^\beta : i \in u_\zeta^\alpha\}, \cup \{a_i^\alpha : i < i_\alpha\}) \) does not fork over \( A \cup \{a_i^\alpha : i < i_\alpha\} \). Now find a list \( \langle a_j : j < j^* \rangle \) such that for each \( \zeta, \{j : a_j \in a_i^\alpha : i < i_\alpha \} \) for some \( \alpha < \delta, \epsilon < \zeta \) is an initial segment \( \beta_\zeta \) of \( j^* \) and \( \beta_{\zeta+1} \leq \beta_\zeta + \kappa \).

We use \( \otimes_1 \) for \( \kappa = \aleph_0 \).

So each \( N_n \) is \( \aleph_\epsilon \)-constructible over \( \emptyset \) hence \( \bigcup_{n<\omega} N_n \) is \( \aleph_\epsilon \)-constructible over \( \emptyset \) and also \( N_\omega \) is \( \aleph_\epsilon \)-constructible over \( \bigcup_{n<\omega} N_n \) hence \( N_\omega \) is \( \aleph_\epsilon \)-constructible over \( \emptyset \). But \( N_\omega \) is \( \aleph_\epsilon \)-saturated hence \( N_\omega \) is \( \aleph_\epsilon \)-primary over \( \emptyset \). Similarly we use: if \( B \) is \( \mathbf{F}_\kappa^a \)-constructible over \( A, \kappa \geq \kappa_\tau(T) \) and \( I_1 \) is indiscernible over \( A, |I_1| > \kappa \) then for some \( J \subseteq I_1 \) of cardinality \( \leq \kappa, I_1 \cap J \) is an indiscernible set over \( B \).

3) Suppose \( N'_2 \) is \( \aleph_\epsilon \)-saturated and \( N_1 + a \subseteq N'_3 \). As \( N_2 \) is \( \aleph_\epsilon \)-prime over \( N_0 + \bar{a} \) and \( N_0 + \bar{a} \subseteq N_1 + \bar{a} \subseteq N'_3 \) we can find an elementary embedding \( f_0 \) of \( N_2 \) into \( N'_3 \) extending \( \text{id}_{N_0+a} \). By [Sh:c, V,3.3], the function \( f_1 = f_0 \cup \text{id}_{N_1} \) is an elementary mapping and clearly \( \text{Dom}(f_1) = N_1 \cup N_2 \). As \( N_3 \) is \( \aleph_\epsilon \)-prime over \( N_1 \cup N_2 \) and \( f_1 \) is an elementary mapping from \( N_1 \cup N_2 \) into \( N'_3 \) which is an \( \aleph_\epsilon \)-saturated model there is an elementary embedding \( f_3 \) of \( N_3 \) into \( N'_3 \) extending \( f_2 \). So as for any such \( N'_3 \) there is such \( f_3 \), clearly \( N_3 \) is \( \aleph_\epsilon \)-prime over \( N_1 + \bar{a} \), as required.

4) Let \( N_0 \) be \( \aleph_0 \)-prime over \( \emptyset \) and let \( \{p_i : i < \alpha\} \subseteq S(N_0) \) be a maximal family of pairwise orthogonal regular types. Let \( I_i = \{\bar{a}_n^i : n < \omega \} \subseteq C \) be a set of elements realizing \( p_i \) independent over \( N_0 \) and let \( I = \bigcup_{i<\alpha} I_i \) and \( N'_i \) be \( \mathbf{F}_{\aleph_0}^a \)-prime over \( N_0 \cup I \).

Now

\[ (*) \text{ if } \bar{a}, \bar{b} \subseteq N'_1 \text{ and } \bar{a}/\bar{b} \text{ is regular (hence stationary), then } \dim(\bar{a}/\bar{b}, N'_1) \leq \aleph_0. \]

[Why? If \( \bar{a}/\bar{b} \perp N_0 \) then \( \dim(\bar{a}/\bar{b}, N'_1) \leq \aleph_0 \) by part (2A) and the choice of the \( p_i \)

and \( I_i \) for \( i < \alpha \). If \( \bar{a}/\bar{b} \perp N_0 \), then for some \( \bar{b}' \perp \bar{a} \subseteq N_0 \) realizing \( \text{stp}(\bar{b}' \perp \bar{a}, \emptyset) \), we have \( \bar{a}'/\bar{b}' + \bar{a}/\bar{b} \) hence \( \dim(\bar{a}/\bar{b}, N'_1) = \dim(\bar{a}'/\bar{b}', N'_1) \), so without loss of generality \( \bar{b}' \perp \bar{a} \subseteq N_0 \), similarly without loss of generality there is \( i(*) < \alpha \) such that \( \bar{a}/\bar{b} \subseteq p_i(*) \) and \( p_i(*) \) does not fork over \( \bar{b} \) now easily \( \dim(\bar{a}/\bar{b}, N'_1) = \dim(\bar{a}/\bar{b}, N_0) + \dim(p_i(*), N_0) \leq \aleph_0 + \aleph_0 = \aleph_0 \) (see [Sh:c, V,1.6](3)). So we have proved \( (*) \).]
Now use 1.17(1) to deduce: $N_1'$ is $\mathbf{F}_{\aleph_1}^a$-prime over $\emptyset$ hence (by uniqueness of $\aleph_c$-prime model, 1.17(2)) $N_1' \cong N_1$.

By renaming without a loss of generality $N_1' = N_1$. Now

$$\text{(**)(a) } (N_1, c)_{c \in N_0}, (N_2, c)_{c \in N_0} \text{ are } \aleph_c\text{-saturated and }$$

$$(\beta) \text{ if } \bar{a} \in \mathcal{C}, \bar{b} \in N_\ell, \bar{a}/\bar{b} \text{ a regular type and } \bar{a}/\bar{b} \text{ (for } \ell = 1 \text{ or } \ell = 2),$$

then

$$\dim(\bar{a}/(\bar{b} \cup N_0), N_\ell) = \aleph_0.$$

[Why? Remember that we work in $(C^\text{eq}, c)_{c \in N_0}$. The “$\aleph_c$-saturated” follows from the second statement.

Note: $\dim(\bar{a}/(\bar{b} \cup N_0), N_\ell) \leq \dim(\bar{a}/\bar{b}, N_\ell) \leq \aleph_0$ (first inequality by monotonicity, second inequality by 1.17(1) and the assumption “$N_\ell$ is $\aleph_c$-prime over $\emptyset$”). If $\bar{a}/\bar{b}$ is not orthogonal to $N_0$ then for some $i < \alpha$ we have $p_i \pm (\bar{a}/\bar{b})$ so easily (using “$N_\ell$ is $\aleph_c$-saturated”) we have $\dim(\bar{a}/(\bar{b} \cup N_0), N_\ell) = \dim(p_i, N_\ell) \geq ||I_i|| = \aleph_0$; so together with the previous sentence we get equality. Lastly, if $\bar{a}/\bar{b} \subseteq N_0$ by part (2B) of 1.18, we have $\dim(\bar{a}/(\bar{b} \cup N_0), N_\ell) < \aleph_0 \Rightarrow \dim(\bar{a}/\bar{b}, N_\ell) < \aleph_0$ which contradicts the assumption “$\aleph_c$-saturated”]. So we have proved (**) hence by 1.17(1) we get “$N_1, N_2$ are isomorphic over $N_0'$” as required.

5) This is proved similarly as if $N$ is $\aleph_c$-prime over $A$ and $B \subseteq N$ is $\varepsilon$-finite then $N$ is $\aleph_c$-prime over $A + B$ and also over $A'$ if $A + B \subseteq A' \subseteq \text{acl}(A + B)$, see part (10).

6) By [Sh:c, V.3.2].

7) First assume that $A_2^c \subseteq N_1$ and $a/N_1$ is regular. As $N_1$ is $\aleph_c$-prime over $N_0 \cup N_1'$ and as $T$ has NDOP (i.e. does not have DOP) we know (by [Sh:c, X.2.1.2.2, p.512]) that $N_1$ is $\aleph_c$-minimal over $N_0 \cup N_1'$ and $a/N_1$ is not orthogonal to $N_0$ or to $N_1'$. But $a/N_1 \perp N_0$ by an assumption, so $a/N_1$ is not orthogonal to $N_1'$ hence there is a regular $p' \in S(N_1')$ not orthogonal to $\frac{a}{N_1}$ hence (by [Sh:c, V.1.12, p.236]) $p'$ is realized say by $a' \in N_2$. By [Sh:c, V.3.3], we know that $N_2$ is $\aleph_c$-prime over $N_1 + a'$. We can find $N_2'$ which is $\aleph_c$-prime over $N_1' + a'$ and $N_2''$ which is $\aleph_c$-prime over $N_1 \cup N_2$ hence by part (3) of 1.18 we know that $N_2''$ is $\aleph_c$-prime over $N_1 + a'$ so by uniqueness, i.e. 1.17(1), without loss of generality $N_2'' = N_2$ hence we are done. In general by induction on $\alpha$ choose $N_{2,0}^\alpha$ such that $N_{2,0}^\alpha$ is $\aleph_c$-prime over $N_1' \cup A_2^c, N_{2,0}^\alpha$ is increasing with $\alpha$ and $N_1 \cup N_{2,0}^\alpha$. Easily for some $\alpha, N_{2,0}^\alpha$ is defined $N_1'$ but not $N_{2,0}^\alpha+1$. Necessarily $N_2$ is $\aleph_c$-prime over $N_1' \cup N_{2,0}^\alpha$. Lastly let $a' \in N_{2,0}^\alpha$ be such that $\text{tp}(a', N_1 \cup N_{2,0}^\alpha)$ does not fork over $N_1 + a'$. Easily $N_{2,0}^\alpha$ is $\aleph_c$-prime over $N_1' + a'$ by (1.17(1)).

8) Similar easier proof.

9) Let $N_0'$ be $\aleph_c$-prime over $A$ such that $B \cup N_0'$, and let $N_1'$ be $\aleph_c$-prime over
\[N_0' \cup B.\] By 1.18(1), we know that \(N_0'\) is \(R_e\)-prime over \(\emptyset\), and by 1.18(10) below \(N_1'\) is \(R_e\)-prime over \(A \cup B\), hence by 1.17(2) we know that \(N_1', N_1\) are isomorphic over \(A \cup B\) hence without loss of generality \(N_1' = N_1\) and so \(N_0 = N_0'\) is as required.  


\[\text{□}_{1.18}\]

1.20 Fact. Assume \(\langle N_1^\eta, a_\eta : \eta \in I \rangle \leq_{\text{direct}} \langle N_2^\eta, a_\eta : \eta \in I \rangle\) (see Definition 1.16) and \(A \subseteq B \subseteq N_{1,0}^\eta\) and \(\bigwedge_{\eta \in I} N_2^\eta < M\).

(1) If \(\nu = \eta^\ast \langle \alpha \rangle \in I\), then \(N_2^\eta \bigcup N_1^\nu\) and even \(N_2^\eta \bigcup \left( \bigcup_{\rho \in I} N_1^\rho \right)\); and

\[\eta \triangleleft \nu \in I \text{ implies } N_2^\nu \bigcup \left( \bigcup_{\rho \in I} N_1^\rho \right)\].

(2) \(\langle N_2^\eta, a_\eta : \eta \in I \rangle\) is an \(R_e\)-decomposition inside \(M\) above \(\langle B_A \rangle\) iff \(\langle N_1^\eta, a_\eta : \eta \in I \rangle\) is an \(R_e\)-decomposition inside \(M\) above \(\langle B_A \rangle\).

(3) Similarly replacing “\(R_e\)-decomposition inside \(M\) above \(\langle B_A \rangle\)” by “\(R_e\)-decomposition of \(M\) above \(\langle B_A \rangle\)”.

\[\text{Proof.}\] 1) We prove the first statement by induction on \(\ell g(\eta)\). If \(\eta = < >\) this is clause (b) by the Definition 1.16 and clause (d) of Definition 1.11(1) (and [Sh:c, V.3.2]). If \(\eta \\

\[\leq_{\text{direct}}\langle \eta^{-} \rangle\) (by condition (e) of Definition 1.11(1)). By the induction hypothesis \(N_2^\eta \bigcup N_1^\eta\) and we know \(N_2^\eta\) is \(R_e\)-primary over \(N_1^\eta\).

\[\eta^{-} \cup N_1,\] we know this implies that no \(p \in S(N_1^\eta)\) orthogonal to \(N_1^{-}\) is realized in \(N_2^\eta\) hence \(\bar{a}_{N_1^\eta} \perp \bar{N}_2^\eta\), so \(\bar{a}_{N_1^\eta} \perp \bar{a}_{N_2^\eta}\) hence \(N_1^\eta \perp N_2^\eta\) hence \(N_1^\nu \bigcup N_2^\nu\) as required.

The other statements hold by the non-forking calculus (remember if \(\eta = \nu^{-} \langle \alpha \rangle \in I\) then use \(\text{tp}(\bigcup \{N_2^\eta : \eta \leq \rho \in I \}, N_1^\eta)\) is orthogonal to \(N_1^\nu\) or see details in the proof of 1.21(1)(a)).

2) By Definition 1.16, for \(\ell = 1, 2\) we have: \(\langle N_2^\eta, a_\eta : \eta \in I \rangle\) is a decomposition inside \(C\) and by assumption \(\bigwedge_{\eta \in I} N_1^\eta < N_2^\eta < M\). So for \(\ell = 1, 2\) we have to prove

\[\langle N_2^\eta, a_\eta : \eta \in I \rangle\] is an \(R_e\)-decomposition inside \(M\) for \(\langle B_A \rangle\)” assuming this holds for

\(1 - \ell\). We have to check Definition 1.11(1).
Clauses 1.5(1)(a),(b) for \( \ell \) holds because they hold for \( 1 - \ell \).

Clause 1.5(1)(c) holds as by the assumptions \( A \subseteq B \subseteq N^1_{\prec <0>} < N^2_{\prec <0>}, A \subseteq N^1_{\prec <\ell>} \) and \( N^1_{\prec <0>} \bigcup N^2_{\prec <\ell>} \).

Clauses 1.5(1)(d),(e),(f),(h) holds as \( \langle N^1_{\eta}, a_\eta : \eta \in I \rangle \) is a decomposition inside \( C \) (for \( \ell = 1 \) given, for \( \ell = 2 \) easily checked).

Clause 1.5(1)(g) holds as \( \bigwedge_{\eta} N^1_{\eta} < N^2_{\eta} < M \) is given and \( M \) is \( \aleph_\nu \)-saturated.

3) First we do the “only if” direction; i.e. prove the maximality of \( \langle N^1_{\eta}, a_\eta : \eta \in I \rangle \) as an \( \aleph_\nu \)-decomposition inside \( M \) for \( (\mathcal{R}) \) (i.e. condition (i) from 1.11(2)), assuming it holds for \( \langle N^2_{\eta}, a_\eta : \eta \in I \rangle \). If this fails, then for some \( \eta \in I \setminus \{<>\} \) and \( a \in M, \{a_{\eta^{-}\langle<>\alpha\rangle} : \eta^{-} < \alpha > \in I\} \cup \{a\} \) is independent over \( N^1_{\eta} \) and \( a \not\in \{a_{\eta^{-}\langle<>\alpha\rangle} : \eta^{-} < \alpha \} \) and \( \alpha_{\eta^{-}} \perp N^1_{\eta} \). Hence, if \( \eta^{-}\langle<>\alpha\rangle \in I \) for \( \ell < k \) then \( \bar{a} = \langle a^\ell\rangle^{-}\langle<>\alpha_{\ell}\rangle : \ell < k \) realizes over \( N^1_{\eta} \) a type orthogonal to \( N^1_{\eta} \), but \( N^1_{\eta} \prec N^1_{\eta}, N^1_{\eta} \prec N^2_{\eta} \) and \( N^1_{\eta} \bigcup N^2_{\eta} \) (see 1.20(1), hence (by [Sh:c, V,2.8])

\[
\text{tp}(\bar{a}, N^2_{\eta}) \perp N^2_{\eta} \quad \text{hence \{a\} \cup \{a_{\eta^{-}\langle<>\alpha\rangle} : \ell < k \} \text{ is independent over } N^2_{\eta} \text{ but } k, \eta^{-}\langle<>\alpha_{\ell}\rangle \in I \text{ for } \ell < k \text{ were arbitrary so } \{a\} \cup \{a_{\eta^{-}\langle<>\alpha\rangle} : \eta^{-}\langle<>\alpha\rangle \in I\} \text{ is independent over } N^2_{\eta} \text{ contradicting condition (i) from Definition 1.11(2) for } \langle N^2_{\eta}, a_\eta : \eta \in I \rangle.
\]

For the other direction use: if the conclusion fails, then for some \( \eta \in I \setminus \{<>\} \) and \( a \in M \setminus N^2_{\eta} \{a_{\eta^{-}\langle<>\alpha\rangle} : \eta^{-} < \alpha > \in I\} \) the set \( \{a_{\eta^{-}\langle<>\alpha\rangle} : \eta^{-}\langle<>\alpha\rangle \in I\} \cup \{a\} \) is independent of \( N^2_{\eta} \) and \( \text{tp}(a, N^2_{\eta}) \) is orthogonal to \( N^2_{\eta} \); let \( N' \prec M \) be \( \aleph_\nu \)-prime over \( N^2_{\eta} + a \). But \( N^2_{\eta} \) is \( \aleph_\nu \)-prime over \( N^1_{\eta} \cup N^2_{\eta} \) (by the definition of \( \prec \)-direct) so by NDOP \( \text{tp}(a, N^2_{\eta}) \pm N^1_{\eta} \) hence there is a regular \( q \in S(N^1_{\eta}) \) such that \( q \pm \text{tp}(a, N^2_{\eta}) \). Hence some \( a' \in N' \) realizes \( q \), clearly \( \{a_{\eta^{-}\langle<>\alpha\rangle} : \eta^{-} < \alpha > \in I\} \cup \{a'\} \) is independent over \( N^2_{\eta} \) (and \( a' \not\in \{a_{\eta^{-}\langle<>\alpha\rangle} : \eta^{-} < \alpha \} \)) hence over \( (N^2_{\eta}, N^1_{\eta}) \) and easily we get contradiction. \( \square_{1.20} \)

1.21 Fact. Assume \( \langle N^1_{\eta}, a^1_\eta : \eta \in I \rangle \) is an \( \aleph_\nu \)-decomposition inside \( M \).

1) If \( N^1_{\eta} < N^2_{\eta} < M, N^2_{\eta} \) is \( \aleph_\nu \)-prime over \( \emptyset \) and \( N^2_{\eta} \bigcup \{a_{\eta^{-}\langle<>\alpha\rangle} : \eta^{-} < \alpha > \in I\} \) then

\[
(a) \begin{bmatrix}
N^2_{\eta} \\
N^1_{\eta} \bigcup_{\eta \in I} N^1_{\eta}
\end{bmatrix}
\]

and
(β) we can find \( N^2_\eta (\eta \in I \backslash \{<>\}) \) such that \( N^2_\eta \prec M \), and
\[
\langle N^1_\eta, a^1_\eta : \eta \in I \rangle \leq_{\text{direct}} \langle N^2_\eta, a^1_\eta : \eta \in I \rangle.
\]

2) If \( C_b \frac{a^1_\eta}{N^1_\eta} \subseteq N^0_\eta < N^1_\eta \) or at least \( N^0_\eta < N^1_\eta \) and \( \frac{a^1_\eta}{N^1_\eta} \pm N^0_\eta \) whenever \( \alpha > \in I \) then we can find \( N^0_\eta \prec M \) and \( a^0_\eta \in N_\eta \) (for \( \eta \in I \backslash \{<>\} \)) such that
\[
\langle N^0_\eta, a^0_\eta : \eta \in I \rangle \leq_{\text{direct}} \langle N^1_\eta, a^0_\eta : \eta \in I \rangle.
\]

3) In part (2), if in addition we are given \( (B^*_\eta : \eta \in I) \) such that \( B^*_\eta \) is an \( \varepsilon \)-finite subset of \( N_\eta \), \( \text{tp}(B^*_\eta, N_\eta) \) does not fork over \( N^0_\eta \) and \( B^*_\eta \subseteq N^0_\eta \) then we can demand in the conclusion that \( \eta \in I \Rightarrow B^*_\eta \subseteq N^0_\eta \).

Proof. 1) For proving (α) let \( \{\eta_i : i < i^*\} \) list the set \( I \) such that \( \eta_i \prec \eta_j \Rightarrow i < j \), so \( \eta_0 = <> \) and without loss of generality for some \( \alpha^* \) we have
\( \eta_i \in \{< \alpha > : < \alpha > \in I \} \Leftrightarrow i \in [1, \alpha^*) \). Now we prove by induction on \( \beta \in [1, i^*) \) that \( N^2_<> \cup \{N^1_\eta : i < \beta \} \). For \( \beta = 1 \) this is assumed. For \( \beta \) limit use the local character of non-forking.

If \( \beta = \gamma + 1 \in [1, \alpha^*) \), then by repeated use of [Sh:c, V,3.2] (as \( \{a_{\eta_j} : j \in [1, \beta]\} \) is independent over \( (N^1_<>), N^2_<> \) and \( N^1_<> \) is \( \aleph \)-saturated and \( N^1_\eta (j \in [1, \gamma]) \) is \( \aleph \)-prime over \( N^1_\gamma + a_{\eta_j} \)) we know that \( \text{tp}(a_{\eta_\gamma}, N^2_<> \cup \bigcup_{\eta_i < \gamma} N^1_{\eta_i}) \) does not fork over \( N^1_<> \). Again by [Sh:c, V,3.2], the type \( \text{tp}(N^1_{\eta_\gamma}, N^2_<> \cup \bigcup_{\eta_i < \gamma} N^1_{\eta_i}) \) does not fork over \( N^1_<> \) hence \( \bigcup_{\eta_i < \beta} N^1_{\eta_i} \cup N^2_<> \) and use symmetry.

Lastly, if \( \beta = \gamma + 1 \in [\alpha^*, i^*) \), \( \text{tp}(a_{\eta_\gamma}, N_{\eta_\gamma}) \) is orthogonal to \( N^1_<> \) and even to \( N^1_{(\eta_\gamma)_-} \) so again by non-forking and [Sh:c, V,3.2] we can do it, so clause (α) holds.

For clause (β), we choose \( N^2_{\eta_i} \) for \( i \in [1, i^*) \) by induction on \( i < i^* \) such that \( N^2_{\eta_i} \prec M \) is \( \aleph \)-prime over \( N^2_{\eta_i} \cup N^1_{\eta_i} \). By the non-forking calculus we can check Definition 1.7.

2) We let \( \{\eta_i : i < i^*\} \) be as above, now we choose \( N^0_{\eta_i}, a^0_{\eta_i} \) by induction on \( i \in [1, i^*) \) such that:
\[
(\ast) \quad N^0_{\eta_i} < N^1_{\eta_i} \quad \text{and} \quad \bigcup_{N^0_{\eta_i}} N^1_{\eta_i} \quad \text{and} \quad N^1_{\eta_i} \text{ is } \aleph \text{-prime over } N^0_{\eta_i} \cup N^1_{\eta_i}
\]
\[
(\ast\ast) \quad a^0_{\eta_i} \in N^0_{\eta_i} \quad \text{and} \quad N^0_{\eta_i} \text{ is } \aleph \text{-prime over } N^0_{\eta_i} + a^0_{\eta_i}.
\]

The induction step has already been done: if \( \ell g(\eta_i) > 1 \) by 1.18(7) and if \( \ell g(\eta_i) = 1 \)
Proof of 1.14. □

1.22 Fact. 1) If \( \langle N^1_\eta, a_\eta : \eta \in I \rangle \leq^*_{\text{direct}} \langle N^2_\eta, a_\eta : \eta \in I \rangle \) and both are \( \aleph \)-decompositions of \( M \) above \( \langle B^2_\eta \rangle \), then
\[
\mathcal{P}(\langle N^1_\eta, a_\eta : \eta \in I \rangle, M) = \mathcal{P}(\langle N^2_\eta, a_\eta^2 : \eta \in I \rangle, M).
\]

Proof. By Definition 1.11(5) it suffices to prove, for each \( \eta \in I \setminus \{<>\} \) that

\( (*) \) for regular \( p \in S(M) \) we have
\[
p \perp N^1_\eta^- \& \ p \pm N^1_\eta \Leftrightarrow p \perp N^2_\eta^- \& \ p \pm N^2_\eta.
\]

Now consider any regular \( p \in S(M) \): first assume \( p \perp N^1_\eta^- \& \ p \pm N^1_\eta \) where \( \eta \in I \setminus \{<>\} \) so \( p \pm N^2_\eta \) (as \( N^1_\eta < N^2_\eta \) and \( p \pm N^1_\eta \)) and we can find a regular \( q \in S(N^1_\eta) \) such that \( q \pm p \); so as \( p \perp N^1_\eta^- \) also \( q \perp N^1_\eta^- \), now \( q \perp N^2_\eta^- \) (as \( N^1_\eta \bigcup N^2_\eta^- \) and \( q \perp N^1_\eta \) see [Sh:c, V,2.8]) hence \( p \perp N^2_\eta^- \).

Second, assume \( p \perp N^2_\eta^- \& \ p \pm N^2_\eta \) where \( \eta \in I \setminus \{<>\} \); remember \( N^1_\eta^- \), \( N^1_\eta \), \( N^2_\eta \), \( N^3_\eta \) are \( \aleph \)-saturated, \( N^1_\eta \bigcup N^2_\eta^- \) and \( N^2_\eta \) is \( \aleph \)-prime over \( N^1_\eta \cup N^2_\eta^- \) and \( T \) does not have \( N^1_\eta^- \)

DOP. Hence \( N^2_\eta \) is \( \aleph \)-minimal over \( N^1_\eta \cup N^2_\eta^- \) and every regular \( q \in S(N^2_\eta) \) is not orthogonal to \( N^1_\eta \) or to \( N^2_\eta^- \). Also as \( p \pm N^2_\eta \) there is a regular \( q \in S(N^2_\eta) \) not orthogonal to \( p \), so as \( p \perp N^2_\eta^- \) also \( q \perp N^2_\eta^- \); hence by the previous sentence \( q \pm N^1_\eta \) hence \( p \perp N^1_\eta \). Lastly, as \( p \perp N^2_\eta^- \) and \( N^1_\eta^- < N^2_\eta^- \) clearly \( p \perp N^1_\eta^- \), as required. □

At last we start proving 1.14.

Proof of 1.14. 1) Let \( N^0 < \mathcal{C} \) be \( \aleph \)-primary over \( A \), without loss of generality \( N^0 \bigcup B \) (but not necessarily \( N^0 < M \)), and let \( N^1 \) be \( \aleph \)-primary over \( N^0 \cup B \).

Now by 1.18(0) the model \( N^0 \) is \( \aleph \)-primary over \( \emptyset \) and by 1.18(1) the model \( N^1 \) is \( \aleph \)-primary over \( \emptyset \) hence (by 1.18(10)) is \( \aleph \)-primary over \( B \), hence without loss of generality \( N^1 < M \). Let \( N_{<>} := N^0, N_{<0>} = N^1, I = \{<>, <0>\} \) and \( a_{<>} = B \). More exactly \( a_\eta \) is such that \( \text{dcl}\{a_\eta\} = \text{dcl}(B) \). Clearly \( \langle N_\eta, a_\eta : \eta \in I \rangle \) is an \( \aleph \)-decomposition inside \( M \) above \( \langle B^1_A \rangle \). Now apply part (2) of 1.14 proved below.
2) By 1.13(4) we know $\langle N_\eta, a_\eta : \eta \in \ell \rangle$ is an $\aleph_\ell$-decomposition inside $M$, by 1.18(2) we then find $J \supseteq I$ and $N_\eta, a_\eta$ for $\eta \in J \setminus I$ such that $\langle N_\eta, a_\eta : \eta \in I \rangle$ is an $\aleph_\ell$-decomposition of $M$. By 1.18(3), $\langle N_\eta, a_\eta : \eta \in J' \rangle$ is an $\aleph_\ell$-decomposition of $M$ above $\langle \beta \rangle$ where $J' = \{ \eta \in J : \eta \in \eta \}$.

3) Part (a) holds by 1.13(2),(3). As for part (b) by 1.13(2) there is $\langle N_\eta, a_\eta : \eta \in J \rangle$, an $\aleph_\ell$-decomposition of $M$ with $I \subseteq J$; easily $[\{0\} \subseteq J \Rightarrow \eta \in I]$. $\Box$

1.23 Fact. If $\langle N_\eta^\ell, a_\eta^\ell : \eta \in \ell \rangle$ are $\aleph_\ell$-decompositions of $M$ above $\langle \beta \rangle$, for $\ell = 1, 2$ and $N_{\ell \gg} = N_{\ell \gg}^2$, then $\mathcal{P}(\langle N_\eta^1, a_\eta^1 : \eta \in I^1 \rangle, M) = \mathcal{P}(\langle N_\eta^2, a_\eta^2 : \eta \in I^2 \rangle, M)$.

Proof. By 1.14(3)(b) we can find $J^1 \supseteq I^1$ and $N_\eta^1, a_\eta^1$ for $\eta \in J^1 \setminus I^1$ such that $\langle N_\eta^1, a_\eta^1 : \eta \in J^1 \rangle$ is an $\aleph_\ell$-decomposition of $M$ and moreover we have $\eta \in J^1 \setminus I^1 \Leftrightarrow \eta \neq \{\} \& \neg \{\{0\} \neq \eta \}$. Let $J^2 = I^2 \cup (J^1 \setminus I^1)$ and for $\eta \in J^2 \setminus I^2$ let $a_\eta^2 := a_\eta^1$, $N_\eta^2 := N_\eta^1$. Easily $\langle N_\eta^2, a_\eta^2 : \eta \in J^2 \rangle$ is an $\aleph_\ell$-decomposition of $M$. By 1.13(6) we know that for every regular $p \in S(M)$ there is (for $\ell = 1, 2$) a unique $\eta(p, \ell) \in J^\ell$ such that $p \pm N\eta(p, \ell) \& p \preceq N\eta(p, \ell)$ (note $\langle \rangle$ – meaningless). By the uniqueness of $\eta(p, \ell)$, if $\eta(p, 1) \in J^1 \setminus I^1$ then as it can serve as $\eta(p, 2)$ clearly it is $\eta(p, 2)$ so $\eta(p, 2) = \eta(p, 1) \in J^1 \setminus I^1 = J^2 \setminus I^2$; similarly $\eta(p, 2) \in J^2 \setminus I^2 \Rightarrow \eta(p, 1) \in J^1 \setminus I^1$ and $\eta(p, 1) = \{\} \Leftrightarrow \eta(p, 2) = \{\}$. So

(*) $\eta(p, 1) \in I^1 \setminus \{\} \Leftrightarrow \eta(p, 2) \in I^2 \setminus \{\}.$

But

(**) $\eta(p, \ell) \in I^\ell \setminus \{\} \Rightarrow p \in \mathcal{P}(\langle N_\eta^\ell, a_\eta^\ell : \eta \in I^\ell \rangle, M).$

Together we finish. $\Box$

We continue proving 1.14.

Proof of 1.14(4). Let $A^* \subseteq M^-$ be $\varepsilon$-finite, so we can find an $\varepsilon$-finite $B^* \subseteq \cup\{N_\eta : \eta \in I\}$ such that $\text{stp}(A^*, B^*) \vdash \text{stp}(A^*, \cup\{N_\eta : \eta \in I\})$. Hence, there is a finite non empty $I^* \subseteq I$ such that $\langle \rangle \in I^*$, $I^* \cup \langle \rangle$ is closed under initial segments and $B^* \subseteq \cup\{N_\eta : \eta \in I^*\}$, so of course $\text{stp}_\nu(A^*, \cup\{N_\eta : \eta \in I^*\}) \vdash \text{stp}(A^*, \cup\{N_\eta : \eta \in I\})$.

We can also find $\langle B_\eta^* : \eta \in I^* \rangle$ such that $B_\eta^*$ is an $\varepsilon$-finite subset of $\{N_\eta : B_\eta = \text{acl}(B_\eta^*) \cup B^* \subseteq \{B_\eta^* : \eta \in I^*\}, \eta \neq \langle \rangle \Rightarrow a_\eta \subseteq B_\eta^* \text{ and if } \eta < \nu \in I^* \text{ then } B_\eta^* \subseteq B_\nu^* \text{ and } \text{tp}_\nu(B_\nu^*, N_\nu) \text{ does not fork over } B_\eta^* \text{. Without loss of generality } B \subseteq B_{\angle \angle}^*.$

For $\eta \in I \setminus I^*$ let $B_\eta^* = B_\eta^* \setminus \{\ell \} \text{ where } \ell < \ell \text{ is maximal such that } \eta \mid \ell \in I^*, \text{ such } \ell \text{ exists if } \ell g(\eta) \text{ is finite and } \langle \rangle \in I^*$. (401)
Let $N^1_\eta = N_\eta$ and $a^1_\eta = a_\eta$ for $\eta \in I$ and without loss of generality $J \neq I$ hence $J \setminus I \neq \emptyset$.

Let $N^2_\eta \prec M$ be $\mathcal{K}_\varepsilon$-prime over $\bigcup \nu \in J \setminus I N_\nu$; letting $J \setminus I = \{ \eta_1 : i < i^* \}$ be such that $[\eta_i < \eta_j \Rightarrow i < j]$ we can find $N^2_{\eta_i,i} \prec M$ (for $i \leq i^*$) increasing continuous, $N^2_{\eta_i,0} = N^2_\eta$ and $N^2_{\eta_i,i+1} \prec N^2_{\eta_i,i} \cup N_\eta$ hence over $N^2_{\eta_i,i} + a_\eta$. Lastly, without loss of generality $N^2_{\eta_i,i^*} = N^2_\eta$.

By 1.18(1), (2) we know $N^2_\eta$ is $\mathcal{K}_\varepsilon$-primary over $\emptyset$ and (using repeatedly 1.18(6) + finite character of forking) we have $N^2_\eta \bigsqcup a_{<0}$. By 1.18(4) 

$N^0_\eta, N^0_{\eta'} \prec N^0_\eta, N^0_{\eta'}$ is $\mathcal{K}_\varepsilon$-primary over $\emptyset$ and $N^1_\eta, N^2_\eta$ are isomorphic over $N^0_\eta$. By 1.21(1) we can for $\eta \in I$ choose $N^2_\eta \prec M$ with $N^1_\eta \subset N^2_\eta$ and $(N^1_\eta, a^0_\eta : \eta \in I) \leq^{\text{direct}} (N^2_\eta, a^0_\eta : \eta \in I)$. Similarly, by 1.21(2) (here $\text{Succ}(\prec) = \{ \emptyset \}$) we can choose an $\mathcal{K}_\varepsilon$-decomposition $\langle N^0_\eta, a^0_\eta : \eta \in I \rangle$ with $\langle N^0_\eta, a^0_\eta : \eta \in I \rangle \leq^{\text{direct}} (N^1_\eta, a^1_\eta : \eta \in I)$. Moreover, we can demand $\eta \in I^* \Rightarrow B^* \subseteq N^0_\eta$ using 1.21(3). By 1.13(12) + 1.14(3) we know that $\langle N^1_\eta, a^0_\eta : \eta \in I \rangle$ is an $\mathcal{K}_\varepsilon$-decomposition of $M^-$ and easily $\langle N^2_\eta, a^0_\eta : \eta \in I \rangle$ is an $\mathcal{K}_\varepsilon$-decomposition of $M$. Now choose by induction on $\eta \in I$ an isomorphism $f_\eta$ from $N^1_\eta$ onto $N^2_\eta$ such that $\nu \vDash f_\nu \subseteq f_\eta$ and $\eta \in I^* \Rightarrow f_\eta \upharpoonright B^*_\eta = \text{id}_{B^*_\eta}$. For $\eta = \emptyset$ we have chosen $N^0_\eta$ such that $N^1_\eta, N^2_\eta$ are isomorphic over $N^0_\eta$. For the induction step note that $f_\eta \cup \text{id}_{N^0_\eta}$ is an elementary mapping as $N^2_{\eta^*} \bigsqcup N^0_\eta$ and $f_\eta \cup \text{id}_{N^0_\eta}$ can be extended to an isomorphism $f_\eta$ from $N^1_\eta$ onto $N^2_\eta$ as $N^\ell_\eta$ is $\mathcal{K}_\varepsilon$-primary (in fact even $\mathcal{K}_\varepsilon$-minimal) over $N^\ell_{\eta^*} \cup N^0_{\eta^*}$ for $\ell = 1, 2$ (which holds easily). If $\eta \in I^*$ there is no problem to add $f_\eta \upharpoonright B^*_\eta = \text{id}_{B^*_\eta}$. Now by 1.13(3) the model $M^-$ is $\mathcal{K}_\varepsilon$-saturated and $\mathcal{K}_\varepsilon$-primary and $\mathcal{K}_\varepsilon$-minimal over $\bigcup_{\eta \in J} N_\eta = \bigcup_{\eta \in I} N^1_\eta$; similarly $M$ is $\mathcal{K}_\varepsilon$-primary over $\bigcup_{\eta \in I} N^2_\eta$. Now $f_\eta$ is an elementary mapping from $\bigcup_{\eta \in I} N^1_\eta$ onto $\bigcup_{\eta \in I} N^2_\eta$ hence can be extended to an isomorphism $f$ from $M^-$ into $M$. Moreover as $\text{stp}_{\ast}(A^*, \cup \{ B^*_\eta : \eta \in I^* \}) \models \text{stp}(A^*, \{ N^1_\eta : \eta \in I \})$, by [Sh:c, CH.XII, §4] we have $\text{tp}_{\ast}(A^*, \cup \{ B^*_\eta : \eta \in I^* \}) \models \text{tp}(A^*, \{ N^1_\eta : \eta \in I \})$ hence $\text{tp}_{\ast}(A^*, \cup \{ B^*_\eta : \eta \in I^* \})$ has a unique extension as a complete type over $\cup \{ N^1_\eta : \eta \in I \}$ hence over $\cup \{ N^2_\eta : \eta \in I \}$.
so without loss of generality $f \upharpoonright A^* = \text{id}_{A^*}$. By the $\aleph_\epsilon$-minimality of $M$ over $\bigcup_{\eta \in I} N_\eta$ (see 1.13(3)), $f$ is onto $M$, so $f$ is as required.

We delay the proof of 1.14(5).

Proof of 1.14(6). Let $\langle N^\ell, a^\ell_\eta : \eta \in I^\ell \rangle$ for $\ell = 1, 2$, be $\aleph_\epsilon$-decompositions of $M$ above $(S^\ell)$, so $\text{dcl}(a^\ell_\eta) = \text{dcl}(B)$. Let $p \in S(M)$, and assume that $p \in \mathcal{P}(\langle N^1_\eta, a^1_\eta : \eta \in I^1 \rangle, M)$, i.e., for some $\eta \in I^1 \setminus \{< >\}$, $(p_\eta \perp N_{\eta^\perp})$ and $p_\eta \pm N_\eta$. We shall prove that the situation is similar for $\ell = 2$; i.e., $p \in \mathcal{P}(\langle N^2_\eta, a^2_\eta : \eta \in I^2 \rangle, M)$; by symmetry this suffices.

Let $n = \ell g(\eta)$, choose $\langle B_\ell : \ell \leq n \rangle$ and $d$ such that:

(a) $A \subseteq B_0$,

(b) $B \subseteq B_1$,

(c) $a^\ell_\eta \upharpoonright \ell \subseteq B_\ell \subseteq N^1_\eta \upharpoonright \ell$, for $\ell \leq n$

(d) $B_{\ell+1} \cup \bigcup_{N^1_\eta \upharpoonright \ell} B_\ell$

(e) $\frac{B_{\ell+1}}{B_{\ell+1} \upharpoonright N^1_\eta \upharpoonright \ell} \upharpoonright N^{1+1}_\eta \upharpoonright \ell$

(\zeta) $d \in B_n$, $a^\ell_\eta \upharpoonright d$ is regular $\pm p$, (hence $\perp B_{n-1}$)

(\eta) $B_\ell$ is $\epsilon$-finite.

[Why such $\langle B_\ell : \ell \leq n \rangle$ exists? We prove by induction on $n$ that for any $\eta \in I$ of length $n$ and $\epsilon$-finite $B' \subseteq N_\eta$, there is $\langle B_\ell : \ell \leq n \rangle$ satisfying (a) – (e), (\eta) such that $B' \subseteq B_n$. Now there is $p' \in S(N^1_\eta)$ regular not orthogonal to $p$, let $B^1 \subseteq N^1_\eta$ be an $\epsilon$-finite set extending $Cb\left(p'\right)$. Applying the previous sentence to $\eta$, $B^1$ we get $\langle B_\ell : \ell \leq n \rangle$, let $d \in N_\eta$ realize $p' \upharpoonright B_n$.

Now as $n > 0$, $\text{tp}(d, B_n) \perp N_\eta$ hence $\text{tp}(d, B_n) \perp B_{n-1}$, hence $\text{tp}(d, B_n) \perp \text{tp}(N_{\eta^\perp}, B_n)$, hence as $\text{tp}(d, B_n)$ is stationary, by [Sh:c, V.1.2](2), p.231, the types $\text{tp}(d, B_n), \text{tp}(N_{\eta^\perp}, B_n)$ are weakly orthogonal so $\text{tp}(d, B_n) \vdash \text{tp}(d, N_{\eta^\perp} \cup B_n)$ hence $\frac{B_n + d}{B_{n-1} + a^\eta_\eta} \vdash \frac{B_n + d}{N^{1+1}_\eta + a^\eta_\eta}$.

Now replace $B_n$ by $B_n \cup \{d\}$ and we finish.

Note that necessarily

(\delta) $\frac{B_n \bigcup N^1_\eta \upharpoonright m \text{ for } m \leq n.}$

[Why? By the non-forking calculus].

(\epsilon) $\frac{B_n}{B_{m} \upharpoonright \ell \upharpoonright \eta} \perp_{a} B_m$ for $m < n$.

[Why? As $N^1_\eta \upharpoonright m$ is $\aleph_\epsilon$-saturated].
Choose \( D^* \subseteq N^2_{<\eta} \) finite such that \( \frac{B_n}{N^2_{<\eta} + B} \) does not fork over \( D^* + B \).

[Note: we really mean \( D^* \subseteq N^2_{<\eta} \), not \( D^* \subseteq N^1_{<\eta} \).

We can find \( N^3_{<\eta}, \aleph \)-prime over \( \emptyset \) such that \( A \subseteq N^3_{<\eta} < N^2_{<\eta} \) and \( D^* \cup A N^3_{<\eta} \) and \( N^2_{<\eta} \) is \( \aleph \)-prime over \( N^3_{<\eta} \cup D^* \) (by 1.18(9)). Hence \( B_n \cup A N^3_{<\eta} \) and \( B_n \cup B N^3_{<\eta} \) (by the non-forking calculus). As \( tp(B, N^2_{<\eta}) \) does not fork over \( A \subseteq N^3_{<\eta} \subseteq N^2_{<\eta} \) by 1.21(2) we can find \( N^3_{\eta}, a^3_{\eta} \) (for \( \eta \in I^2 \setminus \{<\eta>\} \), such that \( \langle N^3_{\eta}, a^3_{\eta} : \eta \in I \rangle \) is an \( \aleph \)-decomposition inside \( M \) above \( (B_n^I)^{\perp} \) and \( \langle N^3_{\eta}, a^3_{\eta} : \eta \in I^2 \rangle \) \( \leq \) \( \aleph \) \( \langle N^2_{\eta}, a^3_{\eta} : \eta \in I^2 \rangle \) and \( a^3_{<0>} = a^3_{<1>} \) (remember \( dcl(a^3_{<0>}) = dcl(B) \)). By 1.20(2) we know \( \langle N^3_{\eta}, a^3_{\eta} : \eta \in I^2 \rangle \) is an \( \aleph \)-decomposition of \( M \) above \( (B_n)^{\perp} \).

By 1.22 it is enough to show \( p \in \mathcal{P}((N^3_{\eta}, a^3_{\eta} : \eta \in I^2), M) \). Let \( N^4_{<\eta} < N^2_{<\eta} \) be \( \aleph \)-prime over \( N^3_{<\eta} \cup B_0 \). Now by the non-forking calculus \( B \cup (N^3_{<\eta} \cup B_0) \) [why? because

\[(a) \text{ as said above } B_n \cup B N^3_{<\eta} \text{ but } B_0 \subseteq B_n \text{ so } B_0 \cup B N^3_{<\eta} \text{, and}
\]

\[(b) \text{ as } B \cup N^1_{<\eta} \text{ and } B_0 \subseteq N^1_{<\eta} \text{ we have } B \cup B_0 \text{ so } B_0 \cup B
\]

hence (by (a) + (b) as \( A \subseteq B \))

\[(c) \frac{B_0}{N^3_{<\eta} + B} \text{ does not fork over } A,
\]

also

\[(d) B \cup A N^3_{<\eta} \text{ (as } A \subseteq N^3_{<\eta} \subseteq N^2_{<\eta} \text{ and } tp(B, N^2_{<\eta}) \text{ does not fork over } A)
\]

putting (c) and (d) together we get

\[(e) \cup \langle N^4_{<\eta}, B, N^3_{<\eta} \rangle \]

hence the conclusion].

Hence \( B \cup B_0 \) so \( B \cup N^4_{<\eta} \) (by 1.18(6)) and so \( (N^3_{<\eta}) \) is \( \aleph \)-prime over \( N^3_{<\eta} + dcl(a^3_{<0>}) = N^3_{<\eta} + dcl(B) \) we have \( N^4_{<\eta} \cup N^3_{<\eta} \) and by 1.21(1) we can choose \( N^4_{<\eta} < M \) (for \( \eta \in I^2 \setminus \{<\eta>\} \), such that \( \langle N^4_{<\eta}, a^3_{<\eta} : \eta \in I^2 \rangle \geq \aleph \) \( \langle N^3_{<\eta}, a^3_{<\eta} : \eta \in I^2 \rangle \).

So by 1.20(1) \( \langle N^4_{<\eta}, a^3_{<\eta} : \eta \in I^2 \rangle \) is an \( \aleph \)-decomposition of \( M \) above \( (B_n)^{\perp} \) hence \( a^3_{<0>}/N^4_{<\eta} \) does not fork over \( A \) but \( A \subseteq B_0 \subseteq N^4_{<\eta} \) so \( a^3_{<\eta>/N^4_{<\eta}} \) does not fork over \( B_0 \)
and by 1.22 it is enough to prove \( p \in \mathcal{P}(\langle N_{\eta}^4, a_{\eta}^3 : \eta \in I^2 \rangle, M) \). Now as said above \(
abla B \cup N_{\eta}^4 \) and \( B \cup N_{\eta}^3 \) so together \( B \cup N_{\eta}^4 \) also we have \( A \subseteq B_0 \subseteq N_{\eta}^4 \), hence \( B \cup N_{\eta}^4 \) and

\[
\frac{B_n}{B_0 + B} \equiv \frac{B_n}{B_0 + a_{\eta}^0} \perp_a B_0 \text{ (by } (\epsilon)^+ \text{ above) but } a_{\eta}^3 \cup N_{\eta}^4 \text{ hence } \frac{B_n}{N_{\eta}^4 + a_{\eta}^0} \text{ is }
\]

\( \kappa \)-isolated. Also letting \( B'_n = B_n \setminus \{d\} \) we have \( \frac{B_n}{N_{\eta}^4 + a_{\eta}^0} \) is \( \kappa \)-isolated and \( \frac{B_n}{B_0} \) (by clause \( (\zeta) \)), and clearly \( d \cup (N_{\eta}^4 \cup B'_n) \) so \( \frac{d}{B_n} \perp N_{\eta}^4 \). Hence we can find \( \langle N_{\eta}^5, a_{\eta}^5 : \eta \in I^5 \rangle \) an \( \kappa \)-decomposition of \( M \) above \( \langle B_n \rangle \) such that \( N_{\eta}^5 = N_{\eta}^4 \), \( \text{dcl}(B_n) = \text{dcl}(a_{\eta}^5) \), \( B_n \setminus \{d\} \subseteq N_{\eta}^5 \) and \( d = a_{\eta}^5 \) (on \( d \) see clause \( (\zeta) \) above) so \( \frac{d}{B_n} \cup N_{\eta}^5 \).

By 1.23 it is enough to show \( p \in \mathcal{P}(\langle N_{\eta}^5, a_{\eta}^5 : \eta \in I^5 \rangle, M) \) which holds trivially as \( \text{tp}(d, B_n \setminus \{d\}) \) witness. \( \square_{1.14(6)} \)

**Proof of 1.14(5).** By 1.8, with \( A, B, A_1, B_1 \) here standing for \( A_1, B_1, A_2, B_2 \) there, we know that there are \( \langle B'_\ell : \ell \leq n \rangle, \langle c_\ell : 1 \leq \ell < n \rangle \) as there. By 1.18(9) we can choose \( N_{\eta}^4 \) such that \( B_0 \subseteq N_{\eta}^4 \cup B_1, N_{\eta}^4 \) is \( \kappa \)-primary over \( \emptyset \). Then

we choose \( \langle N_{\eta}^4, a_{\eta}^4 : \eta \in \{<, <, 0, 0, <, 0, 0, <, 0, 0, \ldots, <, 0, 0, \ldots, 0 > \} \rangle \), (where

\[
N_{\eta}^4_{0, 0, \ldots, 0} \cap M \text{ and } \ell > 0 \Rightarrow a_{\eta}^4_{0, 0, \ldots, 0} = c_\ell B'_{\eta g(\eta)} \subseteq N_{\eta}^4 \text{ and we choose }
\]

\( N_{\eta}^4 \) by induction on \( \ell g(\eta) \) being \( \kappa \)-prime over \( N_{\eta}^4 \cup a_{\eta}^4 \) hence \( a_{\eta}^4 / N_{\eta}^4 \) does not fork over \( B'_{\eta g(\eta)} \) hence \( N_{\eta}^4 \) is \( \kappa \)-prime also over \( N_{\eta}^4 \cup B'_{\eta g(\eta)} \). So \( \langle N_{\eta}^4, a_{\eta}^4 : \eta \in \{<, \ldots, 0 \} \rangle \) is an \( \kappa \)-decomposition inside \( M \) for \( \langle B_1 \rangle \). Now apply first 1.14(2) and then 1.14(6).

**Proof of 1.14(7).** Should be easy. Note that

\((*)_1\) for no \( \langle B'_n \rangle \) do we have \( \langle B'_n \rangle \leq a \langle B'_m \rangle \)

Why? By the definition of Depth zero.

\((*)_2\) if \( \langle B'_n \rangle < a \langle B'_m \rangle \), then also \( \langle B'_n \rangle \) satisfies the assumption.

Hence
(**) for no \((B_1^{A_1}), (B_2^{A_2})\) do we have
\[
\begin{align*}
(B/A) \leq_a (B_1/A_1) <_b (B_2/A_2).
\end{align*}
\]
[Why? As also \((B_1^{A_1})\) satisfies the assumption].

Now we can prove the statement by induction on \(\alpha\) for all pairs \((B/A)\) satisfying the assumption. For \(\alpha = 0\) the statement is a tautology. For \(\alpha\) limit ordinal reread clause (c) of Definition 1.9(1). For \(\alpha = \beta + 1\), reread clause (b) of Definition 1.9(1): on \(\text{tp}_\beta((B/A), M)\) use the induction hypothesis also for computing \(Y^{1,B}_{A,B,M}\) (and reread the definition of \(\text{tp}_0\), in Definition 1.9(1), clause (a)). Lastly \(Y^{2,\beta}_{A,B,M}\) is empty by (*) above.

**Proof of 1.14(8), (9).** Read Definition 1.9. \(\square_{1.14(5), (7), (8), (9)}\)

**Discussion.** In particular, the following Claim 1.26 implies that if \(\langle N_\eta, a_\eta : \eta \in I \rangle\) is an \(\kappa_\varepsilon\)-decomposition of \(M\) above \((B/A)\) and \(M^-\) is \(\kappa_\varepsilon\)-prime over \(\cup\{N_\eta : \eta \in I\}\), then \((B/A)\) has the same \(\text{tp}_\alpha\) in \(M\) and \(M^-\).

**1.24 Claim.** 1) Assume that \(M_1 \prec M_2\) are \(\kappa_\varepsilon\)-saturated, \((B/A) \in \Gamma(M_1).\) Then the following are equivalent:

\(\begin{align*}
(a) & \text{ if } p \in \mathcal{P}((B/A), M_1) \\
& \text{(see 1.14(6) for definition; so } p \in S(M_1) \text{ is regular) then } p \text{ is not realized in } M_2 \\
(b) & \text{ there is an } \kappa_\varepsilon \text{-decomposition of } M_1 \text{ above } (B/A), \text{ which is also an } \kappa_\varepsilon \text{-decomposition of } M_2 \text{ above } (B/A) \\
(c) & \text{ every } \kappa_\varepsilon \text{-decomposition of } M_1 \text{ above } (B/A) \text{ is also an } \kappa_\varepsilon \text{-decomposition of } M_2 \text{ above } (B/A).
\end{align*}\)

2) If \(M\) is \(\kappa_\varepsilon\)-saturated, \((B_1^{A_1}) \leq^* (B_2^{A_2})\) are both in \(\Gamma(M)\) then \(\mathcal{P}((B_2/A_2), M) \subseteq \mathcal{P}((B_1/A_1), M)\).

3) The conditions in 1.24(1) above implies
\(\begin{align*}
(d) & \text{ } p \in \mathcal{P}((B/A), M_2) \Rightarrow p \pm M_1.
\end{align*}\)
Proof. 1) (c) ⇒ (b).

By 1.14(1) there is an \( \mathcal{N}_\varepsilon \)-decomposition of \( M_1 \) above \( (B_1)^A \). By clause (c) it is also an \( \mathcal{N}_\varepsilon \)-decomposition of \( M_2 \) above \( (B_1)^A \), just as needed for clause (b).

(b) ⇒ (a)

Let \( \langle N_\eta, a_\eta : \eta \in I \rangle \) be as said in clause (b). By 1.14(3)(b) we can find \( J_1, I \subseteq J_1 \) and \( N_\eta, a_\eta \) (for \( \eta \in J_1 \setminus I \)) such that \( \langle N_\eta, a_\eta : \eta \in J_1 \rangle \) is an \( \mathcal{N}_\varepsilon \)-decomposition of \( M_1 \) and \( \nu \in J_1 \setminus I \Rightarrow \nu(0) > 0 \). Then we can find \( J_2, J_1 \subseteq J_2 \) and \( N_\eta, a_\eta \) (for \( \eta \in J_2 \setminus J_1 \)) such that \( \langle N_\eta : \eta \in J_2 \rangle \) is an \( \mathcal{N}_\varepsilon \)-decomposition of \( M_2 \) (by 1.14(2)). By 1.14(3)(b), \( \nu \in J_2 \setminus I \Rightarrow \nu(0) > 0 \). So \( \eta \in I \setminus \{\langle \rangle \} \Rightarrow \text{Suc}_j(\eta) = \text{Suc}_j(\eta) \), hence

\[(*) \text{ if } \eta \in I \setminus \{\langle \rangle \} \text{ and } q \in S(N_\eta) \text{ is regular orthogonal to } N_\eta^- \text{ then the stationarization of } q \text{ in } S(M_1) \text{ is not realized in } M_2.\]

Now if \( p \in \mathcal{P}(\langle B_1 \rangle, M_1) \) then \( p \in S(M_1) \) is regular and (see 1.14(1), 1.11(5)) for some \( \eta \in I \setminus \{\langle \rangle \}, p \perp N_\eta^- \), \( p \perp N_\eta \), so there is a regular \( q \in S(N_\eta) \) not orthogonal to \( p \). Now no \( c \in M_2 \) realizes the stationarization of \( q \) over \( M_1 \) (by \( * \) above), hence this applies to \( p \), too.

(a) ⇒ (c)

Let \( \langle N_\eta, a_\eta : \eta \in I \rangle \) be an \( \mathcal{N}_\varepsilon \)-decomposition of \( M_1 \) above \( (B_1)^A \). We can find \( \langle N_\eta, a_\eta : \eta \in J \rangle \) an \( \mathcal{N}_\varepsilon \)-decomposition of \( M_1 \) such that \( I \subseteq J \) and \( \nu \in J \setminus I \Rightarrow \nu(0) > 0 \) (by 1.14(3)(b)), so \( M \) is \( \mathcal{N}_\varepsilon \)-prime over \( \cup\{N_\eta : \eta \in J\} \). We should check that \( \langle N_\eta : a_\eta : \eta \in I \rangle \) it is also an \( \mathcal{N}_\varepsilon \)-decomposition of \( M_2 \) above \( (B_1)^A \), i.e. Definition 1.11(1),(2). Now in 1.11(1), clauses (a)-(h) are immediate, so let us check clause (i) (in 1.11(2)). Let \( \eta \in I \setminus \{\langle \rangle \} \), now is \( \{a_{\eta^-(a)} : \eta^-(a) \in I\} \) really maximal (among independent over \( N_\eta \) sets of elements of \( M_2 \) realizing a type from \( \mathcal{P}_\eta = \{p \in S(N_\eta) : p \text{ orthogonal to } N_\eta^-\}\)?)? This should be clear from clause (a) (and basic properties of dependencies and regular types).

2) By 1.14(5).
3) Left to the reader. \( \square_{1.24} \)

1.25 Conclusion: Assume \( M_1 \prec M_2 \) are \( \mathcal{N}_\varepsilon \)-saturated and \( (B_1)^A \leq^* (B_2)^A \) both in \( \Gamma(M_1) \). If clause (a) (equivalently (b) or (c)) of 1.24 holds for \( (B_1)^A, M_1, M_2 \) then they hold for \( (B_2)^A, M_1, M_2 \).

Proof. By 1.24(2), clause (a) for \( (B_1)^A, M_1, M_2 \) implies clause (a) for \( (B_2)^A, M_1, M_2 \). \( \square_{1.25} \)
1.26 Claim. If \((B_1, A_1) \in \Gamma(M)\) and \((N_\eta, a_\eta : \eta \in I)\) is an \(\aleph_\epsilon\)-decomposition of \(M\) above \((B_1, A_1)\) and \(M^- \subseteq M\) is \(\aleph_\epsilon\)-saturated and \(\bigcup_{\eta \in I} N_\eta \subseteq M^-\) and \(\alpha\) is an ordinal then
\[
\text{tp}_\alpha \left[ \left(\frac{B_1}{A_1}\right), M \right] = \text{tp}_\alpha \left[ \left(\frac{B_1}{A_1}\right), M^- \right]
\]

Proof. We prove this by induction on \(\alpha\) (for all \(B, A, (N_\eta, a_\eta : \eta \in I), I, M\) and \(M^-\) as above). We can find an \(\aleph_\epsilon\)-decomposition \((N_\eta, a_\eta : \eta \in J)\) of \(M\) with \(I \subseteq J\) (by 1.13(4) + 1.13(2)) such that \(\eta \in J \setminus I \iff \eta \neq \langle \rangle\) and \(\neg\langle 0 \rangle \leq \eta\) and so \(M\) is \(\aleph_\epsilon\)-prime over \(\bigcup_{\eta \in J} N_\eta\) and also over \(M^- \cup \{N_\eta : \eta \in J \setminus I\}\).

Case 0: \(\alpha = 0\).
Trivial.

Case 1: \(\alpha\) is a limit ordinal.
Trivial by induction hypothesis (and the definition of \(\text{tp}_\alpha\)).

Case 2: \(\alpha = \beta + 1\).
We can find \(M^* < M^-\) which is \(\aleph_\epsilon\)-prime over \(\bigcup_{\eta \in I} N_\eta\), so as equality is transitive it is enough to prove
\[
\text{tp}_\alpha \left[ \left(\frac{B_1}{A_1}\right), M^* \right] = \text{tp}_\alpha \left[ \left(\frac{B_1}{A_1}\right), M^- \right]
\]
and
\[
\text{tp}_\alpha \left[ \left(\frac{B_1}{A_1}\right), M^* \right] = \text{tp}_\alpha \left[ \left(\frac{B_1}{A_1}\right), M \right].
\]
By symmetry, this means that it is enough to prove the statement when \(M^-\) is \(\aleph_\epsilon\)-prime over \(\bigcup_{\eta \in I} N_\eta\).

Looking at the definition of \(\text{tp}_{\beta+1}\) and remembering the induction hypothesis our problems are as follows:

First component of \(\text{tp}_\alpha\):
given \((B_1)_{A_1} \leq (B_2)_{A_2}\), \(B_2 \subseteq M\), it suffices to find \((B_3)_{A_3}\) such that:

\[
(*) \text{ there is } f \in \text{AUT}(C) \text{ such that: } f \upharpoonright B_1 = \text{id}_{B_1}, f(A_2) = A_3, \\
f(B_2) = B_3 \text{ and } B_3 \subseteq M^- \text{ and } tp_\beta[(B_2)_{A_2}, M] = tp_\beta[(B_3)_{A_3}, M^-] \\
\text{(pedantically we should replace } B_\ell, A_\ell \text{ by indexed sets).}
\]

We can find \(J', M'\) such that:

(i) \(I \subseteq J' \subseteq J, |J'| < \aleph_0, J' \text{ closed under initial segments,}

(ii) \(M' < M\) is \(\aleph_\epsilon\)-prime over \(M^- \cup \{N_\eta : \eta \in J' \setminus I\}\)

(iii) \(B_2 \subseteq M'\).

The induction hypothesis for \(\beta\) applies, and gives

\[
\text{tp}_\beta[(B_2)_{A_2}, M] = \text{tp}_\beta[(B_2)_{A_2}, M'].
\]

By 1.14(4) there is \(g\), an isomorphism from \(M'\) onto \(M^-\) such that \(g \upharpoonright B_1 = \text{id}\).

So clearly \(g(B_2) \subseteq M^-\) hence

\[
\text{tp}_\beta[(B_2)_{A_2}, M'] = \text{tp}_\beta[(g(B_2))_{g(A_2)}, M^-].
\]

So \((B_3)_{A_3} =: g(A_2)_{B_2}\) is as required.

Second component for \(\text{tp}_\alpha\):

So we are given \(\Upsilon\), a \(\text{tp}_\beta\) type, (and we assign the lower part as \(B\)) and we have to prove that the dimension in \(M\) and in \(M^-\) are the same, i.e.

\[
\dim(I, M) = \dim(I^-, M), \text{ where: } I = \{c \in M : \Upsilon = \text{tp}_\beta\((c)_{B_1}, M)\} \text{ and } \\
I^- = \{c \in M^- : \Upsilon = \text{tp}_\beta\((c)_{B_1}, M^-)\}.
\]

Let \(p\) be such that:

\[
\text{tp}_\beta\((c)_{B_1}, M) = \Upsilon \Rightarrow p = \frac{c}{B_1}.\text{ Necessarily } p \perp A_1 \text{ and } p \text{ is regular (and stationary)}. \\
\text{Clearly } I^- \subseteq I, \text{ so without loss of generality } I \neq \emptyset \text{ hence } p \text{ is really well defined, now}
\]

\[
(*) \text{ for every } c \in I \text{ for some } k < \omega, c'_\ell \in M^- \text{ realizing } p \text{ for } \ell < k \text{ we have } c \\
\text{depends on } \{c'_0, c'_1, \ldots, c'_{k-1}\} \text{ over } B_1.
\]

[Why? Clearly \(p \perp N_{<}\) (as \(B_1 \bigcup N_{<}\) and \(p \perp A_1\)) hence

\[
\text{tp}_\ast\left( \bigcup_{\eta \in J' \setminus I} N_\eta, N_{<}\right) \perp p \text{ hence}
\]
tp_{\ast}( \bigcup_{\eta \in J \setminus I} N_{\eta}, M^-) \perp p, \text{ but } M \text{ is } \aleph_\epsilon\text{-prime over } M^- \cup \bigcup_{\eta \in J \setminus I} N_{\eta}\text{ hence by } [Sh:c, V.3.2, p.250] \text{ for no } c \in M \setminus M^- \text{ is } tp(c, M^-) \text{ a stationarization of } p \text{ hence by } [Sh:c, V.1.16](3) \text{ clearly } (\ast) \text{ follows}.

If the type } p \text{ has depth zero, then by } 1.14(7):

\[ I = \{ c \in M : tp(c, B) = p \} \text{ and } \]
\[ I^- = \{ c \in M^- : tp(c, B) = p \}. \]

Now we have to prove \( \dim(I, A) = \dim(I^-, A) \), as \( A \) is \( \varepsilon \)-finite and \( M, M^- \) are \( \aleph_\varepsilon \)-saturated and \( I^- \subseteq I \) clearly \( \aleph_0 \leq \dim(I^-, A) \leq \dim(I, A) \). Now the equality follows by \( (\ast) \) above.

So we can assume “\( p \) has depth > zero”, hence (by \([Sh:c, X.7.2]\)) that the type \( p \) is trivial; hence, see \([Sh:c, X.7.3]\), in \( (\ast) \) without loss of generality \( k = 1 \) and dependency is an equivalence relation, so for “same dimension” it suffices to prove that every equivalence class (in \( M \) i.e. in \( I \)) is representable in \( M^- \) i.e. in \( I^- \). By the remark on \( (\ast) \) in the previous sentence (\( \forall d_1 \in I)(\exists d_2 \in I^-)[\neg d_1 \cup d_2] \). So it is enough to prove that:

\[ \bigotimes \text{ if } d_1, d_2 \in M \text{ realize same type over } B_1, \text{ which is (stationary and) regular, and are dependent over } B_1 \text{ and } d_1 \in M^- \text{ then there is } d_2' \in M^- \text{ such that } \]
\[ \frac{d_2'}{B_1 + d_2} = \frac{d_2}{B_1 + d_1} \text{ and } tp_{\beta}((B_1 + d_2'), B_1) = tp_{\beta}((B_1 + d_2), B_1). \]

Let \( M_0 = N_{\langle \rangle} \). There are \( J', M_1, M_1^+ \) such that

\( (\ast)_{1(i)} J' \subseteq J \) is finite (and of course closed under initial segments)

\( (ii) \) \( \langle \rangle \in J', \langle 0 \rangle \notin J' \)

\( (iii) \) \( M_1 \prec M \) is \( \aleph_\varepsilon \)-prime over \( \cup \{ N_{\eta} : \eta \in J' \} \)

\( (iv) \) \( M_1^+ \prec M \) is \( \aleph_\varepsilon \)-prime over \( M_1 \cup M^- \) (and \( M_1 \bigcup M_0 M^- \))

\( (v) \) \( d_2 \in M_1^+ \).

Now the triple \((B_1 + d_2), M_1, M\) satisfies the demand on \((B_1), M^-, M\) (because \((B_1) \leq star (B_1 + d_2), M^-\) by 1.25. Hence by the induction hypothesis we know that

\[ tp_{\beta}((B_1 + d_2), B_1) = \]
By 1.29(4) there is an isomorphism $f$ from $M_1^+$ onto $M^-$ which is the identity on $B_1 + d_1$; let $d_2' = f(d_2)$ so:

$$tp_\beta \left[ \left( \frac{B_1 + d_2}{B_1} \right), M_1^+ \right] = tp_\beta \left[ \left( \frac{B_1 + d_2'}{B_1} \right), M^- \right].$$

Together

$$tp_\beta \left[ \left( \frac{B_1 + d_2}{B_1} \right), M \right] = tp_\beta \left[ \left( \frac{B_1 + d_2'}{B_1} \right), M^- \right].$$

As $\{d_1, d_2\}$ is not independent over $B_1$, also $\{f(d_1), f(d_2)\} = \{d_1, f(d_2)\}$ is not independent over $B_1$, hence, as $p$ is regular

$$(*) \{d_2, f(d_2)\} \text{ is not independent over } B_2.$$  
Together we have proved $\bigoplus$, hence finished proving the equality of the second component.

Third component: Trivial.
So we have finished the induction step, hence the proof. \(\square\)

1.27 Claim. 1) Suppose $M$ is $\aleph_\epsilon$-saturated, $A \subseteq B \subseteq M$, $(\binom{B}{A}) \in \Gamma$, $\bigwedge_{\ell=1}^2 [A \subseteq A_\ell \subseteq M]$, $A = acl(A), A_\ell$ are $\epsilon$-finite, $\binom{A}{A_1} = \binom{B}{A}, B \uplus A_1$ and $B \uplus A_2$.

Then $tp_\alpha \left[ \left( \frac{A_1 \uplus B}{A_1} \right), M \right] = tp_\alpha \left[ \left( \frac{A_2 \uplus B}{A_2} \right), M \right]$ for any ordinal $\alpha$.

2) Suppose $M$ is $\aleph_\epsilon$-saturated, $B \subseteq M$, $(\binom{B}{A}) \in \Gamma, \bigwedge_{\ell=1}^2 [A \subseteq A_\ell \subseteq M], A = acl(A),$ $B = acl(B), A_\ell = acl(A_\ell), A_\ell$ is $\epsilon$-finite, $\binom{A}{A_1} = \binom{A}{A_2}, B \uplus A_1, B \uplus A_2, f : A_1 \overset{\text{onto}}{\to} A_2$ an elementary mapping, $f \restriction A = id_A, g \supseteq f \cup id_B, g$ elementary mapping from $B_1 = acl(B \cup A_1)$ onto $B_2 = acl(B \cup A_2)$.

Then $g \left( tp_\alpha \left[ \left( \frac{B_1}{A_1} \right), M \right] \right) = tp_\alpha \left[ \left( \frac{B_2}{A_2} \right), M \right]$ for any ordinal $\alpha$.

3) Assume that

(a) $A_\ell = acl(A_\ell) \subseteq B_\ell = acl(B_\ell) \subseteq M^\ell$ for $\ell = 1, 2$

(b) $A_\ell \subseteq A_\ell^+ \subseteq acl(A_\ell^+) \subseteq M^\ell$ for $\ell = 1, 2$
(c) \( B_\ell \bigcup_{A_\ell} A_\ell^- \) for \( \ell = 1, 2 \)

(d) \( f \) is an elementary mapping from \( A_1 \) onto \( A_2 \)

(e) \( g \) is an elementary mapping from \( A_1^+ \) onto \( A_2^+ \)

(f) \( f \upharpoonright A_1 = g \upharpoonright A_1 \)

(g) \( h \) is an elementary mapping from \( B_1^+ = acl(B_1 \cup A_1^+) \) onto \( B_2^+ = acl(B_2 \cup A_2^+) \)

(h) \( f(tp_A[(B_1^+, A_1)], M_1]) = tp_A[(B_2^+, A_2)], M_2]. \)

Then \( h(tp_A[(B_1^+, A_1^+)], M_1]) = tp_A[(B_2^+, A_2^+)], M_2]. \)

Proof. 1) Follows from part (2).

2) We can find \( A_3 \subseteq M \) such that:

(i) \( \frac{A_3}{A} = \frac{A}{A} \)

(ii) \( A_3 \bigcup (B \cup A_1 \cup A_2) \).

Hence without loss of generality \( A_1 \bigcup A_2 \) and even \( \bigcup \{B, A_1, A_2\} \). Now we can find \( N_{>\ell}, \) an \( \mathcal{N}_e \)-prime model over \( \emptyset, N_{<\ell} < M, A \subseteq N_{<\ell} \) and \( (B \cup A_1 \cup A_2) \bigcup N_{<\ell} \)

(e.g. choose \( \{A_1^\alpha \cup A_2^\alpha \cup B^\alpha : \alpha \leq \omega\} \subseteq M \) indiscernible over \( A, A_1^\alpha = A_1, A_2^\alpha = A_2, B^\omega = B \) and let \( N_{<\ell} < M \) be \( \mathcal{N}_e \)-primary over \( \bigcup_{\eta \in \omega} (A_1^\eta \cup A_2^\eta \cup B^\eta \cup A) \).

Now find \( \langle N_\eta, a_\eta : \eta \in I \rangle \) an \( \mathcal{N}_e \)-decomposition of \( M \) with \( acl(a_{<0}) = acl(B), acl(a_{<1}) = acl(A_1), acl(a_{<2}) = acl(A_2). \)

Let \( I = \{\eta \in J : \eta = <\ell \text{ or } < 0 > \subseteq \eta \} \) and \( J' = I \cup \{< 1 >, < 2 >\} \). Let \( N_\eta > M^* \) be \( \mathcal{N}_e \)-prime over \( N_{<1}, N_{<2} \). By 1.12 there is \( \langle N_\eta, a_\eta : \eta \in I \rangle \) an \( \mathcal{N}_e \)-decomposition of \( M > B \) such that \( \langle N_\eta, a_\eta : \eta \in I \rangle \) is direct \( \langle N_\eta, a_\eta : \eta \in I \rangle \).

Let \( M' \prec M \) be \( \mathcal{N}_e \)-prime over \( \bigcup_{\eta \in I} N_\eta \) and \( M' < M \). Let \( M' \prec M \) be \( \mathcal{N}_e \)-prime over \( \bigcup_{\eta \in I} N_\eta \).

Now by 1.26 we have \( \langle B, A \rangle, M = \langle B, A \rangle, M \rangle \) for \( \ell = 1, 2 \) hence it suffices to find an automorphism of \( M' \) extending \( g \). Let \( B^+ = acl(N_{<0} \cup B), A_1^+ = acl(B' \cup A_1) \), let \( \bar{a}_1 \) list \( A_1^* \) be such that \( \bar{a}_2 = g(\bar{a}_1) \).

Clearly \( tp(\bar{a}_1, B^+) \) does not fork over \( A \subseteq B \) and \( acl(B) = B \) and so \( stp(\bar{a}_1, B^+) = stp(\bar{a}_2, B^+) \). Also \( tp(A_2, B^+ \cup A_1) \) does not fork over \( A \) hence \( tp(\bar{a}_2, B^+ \cup \bar{a}_1) \) does not fork over \( A \) hence \( \{\bar{a}_1, \bar{a}_2\} \) is independent over \( B^+ \) hence there is an elementary mapping \( g^+ \) from

\( A \) to
acl\(B^+ \cup \bar{a}_1\) onto acl\((B^+ \cup \bar{a}_2), g^+ \supseteq \text{id}_{\text{acl}}\) and even \(g' = g^+ \cup (g^+)\) is an elementary embedding.

Let \(\bar{a}_1^1\) lists acl\((N_{<\alpha} \cup A_1)\) so clearly \(\bar{a}_2 =: g^+(\bar{a}_1^1)\) list acl\((N_{<\alpha} \cup A_2)\). Clearly \(g' \upharpoonright (\bar{a}_1^1 \cup \bar{a}_2^1)\) is an elementary mapping from \(\bar{a}_1^1 \cup \bar{a}_2^1\) onto itself. Now \(N_{<\alpha}^2\) is \(\aleph\)-primary over \(N_{<\alpha} \cup A_1 \cup A_2\) and \(N_{<\alpha} \cup A_1 \cup A_2 \subseteq \bar{a}_1^1 \cup \bar{a}_2^1 \subseteq \text{acl}(N_{<\alpha} \cup A_1 \cup A_2)\) so by 1.18(10) \(N_{<\alpha}^2\) is \(\aleph\)-primary over \(N_{<\alpha} \cup \bar{a}_1^1 \cup \bar{a}_2^1\) hence we can extend \(g' \upharpoonright (\bar{a}_1^1 \cup \bar{a}_2^1)\) to an automorphism \(h_{<\alpha}\) of \(N_{<\alpha}^2\) so clearly \(h_{<\alpha} \upharpoonright N_{<\alpha} = \text{id}_{N_{<\alpha}}\). Let \(\bar{a}_1^1\) list acl\((B^+ \cup A_1)\) and \(\bar{a}_2^2 = g^+(\bar{a}_1^1)\). So \(\text{tp}(\bar{a}_1^1, N_{<\alpha}^2)\) does not fork over \(\bar{a}_1^1 \subseteq N_{<\alpha}^2\) and acl\((\bar{a}_1^1)\) is an elementary mapping from \(\bar{a}_1^1\) onto \(\text{acl}(\bar{a}_1^1)\) (as \(\aleph\)-primary and \(\aleph\)-minimal over the former. Hence \(h_{<\alpha} \cup g^+ \cup \text{id}_{N_{<\alpha}}\) can be extended to an automorphism of \(N_{<\alpha}^2\) which we call \(h_{<\alpha}\).

Now we define by induction on \(n \in [2, \omega)\) for every \(\eta \in I\) of length \(n\), an automorphism \(h_{\eta}\) of \(N_{\eta}^2\) extending \(h_{\eta-} \cup \text{id}_{N_{\eta-}}\), which exists as \(N_{\eta}^2\) is \(\aleph\)-primary over \(N_{\eta-} \cup N_{\eta}\) (and \(N_{\eta}^2 \subseteq \bigcup_{\eta \in I} N_{\eta}\)). Now \(\bigcup_{\eta \in I} N_{\eta}\) is an elementary mapping (as \(\langle N_{\eta}^2 : \eta \in I \rangle\) is a non-forking tree; i.e. 1.13(10)), with domain and range \(\bigcup_{\eta \in I} N_{\eta}^2\) hence can be extended to an automorphism \(h^*\) of \(M'\), (we can demand \(h^* \upharpoonright M^- = \text{id}_{M^-}\) but not necessarily). So as \(h^*\) extends \(g\), the conclusion follows. 3) Similarly to (2).

\[ \square_{1.27} \]

1.28 Claim. 1) For every \(\Upsilon = \text{tp}_\delta([\bar{a}, M], \bar{a}, \bar{b}\) listing \(A, B\) respectively there is \(\psi = \psi(\bar{x}_A, \bar{x}_B) \in \text{L}_{\infty, \aleph_\omega(q.d.)}\) of depth \(\delta\) such that:

\[ \text{tp}_\delta\left([\bar{A}_B], M\right) = \Upsilon \iff M \models \psi(\bar{a}, \bar{b}). \]

2) Assume \(\otimes_{M_1, M_2}\) of 1.4 holds as exemplified by the family \(\mathcal{F}\) and \(\langle \bar{a} \rangle \in \Gamma(M_1)\) and \(g \in \mathcal{F}, \text{Dom}(g) = B; \text{ and } \alpha\) an ordinal then

\[ \text{tp}_\alpha\left([\bar{B}_A], M\right) = \text{tp}_\alpha\left([\bar{g}(B)], M_2\right). \]

3) Similarly for \(\text{tp}_\alpha([A], M), \text{tp}_\alpha[M]\).
Proof. Straightforward (remember we assume that every first order formula is equivalent to a predicate). □

1.29 Proof of Theorem 1.2. [The proof does not require that the $M^\ell$ are $\aleph_\epsilon$-saturated, but only that 1.27, 1.28 hold except in constructing $g_{\alpha(*)}$ (see $\otimes_{14}, \otimes_{15}$ in 1.30(E), we could instead use NOTOP].

So suppose

\((*)_0\) $M^1 \equiv_{L_{\infty,\kappa}, (d.q.)} M^2$ or (at least) $\otimes_{M^1,M^2}$ from 1.4 holds.

We shall prove $M^1 \cong M^2$. By 1.28 (i.e. by 1.28(1) if the first possibility in $(*)_0$ holds and by 1.28(2) if the second possibility in $(*)_0$ holds)


So it suffices to prove:

1.30 Claim. Assume that $T$ is countable. If $M^1, M^2$ are $\aleph_\epsilon$-saturated models (of $T, T$ as in 1.5), then:

\((*)_1\) $M^1 \cong M^2$.

Proof. Let $\langle W_k, W'_k : k < \omega \rangle$ be a partition of $\omega$ to infinite sets (so pairwise disjoint).

1.31 Explanation: (If seems opaque, the reader may return to it after reading parts of the proof).

We shall now define an approximation to a decomposition. That is we are approximating a non-forking tree $\langle N^\ell_\eta, a^\ell_\eta : \eta \in I^* \rangle$ of countable elementary submodels of $M^\ell$ for $\ell = 1, 2$ and $\langle f^*_\eta : \eta \in I^* \rangle$ such that $f^*_\eta$ an isomorphism from $N^1_\eta$ onto $N^2_\eta$ increasing with $\eta$ such that $M^\ell$ is $\aleph_\epsilon$-prime over $\bigcup_{\eta \in I^*} N^\ell_\eta$.

In the approximation $Y$ we have:

\((\alpha)\) $I$ approximating $I^*$, 

[it will not be $I^* \cap^{\geq} \text{Ord}$ but we may “discover” more immediate successors to each $\eta \in I$; as the approximation to $N_\eta$ improves we have more regular types, but some member of $I$ will be later will drop]

\((\beta)\) $A^\ell_\eta$ approximates $N^\ell_\eta$ and is $\epsilon$-finite

\((\gamma)\) $a^\ell_\eta$ is the $a^\ell_\eta$ (if $\eta$ survives, i.e. will not be dropped)
(δ) $B^\ell_\eta, b^\ell_\eta,m$ expresses commitments on constructing $A^\ell_\eta$: we “promise”
$B^\ell_\eta \subseteq N^\ell_\eta$ and $B^\ell_\eta$ is countable; $b^\ell_\eta,m$ for $m < \omega$ list $B^\ell_\eta$ (so in the choice
$B^\ell_\eta \subseteq M^\ell$ there is some arbitrariness).

(ε) $f_\eta$ approximate $f^*_\eta$

(ζ) $p^\ell_\eta,m$ also expresses commitments on the construction.

Since there are infinitely many commitments that we must meet in a construction
of length $\omega$ and we would like many chances to meet each of them, the sets $W_k, W'_k$
are introduced as a further bookkeeping device. At stage $n$ in the construction we
will deal e.g. with the $b^\ell_\eta,m$ for $\eta$ that are appropriate and for $m \in W_k$ for some
$k < n$ and analogously for $p^\ell_\eta,m$ and the $W'_k$.

Note that while the $A^\ell_\eta$ satisfy the independence properties of a decomposition,
the $B^\ell_\eta$ do not and may well intersect non-trivially. Nevertheless, a conflict arises
if an $a^\ell_\eta,<_i \in I$ falls into $B^\ell_\eta$ since the $a^\ell_\eta,<_i$ are supposed to represent independent
elements realizing regular types over the model approximated by $A^\ell_\eta$ but now $a^\ell_\eta,<_i \in I$
is that model. This problem is addressed by pruning $\eta^* < i >$ from the tree $I$.

1.32 Definition. An approximation $Y$ to an isomorphism consist of:

(a) natural numbers $n, k^*$ and index set: $I \subseteq ^n \geq \text{Ord}$
(and $n$ minimal)

(b) $\langle A^\ell_\eta, B^\ell_\eta, a^\ell_\eta, b^\ell_\eta,m : \eta \in I \text{ and } m \in \bigcup_{k < k^*} W_k \rangle$ for $\ell = 1, 2$ (this is an approxi-
imated decomposition)

(c) $\langle f_\eta : \eta \in I \rangle$

(d) $\langle p^\ell_\eta,m : \eta \in I \text{ and } m \in \bigcup_{k < k^*} W'_k \rangle$

such that:

(1) $I$ closed under initial segments

(2) $\ll \in I$

(3) $A^\ell_\eta \subseteq B^\ell_\eta \subseteq M^\ell$, $A^\ell_\eta$ is $\epsilon$-finite, $acl(A^\ell_\eta) = A^\ell_\eta$, $B^\ell_\eta$ is countable,
$B^\ell_\eta = \{b^\ell_\eta,m : m \in \bigcup_{k < k^*} W_k \}$

(4) $A^\ell_\nu \subseteq A^\ell_\eta$ if $\nu < \eta \in I$
(5) if \( \eta \in I \setminus \langle \rangle \), then \( a^\ell_\eta \) is a (stationary) regular type and \( a^\ell_\eta \in A^\ell_\eta \); if in addition \( \ell g(\eta) > 1 \) then \( \frac{a^\ell_\eta}{A^\ell_{\eta^-}} \perp A^\ell_{\langle \eta^- \rangle} \) (note that we may decide \( a^\ell_{\langle \rangle} \) be not defined or \( \in A^\ell_{\langle \rangle} \))

(6) \( \frac{A^\ell_\eta}{A^\ell_{\eta^-} + a_\eta} \perp_A \) if \( \eta \in I, \ell g(\eta) > 0 \)

(7) if \( \eta \in I, \) not \(<\)-maximal in \( I, \) then the set \( \{ a^\ell_\nu : \nu \in I \text{ and } \nu^- = \eta \} \) is a maximal family of elements realizing over \( A^\ell_\eta \) regular types \( \perp A^\ell_{\langle \eta^- \rangle} \) (when \( \eta^- \) is defined), independent over \( (A^\ell_\eta, B^\ell_\eta) \), (and we can add: if \( \nu_1^- = \nu_2^- = \eta \) and \( a^\ell_{\nu_1} \pm a^\ell_{\nu_2} \) then \( a^\ell_{\nu_1} \mid A_\eta = a^\ell_{\nu_2} \mid A_\eta \))

(8) \( f_\eta \) is an elementary map from \( A^1_\eta \) onto \( A^2_\eta \)

(9) \( f_{\langle \eta^- \rangle} \subseteq f_\eta \) when \( \eta \in I, \ell g(\eta) > 0 \)

(10) \( f_\eta(a^\ell_\eta) = a^2_\eta \)

(11) \( f_\eta \left( \text{tp}_\infty \left[ \left( A^1_\eta \right)^1, M^1 \right] \right) = \text{tp}_\infty \left[ \left( A^2_\eta \right)^2, M^2 \right] \) when \( \eta \in I \setminus \{ \langle \rangle \} \)

(\( \beta \)) \( f_{\langle \rangle} \left( \text{tp}_\infty \left[ A^1_{\langle \rangle}, M^1 \right] \right) = \text{tp}_\infty \left[ A^2_{\langle \rangle}, M^2 \right] \)

(12) \( B^\ell_\eta \prec M^\ell \), moreover, \( B^\ell_\eta \subseteq n a \) \( M^\ell \), i.e., if \( \bar{a} \subseteq N^\ell_\eta, b \in M^\ell \setminus B^\ell_\eta \) and \( M^\ell \models \varphi(b, \bar{a}) \) there for some \( b' \in B^\ell_\eta, | = \varphi(b', \bar{a}) \) and \( b \notin acl(\bar{a}) \Rightarrow b' \notin acl(A) \)

(13) \( \langle p^\ell_{\eta, m} : m \in \bigcup_{k < k^*} W_k \rangle \) is a sequence of types over \( A^\ell_\eta \) (so \( \text{Dom}(p^\ell_{\eta, m}) \) may be a proper subset of \( A^\ell_\eta \)).

1.33 Notation. We write \( n = n_Y = n[Y], I = I_Y = I[Y], A^\ell_\eta = A^\ell_\eta[Y], B^\ell_\eta = B^\ell_\eta[Y], f_\eta = f^Y_\eta = f_\eta[Y], a^\ell_\eta = a^\ell_\eta[Y], b^\ell_\eta = b^\ell_\eta[Y], k^* = k^*[Y] \) and \( p^\ell_{\eta, m} = p^\ell_{\eta, k[Y]} \).

Remark. We may decide to demand: each \( \frac{a^\ell_{\langle \rangle}}{A^\ell_\eta} \) is strongly regular; also: if two such types are not orthogonal then they are equal (or at least have same witness \( \varphi \) for \( (\varphi, \frac{a^\ell_{\langle \rangle}}{A^\ell_\eta}) \) regular). This is easy here as the models are \( \aleph_c \)-saturated (so take \( p' \pm p, \text{rk}(p') \) minimal).

1.34 Observation. \((*)_1 \) implies that there is an approximation, (see 1.29).
Proof. Let $I = \{<>\}, A^<_{\ell>} = ac\ell(\emptyset), k^* = 1$ and then choose countable $B^<_{\ell}$ to
satisfy condition (12) and then choose $f_{\eta}, p^\ell_{\eta,k}, b^\ell_{\eta,m}$ (for $k \in W'_0$ and $m \in W_0$) as required.

1.35 Main Fact. For any approximation $Y, i \in \bigcup_{k<k^*_Y} (W_k \cup W'_k)$ and $m \leq n_Y$ and
$\ell(*) \in \{1, 2\}$ we can find an approximation $Z$ such that:

$\Box(a)$ $n_Z = \max\{m+1, n_Y\}, I_Z \cap m^\geq \Ord = I_Y \cap m^\geq \Ord,$
we mean $m$ not $n_Y$ and $k^*_Z = k^*_Y + 1$

$\Box(b)$ if $\eta \in I_Y, \ell g(\eta) < m$ then

$A^\ell_{\eta}[Z] = A^\ell_{\eta}[Z], a^\ell_{\eta}[Z] = a^\ell_{\eta}[Z]$

$B^\ell_{\eta}[Z] = B^\ell_{\eta}[Y]$

$\Box(c)$ if $\eta \in I_Y \cap I_Z, k < k^*_Y$ and $j \in W'_k$ then

$p^\ell_{\eta,j}[Z] \equiv p^\ell_{\eta,j}[Y]$

$\Box(d)$ if $\eta \in I_Y \cap I_Z, k < k^*_Y$ and $j \in W_k$ then

$b^\ell_{\eta,j}[Z] = b^\ell_{\eta,j}[Y]$

$\Box(e)$ if $\eta \in I_Y, \ell g(\eta) = m, k < k^*_Y$ and $i \in W_k$ and the element $b \in M^\ell(*)$ satisfies clauses (a), (b) below then for some such $b$ we have: $A^\ell(*)[Z] = ac\ell(A^\ell(*)[Y] \cup \{b\});$ where

$\Box(f)$ one of the conditions (i),(ii) listed below holds for $b$

$\Box(g)$ for no $b$ is (i) satisfied (so $\ell g(\eta) \geq 0$ and $b \in M^\ell(*)$,

$b^\ell_{\eta,i} \not\in A^\ell(*)[Y]$ and $\ell g(\eta) > 0 \Rightarrow \frac{b}{A^\ell(*)[Y]} \perp_a A^\ell(*)[Y]$

(iii) for no $b$ is (i) satisfied (so $\ell g(\eta) > 0$) and $b \in M^\ell(*)$,

$b^\ell_{\eta,i} \bigcup_{\ell g(\eta) > 0} \perp_a A^\ell(*)[Y]$

\[\text{recall that } i \text{ is part of the information given in the main fact, and of course, } k \text{ is uniquely determined by } i.\]
(γ)² assume η ∈ IY, ℓg(η) = m, k < k_Y and i ∈ W_k^\prime then we have:

(a) if p_{η,i}^{\ell(*)} is realized by some b ∈ M^{\ell(*)} such that

\[
\text{Rk}
\left(\frac{b}{A_\eta^{\ell(*)}[Y]}, L, \infty\right) = \text{R}
\left(p_{η,i}^{\ell(*)}, L, \infty\right)
\]

then for some such b we have

\[A_\eta^{\ell(*)}[Z] = acl\left(A_\eta^{\ell(*)}[Y] \cup \{b\}\right)\]

(b) if the assumption of clause (a) fails but p_{η,i}^{\ell(*)} is realized
by some b ∈ M^{\ell(*)} \setminus A_\eta^{\ell(*)} such that

\[\left[\ellg(\eta) > 0 \Rightarrow \frac{b}{A_\eta^{\ell(*)}[Y]} \perp_a A_\eta^{\ell(*)}[Y]\right]\]

then for some such b we have

\[A_\eta^{\ell(*)}[Z] = acl\left(A_\eta^{\ell(*)}[Y] \cup \{b\}\right)\]

(δ) If η ∈ IY and ℓg(η) = m, then \(B_\eta^{\ell}[Z] = \{b_{\eta,j}^{\ell}[Y] : j ∈ \{W_k : k < k_Z^\prime\}\}\)
is a countable subset of M^\ell, containing \(B_\nu^\ell[Z] : \nu ≤ η\) and \(ν ∈ Y\) ∪ \(B_\eta^{\ell}[Y]\),
with \(B_\eta^{\ell}[Z] < M^\ell\) moreover \(B_\eta^{\ell}[Z] \subseteq na M^\ell\) i.e. if \(\bar{a} ⊆ B_\eta^{\ell}[Z], \varphi(x, \bar{y})\) is
first order and \(∃x ∈ M^\ell ∩ acl(\bar{a})\) \(\varphi(x, \bar{a})\) then \(∃x ∈ B_\eta^{\ell}[Z] ∩ acl(\bar{a})\) \(\varphi(x, \bar{a})\)
and \(\{a_\eta^{\ell} <_\alpha [Y] : η^\prime(\alpha) ∈ IY\) and \(a_\eta^{\ell} <_\alpha [Y] \notin B_\eta^{\ell}[Z]\}\) is independent over
\(\langle B_\eta^{\ell}[Z], A_\eta^{\ell}[Y]\rangle\)

(ε) if η ∈ IY, ℓg(η) > m, then η ∈ IZ ⇔ a_{η|\{m+1\}}^{\ell}[Y] \notin B_\eta^{\ell}[m][Z]

(ζ) if η ∈ IY ∩ IZ, ℓg(η) > m then \(A_\eta^{\ell}[Z] = acl(A_\eta^{\ell}[Y] ∪ A_\eta^{\ell}[m][Z]\) and

\(B_\eta^{\ell}[Z] = B_\eta^{\ell}[Y]\)

(η) if η ∈ IZ \setminus IY then η⁻ ∈ IY and ℓg(η) = m + 1

(θ) \(\{p_{η,i}^{\ell}[Z] : i ∈ W_{k_Z^\prime−1}^\prime\}\) is “rich enough”, e.g. include all finite types over \(A_\eta^{\ell}\)

(ι) \(\{b_{η,i}^{\ell} : i ∈ W_{k_Z^\prime−1}^\prime\}\) list \(B_\eta^{\ell}[Z]\), each appearing infinitely often.
Proof. First we choose $A_{\eta}^{\ell([Z]}$ for $\eta \in I$ of length $m$ according to condition $(\gamma) = (\gamma)^1 + (\gamma)^2$. (Note: one of the clauses $(\gamma)^1, (\gamma)^2$ necessarily holds trivially as $\bigcup_{k} W_k \cap \bigcup_{k} W'_k = \emptyset$).

Second, we choose (for such $\eta$) an elementary mapping $f^Z_\eta$ extending $f^Y_\eta$ and a set $A_{\eta}^{3-\ell([Z]} \subseteq M^{3-\ell([Z]}$ satisfying “$f^Z_\eta$ is from $A_{\eta}^{1}[Z]$ onto $A_{\eta}^{3-\ell([Z]}$” such that

\[(*)_2 \text{ if } m > 0, \text{ then } f^Z_\eta (\text{tp}_\infty (\langle A_{\eta}^{1}[Y], M_1 \rangle)) = \text{tp}_\infty (\langle A_{\eta}^{2}[Y], M_2 \rangle)\]

\[(*)_3 \text{ if } m = 0, \text{ then } f^Z_\eta (\text{tp}_\infty (A_{\eta}^{1}[Z], M_1)) = \text{tp}_\infty (A_{\eta}^{2}[Z], M_2).\]

[Why possible? If we ask just the equality of tp$_\alpha$ for an ordinal $\alpha$, this follows by the first component of tp$_{\alpha+1}$. But (overshooting) for $\alpha \geq \lceil (||M_1|| + ||M_2||)/\omega \rceil$, equality of tp$_\alpha$ implies equality of tp$_\infty$.]

Third, we choose $B_{\eta}^{\ell}[Z]$ for $\eta \in I_Y$, $\ell \eta(\eta) = m$ according to condition $(\delta)$ (here we use the countability of the language, you can do it by extending it $\omega$ times) in both sides, i.e. for $\ell = 1, 2$.

Fourth, let $I' = \{ \eta \in I : \text{ if } \ell \eta(\eta) > m \text{ then } a^{\ell_{\eta(m)+1}}[Y] \notin B_{\eta^\ell m}[Z] \}$ (this will be $I_Y \cap I_Z$).

Fifth, we choose $A_{\eta}^{\ell}[Z]$ for $\eta \in I'$: if $\ell \eta(\eta) < m$, let $A_{\eta}^{\ell}[Z] = A_{\eta}^{\ell}[Y]$, if $\ell \eta(\eta) = m$ this was done, lastly if $\ell \eta(\eta) > m$, let $A_{\eta}^{\ell}[Z] = \text{act} (A_{\eta}^{\ell}[Y] \cup A_{\eta}^{\ell}[m])[Z]$.

Sixth, by induction on $k \leq n_Y$ we choose $f^Z_\eta$ for $\eta \in I'$ of length $k$: if $\ell \eta(\eta) < m$, let $f^Z_\eta = f^Y_\eta$, if $\ell \eta(\eta) = m$ this was done, lastly if $\ell \eta(\eta) > m$ choose an elementary mapping from $A_{\eta}^{1}$ onto $A_{\eta}^{2}$ extending $f^Y_\eta \cup f^Z_\eta$ (possible as $f^Y_\eta \cup f^Z_\eta$ is an elementary mapping and Dom$(f^Y_\eta) \cap$ Dom$(f^Z_\eta) = A_{\eta}^{\ell([Z]}$, Dom$(f^Y_\eta) \cup$ Dom$(f^Z_\eta) \subseteq A_{\eta}^{\ell([Z]}$)

and $A_{\eta}^{\ell([Z]} = \text{act}(A_{\eta}^{\ell([Z]}).$ Now $f^Z_\eta$ satisfies clause (11) of Definition 1.32 when $\ell \eta(\eta) > m$ by applying 1.27(3).

Seventh, for $\eta \in I'$, of length $m < n_Z$, let $v_\eta =: \{ \alpha : \eta^<\alpha(\alpha) \in I \}$, and we choose $\{a^{\eta^<\alpha}[Z] : \alpha \in u_\eta \}$, $[\alpha \in u_\eta \Rightarrow \eta^<\alpha(\alpha) \notin I]$, a set of elements of $M^1$ realizing (stationary) regular types over $A^{1}_{\eta}[Z]$, orthogonal to $A_{\eta}[Y]$ when $\ell \eta(\eta) > 0$, such that it is independent over $(\bigcup \{a^{\eta^<\alpha}[Y] : \eta^<\alpha(\alpha) \in I' \} \cup B^{1}_{\eta}[Z], A^{1}_{\eta}[Z])$ and maximal under those restrictions. Without loss of generality $\sup(v_\eta) < \min(u_\eta)$ and for $\alpha_1 \in v_\eta \cup u_\eta$ and $\alpha_2 \in u_\eta$ we have:
\[ (\ast)_1 \text{ if (for the given } \alpha_2 \text{ and } \eta \text{) } \alpha_1 \text{ is minimal such that } \]
\[ \frac{a_{n_i}^{\langle \alpha_1 \rangle} \cap [Z]}{A_{n_i}^1[Z]} \perp \frac{a_0^{\langle \alpha_2 \rangle} \cap [Z]}{A_{0}^1[Z]} \]
\[ \frac{a_0^{\langle \alpha_1 \rangle} \cap [Z]}{A_{0}^1[Z]} = \frac{a_0^{\langle \alpha_2 \rangle} \cap [Z]}{A_{0}^1[Z]} \]
\[ (\ast)_2 \text{ if } \alpha_1 < \alpha_2, \text{ and } a_{n_i}^{\langle \alpha_1 \rangle} \cap [Z]/A_{n_i}^1[Z] = a_{n_i}^{\langle \alpha_2 \rangle} \cap [Z]/A_{n_i}^1[Z] \text{ and for some } b \in M^1 \]
\[ \text{realizing } \frac{a_{n_i}^{\langle \alpha_1 \rangle} \cap [Z]}{A_{n_i}^1[Z]} \text{ we have} \]
\[ b \uplus a_{n_i}^{\langle \alpha_2 \rangle} \text{ and} \]
\[ \text{tp}_{\infty} \left[ \left( a_{n_i}^{\langle \alpha_1 \rangle} \cap [Z], M \right) \right] = \text{tp}_{\infty} \left[ \left( a_{n_i}^{\langle \alpha_2 \rangle} \cap [Z], M \right) \right] \]
\[ \text{and } \alpha_1 \text{ is minimal (for the given } \alpha_2 \text{ and } \eta \text{) then} \]
\[ \text{tp}_{\infty} \left[ \left( a_{n_i}^{\langle \alpha_1 \rangle} \cap [Z], M \right) \right] = \text{tp}_{\infty} \left[ \left( a_{n_i}^{\langle \alpha_2 \rangle} \cap [Z], M \right) \right] . \]

Easily (as in [Sh: c, X]) if \( \alpha \in u_{\eta} \) and \( \eta^{\langle \beta \rangle} \in I' \) then \( a_{n_i}^{\langle \alpha \rangle} \cap [Z] \perp \frac{a_{n_i}^{\langle \alpha_2 \rangle} \cap [Y]}{A_{n_i}^1[Y]} \).

For \( \alpha \in u_{\eta} \) let \( A_{n_i}^1 \cap [Z] = \text{acl} \left( A_{n_i}^1[Y] \cup \{ a_{n_i}^{\langle \alpha \rangle} \cap [Z] \} \right) \).

Eighth, by the second component in the definition of \( \text{tp}_{\alpha+1} \) (see Definition 1.9) we can choose (for \( \alpha \in u_{\eta} \)) \( a_{n_i}^{\langle \alpha \rangle} \cap [Z], A_{n_i}^2 \cap [Z] \) and then \( f_{n_i}^{\langle \alpha \rangle} \) as required (see (7) of Definition 1.32).

Ninth and lastly, we let \( I_{\eta} = I' \cup \eta^{\langle \beta \rangle} ; \eta \in I', \ell g(\eta) = m < n_{\eta} \text{ and } \alpha \in u_{\eta} \) and we choose \( B_{\eta}^{i} \) for \( \eta \in I_{\eta} \setminus I_{Y} \) and the \( p_{n_i}^{i}, b_{n_i}^{i} \) as required (also in other case left).

\[ \square_{1.35} \]

1.36 Finishing the Proof of 1.11. We define by induction on \( n < \omega \) an approximation \( Y_0 = Y(n) \). Let \( Y_0 \) be the trivial one (as in observation 1.30(C)).

\( Y_{n+1} \) is gotten from \( Y_n \) as in 1.35 for \( m_n, i_n \leq n, \ell_n(\ast) \in \{1, 2\} \) defined by reasonable bookkeeping (so \( i_n \in \bigcup_{k < k_{n}^{Y(n)}} (W_k \cup W'_k) \)) such that any triples appear infinitely often; without loss of generality: if \( n_1 < n_2 \) & \( \eta \in I_{n_1}^{i} \cap I_{n_2}^{i} \) then \( \eta \in \bigcap_{n=n_1}^{n_2} I_{n}^{i} \).

Let \( I^* = (I^*[\eta]) =: \{ \eta : \text{ for every large enough } n, \eta \in I_{n} \} ; \)

for \( \eta \in I^* \) let: \( A_{\eta}^{i} = \bigcup_{n<\omega} A_{\eta}^{i}[Y_{n}], f_{\eta}^{i} = \bigcup_{n<\omega} f_{\eta}^{Y(n)} \) and

\[ \text{for } \eta \in I^* \text{ let: } A_{\eta}^{i} = \bigcup_{n<\omega} A_{\eta}^{i}[Y_{n}], f_{\eta}^{i} = \bigcup_{n<\omega} f_{\eta}^{Y(n)} \text{ and} \]
$B_{\eta}^\ell[*] = \bigcup_{n<\omega} B_{\eta}^\ell[Y_n]$.

Easily

1. $<> \in I^*$ and $I^* \subseteq \omega^>\text{Ord}$ is closed under initial segments

2. for $\eta \in I^*$, $\left< B_{\eta}^\ell[Y_n] : n < \omega \text{ and } \eta \in I[Y_n] \right>$ is an increasing sequence of $\subseteq_{\text{na}}$-elementary submodels of $M^\ell$

[Why? By clause (12) of Definition 1.32, Main Fact 1.35, clauses $(\beta)(a), (\delta), (\zeta)$.]

hence

3. for $\eta \in I^*$, $B_{\eta}^\ell[*] \subseteq_{\text{na}} M^\ell$.

Also

4. $\nu \triangleleft \eta \in I^* \Rightarrow B_{\nu}^\ell[*] \subseteq B_{\eta}^\ell[*]$.

[Why? Because for infinitely many $n, m = \ell g(\eta)$ and clause $(\delta)$ of Main Fact 1.35].

5. if $\eta \in I[Y_{n_1}] \cap I^*$, $\eta^- = \nu$ and $n_1 \leq n_2$ then

$$A_{\eta}^\ell[Y_{n_1}] \bigcup_{A_{\nu}^\ell[Y_{n_2}]}$$

[Why? Prove by induction on $n_2$ (using the non-forking calculus), for $n_2 = n_1$ this is trivial, so assume $n_2 > n_1$. If $m(n_2-1) > \ell g(\nu)$ we have $A_{\nu}^\ell[Y_{n_2}] = A_{\nu}^\ell[Y_{n_2-1}]$ (see 1.35, clause $(\beta)(a)$ and we have nothing to prove. If $m(n_2-1) < \ell g(\nu)$ then we note that $A_{\nu}^\ell[Y_{n_2}] = \text{acl}(A_{\nu}^\ell[Y_{n_2-1}] \cup A_{\nu}^\ell[Y_{m(n_2-1)}])$ and $A_{\nu}^\ell[Y_{n_2}] = \bigcup_{A_{\nu}^\ell[Y_{m(n_2-1)}]}$ (as $\nu \in I[Y_{n_2}]$, by 1.35 clause $(\delta)$ last phrase) and now use clauses (5), (6) of Definition 1.35. Lastly if $m(n_2-1) = \ell g(\nu)$ again use $\nu \in I[Y_{n_2}]$ by 1.35, clause $(\delta)$, last phrase].

6. if $\eta \in I[Y_{n_1}] \cap I^*$, $\eta^- = \nu$ and $n_1 \leq n_2$ then

$$A_{\eta}^\ell[X_{n_1}] \perp_a A_{\nu}^\ell.$$  

[Why? By clause (6) of Definition 1.32, and orthogonality calculus].

7. if $\eta \in I^*$, then $A_{\eta}^\ell[*] \subseteq B_{\eta}^\ell[*] < M^\ell$ moreover

8. $A_{\eta[*]} \subseteq_{\text{na}} B_{\eta[*]} \subseteq_{\text{na}} M^\ell$. 

(401)
We can now use the induction hypothesis (and \([\text{BeSh 307}, 5.3, p. 292]\)).

Why? The second relation holds by \(\otimes_2\). The first relation we prove by induction on \(\ell g(\eta)\); clearly \(A^\ell_\eta[\ast] = acl(A^\ell_\eta[\ast])\) because \(A^\ell_\eta[Y_n]\) increases with \(n\) by 1.35 and \(A^\ell_\eta[Y_n] = acl(A^\ell_\eta[Y_n])\) by clause (3) of Definition 1.32. We prove \(\langle A^\ell_\eta\rangle[\ast] \subseteq B^\ell_\eta[\ast]\) by induction on \(m = \ell g(\eta)\), so suppose this is true for every \(m' < m\), \(m = \ell g(\eta), \eta \in I^*\), let \(\varphi(x)\) be a formula with parameters in \(A^\ell_\eta[\ast]\) realized in \(M^\ell\) as above say by \(b \in M^\ell\). As \(A^\ell_\eta[Y_n] : n < \omega, \eta \in Y_n\) is increasing with union \(A^\ell_\eta[\ast]\), clearly for some \(n\) we have \(b \cup A^\ell_\eta[\ast]\).

So \(\{\varphi(x)\} = p^\ell_{\eta,i}\) for some \(i\) and for some \(n' > n\) defining \(Y_{n' + 1}\) we have used 1.35 with \((\ell(\ast), i, m)\) there being \((\ell, i, \ell g(\eta))\) here, hence we consider clause (\(\gamma\))^2 of 1.35. So the case left is when the assumption of both clauses (a) and (b) of (\(\gamma\))^2 fail, so we have \(\ell g(\eta) > 0\) and

\[
\begin{align*}
b' \notin A^\ell_\eta[Y_{n'}], b' \in M^\ell \models \varphi[b'] \Rightarrow \frac{b'}{A^\ell_\eta[Y_{n'}]} \pm A^\ell_\eta[\ast].
\end{align*}
\]

We can now use the induction hypothesis (and [\text{BeSh 307, 5.3, p.292}]).

\(\bigoplus_{\alpha} \) if \(\eta \in I^* \) and \(\ell = 1, 2\), then
\(\{a^\ell_{\eta, <\alpha}[\ast] : \eta^\ast(\alpha) \in I^*\}\) is a maximal subset of
\[
\{c \in M^\ell : \frac{c}{A^\ell_\eta[\ast]} \text{ regular, } c \bigcup A^\ell_\eta[\ast] \text{ and } \ell g(\eta) > 0 \Rightarrow \frac{c}{A^\ell_\eta[\ast]} \perp A^\ell_\eta[\ast]\}
\]

independent over \((A^\ell_\eta[\ast], B^\ell_\eta[\ast]).\)

[Why? Note clause (7) of Definition 1.32 and clause (\(\delta\)) of Main Fact 1.35.]

\(\otimes_{\alpha} \) \(A^\ell_{\langle \ast \rangle} = B^\ell_{\langle \ast \rangle}\)

[Why? By the bookkeeping every \(b \in B^\ell_{\langle \ast \rangle}\) is considered for addition to \(A^\ell_{\langle \ast \rangle}\) see 1.35, clause (\(\gamma\))^1, subclause (b)(i) and for () there is nothing to stop us.]

\(\bigotimes_{\alpha} \) \(\otimes_{\alpha} \) \(A^\ell_{\langle \ast \rangle} = B^\ell_{\langle \ast \rangle}\)

[Why? If not, as \(A^\ell_\eta[\ast] \subseteq B^\ell_\eta[\ast]\) by [\text{BeSh 307, Th.B, p.277}] there is \(c \in B^\ell_\eta[\ast] \setminus A^\ell_\eta[\ast]\) such that: \(\frac{c}{A^\ell_\eta[\ast]} = p\). As \(c \in B^\ell_\eta[\ast] = \bigcup B^\ell_\eta[Y_n]\), for every \(n < \omega\) large enough \(c \in B^\ell_\eta[Y_n]\), and \(p\) does not fork over \(A^\ell_\eta[Y_n]\). So for some such \(n\)
the triple \((i_n, \ell_n, m_n)\) is such that \(\ell_n = \ell, m_n = \ell g(\eta)\) and \(b^\ell_{n,i_n} = c\), so by clause 
\((\gamma)^2(b)(ii)\) of 1.35 we have \(c \in A^\ell_{\eta}[Y_n] \subseteq A^\ell_{\eta}[\ast].\]

\[\bigotimes_{11} \text{ if } \eta \in I^*, \ell \in \{1, 2\} \text{ then } \{a^\ell_{\eta^\ell < \alpha} : \eta^\ell(\alpha) \in I^*\} \text{ is a maximal subset of } \{c \in M^\ell : \frac{c}{A^\ell_{\eta}[\ast]} \text{ regular, } \perp A^\ell_{\eta^\ell}[\ast] \text{ when meaningful}\} \text{ independent over } A^\ell_{\eta}[\ast].\]

[Why? If not, then for some \(c \in M, \{a^\ell_{\eta^\ell(\alpha)} : \eta^\ell(\alpha) \in I^*\} \cup \{c\} \text{ is independent over } A^\ell_{\eta}[\ast] \text{ and } \text{tp}(c, A^\ell_{\eta}[\ast]) \text{ is regular (and stationary). Hence by } \bigotimes_{10} \text{ we have } \{a^\ell_{\eta}[Y_n] : \eta^\ell(\alpha) \in I^*\} \cup \{c\} \text{ is independent over } (A^\ell_{\eta}[\ast], B^\ell_{\eta}[\ast]). \text{ Now for large enough } n \text{ we have } c \bigcup A^\ell_{\eta}[Y_n] \text{ and by } \bigotimes_{10} \text{ we have } c \bigcup B^\ell_{\eta}[Y_n], \text{ hence } A^\ell_{\eta}[\ast] \text{ not independent over } (A^\ell_{\eta}[Y_n], B^\ell_{\eta}[Y_n]), \text{ but } \{a^\ell_{\eta^\ell(\alpha)}[Y_n] : \eta^\ell(\alpha) \in I[Y_n]\} \text{ is independent over } (A^\ell_{\eta}[Y_n], B^\ell_{\eta}[Y_n]). \text{ So there is a finite set } w \text{ of ordinals such that } \alpha \in w \Rightarrow \eta^\ell(\alpha) \in I[Y_n] \text{ and } \{c\} \cup \{a^\ell_{\eta^\ell(\alpha)}[Y_n] : \alpha \in w\} \text{ is not independent over } (A^\ell_{\eta}[Y_n], B^\ell_{\eta}[Y_n]), \text{ and without loss of generality } w \text{ is minimal. Let } n_1 \in [n, \omega) \text{ be such that } \alpha \in w \Rightarrow a^\ell_{\eta^\ell(\alpha)} \in B^\ell_{\eta}[\ast] \Rightarrow a^\ell_{\eta}[Y_n]; \text{ clearly exist as } w \text{ is finite and let } u = \{\alpha \in w : a^\ell_{\eta^\ell(\alpha)} \notin B^\ell_{\eta}[\ast]\}; \text{ clearly } \alpha \in u \Rightarrow \eta^\ell < \alpha \in I^*. \text{ Now } \{a^\ell_{\eta^\ell(\alpha)}[\ast] : \eta^\ell(\alpha) \in I^*\} \cup B^\ell_{\eta}[\ast] \text{ includes } \{a^\ell_{\eta^\ell(\alpha)}[Y_n] : \alpha \in w\}, \text{ easy contradiction to the second sentence above.}]

\[\bigoplus_{12} f^\ast_\eta = \bigcup_{m < \omega} f^\ast_m[Y_n] \text{ (for } \eta \in I^*) \text{ is an elementary map from } A^\ell_{\eta}[\ast] \text{ onto } A^\ell_{\eta}[\ast],\]

[Easy].

\[\bigoplus_{13} f^\ast =: \bigcup_{\eta \in I^*} f^\ast_{\eta} \text{ is an elementary mapping from } \bigcup_{\eta \in I^*} A^\ell_{\eta}[\ast] \text{ onto } \bigcup_{\eta \in I^*} A^\ell_{\eta}[\ast].\]

[Clear using by \(\otimes_5 \oplus \otimes_6 \oplus \bigoplus_{12} \) and non-forking calculus].

\[\bigoplus_{14} \text{ We can find } \langle d^\ell_\alpha : \alpha < \alpha(\ast) \rangle \text{ such that:}\]

\(a\) \(d^\ell_\alpha \in M^\ell, \beta < \alpha \Rightarrow d^\ell_\beta \neq d^\ell_\alpha \)
\(\text{tp}(d^\ell_\alpha, \bigcup_{\eta \in I^*} A^\ell_{\eta}[\ast] \cup \{d^\ell_\beta : \beta < \alpha\}) \text{ is } \aleph_1\text{-isolated and } \mathbf{F}_{\aleph_0}^\ell\text{-isolated, and}\)

\(b\) \(g_{\alpha} = \bigcup_{\eta \in I^*} f^\ast_\eta \cup \{(d^1_\alpha, d^2_\alpha) : \alpha < \alpha(\ast)\} \text{ is an elementary mapping,}\)
(c) $\alpha(*)$ is maximal, i.e., we cannot find $d^1_{\alpha(*)}$ such that the demand in 
(a) holds for $\alpha(*) + 1$.

[Why? We can try to choose by induction on $\alpha$, a member $d^1_\alpha$ of $M^1 \setminus \bigcup_{\eta \in I[\alpha]} A^\ell_\eta \cup \{d^1_\beta : \beta < \alpha\}$ such that tp($d^1_\alpha$, $\bigcup_{\eta \in I[\alpha]} A^\ell_\eta \cup \{d^1_\beta : \beta < \alpha\}$) is $\aleph_\varepsilon$-isolated and $F^\ell_{\aleph_0}$-isolated.

So for some $\alpha(*)$, $d^1_\alpha$ is well defined iff $\alpha < \alpha(*)$ (as $\beta < \alpha \Rightarrow d^1_\beta \neq d^1_\alpha \in M^1$). Now choose by induction on $\alpha < \alpha(*)$, $d^2_\alpha \in M^2$ as required above, possible by $^*M^2_\ell$ being $\aleph_\varepsilon$-saturated (see [Sh:c, XII.2.1,p.591], [Sh:c, IV.3.10,p.179].]

$\otimes_{15}$ Dom($g_{\alpha(*)}$), Rang($g_{\alpha(*)}$) are universes of elementary submodels of $M^1$, $M^2$ respectively called $M^1_\ell$, $M^2_\ell$ respectively.

[Why? See [Sh:c, XII.1.2](2),p.591 and the proof of $\otimes_{14}$.

Alternatively, choose a formula $\psi(x, \vec{a})$ such that:

(a) $\vec{a} \subseteq$ Dom($g_{\alpha(*)}$) and $\models \exists x \psi(x, \vec{a})$ but no $b \in$ Dom($g_{\alpha(*)}$) satisfy $\varphi(x, \vec{a})$

(b) under clause (a), Rk($\psi(x, \vec{a})$, $\mathbb{L}_{\tau(T)}$, $\infty$) is minimal

(or just has no extension in $S($Dom($g_{\alpha(*)})$) forking over $\vec{a}$).

Let $\{\varphi_\ell(x, \vec{y}) : \ell < \omega\}$ list that $\mathbb{L}_{\tau(T)}$-formulas and we choose by induction on $\ell$ as formula $\psi_n(x, \vec{a}_n)$ such that:

(i) $\vec{a} \subseteq$ Dom($g_{\alpha(*)}$)

(ii) $\models (\exists x)\psi_n(x, \vec{a}_n)$

(iii) $\psi_{n+1}(x, \vec{a}_{n+1}) \models \psi_n(x, \vec{a}_n)$

(iv) $\psi_0(x, \vec{a}_0) = \psi(x, \vec{a})$

(v) for any formula $\psi'(x, \vec{a}')$ satisfying the demands on $\psi_{n+1}(x, \vec{a}_{n+1})$ we have

Rk($\psi_{n+1}(x, \vec{a}_{n+1})$, $\{\varphi_n(x, \vec{y}_n)\}$, 2) $< \operatorname{Rk}(\psi'(x, \vec{a}), \{\varphi_n(x, \vec{y})\}$, 2)

(on this rank see [Sh:c, II,§2]).

So $p = \{\psi_n(x, \vec{a}_n) : n < \omega\}$ has an extension in $S($Dom($g_{\alpha(*)})$) call it $q$. Now $q$ is $\aleph_\varepsilon$-isolated because $\psi(x, \vec{a}) \in q \in S($Dom($g_{\alpha(*)})$. For every $n$, $\psi_{n+1}(x, \vec{a}_n) \models q \models \{\varphi_n(x, \vec{y}_n)\}$ by clause (v) above so as $\psi_{n+1}(x, \vec{a}_n) \in q$ and this holds for every $n$

clearly $q$ is $F_{\aleph_0}^\ell$-isolated.

$\otimes_{16}$ If $M^\ell \neq M^\ell_\ell$ then for some $d \in M_\ell \setminus M^\ell_\ell$, $\frac{d}{M^\ell_\ell}$ is regular.
[Why? By [BeSh 307, Th.5.9,p.298] as $N^\ell_\eta \subseteq M^\ell$ by $\otimes_7$.]

$\otimes_7$ if $M^\ell \neq M'_\ell$ then for some $\eta \in I^*$, there is $d \in M^\ell \setminus M'_\ell$ such that $\frac{d}{A^\ell_\eta[*]}$ is regular, $d \bigcup A'_\eta[*]$ and $\left[ \ell g(\eta) > 0 \Rightarrow \frac{d}{A^\ell_\eta[*]} \perp A'_\eta[*] \right]$.

[Why? By [Sh:c, XII,1.4,p.529] every non-algebraic $p \in S(M'_\ell)$ is not orthogonal to some $A^\ell_\eta[*]$ so by $\otimes_16$ we can choose $\eta \in I^*$ and $d \in M^\ell \setminus M'_\ell$ such that $\frac{d}{A^\ell_\eta[*]}$ is regular $\pm A^\ell_\eta[*]$; without loss of generality $\ell g(\eta)$ is minimal, now $A^\ell_\eta[*] \subseteq M^\ell$ and by [BeSh 307, 4.5,p.290] without loss of generality $d \bigcup A'_\eta[*]$; the last clause is by $\otimes_16$.

"$\ell g(\eta)$ minimal"].

$\oplus_7 M^\ell = M'_\ell$.

[Why? By $\oplus_11 + \oplus_17$.

$\oplus_7$ there is an isomorphism from $M_1$ onto $M_2$ extending $\bigcup f^*_\eta$.

[Why? By $\oplus_7 + \otimes_15$ we have $M'_1 \cong M'_2$, so by $\otimes_18$ we are done]. $\square_1.36 \square_1.30$

**1.37 Lemma.** Assume $B \bigcup C, A = acl(A) = B \cap C$ and $A, B, C$ are $\epsilon$-finite,

$A \bigcup B \bigcup C \subseteq M, M$ an $\aleph_\epsilon$-saturated model of $T$. For notational simplicity make $A$ a set of individual constants.

Then $\text{tp}_{L_\infty, \aleph_\epsilon,d.q.}(B + C; M) = \text{tp}_{L_\infty, \aleph_\epsilon,d.q.}(B; M) + \text{tp}_{L_\infty, \aleph_\epsilon,d.q.}[C; M]$ where

**1.38 Definition.** 1) For any logic $L$ and $\vec{b}$ a sequence from a model $M$, let

$$\text{tp}_{L}(\vec{b}; M) = \left\{ \varphi(\vec{x}) : M \models \varphi[B], \varphi \text{ a formula in the vocabulary of } M, \right.$$ from the logic $L$ (with free variables from $\vec{x}$, where $\vec{x} = \langle x_i : i < \ell g(\vec{b}) \rangle$).

2) Replacing $\vec{b}$ by a set $B$ means we use the variables $\langle x_b : b \in B \rangle$.

3) Saying $p_1 = p_2 + p_3$ in 1.37 means that we can compute $p_1$ from $p_2$ and $p_3$ (and the knowledge how the variables fit and the knowledge of $T$, of course).
Proof of the Lemma 1.37.
It is enough to prove:

1.39 Claim. Assume

(a) $M^1, M^2$ are $\kappa$-saturated and
(b) $A^i_1 \cup A^i_2$ for $i = 1, 2$
(c) $A^i_1 = ac\ell(A^i_0)$ and $A^i_m$ is $\epsilon$-finite for $i = 1, 2$ and $m < 3$
(d) for $m = 0, 1, 2$ we have $f_m : A^1_m \rightarrow A^2_m$ is an elementary mapping preserving $tp_\infty$ (in $M^1, M^2$ respectively) and
(e) $f_0 \subseteq f_1, f_2$.

Then there is an isomorphism from $M^1$ onto $M^2$ extending $f_1 \cup f_2$.

Proof of 1.39. Repeat the proof of 1.5, but starting with $Y_0$ such that $A^\ell_{<\cdot}[Y_0] = A^\ell_0, A^\ell_{<\cdot}[Y_0] = A^\ell_1, A^\ell_{<\cdot}[Y_0] = A^\ell_2, f^\ell_{<\cdot} = f_0, f^\ell_{<0} = f_1, f^\ell_{<1} = f_2$ and that $\langle \cdot \rangle, \langle 0 \rangle, \langle 1 \rangle$ belongs to all $I[Y_0]$. During the construction we preserve $\langle 0 \rangle, \langle 1 \rangle \in I[Y_n]$ and for helping to preserve this we add also the demand

$\oplus_{2,m} B^\ell_{<\cdot}[Y_n] \cup A^\ell_1 \cup A^\ell_2$.

During the proof, when we have to increase $B^\ell_{<\cdot}$, we use 1.18(1) + 1.16(1).

Discussion: A natural version of 1.39 to say is the conclusion only that $tp_\alpha[A_{1}^1 \cup A_{1}^2, M^1] = tp_\alpha[A_{1}^2 \cup A_{2}^2, M^2]$ and to prove this by induction on $\alpha$. The case $\alpha = 0$ and $\alpha$ limit are obvious. If $\alpha = \beta + 1$, for the condition of $\leq_a$, we use the induction hypothesis and claim 1.27(1). The condition involving $\leq_b$ is similar but harder.
§2 Finer Types

We shall use here alternative types showing us probably a finer way to manipulate $tp$.

2.1 Convention. $T$ is superstable, NDOP $M,N$ are $\aleph_\epsilon$-saturated $\prec C^{eq}$.

2.2 Definition. $\Gamma_3 = \left\{ \left( \bar{b} / \bar{a} \right) : \bar{a} \subseteq \bar{b} \text{ are } \epsilon\text{-finite} \right\}$

$\Gamma_1 = \left\{ \left( \bar{p} / \bar{a} \right) : \bar{a} \text{ is } \epsilon\text{-finite, } p \in S(\bar{a}) \text{ is regular (so stationary)} \right\}$

$\Gamma_2 = \left\{ \left( \bar{p}, r / \bar{a} \right) : \bar{a} \text{ is } \epsilon\text{-finite, } p \text{ is a regular type of depth } > 0, \right.$

$p \pm \bar{a}$ (really only the equivalence class $p/\pm$ matters),

$r = r(x, \bar{y}) \in S(\bar{a})$ is such that for $(c, \bar{b})$ realizing $r$, $c/(\bar{a} + \bar{b})$ is regular $\pm p$, and $\bar{b} / \bar{a} = (r | \bar{y}) \perp p \right\}$.

We may add (to $\Gamma_x$) superscripts:

$(\alpha) f$ if $\bar{a}$ (or $\bar{a} \cdot \bar{b}$) is finite

$(\beta) s$ : for $\Gamma_3$ if $\bar{b} / \bar{a}$ is stationary, for $\Gamma_1$ if $p$ is stationary which holds always and
do $\Gamma_2$ if $r$ is stationary and every automorphism of $C$ over $\bar{a}$ fix $p/\pm$

$(\gamma) c$ if $\bar{a}$ (or $\bar{a}, \bar{b}$) are algebraically closed.

2.3 Claim. If $p$ is regular of depth $> 0$ and $p \pm \bar{a}$ and $\bar{a}$ is $\epsilon$-finite then for some $\bar{a}', \bar{a} \subseteq \bar{a}' \subseteq acl(\bar{a})$ and for some $q$ we have $\left( \bar{p}, q / \bar{a}' \right) \in \Gamma^s_2$.

Proof. Use, e.g., [Sh:c, V,4.11,p.272], assume $\bar{b} / \bar{a} \pm p$; we can define inductively equivalence relations $E_n$, with parameters from $acl(\bar{a}')$, $\bar{a}' = \bar{a} \cdot (\bar{b} / E_0) \cdot \cdots \cdot (\bar{b} / E_{n-1})$, such that $tp(\bar{b} / E_n, acl(\bar{a}''))$ is semi-regular. By superstability this stop for some $n$ hence $\bar{b} \subseteq acl(\bar{a}')$. For some first $m$ $tp(\bar{b} / E_m, acl(\bar{a}''))$ is $\pm p$, by [Sh:c, X,7.3](5),p.552 the type is regular (as because $p$ is trivial having depth $> 0$; see [Sh:c, X,7.2,p.551]). $\Box_{2.3}$
2.4 Definition. We define by induction on an ordinal \( \alpha \) the following (simultaneously): note — if a definition of something depends on another which is not well defined, neither is the something

\[
\begin{align*}
\text{tp}_\alpha^1 \left[ \left( \frac{p}{\bar{a}} \right) \right], M & \quad \text{for} \quad \left( \frac{p}{\bar{a}} \right) \in \Gamma_1, \bar{a} \subseteq M \\
\text{tp}_\alpha^2 \left[ \left( \frac{p, r}{\bar{a}} \right) \right], M & \quad \text{for} \quad \left( \frac{p, r}{\bar{a}} \right) \in \Gamma_1, \bar{a} \subseteq M \\
\text{tp}_\alpha^3 \left[ \left( \frac{\bar{b}}{\bar{a}} \right) \right], M & \quad \text{for} \quad \left( \frac{\bar{b}}{\bar{a}} \right) \in \Gamma_3^e, \bar{a} \subseteq \bar{b} \subseteq M
\end{align*}
\]

**Case A** \( \alpha = 0 \):

\[
\begin{align*}
\text{tp}_\alpha^1 \left[ \left( \frac{p}{\bar{a}} \right) \right], M & \quad \text{is} \quad \text{tp}((c, \bar{a}), \emptyset) \quad \text{for any} \quad c \text{ realizing } p. \\
\text{tp}_\alpha^2 \left[ \left( \frac{p, r}{\bar{a}} \right) \right], M & \quad \text{is} \quad \text{tp}((c, \bar{b}, \bar{a}), \emptyset) \quad \text{for any} \quad (c, \bar{b}) \text{ realizing } r \\
\text{tp}_\alpha^3 \left[ \left( \frac{\bar{b}}{\bar{a}} \right) \right], M & \quad \text{is} \quad \text{tp}((\bar{b}, \bar{a}), \emptyset)
\end{align*}
\]

(i.e., the type and the division of the variables between the sequences).

**Case B** \( \alpha = \beta + 1 \):

(a) \( \text{tp}_\alpha^1 \left[ \left( \frac{p}{\bar{a}} \right) \right], M \) is:

**Subcase a1**: if \( p \) has depth zero, it is \( w_p(M/\bar{a}) \) (the \( p \)-weight, equivalently, the dimension)

**Subcase a2**: if \( p \) has depth > 0 (hence is trivial), then it is \( \{ \langle y, \lambda_{\bar{a}, p}^y \rangle : y \} \) where

\[
\lambda_{\bar{a}, p}^y = \dim(I_{\bar{a}, p}[M], a) \quad \text{where} \quad I_{\bar{a}, p}[M] = \left\{ c \in M : c \text{ realize } p \text{ and } y = \text{tp}_\beta^3 \left[ \left( \frac{a\ell(a+c)}{a\ell(a)} \right) , M \right] \right. \\
\left. \begin{array}{l}
\text{where } \bar{a}^* \text{ list } acl(\bar{a}) \text{ and } \bar{c}^* \text{ list } acl(\bar{a} + c) \end{array} \right\}
\]

an alternative probably more transparent and simpler in use is:
\[ \lambda_{\bar{a},p} = \dim \left\{ c \in M : c \text{ realizes } p \text{ and} \right. \]
\[ y = \{ \text{tp}_3^\beta \left[ \frac{ac\ell(\bar{a} + c')}{ac\ell(\bar{a})} \right], M \} : c' \in p(M) \text{ and } c' \bigcup c \} \]
\[ \text{pedantically } y = \{ \text{tp}_3^\beta \left[ \frac{<c' > \bar{a}^*}{\bar{a}^* \bar{a}} \right], M \} \text{ where} \]
\[ \bar{a}^* \text{ list } acl(\bar{a}) \text{ and} \]
\[ \bar{c}^* \text{ list } acl(\bar{a} + c'), c' \in p(M) \text{ and } c' \bigcup c \} \}

(b) \text{tp}_a^2 \left[ (\bar{p},r) \right], M \text{ is:} \]
\[ \text{tp}_a^1 \left[ \frac{c}{b^+} \right], M \text{ for any } (c, \bar{b}) \text{ realizing } r, b^+ = ac\ell(\bar{a} + \bar{b}), \text{i.e., } b^+ \text{ lists } acl(\bar{a} + \bar{b}) \text{ (so not well defined if we get at least two different cases; so remember } c/b^+ \in S(b^+)). \]

(c) \text{tp}_a^3 \left[ \frac{\bar{b}}{\bar{a}} \right], M \text{ is} \]
\[ \left\{ \langle p, \text{tp}_a^2 \left[ (\bar{p},r) \right], M \rangle : (\bar{p},r) \in \Gamma_2^* \text{ and } p \perp \bar{a} \right\} . \]

Case C: \( \alpha \text{ limit} \): For any \( \ell \in \{1,2,3\} \) and suitable object OB:
\[ \text{tp}_a^\ell[OB,M] = \langle \text{tp}_b^\ell[OB,M] : \beta < \alpha \rangle . \]

2.5 Definition. 1) For \( \left( \bar{p} \right)_a \in \Gamma_1 \) where \( \bar{a} \in M \), let (remembering 1.14(8)):
\[ \mathcal{P}_M^{\left( \bar{a} \right)} = \left\{ q \in S(M) : q \text{ regular and } : q \pm p \text{ or for some } \right. \]
\[ c \in p(M) \text{ we have } q \in \mathcal{P}_M \left( \bar{a} \right) \} \} . \]

2) For \( \left( \bar{p},r \right)_a \in \Gamma_2 \) let
\[ \mathcal{P}_M^{\left( \bar{p},r \right)_a} = \left\{ q \in S(M) : q \text{ regular and } : q \pm p \text{ or for some } \right. \]
\[ (c,\bar{b}) \in r(M), q \in \mathcal{P}_M \left( \bar{a} + \bar{b} \right) \} . \]
3) For a set $\mathcal{P}$ of (stationary) regular types not orthogonal to $M_1$, let $M_1 \leq \mathcal{P} M_2$ means $M_1 \prec M_2$ and for every $p \in \mathcal{P}$ and $\bar{c} \in M_2, M_1 \perp p$.

4) If (in (3)), $\mathcal{P} = \mathcal{P}^{M_1}_{(\bar{a})}$ we may write $(\bar{p})$ instead $\mathcal{P}$, similarly if $\mathcal{P} = \mathcal{P}^{M_1}_{(\bar{a}, \bar{r})}$ we may write $(\bar{p}, \bar{r})$.

2.6 Claim. :
1) From $tp_{\alpha}^{1}(\bar{p}, M)$ we can compute $tp_{\alpha}^{1}(\bar{p}, M)$ if $Dp(p) < \alpha$.

2) From $tp_{\alpha}^{2}(\bar{p}, q), M$ we can compute $tp_{\alpha}^{2}(\bar{p}, q), M$ if $Dp(p) < \alpha$.

3) From $tp_{\alpha}^{3}(\bar{b}, M)$ we can compute $tp_{\alpha}^{3}(\bar{b}, M)$ if $Dp(\bar{b}/\bar{a}) < \alpha$.

4) If Definition 2.5(2) we can replace “some $(c, \bar{b}) \in r(M)$” by “every $(c, \bar{b}) \in r(M)$”.

Proof. 1),2),3) We prove this by induction on $\alpha$. By the definition.
4) Left to the reader.

2.7 Observation. From $tp_{\alpha}(OB, M)$ we can compute $tp_{\beta}(OB, M)$ and $tp_{\beta}[OB, M]$ is well defined if $\beta \leq \alpha$ and the former is well defined.

2.8 Lemma. For every ordinal $\alpha$ the following holds:

1) $tp_{\alpha}^{1}$ is well defined
2) $tp_{\alpha}^{2}$ is well defined.
3) $tp_{\alpha}^{3}$ is well defined.
4) If $\bar{a} \in M_1, (\bar{p}) \in \Gamma_1, M_1 \leq (\bar{a}) M_2$ then

$$tp_{\alpha}^{1}[\bar{p}], M_1 = tp_{\alpha}^{1}[\bar{p}], M_2.$$ 

5) If $\bar{a} \in M_1, (\bar{p}, \bar{r}) \in \Gamma_2, M_1 \leq (\bar{a}) M_2$ then

$$tp_{\alpha}^{2}[\bar{p}, \bar{r}], M_1 = tp_{\alpha}^{2}[\bar{p}, \bar{r}], M_2.$$ 

6) If $\bar{a} \subseteq \bar{b} \subseteq M_1, (\bar{b}) \in \Gamma_3, M_1 \leq (\bar{a}) M_2$ then

$$tp_{\alpha}^{3}[\bar{b}], M_1 = tp_{\alpha}^{3}[\bar{b}], M_2.$$ 

\[5\text{i.e. in all the cases we have tried to define it in Definition 2.9}\]
Proof. We prove it, by induction on $\alpha$, simultaneously (for all clauses and parameters).

If $\alpha$ is zero, they hold trivially by the definition.

If $\alpha$ is limit, they hold trivially by the definition and induction hypothesis. So for the rest of the proof let $\alpha = \beta + 1$.

Proof of (1)$_\alpha$. If $p$ has depth zero — check directly.

If $p$ has depth $> 0$ - by (3)$_\beta$

(i.e. induction hypothesis) no problem.

Proof of (2)$_\alpha$. Like 1.27 (and (4)$_\alpha$).

Proof of (3)$_\alpha$. Like (2)$_\alpha$.

Proof of (4)$_\alpha$. Like 1.26 (and (3)$_\beta$,(6)$_\beta$).

Proof of (5)$_\alpha$. By (2)$_\alpha$ we can look only at $(c, b^+)$ in $M_1$, then use (4)$_\alpha$.

Proof of (6)$_\alpha$. By (5)$_\alpha$. $\square_{2,8}$

2.9 Lemma. For an ordinal $\alpha$ restricting ourselves to the cases (the types $p, p_1$ being) of depth $< \alpha$:

(A1) Assume $\binom{p}{\vec{a}} \in \Gamma_1$, $\vec{a} \subseteq \vec{a}_1 \subseteq M, \vec{a}_1$ is $\epsilon$-finite, $\vec{a}_1 \perp p$ and $p_1$ is the station-

arization of $p$ over $\vec{a}_1$.

Then from $tp^{1}_\alpha[(\binom{p}{\vec{a}}), M]$ we can compute $tp^{1}_\alpha[(\binom{p_1}{\vec{a}_1}), M]$.

(A2) Under the assumption of (A1) also the inverse computations are O.K.

(A3) Assume $\binom{p_1}{\vec{a}_1} \in \Gamma_1$ for $\ell = 1, 2, \vec{a} \subseteq M$ and $p_1 \pm p_2$.

Then from $tp^{1}_\alpha[(\binom{p_1}{\vec{a}_1}), M]$ (and $tp((\vec{a}, c_1, c_2), \emptyset)$ where $c_1, c_2$ realizes $p_1, p_2$ respectively, of course) we can compute $tp^{1}_\alpha[(\binom{p_2}{\vec{a}_1}), M]$.

(B1) Assume $\binom{p_\ell,r_\ell}{\vec{a}_1} \in \Gamma_2^{sc}$ for $\ell = 1, 2, \vec{a} \subseteq M$ and $p_1 \pm p_2$.

Then (from the first order information on $\vec{a}, p_1, p_2, r_1, r_2$, of course, and)

$tp^{2}_\alpha[(\binom{p_1,r_1}{\vec{a}_1}, M)]$ we can compute $tp^{2}_\alpha[(\binom{p_2,r_2}{\vec{a}_1}, M)]$.

(B2) Assume $\vec{a} \subseteq \vec{a}_1 \subseteq M, \vec{a}_1 \perp p, \binom{p,r}{\vec{a}} \in \Gamma_2^{c}, r \subseteq r_1 \in S(\vec{a}_1), r_1$ does not fork over $\vec{a}$, ( so $\binom{p_1}{\vec{a}_1} \in \Gamma_2$).

Then from $tp^{2}_\alpha[(\binom{p_1}{\vec{a}_1}, M)]$ we can compute $tp^{2}_\alpha[(\binom{p_2}{\vec{a}_1}, M)]$. 

(B3) Under the assumption of (B2), the inverse computation are O.K.

(C1) Assume \(\bar{b} /\bar{a} \in \Gamma_3, \bar{a} \subseteq \bar{b} \subseteq M, \bar{a} \subseteq \bar{a}_1, \bar{b} \cup \bar{a}_1, \bar{a}_1 = acl(\bar{a}_1 + \bar{b})\).

Then from \(tp^3_\alpha[(\bar{b} /\bar{a}), M]\) we can compute \(tp^3_\alpha[(\bar{a}_1 /\bar{a}), M]\).

(C2) Under the assumptions of (C1) the inverse computation is O.K.

(C3) Assume \(\bar{b} /\bar{a} \in \Gamma_3, \bar{b} \subseteq \bar{b}^*, \bar{b}^* \perp_{\bar{a}} \bar{a}^* = acl(\bar{b}^*)\). Then from \(tp^3_\alpha[(\bar{b} /\bar{a}), M]\) we can compute

\[
\left\{tp^3_\alpha[(\bar{b}^* /\bar{a}), M] : \bar{b} \subseteq \bar{b}^* \subseteq M \text{ and } \bar{b}^* /\bar{b} = \bar{b}^* /\bar{b}\right\}.
\]

Proof. We prove it, simultaneously, for all clauses and parameters, by induction on \(\alpha\) and the order of the clauses.

For \(\alpha = 0\): easy.

For \(\alpha\) limit: very easy.

So assume \(\alpha = \beta + 1\).

Proof of (A1)\(\alpha\). As \(p\) is stationary \(\perp \bar{a}_1\), for every \(c \in p(M), c /\bar{a} \vdash c /\bar{a}_1\), which necessarily is \(p_1\), hence \(p(M) = p_1(M)\). Also the dependency relation on \(p(M)\) is the same over \(\bar{a}_1\), hence dimension. So it suffices to show:

\((\ast)\) for \(c \in p(M)\), from \(tp^3_\beta[(\frac{acl(\bar{a}+c)}{acl(\bar{a})}), M]\) we can compute \(tp^3_\beta[(\frac{acl(\bar{a}_1+c)}{acl(\bar{a}_1)}), M]\).

But this holds by (C1)\(\beta\).

Proof of (A2)\(\alpha\). Similar using (C2)\(\beta\).

Proof of (A3)\(\alpha\). If \(p_1\) (equivalently \(p_2\)) has depth zero — the dimensions are equal.

Assume they have depth \(> 0\) hence are trivial and dependency over \(\bar{a}\) is an equivalence relation on \(p_1(M) \cup p_2(M)\).

Now for \(c_1 \in p_1(M)\), from \(tp^3_\beta[(\frac{acl(\bar{a}+c_1)}{acl(\bar{a})}), M]\) we can compute for every complete type over \(acl(\bar{a} + c_1)\) not forking over \(\bar{a}\), and \(\bar{d}\) realizing \(r\), \(tp^3_\beta[(\frac{acl(\bar{a}_1+\bar{c}_1)}{acl(\bar{a}_1)}, M)]\) — by (C1)\(\beta\), then we can compute for each such \(r, \bar{d}\)
\[
\left\{ \text{tp}_\beta^3 \left[ \left( \frac{\text{acl}(\bar{a} + \bar{d} + c_2)}{\text{acl}(\bar{a} + \bar{d})}, M \right) \right] : c_2 \in p_2(M) \text{ and } \frac{c_2}{\text{acl}(\bar{a} + \bar{d} + c_1)} \perp_a (\bar{a} + \bar{d}) \right\}
\]

(necessarily \(c_2 \cup \bar{d}\))

(this by \((C3)_\beta\)).

**Proof of \((B1)_\alpha\).** As in earlier cases we can restrict ourselves to the case \(\text{Dp}(p_\ell) > 0\). We can find \((c_\ell, b_\ell) \in r_\ell(M), \bar{b}_1 \cup \bar{d}_1, c_1 \bar{b}_1 \cup \bar{b}_2\) (by [Sh:c, X,7.3](6)). By 2.8(2) (and the definition) from \(\text{tp}_\alpha^1\left[\left(\frac{p_{r_1}}{\text{acl}(\bar{a} + b_1)}, M\right)\right]\) we can compute that it is equal to \(\text{tp}_\alpha^1\left[\left(\frac{c_1}{\text{acl}(\bar{a} + b_1)}\right), M\right]\).

By \((A1)_\alpha\) we can compute \(\text{tp}_\alpha^1\left[\left(\frac{c_2}{\text{acl}(\bar{a} + b_1 + b_2)}\right), M\right]\) hence by \((A3)_\alpha\) we can compute \(\text{tp}_\alpha^1\left[\left(\frac{c_2}{\text{acl}(\bar{a} + b_1 + b_2)}\right), M\right]\).

Now use \((A2)_\alpha\) to compute \(\text{tp}_\alpha^1\left[\left(\frac{c_2}{\text{acl}(\bar{a} + b_2)}\right), M\right]\) and by 2.8(2), 2.4(2) it is equal to \(\text{tp}_\alpha^2\left[\left(\frac{p_{r_2}}{\bar{a}}\right), M\right]\).

**Proof of \((B2)_\alpha\).** Choose \((c, \bar{b}) \in r(M)\) such that \(c \bar{b} \cup \bar{a}_1\).

From \(\text{tp}_\alpha^2\left[\left(\frac{p_{r_1}}{\bar{a}}\right), M\right]\) we can compute \(\text{tp}_\alpha^1\left[\left(\frac{c}{\bar{a} + \bar{b}}\right), M\right]\) (just — see 2.8(2) and Definition 2.4), from it we can compute \(\text{tp}_\alpha^1\left[\left(\frac{c}{\bar{a} + \bar{b} + \bar{a}_1}\right), M\right]\) by \((A1)_\alpha\), from it we can compute \(\text{tp}_\alpha^2\left[\left(\frac{p_{r_2}}{\bar{a}_1}\right), M\right]\) (see 2.8(2) and Definition 2.4).

**Proof of \((B3)_\alpha\).** Let \(\left(\frac{p_{r_1}}{b_1}\right) \in \Gamma_\ell, p \perp \bar{a}_1\) be given. So necessarily \(\frac{\bar{a}_1}{\bar{a}} = p\) (this to enable us to use \((B2), 3\)). It suffices to compute \(\text{tp}_\alpha^2\left[\left(\frac{p_{r_1}}{b_1}\right), M\right]\) and we can discard the case \(\text{Dp}(p) = 0\).

So \(p\) is regular \(\pm \bar{a}_1, \perp \bar{a}_1\), hence \(p \pm \bar{b}_1 \perp \bar{a}_1\), and as \(\bar{a} \subseteq \bar{b}, \bar{a} = \text{acl}(\bar{b})\) we can find \((\frac{p_{r_1}}{\bar{b}_1}) \in \Gamma_2\), (see 2.3) and we know \(\text{tp}_\alpha^2\left[\left(\frac{p_{r_1}}{\bar{b}_1}\right), M\right]\), and we can find \(r_2\), a complete type over \(\bar{b}_1\) extending \(r_1\) which does not fork over \(\bar{b}_1\). From \(\text{tp}_\alpha^2\left[\left(\frac{p_{r_1}}{\bar{b}_1}\right), M\right]\) we can compute \(\text{tp}_\alpha^2\left[\left(\frac{p_{r_2}}{\bar{b}_1}\right), M\right]\) by \((B2)_\alpha\), and from it \(\text{tp}_\alpha^2\left[\left(\frac{p_{r_1}}{b_1}\right), M\right]\) by \((B1)_\alpha\).

**Proof of \((C2)_\alpha\).** Similar, use \((B3)_\alpha\) instead of \((B2)_\alpha\).
Proof of $(C3)_\alpha$. Without loss of generality $\frac{b^*}{b}$ is semi regular, let $p^*$ be a regular type not orthogonal to it and without loss of generality $\text{Dp}(p^*) > 0 \Rightarrow \frac{b^*}{b}$ regular (as in 2.3).

If $p^*$ has depth zero, then the only $p$ appearing in the definition $\text{tp}_\alpha^3(\frac{b^*}{b}, M)$ is $p^*$ (up to $\pm$) and this is easy. Then $\text{tp}_\alpha^2$ is just the dimension and we have no problem.

So assume $p^*$ has depth $> 0$. We can by $(B1)_\alpha, (B2)_\alpha$ compute $\text{tp}_\alpha^2 [(p', q'), M]$ when $p' \pm \bar{b}, p' \pm p^*$ (regardless of the choice of $\bar{b}^*$). Next assume $p' \pm p^*$; by $(B1)_\alpha$ without loss of generality $q'$ does not fork over $\bar{b}$. As $\text{Dp}(p^*) > 0$, it is trivial (and we assume $w_p(\bar{b}^*, \bar{b}) = 1$) hence $\bar{b}^*/\bar{b}$ is regular so in $\text{tp}_\alpha^2 [(p', q'), M]$ we just lose a weight 1 for one specific $\text{tp}_\alpha^3$ type: the one $\bar{b}^*$ realizes concerning which we have a free choice. We are left with the cases $p' \pm \bar{b}, p' \pm p^*$; well we know $\text{tp}_\alpha^3$, but we have to add $\text{tp}_\alpha^3$? Use Claim 2.6(3) (and $(A1)_\alpha$ as we add a parameter).

2.10 Claim. $\text{tp}_\gamma^3 [\frac{b^*}{b}, a, M], \text{tp}_\gamma^3 [\bar{b}, a, M], \text{tp}_\gamma^3 [M]$ are expressible by formulas in $L_{\gamma^\infty, \aleph_{\gamma}}^\gamma$ (d.q.).

By 2.9 we have

2.11 Conclusion. If $\text{Dp}(T) < \infty$ then:
1) From $\text{tp}_\infty^3 [(\frac{B}{A}), M]$ we can compute $\text{tp}_\infty^3 [(\frac{B}{A}), M]$ (the type from §1).
2) Similarly from $\text{tp}_\infty^3 [A, M]$ we can compute $\text{tp}_\infty^3 [(A), M]$.

From 2.6, 2.10, 2.11 and 1.30 we get

2.12 Corollary. If $\gamma = \text{Dp}(T)$ and $M, N$ are $\aleph_\gamma$-saturated, then

$$M \cong N \iff \text{tp}_\gamma^3 [M] = \text{tp}_\gamma^3 [N] \iff M \cong L_{\gamma^\infty, \aleph_{\gamma}}^\gamma (d.q.) N.$$
APPENDIX

The following clarifies several issues raised by Baldwin. A consequence of

the existence of nice invariants for characterization up to isomorphism (or
characterization of the models up to isomorphism by their \( \mathcal{L} \)-theory for
suitable logic \( \mathcal{L} \))

naturally give absoluteness, e.g. extending the universe say by nice forcing preserve
non-isomorphism. So negative results for

\((*)\) is non-isomorphism (of models of \( T \)) preserved by forcing by “nice forcing
notions”?

implies that we cannot characterize models up to isomorphism by their \( \mathcal{L} \)-theory
when the logic \( \mathcal{L} \) is “nice”, i.e. when \( \text{Th}_{\mathcal{L}}(M) \) preserved by nice forcing notions.
So coding a stationary set by the isomorphism type can be interpreted as strong
evidence of “no nice invariants”, see [Sh 220]. Baldwin, Laskowski, Shelah [BLSh
464] show that not only for every unsuperstable; but also for some quite trivial
superstable (with NDOP, NOTOP) countable \( T \), there are non-isomorphic models
which can be made isomorphic by some ccc (even \( \sigma \)-centered) forcing notion. This
shows that the lack of a really finite characterization is serious.
Can we still get from the characterization in this paper an absoluteness result?
Note that for preserving \( \aleph_{\varepsilon} \)-saturation (for simplicity for models of countable \( T \))
we need to add no reals\(^6\), and in order not to erase distinction of dimensions we
want not to collapse cardinals, so the following questions is natural, for a first order
(countable) complete \( T \):

\((*)_{T} \) assume \( V_1 \subseteq V_2 \) are transitive models of ZFC with the same cardinals
and reals, the theory \( T \in V_1 \). If the models \( M_1, M_2 \) are from \( V_1 \) and they are
models of \( T \) not isomorphic in \( V_1 \); must they still be not isomorphic in \( V_2 \)\(^7\)

\((*)_{T,\kappa} \) like \((*)_{T}^{1} \) we assume in addition \( \mathcal{P}(\kappa)^{V_1} = \mathcal{P}(\kappa)^{V_2} \).

2.13 Theorem. 1) For countable first order complete \( T \) the answer to \((*)_{T} \) and
\((*)_{T,\kappa} \) for any \( \kappa \) is negative except when possibly \( T \) is superstable, NDOP, NOTOP.
2) For any first order complete \( T \) for the class of \( \aleph_{\varepsilon} \)-saturated models, the answer
to \((*)_{T,[T]} \) is negative except possible when \( T \) is superstable with NDOP.

\(^6\) (the set of \( \{\text{acl}(\bar{a}) : \bar{a} \in \omega^\omega M\} \) is absolute but the set of their enumeration and of the
\( \{f \upharpoonright (\text{acl}(\bar{a})) : f \in \text{AUT}(\mathcal{C}), f(\bar{a}) = \bar{a}\} \) is not).

\(^7\) Note we did not say they have the same \( \omega \)-sequences of ordinals, e.g. if \( V_2 = V_1^{\mathcal{P}, P \text{ Prikry}} \)
forcing, then the assumption of \((*)_{T} \) holds though a new \( \omega \)-sequence of ordinals was added. So
for \( V_1 \subseteq V_2 \) as in \((*)_{T} \), the \( \mathcal{L}_{\infty,\aleph_1} \)-theory is not necessarily preserved.
Proof. By quoting.

So we restrict ourselves to these. It should be quite transparent that $L_{\infty, \kappa, (q.d.)}$-theory is preserved from $V_1$ to $V_2$ (as well as the set of sentences in the logic).

Hence

2.14 Theorem. For the class of $\kappa$-saturated models of superstable NDOP, $NTO$-TOP theory $T$ the answer to $(*)_{T, |T|}$ is yes.

Proof. In $V_1$ for $\ell = 1, 2$ let $\sA_\ell = \{ A \subseteq M^\eq_\ell : A$ is $\varepsilon$-finite and $\acl(A) = A \}$. Clearly the same definition gives $\sA_\ell$ in $V_2$ and $M_1, M_2$ are $\kappa$-saturated also in $V_2$.

$A \in \sA_\ell \Rightarrow |A| \leq |T|$ and let $\sF^* = \{ f :$ for some $A_1 \in \sA_2$ and $A_2, \sA_2, f$ is a one-to-one function from $A_1$ onto $A_2$ which is an $(M_1, M_2)$-elementary mapping$\}$. Again this definition gives the same set. We can define a rank function $\rk: \sF^* \to \Ord \cup \{ \infty \}$ such that $\rk(f) = \infty$ iff $(M_1, a)_{a \in \Dom(f)} \equiv_{L_{\infty, \kappa, (d.q.)}} (M_2, f(a))_{a \in \Dom(f)}$ and it too is absolute.

Easily in both universes

(a) $M_1 \cong M_2$ iff $M_1 \equiv_{L_{\infty, \kappa, (d.q.)}} M_2$.

[Why? By Theorem 1.5.]

(b) $M_2 \equiv_{L_{\infty, \kappa_0, (d.q.)}} M_2$ iff there is $\sF \subseteq \sF^*$ as in $\otimes M_1, M_2$ from 1.4.

[Why? By 1.4.]

(c) there is $\sF \subseteq \sF^*$ as in $\otimes M_1, M_2$ from 1.4 iff $\infty$ belongs to the range of the rank function.

[Why? Well known.]
REFERENCES.


