

# Non-existence of Universal Orders in Many Cardinals

Menachem Kojman and Saharon Shelah†

Department of Mathematics, Hebrew University

91904 Jerusalem, Israel

## ABSTRACT

Our theme is that not every interesting question in set theory is independent of *ZFC*. We give an example of a first order theory  $T$  with countable  $D(T)$  which cannot have a universal model at  $\aleph_1$  without CH; we prove in *ZFC* a covering theorem from the hypothesis of the existence of a universal model for some theory; and we prove — again in *ZFC* — that for a large class of cardinals there is no universal linear order (e.g. in every  $\aleph_1 < \lambda < 2^{\aleph_0}$ ). In fact, what we show is that if there is a universal linear order at a regular  $\lambda$  and its existence is not a result of a trivial cardinal arithmetical reason, then  $\lambda$  “resembles”  $\aleph_1$  — a cardinal for which the consistency of having a universal order is known. As for singular cardinals, we show that for many singular cardinals, if they are not strong limits then they have no universal linear order. As a result of the non-existence of a universal linear order, we show the non-existence of universal models for all theories possessing the strict order property (for example, ordered fields and groups, Boolean algebras, p-adic rings and fields, partial orders, models of PA and so on).

**Key words:** Universal model, Linear Order, Covering Numbers, Club guessing, Strict Order Property

**Subject Classification:** Model Theory, Set Theory, Theory of Orders

## 0. Introduction

### General Description

This paper consists entirely of proofs in *ZFC*. We can even dare to recommend reading it to anybody who is interested in linear orders or partial orders in themselves, and to whom axiomatic set theory and model theory are of less interest. Such a reader should, though, consult the appendix to this paper, or a standard textbook like [CK] for the notion of “elementary submodel”, and confine his reading to sections 3, 4 and 5.

The general problem addressed in this paper is the computation of the universal spectrum of a theory (or a class of models), namely the class of cardinals in which the theory (the class) has a universal model. (A definition of “universal model” is found below). As the universal spectrum of a theory usually depends on cardinal arithmetic, and even on the particular universe of set theory in which a given cardinal arithmetic holds (see below), the problem of determining the universal spectrum of a theory must be rephrased as: under which cardinal arithmetical assumptions *can* a given theory (class) possess a universal model in a given cardinality  $\lambda$ ?

All results in this paper are various *negative* answers in *ZFC* to this question, namely theorems of the form “if  $C(\lambda)$  (some cardinal arithmetical condition on a cardinal  $\lambda$ ) then there is *no* universal model of  $T$  at cardinality  $\lambda$ ”. In general, it is harder to prove such theorems when the cardinal  $\lambda$  in question is singular. Such theorems are first proved for the case where  $T$  is the theory of linear orders, and then are shown to hold also for a larger class of theories, including the theory of Boolean algebras, the theory of ordered fields, the theory of partial orders and others.

---

† Supported by a BSF grant. Publication No. 409

## Background and detailed content

A universal model at power  $\lambda$ , for a class of models  $K$ , is a model  $M \in K$  of cardinality  $\lambda$  with the property that for all  $N \in K$  such that  $|N| \leq \lambda$  there is an embedding of  $N$  into  $M$ . At this point let us clarify what “embedding” means in this paper. If  $K = MOD(T)$  is the class of models of a first order theory  $T$ , then “embedding” should be understood as “elementary embedding” when  $T$  is complete, and “universal” is with respect to elementary embeddings; when  $T$  is not complete (e.g. the theory of linear orders, the theory of graphs or the theory of Boolean algebras), “embedding” is an ordinary embedding, namely a 1-1 function which preserves all relations and operations, and “universal” is with respect to ordinary embeddings. This distinction is necessary, because there are theories for which universal models in the sense of an ordinary embedding exist, whereas universal models in the sense of an elementary embedding do not exist (see appendix for such an example).

Although the notion “universal model” is older than its relative, “saturated model”, and arises more often and more naturally in other branches of mathematics, it has won less attention, perhaps because answers to questions involving the former notion were harder to get. As one example of a contribution to the theory of universal models we can quote [GrSh 174], in which it is shown that the class of locally finite groups has a universal model in any strong limit of cofinality  $\aleph_0$  above a compact cardinal. The class  $\{\lambda : T \text{ has a saturated model of cardinality } \lambda\}$  has been characterized for a first order theory  $T$  (See the situation — with history — in [Sh-a] or [Sh-c], VIII.4).

Saturated models are universal, and their existence is known at cardinals  $\lambda$  such that  $\lambda = \lambda^{<\lambda} > |T|$  or just  $\lambda = \lambda^{<\lambda} \geq |D(T)|$  ( $D(T)$  is defined below) for every  $T$ ; furthermore, when  $\lambda = 2^{<\lambda}$ , essentially the same proof gives a “special model”, which is also universal (for these results see [CK]). Therefore the problem of the existence of a universal model for a first order theory remains unsettled in classical model theory only for cardinals  $\lambda < 2^{<\lambda}$ .

The consistency of *not* having universal models at such  $\lambda$ 's for all theories which do not have one at every infinite power is very easy (see appendix). In the other direction, the second author proved in [Sh 100] the consistency of the existence of a universal order at  $\aleph_1$  with the negation of CH, and, in [Sh 175], [Sh 175a], proved the consistency of the existence of universal graph at  $\lambda$ , if there is a  $\kappa$  such that  $\kappa = \kappa^{<\kappa} < \lambda < 2^\kappa = \text{cf}(2^\kappa)$ . One could expect at that point to prove that every theory  $T$  which has no trivial reason for not having a universal model at  $\aleph_1 < 2^{\aleph_0}$ , can have one. (By a “trivial reason” we mean an uncountable  $D(T)$ .  $D(T)$  is the set of all complete  $n$ -types over the empty set,  $n < \omega$ ; it is known and easy to prove that if  $D(T)$  is uncountable, it is of size  $2^{\aleph_0}$ ; and every type in  $D(T)$  must be realized in a universal model). But this is not the case. In Section 1. we show that there is a first order theory  $T$  with  $|D(T)| = \aleph_0$  (which is even  $\aleph_0$ -categorical) which has a universal model in  $\aleph_1$  iff CH.

An attempt to characterize the class of theories for which it is *consistent* to have a universal

model at  $\aleph_1 < 2^{\aleph_0}$  was done by Mekler. Continuing [Sh 175] he has shown in [M] that it is consistent with the negation of CH that every universal theory of relational structures with the joint embedding property and amalgamation for  $P^-(3)$ -diagrams and only finitely many isomorphism types at every finite power, has a universal model at  $\aleph_1$ . He has also shown, continuing [Sh 175a], that it is consistent with  $\kappa^{<\kappa} = \kappa < \text{cf}(2^\kappa) < \lambda < 2^\kappa$  that every 4-amalgamation class, which in every finite power has only finitely many isomorphism types, has a universal structure in power  $\lambda$ .

In Section 2 we prove a covering theorem which shows, as one corollary, that if  $2^{\aleph_0} = \aleph_{\omega_1}$ , there are no universal models for non- $\omega$ -stable theories in every regular  $\lambda$  below the continuum.

In Section 3 we prove in *ZFC* several non-existence theorems for universal linear orders in regular cardinals. We show that there can be a universal linear order at a regular cardinal  $\lambda$  only if  $\lambda = \lambda^{<\lambda}$  or if  $\lambda = \mu^+$  and  $2^{<\mu} \leq \lambda$ . In Section 5 we prove non-existence theorems for universal linear orders in singular cardinals. For example, if  $\mu$  is not a strong limit and is not a fix point of the  $\aleph$  function, then there is no universal linear order in  $\mu$ .

In Section 5 we reduce the existence of a universal linear order in cardinality  $\lambda$  to the existence of a universal model for any theory possessing the strict order property. Thus the non existence theorems from Sections 3 and 4 which were proven for linear orders are shown to hold for a large collection of theories.

The combined results from Sections 3, 4 and 5 show that it is impossible to generalize [Sh 100] in the same fashion that [Sh 175a] and [M] generalize [Sh 175]: While the proof of the consistency of having a universal graph in  $\aleph_2 < 2^{\aleph_0}$  generalizes the proof for the case  $\aleph_1 < 2^{\aleph_0}$ , the consistency of universal linear order is true for the former case and is *false* for the latter. This points out an interesting difference between the theory of order and the theory of graphs.

The second author is interested in the classification of unstable theories (see [Sh 93]). With respect to the problem of determining the stability spectrum of a theory  $T$  (namely the class  $K_T = \{\lambda : T \text{ has a universal model at } \lambda\}$ ), there are several more results which were obtained, in addition to what is published here: the main one is a satisfactory distinction between superstable to stable-unsuperstable theories.

**Notation and Terminology:** By “order” we shall mean linear order.  $|M|$  denotes the universe of a model  $M$  and  $||M||$  denotes its cardinality.

### Section 1: A theory without universal models in $\aleph_1 < 2^{\aleph_0}$ .

We present a theory  $T$ . In the language  $L(T)$  there are two  $n$ -ary relation symbols,  $R_n(\dots)$  and  $P_n(\dots)$  for every natural number  $n \geq 2$ .  $T$  has no constants or function symbols. The axioms of  $T$  are:

1. The sentences saying that  $P_n$  and  $R_n$  are invariant under permutation of arguments and that  $P_n(x_1, \dots, x_n)$  and  $R_n(x_1, \dots, x_n)$  do not hold if for some  $1 \leq i < j \leq n$ ,  $x_i = x_j$ , for all  $n \geq 2$

2. For each  $n$  the sentence saying that there are no  $2n-1$  distinct elements,  $x_1, \dots, x_n, y_1, \dots, y_{n-1}$  such that  $P_n(x_1, \dots, x_n)$  and for all  $1 \leq i \leq n$ ,  $R_n(y_1, \dots, y_{n-1}, x_i)$

**Fact 1.1:** 1. There are only finitely many quantifier free  $n$ -types of  $T$  for every finite  $n$ .

2.  $T$  has the joint embedding property and the amalgamation property.

**Proof:** 1. is obvious. Suppose that  $M, N$  are two models of  $T$  which agree on their intersection. As  $T$  is universal, the intersection is also a model of  $T$ . Define a model  $M'$  such that  $|M'| = |M| \cup |N|$  and such that  $P_n^{M'}, R_n^{M'}$  equal, respectively,  $P_n^M \cup P_n^N$  and  $R_n^M \cup R_n^N$ . Suppose to the contrary that  $M'$  does not satisfy  $T$ . So for some  $n$  there are  $a_1, \dots, a_n, b_1, \dots, b_{n-1}$  which realize the forbidden type. Certainly,  $\{a_1, \dots, a_n, b_1, \dots, b_{n-1}\} \not\subseteq M$ , as  $M \models T$ , and  $\{a_1, \dots, a_n, b_1, \dots, b_{n-1}\} \not\subseteq N$  as  $N \models T$ . So either

- (a) there is some  $a_i \notin M$  and  $a_j \notin N$ ,
- (b) there is some  $a_i \notin M$  and  $b_j \notin N$  or
- (c) there is some  $b_i \notin M$  and  $b_j \notin N$ .

If (a) holds this contradicts  $P(a_1, \dots, a_n)$ ; if (b) holds this contradicts  $R(b_1, \dots, b_{n-1}, a_i)$ ; and if (c) holds this contradicts  $R(b_1, \dots, b_{n-1}, a_1)$ . ⊥

**Fact 1.2**  $T$  has a universal homogeneous model  $M$  at  $\aleph_0$ .

This should be well known, but for completeness of presentation's sake we give:

**Proof:** Construct an increasing sequence of finite models  $M_n$ :

1.  $M_n \models T$  and  $M_n$  is finite.
2.  $M_n \subset M_{n+1}$ .
3. In  $M_{n+1} \setminus M_n$  all quantifier free types (of  $T$ ) over  $M_n$  are realized.

As  $T$  has only finitely many quantifier free types over every finite set, and because of fact 1.1, this construction is possible. The model  $M = \cup M_n$  clearly satisfies  $T$ .

Suppose that  $h$  is a finite embedding from any other model  $N$  into  $M$  and that  $a \in N \setminus \text{dom}(h)$ . There is some  $n_0$  such that  $\text{ran}(h) \subseteq M_{n_0}$ . In  $M_{n_0+1}$  there is some  $b$  such that its relational type over  $\text{ran}(h)$  (in  $M$ ) equals the relational type of  $a$  over  $\text{dom}(h)$  (in  $N$ ). Set  $h' = h \cup \langle a, b \rangle$  to obtain an embedding with  $a$  in its domain. By this observation it is immediate that every countable model of  $T$  is embeddable into  $M$ . Hence the universality of  $M$  in  $\aleph_0$ .

As there are no unary relation symbols in  $T$ , any  $h = \{\langle a_1, a_2 \rangle\}$ , where  $a_1, a_2 \in M$  is an embedding. Suppose that  $h$  is a finite embedding,  $\text{dom}(h), \text{ran}(h) \subseteq M$  and that  $b \in M \setminus \text{ran}(h)$ . Pick, as before, some  $a \in M \setminus \text{dom}(h)$  such that its relational type over  $\text{dom}(h)$  equals the relational type of  $b$  over  $\text{ran}(h)$ , and extend  $h$  to include  $b$  in its range. These observations show that for every two sequences  $\bar{a}, \bar{b} \in M^n$  there is an automorphism  $f$  of  $M$  with  $f(\bar{a}) = \bar{b}$ . Hence  $M$  is homogeneous. ⊥

Denote by  $T_1$  the theory  $Th(M)$ , the theory of the model  $M$ . Then

**1.3 Fact**  $T_1$  is a complete theory extending  $T$ , which admits elimination of quantifiers and is  $\aleph_0$ -categorical.

**Proof** Clearly, every simple existential formula is equivalent to a quantifier free formula in  $T_1$ . Hence the elimination of quantifiers. By fact 1.1 there are only finitely many  $n$ -types of  $T$  over the empty set. Therefore  $T$  is  $\aleph_0$ -categorical (see [CK] for details).  $\dashv$

**1.4 Fact** (1) For every infinite model  $M \models T$  there is a model  $M'$  such that  $M \subseteq M' \models T_1$  and  $\|M\| = \|M'\|$ .

(2)  $T$  and  $T_1$  have the same spectrum of universal models, namely for every cardinal  $\lambda$ ,  $T$  has a universal model in  $\lambda$  (with respect to ordinary embeddings) iff  $T_1$  has a universal model in  $\lambda$  (with respect to elementary embeddings).

**Proof:** (1) follows by the compactness theorem, as every finite submodel of  $M$  (even countable) satisfies  $T$ , and is therefore embeddable into the countable model of  $T_1$ . For (2), we may forget about finite cardinals, as neither of both theories has universal finite models (in fact  $T_1$  has no finite models at all). Suppose first that  $M \models T$  is universal for  $T$  in power  $\lambda$ . Then by 1. there is some  $M' \supseteq M$ , a model of  $T_1$  of the same cardinality. Let  $N \models T_1$  be arbitrary of power  $\lambda$ . As  $N \models T$ , there is an embedding  $h : N \rightarrow M$ .  $h$  is also an embedding into  $M'$ . As  $T_1$  has elimination of quantifiers,  $h$  is an elementary embedding of  $N$  into  $M'$ . So  $M'$  is a universal model of  $T_1$  in  $\lambda$ . Conversely, suppose that  $M \models T_1$  is universal for  $T_1$  in power  $\lambda$ . In particular,  $M \models T$ . Let  $N \models T$  be arbitrary of power  $\lambda$ . By (1) there is some  $N' \supseteq N$  of cardinality  $\lambda$ ,  $N' \models T_1$ . Let  $h : N' \rightarrow M$  be an elementary embedding.  $h \upharpoonright N$  is an embedding of  $N$  into  $M$ . So  $M$  itself is universal for  $T$ .  $\dashv$

**1.5 Remark:** 1.  $T$  does not satisfy the 3-amalgamation property, as seen by a simple example.

2. Also  $T_1$  has the joint embedding property.

**1.6 Theorem**  $T$  has a universal model in  $\aleph_1$  iff  $\aleph_1 = 2^{\aleph_0}$ .

**Proof:** If  $\aleph_1 = 2^{\aleph_0}$  then all countable theories have universal models in  $\aleph_1$ . We proceed now to prove that  $2^{\aleph_0} > \aleph_1$  implies that  $T$  has no universal model at  $\aleph_1$ . Suppose to the contrary that CH fails, but that  $M$  is a universal model at  $\aleph_1$ . Without loss of generality,  $|M| = \omega_1$ . We define now  $2^{\aleph_0}$  models of  $T$ : for each  $\eta \in {}^\omega 2$  let  $M_\eta$  be a model with universe  $\omega_1$  such that

- a .  $P_n^{M_\eta} = [\omega_1]^n$  iff  $R_n^{M_\eta} = \emptyset$  iff  $\eta(n) = 0$
- b .  $R_n^{M_\eta} = [\omega_1]^n$  iff  $P_n^{M_\eta} = \emptyset$  iff  $\eta(n) = 1$

For each  $\eta$  the model  $M_\eta$  trivially satisfies  $T$ .

As  $M$  is universal, we can choose for each  $\eta$  an embedding  $h_\eta : M_\eta \rightarrow M$ . Let  $M_\eta^*$  be the model obtained from  $M$  by enriching it with the relations of  $M_\eta$  and the function  $h_\eta$ . Let  $C_\eta$  be the closed unbounded set  $\{\delta \in \omega_1 : M_\eta^* \upharpoonright \delta \prec M_\eta^*\}$ , and let  $\delta_\eta \in C_\eta$ .

As we have  $2^{\aleph_0}$   $\eta$ 's, by the pigeon hole principle there are more than  $\aleph_1$  sequences,  $\langle \eta_i : i < i(*) \rangle$ , such that for all  $i < i(*)$   $\delta_{\eta_i} = \delta_0$ . As there are only  $\aleph_1$  possible values to  $h_{\eta_i}(\delta_0)$ , we may

assume that for all  $i < i^*$ ,  $h_{\eta_i}(\delta_0) = \gamma_0$  for some fixed  $\gamma_0$  and that  $\eta_i \upharpoonright 2$  is fixed. (Note that by elementarity, and as  $h$  is one to one,  $\gamma_0 \geq \delta_0$ ). Pick now  $i < j < i^*$ . There exists an  $n > 1$  with  $\eta_i(n) \neq \eta_j(n)$ , and assume by symmetry  $\eta_i(n) = 0$ . This means that every  $n$ -tuple of distinct members of the range of  $h_{\eta_i}$  satisfies the relation  $P_n^M$ , while every  $n$ -tuple of distinct members of the range of  $h_{\eta_j}$  satisfies the relation  $R_n^M$ . We intend now to derive a contradiction by constructing the forbidden type inside  $M$ . Pick any  $n - 1$  points,  $b_1, \dots, b_{n-1} \in \delta_0$  in the range of  $h_{\eta_j}$ . Notice that  $M \models R_n(b_1, \dots, b_{n-1}, \gamma_0)$ . Work now in  $M_{\eta_i}^*$ .

$M_{\eta_i}^* \models \exists(x)R_n(b_1, \dots, b_{n-1}, h_{\eta_i}(x))$  as  $\delta_0$  witnesses this.

So by elementarity there is such an  $c_1$  below  $\delta_0$  with  $a_1 = h_{\eta_i}(c_1)$  also below  $\delta_0$ .

$M_{\eta_i}^* \models \exists(x \neq c_1)R_n(b_1, \dots, b_{n-1}, h_{\eta_i}(x))$  as  $\delta_0$  witnesses this.

So by elementarity we can find  $c_2, a_2$  below  $\delta_0$ . We proceed by induction, each time picking  $a_{i+1}$  different from all the previous  $a$ 's. So when  $i = n$  we have constructed the forbidden type, as  $a_1, \dots, a_n$ , being in the range of  $h_{\eta_i}$ , satisfy the relation  $P_n^M$ . This contradicts  $M \models T$ .  $\dashv$

The proof above tells us a bit more than is stated in theorem 1.6: what was actually done, was to construct  $2^{\aleph_0}$  models of  $T$ , each of size  $\aleph_1$ , such that no  $2^{\aleph_0}$  can be embedded into a single model of  $T$ . But this construction uses no special feature of  $\aleph_1$  and the models as defined above can be defined in any cardinality. Let us state the following.

**1.7 Theorem:** Let  $T$  be the theory in 1.6. If  $\aleph_0 < \lambda = \text{cf}\lambda < 2^{\aleph_0}$  and  $\mu < 2^{\aleph_0}$  are cardinals, then for every family of models of  $T$   $\{M_i : i < \mu\}$ , each  $M_i$  of cardinality  $\lambda$ , there is a model  $M$  of  $T$  which cannot be embedded into any  $M_i$  in the family.

Proof: Suppose such a family is given. As in the previous proof, there are  $2^{\aleph_0}$  trivial models of  $T$ , each with universe  $\lambda$ . Suppose that each of them is embedded into some member of the family. As  $\mu < 2^{\aleph_0}$ , there must be a fixed member  $M_{i^*}$  of the family into which more than  $\lambda$  such models are embedded. The contradiction follows now as above.  $\dashv$

## Section 2: A covering theorem

We prove here a theorem that as one consequence puts a restriction on the cofinality of  $2^\lambda$  — provided there is a universal model for a suitable theory in some cardinality  $\kappa \in (\lambda, 2^\lambda)$ .

**2.1 Theorem** Let  $T$  be a first order theory,  $\lambda < \kappa < \mu$  cardinals. Suppose that  $T$  has a universal model at  $\kappa$  and that there is a model  $M$  of  $T$ ,  $|M| = \mu$ , with a subset  $A \subseteq \mu$ ,  $|A| = \lambda$  such that  $|S(A)| \geq \mu$ , namely there are  $\mu$  complete 1-types over  $A$ . Then there is a family  $\langle B_i : i < \kappa^\lambda \rangle \subseteq [\mu]^\kappa$  which covers  $[\mu]^\kappa$ , namely for every  $C \in [\mu]^\kappa$  there is an  $i < \kappa$  such that  $C \subseteq B_i$ .

As corollaries we get

**2.2 Corollary :** If  $\text{cf}(2^\lambda) \leq \kappa < 2^\lambda$  and  $T$  is a first order theory possessing the independence property then  $T$  has no universal model at  $\kappa$ .

**2.3 Corollary :** Suppose that  $2^{\aleph_0} = \aleph_{\omega_1}$ . Then all theories unstable in  $\aleph_0$  (e.g. the theory of graphs, the theory of linear order and so on) do not have a universal model in any cardinal  $\kappa \in [\aleph_1, \aleph_{\omega_1})$ .

**Proof of Corollary 2.2 from Theorem 2.1:** If  $T$  possesses the independence property, then there is a model of  $T$  in which  $2^\lambda$  types over a set of size  $\lambda$  are realized. By Theorem 2.1, if there were a universal model for  $T$  at  $\kappa$ , then there would be a covering family of  $[2^\lambda]^\kappa$  of size  $2^\lambda$ ; but as  $cf(2^\lambda) \leq \kappa$ , this is clearly impossible.  $\dashv$

**Proof of Corollary 2.3 from Theorem 2.1:** If a countable theory  $T$  is not stable at  $\aleph_0$  then  $T$  has a model in which  $2^{\aleph_0}$  complete 1-types over a countable set are realized. So a universal model at  $\kappa \in [\aleph_1, 2^{\aleph_0})$  would imply, by Theorem 2.1, that there is a covering family of  $[2^{\aleph_0}]^\kappa$  of size  $2^{\aleph_0}$ , which is impossible as  $cf(2^{\aleph_0}) \leq \kappa$ .  $\dashv$

**Proof of theorem 2.1:** Let  $U$  be a universal model for  $T$  with universe  $\kappa$  and let  $M$  be a model of  $T$  with a subset  $A \subset |M|$  of size  $\lambda$  with  $\langle p_i \in S(M) : i < \mu \rangle$  a sequence of distinct complete types over  $A$ . Without loss of generality  $M$  is of size  $\mu$  and in  $M$  all  $p_i$ 's are realized. We can further assume (by enumerating  $|M|$ ) that  $|M| = \mu + \mu$ , that  $p_i$  is realized by the element  $i$  and that  $A = \{\alpha : \mu \leq \alpha < \mu + \lambda\}$ . For each submodel  $N$ ,  $A \cup B \subseteq N \prec M$  of size  $\kappa$  pick an embedding  $h_N : N \rightarrow U$ . For each function  $f : A \rightarrow \kappa$  let  $C_f$  be the set of submodels  $\{N \subseteq M : |N| \leq \kappa, A \subseteq N, \text{ and } h_N \upharpoonright A = f\}$ .

**2.4 Claim:** For each  $f \in \kappa^A$ ,  $|\cup C_f \cap \mu| \leq \kappa$ .

**Proof:** Enumerate all members of  $C_f$  in a sequence  $\langle N_\alpha : \alpha < \alpha(*) \rangle$  and define a function  $g : \cup C_f \cap \mu \rightarrow \kappa$  by induction on  $\alpha$  as follows:  $g \upharpoonright (N_\alpha \setminus \cup_{\beta < \alpha} N_\beta) = h_{N_\alpha} \upharpoonright (N_\alpha \setminus \cup_{\beta < \alpha} N_\beta)$ . We are done if  $g$  is a 1-1 function. This follows from

**2.5 Fact** If  $i \in N_\alpha$ ,  $j \in N_\beta$ , and  $i < j < \mu$  then  $h_{N_\alpha}(i) \neq h_{N_\beta}(j)$ .

**Proof of Fact:** As both embeddings agree on the image of  $A$  and  $i, j$  realize different types over  $A$ , the fact is immediate.

**2.6 Claim:** The family  $\{(\cup C_f) \cap \mu : f \in {}^A \kappa\}$  is a covering family of  $[\mu]^\kappa$  of size  $\lambda^\kappa$ .

**Proof of claim:** Clearly the size of the family is as stated. Let  $B \in [\mu]^\kappa$  be any set. Then it is a subset of some elementary submodel  $N \subseteq M$  which contains  $A$  as a subset. So it is a subset of  $\cup C_{f_N \upharpoonright A} \cap \mu$ .  $\dashv$

### Section 3: Non-existence of universal linear orders

In this section we prove some non-existence theorem for universal linear orders in regular cardinals. We start by showing that there is no universal linear order in a regular cardinal  $\lambda$  if  $\aleph_1 < \lambda < 2^{\aleph_0}$ . We shall generalize this for more regular cardinals later in this section. The combinatorial tool which enables these theorems is the guessing of clubs which was introduced

in [Sh-e] and can be found also in [Sh-g], which will, presumably, be available sooner. Proofs of the relevant combinatorial principles are repeated in the appendix to this paper for the reader's convenience.

### 3.1 Definitions

1. If  $C$  is a set of ordinals,  $\delta$  an ordinal, we denote by  $\delta_C^s$  the element  $\min\{C \setminus (\delta + 1)\}$  when it exists.
2. a *cut*  $D$  of a linear order  $O$  is pair  $\langle D_1, D_2 \rangle$  such that  $D_1$  is an initial segment of  $O$ , namely  $D_1 \subseteq |O|$  and  $y < x \in D \Rightarrow y \in D_1$ ,  $D_2$  is an end segment, namely  $D_2 \subseteq |O|$  and  $y > x \in D_2 \Rightarrow y \in D_2$ ,  $D_1 \cap D_2 = \emptyset$  and  $D_1 \cup D_2 = |O|$ . If  $O_1 \subseteq O_2$  are linear orders, then an element  $x \in O_2 \setminus O_1$  *realizes* a cut  $D$  of  $O_1$  if  $D_1 = \{y \in O_1 : y < x\}$ .
3. Let  $O = \cup_{j < \lambda} O_j$  be an increasing continuous union of linear orders, let  $\delta \in \lambda$  be limit, and let  $C \subseteq \delta$  be unbounded in  $\delta$ . Let  $x \in (O \setminus \cup_{j < \delta} O_j)$ . Define  $\text{Inv}_O(C, \delta, x)$ , *the invariance of  $x$  in  $O$  with respect to  $C$* , as  $\{\alpha \in C : \exists y \in O_{\alpha_C^s} \text{ such that } y \text{ and } x \text{ realize the same cut of } O_\alpha\}$ . Note that this definition is applicable also to cuts (rather than only to elements).
4. A  $\kappa$ -*scale* for  $\lambda$  is a sequence  $\overline{C} = \langle c_\delta : \delta \in S \rangle$  where  $S = \{\alpha < \lambda : \text{cf}\alpha = \text{cf}\kappa\}$  and for every  $\delta \in S$ ,  $c_\delta$  is a club of  $\delta$  or order type  $\kappa$ ,  $c_\delta = \langle \alpha_j^\delta : j < \kappa \rangle$  is an increasing enumeration of  $c_\delta$ . If  $O = \cup_{i < \lambda} O_i$  is a linear order represented as a continuous increasing union of smaller orders, and  $\overline{C}$  is a  $\kappa$ -scale for some  $\kappa < \lambda$ , let  $\text{INV}(O, \overline{C}) =^{def} \{X \subseteq \kappa : (\exists \delta \in S)(\exists x > \delta) \text{Inv}(c_\delta, \delta, x) = \{\alpha_j^\delta : \alpha \in X\}\}$ . So  $\text{INV}(O, \overline{C})$  is the set of all subsets of  $\kappa$  which are obtained as an invariance of some element in  $O$  with respect to some  $c_\delta$  in the scale.

**3.2 claim:** Suppose  $h : O_1 \rightarrow O_2$  is an embedding of linear orders,  $\|O_1\| = \|O_2\| = \lambda = \text{cf}\lambda > \aleph_0$ . Then for any representations  $O_i = \cup_{j < \lambda} O_j^i$ ,  $i = 1, 2$ , the union increasing and continuous and each  $|O_j^i| < \lambda$ , there exists a club  $E \subseteq \lambda$  such that for any  $\delta < \lambda$  and  $C \subseteq \delta$  a club of  $\delta$  which satisfies  $C \subseteq E$ , we have

$$(*) \quad (\forall x \in O_1 \setminus O_\delta) (\text{Inv}_{O_1}(C, \delta, x) = \text{Inv}_{O_2}(C, \delta, h(x)))$$

**Proof of Claim:** without loss of generality we may assume that  $|O_1| = |O_2| = \lambda$ . Define the model  $M = \langle \lambda, <_{O_1}, <_{O_2}, \in, h \rangle$ . Let  $E = \{\delta < \lambda : M \upharpoonright \delta \prec M \text{ and for all } \delta \in E, \delta = \cup_{j < \delta} O_j^i, i = 1, 2\}$ . Let  $x \in (O_1 \setminus O_\delta^1)$ . Note that by elementarity  $h(x) \in (\lambda \setminus \delta)$ . Suppose first that  $\alpha$  belongs to the left hand side of the equality in (\*) and let  $y \in [\alpha, \alpha_C^s)$  demonstrate this. So  $x$  and  $y$  realize the same cut of  $O_1 \upharpoonright \alpha$ . As  $h$  is an embedding,  $h(x), h(y)$  satisfy the same cut of  $h''(O_1 \upharpoonright \alpha)$  (which equals, by elementarity,  $(h''(O_1)) \upharpoonright \alpha$ ). If  $h(x), h(y)$  satisfy also the same cut of  $O_2 \upharpoonright \alpha$  we are done, but the problem is, of course, that  $h$  is not necessarily onto. Otherwise suppose that (w.l.o.g)  $h(y) < h(x)$  and that there is an element  $z \in O_2 \upharpoonright \alpha$  such that  $h(y) <_{O_2} z <_{O_2} h(x)$ . Define in  $M$  the set  $D = \{t : \text{there is no } q \text{ such that } z \leq_{O_2} h(q) \leq_{O_2} t\}$ .  $D$  is definable in  $M$  with parameters in  $M \upharpoonright \alpha^s$ . By elementarity the definition is absolute between  $M$  and  $M \upharpoonright \alpha^s$ , that is  $D \cap \alpha^s$  is the same as  $D$  interpreted in  $M \upharpoonright \alpha^s$ .  $D$  is a cut of  $O_2 \upharpoonright \alpha$ . Let  $D'$  be  $D \cap \alpha$ .  $D'$  is definable in  $M \upharpoonright \alpha^s$ .



**3.2.1 subclaim:**  $h(x)$  satisfies the cut  $D'$  determined by  $D$ .

**Proof of Subclaim:** let  $z <_{O_2} \beta <_{O_2} h(x)$ . As there are no points in the range of  $h \upharpoonright \alpha$  between  $z$  and  $h(x)$ , there are certainly none between  $z$  and  $\beta$ . So  $\beta \in D'$ . Conversely, suppose  $h(x) <_{O_2} \beta$ . Then  $M$  satisfies that there is an image under  $h$  (namely  $h(x)$ ) between  $z$  and  $\beta$ . By elementarity there is such an image  $h(x')$  where  $x' \in \alpha$ . So  $\beta \notin D'$ .

As  $D'$  is a cut of  $O' \upharpoonright \alpha$  definable in  $M \upharpoonright \alpha^s$  which is realized by  $h(x)$ , elementarity assures us that it is realized by some  $y' \in [\alpha, \alpha^s]$ . So  $\alpha$  belongs to the right hand side of (\*).

Assume that  $\alpha$  belongs to the right hand side of (\*). Then there is an element  $y \in [\alpha, \alpha^s]$  which satisfies the same cut of  $O_2 \upharpoonright \alpha$  as  $h(x)$ . If  $y = h(y')$  for some  $y'$  we are done. Else, we note that the cut of  $O_2 \upharpoonright \alpha$  which  $y$  determines is definable in  $M \upharpoonright \alpha^s$ . Now clearly  $h(x)$  and  $y$  satisfy the same cut of  $O_2 \upharpoonright \alpha$ . By elementarity there is an element  $y'$  such that  $h(y')$  satisfies the same cut as  $y$ , therefore as  $h(x)$ . In other words,  $\alpha$  belongs to the left hand side of (\*).  $\dashv$

**3.3 Fact** If  $O$  is an order with universe  $\lambda$  and  $\overline{C}$  is a  $\kappa$ -scale, then  $|\text{INV}(O, \overline{C})| \leq \lambda$

**Proof:** Trivial.

**3.4 Lemma** (the construction lemma): If  $\lambda < 2^{\aleph_0}$  is a regular uncountable cardinal,  $\overline{C}$  is an  $\omega$ -scale, and  $A \subseteq \omega$  is given, then there is an order  $O$  with universe  $\lambda$ ,  $O = \cup_{i < \lambda} O_i$ , increasing continuous union of smaller orders, such that for every  $\delta < \lambda$  with  $\text{cf} \delta = \aleph_0$ ,  $\text{Inv}(c_\delta, \delta, \delta) = \{\alpha_n^\delta : n \in A\}$ .

**Proof:** We define by induction on  $0 < \alpha < \lambda$  an order  $O_\alpha$  with the properties listed below. We denote by  $Q$  the order of the rationals. If  $O_1 \subseteq O_2$  are linear orders,  $D$  a cut of  $O_1$  and  $D'$  a cut of  $O_2$ , we say that  $D'$  extends  $D$  if  $D'_1 \upharpoonright |O_1| = D_1$  and  $D'_2 = D_2$ . Also note that if  $O_1 \subseteq O_2$  are linear orders and  $D^1$  is a cut of  $O_1$  which is not realized in  $O_2$  then it corresponds naturally to a cut  $D^2$  of  $O_2$ . In such a case we say that  $D^1$  is (really) also a cut of  $O_2$ .

1.  $O_\alpha$  has universe  $|O_\alpha| \in \lambda$ .
2. If  $\beta + 1 = \alpha$  and  $x \in (O_\alpha \setminus O_\beta)$ , then  $\{y \in O_\alpha \setminus O_\beta : x \text{ and } y \text{ satisfy the same cut of } O_\beta\}$  has order type  $Q$ .
3. If  $\alpha < \beta < \gamma$ ,  $\gamma$  is a successor, and there is a cut  $D$  of  $O_\alpha$  which is realized by an element of  $O_\beta$  but is not realized by no element of  $O_\nu$  for  $\alpha < \nu < \beta$ , then there is a cut  $D'$  of  $O_\beta$ , which extends  $D$ , which is realized in  $O_\gamma$  but is not realized in  $O_\nu$  for all  $\beta < \nu < \gamma$ . Also for every successor  $\alpha$  there is a cut of  $O_0$  which is realized in  $O_\alpha$  but is not realized in  $O_\beta$  for every  $\beta < \alpha$ .
4. If  $\alpha$  is limit then  $O_\alpha = \cup_{\beta < \alpha} O_\beta$ .
5. If  $\text{cf} \delta = \aleph_0$  and for all  $\beta \in C_\alpha$ ,  $|O_\beta| = \beta$ , then  $\text{Inv}_{O_{\delta+1}}(c_\delta, \delta, \delta) = \{\alpha_n^\delta : n \in A\}$ .

There should be no problem taking care of 1–4. Assume that the conditions of 5. are satisfied. We wish to define the order  $O_{\alpha+1}$ . Let  $C_\alpha = \langle \beta_n : n < \omega \rangle$ . By induction on  $A = \langle a_n : n < \omega \rangle$  define an increasing sequence of cuts,  $\langle D_{a_n} : n \in \omega \rangle$  such that  $D_{a_n}$  is a cut of  $O_{a_n}$  which is realized for the first time in  $O_{a_{n+1}}$ . Demand 3. enables this. In  $O_{\alpha+1}$  let  $\alpha$  satisfy  $\bigcup D_n$  to get 5.  $\dashv$

We are almost ready to prove the non existence of a universal order in a regular  $\lambda$ ,  $\aleph_1 < \lambda < 2^{\aleph_0}$ . We recall from [Sh-e] chapter III.7.8 (see also [Sh-g]),

**3.5 Fact:** If  $\lambda > \aleph_1$  is regular, then there is a sequence  $\overline{C} = \langle c_\delta : \delta < \lambda, \text{cf}\delta = \aleph_0 \rangle$ , such that  $c_\delta \subseteq \delta$  is a club of  $\delta$  of order type  $\omega_0$ , with the property that for every club  $E \subseteq \mu$  the set  $S_E = \{\delta < \lambda : \text{cf}\delta = \aleph_0 \text{ and } c_\delta \subseteq E\}$  is stationary.  $\dashv$

A proof of this fact is found in the appendix.

**3.6 Theorem** If  $\aleph_1 < \lambda = \text{cf}\lambda < 2^{\aleph_0}$ , then there is no universal order in cardinality  $\lambda$ .

**Proof:** Suppose to the contrary that  $UO$  is a universal order in cardinality  $\lambda$ . Without loss of generality,  $|UO| = \lambda$ . Fix some club guessing sequence  $\overline{C} = \langle c_\delta : \delta < \lambda, \text{cf}\delta = \aleph_0 \rangle$ . This is known to exist by the previous fact. As  $|\text{INV}(UO, \overline{C})| \leq \lambda$ , there is some  $A \subseteq \omega$ ,  $A \notin \text{INV}(UO, \overline{C})$ . Use the construction lemma to get an order  $M$  with universe  $\lambda$  and with the property that for every  $\delta < \lambda$ ,  $\text{cf}\delta = \aleph_0$  implies that  $\text{Inv}_M(c_\delta, \delta, \delta) = \{\alpha_n^\delta : n \in A\}$ . Let  $h : M \rightarrow UO$  be an embedding. Let  $E_h$  be the club given by 3.2. As  $\overline{C}$  guesses clubs, there is some  $\delta(*)$  with  $c_{\delta(*)} \subseteq E_h$ . Therefore, by 3.2,  $\text{Inv}_M(c_{\delta(*)}, \delta(*), \delta(*)) = \text{Inv}_{UO}(c_{\delta(*)}, \delta(*), h(\delta(*)))$ . But  $\text{Inv}_M(\gamma_{\delta(*)}, \delta(*), \delta(*)) = \{\alpha_n^\delta : n \in A\}$ . This means that  $A \in \text{INV}(UO, \overline{C})$ , a contradiction to the choice of  $A \notin \text{INV}(UO, \overline{C})$ .  $\dashv$

We wish now to generalize Theorem 3.6 by replacing  $\omega_0$  by a more general  $\kappa$ . As the proof of 3.6 made use of both club guessing and the construction lemma, we should see what remains true of these two facts for  $\kappa > \aleph_0$ . The proof of the construction lemma does not work when replacing  $\aleph_0$  by some other cardinal. We need some extra machinery to handle the limit points below  $\kappa$ .

**3.7 Lemma** (the second construction lemma) Suppose  $\kappa < \lambda = \text{cf}\lambda$  are cardinals,  $2^\kappa \geq \lambda$  and that there is a stationary  $S \subseteq \lambda$  and sequences  $\langle c_\delta : \delta \in S \rangle$  and  $\langle P_\alpha : \alpha < \lambda \rangle$  which satisfy:

- (1)  $\text{otp}c_\delta = \kappa$  and  $\sup c_\delta = \delta$ ;
- (2)  $P_\alpha \subseteq \mathcal{P}(\alpha)$  and  $|P_\alpha| < \lambda$ ;
- (3) if  $\alpha \in \text{nacc}c_\delta$  then  $c_\delta \cap \alpha \in \cup_{\beta < \alpha} P_\beta$ ,

THEN when given such sequences and a closed  $A \subseteq \text{Lim}\kappa$  there is a linear order  $O$  with universe  $\lambda$  with the property that for every  $\delta \in S$ ,  $\text{Inv}(c_\delta, \delta, \delta) = A_\delta$ , where  $A_\delta$  is the subset of  $c_\delta$  which is isomorphic to  $A$ .

**Proof** We pick some linear order  $L$  of cardinality smaller than  $\lambda$  which has at least  $\lambda$  cuts. We assume, without loss of generality, that  $P_\alpha \subseteq P_\beta$  whenever  $\alpha < \beta$ , that for limit  $\alpha$   $P_\alpha = \cup_{\beta < \alpha} P_\beta$  and that if  $\alpha \in \text{nacc}c_\delta$  then  $A_\delta \cap \alpha \in P_\alpha$ . Next we construct by induction on  $\alpha < \lambda$  an order  $O_\alpha$  and a partial function  $F$  with the following demands:

- (1) the universe of  $O_\alpha$  is an ordinal below  $\lambda$ .
- (2)  $\alpha < \beta \Rightarrow O_\alpha \subseteq O_\beta$ , and if  $\alpha$  is limit, then  $O_\alpha = \cup_{\beta < \alpha} O_\beta$ .
- (3) If  $x \in O_\beta \setminus O_\alpha$ , then the order type of  $\{y \in O_\beta : x \text{ and } y \text{ satisfy the same cut of } O_\alpha\}$  contains  $L$  as a suborder. Also, if  $\alpha$  is a successor, then there is an element in  $O_\alpha$  which satisfies a new cut of  $O_0$ .

- (4) If  $\alpha < \beta < \gamma$  and  $\gamma$  is a successor, then if  $D$  is a cut of  $O_\alpha$  which is realized in  $O_\beta$  but not in an earlier stage, then there is a cut  $D'$  of  $O_\beta$  which extends  $D$  and is realized in  $O_\gamma$  but is not realized in  $O_\delta$  for any  $\delta < \gamma$ .
- (5)  $F$  is a partial function,  $\text{dom}F \subseteq S \times (\lambda \setminus \text{Lim } \lambda)$ . A pair  $\langle \delta, \alpha \rangle \in \text{dom}F$  iff  $\alpha < \delta$ ,  $\emptyset \neq A_\delta \cap \alpha \in P_\alpha \setminus P_{\alpha-1}$ .  $F(\delta, \alpha)$  is a pair  $\langle \beta(\delta, \alpha), D(\delta, \alpha) \rangle$ , where  $\beta < \alpha$  and  $D$  is a cut of  $O_\beta$  which is realized in  $O_\alpha$ . If  $\beta$  is not a limit of  $A_\delta$  then  $D$  is not realized in  $O_\gamma$  for any  $\gamma < \alpha$ .  $F(\delta, \alpha)$  depends only on  $A_\delta \cap \alpha$ , namely if  $A_{\delta_1} \cap \alpha = A_{\delta_2} \cap \alpha$  then  $F(\delta_1, \alpha) = F(\delta_2, \alpha)$ . If  $\alpha < \gamma$  and  $F(\delta, \alpha), F(\delta, \gamma)$  are both defined, then  $\beta(\delta, \alpha) < \beta(\delta, \gamma)$  and  $D(\delta, \gamma)$  extends  $D(\delta, \alpha)$ .
- (6) If  $\delta \in S$  then  $\text{Inv}_{O_{\delta+1}}(c_\delta, \delta, \delta) = A_\delta$ .

As  $O_0$  we pick  $L$ . When  $\alpha$  is limit, we define  $O_\alpha$  as the union of previous orders. When  $\alpha$  is a successor we add less than  $\lambda$  elements to take care of demands 3 and 4. If  $A_\delta \cap \alpha \in P_\alpha \setminus \alpha - 1$  we must define  $F(\delta, \alpha)$ . If  $A_\delta \cap \alpha$  contains exactly one member, let  $\beta(\delta, \alpha) = 0$  and as  $D(\delta, \alpha)$  pick (by(3)) a cut of  $O_0$  which is realized in  $O_\alpha$  but is not realized in  $O_\gamma$  for any  $\gamma < \alpha$ . In case the order type of  $\{\gamma < \alpha : F(\delta, \gamma) \text{ is defined}\}$  is limit, we let  $D(\delta, \alpha) = \cup_{\gamma < \alpha} D(\delta, \gamma)$  and  $\beta(\delta, \alpha) = \cup_{\gamma < \alpha} \beta(\delta, \gamma)$ . Note that  $\beta(\delta, \alpha) < \alpha$ , because it is limit. Add more elements to  $O_\alpha$  to realize  $D$ . Since  $|P_\alpha| < \lambda$ , this requirement of addition of elements is satisfied by adding less than  $\lambda$  new elements. In case there is a last  $\gamma < \alpha$  for which  $F(\delta, \gamma)$  is defined, let this  $\gamma$  be  $\beta(\delta, \alpha)$  and pick (by (4)) a cut  $D$  of  $O_\gamma$  which extends  $D(\delta, \gamma)$  and is realized in  $O_\alpha$ , but is not realized earlier, as  $D(\delta, \alpha)$ . When  $\alpha = \delta + 1$  and  $\delta \in S$ , let the element  $\delta$  realize, in  $O_\alpha$  the cut  $\cup_{\gamma < \alpha} F(\delta, \gamma)$ .

Having added less than  $\lambda$  new elements, we fulfill demand (1). (2) and (3) are obvious, and (4) and (5) have been taken care of.

**Claim:** demand (6) holds.

**Proof:** Suppose that  $\delta \in S$ . We show by induction that for every  $x \in A_\delta$ , for every  $y \leq x$  in  $c_\delta$  there is some  $\gamma < y_{c_\delta}^s$  which satisfies the cut of  $\delta$  over  $O_\gamma$  iff  $y \in A_\delta$ . Suppose  $x$  is the first member of  $A_\delta$ . Then the first  $\gamma$  for which  $F(\delta, \gamma)$  is defined satisfies  $x < \gamma < x_{c_d}^s$  by the assumptions on  $\langle P_\alpha : \alpha < \lambda \rangle$ .  $F(\delta, \gamma)$  is a cut of  $O_0$  which is realized in  $O_\gamma$  but not before. If  $y \in c_\delta \cap x$ , as  $x$  is a limit of  $c_\delta$ ,  $y_{c_d}^s < x$ . The cut of  $\delta$  over  $O_0$  is  $F(\delta, \gamma)$ . And the cut of  $\delta$  over  $O_y$  extends this cut. As  $F(\delta, \alpha)$  is not realized by the stage  $O_y$ , certainly the cut of  $\delta$  over  $O_y$  is not realized by this stage either. So  $y \notin \text{Inv}(C_\delta, \delta, \delta)$ . As the cut of  $\delta$  over  $O_0$  is not realized in  $O_x$ , it is really also a cut of  $O_x$ . This cut is realized in  $O_\gamma$ , where  $\gamma < x_{c_d}^s$ . So by definition,  $x \in \text{Inv}(c_\delta, \delta, \delta)$ .

In the case  $x$  is a successor of  $A_\delta$ , denote by  $z$  its predecessor in  $A_\delta$ . The minimal  $\gamma$  above  $x$  for which  $F(\delta, \gamma)$  is defined is smaller than  $X_{\gamma_\delta}^s$ , and  $\beta(\delta, \gamma)$  is in the interval  $(z, z_{c_d}^s)$ . The same argument as in the previous case shows that for every  $y \in (z, x]$ ,  $y \in \text{Inv}(\gamma_\delta, \delta, \delta)$  iff  $y \in A_\delta$ . When  $x$  is a limit of  $A_\delta$ , by the induction hypothesis, for every  $y < x$  the required holds. As for  $x$  itself, if  $\gamma$  is the minimal above  $x$  for which  $F(\delta, \gamma)$  is defined,  $\gamma < x_{c_\delta}^s$  and  $F(\delta, \gamma)$  is realized in  $O_\gamma$ . Therefore  $x \in \text{Inv}(c_\delta, \delta, \delta)$ . +

By [Sh 420] we know:

**3.8 Fact** If  $\kappa$  is a cardinal and  $\kappa^+ < \lambda = \text{cf}\lambda$ , then there is a stationary set  $S$  and sequences  $\langle c_\delta : \delta \in S \rangle$ ,  $\langle P_\alpha : \alpha \in \lambda \rangle$  as in the assumptions of 3.7.

What is still lacking is the appropriate club guessing fact, which we quote now from [Sh g]:

**3.9 Fact:** If  $\kappa$  is a cardinal,  $\kappa^+ < \lambda = \text{cf}\lambda$  and there is a stationary set  $S \subseteq \lambda$  and sequences  $\langle c_\delta : \delta \in S \rangle$ ,  $\langle P_\alpha : \alpha < \lambda \rangle$  as in 3.7, then there are such with the additional property that  $\langle c_\delta : \delta \in S \rangle$  guesses clubs.  $\dashv$

**3.10 Theorem** Suppose  $\lambda = \text{cf}\lambda$  and there is some cardinal  $\kappa$  such that  $\kappa^+ < \lambda < 2^\kappa$ , then there is no universal linear order in cardinality  $\lambda$ .  $\dashv$

**Proof:** Suppose  $O$  is any order of cardinality  $\lambda$ , and assume without loss of generality that its universe is  $\lambda$ . Pick a stationary set  $S$  and sequences as in 3.7, with the a property that  $\overline{C} = \langle c_\delta : \delta \in S \rangle$  guesses clubs. Pick a closed set  $A \subseteq \text{Lim } \kappa$  which is not in  $\text{INV}(O, \overline{C})$ . Use 3.7 to construct an order  $O'$  with universe  $\lambda$  and the property that for every  $\delta \in S$ ,  $\text{Inv}_{O'}(c_\delta, \delta, \delta) \simeq A$ . If  $O'$  were embedded into  $O$ , some  $c_\delta$  would guess the club of the embedding, what would lead to a contradiction. So  $O'$  is not embeddable into  $O$ , and therefore there is no universal linear order in  $\lambda$ .  $\dashv$

#### Section 4: Singular cardinals

We shall state now a theorem which concerns the non-existence of universal linear orders in singular cardinals. Let us note, though, the following well known fact first:

**4.1 Fact** If  $\mu$  is a strong limit, then for every first order theory  $T$  such that  $|T| < \mu$  there is a special model of size  $\mu$ , and therefore also a universal model in  $\mu$ .  $\dashv$

For the definition of special model see the appendix. A special model is universal. For more details see [CK] p. 217.

This means that for non existence of universal models we must look at singulars which are not strong limits. We will see at the end of this section that if, e.g.,  $\aleph_\omega$  is not a strong limit, then there is no universal linear order at  $\aleph_\omega$ .

We recall from [Sh-g 355,5]

**4.2 Definition**  $\text{cov}(\lambda, \mu, \theta, \sigma)$  is the minimal size of a family  $A \subseteq [\lambda]^{<\mu}$  which satisfies that for all  $X \in [\lambda]^{<\theta}$  there are less than  $\sigma$  members of  $A$  whose union covers  $X$ .

**4.3 Theorem:** Suppose  $\theta = \text{cf}\theta < \theta^+ < \kappa$  are regular cardinals,  $\kappa < \mu$  and there is a binary tree  $T \subseteq {}^{<\theta}2$  of size  $< \kappa$  with  $> \mu^* := \text{cov}(\mu, \kappa^+, \kappa^+, \kappa)$  branches of length  $\theta$ . Then

(\*) $_{\mu, \kappa}$  There is no linear order of size  $\mu$  which is universal for linear orders of size  $\kappa$  (namely that every linear order of size  $\kappa$  is embedded in it).

**Proof:** Let  $\overline{A} = \langle A_i : i < \mu^* \rangle \subseteq [\mu]^{<\kappa^+}$  demonstrate the definition of  $\mu^*$ . Without loss of generality,  $|A_i| = \kappa$  for all  $i$ . Suppose to the contrary that there is an order  $UO = \langle \mu, <_{UO} \rangle$  into

which every order of size  $\kappa$  is embedded. Let  $M_i$  be  $UO \upharpoonright A_i$  for every  $i < \mu^*$ . Then every  $M_i$  is isomorphic to some  $M'_i$  with universe  $\kappa$ , and for every order  $O$  of size  $\kappa$  there is a set  $J \subseteq \mu^*$ ,  $|J| < \kappa$  such that  $O$  is embedded into  $\cup_{i \in J} M_i$ .

We fix a club guessing sequence,  $\overline{C} = \langle c_\delta : \delta < \mu \text{ and } \text{cf} \delta = \theta \rangle$ , and an increasing continuous sequence  $\langle P_\alpha : \alpha < \kappa \rangle$  such that  $P_\alpha$  is a family of subsets of  $\alpha$ ,  $|P_\alpha| < \kappa$  and for all  $\alpha < \mu$  if  $\delta \in S$  and  $\alpha \in \text{nacc } c_\delta$ , then  $(c_\delta \cap \alpha) \in P_\alpha$ . For the existence of these, see [Sh 420].

For each  $\delta \in S$  enumerate  $c_\delta$  as  $\langle a_i^\delta : i < \theta \rangle$  in an increasing continuous fashion. Now  $T$  can be viewed as  $T_\delta$ , a tree of subsets of  $c_\delta$ . Under the assumptions we already have, it is no lose of generality to assume that for every  $\alpha \in \kappa$ , if  $\alpha \in \text{nacc}(c_\delta)$ , then  $T_\delta \cap \mathcal{P}(\alpha) \subseteq P_\alpha$ . The reason is that there are  $\theta$  possibilities for the unique  $i$  such that  $\alpha = a_i^\delta$ , and for each such possibility there are  $< \kappa$  subsets in  $T \cap \mathcal{P}(i)$ ; So we can add all the required sets into  $P_\alpha$  without changing the fact that  $|P_\alpha| < \kappa$ .

So by now we have the assumptions of 3.7. Using it we construct a linear order  $O$  on  $\kappa$ , with  $A \subseteq \kappa$  not in  $\{\text{Inv}_{M'_i}(c_\delta, \delta, x) : i < \mu^*, x \in \kappa\}$ .

Suppose now that there is an embedding  $h : O \rightarrow UO$ . The image of  $h$  is covered by  $\cup\{O_i : j \in J\}$  for some  $J$  of size  $< \kappa$ . Let  $S_j = \{x \in \kappa : h(x) \in M_j\}$ . Then there is some  $j_0$  such that  $S_{j_0} \notin \overline{id}^a(\overline{C})$ , (the latter is the ideal of non-guessing, namely  $X \in \overline{id}^a(\overline{C})$  iff there is a club  $E$  such that  $\forall(\delta \in S \cap X)(c_\delta \not\subseteq E)$ . This ideal is clearly  $\kappa$ -complete.)

Let  $O'$  be  $O \upharpoonright S_{j_0}$ . Let  $O'_i = O' \upharpoonright i$  for  $i < \kappa$ . Then this is a presentation of  $L'$  as an increasing continuous union of small orders. By 3.2, and the fact that the identity map embeds  $O'$  into  $O$ , almost everywhere the invariance with relation to  $O$  is the same as with relation to  $O'$ . So we can get again the same contradiction as in previous proofs by inspecting the embedding  $h \upharpoonright O'$ .  $\dashv$

We wish now to obtain the same results using more concrete assumptions. We first review some facts concerning covering numbers.

Recall the well known (see e.g. [Sh-g 355,5])

**4.4 Fact** If  $\delta < \kappa = \text{cf} \kappa < \mu = \aleph_\delta$  then  $\text{cov}(\mu, \kappa^+, \kappa^+, \kappa) = \mu$ .

**Proof:** By induction on  $\chi$ , a cardinal,  $\kappa < \chi \leq \mu$ .

(a)  $\chi = \theta^+$ .

For every  $\alpha < \chi$  fix a family  $P_\alpha \subseteq [\alpha]^\kappa$  with the proprty that for every set  $A \in [\alpha]^\kappa$  there is a set  $X \subseteq P_\alpha$ ,  $|X| < \kappa$  and  $A \subseteq \cup X$ . Let  $P$  be the union of  $P_\alpha$  for  $\alpha < \chi$ . The size of  $P$  is clearly  $\chi$ , and clearly for every set  $A \subseteq \chi$  of size  $\kappa$  there is a covering of  $A$  by less than  $\kappa$  members of  $P$ .

(b)  $\chi = \aleph_\beta$  is a limit cardinal.

As  $\mu = \aleph_\delta$  with  $\delta < \kappa$ , certainly  $\beta < \kappa$ . Let  $\langle \chi_i : i < \text{cf} \beta \rangle$  be increasing and unbounded below  $\chi$ . Let  $P_i$  demonstrate that  $\text{cov}(\chi_i, \kappa^+, \kappa^+, \kappa) = \chi_i$ , and let  $P = \cup_i P_i$ . Then  $|P| = \chi$ . If  $A \in [\chi]^\kappa$ , cover  $A \cap \chi_i$  by less than  $\kappa$  members of  $P$ . Thus to cover  $A$  we need less than  $\kappa$  members of  $P$ .

#### 4.5 Improved fact

If  $\mu$  is a fix point of the first order (i.e.  $\lambda = \aleph_\lambda$ ), but not of the second order, i.e.  $|\{\lambda < \mu : \lambda = \aleph_\lambda\}| = \sigma < \mu$ , and  $\sigma + \text{cf}\mu < \kappa < \mu$ , then  $\text{cov}(\mu, \kappa^+, \kappa^+, \kappa) = \mu$ .

**Proof:** Suppose that  $\kappa < \chi < \mu$  and  $\text{cf}\chi = \kappa$ . By the assumptions,  $\chi \neq \aleph_\chi$ , say  $\chi = \aleph_\delta$ . By [Sh 400], section 2,  $\text{pp}(\chi) < \aleph_{|\delta|+4} < \mu$ . By [Sh-g 355, 5.4], and the fact that  $\chi$  is arbitrarily large below  $\mu$ , we are done.  $\dashv$

We see that we can have arbitrarily large  $\kappa$  below a singular  $\mu$  with  $\mu = \text{cov}(\mu, \kappa^+, \kappa^+, \kappa)$ , when  $\mu$  is a limit which is not a second order fix point of the  $\aleph$  function. But for applying theorem 4.3 we need also a binary tree of height and size  $< \mu$  with  $> \mu$  branches. This happens if there is some  $\sigma < \mu$  with  $2^{<\sigma} < \mu$  and  $2^\sigma > \mu$ . So we can state

**4.6 Corollary:** If  $\mu$  is a singular cardinal which is not a second order fix point, and there is some  $\sigma < \mu$  such that  $2^{<\sigma} < \mu < 2^\sigma$ , then there is no universal linear order of power  $\mu$ .

**Proof:** Let  $T$  be the tree  ${}^{<\sigma}2$ . By the fact on covering numbers, pick  $\sigma < \sigma^+ < \kappa$  such that  $\text{cov}(\mu, \kappa^+, \kappa^+, \kappa) = \mu$ , and apply theorem (\*).

As for no  $\mu$  with  $\text{cf}\mu = \aleph_0$  is there a  $\sigma < \mu$  with  $2^\sigma = \mu$ , we can weaken the assumptions to get

**4.7 Corollary** If  $\aleph_\mu > \mu$  or  $\mu$  is not a second order fix point,  $\text{cf}\mu = \aleph_0$  or  $2^{<\text{cf}\mu} < \mu$ , and  $\mu \neq 2^{<\mu}$ , then there is no universal linear order at  $\mu$ .  $\dashv$

### Section 5 Generalizations:

In this section we prove that if there is no universal linear order in a cardinal  $\lambda$  then there is no universal model in  $\lambda$  for any countable theory  $T$  possessing the strict order property (e.g. there is no universal Boolean algebra in  $\lambda$ ). This means that all the non-existence theorems in Section 3 and Section 4 hold for a large class of theories.

**5.1 Definition 1.** A formula  $\varphi(\bar{x}; \bar{y})$  has the *strict order property* if for every  $n$  there are  $\bar{a}_l$  ( $l < n$ ) such that for any  $k, l < n$ ,

$$\models (\exists x)[\neg\varphi(\bar{x}; \bar{a}_k) \wedge \varphi(\bar{x}; \bar{a}_l)] \Leftrightarrow k < l$$

2. A theory  $T$  has the *strict order property* if some formula  $\varphi(\bar{x}; \bar{y})$  has the strict order property.

This definition appears in in [Sh-a] p. 68, or [Sh-c] p. 69. Every unstable theory possesses the strict order property or possesses the independence property (or both). For details see [Sh-a] or [Sh-c].

**5.2 Fact 1.** Suppose  $T$  has the strict order property, with  $\varphi(\bar{x}; \bar{y})$  witnessing this, and let  $M \models T$ . Then  $\varphi$  defines a partial order  $P_M = \langle |M|^n, \leq_\varphi \rangle$ , where  $n = \text{lg}(\bar{y})$ , the length of  $\bar{y}$ , the order being given by  $\bar{y}_1 \leq_\varphi \bar{y}_2 \Leftrightarrow M \models \varphi(\bar{x}; \bar{y}_1) \rightarrow \varphi(\bar{x}; \bar{y}_2)$ . In this order there are arbitrarily long chains.

2. If  $h : M_1 \rightarrow M_2$  is an embedding between two models of  $T$  which preserves  $\varphi$ , then  $h' : P_{M_1} \rightarrow P_{M_2}$  is an embedding of partial orders, where  $h'(x_1, \dots, x_n) = (h(x_1), \dots, h(x_n))$ .

**Proof:** Immediate from the definition. ⊢

**5.3 Lemma** Suppose  $T$  has the strict order property, with  $\varphi(\bar{x}; \bar{y})$  witnessing this. If  $L$  is a given linear order, then there is a model  $M$  of  $T$  such that  $L$  is isomorphic to a suborder of  $P_M$  and  $\|M\| = |L| + \aleph_0$ .

**Proof:** As there are arbitrarily long chains with respect to  $\leq_\varphi$ , this is an immediate corollary of the compactness theorem. ⊢

**5.4 Lemma** If there is a partial order of size  $\lambda$  with the property that every linear order of size  $\lambda$  can be embedded into it, then there is a universal linear order in power  $\lambda$ .

**Proof:** Suppose that  $P = \langle |P|, \leq \rangle$  is a partial order of size  $\lambda$  with this property. Divide by the equivalence relation  $x \sim y \Leftrightarrow x \leq y \wedge y \leq x$ , to obtain  $P'$ , a partial order in the strict sense. There is some linear order  $<$  on  $|P'|$  which extends  $\leq$ . Let  $UO = \langle |P'|, < \rangle$ . Let  $L$  be any linear order, and let  $h : L \rightarrow P$  be an embedding. Whenever  $x \neq y$  are elements of  $L$ ,  $h(x) \not\leq h(y)$  in  $P$ . Therefore  $h' : L \rightarrow P'$  defined by  $h'(x) = [h(x)]$  is still an embedding.  $h'$  is an embedding of  $L$  into  $UO$ . So  $UO$  is universal. ⊢

**5.5 Theorem** Suppose that  $T$  has the strict order property,  $\lambda$  is a cardinal, and that  $T$  has a universal model (with respect to elementary embeddings) in cardinality  $\lambda$ . Then there exists a universal linear order in cardinality  $\lambda$ .

**Proof:** By 5.4 it is enough to show that there is a partial order of size  $\lambda$  which is universal for linear orders, namely that every linear order of the same size is embedded into it. Let  $M$  be a universal model of  $T$ , and let  $\varphi$  witness the strict order property. We check that  $P_M$  is a partial order universal for linear orders. Suppose that  $L$  is some linear order of size  $\lambda$ . By 5.3  $L$  is isomorphic so some suborder of  $P_{M_L}$ . Pick an elementary embedding  $h : M_L \rightarrow M$ . In particular,  $h$  preserves  $\varphi$ . So by 5.2, there is an embedding of  $P_{M_L}$  into  $P_M$ . The restriction of this embedding to (the isomorphic copy of)  $L$  is the required embedding. ⊢

**5.6 Remark** If there is a quantifier free formula in  $T$  which defines a partial order on models of  $T$  with arbitrarily long chains, then also a universal model of  $T$  in power  $\lambda$  in the sense of *ordinary embedding* implies the existence of a universal linear order in  $\lambda$ .

**5.7 Conclusions** Under the hypotheses of 3.6, there are no universal models in  $\lambda$  for the following theories:

- Partial orders (ordinary embeddings)
- Boolean algebras (ordinary embeddings).
- lattices (ordinary embedding)
- ordered fields (ordinary embeddings)

ordered groups (ordinary embeddings)  
 number theory (elementary embeddings)  
 the theory of  $p$ -adic rings (elementary embeddings)

**Proof** All these theories have the strict order property, and most have a definable order via a quantifier free formula. ⊣

## Appendix

We review here several notions and definitions from set theory and model theory.

### Set theory

A set of ordinals  $C$  is *closed* if  $\sup(C \cap \alpha) = \alpha$  implies  $\alpha \in C$ . A *club* of a cardinal  $\lambda$  is a closed unbounded set of  $\lambda$ . When  $\lambda$  is an uncountable cardinal, the intersection of two clubs contains a club. For details see any standard textbook like [Le].

### Model theory

A model  $M$  is a *submodel* of  $N$  if  $|M| \subseteq |N|$  (the universe of  $M$  is a subset of the universe of  $N$ ) and every relation or function of  $M$  is the restriction of the respective relation or function of  $N$ . A model  $M$  is an *elementary submodel* of  $N$  if it is a submodel of  $N$ , and for every formula  $\varphi$  with parameters from  $M$ ,  $M \models \varphi \Leftrightarrow N \models \varphi$ . An embedding of models  $h : M \rightarrow N$  is an *elementary embedding* if its image is an elementary submodel. A model  $M$  is  $\lambda$ -*saturated* if for every set  $A \subseteq |M|$  with  $|A| < \lambda$  and a type  $p$  over  $A$ ,  $p$  is realized in  $M$ . A model  $M$  is *saturated* if it is  $\|M\|$ -saturated. A model  $M$  is *special* if there is an increasing sequence of elementary submodels  $\langle M_\lambda : \lambda < \|M\| \text{ is a cardinal} \rangle$ ,  $M = \cup_\lambda M_\lambda$  and every  $M_\lambda$  is  $\lambda^+$ -saturated.

A formula  $\varphi(\bar{x}; \bar{y})$  has the *independence property* if for every  $n < \omega$  there are sequences  $\bar{a}_l$  ( $l < n$ ) such that for every  $w \subseteq n$ ,  $\models (\exists \bar{x}) \bigwedge_{l < n} \varphi(\bar{x}; \bar{a}_l)^{\text{if } l \in w}$

A first order theory  $T$  has the *independence property* if some formula  $\varphi(\bar{x}; \bar{y})$  has the independence property.

### A theory with universal models with respect to ordinary but not elementary embeddings

Let  $T$  be the theory of the following model  $M$ : the universe is divided into two infinite parts, the domain of the unary predicate  $P$  and its complement. The domain of  $P$  is the set  $\langle x_i : i < \omega + \omega \rangle$ . There are two binary relations  $R_1, R_2$ . For every pair  $\langle \alpha, \beta \rangle$  of ordinals below  $\omega + \omega$  there is a unique element  $y \in M$  such that  $\neg P(y)$  (think of  $y$  as an ordered pair) which satisfies  $R_1(y, x_\alpha) \wedge R_2(y, x_\beta)$  iff  $\alpha < \beta < \omega$  or  $\alpha < \omega \leq \beta$  or  $\alpha \geq \omega$  and  $\beta \geq \omega$ . So for the elements above  $\omega$  in the  $P$  part all possible ordered pairs exist, while those below  $\omega$  are linearly ordered by the existence of ordered pairs. Let  $\varphi(x_1; x_2) = P(x_1) \wedge P(x_2) \wedge (\exists y)(\neg P(y) \wedge R_1(y, x_1) \wedge R_2(y, x_2))$ . This formula witnesses that  $T$  has the strict order property. By Theorem 3.5 and Theorem 5.5  $T$  has no universal model with respect to elementary embeddings in any regular  $\lambda$ ,  $\aleph_1 < \lambda < \aleph_0$ . But with respect to ordinary embeddings  $T$  has a universal model in every infinite cardinality  $\lambda$ :



let  $M$  be any model of  $T$  of cardinality  $\lambda$  in which there is a set  $X = \{x_i : i < \lambda\}$  of elements in the domain of  $P$  for which all possible ordered pairs exist. If  $M'$  is any other model of cardinality  $\lambda$ , and  $h$  is a 1-1 function which maps the domain of  $P$  in  $M'$  into  $X$ , then  $h$  can be completed to an embedding of  $M'$  into  $M$ .  $\dashv$

**The consistency of not having universal models** Let us state it here, for the sake of those who read several times that was is easy but are still interested in the details:

**Fact** If  $\lambda$  regular,  $V \models \text{GCH}$ , for simplicity, and  $P$  is a Cohen forcing which adds  $\lambda^{++}$  Cohen subsets to  $\lambda$ , then in  $V^P$  there is no universal graph (linear order, model of a complete first order  $T$  which is unstable in  $\lambda$ ) in power  $\lambda^+$ .

**Proof** Suppose to the contrary that there is a universal graph  $G^*$  of power  $\lambda^+$ . We may assume, without loss of generality, that its universe  $|G^*|$  equals  $\lambda^+$ . As  $G$  is an object of size  $\lambda^+$ , it is in some intermediate universe  $V'$ ,  $V \subseteq V' \subseteq V^P$ , such that there are  $\lambda^{++}$  Cohen subsets outside of  $V'$ . So without loss of generality,  $G \in V$ . Let  $G$  be the graph with universe  $\lambda^+$  defined as follows: fix a 1-1 enumeration  $\langle A_\alpha : \lambda \leq \alpha < \lambda^+ \rangle$  of  $\lambda^+$  Cohen subsets of  $\lambda$ . A pair  $\alpha, \beta$  is joined by an edge iff  $\beta \geq \lambda$  and  $\alpha \in A_\beta$ , or  $\alpha \geq \lambda$  and  $\beta \in A_\alpha$ . By the universality of  $G^*$ , there should exist an embedding  $h : G \rightarrow G^*$ . Consider  $h \upharpoonright \lambda$ . This is an object of size  $\lambda$ , and therefore is in some  $V'$ ,  $V \subseteq V' \subseteq V^P$ , where in  $V'$  there are at most  $\lambda$  of the Cohen subsets. For every  $y \in G^*$ , the set  $\{x \in \lambda : h(x) \text{ is joined by an edge to } y\}$  is in  $V'$ . Pick an  $\alpha \geq \lambda$  such that  $A_\alpha$  is not in  $V'$  and set  $y = h(\alpha)$  to get a contradiction.  $\dashv$

The same proof is adaptable to the other cases.

Guessing clubs

**Proof of Fact 3.3** Let  $S_0$  be  $\{\delta < \lambda : \text{cf}\delta = \aleph_0\}$ . Suppose to the contrary that for every sequence  $\overline{C}$  as above there is a club  $C \subseteq \lambda$  such that for every  $\delta \in S_0 \cap C$ ,  $c_\delta \not\subseteq C$ . We construct by induction on  $\beta < \aleph_1$   $\overline{C}_\beta, C_\beta$ .  $\overline{C}_\beta = \langle c_\delta^\beta : \delta \in S_0 \rangle$  is such that for every  $\delta \in C_\beta \cap S_0$ ,  $c_\delta^\beta$  is a club of  $\delta$  of order type  $\omega$ , and  $c_\delta^\beta \not\subseteq C_\beta$ . Furthermore, letting  $c_\delta^\beta = \langle \alpha_n^{\beta, \delta} : n < \omega \rangle$ , where  $\alpha_n^{\beta, \delta} \leq \alpha_m^{\beta, \delta}$  if  $n < m$ , we demand that  $\alpha_n^{\beta+1, \delta} = \sup\{\alpha_n^{\beta, \delta} \cap C_\beta\}$  if this intersection is non-empty, and  $\alpha_n^{\beta+1, \delta} = 0$  otherwise. When  $\beta$  is limit, we demand that  $\alpha_n^{\beta, \delta} = \min\{\alpha_n^{\gamma, \delta} \text{ for } \gamma < \beta\}$ . Let  $\overline{C}_0$  be arbitrary. At the induction step pick  $C_\beta$  which demonstrate that  $\overline{C}_\beta$  is not as required by the fact and define each  $c_\delta^{\beta+1}$  by the demand above. Note that for club many  $\delta$ 's the resulting  $c_\delta^{\beta+1}$  is cofinal in  $\delta$ , so without loss of generality this is so for every  $\delta \in C'_{\beta+1}$ .

It is straightforwaed to verify that for all  $\beta < \gamma < \omega_1$ ,  $\delta \in S_0$

1. For all  $\delta \in S_0 \cap C_\beta$ ,  $c_\delta^{\beta, \delta} \subseteq \delta$  is a club of  $\delta$  of order type  $\omega$ .
2. For all  $\delta \in S_0$ ,  $c_\delta^\beta \setminus \{0\} \subseteq C_{\beta+1}$ .
3. For all  $\delta \in S_0 \cap C_\beta$ ,  $c_\delta^\beta \not\subseteq C_\beta$
4.  $\alpha_n^{\gamma, \delta} \leq \alpha_n^{\beta, \delta}$

Let  $C = \bigcap_{\beta < \omega_1} C_\beta$ . Pick  $\delta_0 \in C \cap S_0$ . Then  $C \cap \delta$  is unbounded in  $\delta$  and of order type  $\omega$ .

Furthermore, for every  $\beta < \omega_1$ ,  $c_{\delta_0}^\beta \not\subseteq C_\beta$ .

But on the other hand, there is a  $\beta_0$  such that for all  $\beta_0 < \beta < \gamma$ ,  $c_{\delta_0}^\beta = c_{\delta_0}^\gamma$ , because of 4. - a contradiction.  $\dashv$

Now for the case of uncountable cofinality. We recall

**3.4 Fact:** If  $\text{cf}\kappa = \kappa < \kappa^+ < \lambda = \text{cf}\lambda$ , then there is a sequence  $\overline{C} = \langle c_\delta : \delta \in S_\kappa^\lambda \rangle$ ,  $S_\kappa^\lambda$  being the set of members of  $\lambda$  with cofinality  $\kappa$ , where  $c_\delta$  is a club of  $\delta$  of order type  $\kappa$  with the property that for every club  $E \subseteq \lambda$  the set  $S_E = \{\delta \in S_\kappa^\lambda : c_\delta \subseteq E\}$  is stationary. Furthermore, if there is a square sequence on  $S_\kappa^\lambda$ ,  $\overline{C}$  can be chosen to be a square sequence.

**Proof:** This proof is actually simpler than that of the previous fact. Start with any sequence  $\overline{C}_0 = \langle c_\delta^0 : \delta \in S_\kappa^\lambda \rangle$ . By induction on  $i < \kappa^+$  define  $\overline{C}_i$  as follows: if  $\overline{C}_i$  has the property of guessing clubs, we stop. Otherwise there is a club  $E_i$  such that  $E_i$  is not guessed stationarily often. This means there is a club  $C_i$  such that  $\delta \in C_i$  implies that  $c_{i_\delta} \not\subseteq E_i$ . We may assume that  $C_i = E_i$ . Let  $c_\delta^{i+1} = c_\delta^i \cap E_i$ . If  $i$  is limit,  $c_\delta^i = \bigcap_{j < i} c_\delta^j$ . Suppose the induction goes on for  $\kappa^+$  steps. Let  $E = \bigcap E_i$ . For every  $\delta \in E \cap S_\kappa^\lambda$ ,  $E \cap C_\delta^0$  is a club of  $\delta$ . Therefore for stationarily many points  $\delta$ ,  $c^{\kappa^+}$  is a club of  $\delta$ , say that this holds for all  $\delta \in S \subseteq S_\kappa^\lambda$ . Clearly, for every  $\delta$  there is an  $i$  such that for every  $i < j$   $C_\delta^i = C_\delta^j$ , because the size of  $c_\delta^0$  is  $\kappa$  and  $\kappa^+$  is regular. But on the other hand, for every  $\delta \in S$  and  $\zeta < \kappa^+$ ,  $c_\delta^\zeta \not\subseteq E_i$ , while  $c_\delta^{\zeta+1} \subseteq E_i$ . A contradiction. Thus the induction stops before  $\kappa^+$ , and the resulting sequence guesses clubs.

What about the square property? If there is a square sequence on  $S_\kappa^\lambda$ , let in the proof  $\overline{C}_0$  be a square sequence. Notice that the operation of intersecting the  $c_\delta$  with a club  $E$  preserves the property: suppose  $\delta_1 < \delta_2$  and  $\delta_1 \in \text{acc}(\gamma_{\delta_2}) \cap E$ . Then clearly  $\delta_1 \in \text{acc}(c_{\delta_2})$ . Therefore, by the square property,  $c_{\delta_1} = \gamma_{\delta_2} \cap \delta_1$ . Intersecting both sides of the equation with  $E$  yields that  $c_{\delta_1} \cap E = c_{\delta_2} \cap E \cap \delta_1$ . Therefore the proof is complete.  $\dashv$

## References

- [Le] Azriel Levy, **Basic Set Theory**, Springer Verlag, 1979.
- [CK] C. C Chang and J. J. Keisler **Model Theory**, North Holland 1973.
- [Sh-a] Saharon Shelah, **Classification Theory and the Number of non-Isomorphic Models**, North Holland, 1978.
- [Sh-c] Saharon Shelah, **Classification Theory and the Number of non-Isomorphic Models, Revised**, North Holland Publ. Co., 1990.
- [GrSh 174] R. Grossberg and S. Shelah, *On universal locally finite groups*, Israel J. of Math. **44** , (1983), 289-302.
- [Sh-g] Saharon Shelah, **Cardinal Arithmetic**, to appear.

[Sh 93] Saharon Shelah, *Simple Unstable Theories*, Annals of Math. Logic **19** (1980), 177–204.

[Sh 175] Saharon Shelah, *On Universal graphs without instances of CH* Annals of Pure and Applied Logic **26**, (1984), 75–87.

[Sh 175a] Saharon Shelah, *Universal graphs without instances of CH, revised*, Israel J. Math, **70** (1990) 69–81.

[M] Alan H. Mekler, *Universal Structures in Power  $\aleph_1$*  Journal of Symbolic Logic, accepted.